

# Constructing quotients of algebraic varieties by linear algebraic group actions

Gergely Bérczi, Thomas Hawes, Frances Kirwan\*

Mathematical Institute, University of Oxford, Andrew Wiles Building, Oxford. OX2 6GG. UK.

Emails: `berczi@maths.ox.ac.uk`, `thomas.hawes01@gmail.com`, `kirwan@maths.ox.ac.uk`

Brent Doran<sup>†</sup>

Departement Mathematik, Eidgenössische Technische Hochschule, 8092 Zürich, Switzerland.

Email: `brent.doran@math.ethz.ch`

## Abstract

In this article we review the question of constructing geometric quotients of actions of linear algebraic groups on irreducible varieties over algebraically closed fields of characteristic zero, in the spirit of Mumford's geometric invariant theory (GIT). The article surveys some recent work on geometric invariant theory and quotients of varieties by linear algebraic group actions, as well as background material on linear algebraic groups, Mumford's GIT and some of the challenges that the non-reductive setting presents. The earlier work of two of the authors in the setting of unipotent group actions is extended to deal with actions of any linear algebraic group. Given the data of a linearisation for an action of a linear algebraic group  $H$  on an irreducible variety  $X$ , an open subset of stable points  $X^s$  is defined which admits a geometric quotient variety  $X^s/H$ . We construct projective completions of the quotient  $X^s/H$  by considering a suitable extension of the group action to an action of a reductive group on a reductive envelope and using Mumford's GIT. In good cases one can also compute the stable locus  $X^s$  in terms of stability (in the sense of Mumford for reductive groups) for the reductive envelope.

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# 1 Introduction

Group actions are ubiquitous within algebraic geometry. Many spaces that one might want to understand arise naturally as the quotient of a variety by a group action, with moduli spaces giving some of the most prominent examples [MumFK94, Ne78, Gi83, KiPT01]. Given a variety  $X$  over an algebraically closed field  $\mathbb{k}$  of characteristic zero<sup>1</sup> and a linear algebraic group  $H$  acting on  $X$ , a basic question to ask therefore is how one can construct the quotient  $X/H$  and study it. By ‘quotient’ here we mean more precisely a *geometric quotient*, in the sense of [MumFK94]: this is a variety  $X/H$  with an  $H$ -invariant morphism  $X \rightarrow X/H$  that, amongst other properties, is universal with respect to  $H$ -invariant morphisms from  $X$ , and whose fibres are the orbits of the action on  $X$ . As is well known, there are lots of cases of interest where a geometric quotient for an action cannot possibly exist in the category of varieties. One way to address this is to enlarge one’s category and work with more general geometric objects, such as algebraic spaces [Ar71, Kn71] or even stacks [DelM69, LaMB00]. Another approach is to instead look for nonempty open subsets of  $X$  that admit a geometric quotient variety; such open subsets are guaranteed to exist by a theorem of Rosenlicht [Ro63]. It is this second approach with which we will be concerned in this article.

In the case where  $H = G$  is a reductive linear algebraic group this second approach was studied by Mumford in the first edition of [MumFK94], resulting in his *geometric invariant theory* (GIT). (In this context see also the work of Seshadri [Ses63, Ses72] and also [Ses77] which is valid over an arbitrary base.) Mumford’s GIT works particularly well in the case where  $X$  is projective. Given the additional choice of an ample linearisation  $L \rightarrow X$  (that is, an ample line bundle  $L \rightarrow X$  with a lift of the action of  $G$  to  $L$ ) Mumford defines a  $G$ -invariant open subset of stable points  $X^s$  in  $X$ , which has a geometric quotient variety  $X^s/G$ . This is contained in the  $G$ -invariant open subset of semistable points  $X^{ss}$  in  $X$  and there is a natural surjective  $G$ -invariant map from  $X^{ss}$  onto a projective variety  $X//G$  (canonical to the choice of  $L$ ) which can be described as  $\text{Proj}(S^G)$ , where  $S = \bigoplus_{r \geq 0} H^0(X, L^{\otimes r})$  and  $S^G$  is the ring of invariant sections of non-negative tensor powers of  $L \rightarrow X$ . Thus (for projective  $X$  and ample  $L$ ) there is a diagram

$$\begin{array}{ccccc} X^s & \subseteq & X^{ss} & \subseteq & X \\ \downarrow \text{geo} & & \downarrow & & \downarrow \\ X^s/G & \subseteq & X//G & = & \text{Proj}(S^G). \end{array}$$

The variety  $X//G$ , which is often referred to as the *GIT quotient*, provides a natural projective completion of the quotient of the stable set. Tools abound for studying the spaces in this diagram. In [MumFK94] Mumford gave numerical criteria for computing the sets of stable and semistable points in terms of the actions of one-parameter subgroups of  $G$ . When  $X$  is smooth the local geometry of the orbits in  $X^{ss}$  can be studied with the slice theorem of Luna [Lu73] and, in the case where the ground field is the complex numbers  $\mathbb{C}$ , links with symplectic geometry yield ways to compute the (rational intersection) cohomology of  $X//G$  [Ki84, Ki85, Ki86, Ki87, JK95, JKKW03]. Moreover, as studied in the work of ‘variation of GIT’ (VGIT) by Thaddeus

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<sup>1</sup>We assume characteristic zero throughout this article, although Mumford’s GIT has been extended to algebraically closed fields of arbitrary characteristic; cf. [MumFK94] Appendices A and C.

[T96], Dolgachev and Hu [DolH98] and Ressayre [Re00], the GIT quotients undergo birational transformations when the linearisation varies, which can be described explicitly as certain kinds of flips.

Various authors have considered the question of finding open subsets of ‘stable’ points that admit geometric quotients in the case where  $H$  is not reductive. This problem is very much more challenging, in essence because the representation theory of a non-reductive group is not as complete or well-behaved as in the reductive case. Fauntleroy [Fa83, Fa85] and Dixmier and Raynaud [DiR81] give geometric descriptions of open subsets that admit geometric quotients, but these are typically difficult to find in practice (requiring knowledge of which points are separated by invariant functions) and often some extra condition on  $X$ , such as normality or quasi-factoriality, needs to be imposed. More algorithmic approaches have been taken in [GeP93, GeP98, vdE93, Sa00], though here the geometric picture is somewhat more obscure in favour of computation. Other progress, this time in the algebraic side of the subject, involves the search for *separating sets of invariants* to construct quotient morphisms of affine varieties  $X$ , made popular by Derksen and Kemper in [DerK02] and pursued recently in the work of Dufresne and others, [Du13, DuES14, DuJ15] and [DuK15, Section 4]. (The use of separating sets of (rational) invariants seems to in fact go back to Rosenlicht [Ro56] and is used in the proof of his aforementioned theorem; see [PV94].) A key ingredient here is the observation that one can find a *finite* set of invariants  $\mathcal{S} \subseteq \mathcal{O}(X)^H$  such that two points in  $X$  get separated by the natural map  $X \rightarrow \operatorname{Spec}(\mathcal{O}(X)^H)$  if, and only if, there is an element of  $\mathcal{S}$  that separates the points. Therefore one does not need to find a full generating set for  $\mathcal{O}(X)^H$  to describe quotient maps.

Any linear algebraic group  $H$  has a canonical normal unipotent subgroup  $H_u$ , called the unipotent radical of  $H$ , such that  $H/H_u$  is reductive, thus constructing quotients for  $H$  can, in principle, be reduced to studying the actions of unipotent groups. The case where  $H$  is a unipotent group acting on an irreducible projective variety  $X$  with ample linearisation  $L \rightarrow X$  was studied in [DorK07], building on the work in [Fa83, Fa85, GeP93, GeP98, Wi03]. The overarching idea in that paper was to consider various notions of ‘stability’, ‘semistability’ and ‘quotient’ that are intrinsic to the linearisation  $L \rightarrow X$ , and relate these to the GIT of [MumFK94] of certain reductive linearisations associated to  $L \rightarrow X$ . The main appeal of this approach is that it gives ways to use the tools available in the reductive setting to study quotients of unipotent group actions on (open subsets of) projective varieties. A summary of the main results and definitions from [DorK07] will be given in the upcoming Section 2. These techniques have been used in practice in a number of settings where unipotent actions arise naturally [AsD07, AsD08, AsD09, Ki09, Ki11, WaDGC13, DorG14, DorH15, DaKS13, DaKS14].

The purpose of the present article is to generalise the material of [DorK07] to the case where  $H$  is any linear algebraic group, not necessarily unipotent or reductive. Thus we develop a theoretical framework for studying non-reductive group actions that is in the same spirit as Mumford’s GIT, with the basic guiding goal of obtaining results that are as close as possible to the earlier diagram relating the stable and semistable sets and GIT quotient in the reductive setting. Indeed, our constructions reduce to Mumford’s theory when  $H$  is a reductive group and  $L \rightarrow X$  is an ample linearisation over a projective variety. The way we extend the work in [DorK07] is to make use of natural residual actions of the reductive group  $H_r := H/H_u$  and

take quotients in stages—first by  $H_u$ , then by  $H_r$ .

Let us now give a summary of the contents of the rest of this article. We begin in Section 2 by recalling background material on linear algebraic groups and their quotients. We discuss various notions of ‘quotient’ in the category of algebraic varieties and recall the concept of a linearisation, as introduced by Mumford in the first edition of [MumFK94]. Some of the main differences between actions of reductive groups and unipotent groups are also highlighted. We then recall the main theorems of GIT for reductive groups in [MumFK94], paying particular attention to the case of ample linearisations over projective varieties. A summary of the work on GIT for unipotent groups in [DorK07] is given, describing more fully the various notions of ‘stability’ and ‘semistability’ considered there, as well as the definition of the *enveloping quotient* and *enveloped quotient* and the construction of *reductive envelopes*.

In Section 3 we begin the work of extending the theory of [DorK07] to more general linear algebraic groups  $H$ , focussing on constructing objects from the data of a linearisation  $L \rightarrow X$ . Unlike in [DorK07], we do not assume  $X$  is projective or irreducible, or that the linearisation  $L$  is ample. We start by considering the natural  $H$ -invariant rational map

$$q : X \dashrightarrow \mathrm{Proj}(S^H)$$

to a scheme  $\mathrm{Proj}(S^H)$  which is not necessarily noetherian. The maximal domain of definition contains the open subsets  $X_f$  given by the non-vanishing of invariant sections  $f$  of positive tensor powers of  $L \rightarrow X$ , and imposing various conditions on the sections  $f$  yields different  $H$ -invariant open subsets of  $X$ . In this way, a subset  $X^{\mathrm{ss}, \mathrm{fg}}$  of *finitely generated semistable* points of  $X$  is defined in Definition 3.1.1 such that  $q$  maps  $X^{\mathrm{ss}, \mathrm{fg}}$  into an open subscheme  $X \wr H$  of  $\mathrm{Proj}(S^H)$ , locally of finite type over  $\mathbb{k}$ , called the *enveloping quotient* (Definition 3.1.6). While this looks similar to Mumford’s GIT quotient in the reductive setting, in general there are two key differences to note. Firstly, the enveloping quotient  $X \wr H$  is not a quasi-projective variety in general, although when  $X$  is projective,  $L \rightarrow X$  is ample and  $S^H$  is finitely generated then  $X \wr H = \mathrm{Proj}(S^H)$  is in fact the projective variety associated to the graded algebra  $S^H$ . Secondly, the map  $q : X^{\mathrm{ss}, \mathrm{fg}} \rightarrow X \wr H$  is not surjective in general; instead the image  $q(X^{\mathrm{ss}, \mathrm{fg}})$  is a dense constructible subset of  $X \wr H$  called the *enveloped quotient*. To address the fact that the enveloping quotient is only a scheme *locally* of finite type in general, we introduce *inner enveloping quotients* in Definition 3.1.12 as quasi-compact open subschemes of the enveloping quotient that contain the enveloped quotient  $q(X^{\mathrm{ss}, \mathrm{fg}})$ . Inner enveloping quotients are not canonical to the linearisation, but have the advantage of being quasi-projective varieties; this is shown in Proposition 3.1.14. A way in which the collection of inner enveloping quotients can be thought of as ‘universal’ with respect to  $H$ -invariant morphisms from the finitely generated semistable locus is discussed. We also compare the framework developed here with Mumford’s reductive GIT in the case where  $H = G$  is reductive. Building on the notion of stability in [DorK07], we define the *stable locus*  $X^s$  for a general linearisation of a linear algebraic group over an irreducible variety in Definition 3.3.2. When the group  $H$  is reductive or unipotent our definition reduces to that of Mumford or [DorK07], respectively. For any choice of inner enveloping quotient  $\mathcal{U} \subseteq X \wr H$ , we show that the natural map  $q : X^{\mathrm{ss}, \mathrm{fg}} \rightarrow \mathcal{U}$  restricts to define a geometric

quotient on the stable locus, thus obtaining a diagram

$$\begin{array}{ccccc} X^s & \subseteq & X^{\text{ss,fg}} & \subseteq & X \\ \downarrow \text{geo} & & \downarrow & & \downarrow \text{---} \\ X^s/H & \subseteq & \mathcal{U} & \subseteq & \text{Proj}(S^H) \end{array}$$

(see Theorem 3.4.2). This is analogous to Mumford's in the reductive case, but in contrast the map  $q : X^{\text{ss,fg}} \rightarrow \mathcal{U}$  is not necessarily surjective, while there are many choices of inner enveloping quotient  $\mathcal{U}$  containing  $X^s/H$  and such a  $\mathcal{U}$  is not necessarily a projective variety.

The possible lack of projectivity of an inner enveloping quotient  $\mathcal{U}$  naturally motivates the construction of their projective completions  $\overline{\mathcal{U}}$ . Any such completion contains the enveloped quotient  $q(X^{\text{ss,fg}})$  as a dense constructible subset, so we refer to  $\overline{\mathcal{U}}$  as a *projective completion of the enveloped quotient* (Definition 4.0.1). In Section 4 we extend the theory of reductive envelopes from [DorK07] to give ways of constructing projective completions of the enveloped quotient. We consider the formation of fibre spaces  $G \times^{H_u} X$  defined by homomorphisms  $H \rightarrow G$ , with  $G$  a reductive group, which restrict to give embeddings of the unipotent radical  $H_u \hookrightarrow G$ . Such homomorphisms give a diagonal action of the reductive group  $H_r = H/H_u$  on  $G \times^{H_u} X$  that commutes with the  $G$ -action, so  $G \times^{H_u} X$  is a  $G \times H_r$ -variety that comes equipped with a canonical  $G \times H_r$ -linearisation. Various kinds of *reductive envelope*  $(\overline{G \times^{H_u} X}, L')$  are defined in Definitions 4.1.4 and 4.2.1, where  $\overline{G \times^{H_u} X}$  is an equivariant completion of  $G \times^{H_u} X$  and  $L' \rightarrow \overline{G \times^{H_u} X}$  is an extension of the  $G \times H_r$ -linearisation over  $G \times^{H_u} X$ , by requiring that invariant sections of certain choices of linear systems over  $X$  extend to linear systems over the reductive envelope, satisfying assumptions of varying strength. In Theorem 4.1.14 we show that when the line bundle  $L' \rightarrow \overline{G \times^{H_u} X}$  in the reductive envelope is ample, then the reductive GIT quotient  $\overline{G \times^{H_u} X} //_{L'} (G \times H_r)$  gives a projective completion of the enveloped quotient, and there is a chain of inclusions

$$X \cap (\overline{G \times^U X})^{s(L')} \subseteq X^s \subseteq X^{\text{ss,fg}} \subseteq X \cap (\overline{G \times^U X})^{\text{ss}(L')}.$$

When the reductive envelope  $(\overline{G \times^{H_u} X}, L')$  is *strong* (Definition 4.2.1) then in Proposition 4.2.2 we obtain equalities

$$X^s = X \cap (\overline{G \times^U X})^{s(L')}, \quad X^{\text{ss,fg}} = X \cap (\overline{G \times^U X})^{\text{ss}(L')},$$

which thus provides a way to compute the intrinsically defined stable locus and finitely generated semistable locus using methods from reductive GIT. The existence of strong reductive envelopes with ample  $L'$  is therefore especially good for the purposes of computation in our non-reductive geometric invariant theory. In relation to this, we show that when the  $H$ -linearisation over  $X$  extends to one of the reductive group  $G \times H_r$  in an appropriate way, the arguments in [DorK07] can be extended to reduce the construction of strong reductive envelopes with ample  $L'$  to a study of the homogeneous space  $G/H_u$ . (Such homogeneous spaces were first considered by [BiBHM63] and studied by Grosshans in [Gros73, Gros97].) This set-up works out particularly well when  $H_u$  is a *Grosshans subgroup* of  $G$ , for then  $S^H$  is a finitely generated algebra and an explicit descriptions of  $X^s$ ,  $X^{\text{ss,fg}}$  and  $X \curvearrowright H = \text{Proj}(S^H)$  can be obtained in terms of the reductive GIT of  $L' \rightarrow \overline{G \times^{H_u} X}$  (Corollary 4.2.10).

In Section 5 we study the space of  $n$  unordered points on  $\mathbb{P}^1$  under the action of a Borel subgroup in  $\mathrm{SL}(2, \mathbb{k})$ . This serves both to illustrate the use of strong reductive envelopes for performing computations and to give an informal look at the potential for studying the variation of non-reductive quotients in certain good cases. The final section contains a brief outline of ongoing research applying the theory developed in the article to moduli spaces occurring naturally in algebraic geometry.

To supplement the main text, and for the convenience of the reader, we have also included a proof of a well-known result concerning the GIT quotient of a product of reductive groups in Appendix A.

We would like to thank the referee for some very helpful suggestions (including the inclusion of an index) and corrections, and to apologise for any remaining errors, misprints and sources of confusion which are entirely our responsibility.

## 1.1 Notation and Conventions

We work over a ground field  $\mathbb{k}$  that is algebraically closed and of characteristic zero. By ‘variety’ we mean a reduced, separated scheme of finite type over  $\mathbb{k}$ ; note that we do not assume varieties are irreducible unless otherwise stated, but do insist they are separated. By a ‘point’ in a scheme we will always mean a closed  $\mathbb{k}$ -valued point. A projective completion  $X \hookrightarrow \overline{X}$  of a variety  $X$  is a dominant open immersion into a projective variety  $\overline{X}$ . If a topological space satisfies the condition that every cover of it by open sets admits a finite subcover then we say it is ‘quasi-compact’. A scheme is quasi-compact if and only if it is the union of finitely many open affine subschemes; any scheme which is complete or quasi-projective is quasi-compact.

When we talk about actions of groups on varieties or vector spaces, we always mean a *left action*, unless stated otherwise.

When talking about line bundles we will usually be referring to the total space of an invertible sheaf of modules; the sheaf of sections of a line bundle  $L$  is denoted by  $\underline{L}$ , so that  $L = \mathbf{Spec}(\underline{L}^*)$ . An exception to this is when talking about twisting sheaves  $\mathcal{O}(n)$  on varieties—here we don’t make any notational distinction between the sheaf and its total space. Given a linearisation  $L \rightarrow X$  of a group  $H$  and a character  $\chi$  of  $H$ , the *twist*  $L^{(\chi)} \rightarrow X$  of  $L \rightarrow X$  by  $\chi$  is the linearisation obtained by multiplying the fibres of the linearisation  $L \rightarrow X$  by the character  $\chi^{-1}$ ; this will also be emphasised in Section 2.1.2 of the main text.

If  $\phi : X \rightarrow Y$  is a morphism of schemes, then the natural pullback morphism of structure sheaves is denoted  $\phi^\# : \mathcal{O}_Y \rightarrow \phi_* \mathcal{O}_X$ . On the other hand, if  $L \rightarrow Y$  is a line bundle then we use the notation  $\phi^*$  to denote pull-back  $\underline{L} \rightarrow \phi_*(\phi^* \underline{L})$ . Given an  $\mathcal{O}_X$ -module  $\mathcal{F}$  and an  $\mathcal{O}_Y$ -module  $\mathcal{G}$ , then  $\mathcal{F} \boxtimes \mathcal{G} := (\mathrm{pr}_X^* \mathcal{F}) \otimes_{\mathcal{O}_{X \times Y}} (\mathrm{pr}_Y^* \mathcal{G})$ , where  $\mathrm{pr}_X : X \times Y \rightarrow X$  and  $\mathrm{pr}_Y : X \times Y \rightarrow Y$  are the projections.

