

EXTENDED CREDITGRADES WITH LOCAL
VOLATILITY

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Introduction

The development of liquid markets in credit products has made it clear that stock options and credit default swaps (CDS) written on the same reference company can be valued and estimated jointly. Basically, there are two types of credit models: intensity based model, which models default via a Cox process and structural model, which models default when the firm value hitting default barrier. In this paper, we build up a new structural model extending the CreditGrades model with Local Volatility. This new model fits market implied volatility surface better than classic Merton's Model, and CDS spreads could be solved numerically from option data.

In the first section, we reviewed the historic development on structural model. Since Merton first raised structural model in 1974, John Hull et al have extended it with various dynamics such as stochastic volatility and jump diffusions. Furthermore, Finger et al introduced the CreditGrades model in which default barriers are uncertain.

In the second section, we introduced our extended CreditGrades model with local volatility dynamics. We discussed the model assumptions and provided a calibration formula using market option data. Survival Probability and CDS spreads can be solved under our model via the PDE in this chapter, although it seems numerical approach is necessary.

Empirical study on General Motors has been shown in the third section. We calibrated our model to the market implied volatility of General Motors from year 2005. Figures of local volatility surfaces and survival probabilities demonstrated certain features of our model, which we discussed carefully in that chapter.

Finally, we put out mathematical deduction on calibration formula and Matlab codes, as well as original market data in appendix.

Chapter 1

Literature Review

The structural model is initially introduced by Merton(1974) in which the equity is modeled as a call option on the firm value, and the firm value is assumed to evolve as pure diffusion. Default happens when the firm value hits the default barrier, and at the same time as the equity price goes to zero. This model provides a very intuitive link between a firm's fundamental conditions and the value of its derivatives, and incorporates volatility skewness implicitly via leverage effect, which will be further discussed in Model Setup section.

Empirical studies showed its failure to explain short-term credit spreads; because in Merton's Model, it always takes some time for the firm value to diffuse to default barrier (Eg. Peter Carr, 2009). Hence, Finger et al (2002) introduced The CreditGrades Model where they made the default barrier uncertain. Sepp (2006) equipped the underlying firm value process with stochastic variance and jumps. These approaches all somewhat resolved the problem of short term credit spread. More involved implementation and calibration schemes are needed to fit each model.

We introduce the model setup for each model in the first section. In the second section, we discuss the option pricing and survival probability calculation under different frameworks. Finally, we include several calibration methods to fit these models to market data.

1.1 Model Setup

A. Geometric Brownian Motion and CreditGrades

Let $V(t)$ be firm value process, following driftless Geometric Brownian

Motion,

$$dV_t = V_t \sigma_V dW_t,$$

the σ And default barrier $D = R \cdot B$. Here B refers to the corresponding firm debt, and R is the recovery rate. In Hull and White's paper (2003), default events are defined only at maturity time T . Default happens as soon as the firm value hits the barrier and S_t remains 0 afterwards. D becomes an absorbing barrier in this case.

If R is reduced to a constant, then this is the classic Merton's model, where the equity may be regarded as a European call option. The CreditGrades Model models the recovery rate R to be a lognormally-distributed random variable with mean \bar{R} and standard deviation λ so that

$$RB = \bar{R}B \exp(\lambda Z - \lambda^2/2),$$

where Z is a standard normal r.v. independent of W_t . Therefore, equity is equivalent to a down-and-out barrier call option written on the firm value. The uncertainty of the recovery rate is consistent with reality and it compensates for low short-term credit spread at the same time. Model implied volatility of equity process is $\frac{S(t)+D(t)}{S(t)} \cdot \sigma$. This is the leverage effect which contributes to volatility skewness.

B. Stochastic Volatility

One disadvantage of CreditGrades with regular diffusion is that it fails to capture the full picture of volatility structure due to a constant σ . To extend Finger's CreditGrades (2002), Sepp (2005) wrote the firm value process under the stochastic volatility dynamics based on Heston's model(1993):

$$\begin{cases} dV(t)/V(t) = (r(t) - d(t))dt + \sqrt{v(t)}dW(t) \\ dv(t) = \kappa_v(v_\infty - v(t))dt + \varepsilon_v\sqrt{v(t)}dZ(t) \\ V(0) = S(0) + D(0), v(0) = v_0, \end{cases}$$

where $v(t)$ is the variance of the firm's return, $d(t)$ is dividend rate, $r(t)$ is interest rate, v_∞ is the long-term variance, κ_v is a mean-reverting rate, ε_v is a volatility of instantaneous variance. And W_t and Z_t are two independent Brownian motion.

The corresponding stock price process is:

$$\begin{cases} dS(t) = (r(t) - d(t))S(t)dt + \sqrt{v(t)}(S(t) + D(t))dW(t) \\ dv(t) = \kappa_v(v_\infty - v(t))dt + \varepsilon_v\sqrt{v(t)}dZ(t), v(0) = v_0. \end{cases}$$

One advantage of this model is that the independent Brownian motion of volatility process introduces volatility smile; and that the stochastic volatility process combined with leverage effect makes volatility smile asymmetric.

This asymmetric volatility smile agrees with empirical observations.

C. Double-Exponential Jump-Diffusion

Sepp introduced a Jump-Diffusion model in the same paper as above:

$$\begin{cases} dV(t)/V(t) = (r(t) - d(t) - \lambda\alpha v(t))dt + \sqrt{v(t)}dW(t) + (e^J - 1)dN(t), \\ V(0) = S(0) + D(0), \end{cases}$$

where $N(t)$ is a Poisson process with deterministic intensity $\lambda v(t)$, and J is a random jump with double-exponential distribution. α is chosen to make the discounted firm value process a martingale.

The SDE for the stock price is then:

$$dS(t) = ((r(t) - d(t))S(t) - \lambda\alpha v(t)(S(t) + D(t)))dt + \sqrt{v(t)}(S(t) + B(t))dW(t) + (S(t) + D(t))(e^J - 1)dN(t).$$

The solution to the above SDE is:

$$S(t) = (S(0) + D(0))e^{\int_0^t (r(s) - d(s) - (\lambda\alpha + \frac{1}{2})v(s))ds + \int_0^t \sqrt{v(s)}dW(t) + \sum_{n=1}^{N(t)} J_n} - D(t).$$

The introduction of jumps further contribute to the short-term default probability, although not all model parameters are directly observable in the market.

1.2 Survival Probability and Credit Pricing

Various methods are provided by many researchers to solve the different structural models listed above. Here we briefly introduce the main contributions and solutions.

A. Geometric Brownian Motion and Credit Grades

For the simple version of Merton's model (default happens only at maturity and recovery is constant), solutions are very easy to get. The equity is a call on firm's value, with strike D , and has payoff $S_T = \max(V_T - D, 0)$. Hence, we can write

$$E_0 = V_0 N(d_1) - De^{-rT} N(d_2),$$

where

$$d_1 = \frac{\log(V_0 e^{rT}/D)}{\sigma_V \sqrt{T}} + 0.5\sigma_V \sqrt{T}; \quad d_2 = d_1 - \sigma_V \sqrt{T}.$$

The probability of default is given by $N(-d_2)$. In addition, the option on the equity could be regarded as a compound option. Hull and White (2003) shows the time zero value of a vanilla put on the equity with strike K and expiry time $\tau < T$ on the equity is

$$P = De^{-rT}M(-a_2, d_2; -\sqrt{\tau/T}) - V_0M(-a_1, d_1; -\sqrt{\tau/T}) + Ke^{-rT}N(-a_2), \quad (1.1)$$

where

$$a_1 = \frac{V_0/V_\tau^*e^{-r\tau}}{\sigma_V\sqrt{\tau}} + \frac{1}{2}\sigma_V\sqrt{\tau}; \quad a_2 = a_1 - \frac{1}{2}\sigma_V\sqrt{\tau},$$

M is the cumulative bivariate normal distribution function, and V_τ^* is the corresponding asset value at time τ , when the equity value is exactly equal to K .

For CreditGrades, default doesn't occur only if firm value stays above the uncertain barrier, i.e. for any $0 < t < T$,

$$V_0e^{\sigma W_t - \sigma^2 t/2} > \bar{L}De^{\lambda Z - \lambda^2/2}$$

CreditGrades can be solve by a change of variable. Define $X_t = \sigma W_t - \lambda Z - \frac{\sigma^2 t}{2} - \frac{\lambda^2}{2}$, then the survival condition becomes

$$X_t > \log\left(\frac{\bar{L}D}{V_0}\right) - \lambda^2.$$

Using the Reflection Principle, the survival probability until time t is then

$$P(t) = N\left(-\frac{A_t}{2} + \frac{\log(d)}{A_t}\right) - d \cdot N\left(-\frac{A_t}{2} - \frac{\log(d)}{A_t}\right),$$

where

$$d = \frac{V_0e^{\lambda^2}}{LD}, \quad A_t^2 = \sigma^2 t + \lambda^2.$$

Similarly, the vanilla call option on equity is nothing else but a down-and-out barrier call option:

$$C(t, S_t) = C_{BS}(t, T, S(t)+D(t), K+D(T)) - \frac{S(t) + D(t)}{D(t)} C_{BS}(t, T, \frac{(D(t))^2}{S(t) + D(t)}, K+D(T)),$$

where $C_{BS}(t, T, S, K)$ is the standard Black-Scholes price of a call option with same maturity, interest rate, and other model parameters.

B. Stochastic Volatility

By Feynman-Kac, the call option price $C(t, S)$ solves the following PDE under the stochastic variance model:

$$\begin{cases} \frac{\partial C}{\partial t} + \frac{1}{2}v(t)(S + D(t))^2 \frac{\partial^2 C}{\partial S^2} + (r(t) - d(t))S \frac{\partial C}{\partial S} + \kappa_v(v_\infty - v)C_v + \frac{1}{2}\varepsilon_v v \frac{\partial^2 C}{\partial v^2} - r(t)C = 0, \\ C(t, 0) = 0, \quad C(T, S) = \max(S - K, 0). \end{cases}$$

Sepp (2006) solved this PDE by a combination of Fourier transforms and the method of images. The final result is:

$$C(t, S_t) = (D(T) + K)e^{-\int_t^T r(s)ds} Z(\tau, y),$$

where

$$\begin{aligned} Z(\tau, y) &= e^y - e^b - \frac{e^{\frac{1}{2}y}}{\pi} \int_0^\infty \frac{e^{A(t,k)+B(t,k)v} (\cos(yk) - \cos((y-2b)k))}{k^2 + \frac{1}{4}} dk, \\ y &= \ln\left(\frac{S + D(t)}{D(T) + K}\right) + \int_t^T (r(s) - d(s))ds, \\ b &= \ln\left(\frac{D(t)}{D(T) + K}\right) + \int_t^T (r(s) - d(s))ds, \\ B(t, k) &= -(k^2 + \frac{1}{4}) \frac{1 - e^{-(T-t)\zeta}}{\psi_- + \psi_+ e^{-(T-t)\zeta}}, \\ A(t, k) &= -\frac{\kappa_v v_\infty}{\varepsilon_v^2} \left[(T-t)\zeta_+ + 2 \ln\left(\frac{\psi_- + \psi_+ e^{-(T-t)\zeta}}{2\zeta}\right) \right], \\ \psi_\pm &= \mp \kappa_v + \zeta, \quad \zeta = \sqrt{\kappa_v^2 + \varepsilon_v^2(k^2 + \frac{1}{4})}. \end{aligned}$$

Under this model, the survival probability $Q(t, S_t; T)$ can also be solved via Fourier transforms and method of images for the following PDE:

$$\begin{cases} Q_t + \frac{1}{2}v(t)(S + D(t))^2 Q_{SS} + (r(t) - d(t))S Q_S + \kappa_v(v_\infty - v)Q_v + \frac{1}{2}\varepsilon_v v Q_{vv} = 0, \\ Q(t, 0) = 0, \quad Q(T, S) = 1. \end{cases}$$

Relevant results and details could be found at Sepp's (2006) paper.