Unless indicated otherwise, graded rings  $R$  are always non-negatively  $\mathbb{Z}$ -graded, i.e. if  $R$  is a graded ring then the degree  $d \in \mathbb{Z}$  piece  $S_d$  is trivial whenever  $d < 0$ . If  $f \in R$  a non-zero homogeneous element then  $R_{(f)}$  is the subring of the localisation  $R_f$  consisting of degree 0 elements. Similarly, if  $M$  a graded  $R$ -module and  $f \in R$  a non-zero homogeneous element, then  $M_{(f)}$  is the  $R_{(f)}$ -submodule of the localisation  $M_f$  consisting of degree 0 elements. If  $r \in \mathbb{Z}$  is positive then  $R^{(r)}$  denotes the Veronese subring of  $R$  whose degree  $m$  piece is  $(R^{(r)})_m = R_{mr}$ .

Associated to a vector space  $V$  we understand the projective space  $\mathbb{P}(V)$  to be the space whose points correspond to one-dimensional subspaces of  $V$ . Another way to say this is that  $\mathbb{P}(V) = \text{Proj}(\text{Sym}^\bullet(V^*))$ , where  $\text{Sym}^\bullet(V^*)$  is the symmetric algebra  $\bigoplus_{m \geq 0} \text{Sym}^m(V^*)$ . With these conventions, if  $L \rightarrow X$  is a line bundle on a scheme  $X$  with a basepoint-free linear system  $V \subseteq H^0(X, L)$ , then there is a canonical morphism  $X \rightarrow \mathbb{P}(V^*)$ .

Finally, for basic facts in algebraic geometry we refer the reader to [Har77] and [St15]. The latter is particularly useful for results regarding schemes that are not necessarily noetherian.

## 2 Background: Quotients of Varieties and Geometric Invariant Theory

In this section we recall some background material that will be used in subsequent sections. We begin in Section 2.1 by recalling basic definitions concerning linear algebraic groups, then discuss various kinds of quotient in the category of varieties and review the concept of a linearisation of an action. We also recall the definitions of reductive groups and unipotent groups and compare them from the point of view of the geometry of their actions and their invariant theory. In Section 2.2 we give a summary of the main results from Mumford's GIT for reductive groups, paying particular attention to the case of ample linearisations over projective varieties. Finally, in Section 2.3 we recall the main definitions and results of [DorK07], which will form the basis for our development of a geometric invariant theoretic approach to studying actions of more general linear algebraic groups in later sections.

The material on linear algebraic groups is taken from [Bo91, Sp94] and our main references for the material on quotients are [Ser58, Ses72, MumFK94]. For reductive GIT we have mostly used [MumFK94]; see also [Ne78, Dol03, Muk03, Sc08].

### 2.1 Basics of Group Actions and Quotients

#### 2.1.1 Linear Algebraic Groups and Quotients

We begin by recalling some of the basic theory of algebraic groups. Following [Bo91, Chapter 1] we define an *algebraic group* as a variety  $H$  equipped with a group structure such that the multiplication map  $H \times H \rightarrow H$  and inversion map  $H \rightarrow H$  are morphisms of varieties. We write  $e \in H$  for the identity element of  $H$ . A *homomorphism* of algebraic groups  $H_1 \rightarrow H_2$  is a morphism of varieties that is also a homomorphism of the group structures. If  $H_1 \hookrightarrow H_2$  is a homomorphism that is also a closed immersion then we say that  $H_1$  is a *closed subgroup* of  $H_2$ . A first example of an algebraic group is the general linear group  $\text{GL}(n, \mathbb{k})$ , for any integer  $n \geq 0$ . A *linear algebraic group* is an algebraic group that is a closed subgroup of  $\text{GL}(n, \mathbb{k})$ , for some  $n \geq 0$ . It is these groups we will be concerned with in this article; see [Bri09, Bri11, BriSU13, Bri15b] for work on the structure and geometry of more general algebraic groups. A basic result in the theory of algebraic groups says that every *affine* algebraic group (i.e. an algebraic group that is also an affine variety) is isomorphic to a linear algebraic group [Bo91, Proposition 1.10], thus one can identify and work with linear algebraic groups in a more intrinsic fashion.



*Example 2.1.1.* Any finite group is a linear algebraic group. The group  $\mathbb{G}_m := (\mathbb{k} \setminus \{0\}, \times)$  of non-zero elements of  $\mathbb{k}$  under multiplication is a linear algebraic group (indeed, it is just  $\mathrm{GL}(1, \mathbb{k})$ ). The group  $\mathbb{G}_a := (\mathbb{k}, +)$  of elements of  $\mathbb{k}$  under addition is also a linear algebraic group: it is isomorphic to the group  $\mathrm{U}_2 \subseteq \mathrm{GL}(2, \mathbb{k})$  of upper-triangular matrices via  $a \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ .

*Example 2.1.2.* (Operations on linear algebraic groups [Bo91, Chapters 1 and 6].) A *normal subgroup*  $N$  of an affine algebraic group  $H$  is a closed subgroup that is normal as an abstract group. If  $H$  is a linear algebraic group and  $N$  a normal subgroup, then  $H/N$  is a linear algebraic group. Products of linear algebraic groups are also linear algebraic groups.

Now let  $X$  be a variety over  $\mathbb{k}$  and  $H$  a linear algebraic group. An *action* of  $H$  on  $X$  is a (left) action  $H \times X \rightarrow X$  that is also a morphism of varieties. In this case we will often refer to  $X$  as an “ $H$ -variety” and sometimes write  $H \curvearrowright X$  to indicate the given action. We usually write the morphism of an action as

$$H \times X \rightarrow X, \quad (h, x) \mapsto h \cdot x \quad (\text{or } (h, x) \mapsto hx)$$

if no confusion is likely to occur. Given a point  $x \in X$ , we write

$$H \cdot x := \{hx \mid h \in H\}$$

for the *orbit* of  $x$  under the action of  $H$  and

$$\mathrm{Stab}_H(x) := \{h \in H \mid hx = x\}$$

for the *stabiliser* of  $x$ . Given a subset  $Z \subseteq X$ , we say  $Z$  is  *$H$ -stable*, or  *$H$ -invariant*, if  $H \cdot z \subseteq Z$  for all  $z \in Z$ .

Note that in the case where  $X = V$  is a finite-dimensional vector space with an action of  $H$  that is algebraic,  $V$  is also called a *rational  $H$ -module*.

*Remark 2.1.3.* Throughout this article we shall assume for simplicity that all actions are such that stabilisers of general points are finite.

Given a homomorphism of linear algebraic groups  $\rho : H_1 \rightarrow H_2$ , an  $H_1$ -variety  $X$  and an  $H_2$ -variety  $Y$ , we say a morphism  $\phi : X \rightarrow Y$  is *equivariant (with respect to  $\rho : H_1 \rightarrow H_2$ )* if  $\phi(hx) = \rho(h)\phi(x)$  for all  $h \in H_1$  and all  $x \in X$ . If  $H = H_1 = H_2$  and  $\rho$  is the identity homomorphism, then we simply say  $\phi$  is  *$H$ -equivariant*, and if furthermore  $H$  acts trivially on  $Y$  (so that  $\phi(hx) = \phi(x)$  for all  $h \in H$  and all  $x \in X$ ) then we say  $\phi$  is  *$H$ -invariant*.

If a linear algebraic group  $H$  acts on a variety  $X$  then a fundamental question, if vaguely stated, is to ask: does there exist a variety  $Y$  that is a ‘quotient’ of  $X$  by the action of  $H$ ? There are various definitions to make the term ‘quotient’ more precise, with varying agreement with one’s geometric intuition. We will recall the kinds of ‘quotient’ we shall be concerned with momentarily. Before doing so, note that given an  $H$ -variety  $X$  and an  $H$ -stable open subset  $U \subseteq X$ , there is a canonically induced action of  $H$  on the ring of regular functions:

$$(h \cdot f)(x) := f(h^{-1}x) \quad \text{for all } x \in U, f \in \mathcal{O}(U), h \in H, \quad (2.1)$$

and one can consider the subring of invariant functions:

$$\mathcal{O}(U)^H = \{f \in \mathcal{O}(U) \mid h \cdot f = f \text{ for all } h \in H\}.$$

**Definition 2.1.4.** Let  $H$  be a linear algebraic group acting on a variety  $X$ .

1. A *categorical quotient* is a variety  $Y$  together with an  $H$ -invariant morphism  $\phi : X \rightarrow Y$  satisfying the following universal property: any other  $H$ -invariant morphism  $X \rightarrow Z$  admits a unique factorisation through  $\phi$ .
2. A *good quotient* is an  $H$ -invariant morphism  $\phi : X \rightarrow Y$  satisfying the following properties:
  - (a) the morphism  $\phi$  is surjective and affine;
  - (b) the pull-back map  $\phi^\# : \mathcal{O}_Y \rightarrow \phi_* \mathcal{O}_X$  induces an isomorphism of sheaves  $\mathcal{O}_Y \cong (\phi_* \mathcal{O}_X)^H$ , where  $(\phi_* \mathcal{O}_X)^H(U) = \mathcal{O}_X(\phi^{-1}(U))^H$  for each open subset  $U \subseteq Y$ ; and
  - (c) if  $W_1, W_2$  are disjoint  $H$ -invariant closed subsets of  $X$ , then  $\phi(W_1)$  and  $\phi(W_2)$  are disjoint closed subsets of  $Y$ . (Note this implies  $\phi : X \rightarrow Y$  is a submersion [MumFK94, Chapter 0, §2, Remark 6].)
3. A *geometric quotient* is a good quotient  $\phi : X \rightarrow Y$  that is also an *orbit space*; i.e.  $\phi^{-1}(y)$  is a single  $H$ -orbit for each  $y \in Y$ . In this case we write  $Y = X/H$ .
4. A *principal  $H$ -bundle* (or a *locally isotrivial quotient*) is an  $H$ -invariant morphism  $\phi : X \rightarrow Y$  such that, for every point  $y \in Y$ , there is a Zariski-open neighbourhood  $U_y \subseteq Y$  of  $y$  and a finite étale morphism  $\widetilde{U}_y \rightarrow U_y$  such that there exists an  $H$ -equivariant isomorphism  $H \times \widetilde{U}_y \cong \widetilde{U}_y \times_Y X$ , where the fibred product  $\widetilde{U}_y \times_Y X$  has the canonical  $H$ -action and  $H \times \widetilde{U}_y$  has the *trivial  $H$ -bundle* action, induced by left multiplication by  $H$  on itself:

$$H \times (H \times \widetilde{U}_y) \rightarrow H \times \widetilde{U}_y, \quad (h, h_0, u) \mapsto (hh_0, u).$$

Definition 2.1.4, 1 is taken from [MumFK94, Definition 0.5], while 2–3 are from [Ses72, Definitions 1.4 and 1.5] and 4 is [Ser58, Definition 2.2].

*Remark 2.1.5.* Because we work exclusively with linear algebraic groups, by a result of Grothendieck [Grot60, Page 326] we may equivalently work with quotients that are locally trivial in the fppf-topology in Definition 2.1.4, 4 (the reader may also consult [Sc08, Remark 2.1.1.6] for a justification of this).

In general we have the following chain of implications: principal bundle  $\implies$  geometric quotient  $\implies$  good quotient  $\implies$  categorical quotient. (The main non-trivial implication is the last one, whose proof may be found in [MumFK94, Chapter 0, §2, Remark 6]. Accordingly, one often refers to a good quotient as a “good categorical quotient”.) However, none of the reverse implications hold.

*Example 2.1.6.* (Good quotient  $\nRightarrow$  geometric quotient.) Let  $\mathbb{G}_m$  act on  $X = \mathbb{k}^n$  by the usual scaling action,  $t \cdot (x_1, \dots, x_n) = (tx_1, \dots, tx_n)$ . Then the unique map  $\mathbb{k}^n \rightarrow \text{pt} := \text{Spec } \mathbb{k}$  is a good quotient for this action, but is clearly not a geometric quotient: the preimage of  $\text{pt}$  consists of many orbits.

*Example 2.1.7.* (Geometric quotient  $\nRightarrow$  principal bundle.) Let  $\mathbb{G}_m$  act on  $\mathbb{k}^n \setminus \{0\}$  ( $n > 1$ ) by the action  $t \cdot (x_1, \dots, x_n) = (t^r x_1, \dots, t^r x_n)$ , where  $r \geq 2$  is an integer. Then the usual projection

$\mathbb{A}^n \setminus \{0\} \rightarrow \mathbb{P}^{n-1}$  is a geometric quotient which is not a principal  $\mathbb{G}_m$ -bundle, because the action is not set-theoretically free. (Thus, it is possible for geometric quotients to exist for actions where some stabilisers are non-trivial.)

*Example 2.1.8.* (Categorical quotient  $\nRightarrow$  good quotient.) Such examples are more difficult to come by, but do exist. The interested reader can refer to [ACNH99, ACNH01].

A very useful property of good and geometric quotients is that they are determined locally on the base variety. That is [Ne78, Proposition 3.10],

- an  $H$ -invariant morphism  $\phi : X \rightarrow Y$  is a good (respectively, geometric) quotient if, and only if, there is an open cover  $\{U_i\}$  of  $Y$  such that each restriction  $\phi : \phi^{-1}(U_i) \rightarrow U_i$  is a good (respectively, geometric) quotient of  $H \curvearrowright \phi^{-1}(U_i)$ ; and
- if  $\phi : X \rightarrow Y$  is a good (respectively, geometric) quotient, then for each open subset  $U \subseteq Y$  the restriction  $\phi : \phi^{-1}(U) \rightarrow U$  is a good (respectively, geometric) quotient of  $H \curvearrowright \phi^{-1}(U)$ .

Given an action of  $H$  on  $X$ , there are certain topological restrictions on the action that must be fulfilled if a geometric quotient or a principal bundle structure is to exist. If a geometric quotient for  $H \curvearrowright X$  exists then the action must be *closed*: that is, for each point  $x \in X$  the orbit  $H \cdot x$  is a closed subset of  $X$ . Furthermore, by [MumFK94, Proposition 0.9] and [Ses72, Theorem 6.1] a geometric quotient  $X \rightarrow X/H$  has the structure of a principal  $H$ -bundle if, and only if, the action of  $H$  on  $X$  is *free*: that is, the graph morphism

$$H \times X \rightarrow X \times X, \quad (h, x) \mapsto (hx, x)$$

of the action morphism is a closed immersion. Checking freeness of an action can be made simpler by the following lemma.

**Lemma 2.1.9.** [EG98, §6.3, Lemma 8] *An action of a linear algebraic group  $H$  on  $X$  is free (in the above sense) if, and only if, it is set-theoretically free and proper (that is, the graph morphism  $H \times X \rightarrow X \times X$  of the action is a proper morphism).*

Example 2.1.6 shows that not every action of a linear algebraic group on a variety need admit a geometric quotient. More generally, there are actions which do not admit even a categorical quotient; see [ACNH00] for examples in the context of toric varieties under actions of subtori. From here, there are a couple of possible ways to proceed if one wants to construct a quotient for the action. One way is to try to enlarge the category in which one works so that it contains a quotient object for the action. For example, any proper action with finite stabilisers has an *algebraic space* that is a geometric quotient [Ar71, Kn71, Ko97, KeeM97]. More generally, one can use the category of stacks [DelM69, LaMB00], where every action of a linear algebraic group on a variety has a quotient stack. An alternative approach, which we adopt in this article, is to look for nonempty invariant open subsets that admit a geometric quotient. This approach is validated by the following result of Rosenlicht.

**Theorem 2.1.10.** [Ro63]<sup>2</sup> Let  $H$  be a linear algebraic group acting on an irreducible variety  $X$ . Then there is a nonempty  $H$ -invariant open subset  $U \subseteq X$  admitting a quasi-projective geometric quotient  $U/H$ .

Rosenlicht’s proof of Theorem 2.1.10 is non-constructive, so the question remains of how to explicitly find nonempty open subsets—ideally as large as possible—that admit geometric quotients. This is the basic task of *geometric invariant theory*. We will discuss ways in which this has been done for certain kinds of linear algebraic group in the upcoming Sections 2.2 and 2.3.

### 2.1.2 Linearisations of Actions

A natural way to try and construct open subsets of  $X$  that admit geometric quotients is to glue together quotients of smaller open subsets which are easier to understand (for example, affine open subsets) and appeal to the fact that geometric quotients are local on the base. Such a strategy in general runs the risk of resulting in non-separated quotient schemes. This can be avoided by considering open subsets  $X_f$  defined by the non-vanishing of some invariant rational function  $f$  and gluing the maps  $X_f \rightarrow \operatorname{Spec}(\mathcal{O}(X_f)^H)$ , for then orbits in  $X_f$  are separated by orbits outside of  $X_f$  by  $f$ . This is essentially what a *linearisation* achieves for us, a notion due to Mumford [MumFK94, Definition 1.6] that is fundamental for what follows.

**Definition 2.1.11.** Let  $H$  be a linear algebraic group acting on a variety  $X$ . A *linearisation* of the action is a line bundle  $L \rightarrow X$  together with a choice of  $H$ -action on  $L$  such that

1. the bundle projection  $L \rightarrow X$  is  $H$ -equivariant; and
2. for each  $h \in H$  and  $x \in X$ , the induced map between the fibres

$$L|_x \rightarrow L|_{hx}, \quad l \mapsto hl$$

is linear.

*Remark 2.1.12.* If  $L \rightarrow X$  is a linearisation for the action of  $H$  on  $X$ , we will often represent this using the notation  $H \curvearrowright L \rightarrow X$ , or say that  $L \rightarrow X$  is an “ $H$ -linearisation” for short. In general we will not distinguish between the line bundle and the linearisation in our notation, unless this is likely to lead to confusion.

For practical purposes (e.g. the study of moduli problems) the following two classes of examples frequently arise.

*Example 2.1.13.* Consider the case where  $X = \operatorname{Spec} A$  affine and  $L = \mathcal{O}_X = X \times \mathbb{k}$  is the trivial line bundle. Then a linearisation of  $H$  on  $\mathcal{O}_X$  corresponds to a choice of character  $\chi : H \rightarrow \mathbb{G}_m$  [Dol03, Theorem 7.1 and Corollary 7.1] via

$$H \times (X \times \mathbb{k}) \rightarrow X \times \mathbb{k}, \quad (h, x, t) \mapsto (hx, \chi(h)t).$$

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<sup>2</sup>For slightly more modern treatments, see also [PV94, Theorem 4.4] and [Dol03, Theorem 6.2].

*Example 2.1.14.* Let  $V$  be a finite dimensional vector space over  $\mathbb{k}$  and  $\rho : H \rightarrow \mathrm{GL}(V)$  a homomorphism. Then  $H$  acts on  $\mathbb{P}(V)$  in the obvious manner, and  $\rho$  defines a canonical choice of linearisation on the tautological line bundle  $\mathcal{O}(-1) \rightarrow \mathbb{P}(V)$ . This dually defines a linearisation on  $\mathcal{O}(1) \rightarrow \mathbb{P}(V)$  (see below).

A linearisation  $H \curvearrowright L \rightarrow X$  gives us a natural action on the sections of  $L \rightarrow X$  over invariant open subsets  $U \subseteq X$ , by the formula

$$(h \cdot f)(x) = hf(h^{-1}x) \quad \text{for all } x \in U, f \in H^0(U, L), h \in H. \quad (2.2)$$

Given any invariant open subset  $U \subseteq X$  we write

$$H^0(U, L)^H := \{f \in H^0(U, L) \mid h \cdot f = f \text{ for all } h \in H\}$$

for the sections invariant under the action (2.2). Elements of  $H^0(U, L)^H$  are called *invariants* for the linearisation  $L|_U \rightarrow U$ .