C. Double-Exponential Jump-Diffusion

Under this process, $Q(t, S_t; T)$ solves the following PIDE:

$$\begin{cases} Q_t + \frac{1}{2}v(t)(S + D(t))^2 Q_{SS} + (r(t) - d(t))S Q_S + \\ + \lambda v(t) \int_{-\infty}^\infty [P((S + D(t))e^J) - P(S + D(t))] \varpi(J) dJ = 0, \\ Q(t, 0) = 0, \quad Q(T, S) = 1, \end{cases}$$

and $C(t, S_t)$ solves the following PIDE:

$$\begin{cases} C_t + \frac{1}{2}v(t)(S + D(t))^2C_{SS} + (r(t) - d(t))SC_S + \\ + \lambda v(t) \int_{-\infty}^{\infty} [C((S + D(t))e^J) - C(S + D(t))] \varpi(J)dJ - r(t)C = 0, \\ C(t, 0) = 0, C(T, S) = \max\{S - K, 0\}. \end{cases}$$

Both formulas are solved in the same paper of Sepp (2006) by Laplace transform and closed-form solutions are obtainable similar to the stochastic volatility case in the above of jumps.

1.3 Calibration Methods

A. Geometric Brownian Motion and CreditGrades

In Merton's model, the credit spread is given by $s = \ln [N(d_2) + N(-d_1)]/T$, where d_1, d_2 are defined as standard Black-Scholes formula. That is to say, s is totally dependent on the leverage L , firm volatility σ_A , and time to maturity T . L can be read from the firm's balance sheet and T is apparent. The firm's volatility σ_A is deduced directly from Ito's formula and equity volatility:

$$\sigma_E = \frac{\sigma_A N(d_1)}{N(d_1) - LN(d_2)}.$$

Hull and White (2003) improved this calibration method by using option implied volatility. Formula (1.1) combined with the BS-formula defines a relationship between T, L, σ_A and implied volatility. Thus two pairs of T and implied volatilities indicate the firm's leverage and volatility.

Hull and White's scheme outperforms the traditional one. However, due to the volatility skewness, different combinations of maturities and volatility may result in different solutions.

Finger and Stamicar (2005) extended Hull and White's calibration methods to CreditGrades in three ways. The first scheme is estimating debt-per-share from balance sheet data and firm volatility from at-the-money volatility. The second method is calculating both asset volatility and leverage from option implied volatilities. The last method adopts one CDS spread and an at-the-money volatility to calibrate model parameters. These methods provide choices when calibrating market data, while none can fully include the whole volatility surface.

B and C. Stochastic Volatility and Jump Diffusion

Simple stochastic volatility or jump diffusion models are hard to calibrate, because the equivalent risk-neutral martingales are not unique. Thus, Sepp assumes a time-inhomogeneous variance process as:

$$v(t) = v_\infty + (v_0 - v_\infty)e^{-\kappa_v t},$$

and calibrates it to market data via optimization.

Chapter 2

CreditGrades with Local Volatility

In this chapter, we build up a new model, incorporating local volatility structure into CreditGrades. For all structural models, there are always two problems to be considered. One is the ability to fully recover the volatility surface; the other is fitting short-term credit spreads correctly. Here I deduce a calibration formula similar to Dupire's method to incorporate the whole volatility surface to my model; and assume random default barrier to increase short term default probability.

We can obtain CDS spreads from all put option prices. This makes sense to the real markets as well, because put options may imply the information on default risk, and actually people do use deep out-of-money put as an against default, especially when a CDS is unavailable. In order to get CDS spreads, firstly I calibrate the firm's local volatility to vanilla put data using put options with different strikes and maturities; then I calculate the survival probability by solving the PDE numerically. Finally, credit spreads are obtained by comparing fixed legs and floating legs in a CDS contract.

2.1 Model Setup

The firm value is modelled as:

$$dV(t)/V(t) = (r_t - q_t)dt + \sigma(t, V_t)dW_t,$$

where $V(t)$ is the firm value process, $r(t)$ and $q(t)$ are the interest rate and dividend rate respectively, and $\sigma(t, V_t)$ is the local volatility of firm value process. Here I assume firm value process follows geometric Brownian Motion with time dependent interest rate and dividend rate as well as volatility

depending on the time t and stock price S_t .

The default barrier is defined as:

$$D(t) = RB(t) = B_0 \exp(\int_0^t (r_s - q_s) ds),$$

where $B(t)$ is the firm's liability or debt, and R is the recovery rate. Thus, $D(t)$ is the default barrier, which we assume increasing at the same rate as firm value. The reason behind this is to make the firm's leverage ratio constant, which is consistent with reality and eases the calculation later on.

The firm value is

$$V(t) = S(t) + D(t), \quad \text{given } \tau > t$$

where the equity value $S(t)$ is defined as the residual part of the firm value when we deduct the recovery part before default. Once the firm value hits the barrier, equity becomes zero and stays zero forever.

The recovery rate is defined as:

$$RB = \bar{R}B \exp(\lambda Z - \lambda^2/2)$$

If R is random, then $D(t)$ becomes a random barrier and CreditGrades follows. Here I assume R is log normal distributed, with mean \bar{R} and variance λ .

The default time is

$$\tau := \inf\{u : V(u) < D(u)\},$$

i.e. the first time firm value hits the debt barrier $D(t)$; τ is a stopping time.

2.2 Recovering Local Vol from option markets

We make two additional assumptions here, in order to get a closed-form calibration formula for local volatility:

Assumption 2.1. *For a put option expires at time T , default of the underlying firm only happens at time T .*

This is the same assumption Hull and White (2003) make. Although there is before T version of default definition, that default happens whenever the barrier is touched, we can prove that when combined with the Second assumption, the solution of time T default is consistent with default at any time.