*Remark 2.1.15.* We should point out here that saying a section  $f \in H^0(U, L)$  is invariant in this sense is the same as saying, in the terminology of Section 2.1.1, it is *H-equivariant* as a morphism  $f : U \rightarrow L$ , rather than necessarily invariant. When talking about sections of line bundles we always take invariance to be with respect to the action (2.2). Unfortunately both uses of the term ‘invariant’ are commonplace.

There are also various natural operations on linearisations over an  $H$ -variety arising from the standard operations on line bundles. Given an  $H$ -linearisation  $L \rightarrow X$ , the dual line bundle  $L^* \rightarrow X$  has a canonical linearisation, defined fibre-wise by pulling back linear maps along the action of  $H$  i.e. for any  $x \in X$ , an element  $h \in H$  acts via

$$(L|_x)^* \rightarrow (L|_{hx})^*, \quad (h, \alpha) \mapsto \alpha \circ h^{-1} : L|_{hx} \rightarrow \mathbb{k}.$$

(Note the use of  $h^{-1}$  is to ensure the resulting  $H$ -action is a left action.) Also, given two  $H$ -linearisations  $L_1 \rightarrow X$  and  $L_2 \rightarrow X$ , there is a canonical linearisation on the tensor product  $L_1 \otimes L_2 \rightarrow X$ , induced by the map on fibres

$$((L_1)|_x \otimes (L_2)|_x) \rightarrow (L_1)|_{hx} \otimes (L_2)|_{hx}, \quad (h, l_1 \otimes l_2) \mapsto (hl_1) \otimes (hl_2),$$

for any  $x \in X$  and  $h \in H$ .

A *character* of  $H$  is simply a group homomorphism  $H \rightarrow \mathbb{G}_m$ . Given a linearisation  $L \rightarrow X$  and a character  $\chi \in \mathrm{Hom}(H, \mathbb{G}_m)$ , we define a linearisation  $L^{(\chi)} \rightarrow X$ , which is said to be the result of *twisting*  $L$  by the character  $\chi$ , as the linearisation  $L \otimes \mathcal{O}_X^{(\chi)}$ , where  $\mathcal{O}_X^{(\chi)}$  is the linearisation of the trivial bundle  $\mathcal{O}_X = X \times \mathbb{k}$  defined by  $\chi^{-1}$  (see Example 2.1.13). In other words,  $L^{(\chi)} \rightarrow X$  is obtained by multiplying the fibres of the linearisation  $L \rightarrow X$  by the character  $\chi^{-1}$ .

Finally, if  $\phi : X \rightarrow Y$  is an equivariant morphism between two  $H$ -varieties and  $H \curvearrowright L \rightarrow Y$  is a linearisation, then there is a unique linearisation on the pullback line bundle  $\phi^*L \rightarrow X$  making the natural bundle map  $\phi^*L \rightarrow L$  equivariant. This linearisation makes the pullback map  $\phi^* : H^0(Y, L) \rightarrow H^0(X, \phi^*L)$  an  $H$ -equivariant linear map with respect to the actions defined as in (2.2).

Given a line bundle  $L \rightarrow X$ , define the *section ring* of  $L \rightarrow X$  to be the commutative graded ring

$$S := \mathbb{k}[X, L] := \bigoplus_{r \geq 0} H^0(X, L^{\otimes r}),$$

where the multiplication is induced by the natural maps

$$H^0(X, L^{\otimes r_1}) \otimes H^0(X, L^{\otimes r_2}) \rightarrow H^0(X, L^{\otimes (r_1 + r_2)}).$$

The action in (2.2) defines a linear action of  $H$  on  $\mathbb{k}[X, L]$  that respects the grading and distributes over the multiplicative structure. Given  $r > 0$  and an invariant global section  $f \in H^0(X, L^{\otimes r})^H$ , the open set

$$X_f := \{x \in X \mid f(x) \neq 0\}$$

is  $H$ -invariant. There is a naturally induced action of  $H$  on  $S_{(f)}$ , and the corresponding action on  $\mathcal{O}(X_f)$  under the canonical isomorphism  $S_{(f)} \cong \mathcal{O}(X_f)$  is the one defined by the formula in (2.1). A linearisation thus gives a way of studying the invariant functions on certain open subsets of  $X$ , which is an important consideration for the construction of geometric quotients (cf. Definition 2.1.4).

*Example 2.1.16.* In the case of Example 2.1.13, where  $X = \operatorname{Spec} A$  is affine,  $L = \mathcal{O}_X$  and the action of  $H$  on  $\mathcal{O}_X$  is defined by a character  $\chi : H \rightarrow \mathbb{G}_m$ , then  $S$  is the graded ring  $\bigoplus_{r \geq 0} A$  (with the grading corresponding to  $r$ ) and the ring of invariants  $S^H$  is the graded subring of *semi-invariants*,

$$\bigoplus_{r \geq 0} A_{\chi^r}^H, \quad A_{\chi^r}^H := \{f \in A \mid f(hx) = \chi(h)^r f(x) \text{ for all } x \in X, h \in H\};$$

see [Muk03, Chapter 6] or [PV94, §3].

*Example 2.1.17.* Suppose now  $X$  is a projective  $H$ -variety and  $L \rightarrow X$  a very ample linearisation (that is, a linearisation which is very ample as a line bundle). Letting  $V = H^0(X, L)^*$ , the natural graded ring map  $\operatorname{Sym}^\bullet H^0(X, L) \rightarrow \mathbb{k}[X, L]$  defines an embedding  $\phi : X \hookrightarrow \mathbb{P}(V)$ . Dualising the action of  $H$  on  $H^0(X, L)$  of (2.2) defines a canonical linearisation  $H \curvearrowright \mathcal{O}_{\mathbb{P}(V)}(1) \rightarrow \mathbb{P}(V)$  as in Example 2.1.14, with respect to which  $\phi$  is  $H$ -equivariant and  $L = \phi^* \mathcal{O}_{\mathbb{P}(V)}(1)$  as linearisations. If  $L \rightarrow X$  is sufficiently positive then the restriction map  $\phi^* : \mathbb{k}[\mathbb{P}(V), \mathcal{O}(1)] \rightarrow \mathbb{k}[X, L]$  is surjective by Serre Vanishing [Har77, Chapter 3, Proposition 5.3], so that  $\mathbb{k}[X, L]^H \cong (\mathbb{k}[\mathbb{P}(V), \mathcal{O}(1)] / \ker(\phi^*))^H$ . Note that in general the induced restriction map on invariants  $\phi^* : \mathbb{k}[\mathbb{P}(V), \mathcal{O}(1)]^H \rightarrow \mathbb{k}[X, L]^H$  is *not* surjective; that is, not every invariant section over  $X$  extends to one over  $\mathbb{P}(V)$ .

*Remark 2.1.18.* When  $X$  is a *normal* quasi-projective variety equipped with an action of a connected linear algebraic group  $H$ , then one can always find an equivariant embedding of  $X$  into some projective space  $\mathbb{P}^m$ , with the  $H$ -action on  $\mathbb{P}^m$  defined by some representation  $H \rightarrow \operatorname{GL}(m+1, \mathbb{k})$  (cf. Example 2.1.14 [MumFK94, Corollary 1.6] when  $X$  is complete, and for the more general case [Su74, Su75]). Hence any normal quasi-projective variety equipped with an action of a connected linear algebraic group admits a very ample linearisation. For work on linearisations of actions on more general varieties see [Bri15a].

We saw earlier that, given a linear algebraic group  $H$  acting on a variety  $X$ , the operations of tensor product and dualising may be applied to  $H$ -linearisations. These give an abelian group structure to the set  $\mathrm{Pic}^H(X)$  of isomorphism classes of  $H$ -linearised line bundles, such that the natural forgetful map  $\mathrm{Pic}^H(X) \rightarrow \mathrm{Pic}(X)$  to the usual Picard group of  $X$  is a homomorphism. We shall see that many constructions in geometric invariant theory are independent of taking positive tensor powers of a linearisation, therefore it is convenient to consider the following notion.

**Definition 2.1.19.** Given a linear algebraic group  $H$  acting on a variety  $X$ , define a *rational linearisation* to be an element of  $\mathrm{Pic}^H(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ .

*Remark 2.1.20.* Given an element  $\mathcal{L} \in \mathrm{Pic}^H(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ , we may write  $\mathcal{L} = \frac{1}{n}L$  for some integer  $n > 0$  and  $H$ -linearisation  $L \in \mathrm{Pic}^H(X)$ , and if  $\tilde{n} \in \mathbb{Z}_{>0}$  and  $\tilde{L} \in \mathrm{Pic}^H(X)$  are another such integer and linearisation then we have  $L^{\otimes \tilde{n}} = \tilde{L}^{\otimes n}$  within  $\mathrm{Pic}^H(X)$ . This observation allows us to define various geometric invariant theoretic notions for rational linearisations.

We conclude this section with a very useful observation regarding linearisations  $L \rightarrow X$ : the induced actions on  $H^0(X, L)$  are *locally finite* (also called *rational* in [Ne78]). In other words, we have the following result (see [MumFK94, Chapter 1, §1, Lemma], or [Bo91, Proposition 1.9] in the case  $L = \mathcal{O}_X$ ).

**Lemma 2.1.21.** *Let  $H$  be a linear algebraic group acting on a variety  $X$  and  $L \rightarrow X$  a linearisation. Given a finite-dimensional linear subspace  $W \subseteq H^0(X, L)$ , there is a finite-dimensional rational  $H$ -module  $V \subseteq H^0(X, L)$  containing  $W$ .*

### 2.1.3 Unipotent Groups and Reductive Groups

It turns out that the problem of constructing quotients and finding invariants for a given linearisation depends very much on the sort of linear algebraic group one is considering. In regards to this, it is helpful to focus one's attention on two particular sub-classes of group: *unipotent* groups and *reductive* groups. These are defined as follows.

**Definition 2.1.22.** Let  $H$  be a linear algebraic group.

1. [Bo91, Chapter 4] We say  $H$  is *unipotent* if there is a closed embedding  $\rho : H \hookrightarrow \mathrm{GL}(n, \mathbb{k})$ , for some  $n \geq 0$ , such that  $\rho(h) - \rho(e)$  is nilpotent in  $\mathrm{GL}(n, \mathbb{k})$  for each  $h \in H$ , i.e.  $(\rho(h) - \rho(e))^m = 0$  for some  $m \geq 0$  (depending on  $h$ ).<sup>3</sup>
2. [Bo91, §11.21] The *unipotent radical*  $H_u$  of  $H$  is the maximal connected<sup>4</sup> normal unipotent subgroup of  $H$ .
3. [Bo91, §11.21] We say  $H$  is *reductive* if  $H_u = \{e\}$ .

The following are well-known examples of reductive and unipotent groups.

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<sup>3</sup>If this is the case, then in fact for *any* closed embedding  $\rho : H \hookrightarrow \mathrm{GL}(\tilde{n}, \mathbb{k})$  into any general linear group  $\mathrm{GL}(\tilde{n}, \mathbb{k})$  one has  $\rho(h) - \rho(e)$  nilpotent for each  $h \in H$ ; cf. [Bo91, Theorem 4.4].

<sup>4</sup>In fact, over characteristic zero all unipotent groups are necessarily connected; see [DemG70, Chapter II, §6.3].

*Example 2.1.23.* (Reductive groups; see [Sp94].) The classical groups  $\mathrm{GL}(n, \mathbb{k})$ ,  $\mathrm{SL}(n, \mathbb{k})$ ,  $\mathrm{Sp}(2n, \mathbb{k})$  and  $\mathrm{SO}(n, \mathbb{k})$  are reductive. Products of reductive groups are again reductive; in particular, groups isomorphic to products of  $\mathbb{G}_m$  (which are called *tori*) are reductive. All finite groups and all semisimple linear algebraic groups are reductive.

*Example 2.1.24.* (Unipotent groups.) The group  $\mathbb{G}_a$  is unipotent. The group

$$\mathbb{U}_n := \{(a_{ij}) \in \mathrm{GL}(n, \mathbb{k}) \mid a_{ii} = 1 \text{ for each } i = 1, \dots, n \text{ and } a_{ij} = 0 \text{ whenever } j < i\}$$

of strictly upper triangular inside  $\mathrm{GL}(n, \mathbb{k})$  is unipotent, for each  $n \geq 1$ . In fact, a group  $H$  is unipotent if, and only if, it is isomorphic to a closed subgroup of  $\mathbb{U}_n$  for some  $n \geq 1$  [Bo91, Theorem 4.8]. Products of unipotent groups are unipotent, and all subgroups of unipotent groups are unipotent.

Given a linear algebraic group  $H$ , its quotient by the unipotent radical,

$$H_r := H/H_u$$

is a reductive group. Moreover, given a variety  $X$  and a normal subgroup  $N$  of  $H$ , it is easy to show that if  $X$  has a geometric  $N$ -quotient  $X/N$  and  $X/N$  has a geometric  $H/N$ -quotient  $(X/N)/(H/N)$ , then the geometric quotient for the  $H$ -action on  $X$  exists, with  $X/H = (X/N)/(H/N)$ . This suggests that a natural way to construct geometric quotients by general linear algebraic groups is to try and understand the construction of quotients for unipotent groups and reductive groups.

Reductive groups and unipotent groups behave rather differently from the point of view of invariant theory. On the one hand, reductive groups have a well behaved representation theory. Any representation of a reductive group can be written as a direct sum of irreducible representations [Sp94, §4.6.6], which has a number of important consequences. Firstly, given a finitely generated  $\mathbb{k}$ -algebra  $A$  and a reductive group  $G$  acting on  $A$  in a locally finite fashion, the ring of invariants  $A^G$  is also finitely generated over  $\mathbb{k}$  by a theorem of Nagata [Na64]. Thus  $\mathrm{Spec}(A^G)$  is an affine *variety*. Secondly, one has the following result (see [Na64, Lemma 5.1.A]): if  $I \subseteq A$  is a  $G$ -invariant ideal, then any invariant element of  $A/I$  lifts to an invariant in  $A$ ; geometrically stated, any invariant regular function on a  $G$ -invariant closed subset  $Z$  of  $\mathrm{Spec} A$  extends to  $G$ -invariant regular function on the whole of  $\mathrm{Spec} A$ . Thirdly, given two distinct ideals  $I, J$  of  $A$  invariant under the  $G$ -action and such that  $I + J = A$ , one can always find an element of  $A^G$  contained in one but not the other (this follows from a result of Haboush [Hab75], see also [Ne78, Lemma 3.3]). Geometrically this says that any two disjoint closed invariant subsets of  $\mathrm{Spec} A$  can be separated by an invariant function—this implies property 2c of Definition 2.1.4 of a good quotient. The upshot is that reductive group actions on affine varieties are amenable to constructing good quotients; this will be the content of Theorem 2.2.1 in the next section.

All three of the above properties fail for non-reductive group actions. The issue of whether the ring of invariants is finitely generated, which is closely related to the fourteenth problem of Hilbert<sup>5</sup>, has arguably received the most attention historically. The following celebrated example of Nagata [Na59] demonstrates that the ring of invariants for a non-reductive group need not be finitely generated. (We follow the exposition of [Dol03, §4.3].)

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<sup>5</sup>A good survey of counterexamples to Hilbert's fourteenth problem from an invariant theoretic perspective is [Fr01].



*Example 2.1.25.* Given  $n > 0$ , let  $X = \mathbb{k}^2 \oplus \cdots \oplus \mathbb{k}^2$  ( $n$  times) and consider the action of  $H'_1 \times \cdots \times H'_n$  on  $X$ , where  $H'_i = \left\{ \begin{pmatrix} c_i & a_i \\ 0 & c_i \end{pmatrix} \mid a_i, c_i \in \mathbb{k}, c_i \neq 0 \right\}$  acts on the  $i$ -th factor of  $\mathbb{k}^2$  in  $X$  by usual matrix multiplication. Let  $H \subseteq H'_1 \times \cdots \times H'_n$  be the subgroup obtained by demanding that  $c_1 \cdots c_n = 1$  and  $(a_1, \dots, a_n)$  satisfy three suitable linear equations  $\sum_j x_{i,j} a_j = 0$ , ( $i = 1, 2, 3$ ). Then for  $n = 16$ , the ring of invariants  $\mathcal{O}(X)^H$  is not finitely generated over  $\mathbb{k}$ . It follows that  $\mathcal{O}(X)^{H_u}$  is not finitely generated over  $\mathbb{k}$ , where  $H_u$  is the unipotent radical of  $H$  defined by  $c_i = 1$  for each  $i = 1, \dots, n$  (see [Na60]).

Despite suffering representation-theoretic deficiencies from not being reductive, unipotent group actions nevertheless have their own distinctive invariant theoretic flavour; indeed, topologically they can be better behaved than reductive groups. For example, every action of a unipotent group on an affine variety is closed. Somewhat more strikingly, every unipotent group is *special*: any principal bundle of a unipotent group is *Zariski*-locally trivial, not just locally trivial in the isotrivial topology [Ser58, Proposition 14]. Finally, affine varieties that admit affine locally trivial quotients are easily recognisable thanks to the next result.

**Proposition 2.1.26.** *Suppose  $X$  is an affine variety acted upon by a unipotent group  $U$  and a locally trivial quotient  $X \rightarrow X/U$  exists. Then  $X/U$  is affine if and only if  $X \rightarrow X/U$  is a trivial  $U$ -bundle.*

*Proof.* This is an immediate consequence of [AsD07, Theorem 3.14], which involves cohomological techniques from [GeP93]. However these techniques are not needed for this result: it is easy to prove by induction on the dimension of  $U$  that every principal  $U$ -bundle over an affine base is trivial. Conversely if  $X \rightarrow X/U$  is a trivial  $U$ -bundle, so that  $X \cong X/U \times U$ , then the unit morphism  $\mathrm{Spec} \mathbb{k} \rightarrow U$  gives us a closed immersion  $X/U \rightarrow X$  and so  $X/U$  is affine.  $\square$

## 2.2 Mumford's Geometric Invariant Theory for Reductive Groups

In the first edition of [MumFK94] Mumford introduced his geometric invariant theory (GIT) to give both a theoretical framework and computational tools for finding invariant open sets of points inside a  $G$ -variety  $X$  that admit geometric quotients, when  $G$  is a reductive linear algebraic group. On the theoretical side, a basic strategy for constructing such open subsets is to patch together quotients of open affines arising from the data of a linearisation. Because the invariant theory of reductive groups is well behaved, such quotients are easy to describe.

**Theorem 2.2.1.** *[MumFK94, Chapter 1, §2]<sup>6</sup> Let  $X = \mathrm{Spec} A$  be an affine variety upon which a reductive group  $G$  acts. Then*

1. *the natural map  $\phi : X \rightarrow \mathrm{Spec}(A^G)$  induced by the inclusion  $A^G \hookrightarrow A$  is a good categorical quotient; and*
2. *the set  $U := \{x \in X \mid G \cdot x \text{ is closed and } \mathrm{Stab}_G(x) \text{ is finite}\}$  is an open subset of  $X$ , and the restriction of  $\phi$  to  $U$  gives a geometric quotient  $U \rightarrow \phi(U)$  for the  $G$ -action on  $U$ , with  $\phi(U)$  open in  $\mathrm{Spec}(A^G)$ .*

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<sup>6</sup>While Theorem 2.2.1, 2 is not stated explicitly in [MumFK94, Chapter 1, §2], it follows easily from the material there, together with the Closed Orbit Lemma [Bo91, Proposition 1.8] and the lower semi-continuity of the function  $x \mapsto \dim(H \cdot x)$ . See [Ne78, Proposition 3.8] for a proof.

To deal with the more general case where  $G$  acts on any variety  $X$ , Mumford used the extra data of a linearisation  $L \rightarrow X$  to define  $G$ -invariant open subsets which are obtained by patching affine open subsets of the form  $X_f$ , for  $f$  an invariant section of a positive tensor power of  $L \rightarrow X$ .