Assumption 2.2. For local volatility at the default barrier, assume $\sigma(t, D_t) \equiv 0$.

The dynamics of S_t are:

$$dS_t = (r(t) - q(t))S_t dt + \sigma(t, S(t) + D(t))(S(t) + D(t))dW(t).$$

Then, $\sigma(t, D_t) = 0$ ensures that whenever default happens, S_t stays at zero till maturity. Hence, we only need to consider the default probability at the terminal time. This explains why Assumptions (2.1) is reasonable.

Under Assumptions (2.1), the price of an equity put option could be written as a function of firm value V_t , in a way slightly different from the classic case:

$$P(T, K) = e^{-\int_0^T r_u du} \cdot \int_{D_T}^{K+D_T} (K+D_T-y) \cdot p(0, T; V_0, y) dy + e^{-\int_0^T r_u du} \cdot \int_0^{D_T} K \cdot p(0, T; V_0, y) dy,$$

where $p(0, T; V_0, V_T)$ is the probability transition density function from V_0 at time 0 to V_T at time T .

We differentiate it with respect to K and T respectively, and by Fokker-Planck equation:

$$\frac{\partial P}{\partial T}(0, T; V_0, y) + \frac{\partial}{\partial y} [(r_T - q_T) \cdot y \cdot p(0, T; V_0, y)] - \frac{1}{2} \frac{\partial^2}{\partial y^2} [\sigma^2(t, y) \cdot y^2 p(0, T; V_0, y)] = 0.$$

We obtain:

$$\begin{aligned} P_T(K, T) = & -r(T)P(K, T) - (r(T) - q(T))K \frac{\partial}{\partial K} P(T, K + D_T) + \\ & + \frac{1}{2} \sigma^2(t, K + D_T)(K + D_T)^2 \frac{\partial^2}{\partial K^2} P(T, K + D_T) - \frac{1}{2} \sigma^2(t, D_T) D_T^2 \frac{\partial^2}{\partial K^2} P(T, D_T). \end{aligned}$$

From Assumption (2.1), we are able to obtain the closed-form calibration formula for $\sigma(t, V_t)$:

$$\sigma^2(T, K + D_T) = \frac{P_T(T, K) + q_T \cdot P(T, K) + K \cdot P_K(T, K) \cdot (r_T - q_T)}{\frac{1}{2} \cdot P_{KK}(T, K) \cdot (K + D_T)^2}. \quad (2.1)$$

The calculation method is similar to Dupire's formula. The derivation is presented in Appendix A.

An alternative calibration method to put price calibration is to use implied volatility $\sigma_{imp}(K, T)$. The formula can be easily deduced from above as:

$$\sigma^2(T, K + D_T) = \frac{\frac{\sigma_{imp}}{T} + 2\frac{\partial\sigma_{imp}}{\partial T} + 2(r(T) - q(T))K\frac{\partial\sigma_{imp}}{\partial x}}{(K + D_T)^2 \left(\frac{\partial^2\sigma_{imp}}{\partial x^2} - d_1\sqrt{T} \left(\frac{\partial\sigma_{imp}}{\partial x} \right)^2 + \frac{1}{\sigma_{imp}} \left(\frac{1}{K\sqrt{T}} + d_1\frac{\partial\sigma_{imp}}{\partial K} \right)^2 \right)}.$$

where

$$d_1 = \frac{\ln(S/K) + \left(r(T) - q(T) + \frac{1}{2}\sigma_{imp}^2 \right) T}{\sigma_{imp}\sqrt{T}}.$$

Since the formula of put price calibration is comparatively simpler, we use the local volatility data generated from put price to continue calculating default probability.

2.3 Default probability and CDS pricing

The survival probability $Q(t, S_t; T)$ is defined as the probability of firm's survival till terminal time T , given stock price S_t at time t . Since $Q(t, S_t; T) = \mathbb{E} [\mathbf{1}_{\{S(u) > 0; t < u < T\}}]$, it's a martingale and by Feynman-Kac,

$$\begin{cases} \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial S} \cdot S_t \cdot (r_t - q_t) + \frac{1}{2} \cdot \frac{\partial^2 Q}{\partial S^2} \cdot \sigma^2(t, S_t + D_t) \cdot (S_t + D_t)^2 = 0 \\ Q(t, 0) = 0; \quad Q(T, S) = 1 \end{cases}$$

Since $\sigma(t, S_t + D_t)$ is dependent on both S_t and t , it may not be possible to get a closed-form solution for survival probability. Instead, we look for a numerical solution using finite difference methods.

Given the survival probability, we can price a CDS:

- A Continuous Payment of Protection
- Starting date t , Maturity time T
- A Fixed CDS spread $\Delta CDS(T)$; Notional N ; Recovery rate R
- Time value of money $B(t, T) = \exp^{-\int_t^T r(u)du}$

Fixed Leg payment is:

$$\mathbb{E} \left[\int_t^T B(t, u) \cdot \Delta CDS(T) \cdot N \mathbf{1}_{\{\tau > u\}} du \right] = \Delta CDS(T) \cdot N \cdot \int_t^T B(t, u) \cdot Q(t, S_t; u) du.$$

And Floating Leg is:

$$\begin{aligned}\mathbb{E} \left[\int_t^T B(t, u)(1 - R)N \mathbf{1}_{\tau \in du} \right] &= - (1 - R)N \int_t^T B(t, u) dQ(t, S_t; u) \\ &= (1 - R)N \left[1 - B(t, T)Q(t, S_t; T) + \int_t^T Q(t, S_t; u) dB(t, u) \right].\end{aligned}$$

By comparing them, we get the CDS spread as:

$$\Delta CDS(T) = (1 - R) \frac{\left[1 - B(t, T)Q(t, S_t; T) + \int_t^T Q(t, S_t; u) dB(t, u) \right]}{\int_t^T B(t, u) Q(t, S_t; u) du},$$

which is explicitly expressed as a function of $Q(t, S_t; T)$, r_t and R .