**Definition 2.2.2.** [MumFK94, Chapter 1, §4] Let  $X$  be a  $G$ -variety and  $L \rightarrow X$  a linearisation. A point  $x \in X$  is called

1. *semistable* if there is an invariant  $f \in H^0(X, L^{\otimes r})^G$ , with  $r > 0$ , such that  $f(x) \neq 0$  and  $X_f$  is affine; and
2. *stable* if there is an invariant  $f \in H^0(X, L^{\otimes r})^G$ , with  $r > 0$ , such that  $X_f$  is affine, the  $G$ -action on  $X_f$  is closed and  $\text{Stab}_G(y)$  is finite for all  $y \in X_f$ .

We denote the subset of semistable (respectively, stable) points by  $X^{\text{ss}(L)}$  (respectively  $X^{\text{s}(L)}$ ), dropping the mention of  $L$  if there is no risk of confusion. Note that we have followed [Ne78, Chapter 3, §5] in requiring finite stabilisers for Definition 2.2.2, 2 of ‘stable’; this corresponds to Mumford’s definition of ‘properly stable’ in [MumFK94, Definition 1.8].

*Remark 2.2.3.* The sets  $X^{\text{ss}}$  and  $X^{\text{s}}$  are  $G$ -invariant open subsets of  $X$  that may be defined for rational linearisations  $\mathcal{L}$ , in the following way: if  $n > 0$  is an integer such that  $L = n\mathcal{L}$  is in  $\text{Pic}^G(X)$ , then define  $X^{\text{ss}(\mathcal{L})} = X^{\text{ss}(L)}$  and  $X^{\text{s}(\mathcal{L})} = X^{\text{s}(L)}$ . Since  $X_f = X_{f^m}$  for any global section  $f$  of a line bundle and any integer  $m > 0$ , this is well-defined by Remark 2.1.20. In fact, for ample  $L \rightarrow X$  the sets  $X^{\text{s}(L)}$  and  $X^{\text{ss}(L)}$  depend only on the *fractional linearisation* class of  $L$ , in the sense of Thaddeus [T96].

The definitions of semistability and stability, respectively, are so specified as to allow one to glue the good or geometric quotients  $X_f \rightarrow \text{Spec}(\mathcal{O}(X_f)^G)$ , respectively, of the affine  $X_f$ . The central result of Mumford’s geometric invariant theory says that these quotients can be glued into *quasi-projective varieties*.

**Theorem 2.2.4.** [MumFK94, Theorem 1.10] Let  $G$  be a reductive group acting on a variety  $X$  and  $L \rightarrow X$  a linearisation for the action. Then

1. the semistable locus  $X^{\text{ss}}$  has a good categorical quotient  $\phi : X^{\text{ss}} \rightarrow X//G$  onto a quasi-projective variety  $X//G$ , and there is an ample line bundle  $M \rightarrow X//G$  pulling back to a positive tensor power of  $L|_{X^{\text{ss}}}$  under  $\phi$ ; and
2. the image of  $X^{\text{s}}$  under  $\phi$  is an open subset of  $X//G$ , and the restriction of  $\phi$  to  $X^{\text{s}}$  gives a geometric quotient  $\phi : X^{\text{s}} \rightarrow \phi(X^{\text{s}})$  for the action of  $G$  on  $X^{\text{s}}$ .

The variety  $X//G$  is called the *GIT quotient* for the linearisation  $G \curvearrowright L \rightarrow X$ . By a result of Seshadri [Ses77, Proposition 9] the GIT quotient  $X//G$  can be regarded topologically as the quotient  $X^{\text{ss}}/\sim$  of  $X^{\text{ss}}$  under the ‘S-equivalence’ relation  $\sim$ , where  $x_1 \sim x_2$  if, and only if, the closures of  $G \cdot x_1$  and  $G \cdot x_2$  in  $X^{\text{ss}}$  intersect nontrivially.

The affine case of Theorem 2.2.1 can be recovered from Theorem 2.2.4 by considering the linearisation of  $L = \mathcal{O}_X \rightarrow X$  defined by the trivial character from Examples 2.1.13 and 2.1.16:

in this case the constant function  $1 \in H^0(X, L)$  is an invariant, so  $X^{\text{ss}(\mathcal{O}_X)} = X$ , and it follows immediately that  $U$  from Theorem 2.2.1 is equal to the stable locus  $X^{\text{s}(\mathcal{O}_X)}$ .

Another important special case of Theorem 2.2.4 is when  $X$  is *projective* and  $L \rightarrow X$  is an *ample* linearised line bundle. In this case the ring of invariant sections  $S^G$  is a finitely generated  $\mathbb{k}$ -algebra by Nagata's theorem [Na64], and  $X//G = \text{Proj}(S^G)$  is the projective variety associated to the graded ring  $S^G$  [MumFK94, Page 40]. The good quotient  $\phi : X^{\text{ss}} \rightarrow X//G$  is a representative of the rational map  $X \dashrightarrow \text{Proj}(S^G)$  induced by the inclusion  $S^G \hookrightarrow S$ , thus we have a commutative diagram, with all inclusions open:

$$\begin{array}{ccccc} X^{\text{s}} & \subseteq & X^{\text{ss}} & \subseteq & X \\ \text{geo} \downarrow \phi & & \text{good} \downarrow \phi & & \downarrow \\ X^{\text{s}}/G & \subseteq & X//G & = & \text{Proj}(S^G) \end{array} \quad (2.3)$$

Note in the case that the GIT quotient  $X//G$  may therefore be regarded as a canonical compactification of the geometric quotient  $X^{\text{s}}/G$  of the stable locus.

Another appealing feature of the case where  $X$  is projective and  $L \rightarrow X$  is ample is that there is an effective way to compute the semistable and stable loci, via the Hilbert-Mumford criteria. For completeness we present here two versions of this result. Firstly, if  $G$  is any reductive group, then a *one-parameter subgroup* of  $G$  (or *1-PS* for short) is simply a non-trivial homomorphism  $\lambda : \mathbb{G}_m \rightarrow G$ . Given a point  $x \in X$  and a one-parameter subgroup  $\lambda$ , the limit<sup>7</sup>  $x_0 := \lim_{t \rightarrow 0} \lambda(t) \cdot x$  is a well-defined point in  $X$ , because  $X$  is proper. The subgroup  $\lambda$  fixes  $x_0$ , hence  $\mathbb{G}_m$  acts on the fibre  $L|_{x_0}$  over  $x_0$  via  $\lambda$ ; define

$$\mu(x, \lambda) := \text{weight for the } \mathbb{G}_m\text{-action on } L|_{x_0}.$$

Then the first form of the Hilbert-Mumford criteria can be stated as follows.

**Theorem 2.2.5.** [MumFK94, Theorem 2.1] *Let  $G$  be a reductive group,  $X$  a projective  $G$ -variety and  $L \rightarrow X$  an ample linearisation. Then for any point  $x \in X$ ,*

$$\begin{aligned} x \in X^{\text{ss}} &\iff \mu(x, \lambda) \geq 0 \text{ for all 1-PS } \lambda : \mathbb{G}_m \rightarrow G; \\ x \in X^{\text{s}} &\iff \mu(x, \lambda) > 0 \text{ for all 1-PS } \lambda : \mathbb{G}_m \rightarrow G. \end{aligned}$$

The second form of the Hilbert-Mumford criteria makes use of an embedding in a projective space. Suppose still that  $X$  is a projective  $G$ -variety, with  $G$  reductive, but now assume  $L \rightarrow X$  is a very ample linearisation. As in Example 2.1.17,  $X$  embeds into the projective space  $\mathbb{P}(V)$  equivariantly, where  $V = H^0(X, L)^*$ . Fix a maximal torus  $T \subseteq G$  and let  $\text{Hom}(T, \mathbb{G}_m)$  be the abelian group of characters of  $T$ . The action of  $T$  on  $V$  is diagonalisable [Bo91, Proposition 8.4], so we may decompose  $V$  into  $T$ -weight spaces:

$$V = \bigoplus_{\chi \in \text{Hom}(T, \mathbb{G}_m)} V_{\chi}, \quad V_{\chi} := \{v \in V \mid t \cdot v = \chi(t)v \text{ for all } t \in T\}.$$

---

<sup>7</sup>By ' $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ ' we mean the value at 0 in  $\mathbb{k}$  of  $\phi : \mathbb{k} \rightarrow X$ , where  $\phi$  is the unique extension of the morphism of varieties  $t \mapsto \lambda(t) \cdot x$  to a morphism on  $\mathbb{k}$ .

Given  $x \in X$ , write  $x = [v] \in \mathbb{P}(V)$  with  $v \in V \setminus \{0\}$  and let  $v = \sum_{\chi} v_{\chi}$  with  $v_{\chi} \in V_{\chi}$ . Define the *weight polytope* of  $x$  to be

$$\Delta_x := \overline{\text{convex hull of } \{\chi \mid v_{\chi} \neq 0\}} \subseteq \text{Hom}(T, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{R},$$

where the closure is taken with respect to the usual Euclidean topology on the vector space  $\text{Hom}(T, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{R}$ . Denote the interior of  $\Delta_x$  inside  $\text{Hom}(T, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{R}$  by  $\Delta_x^{\circ}$ . Then the Hilbert-Mumford criteria can be stated in the following way (see [Dol03, Theorem 9.2] and [Dol03, Theorem 9.3]).

**Theorem 2.2.6.** *Retain the preceding notation.*

1. *A point  $x \in X$  is semistable (respectively, stable) for  $G \curvearrowright L \rightarrow X$  if, and only if, for each  $g \in G$  the point  $gx$  is semistable (respectively, stable) for the restricted linearisation  $T \curvearrowright L \rightarrow X$ .*
2. *For any point  $x \in X$  we have*

$$\begin{aligned} x \text{ is semistable for } T \curvearrowright L \rightarrow X &\iff 0 \in \Delta_x; \\ x \text{ is stable for } T \curvearrowright L \rightarrow X &\iff 0 \in \Delta_x^{\circ}. \end{aligned}$$

Thus we see that in the case where  $X$  is projective and  $L \rightarrow X$  is very ample, semistability and stability can be computed in terms of weights for torus actions.

## 2.3 Geometric Invariant Theory for Unipotent Groups

Given the effectiveness of Mumford's GIT for studying quotients of reductive groups, there has been interest in developing a similar geometric invariant theoretic approach to studying unipotent group actions. Such a programme is taken up in the paper *Towards non-reductive geometric invariant theory* [DorK07], building on previous work such as [Fa83, Fa85, GeP93, GeP98, Wi03]. Given an irreducible projective variety  $X$  with an action of a unipotent group  $U$  and an ample linearisation  $L \rightarrow X$  for the action, the paper considers various notions of 'stability' (intrinsic to the data of the linearisation  $L \rightarrow X$ ) that admit geometric quotients, formulates an analogue of Mumford's reductive GIT quotient in this context, and relates these notions to those of reductive GIT for the purposes of computation. In this section we summarise the main definitions and results presented there, which will form the backbone of our development of a geometric invariant theory for more general linear algebraic groups in upcoming sections. We also take the opportunity to point out some errors in [DorK07], but leave details of how to correct them to Section 3.

We assume for the rest of this section that  $U$  is a unipotent group acting on an irreducible projective variety  $X$  with ample linearisation  $L \rightarrow X$ .

### 2.3.1 Intrinsic Notions of Semistability and Stability

As in Section 2.1.2, let  $S = \mathbb{k}[X, L]$  be the section ring. The inclusion  $S^U \hookrightarrow S$  defines a rational map

$$q : X \dashrightarrow \text{Proj}(S^U)$$

which is  $U$ -invariant on its maximal domain of definition.

**Definition 2.3.1.** [DorK07, Definition 4.1.1] Let  $U$  be a unipotent linear algebraic group acting on an irreducible projective variety  $X$  and  $L \rightarrow X$  an ample linearisation. The *naively semistable locus* is the open subset

$$X^{\text{nss}} := \bigcup_{f \in I^{\text{nss}}} X_f$$

of  $X$ , where  $I^{\text{nss}} := \bigcup_{r>0} H^0(X, L^{\otimes r})^U$  is the set of invariant sections of positive tensor powers of  $L$ .

The rational map  $q$  restricts to define a  $U$ -invariant morphism  $q : X^{\text{nss}} \rightarrow \text{Proj}(S^U)$ . As Nagata showed [Na59], the ring of invariants  $S^U$  need not be finitely generated over  $\mathbb{k}$ , so  $\text{Proj}(S^U)$  is in general a non-noetherian scheme. It can also happen that this map is not surjective, with the image only a dense constructible subset of  $\text{Proj}(S^U)$  in general, even if  $\text{Proj}(S^U)$  is of finite type (an example of this phenomenon—which features later in Example 2.3.19—is given in [DorK07, §6]). To address the first of these issues, it is natural to consider the following subset of  $X^{\text{nss}}$ .

**Definition 2.3.2.** [DorK07, Definition 4.2.6] Let  $U$  be a unipotent linear algebraic group acting on an irreducible projective variety  $X$  and  $L \rightarrow X$  an ample linearisation. The *finitely generated semistable locus* is the open subset

$$X^{\text{ss,fg}} := \bigcup_{f \in I^{\text{ss,fg}}} X_f$$

of  $X^{\text{nss}}$ , where

$$I^{\text{ss,fg}} := \{f \in \bigcup_{r>0} H^0(X, L^{\otimes r})^U \mid \mathcal{O}(X_f)^U \text{ is a finitely generated } \mathbb{k}\text{-algebra}\}.$$

The image of  $X^{\text{ss,fg}}$  under the map  $q : X^{\text{nss}} \rightarrow \text{Proj}(S^U)$  is contained in the open subscheme of  $\text{Proj}(S^U)$  obtained by patching together the affine open subsets  $\text{Spec}(\mathcal{O}(X_f)^U)$  for which  $\mathcal{O}(X_f)^U$  is a finitely generated  $\mathbb{k}$ -algebra.

**Definition 2.3.3.** [DorK07, Definition 4.2.7] Let  $U$  be a unipotent linear algebraic group acting on an irreducible projective variety  $X$  and  $L \rightarrow X$  an ample linearisation. The *enveloping quotient* is the open subscheme

$$X//U := \bigcup_{f \in I^{\text{ss,fg}}} \text{Spec}(\mathcal{O}(X_f)^U) \subseteq \text{Proj}(S^U)$$

of  $\text{Proj}(S^U)$ , together with the canonical map  $q : X^{\text{ss,fg}} \rightarrow X//U$ . The image  $q(X^{\text{ss,fg}})$  of this map is called the *enveloped quotient*.

The enveloping quotient  $X//U$  is canonical to the data of the linearisation and, as we will shortly see, in some sense plays the role of Mumford's reductive GIT quotient [MumFK94]. But there are two significant differences from the reductive case to be aware of (compare with the

discussion after Theorem 2.2.4). Firstly,  $X//U$  is not a projective variety in general (however if  $S^U$  is a finitely generated  $\mathbb{k}$ -algebra then  $X^{\text{ss},\text{fg}} = X^{\text{nss}}$  and  $X//U = \text{Proj}(S^U)$  is a projective variety). Secondly, the map  $q : X^{\text{ss},\text{fg}} \rightarrow X//U$  is not necessarily surjective and the image  $q(X^{\text{ss},\text{fg}})$  is not necessarily a variety (even when  $S^U$  is finitely generated). In particular, neither  $X//U$  nor  $q(X^{\text{ss},\text{fg}})$  are a categorical quotient of  $X^{\text{ss},\text{fg}}$  in general.

*Remark 2.3.4.* The enveloping quotient  $X//U$  is a scheme *locally* of finite type. In [DorK07, Proposition 4.2.9] it is erroneously claimed that  $X//U$  is a quasi-projective variety. The basic problem is that it is not necessarily quasi-compact: the ideal in  $S^U$  generated by  $I^{\text{ss},\text{fg}}$  may not satisfy the ascending chain condition and we cannot guarantee that finitely many of the affine open subsets  $\text{Spec}(\mathcal{O}(X_f)^U)$ , for  $f \in I^{\text{ss},\text{fg}}$ , cover  $X//U$ . (In the proof of [DorK07, Proposition 4.2.9] it is implicitly assumed such a finite cover of  $X//U$  exists in order to construct an embedding of  $X//U$  into a projective space.) Of course  $X^{\text{ss},\text{fg}}$  and  $X^{\text{nss}}$  are quasi-compact because they are quasi-projective. Geometrically speaking, the problem with the enveloping quotient  $X//U$  is that even though finitely many of the open sets  $X_f$ , with  $f \in I^{\text{ss},\text{fg}}$ , cover  $X^{\text{ss},\text{fg}}$ , the enveloping quotient map  $q : X^{\text{ss},\text{fg}} \rightarrow X//U$  is not surjective in general. However, if either  $S^U$  is a finitely generated  $\mathbb{k}$ -algebra or the enveloping quotient map  $q : X^{\text{ss},\text{fg}} \rightarrow X//U$  is surjective then the proof of [DorK07, Proposition 4.2.9] goes through to show that  $X//U$  is a quasi-projective variety.

The finitely generated semistable locus  $X^{\text{ss},\text{fg}}$  is analogous to Mumford's notion of semistability in reductive GIT (cf. Definition 2.2.2, 1), and indeed in [DorK07, Definition 5.3.7] the set  $X^{\text{ss},\text{fg}}$  is dubbed the 'semistable' locus for the linearisation  $U \curvearrowright L \rightarrow X$ . (In this article we will refrain from referring to  $X^{\text{ss},\text{fg}}$  as the 'semistable set' to preserve continuity with the more general non-reductive setting, to be discussed in Section 3.)

Various kinds of 'stable' set are also considered in [DorK07], each of which are subsets of  $X^{\text{ss},\text{fg}}$  whose images under the enveloping quotient map  $q : X^{\text{ss},\text{fg}} \rightarrow X//U$  define geometric quotients. One of the conclusions of that paper is that the following 'locally trivial' version of stability is well suited to studying linearised actions of unipotent groups.

**Definition 2.3.5.** [DorK07, Definition 4.2.6] Let  $U$  be a unipotent linear algebraic group acting on an irreducible projective variety  $X$  and  $L \rightarrow X$  an ample linearisation. The set of *locally trivial stable* points (later called the set of *stable* points in [DorK07, Definition 5.3.7]) is the set

$$X^{\text{s}} = X^{\text{lts}} = \bigcup_{f \in I^{\text{lts}}} X_f,$$

where

$$I^{\text{lts}} := \left\{ f \in \bigcup_{r>0} H^0(X, L^{\otimes m})^U \mid \begin{array}{l} \mathcal{O}(X_f)^U \text{ is a finitely generated } \mathbb{k}\text{-algebra and} \\ q : X_f \rightarrow \text{Spec}(\mathcal{O}(X_f)^U) \text{ is a trivial } U\text{-bundle} \end{array} \right\}.$$

**Proposition 2.3.6.** [DorK07, §4] Let  $U$  be a unipotent linear algebraic group acting on an irreducible projective variety  $X$  and  $L \rightarrow X$  an ample linearisation. The image  $q(X^{\text{s}})$  of  $X^{\text{s}}$  under the enveloping quotient map  $q : X^{\text{ss},\text{fg}} \rightarrow X//U$  is an open subscheme of  $X//U$  that is a

quasi-projective variety, and  $q : X^s \rightarrow q(X^s)$  is a geometric quotient:

$$\begin{array}{ccccc} X^s & \subseteq & X^{ss,fg} & \subseteq & X^{nss} \\ \text{geo} \downarrow & & \downarrow & & \downarrow q \\ q(X^s) & \subseteq & X//U & \subseteq & \text{Proj}(S^U) \end{array}$$

It is helpful to compare this to the case where  $G$  is reductive and  $L \rightarrow X$  is an ample linearisation over a projective  $G$ -variety. The diagram in Proposition 2.3.6 is similar to (2.3), but the unipotence of  $U$  leads to the two main differences mentioned earlier: the enveloping quotient  $X//U$  need not be projective and  $q : X^{ss,fg} \rightarrow X//U$  is not in general a good categorical quotient.

*Remark 2.3.7.* It is clear from Remark 2.1.20 and the definitions that  $X^{nss}$ ,  $X^{ss,fg}$ ,  $X^s$  and  $X//U$  may be defined for rational linearisations.