Chapter 3

Empirical Study

3.1 Interpolating Implied Volatility Surface

The data we choose to implement on Local Volatility CreditGrades Model are option implied volatilities of General Motors. I obtain my data from Sepp's paper (2006) and from Financial Institutions. Time $t = 0$ is November 8, 2005. The spot price is $S(0) = 25.86$, dividend yield is $d = 0.078$. From Stamicar-Finger (2005), the debt-per-share $B(0) = 65$ and recovery rate R is around 0.5.

As we only have discrete implied volatility data (See Appendix C), which is very sparse when strike is low and maturity is close to $t = 0$, we need to interpolate it to get a full surface. One way of doing this is by linear estimation, and the other way is by estimating it as second-order Taylor Expansion.

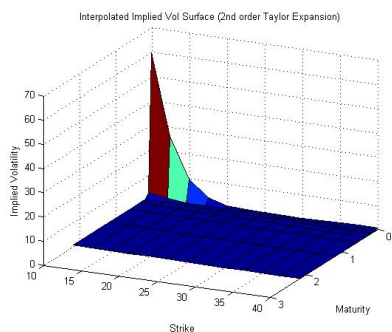


Figure 3.1: Second-order Taylor Expansion

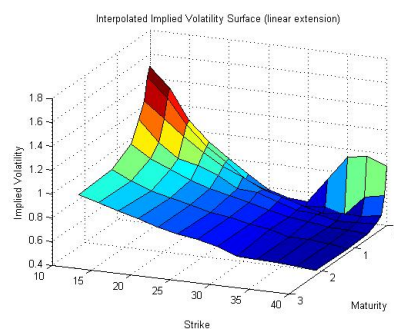


Figure 3.2: Linear Estimation

Comparing between each figure, the volatility in the linear estimation seems more reasonable, within a range from 0.4 to 1.5. So I choose the Figure (2.1) to estimate Local Volatility.

3.2 Calibrating Local Volatility

Before implementation, we first denote $\tilde{\sigma}(t, S_t) = \sigma(t, S_t + D_t)$. Although D_t is random in this case, the following PDE for survival probability only contains $\sigma(t, S_t + D_t)$. This enables us to generate a series of R with iid distribution first, and then fix one R to get $\tilde{\sigma}(t, S_t)$ and do it repeatedly to obtain the whole survival structure by taking average over each R s.

Using Equation (2.1) and discretizing it centrally we get:

$$\tilde{\sigma}^2(T, K) = \frac{q(T) \cdot P(i, j) + K(i) \cdot (r(j) - q(T)) \cdot \frac{P(i+1, j) - P(i-1, j)}{K(i+1) - K(i-1)} + \frac{P(i, j+1) - P(i, j-1)}{T(j+1) - T(j-1)}}{0.5 \cdot (K(i) + D(j))^2 \cdot \frac{P(i+1, j) - 2P(i, j) + P(i-1, j)}{(K(i+1) - K(i-1))^2}}$$

And for grids at boundary, we follow a similar discretization scheme but upwind or downwind. Finally, for $\tilde{\sigma}(t, 0)$, we follow our Assumption (2.2) to set it 0.

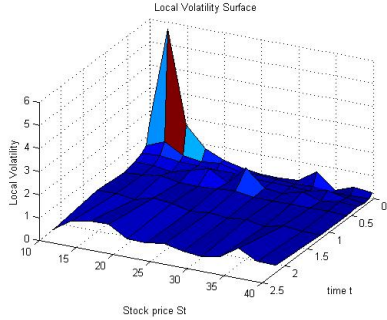


Figure 3.3: Constant $R = 0.5$

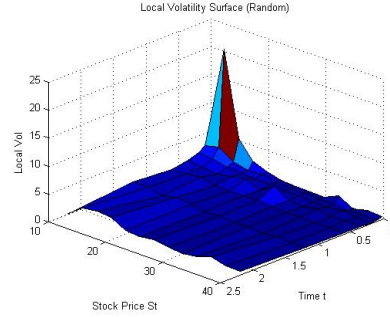


Figure 3.4: Lognormally Distributed R

Both surfaces fit implied volatility surface under their models. As shown by figures (3.3) and (3.4), local volatility for the random barrier case is comparatively higher than constant situation. This is something we can expect, because higher local volatility leads to a higher probability of default, and higher default risk is expected when default barrier is random. Therefore, this satisfies our intention to bring in CreditGrades to enhance short-term credit spreads. We will see this more explicitly after getting survival probability.

3.3 Recovering Survival Probability

Finally, we need to solve Equation (2.3) numerically. Since R is independent of the underlying Brownian Motion of the firm value process, we can seek a numerical method combining Monte Carlo and Finite Difference. First we generate a series of recovery rates R , which are distributed lognormally as required; then for each R , we use an explicit Finite Difference scheme to deduce solution to the PDE.

The explicit scheme is:

$$Q(i, j) = Q(i, j + 1) + \Delta t_j \frac{Q(i + 1, j + 1) - Q(i - 1, j + 1)}{K(i + 1) - K(i - 1)} (r(j) - q(T)) S(i) \\ + \Delta t_j \cdot 0.5 \cdot \frac{Q(i + 1, j + 1) - 2Q(i, j + 1) + Q(i - 1, j + 1)}{(K(i + 1) - K(i))^2} \tilde{\sigma}^2(i, j + 1) (S(i) - D(j))$$

Here $Q(i, j)$ denotes the survival probability at $S = S(i)$ and $t = t(j)$. i is from 1 to 14 and j is from 1 to 8. We set $Q(0, j) \equiv 0$ and $Q(i, 8) \equiv 1$, with the exception that $Q(0, 8) = 0$.

In consideration of sparse data in time (only 7 time steps in all!), I divide each time step into 1000 sub-steps and this enables default risk at boundary to have an influence on the survival probability of higher stock price within finite time steps.