### 2.3.2 Extending to Reductive Linearisations

For the rest of this section, we assume that the linearisation  $U \curvearrowright L \rightarrow X$  is such that  $X//U$  is quasi-projective (see Remark 2.3.4).

A rather helpful approach to studying the  $U$ -linearisation  $L \rightarrow X$ , and the spaces  $X^s$ ,  $X^{ss,fg}$  and  $X//U$  thus arising, is to construct an associated linearisation of a reductive group  $G$  which contains  $U$  as a closed subgroup, by making use of the fibre space associated to the homogeneous space  $G/U$ . We take a moment to recall the general construction of such fibre spaces.

Let  $H_1$  and  $H_2$  be linear algebraic groups and suppose  $H_1 \hookrightarrow H_2$  is a closed embedding. For the moment suppose also that  $X$  is any  $H_1$ -variety. Then we may consider the diagonal action of  $H_1$  on the product  $H_2 \times X$ :

$$H_1 \curvearrowright H_2 \times X, \quad h_1 \cdot (h_2, x) := (h_2 h_1^{-1}, h_1 x),$$

where  $h_1 \in H_1$ ,  $h_2 \in H_2$  and  $x \in X$ . If  $H_1$  is unipotent, or if  $X$  satisfies some mild assumptions—for example, if  $X$  is quasi-projective, or more generally if any finite set of points in  $X$  is contained in an affine open subset—the geometric quotient  $H_2 \times^{H_1} X$  for this action exists as a variety<sup>8</sup> (see [EG98, Proposition 23] in case  $H_1$  is unipotent,<sup>9</sup> or [PV94, Theorem 4.19] otherwise). This quotient is the associated fibre space of the principal  $H_1$ -bundle  $H_2 \rightarrow H_2/H_1$  with fibre  $X$ ; see [Ser58, §3.2]. We shall write points in  $H_2 \times^{H_1} X$  as equivalence classes  $[h_2, x]$  of points  $(h_2, x) \in H_2 \times X$ . The action of  $H_2$  on  $H_2 \times^{H_1} X$  induced by left multiplication of  $H_2$  on itself makes  $H_2 \times^{H_1} X$  into an  $H_2$ -variety. Note there is a natural closed immersion

$$\alpha : X \hookrightarrow H_2 \times^{H_1} X, \quad x \mapsto [e, x],$$

which is  $H_1$ -equivariant with respect to  $H_1$  acting on  $H_2 \times^{H_1} X$  through the action of  $H_2$ .

<sup>8</sup>Indeed,  $H_2 \times X \rightarrow H_2 \times^{H_1} X$  is in fact a principal  $H_1$ -bundle [Ser58, Proposition 4].

<sup>9</sup>When  $H_1$  is unipotent the  $H_1$ -bundle  $H_2 \rightarrow H_2/H_1$  is Zariski locally trivial and thus the geometric quotient  $H_2 \times^{H_1} X$  can be shown to exist by working locally over  $H_2/H_1$ .

Suppose  $L \rightarrow X$  is a linearisation for the  $H_1$ -action on  $X$ . Then this extends to a natural  $H_2$ -linearisation  $H_2 \times^{H_1} L \rightarrow H_2 \times^{H_1} X$  which pulls back to  $L \rightarrow X$  under  $\alpha$ , using the same constructions as above. For brevity, we will usually abuse notation and write  $L \rightarrow H_2 \times^{H_1} X$  for this linearisation instead of  $H_2 \times^{H_1} L$ , unless confusion is likely to arise. Observe that, because the projection  $H_2 \times X \rightarrow H_2 \times^{H_1} X$  is a categorical quotient, pullback along  $\alpha$  induces an isomorphism of graded rings

$$\alpha^* : \mathbb{k}[H_2 \times^{H_1} X, L]^{H_2} \xrightarrow{\cong} \mathbb{k}[X, L]^{H_1}.$$

Let us now return to the setting where  $U$  is a unipotent group acting on an irreducible projective variety  $X$  with ample linearisation  $L \rightarrow X$ . Following [DorK07, §5.1], given a closed embedding of  $U$  into some reductive group  $G$  (e.g.  $G = \mathrm{GL}(n, \mathbb{k})$  for suitable  $n$ ), consider the  $G$ -linearisation  $G \curvearrowright L = G \times^U L \rightarrow G \times^U X$ . This is a linearisation over the quasi-projective variety  $G \times^U X$ , so it makes sense to ask for semistability and stability, in the sense of Mumford's reductive GIT (Definition 2.2.2).

**Definition 2.3.8.** [DorK07, Definition 5.1.6] Let  $U$  be a unipotent group contained in a reductive group  $G$  as a closed subgroup and let  $L \rightarrow X$  be an ample  $U$ -linearisation over a projective  $U$ -variety  $X$ . Define the set of *Mumford stable* points to be

$$X^{\mathrm{ms}} := \alpha^{-1}((G \times^U X)^s)$$

and the set of *Mumford semistable* points to be

$$X^{\mathrm{mss}} := \alpha^{-1}((G \times^U X)^{\mathrm{ss}})$$

where (semi)stability of  $G \times^U X$  is defined as in Definition 2.2.2 with respect to the  $G$ -linearisation  $G \times^U L \rightarrow G \times^U X$ , and  $\alpha : X \hookrightarrow G \times^U X$  is the natural closed immersion.

These sets would appear to depend on the choice of  $G$  and embedding  $U \hookrightarrow G$ , but in fact this is not the case by virtue of the following result.

**Proposition 2.3.9.** [DorK07, Lemma 5.1.7 and Proposition 5.1.10] *Given a unipotent group  $U$ , a reductive group  $G$  containing  $U$  as a closed subgroup and an ample  $U$ -linearisation  $L \rightarrow X$  of a projective  $U$ -variety  $X$ , we have equalities*

$$X^{\mathrm{mss}} = X^{\mathrm{ms}} = X^{\mathrm{lbs}}.$$

There are two main facts used in the proof that  $X^{\mathrm{mss}} = X^{\mathrm{ms}}$ . The first is that any stabiliser of a point with a closed  $G$ -orbit in  $(G \times^U X)^{\mathrm{ss}}$  must have a *reductive* stabiliser (by Matsushima's criterion [PV94, Theorem 4.17]) that is also a subgroup of  $U$ , hence is trivial. The second is that any  $U$ -orbit of a  $U$ -stable affine subvariety is necessarily closed. Both of these rely on the unipotency of  $U$  in an essential way. The equality  $X^{\mathrm{ms}} = X^{\mathrm{lbs}}$  is established by using descent to relate the notions of  $U$ -local triviality of suitable affine open subsets of  $X$  with  $G$ -local triviality of the corresponding subsets in  $G \times^U X$ .

As we observed above,  $G \times^U X$  is only a *quasi*-projective variety, so computing (semi)stability for the linearisation  $G \times^U L \rightarrow G \times^U X$  is difficult in general; on the other hand, reductive GIT



is very effective at dealing with ample linearisations over *projective* varieties. Therefore it is reasonable to study  $G$ -equivariant projective completions  $\overline{G \times^U X}$  of  $G \times^U X$ , together with extensions  $L' \rightarrow \overline{G \times^U X}$  of the linearisation  $L \rightarrow G \times^U X$ , in a bid to compute the stable locus  $X^s = X^{\text{ls}}$  for  $U \curvearrowright L \rightarrow X$  and study completions of the enveloping quotient  $X//U$ . More precisely, the strategy adopted in [DorK07] is to look for  $G \curvearrowright L' \rightarrow \overline{G \times^U X}$  such that

- (1) the pre-image of the stable locus of  $L' \rightarrow \overline{G \times^U X}$  under  $X \hookrightarrow \overline{G \times^U X}$  is contained in  $X^s = X^{\text{ls}}$ ; and
- (2) there is a naturally induced open embedding of  $X//U$  into the GIT quotient  $\overline{G \times^U X} //_{L'} G$ .

The following definition, which can be regarded as an enhanced version of a collection of ‘separating invariants’ in [DerK02, Definition 2.3.8], facilitates this.<sup>10</sup>

**Definition 2.3.10.** Let  $U$  be a unipotent group acting on a projective variety  $X$ , with ample linearisation  $L \rightarrow X$ , and let  $G$  be a reductive group containing  $U$  as a closed subgroup. A finite collection  $\mathcal{A} \subseteq \bigcup_{r>0} H^0(X, L^{\otimes r})^U$  is called a *finite fully separating set of invariants* if

1.  $X^{\text{ns}} = \bigcup_{f \in \mathcal{A}} X_f$  and the set  $\mathcal{A}$  is *separating*: whenever  $x, y \in X^{\text{ns}}$  are distinct points and there exist  $U$ -invariant sections  $g_0, g_1 \in H^0(X, L^{\otimes r})^U$  (for some  $r > 0$ ) such that  $g_0(x) \neq 0$ ,  $g_1(y) \neq 0$  and  $[g_0(x) : g_1(x)] \neq [g_0(y) : g_1(y)]$  (as points in  $\mathbb{P}^1$ ), then there are sections  $f_0, f_1 \in \mathcal{A}$  of some common tensor power of  $L$  such that  $f_0(x) \neq 0$ ,  $f_1(y) \neq 0$  and  $[f_0(x) : f_1(x)] \neq [f_0(y) : f_1(y)]$ ;
2. for every  $x \in X^s$  there is  $f \in \mathcal{A}$  with associated  $G$ -invariant  $F$  such that  $x \in (G \times_U X)_F$  and  $(G \times_U X)_F$  is affine; and
3. we have  $X//U \subseteq \bigcup_{f \in \mathcal{A}} \text{Spec}(\mathcal{O}(X_f)^U) \subseteq \text{Proj}(S^U)$ , and for every  $x \in X^{\text{ss,fg}}$  there is  $f \in \mathcal{A}$  such that  $x \in X_f$  and  $\mathcal{O}(X_f)^U \cong \mathbb{k}[\mathcal{A}]_{(f)}$  (where  $\mathbb{k}[\mathcal{A}]$  is the graded subalgebra of  $S^U = \mathbb{k}[X, L]^U$  generated by  $\mathcal{A}$ ).

The existence of a finite fully separating set of invariants follows by a suitable application of Hilbert’s Basis Theorem inside  $\mathbb{k}[X, L]$  and the quasi-compactness of  $X^{\text{ns}}$ ,  $X^{\text{ss,fg}}$  and  $X//U$  (recall from the assumption made at the start of this section that  $X//U$  is quasi-projective, as are  $X^{\text{ns}}$  and  $X^{\text{ss,fg}}$ ). The salient conditions in Definition 2.3.10 relevant to points (1) and (2) above are the conditions 2 and 3, respectively. The idea now is to consider  $L' \rightarrow \overline{G \times^U X}$  such that some finite fully separating set of invariants  $\mathcal{A} \subseteq \mathbb{k}[X, L]^U$  extends to a collection of  $G$ -invariant sections over  $\overline{G \times^U X}$ , with various further restrictions to increase their effectiveness for studying  $U \curvearrowright L \rightarrow X$ .

**Definition 2.3.11.** [DorK07, Definitions 5.2.4–5.2.7] Let  $U \curvearrowright L \rightarrow X$  be an ample linearisation of a unipotent group over an irreducible projective variety,  $G$  a reductive group containing  $U$  as a closed subgroup and  $\mathcal{A}$  a finite fully separating set of invariants. Suppose  $\beta : G \times^U X \hookrightarrow \overline{G \times^U X}$  is a dominant  $G$ -equivariant open immersion into a projective  $G$ -variety  $\overline{G \times^U X}$  and

<sup>10</sup>The definition we give is stated in a way that corrects a couple of small errors in the original [DorK07, Definition 5.2.2].

$L' \rightarrow \overline{G \times^U X}$  a  $G$ -linearisation that restricts to  $U \curvearrowright L \rightarrow X$  under  $\beta \circ \alpha$ . If every  $f \in \mathcal{A}$  extends to a  $G$ -invariant section of some positive tensor power of  $L'$  over  $\overline{G \times^U X}$ , the pair  $(\overline{G \times^U X}, L')$  is called a *reductive envelope* for  $U \curvearrowright L \rightarrow X$  (with respect to  $\mathcal{A}$ ). Furthermore,

1. if each  $f \in \mathcal{A}$  extends to a  $G$ -invariant  $F$  over  $\overline{G \times^U X}$  such that  $\overline{G \times^U X}_F$  is affine then  $(\overline{G \times^U X}, L')$  is called a *fine reductive envelope*;
2. if  $L'$  is an ample line bundle then  $(\overline{G \times^U X}, L')$  is called an *ample reductive envelope*; and
3. if each  $f \in \mathcal{A}$  extends to a  $G$ -invariant section  $F$  over  $\overline{G \times^U X}$  which vanishes on the codimension 1 part of the boundary  $\overline{G \times^U X} \setminus (G \times^U X)$  then  $(\overline{G \times^U X}, L')$  is called a *strong reductive envelope*.

Clearly any ample reductive envelope is a fine reductive envelope. In [DorK07, Proposition 5.2.8] it is shown that for any ample linearisation  $U \curvearrowright L \rightarrow X$ , there is some positive tensor power  $L^{\otimes r}$  of  $L$  which possesses an ample reductive envelope.

Associated to any reductive envelope  $L' \rightarrow \overline{G \times^U X}$  are the *completely semistable locus*

$$X^{\overline{\text{ss}}} = (\beta \circ \alpha)^{-1}(\overline{G \times^U X}^{\text{ss}(L')})$$

and the *completely stable locus*

$$X^{\overline{\text{s}}} = (\beta \circ \alpha)^{-1}(\overline{G \times^U X}^{\text{s}(L')}).$$

The main theorem concerning reductive envelopes, stated below, says that in the case where  $L' \rightarrow \overline{G \times^U X}$  is fine, the sets  $X^{\overline{\text{ss}}}$  and  $X^{\overline{\text{s}}}$  ‘bookend’ the inclusion  $X^{\text{lts}} \subseteq X^{\text{ss,fg}}$  associated to the  $U$ -linearisation  $L \rightarrow X$ , and the GIT quotient  $\overline{G \times^U X} //_{L'} G$  contains the enveloping quotient  $X // U$ .

**Theorem 2.3.12.** [DorK07, Theorem 5.3.1]<sup>11</sup> *Let  $X$  be an irreducible projective variety with an ample linearisation  $L \rightarrow X$  of a unipotent group  $U$ , and let  $(\overline{G \times^U X}, L')$  be a fine reductive envelope, with open embedding  $\beta : G \times^U X \hookrightarrow \overline{G \times^U X}$ . Let  $\pi : \overline{G \times^U X}^{\text{ss}(L')} \rightarrow \overline{G \times^U X} //_{L'} G$  be the GIT quotient map and suppose  $X // U$  is a quasi-projective variety. Then there is a commutative diagram:*

$$\begin{array}{ccccccc} X^{\overline{\text{s}}} & \subseteq & X^{\text{lts}} = X^{\text{ms}} = X^{\text{mss}} & \subseteq & X^{\text{ss,fg}} & \subseteq & X^{\overline{\text{ss}}} = X^{\text{nss}} \\ \downarrow q & & \downarrow q & & \downarrow q & & \downarrow \pi \circ \beta \circ \alpha \\ q(X^{\overline{\text{s}}}) & \subseteq & q(X^{\text{lts}}) & \subseteq & X // U & \subseteq & \overline{G \times^U X} //_{L'} G \end{array}$$

with all inclusions open.

<sup>11</sup>[DorK07, Theorem 5.3.1] actually says more than presented here and needs a normality assumption on  $X$  to include this extra material. An examination of the proof shows that the version we give here does not need  $X$  to be normal.

*Remark 2.3.13.* Note that Theorem 2.3.12 holds for ample reductive envelopes in particular, and in this case  $\overline{G \times^U X} //_{L'} G$  gives a projective completion of  $X // U$ . If furthermore  $\mathbb{k}[X, L]^U$  is a finitely generated  $\mathbb{k}$ -algebra, then  $X^{\text{ss,fg}} = X^{\text{nss}} = X^{\text{ss}}$  and  $X // U \cong \overline{G \times^U X} //_{L'} G$ .

In the case where the reductive envelope is fine and strong with a completion  $\overline{G \times^U X}$  that is normal, the sets  $X^{\text{s}} = X^{\text{ltss}}$  and  $X^{\text{ss,fg}}$  can be computed via the stable and semistable loci of the reductive envelope:

**Theorem 2.3.14.** *[DorK07, Theorem 5.3.5] Retain the notation of Theorem 2.3.12. If furthermore  $\overline{G \times^U X}$  is normal, and  $(\overline{G \times^U X}, L')$  defines a fine strong reductive envelope, then  $X^{\text{s}} = X^{\text{ltss}}$  and  $X^{\text{ss}} = X^{\text{ss,fg}}$ .*

Given their use for computing the stable locus and the finitely generated semistable locus of a linearisation of a unipotent group, the question of how to construct ample strong reductive envelopes which are normal has special importance. Note that ampleness is desirable, because it means the associated  $X^{\text{ss}}$  and  $X^{\text{s}}$  can be computed via the Hilbert-Mumford criteria applied to  $L' \rightarrow \overline{G \times^U X}$ . For sufficiently nice completions  $\overline{G \times^U X}$  one can turn any  $G$ -linearisation  $L' \rightarrow \overline{G \times^U X}$  into a strong reductive envelope by using the boundary divisors of  $\overline{G \times^U X}$  to modify the line bundle  $L'$  appropriately.

**Definition 2.3.15.** [DorK07, Definition 5.3.8] Let  $X$  be a quasi-projective variety and  $\beta : X \hookrightarrow \overline{X}$  a projective completion of  $X$ . The completion is said to be *gentle* if  $\overline{X}$  is normal and every codimension 1 component of the boundary of  $X$  in  $\overline{X}$  is a  $\mathbb{Q}$ -Cartier divisor.

*Remark 2.3.16.* As was observed in [DorK07, Remark 5.3.11], in general there does not exist a projective completion  $\overline{G \times^U X}$  of  $G \times^U X$  and a  $G$ -linearisation on a line bundle  $L' \rightarrow \overline{G \times^U X}$  extending the induced linearisation on  $G \times^U X$  with the following three properties all holding simultaneously, although we can ensure that any two of them hold together:

- (i)  $L' \rightarrow \overline{G \times^U X}$  is a reductive envelope for  $U \curvearrowright L \rightarrow X$  with respect to some finite fully separating set of invariants;
- (ii) the completion  $\overline{G \times^U X}$  of  $G \times^U X$  is gentle;
- (iii)  $L'$  is ample.

Suppose  $\overline{G \times^U X}$  is a gentle completion of  $G \times^U X$  and  $L' \rightarrow \overline{G \times^U X}$  is *any*  $G$ -linearisation extending  $L \rightarrow G \times^U X$ . Let  $D_1, \dots, D_m \subseteq \overline{G \times^U X}$  be the codimension 1 irreducible components of the complement of  $G \times^U X$  in  $\overline{G \times^U X}$  and define the  $\mathbb{Q}$ -Cartier divisor

$$D := \sum_{i=1}^m D_i.$$

Then for any sufficiently divisible integer  $N > 0$  the divisor  $ND$  is Cartier and defines a line bundle  $\mathcal{O}(ND)$  on  $\overline{G \times^U X}$  which restricts to the trivial bundle on  $G \times^U X$ . Define

$$L'_N := L' \otimes \mathcal{O}(ND) \rightarrow \overline{G \times^U X}.$$

If  $G$  is connected, then the  $G$ -linearisation on  $L \rightarrow G \times^U X$  extends uniquely to a  $G$ -linearisation on  $L'_N$ . The next proposition provides a useful way for turning  $L' \rightarrow \overline{G \times^U X}$  into a strong reductive envelope.