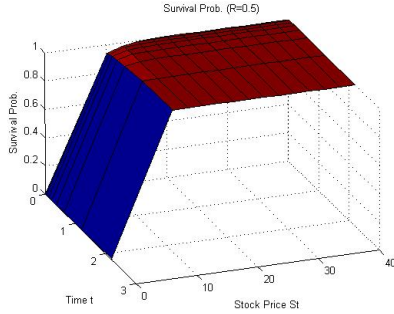


Figure 3.5: Constant $R = 0.5$

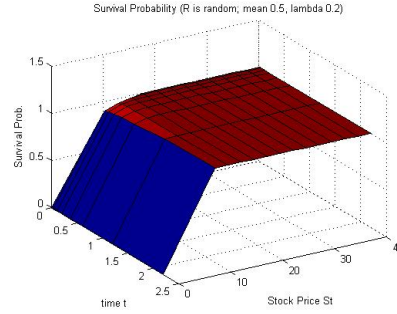


Figure 3.6: Random R , mean 0.5, $\lambda = 0.2$

Comparing Figure (3.5) and (3.6), we illustrate that incorporating the random default barrier using CreditGrades raises the short-term credit spreads. For example, when the stock price is at 10 and the date is Nov-05, the survival probability in the random and constant barrier models are 0.7924 and 0.8132 respectively.

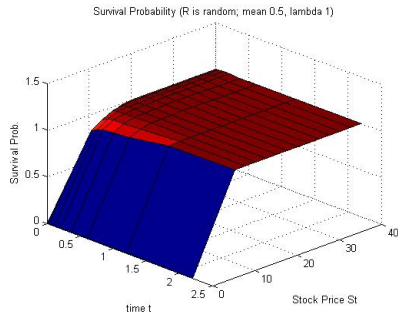


Figure 3.7: Random R , mean 0.5, $\lambda = 1$

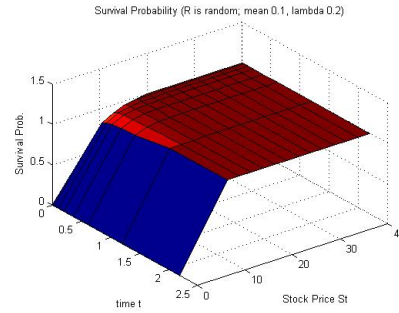


Figure 3.8: Random R , mean 0.1, $\lambda = 0.2$

Comparing Figure (3.6) and (3.7), we see that an increasing in λ also contributes to higher default risk as we can expect.

Finally, slightly out of our expectation, changing the mean of recovery from 0.5 in Figure (3.6) to 0.1 in (3.8) makes no material difference. Even though, the positive relationship between higher recovery rate and default risk still remains. For example, when the stock price is at 10 and the date is Nov-05, survival probabilities are 0.7924 for $\bar{R} = 0.5$ and 0.7932 for $\bar{R} = 0.1$. The reason for this non-material difference may due to the fact that default only happens at extreme cases; hence variance of recovery plays a more important role in default than the mean recovery.

Chapter 4

Conclusion

We have extended classic CreditGrades model with local volatility structures. We have derived the calibration formula using market put prices, and have written out the PDE to solve survival probability under our model. Implementation on General Motors has shown that higher variance of recovery rate will result in higher short-term credit spreads. Our local volatility model fits the market data as well.

Due to the time limitation, we have not computed the CDS term structure. For further study, it may be interesting, to compare our CreditGrades model with local volatility with the intensity model with local volatility by Peter Carr, and with the CreditGrades model with stochastic volatility by Artur Sepp.

Chapter 5

Appendix

5.1 Appendix A. Local Volatility Calibration Formula

First we recall that the put price under the default model is different from classical case.

$$\begin{aligned} P(T, K) &= e^{-\int_0^T r_u du} \cdot \int_{D_T}^{K+D_T} (K + D_T - y) \cdot p(0, T; V_0, y) dy + e^{-\int_0^T r_u du} \cdot \int_0^{D_T} K \cdot p(0, T; V_0, y) dy \\ &= e^{-\int_0^T r_u du} \cdot \int_0^{K+D_T} K \cdot p(0, T; V_0, y) dy + e^{-\int_0^T r_u du} \cdot \int_{D_T}^{D_T+K} (D_T - y) \cdot p(0, T; V_0, y) dy. \end{aligned}$$

We differentiate this with respect to T and K twice,

$$\begin{aligned} \frac{\partial P}{\partial K} &= e^{-\int_0^T r_u du} \cdot \int_0^{K+D_T} p(0, T; V_0, y) dy, \\ \frac{\partial^2 P}{\partial K^2} &= e^{-\int_0^T r_u du} \cdot p(0, T; V_0, K + D_T), \\ \frac{\partial P}{\partial T} &= r(T) \cdot P(K, T) + e^{-\int_0^T r_u du} K \cdot p(0, T; V_0, K + D_T) D_T (r(T) - q(T)) \\ &\quad + e^{-\int_0^T r_u du} \cdot \int_0^K p_T(0, T; V_0, y) dy \\ &\quad + e^{-\int_0^T r_u du} \cdot \int_{D_T}^{K+D_T} D_T (r(T) - q(T)) p(0, T; V_0, y) dy \\ &\quad - e^{-\int_0^T r_u du} K \cdot p(0, T; V_0, K + D_T) \cdot D_T (r(T) - q(T)) \\ &\quad + e^{-\int_0^T r_u du} \int_{D_T}^{K+D_T} (D_T - y) p_T(0, T; V_0, y) dy. \end{aligned} \tag{5.1}$$

By Fokker-Planck equation:

$$\frac{\partial}{\partial T} p(0, T; V_0, y) + \frac{\partial}{\partial y} [(r_T - q_T) \cdot y \cdot p(0, T; V_0, y)] - \frac{1}{2} \frac{\partial^2}{\partial y^2} [\sigma^2(t, y) \cdot y^2 p(0, T; V_0, y)] = 0.$$

Substitute this into part of (5.1), we get

$$\begin{aligned} & e^{-\int_0^T r_u du} \cdot \int_0^K p_T(0, T; V_0, y) dy \\ &= e^{-\int_0^T r_u du} \int_0^{K+D_T} K \cdot (-1) \frac{\partial}{\partial y} [(r(T) - q(T)) y \cdot p(0, T; V_0, y)] dy + \\ & \quad + e^{-\int_0^T r_u du} \int_0^{K+D_T} K \cdot \frac{1}{2} \cdot \frac{\partial^2}{\partial y^2} [\sigma^2(t, y) y^2 p(0, T; V_0, y)] dy \\ &= -K(r(T) - q(T))(K + D(T)) \cdot \frac{\partial^2 P}{\partial K^2} \\ & \quad + \frac{1}{2} K e^{-\int_0^T r_u du} \cdot \frac{\partial}{\partial y} [\sigma^2(t, y) y^2 p(0, T; V_0, y)] |_{K+D_T}, \end{aligned}$$

$$\begin{aligned} & e^{-\int_0^T r_u du} \int_{D_T}^{K+D_T} (D_T - y) p_T(0, T; V_0, y) dy \\ &= e^{-\int_0^T r_u du} \int_{D_T}^{K+D_T} K \cdot (D_T - y) \frac{\partial}{\partial y} [(r(T) - q(T)) y \cdot p(0, T; V_0, y)] dy + \\ & \quad + e^{-\int_0^T r_u du} \int_{D_T}^{K+D_T} (D_T - y) \frac{1}{2} \cdot \frac{\partial^2}{\partial y^2} [\sigma^2(t, y) y^2 p(0, T; V_0, y)] dy \\ &= (r(T) - q(T)) K (K + D_T) \cdot \frac{\partial^2 P}{\partial K^2} \\ & \quad - e^{-\int_0^T r_u du} (r(T) - q(T)) \int_{D_T}^{K+D_T} y \cdot p(0, T; V_0, y) dy \\ & \quad - e^{-\int_0^T r_u du} \frac{1}{2} (-K) \frac{\partial}{\partial y} [\sigma^2(t, y) y^2 p(0, T; V_0, y)] |_{y=K+D_T} \\ & \quad + e^{-\int_0^T r_u du} \frac{1}{2} \sigma^2(t, K + D_T) (K + D_T)^2 p(0, T; V_0, K + D_T) \\ & \quad - e^{-\int_0^T r_u du} \frac{1}{2} \sigma^2(t, D_T) D_T^2 p(0, T; V_0, D_T). \end{aligned}$$

Additionally,

$$P(K, T) - K \cdot \frac{\partial P}{\partial K} = e^{-\int_0^T r_u du} \int_{D_T}^{K+D_T} (y - D_T) p(0, T; V_0, y) dy.$$

Putting everything together, we get the final result as:

$$\frac{\partial P}{\partial T}(K, T) = -r(T)P(K, T) - (r(T) - q(T))K \frac{\partial}{\partial K} P(T, K + D_T) + \frac{1}{2}\sigma^2(t, K + D_T)(K + D_T)^2 \frac{\partial^2}{\partial K^2} P(T, K + D_T) - \frac{1}{2}\sigma^2(t, D_T)D_T^2 \frac{\partial^2}{\partial K^2} P(T, D_T).$$

5.2 Appendix B. Main Code

Code for Calibration using put price:

```
% Code for Calibration using put price
% T is vector of time steps
% P is put price matrix
% K is vector of strikes
% D is vector of Debt barrier
% r is vector interest rate process
function L_Vol=Dupire_put(P,T,K,r,D)
L_Vol=zeros(13,7);
for i=2:12
    for j=2:6
        a=0.078*P(i,j)+K(i)*(r(j)-0.078)*(P(i+1,j)-P(i-1,j))/5
        +(P(i,j+1)-P(i,j-1))/(T(j+1)-T(j-1));
        b=0.5*(K(i)+D(j+1))^2*(P(i+1,j)-2*P(i,j)+P(i-1,j))/6.25;
        L_Vol(i,j)=sqrt(abs(a/b));
    end
end
j=1;
for i=2:12
    a=0.078*P(i,j)+K(i)*(r(j)-0.078)*(P(i+1,j)-P(i-1,j))/5
    +(P(i,j+1)-P(i,j))/T(j+1)-T(j));
    b=0.5*(K(i)+D(j+1))^2*(P(i+1,j)-2*P(i,j)+P(i-1,j))/6.25;
    L_Vol(i,j)=sqrt(abs(a/b));
end
j=7;
for i=2:12
    a=0.078*P(i,j)+K(i)*(r(j)-0.078)*(P(i+1,j)-P(i-1,j))/5
    +(P(i,j)-P(i,j-1))/(T(j)-T(j-1));
    b=0.5*(K(i)+D(j+1))^2*(P(i+1,j)-2*P(i,j)+P(i-1,j))/6.25;
    L_Vol(i,j)=sqrt(abs(a/b));
end
i=1;
for j=2:6
    a=0.078*P(i,j)+K(i)*(r(j)-0.078)*(P(i+1,j)-P(i,j))/2.5
```



```

        +(P(i,j)-P(i,j-1))/(T(j)-T(j-1));
        b=0.5*(K(i)+D(j+1))^2*(P(i+2,j)-2*P(i+1,j)+P(i,j))/6.25;
        L_Vol(i,j)=sqrt(abs(a/b));
    end
    i=13;
    for j=2:6
        a=0.078*P(i,j)+K(i)*(r(j)-0.078)*(P(i,j)-P(i-1,j))/2.5
        +(P(i,j)-P(i,j-1))/(T(j)-T(j-1));
        b=0.5*(K(i)+D(j+1))^2*(P(i,j)-2*P(i-1,j)+P(i-2,j))/6.25;
        L_Vol(i,j)=sqrt(abs(a/b));
    end

    i=1;j=1;
    a=0.078*P(i,j)+K(i)*(r(j)-0.078)*(P(i+1,j)-P(i,j))/2.5
    +(P(i,j+1)-P(i,j))/(T(j+1)-T(j));
    b=0.5*(K(i)+D(j+1))^2*(P(i+1,j)-2*P(i+2,j)+P(i,j))/6.25;
    L_Vol(i,j)=sqrt(abs(a/b));

    i=1;j=7;
    a=0.078*P(i,j)+K(i)*(r(j)-0.078)*(P(i+1,j)-P(i,j))/2.5
    +(P(i,j)-P(i,j-1))/(T(j)-T(j-1));
    b=0.5*(K(i)+D(j+1))^2*(P(i+1,j)-2*P(i+2,j)+P(i,j))/6.25;
    L_Vol(i,j)=sqrt(abs(a/b));

    i=13;j=1;
    a=0.078*P(i,j)+K(i)*(r(j)-0.078)*(P(i,j)-P(i-1,j))/2.5
    +(P(i,j+1)-P(i,j))/(T(j+1)-T(j));
    b=0.5*(K(i)+D(j+1))^2*(P(i,j)-2*P(i-1,j)+P(i-2,j))/6.25;
    L_Vol(i,j)=sqrt(abs(a/b));

    i=13;j=7;
    a=0.078*P(i,j)+K(i)*(r(j)-0.078)*(P(i,j)-P(i-1,j))/2.5
    +(P(i,j)-P(i,j-1))/(T(j)-T(j-1));
    b=0.5*(K(i)+D(j+1))^2*(P(i,j)-2*P(i-1,j)+P(i-2,j))/6.25;
    L_Vol(i,j)=sqrt(abs(a/b));

end