**Proposition 2.3.17.** *[DorK07, Proposition 5.3.10] Suppose  $G$  is a connected reductive group and, as above, suppose  $\overline{G \times^U X}$  is a gentle completion of  $G \times^U X$  and  $L' \rightarrow \overline{G \times^U X}$  is an extension of the  $G$ -linearisation  $L \rightarrow G \times^U X$ . Given a finite fully separating set of invariants  $\mathcal{A}$  on  $X$ , then  $(\overline{G \times^U X}, L'_N)$  is a strong reductive envelope with respect to  $\mathcal{A}$ , for sufficiently divisible integers  $N > 0$ . If in fact  $(\overline{G \times^U X}, L')$  defines a fine reductive envelope with respect to  $\mathcal{A}$ , then  $(\overline{G \times^U X}, L'_N)$  defines a fine strong reductive envelope.*

The above construction is especially simple to describe explicitly when  $X$  is normal and a reductive group  $G$  can be found that contains  $U$  as a closed subgroup in such a way that

- the homogeneous space  $G/U$  can be embedded in a normal affine variety  $\overline{G/U}^{\text{aff}}$  with codimension 2 complement; and
- the  $U$ -linearisation  $U \curvearrowright L \rightarrow X$  extends to a  $G$ -linearisation  $G \curvearrowright L \rightarrow X$ .

(Note that the first of these conditions is equivalent to  $U$  being a *Grosshans subgroup* of  $G$ , about which more will be discussed in Section 4.2.1.) The extension of the linearisation leads to an isomorphism of  $G$ -linearisations

$$\begin{array}{ccc} G \times^U L & \xrightarrow{\cong} & (G/U) \times L & [g, l] \mapsto (gU, gl) \\ \downarrow & & \downarrow & \\ G \times^U X & \xrightarrow{\cong} & (G/U) \times X & [g, x] \mapsto (gU, gx) \end{array}$$

with the corresponding  $G$ -linearisation on the right hand side being the product of the linearisation  $G \curvearrowright L \rightarrow X$  and left multiplication on  $G/U$ . Note that because  $\overline{G/U}^{\text{aff}} \times X$  is normal and  $G$  reductive, the ring of invariants

$$\mathbb{k}[X, L]^U \cong (\mathcal{O}(G/U) \otimes \mathbb{k}[X, L])^G = (\mathcal{O}(\overline{G/U}^{\text{aff}}) \otimes \mathbb{k}[X, L])^G$$

is a finitely generated  $\mathbb{k}$ -algebra. In particular, this implies  $X//U = \text{Proj}(S^U)$  is a projective variety. One can choose a normal projective  $G$ -equivariant completion  $\overline{G/U}$  of  $\overline{G/U}^{\text{aff}}$  whose boundary consists of an effective Cartier divisor  $D_\infty$  (not necessarily prime), and there is a very ample  $G$ -linearisation on the associated line bundle  $\mathcal{O}(D_\infty) \rightarrow \overline{G/U}$  extending the canonical linearisation on  $\mathcal{O}_{G/U} \rightarrow G/U$ . As above, for any  $N > 0$ , let

$$L'_N = \mathcal{O}(ND_\infty) \boxtimes L \rightarrow \overline{G/U} \times X,$$

equipped with its natural  $G$ -linearisation. Then using Proposition 2.3.17 and Theorem 2.3.12, one deduces

**Proposition 2.3.18.** *[DorK07, Lemma 5.3.14] In the above situation, the pair  $(\overline{G/U} \times X, L'_N)$  defines an ample strong reductive envelope for sufficiently large  $N > 0$ , and  $X//U = (X//U \times X)_{/L'_N} G$ .*

*Example 2.3.19.* [DorK07, §6] Let  $U = (\mathbb{C}, +)$ , embedded in  $\mathrm{GL}(2, \mathbb{C})$  as the subgroup of upper triangular matrices, act on  $V = \mathrm{Sym}^n \mathbb{C}^2$  via the standard representation of  $\mathrm{GL}(2, \mathbb{C})$  on  $V$ , and consider the canonical  $U$ -linearisation on  $L := \mathcal{O}(1) \rightarrow X := \mathbb{P}(V)$ . (Note that  $X$  may be regarded as the space of degree  $n$  divisors on  $\mathbb{P}^1$ , and the action of  $U$  on  $X$  corresponds to moving points on  $\mathbb{P}^1$  by the usual translation Möbius transformation.) This linearisation extends to one of  $G = \mathrm{SL}(2, \mathbb{C})$  in the obvious way. The homogeneous space  $G/U$  is isomorphic to  $\mathbb{C}^2 \setminus \{0\}$  via the usual transitive action of  $G$  on  $\mathbb{C}^2 \setminus \{0\}$ , and it has a normal  $G$ -equivariant affine completion  $\mathbb{C}^2$ . Embedding  $\mathbb{C}^2$  into  $\mathbb{P}^2$  by adding a hyperplane at infinity, we arrive in the setting of Proposition 2.3.18, so  $\mathcal{O}_{\mathbb{P}^2}(N) \boxtimes L \rightarrow \mathbb{P}^2 \times X$  defines a strong ample reductive envelope for  $U \curvearrowright L \rightarrow X$ , for sufficiently large  $N > 0$ . Using the Hilbert-Mumford criterion on  $\mathbb{P}^2 \times X$  and Theorem 2.3.14, one sees that

$$\begin{aligned} X^s &= \{\text{divisors } \sum_{i=1}^n p_i \text{ where } < n/2 \text{ of the } p_i \text{ coincide}\}, \\ X^{\mathrm{ss}, \mathrm{fg}} &= \{\text{divisors } \sum_{i=1}^n p_i \text{ where } \leq n/2 \text{ of the } p_i \text{ coincide}\}. \end{aligned}$$

In the case where  $n$  is odd then  $X^s = X^{\mathrm{ss}, \mathrm{fg}}$  and  $X^s/U$  is an open subset of  $X//U = (\mathbb{P}^2 \times X)//G$  with complement given by the reductive GIT quotient  $(\{0\} \times X)//G = X//G$  for the classical action of  $G = \mathrm{SL}(2, \mathbb{C})$  on  $X$  (linearised with respect to  $\mathcal{O}(1) \rightarrow X$ ). In particular, the enveloping quotient map  $X^{\mathrm{ss}, \mathrm{fg}} \rightarrow X//U$  is not surjective.

On the other hand, when  $n$  is even then  $X^s$  is a proper subset of  $X^{\mathrm{ss}, \mathrm{fg}}$  and the image of  $X^{\mathrm{ss}, \mathrm{fg}} \rightarrow X//U$  is not a variety: it is equal to the union of  $X^s/U$  together with the point  $\mathrm{pt} = (X//G) \setminus (X^{s(G)}/G)$  given by the quotient of the strictly semistable set for the  $G$ -linearisation on  $X$ .

### 3 Geometric Invariant Theory for Non-Reductive Groups

In this section we extend the constructions of [DorK07] described in Section 2.3.1 to the case where  $H$  is a linear algebraic group acting on a variety, with  $H$  not necessarily unipotent. Let  $H_u$  be the unipotent radical of  $H$ , so that  $H_u$  is a unipotent normal subgroup of  $H$  and  $H_r = H/H_u$  is reductive. A number of our definitions and results are simple generalisations of those found in [DorK07, §4 and §5.1] to this more general context. Having said this, we also address some errors that occur in [DorK07, §4] and thus our work can be seen as giving some new perspectives on the unipotent picture. A further guiding goal is to develop a theory which reduces to Mumford's GIT [MumFK94] in the case where  $H = G$  is a reductive group acting on a projective variety equipped with an ample linearisation. Throughout this section  $X$  will be a variety with an action of a linear algebraic group  $H$  and  $L \rightarrow X$  a linearisation of the action. We don't necessarily assume  $X$  is projective or irreducible, or  $L$  is ample, unless explicitly stated.

We begin in Section 3.1 by extending the finitely generated semistable locus  $X^{\mathrm{ss}, \mathrm{fg}}$  and the notions of enveloping quotients and enveloped quotients from Section 2.3.1 to the more general non-reductive case (Definitions 3.1.1 and 3.1.6). As a way to address the observation that the enveloping quotient  $X \curvearrowright H$  need not be a variety (see Remark 2.3.4) we introduce the concept of an *inner enveloping quotient* in Definition 3.1.12. These are subvarieties of the enveloping quotient that, in some sense, play the role of the GIT quotient from Mumford's theory for

reductive groups; indeed, in the case where  $H = G$  is reductive,  $X$  is projective and  $L \rightarrow X$  is ample, there is only one inner enveloping quotient—namely, the GIT quotient  $X//G$ . An inner enveloping quotient is in general not intrinsic solely to the data of the linearisation  $L \rightarrow X$ , but instead corresponds to a choice of a certain kind of linear system, called an *enveloping system*, introduced in Definition 3.1.17. We explore ways in which the collection of all inner enveloping quotients gives a certain ‘universality’ with respect to  $H$ -invariant morphisms from  $X^{\text{ss,fg}}$ . In Section 3.2 we examine how the enveloping quotient behaves under naturally induced group actions. In particular, we note some of the difficulties that can arise when trying to take enveloping quotients ‘in stages’: first by a normal subgroup  $N$  of  $H$  and then by the quotient group  $H/N$ . In Section 3.3 we introduce the stable locus  $X^s$  for a general non-reductive linearisation over an irreducible variety  $X$  (Definition 3.3.2). This is an  $H$ -invariant open subset of  $X$  that is intrinsic to the linearisation  $L \rightarrow X$  and admits a geometric quotient under the  $H$ -action. Our notion of stability also reduces to Definition 2.2.2 in the reductive case and to Definition 2.3.5 in the unipotent case. Following the ideas of [DorK07, §5], we relate our definition of stability for  $H \curvearrowright L \rightarrow X$  to stability for a certain reductive linearisation obtained by extension to a reductive structure group, which will be important for the work on reductive envelopes in Section 4. Finally, in Section 3.4 we draw together all our definitions and key results into Theorem 3.4.2, which provides a summary of our geometric invariant theoretic picture for non-reductive groups.

### 3.1 Finitely Generated Semistability and Enveloping Quotients

Let  $H$  be a linear algebraic group acting on a variety  $X$  equipped with a linearisation  $H \curvearrowright L \rightarrow X$ . As in Section 2, we let  $S = \mathbb{k}[X, L] = \bigoplus_{r \geq 0} H^0(X, L^{\otimes r})$  be the graded  $\mathbb{k}$ -algebra of global sections of positive tensor powers of  $L$  and  $S^H$  be the subring of invariant sections under the action (2.2) of Section 2.1.2. The inclusion  $S^H \hookrightarrow S$  defines an  $H$ -invariant rational map of schemes

$$q : X \dashrightarrow \text{Proj}(S^H), \quad (3.1)$$

whose maximal domain of definition contains the open subset of points where some invariant section of a positive tensor power of  $L$  does not vanish. As we have already seen, the basic technique of geometric invariant theory is, roughly speaking, to use the non-vanishing loci  $X_f$  of invariant sections  $f$  to construct  $H$ -invariant open subsets of  $X$  which admit geometric quotients in the category of varieties. Since any such geometric quotient must be a scheme of finite type, it makes sense to restrict which opens  $X_f$  to include in the following manner.

**Definition 3.1.1.** Let  $H$  be a linear algebraic group acting on a variety  $X$  and  $L \rightarrow X$  a linearisation of the action. The *naively semistable locus* is the open subset

$$X^{\text{nss}} := \bigcup_{f \in I^{\text{nss}}} X_f$$

of  $X$ , where  $I^{\text{nss}} := \bigcup_{r > 0} H^0(X, L^{\otimes r})^H$  is the set of invariant sections of positive tensor powers of  $L$ . The *finitely generated semistable locus* is the open subset

$$X^{\text{ss,fg}} := \bigcup_{f \in I^{\text{ss,fg}}} X_f$$

of  $X^{\text{ns}}$ , where

$$I^{\text{ss,fg}} := \{f \in \bigcup_{r>0} H^0(X, L^{\otimes r})^H \mid (S^H)_{(f)} \text{ is a finitely generated } \mathbb{k}\text{-algebra}\}.$$

These definitions generalise Definitions 2.3.1 and 2.3.2 of the naively semistable and finitely generated semistable loci, respectively, from [DorK07]. The finitely generated semistable locus is also closely related to the ‘algebraic locus’ of an affine scheme introduced in [DuK15, Remark 2.3] (see the upcoming Example 3.1.3). They depend on the choice of the linearisation  $L$ ; when necessary, we shall indicate this by writing  $X^{\text{ns}(L)}$  and  $X^{\text{ss,fg}(L)}$ .

*Remark 3.1.2.* It is clear from the definition that for any  $r > 0$  the subset  $X^{\text{ns}}$  is unaffected by replacing the linearisation  $L \rightarrow X$  with  $L^{\otimes r} \rightarrow X$ , and there is a canonical isomorphism  $\text{Proj}(\mathbb{k}[X, L]^H) \cong \text{Proj}(\mathbb{k}[X, L^{\otimes r}]^H)$ . The subset  $X^{\text{ss,fg}}$  is also unaffected by this replacement. Indeed, it is easy to see that  $X^{\text{ss,fg}(L^{\otimes r})} \subseteq X^{\text{ss,fg}(L)}$ . For the reverse containment, note that for any  $f \in H^0(X, L^{\otimes m})^H$  ( $m > 0$ ) with  $(\mathbb{k}[X, L]^H)_{(f)}$  a finitely generated  $\mathbb{k}$ -algebra, we have  $f^r \in \mathbb{k}[X, L^{\otimes r}]^H$  with

$$(\mathbb{k}[X, L^{\otimes r}]^H)_{(f^r)} = (\mathbb{k}[X, L]^H)_{(f^r)} = (\mathbb{k}[X, L]^H)_{(f)}$$

a finitely generated  $\mathbb{k}$ -algebra, so that  $X_f = X_{f^r} \subseteq X^{\text{ss,fg}(L^{\otimes r})}$ . It thus makes sense to define  $X^{\text{ns}}$ ,  $X^{\text{ss,fg}}$  and the scheme  $\text{Proj}(\mathbb{k}[X, L]^H)$  for rational linearisations using Remark 2.1.20.

As we noted in Section 2.1.2, the most common linearisations one comes across are either when  $X$  is affine and  $L = \mathcal{O}_X$  is the trivial bundle, or else when  $X$  is a projective variety and  $L$  is an ample line bundle. We take a moment to consider the rational map (3.1) and Definition 3.1.1 in each of these cases.

*Example 3.1.3.* In the case where  $X = \text{Spec } A$  affine and  $L = \mathcal{O}_X$ , recall from Example 2.1.16 that the linearisation is defined by a character  $\chi : H \rightarrow \mathbb{G}_m$  and that  $S^H$  is the graded subring of semi-invariants,

$$\bigoplus_{r \geq 0} A_{\chi^r}^H, \quad A_{\chi^r}^H := \{f \in A \mid f(hx) = \chi(h)^r f(x) \text{ for all } x \in X, h \in H\}.$$

The rational map  $q : X \dashrightarrow \text{Proj}(\bigoplus_{r \geq 0} A_{\chi^r}^H)$  corresponds to the natural map  $\bigoplus_{r \geq 0} A_{\chi^r}^H \rightarrow A$  induced by the inclusions  $A_{\chi^r}^H \hookrightarrow A$ , and  $X^{\text{ns}}$  is in this case the maximal domain of definition of  $q$ , consisting of points  $x \in X$  where  $f(x) \neq 0$  for some  $f \in A_{\chi^r}^H$  with  $r > 0$ .

In the special case where  $\chi = 1$  is the trivial character, then the ring of semi-invariants is just  $\bigoplus_{r \geq 0} A^H$ , so that  $\text{Proj}(\mathbb{k}[X, L]^H) = \text{Spec}(A^H)$ . Furthermore,  $X^{\text{ns}} = X$  because the constant function  $1 \in H^0(X, L)^H$ , and  $X^{\text{ss,fg}}$  is the union of  $X_f$  with  $f \in A^H$  such that  $(A^H)_f$  is a finitely generated  $\mathbb{k}$ -algebra. (In fact,  $X^{\text{ss,fg}}$  is the preimage of Dufresne and Kraft’s algebraic locus of  $\text{Spec}(A^H)$  under  $q : X \rightarrow \text{Spec}(A^H)$  in this case; see [DuK15, Remark 2.3].)

*Example 3.1.4.* If now  $X$  is projective and  $L$  is ample, then each of the open subsets  $X_f$  arising in Definition 3.1.1 is affine, so the restriction of the rational map  $q : X \dashrightarrow \text{Proj}(S^H)$  to  $X^{\text{ns}}$  and  $X^{\text{ss,fg}}$  defines an *affine* morphism. Moreover, by taking a sufficiently positive tensor power  $L^{\otimes r}$  of  $L$  we may embed  $X$  equivariantly into the projective space  $\mathbb{P}(V^*)$  using the complete linear

system  $V = H^0(X, L^{\otimes r})$ , and the linearisation  $L^{\otimes r}$  extends to  $\mathcal{O}_{\mathbb{P}(V^*)}(1) \rightarrow \mathbb{P}(V^*)$ ; see Example 2.1.17. If  $L$  is very ample (so that we may take  $r = 1$ ),  $I_X$  is the kernel of the restriction map  $\mathbb{k}[\mathbb{P}(V^*), \mathcal{O}(1)] \rightarrow \mathbb{k}[X, L]$  and  $R_X = \mathbb{k}[\mathbb{P}(V^*), \mathcal{O}(1)]/I_X$ , then by [Har77, Chapter 2, Lemma 5.14 and Remark 5.14.1] for some  $m > 0$  the  $m$ -th Veronese subring  $(R_X)^{(m)} \subseteq R_X$  is isomorphic to  $\mathbb{k}[X, L^{\otimes m}]$ . Then  $\text{Proj}(\mathbb{k}[X, L]^H) \cong \text{Proj}((R_X)^H)$ , and in light of Remark 3.1.2 computing  $X^{\text{nss}}$  and  $X^{\text{ss,fg}}$  for  $H \curvearrowright L \rightarrow X$  is essentially equivalent to studying the action of  $H$  on  $R_X$ . Thus when  $L$  is ample and  $X$  is projective we can always reduce to the case where  $H$  acts on a projective space  $\mathbb{P}^n$  via a representation  $H \rightarrow \text{GL}(n+1, \mathbb{k})$  and  $X \subseteq \mathbb{P}^n$  is a closed subvariety invariant under the action.

*Remark 3.1.5.* In general the finitely generated semistable locus  $X^{\text{ss,fg}}$  is strictly contained in  $X^{\text{nss}}$ , due to the fact that the subring of invariant sections can be non-noetherian (even if  $S$  is a finitely generated  $\mathbb{k}$ -algebra). Indeed, when  $X = \text{Spec } A$  is an affine variety and  $L = \mathcal{O}_X$  is equipped with the canonical  $H$ -linearisation (i.e. defined by the trivial character  $1 : H \rightarrow \mathbb{G}_m$ ), then as seen in Example 3.1.3  $X^{\text{nss}} = X$  and  $X^{\text{ss,fg}}$  is the union of all  $X_f$  where  $(A^H)_f$  is finitely generated over  $\mathbb{k}$ . In [DerK08, Proposition 2.10] Derksen and Kemper show that the set

$$I^{\text{ss,fg}} \cup \{0\} = \{f \in A^H \mid (A^H)_f \text{ is finitely generated}\} \cup \{0\}$$

is in fact a radical ideal of  $A^H$ . In [Gros76] Grosshans shows that if  $A$  is an integral domain then there is a nonzero  $f \in A^H$  such that  $(A^H)_f$  is finitely generated, so if  $A^H$  is not finitely generated then  $I^{\text{ss,fg}}$  is a proper nonzero ideal. It follows that any irreducible affine example in which the ring of invariant global functions is not finitely generated will result in  $\emptyset \neq X^{\text{ss,fg}} \neq X^{\text{nss}}$ ; for example, the Nagata counterexample in Example 2.1.25.

The rational map of (3.1) restricts to define a morphism on  $X^{\text{ss,fg}}$  whose image is contained in the following open subscheme of  $\text{Proj}(S^H)$ .

**Definition 3.1.6.** Let  $H$  be a linear algebraic group and  $H \curvearrowright L \rightarrow X$  a linearisation of an  $H$ -variety  $X$ . The *enveloping quotient* is the scheme

$$X \wr H := \bigcup_{f \in I^{\text{ss,fg}}} \text{Spec}((S^H)_{(f)}) \subseteq \text{Proj}(S^H)$$

together with the canonical map  $q : X^{\text{ss,fg}} \rightarrow X \wr H$ . We call the image  $q(X^{\text{ss,fg}})$  of this map the *enveloped quotient*.