```

Code for Explicit Finite Difference Methods Solving Survival PDE:

```

% Code for Explicit Finite Difference Solving Survival PDE
% L_Vol is matrix of local volatility implied from put price
function Q=Survival_Probability(T,K,r,D,L_Vol)
Q=[zeros(14,7),ones(14,1)];

```

```

Q(1,8)=0;
T=[T(1),diff(T)];
K=[0,K];
T=T/1000;
for j=7:-1:1
    QQ=Q(:,j+1);
    for k=1:1000
        for i=3:13
            Q(i,j)=Q(i,j+1)+T(j)*(Q(i+1,j+1)-Q(i-1,j+1))*(r(j)-0.078)*K(i)/5+T(j)
                *0.5/6.25*(Q(i+1,j+1)-2*Q(i,j+1)+Q(i-1,j+1))
                *L_Vol(i-1,j)^2*(K(i)+D(j+1));
        end
        i=2;
        Q(i,j)=Q(i,j+1)+(6/5*(Q(3,j+1)-Q(2,j+1)))/2.5+1/6*(Q(2,j+1)
            -Q(1,j+1))/10)*T(j)*(r(j)-0.078)*K(i)+T(j)*0.5*(10*Q(3,j+1)
            -12.5*Q(2,j+1))/(0.5*2.5*10*12.5)*L_Vol(i-1,j)^2*(K(i)+D(j+1));
        i=14;
        Q(14,j)=Q(i,j+1)+(Q(i,j+1)-Q(i-1,j+1))/2.5*T(j)*(r(j)-0.078)*K(i)+T(j)
            *0.5/6.25*(Q(i,j+1)-2*Q(i-1,j+1)+Q(i-2,j+1))*L_Vol(i-1,j)^2*(K(i)+D(j+1));
        Q(:,j+1)=Q(:,j);
    end
    Q(:,j+1)=QQ;
end

```

5.3 Appendix C. General Motors Option Data

| Expiry month | Nov-05 | Dec-05 | Jan-06 | Mar-06 | Jun-06 | Jan-07 | Jan-08 |
|---------------|--------|--------|--------|--------|--------|--------|--------|
| Maturity | 2w | 1.2m | 2.4m | 4.2m | 7.2m | 1.2y | 2.2y |
| Interest Rate | 0.0392 | 0.0392 | 0.0392 | 0.0405 | 0.0425 | 0.0432 | 0.0437 |
| Strike / Time | 0.0278 | 0.1056 | 0.2000 | 0.3583 | 0.6056 | 1.1972 | 2.1944 |
| 10 | | | 1.2470 | 1.1192 | 0.9620 | 0.8621 | 0.8419 |
| 12.5 | | | 1.1196 | 0.9484 | 0.8588 | 0.7860 | 0.7935 |
| 15 | | 1.0640 | 0.9894 | 0.8321 | 0.7871 | 0.7212 | 0.7451 |
| 17.5 | | 0.9062 | 0.8744 | 0.7901 | 0.7155 | 0.6829 | 0.6967 |
| 20 | | 0.7935 | 0.7779 | 0.7260 | 0.6655 | 0.6402 | 0.6484 |
| 22.5 | 0.6983 | 0.6642 | 0.7015 | 0.6408 | 0.6284 | 0.5965 | 0.6096 |
| 25 | 0.5883 | 0.5960 | 0.6213 | 0.6227 | 0.5913 | 0.5808 | 0.5812 |
| 27.5 | 0.5365 | 0.5351 | 0.5725 | 0.5665 | 0.5461 | 0.5397 | |
| 30 | 0.5259 | 0.5023 | 0.5236 | 0.5155 | 0.5071 | 0.5174 | 0.4806 |
| 32.5 | 0.7341 | 0.5109 | 0.4787 | 0.5044 | 0.4852 | 0.4734 | |
| 35 | 0.9422 | 0.4682 | 0.4995 | 0.4632 | 0.4633 | 0.4719 | 0.4669 |
| 37.5 | | 0.6368 | 0.4957 | | 0.4526 | | |
| 40 | | | 0.5222 | | 0.4419 | 0.4345 | 0.4408 |
| 42.5 | | | 0.5669 | | | | |
| 45 | | | | | | 0.4085 | 0.4237 |
| 50 | | | | | | 0.4136 | 0.4067 |
| 55 | | | | | | 0.4022 | 0.3896 |

Figure 5.1: Market Option Implied Volatility

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