When it is necessary to do so, we will include the data of the linearisation in an enveloping quotient by writing  $X \wr_L H$ . Definition 3.1.6 is simply an extension of the definition of enveloping quotient and enveloped quotient in [DorK07] to the case of linearisations for any linear algebraic group. Observe that we do not use the “//” notation, since one can define the enveloping quotient for a linearisation of a reductive group and in general this is *not* equal to Mumford’s reductive GIT quotient from [MumFK94] (more will be said about this in Section 3.1.2). Observe that the enveloping quotient is a canonically defined reduced, separated scheme locally of finite type over  $\mathbb{k}$ .



*Remark 3.1.7.* As we noted earlier in Remark 2.3.4, the enveloping quotient is only a scheme locally of finite type in general. However, when  $S^H$  is a finitely generated  $\mathbb{k}$ -algebra, or when the enveloping quotient map  $q : X^{\text{ss,fg}} \rightarrow X \wr H$  is surjective, then  $X \wr H$  is noetherian and hence a variety. In particular, if  $X$  is projective,  $L \rightarrow X$  is ample and  $S^H$  finitely generated over  $\mathbb{k}$  then  $X \wr H = \text{Proj}(S^H)$  is a projective variety.

*Remark 3.1.8.* When  $X = \text{Spec } A$  is affine and  $L = \mathcal{O}_X$  has linearisation defined by the trivial character (see Example 2.1.13) the enveloping quotient is precisely the algebraic locus of  $\text{Spec}(A^H)$ , in the sense of [DuK15, Remark 2.3].

In the next lemma we make some initial observations about the rational map  $q : X \dashrightarrow \text{Proj}(S^H)$  of (3.1) associated to the linearisation  $H \curvearrowright L \rightarrow X$ . As well as using standard facts about the Proj construction, we need the following commutative algebra result from [DerK08, Proposition 2.9]: if  $A$  is an integral domain over  $\mathbb{k}$  and  $a, b \in A \setminus \{0\}$  are such that  $A_a$  and  $A_b$  are both finitely generated  $\mathbb{k}$ -algebras and the ideal generated by  $a, b$  is equal to  $A$ , then  $A$  is also a finitely generated  $\mathbb{k}$ -algebra (and in fact  $A = A_a \cap A_b$ ).

**Lemma 3.1.9.** *Suppose  $S^H \neq \mathbb{k}$ , let  $\mathcal{Y} = \text{Proj}(S^H)$  and let  $\underline{L}$  denote the sheaf of sections of  $L$  on  $X$ . Then for each  $r \geq 0$ , pulling back along  $q$  defines inclusions of sheaves  $q^* : \mathcal{O}_{\mathcal{Y}}(r) \hookrightarrow (q_*(\underline{L}^{\otimes r}|_{X^{\text{ss}}}))^H \subseteq q_*(\underline{L}^{\otimes r}|_{X^{\text{ss}}})$ . If  $S$  is furthermore assumed to be an integral domain, then*

1. *for each  $r \geq 0$  the twisting sheaf  $\mathcal{O}_{\mathcal{Y}}(r)$  is identified with  $(q_*(\underline{L}^{\otimes r}|_{X^{\text{ss}}}))^H$  via  $q^*$ ; and*
2. *if  $S$  is finitely generated over  $\mathbb{k}$  then the ideal  $\mathfrak{a} \subseteq S^H$  generated by  $I^{\text{ss,fg}}$  is a non-zero graded radical ideal of  $S^H$  satisfying  $\mathfrak{a} \cap S_r^H = I^{\text{ss,fg}} \cap S_r^H$  for each  $r \geq 0$ . In particular,  $X^{\text{ss,fg}} \neq \emptyset$ .*

*Proof.* Fix  $r \geq 0$  and let  $f \in S_m^H = H^0(X, L^{\otimes m})^H$  for  $m > 0$ . Let  $S(r)$  be the graded  $S$ -module with degree  $d$  piece equal to  $S_{d+r}$  for each  $d \in \mathbb{Z}$  and let

$$M = \bigoplus_{n \geq 0} S(r)_{mn} = \bigoplus_{n \geq 0} H^0(X, L^{\otimes r} \otimes L^{\otimes mn})$$

with its  $S^{(m)} = \mathbb{k}[X, L^{\otimes m}]$ -module structure. Since  $X$  is quasi-compact and separated, by [Har77, Chapter 2, Lemma 5.14 and Remark 5.14.1] there is a canonical identification of  $\mathcal{O}(X_f)$ -modules  $H^0(X_f, L^{\otimes r}) = M_{(f)}$ , where the module structure on the right hand side comes from the identification  $\mathcal{O}(X_f) = (S^{(m)})_{(f)}$ . By definition we have  $H^0(\text{Spec}((S^H)_{(f)}), \mathcal{O}_{\mathcal{Y}}(r)) = (M^H)_{(f)}$  with its  $((S^H)^{(m)})_{(f)}$ -module structure, and the pullback map  $q^* : H^0(\text{Spec}((S^H)_{(f)}), \mathcal{O}_{\mathcal{Y}}(r)) \rightarrow H^0(X_f, L^{\otimes r})$  corresponds to the inclusion  $(M^H)_{(f)} \hookrightarrow (M_{(f)})^H \subseteq M_{(f)}$  under these identifications. Hence

$$q^* : \mathcal{O}_{\mathcal{Y}}(r) \hookrightarrow (q_*(\underline{L}^{\otimes r}|_{X^{\text{ss}}}))^H \subseteq q_*(\underline{L}^{\otimes r}|_{X^{\text{ss}}})$$

is an inclusion of sheaves. If furthermore  $S$  is an integral domain, then in fact  $(M^H)_{(f)} = (M_{(f)})^H$  (with notation as above): for if  $g \in S(r)_{mn}$  is such that  $g/f^n \in (M_{(f)})^H$  and  $h \in H$ , then  $g/f^n = h \cdot (g/f^n) = (h \cdot g)/f^n$  and so  $h \cdot g = g$ . Statement 1 of the lemma follows.

Now we prove 2, assuming  $S$  is an integral domain and finitely generated over  $\mathbb{k}$ . We first show that  $I^{\text{ss,fg}} \neq \emptyset$ . Since  $S^H \neq \mathbb{k}$  we can find a nonzero homogeneous  $f \in S^H$  of positive degree.

Then  $A := S_{(f)}$  is a finitely generated integral domain over  $\mathbb{k}$ , so applying Grosshans' localisation result [Gros76] (see Remark 3.1.5) to  $A = S_{(f)}$  we conclude that there exists  $a \in A^H \setminus \{0\}$  such that  $(A^H)_a$  is finitely generated over  $\mathbb{k}$ . Because  $S$  is an integral domain we have  $A^H = (S^H)_{(f)}$ , so  $a = g/f^m$  for some integer  $m \geq 0$  and  $g \in S^H$  homogeneous of degree equal to  $m \deg f$ , and  $(A^H)_a = (S^H)_{(fg)}$ . Hence  $fg \in I^{\text{ss,fg}}$ . Note this implies  $X^{\text{ss,fg}} \neq \emptyset$ .

It is immediate that  $\mathfrak{a}$  is a graded ideal of  $S^H$ . The fact that it is radical follows from the equality  $(S^H)_{(f)} = (S^H)_{(f^m)}$  for each  $f \in S^H$  homogeneous and each  $m > 0$ . It remains to show  $\mathfrak{a} \cap S_r^H = I^{\text{ss,fg}} \cap S_r^H$  for all  $r \geq 0$ . The inclusions  $\mathfrak{a} \cap S_r^H \supseteq I^{\text{ss,fg}} \cap S_r^H$  are obvious. For the reverse containments, it suffices to show that for any  $g_1, g_2 \in I^{\text{ss,fg}}$  and any  $f \in H^0(X, L^{\otimes r'})^H$  ( $r' > 0$ ), we have  $fg_i \in I^{\text{ss,fg}}$  and  $\tilde{g} := g_1 + g_2 \in I^{\text{ss,fg}}$ . To this end, note that  $fg_i \in I^{\text{ss,fg}}$  because

$$(S^H)_{(fg_i)} = ((S^H)_{(g_i)})_{\frac{f^r}{g_i^{r'}}}$$

is the localisation of a finitely generated algebra. On the other hand, setting  $a_i := g_i/\tilde{g} \in (S^H)_{(\tilde{g})}$ , we see that each  $((S^H)_{(\tilde{g})})_{a_i} = ((S^H)_{(\tilde{g}g_i)})$  is a finitely generated integral domain over  $\mathbb{k}$ . Since  $(a_1, a_2)$  is the unit ideal in  $(S^H)_{(\tilde{g})}$ , the ring  $(S^H)_{(\tilde{g})}$  is therefore finitely generated by the result [DerK08, Proposition 2.9] quoted before the statement of the lemma, so  $\tilde{g} = g_1 + g_2 \in I^{\text{ss,fg}}$ . Thus,  $\mathfrak{a} \cap H^0(X, L^{\otimes r})^H \subseteq I^{\text{ss,fg}} \cap H^0(X, L^{\otimes r})^H$  for each  $r \geq 0$ .  $\square$

*Remark 3.1.10.* When  $X$  is irreducible the ring of sections  $S$  is an integral domain, so for any section  $f \in I^{\text{nss}} = \bigcup_{r>0} H^0(X, L^{\otimes r})^H$  we have a natural identification  $\mathcal{O}(X_f)^H = (S_{(f)})^H = (S^H)_{(f)}$ , by Lemma 3.1.9, 1. Under this identification, the morphism  $q : X_f \rightarrow \text{Spec}((S^H)_{(f)})$  defined by (3.1) corresponds to the natural map  $X_f \rightarrow \text{Spec}(\mathcal{O}(X_f)^H)$  induced by  $\mathcal{O}(X_f)^H \hookrightarrow \mathcal{O}(X_f)$ .

From the lemma we see that for general  $X$  the natural map of sheaves  $q^\# : \mathcal{O}_{\mathcal{Y}} \rightarrow q_* \mathcal{O}_{X^{\text{nss}}}$  is injective with image contained in  $(q_* \mathcal{O}_{X^{\text{nss}}})^H$  and  $q : X^{\text{nss}} \rightarrow \mathcal{Y}$  is a dominant morphism. Similarly, if  $\mathcal{U} \subseteq \mathcal{Y}$  is any nonempty open subscheme (such as the enveloping quotient  $X \wr H$ ) then the sheaves  $\mathcal{O}_{\mathcal{Y}}(r)|_{\mathcal{U}}$  ( $r \geq 0$ ) are quasi-coherent sheaves whose sections are included in the  $H$ -invariant sections of  $L^{\otimes r}|_{q^{-1}(\mathcal{U})}$  under pullback by  $q$ . Notice also that, as a corollary of 2 of Lemma 3.1.9, any situation where a linear algebraic group  $H$  acts on a finitely generated integral  $\mathbb{k}$ -algebra  $S$  such that the ring of invariants  $S^H$  is not finitely generated over  $\mathbb{k}$  will result in an example of a projective variety  $X = \text{Proj } S$  with ample linearisation  $L \rightarrow X$  such that  $X^{\text{nss}} \neq X^{\text{ss,fg}}$  and  $X \wr H \neq \text{Proj}(S^H)$ . This holds, in particular, for the projectivised version of the Nagata example (cf. Example 2.1.25).

### 3.1.1 ‘Universality’ of the Enveloping Quotient

Given a linear algebraic group  $H$  acting on a variety  $X$  with linearisation  $L \rightarrow X$ , it is not necessarily the case that  $q : X^{\text{ss,fg}} \rightarrow X \wr H$  is surjective, thus  $X \wr H$  is not in general a categorical quotient of  $X^{\text{ss,fg}}$ .

*Example 3.1.11.* Let  $X = \text{SL}(2, \mathbb{k})$  and let  $H \subseteq X$  be the subgroup of strictly upper triangular matrices, acting on  $X$  via matrix multiplication. There is a unique linearisation of the trivial

bundle  $\mathcal{O}_X \rightarrow X$ . By [Bo91, Theorem 6.8] the geometric quotient  $X/H$  exists; in fact,  $H$  is precisely the stabiliser of the standard action of  $\mathrm{SL}(2, \mathbb{k})$  on  $\mathbb{k}^2 \setminus \{0\}$ , therefore  $X/H \cong \mathbb{k}^2 \setminus \{0\}$  and  $\mathbb{k}[X, \mathcal{O}_X]^H \cong \mathbb{k}[z_0, z_1]$  is finitely generated. So in this case  $X \wr H = \mathrm{Spec}(\mathbb{k}[X, \mathcal{O}_X]^H) \cong \mathbb{k}^2$ , and the image of  $X^{\mathrm{ss}, \mathrm{fg}} = X$  under the enveloping quotient map is identified with  $\mathbb{k}^2 \setminus \{0\}$ .

An example of the failure of surjectivity of  $q : X^{\mathrm{ss}, \mathrm{fg}} \rightarrow X \wr H$  in the projective case was given in Example 2.3.19. There we also saw examples where the enveloped quotient  $q(X^{\mathrm{ss}, \mathrm{fg}})$  is not a variety, so in general a categorical quotient of  $X^{\mathrm{ss}, \mathrm{fg}}$  need not exist at all. This raises the question of whether there is any sort of way in which to view the enveloping quotient as ‘universal’ for  $H$ -invariant morphisms. Here we give one possible way to answer this. (We should say that our use of the word ‘universal’ here is informal—while we do prove a sort of uniqueness and existence result regarding morphisms induced by certain  $H$ -invariant morphisms from  $X^{\mathrm{ss}, \mathrm{fg}}$ , we don’t formulate this in terms of a universal property within some category, though it is surely possible to do so. This is simply because we won’t have need for such a formal usage in what follows.)

The key observation is that, even though  $X \wr H$  may not be quasi-compact, the enveloped quotient—being the image  $q(X^{\mathrm{ss}, \mathrm{fg}})$  of  $X^{\mathrm{ss}, \mathrm{fg}}$ —is quasi-compact as a subset of  $X \wr H$ . So it is natural to look at *quasi-compact* open subschemes  $\mathcal{U}$  of  $X \wr H$  that contain  $q(X^{\mathrm{ss}, \mathrm{fg}})$ . Observe that it is easy to give examples of such subsets (at least non-constructively): because  $X^{\mathrm{ss}, \mathrm{fg}}$  is quasi-compact there is a finite collection of sections  $f_i \in I^{\mathrm{ss}, \mathrm{fg}}$  such that  $X^{\mathrm{ss}, \mathrm{fg}}$  is covered by the basic opens  $X_{f_i}$  and  $\bigcup_i \mathrm{Spec}((S^H)_{(f_i)})$  is a quasi-compact open subscheme of  $X \wr H$  containing  $q(X^{\mathrm{ss}, \mathrm{fg}})$ . Furthermore, it is easy to see that we have an equality of sets

$$q(X^{\mathrm{ss}, \mathrm{fg}}) = \bigcap \{ \mathcal{U} \mid \mathcal{U} \subseteq X \wr H \text{ is open, quasi-compact and contains } q(X^{\mathrm{ss}, \mathrm{fg}}) \}.$$

In fact, with a bit more work we can see that  $q(X^{\mathrm{ss}, \mathrm{fg}})$  is a constructible subset of  $X \wr H$ . Indeed, any quasi-compact open subscheme  $\mathcal{U}$  of  $\mathrm{Proj}(S^H)$  is of finite type and separated, since  $\mathrm{Proj}(S^H)$  is separated. Choosing such a  $\mathcal{U}$  and restricting attention to the induced morphism between varieties  $q : X^{\mathrm{ss}, \mathrm{fg}} \rightarrow \mathcal{U}$ , we may apply Chevalley’s Theorem [St15, Tag 05H4] to conclude that  $q(X^{\mathrm{ss}, \mathrm{fg}})$  is a constructible subset of  $\mathcal{U}$ , and since  $\mathcal{U}$  is a quasi-compact open subset of  $X \wr H$  it follows that  $q(X^{\mathrm{ss}, \mathrm{fg}})$  is constructible inside  $X \wr H$  too.

This suggests it is natural to study diagrams of the form

$$\begin{array}{ccccc} X^{\mathrm{ss}, \mathrm{fg}} & \subseteq & X^{\mathrm{nss}} & \subseteq & X \\ \swarrow & \downarrow & & \downarrow q & \\ \mathcal{U} \subseteq X \wr H & \subseteq & \mathrm{Proj}(S^H) & & \end{array}$$

where the inclusions are open and the  $\mathcal{U}$  are quasi-compact.

More generally, because  $q : X^{\mathrm{ss}, \mathrm{fg}} \rightarrow X \wr H$  is dominant *any* nonempty open set  $\mathcal{U} \subseteq X \wr H$  intersects  $q(X^{\mathrm{ss}, \mathrm{fg}})$  and is covered by basic affine open subsets of the form  $\mathrm{Spec}((S^H)_{(f)})$  with  $f$  such that  $(S^H)_{(f)}$  is finitely generated. Thus the pre-image of  $\mathcal{U}$  under the enveloping quotient map  $q$  is a nonempty union of the associated open subsets  $X_f$ . So given any open subset  $U \subseteq X$  that is a union of  $X_f$  with  $f \in I^{\mathrm{ss}, \mathrm{fg}}$  we may also consider its image  $q(U)$  as an intersection of those quasi-compact open  $\mathcal{U} \subseteq X \wr H$  containing it. This motivates the following definition.

**Definition 3.1.12.** Let  $H$  be a linear algebraic group acting on a variety  $X$  with linearisation  $L \rightarrow X$  and let  $U \subseteq X^{\text{ss,fg}}$  be a nonempty  $H$ -invariant open subset. An *inner enveloping quotient* of  $U$  is a quasi-compact open subscheme of  $X \wr H$  that contains the image  $q(U)$  of  $U$  under the enveloping quotient map  $q : X^{\text{ss,fg}} \rightarrow X \wr H$ . An inner enveloping quotient of  $U = X^{\text{ss,fg}}$  is simply called an *inner enveloping quotient*.

*Example 3.1.13.* In the case where  $X$  is an irreducible projective  $H$ -variety and  $L \rightarrow X$  an ample linearisation we can intrinsically define a collection of inner enveloping quotients, as follows. The section ring  $S = \mathbb{k}[X, L]$  is an integral domain finitely generated over  $\mathbb{k}$ , so by Lemma 3.1.9, 2 the set  $I^{\text{ss,fg}} \cap H^0(X, L^{\otimes r})^H$  is a finite dimensional vector space over  $\mathbb{k}$ , for each  $r > 0$ . Taking  $r > 0$  such that  $X^{\text{ss,fg}} = \bigcup \{X_f \mid f \in I^{\text{ss,fg}} \cap H^0(X, L^{\otimes r})^H\}$ , the associated open subscheme

$$\mathcal{U}^{(r)} := \bigcup \{\text{Spec}((S^H)_{(f)}) \mid f \in I^{\text{ss,fg}} \cap H^0(X, L^{\otimes r})^H\} \subseteq X \wr H$$

is an inner enveloping quotient: choosing any basis  $\{f_i\}$  of  $I^{\text{ss,fg}} \cap H^0(X, L^{\otimes r})^H$  yields a finite open cover  $\{\text{Spec}((S^H)_{(f_i)})\}$  of  $\mathcal{U}^{(r)}$  by quasi-compact open subsets.

From the discussion above we see it is natural to regard the image of an  $H$ -invariant open subset  $U$  of  $X^{\text{ss,fg}}$  under the enveloping quotient  $q : X^{\text{ss,fg}} \rightarrow X \wr H$  as sitting inside a ‘germ’ of inner enveloping quotients of  $U$ . The following proposition makes this idea more precise in the case where  $U$  is a union of open subsets of the form  $X_f$  with  $f \in I^{\text{ss,fg}}$ .

**Proposition 3.1.14.** *Let  $H$  be a linear algebraic group acting on an irreducible variety  $X$  with linearisation  $L \rightarrow X$  and let  $U = \bigcup_{f \in \mathcal{S}} X_f$ , where  $\mathcal{S}$  is a nonempty subset of  $I^{\text{ss,fg}}$ . Suppose we are given the data of a quasi-projective variety  $Z$  together with a very ample line bundle  $M \rightarrow Z$  and an  $H$ -invariant morphism  $\phi : U \rightarrow Z$  with  $\phi^*M \cong L^{\otimes r}|_U$  for some  $r > 0$ . Then*

1. *there is an inner enveloping quotient  $\mathcal{U}$  of  $U$  and a morphism  $\bar{\phi} : \mathcal{U} \rightarrow Z$  such that  $\phi = \bar{\phi} \circ q|_U$  and  $\bar{\phi}^*M \cong \mathcal{O}_{\mathcal{U}}(r)$ ; and*
2. *if  $\mathcal{U}, \mathcal{U}' \subseteq X \wr H$  are two inner enveloping quotients of  $U$  and  $\psi : \mathcal{U} \rightarrow Z$  and  $\psi' : \mathcal{U}' \rightarrow Z$  two morphisms such that  $\psi \circ q|_U = \psi' \circ q|_U$ , then  $\psi$  and  $\psi'$  agree on  $\mathcal{U}' \cap \mathcal{U}$ .*

*Proof.* (Proof of 1.) Let  $\iota : Z \hookrightarrow \mathbb{P}^n$  be a locally closed immersion defined by sections  $\sigma_0, \dots, \sigma_n \in H^0(Z, M)$ , so that the composition  $\iota \circ \phi : U \rightarrow \mathbb{P}^n$  is defined by the  $H$ -invariant sections  $f_0 = \phi^*\sigma_0, \dots, f_n = \phi^*\sigma_n \in H^0(U, L^{\otimes r})^H$ . Let  $\mathcal{U}_0 = \bigcup_{f \in \mathcal{S}} \text{Spec}((S^H)_{(f)}) \subseteq X \wr H$ . Then  $U = q^{-1}(\mathcal{U}_0)$ , so appealing to Lemma 3.1.9, 1 there are  $g_0, \dots, g_n \in H^0(\mathcal{U}_0, \mathcal{O}(r))$  such that  $q^*g_i = f_i$  for each  $i$ . The sections  $g_0, \dots, g_n$  define a morphism  $\Phi : \mathcal{U} \rightarrow \mathbb{P}^n$  on some nonempty quasi-compact open subscheme  $\mathcal{U} \subseteq \mathcal{U}_0$  that contains  $q(U)$ , since the collection of  $q^*g_i = f_i$  is basepoint free on  $U$ , and by construction  $\Phi \circ q = \iota \circ \phi$ . Because  $q^*$  is injective, any section of a power of  $\mathcal{O}_{\mathbb{P}^n}(1)$  that vanishes on  $Z$  will pull back under  $\bar{\phi}$  to a zero section over  $\mathcal{U}$ , so the image of  $\mathcal{U}$  under  $\Phi$  is contained in the closure  $\overline{\iota(Z)}$  of  $\iota(Z)$  in  $\mathbb{P}^n$ ; by shrinking  $\mathcal{U}$  if necessary we may assume that  $\Phi(\mathcal{U}) \subseteq \iota(Z) \cong Z$ . Then  $\bar{\phi} := \Phi|_{\mathcal{U}} : \mathcal{U} \rightarrow Z$  is a morphism such that  $\bar{\phi} \circ q|_U = \phi$  and  $\bar{\phi}^*M \cong \mathcal{O}_{\mathcal{U}}(r)$ .

(Proof of 2.) Suppose we have  $\psi : \mathcal{U} \rightarrow Z$  and  $\psi' : \mathcal{U}' \rightarrow Z$  with  $q(U) \subseteq \mathcal{U} \cap \mathcal{U}'$  and  $\psi \circ q|_U = \psi' \circ q|_U$ . Then  $q(U)$  is a dense constructible subset of the noetherian space  $\mathcal{U} \cap \mathcal{U}'$ , so the interior  $q(U)^\circ$  is a nonempty dense open subscheme of  $\mathcal{U} \cap \mathcal{U}'$  on which  $\psi$  and  $\psi'$  agree. Since  $X \wr H$  is separated, so too is  $\mathcal{U} \cap \mathcal{U}'$  and thus we have  $\psi = \psi'$  on  $\mathcal{U} \cap \mathcal{U}'$ .  $\square$

*Remark 3.1.15.* If the sections  $\sigma_0, \dots, \sigma_n \in H^0(Z, M)$  defining an embedding  $Z \hookrightarrow \mathbb{P}^n$  in the statement of Proposition 3.1.14 are such that each  $\phi^* \sigma_i$  extends to a global section of  $L^{\otimes r} \rightarrow X$ , then in fact one can prove 1 and 2 for reducible  $X$  and *any*  $H$ -invariant open subset  $U \subseteq X$ .

As a corollary of Proposition 3.1.14, we obtain a sort of universal property for the enveloping quotient  $q : X^{\text{ss}, \text{fg}} \rightarrow X \mathbin{\mathbb{A}} H$  when  $X$  is irreducible (for reducible  $X$  an appropriate statement can be formulated from Remark 3.1.15). Given a quasi-projective variety  $Z$  embedded in some projective space and an  $H$ -invariant morphism  $\phi : X^{\text{ss}, \text{fg}} \rightarrow Z$  defined by sections of some positive power of  $L|_{X^{\text{ss}, \text{fg}}}$ , there is an inner enveloping quotient  $\mathcal{U} \subseteq X \mathbin{\mathbb{A}} H$  of  $X^{\text{ss}, \text{fg}}$  and a morphism  $\bar{\phi} : \mathcal{U} \rightarrow Z$  such that the diagram

$$\begin{array}{ccc} & X^{\text{ss}, \text{fg}} & \\ & \downarrow q \quad \searrow \phi & \\ X \mathbin{\mathbb{A}} H & \supseteq \mathcal{U} \xrightarrow{\bar{\phi}} & Z \end{array}$$

commutes, and any other inner enveloping quotient  $\mathcal{U}'$  and morphism  $\bar{\phi}'$  with  $\bar{\phi}' \circ q = \phi$  defines the same rational map  $X \mathbin{\mathbb{A}} H \dashrightarrow Z$  as  $(\mathcal{U}, \bar{\phi})$ .

*Remark 3.1.16.* The inner enveloping quotient  $\mathcal{U}$  and the map  $\bar{\phi} : \mathcal{U} \rightarrow Z$  constructed above depend on the choice of embedding of  $Z$  into a projective space  $\mathbb{P}^n$ , and the whole construction furthermore relies on the requirement that the morphism  $\phi : X^{\text{ss}, \text{fg}} \rightarrow Z \subseteq \mathbb{P}^n$  is defined by sections of some positive tensor power of  $L \rightarrow X^{\text{ss}, \text{fg}}$  (or for reducible  $X$ , of  $L \rightarrow X$ ). Contrast this to Mumford's GIT quotient arising from a reductive group  $G$  acting on a variety  $X$  with linearisation  $L$ : then the GIT quotient  $X^{\text{ss}} \rightarrow X // G$  is a categorical quotient of the semistable locus  $X^{\text{ss}}$  *in the category of varieties*, so that a  $G$ -invariant morphism  $X^{\text{ss}} \rightarrow Z$  factors uniquely through  $X^{\text{ss}} \rightarrow X // G$  without any further assumptions on  $X^{\text{ss}} \rightarrow Z$ . So we see that the universality of the collection of inner enveloping quotients for a general linear algebraic group  $H$  is a considerably weaker notion than the universal property of a reductive GIT quotient. The reason for this can be traced in large part to the fact that the enveloping quotient  $q : X^{\text{ss}, \text{fg}} \rightarrow X \mathbin{\mathbb{A}} H$  is not surjective. In 5.3.2 of Section 5 we will find examples of enveloping quotients where  $q : X^{\text{ss}, \text{fg}} \rightarrow X \mathbin{\mathbb{A}} H$  is surjective and indeed  $X \mathbin{\mathbb{A}} H$  is a geometric—and hence categorical—quotient of  $X^{\text{ss}, \text{fg}}$ .

Any inner enveloping quotient  $\mathcal{U} \subseteq X \mathbin{\mathbb{A}} H$  is quasi-compact, so for sufficiently large integers  $r > 0$  the twisting sheaf  $\mathcal{O}_{\mathcal{U}}(r)$  defines a line bundle on  $\mathcal{U}$ . We shall soon see that, for  $r$  large enough,  $\mathcal{O}_{\mathcal{U}}(r)$  is in fact very ample. In order to prove this—as well as a similar statement for inner enveloping quotients of more general open subsets of  $X^{\text{ss}, \text{fg}}$ —it is convenient to make the following definition.

**Definition 3.1.17.** Let  $H$  be a linear algebraic group acting on a variety  $X$  and  $L \rightarrow X$  a linearisation. For  $r > 0$  and  $\mathcal{S} \subseteq H^0(X, L^{\otimes r})^H$  a finite subset of invariant sections, we say a linear subspace  $V \subseteq H^0(X, L^{\otimes r})$  is an *enveloping system adapted to  $\mathcal{S}$*  if

1. it is finite dimensional, contains  $\mathcal{S}$  and is stable under the  $H$ -action; and
2. for each  $f \in \mathcal{S}$  the  $\mathbb{k}$ -algebra  $(S^H)_{(f)}$  is finitely generated with generating set  $\{\tilde{f}/f \mid \tilde{f} \in V^H\}$ .

We call  $V$  simply an *enveloping system* if it is an enveloping system adapted to a subset  $\mathcal{S}$  such that  $X^{\text{ss,fg}} = \bigcup_{f \in \mathcal{S}} X_f$ .

*Example 3.1.18.* Suppose  $S^H$  is a finitely generated  $\mathbb{k}$ -algebra. Then there exists  $r > 0$  such that the  $r$ -th Veronese subring  $(S^H)^{(r)}$  is generated by its degree 1 piece  $S_r^H = H^0(X, L^{\otimes r})^H$ . Therefore  $X^{\text{ss,fg}} = X^{\text{nss}}$  is covered by the basic open subsets  $X_f$  with  $f \in H^0(X, L^{\otimes r})^H$ , and for each such  $f$  we have  $(S^H)_{(f)} = ((S^H)^{(r)})_{(f)}$  generated by  $\tilde{f}/f$  with  $\tilde{f} \in H^0(X, L^{\otimes r})^H$ . So  $H^0(X, L^{\otimes r})^H$  is an enveloping system.

The following basic result asserts that finding enveloping systems adapted to finite subsets is essentially equivalent to finding quasi-compact open subschemes of the enveloping quotient  $X \mathbin{\mathbb{H}} H$  and giving ways to embed them into projective spaces.

**Proposition 3.1.19.** *Suppose  $H$  is a linear algebraic group and  $L \rightarrow X$  a linearisation of an  $H$ -variety  $X$ .*

1. *For any quasi-compact open subscheme  $\mathcal{U} \subseteq X \mathbin{\mathbb{H}} H$ , there is an enveloping system  $V \subseteq H^0(X, L^{\otimes r})^H$  adapted to a finite subset  $\mathcal{S} \subseteq H^0(X, L^{\otimes r})^H$  with  $\mathcal{U} = \bigcup_{f \in \mathcal{S}} \text{Spec}((S^H)_{(f)})$ , for some  $r > 0$  such that  $\mathcal{O}_{\mathcal{U}}(r)$  is a very ample line bundle. Moreover, the natural map  $V \rightarrow H^0(\mathcal{U}, \mathcal{O}_{\mathcal{U}}(r))$  defines a locally closed embedding  $\mathcal{U} \hookrightarrow \mathbb{P}(V^*)$ .*
2. *Conversely, suppose  $H^0(X, L^{\otimes r})$  contains an enveloping system  $V$  adapted to a finite subset  $\mathcal{S} \subseteq H^0(X, L^{\otimes r})^H$ , let  $\mathcal{U} = \bigcup_{f \in \mathcal{S}} \text{Spec}((S^H)_{(f)}) \subseteq X \mathbin{\mathbb{H}} H$  and let  $\phi : \mathcal{U} := \bigcup_{f \in \mathcal{S}} X_f \rightarrow \mathbb{P}((V^H)^*)$  be the  $H$ -invariant map defined by the inclusion  $V^H \subseteq H^0(X, L^{\otimes r})$ . Then there is a locally closed embedding  $\bar{\phi} : \mathcal{U} \hookrightarrow \mathbb{P}((V^H)^*)$  such that  $\phi = \bar{\phi} \circ q$  on  $\mathcal{U}$  and  $\bar{\phi}^* \mathcal{O}_{\mathbb{P}((V^H)^*)}(1) = \mathcal{O}_{\mathcal{U}}(r)$ .*
3. *If  $V \subseteq H^0(X, L^{\otimes r})$  is any enveloping system adapted to  $\mathcal{S}$ , then the image of the natural multiplication map  $V^{\otimes n} \rightarrow H^0(X, L^{\otimes rn})$  defines an enveloping system adapted to the set  $\{f^n \mid f \in \mathcal{S}\}$ , for each  $n > 0$ .*

*Proof.* (Proof of 1.) The argument we use can essentially be found in [DorK07, Proposition 4.2.2] and is based on a slight modification of the argument used to prove quasi-projectivity of the GIT quotient in reductive GIT (cf. [MumFK94, Theorem 1.10]). For completeness, it runs as follows. Let  $\mathcal{Y} = \text{Proj}(S^H)$ . Since  $\mathcal{U}$  is quasi-compact, we may find finitely many invariants  $f_1, \dots, f_m \in I^{\text{ss,fg}}$  such that the basic open subsets  $\text{Spec}((S^H)_{(f_i)})$  cover  $\mathcal{U}$ . Using the reducedness of  $S^H$  we can take powers of the  $f_i$  and assume, without loss of generality, that there is  $r_0 > 0$  such that  $f_i \in S_{r_0}^H$  for each  $i$ , so that  $\mathcal{O}_{\mathcal{U}}(r_0)$  is the trivial line bundle over  $\text{Spec}((S^H)_{(f_i)})$ . The  $\mathbb{k}$ -algebras  $(S^H)_{(f_i)}$  have finite generating sets, which we can write as  $\{g_{i1}/(f_i^{r_1}), \dots, g_{in_i}/(f_i^{r_1})\}$  for  $g_{ij} \in S_{r_0 r_1}^H$  and some  $n_i > 0$ , with one common  $r_1 > 0$  working for each  $i = 1, \dots, m$ . Resetting  $f_i = f_i^{r_1}$  for each  $i$  and letting  $\mathcal{S} := \{f_1, \dots, f_m\}$ , we can assume that we have found  $r > 0$  and a set

$$A := \mathcal{S} \cup \{g_{i,j} \mid i = 1, \dots, m, j = 1, \dots, n_i\}$$

of invariant sections such that  $\mathcal{U} = \bigcup_{f \in \mathcal{S}} \text{Spec}((S^H)_{(f)})$ , the sheaf  $\mathcal{O}_{\mathcal{U}}(r)$  is locally free and  $(S^H)_{(f_i)} = \mathbb{k}[g_{i,1}/f_i, \dots, g_{i,n_i}/f_i]$  for each  $i$ . Taking  $V \subseteq S_r^H = H^0(X, L^{\otimes r})^H$  to be the  $\mathbb{k}$ -span of

the elements of  $A$ , we see that  $V$  is an enveloping system adapted to  $\mathcal{S}$ . The image of the natural map  $V \rightarrow H^0(\mathcal{U}, \mathcal{O}_{\mathcal{U}}(r))$  induced by the structure map  $S_r^H \rightarrow H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}(r))$  is basepoint-free on  $\mathcal{U}$ , so  $V \rightarrow H^0(\mathcal{U}, \mathcal{O}_{\mathcal{U}}(r))$  defines a morphism

$$\psi : \mathcal{U} \rightarrow \mathbb{P}(V^*)$$

such that  $\psi^* \mathcal{O}_{\mathbb{P}(V^*)}(1) = \mathcal{O}_{\mathcal{U}}(r)$ . Now  $H^0(\mathcal{U}_{f_i}, \mathcal{O}_{\mathcal{U}}(r)) \cong (S^H)_{(f_i)}$  and the restriction of  $\psi$  to  $\mathcal{U}_{f_i}$  maps into the affine open subset  $\mathbb{P}(V^*)_{f_i}$  of points of  $\mathbb{P}(V^*)$  where  $f_i \in H^0(\mathbb{P}(V^*), \mathcal{O}(1))$  doesn't vanish. So  $\psi : \mathcal{U}_{f_i} \rightarrow \mathbb{P}(V^*)_{f_i}$  corresponds to the natural ring homomorphism

$$(\mathrm{Sym}^\bullet V)_{(f_i)} \rightarrow (S^H)_{(f_i)}$$

given by multiplying sections, which is surjective because the generators  $g_{i,1}/f_i, \dots, g_{n_i,1}/f_i$  of  $(S^H)_{(f_i)}$  are contained in the image. Thus  $\psi : \mathcal{U}_{f_i} \rightarrow \mathbb{P}(V^*)_{f_i}$  is a closed immersion. Since  $\mathcal{U}$  is covered by the  $\mathcal{U}_{f_i}$ , the map  $\psi : \mathcal{U} \rightarrow \mathbb{P}(V^*)$  is a locally closed immersion and  $\mathcal{O}_{\mathcal{U}}(r)$  is very ample.

(Proof of 2.) Suppose  $V \subseteq H^0(X, L^{\otimes r})$  is an enveloping system adapted to  $\mathcal{S} \subseteq H^0(X, L^{\otimes r})^H$  and let  $\mathcal{U} = \bigcup_{f \in \mathcal{S}} \mathrm{Spec}((S^H)_{(f)}) \subseteq \mathcal{Y} = \mathrm{Proj}(S^H)$ . As above, the structure map  $S_r^H \rightarrow H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}(r))$  defines a linear map

$$\alpha : H^0(\mathbb{P}((V^H)^*), \mathcal{O}(1)) = V^H \rightarrow H^0(\mathcal{U}, \mathcal{O}_{\mathcal{U}}(r))$$

such that the composition  $q^* \circ \alpha$  is equal to  $\phi^* : H^0(\mathbb{P}((V^H)^*), \mathcal{O}(1)) \rightarrow H^0(\mathcal{U}, L^{\otimes r})$ . Now  $\mathcal{S} \subseteq V^H$ , so  $\mathcal{O}_{\mathcal{U}}(r)$  is globally generated by the sections in the image of  $\alpha$  and thus  $\alpha$  defines a morphism

$$\bar{\phi} : \mathcal{U} \rightarrow \mathbb{P}((V^H)^*)$$

such that  $\bar{\phi}^* \mathcal{O}_{\mathbb{P}((V^H)^*)}(1) = \mathcal{O}_{\mathcal{U}}(r)$  and  $\phi = \bar{\phi} \circ q$ . By 2 of Definition 3.1.17, for each  $f \in \mathcal{S}$  the algebra  $(S^H)_{(f)}$  is generated by  $\tilde{f}/f$ , where  $\tilde{f} \in V^H$ , and now the argument used in the proof of 1 above shows that  $\bar{\phi}$  is a locally closed immersion.

(Proof of 3.) Given an enveloping system  $V \subseteq H^0(X, L^{\otimes r})$  adapted to  $\mathcal{S}$  and  $n > 0$ , the image  $V'$  of the natural multiplication map  $V^{\otimes n} \rightarrow H^0(X, L^{\otimes nr})$  is a finite dimensional  $H$ -stable subspace of  $H^0(X, L^{\otimes nr})$  that contains the set of  $n$ -fold products of invariant sections  $A' := \{f_1 \cdots f_n \mid \text{each } f_k \in V^H\}$ . For any  $f \in \mathcal{S}$  the algebra  $(S^H)_{(f^n)} = (S^H)_{(f)}$  is generated by  $A'$ , since we have  $\tilde{f}/f = (\tilde{f}f^{n-1})/f^n$  in  $(S^H)_{(f)}$  for all  $\tilde{f} \in V^H$ . Hence  $V'$  is an enveloping system adapted to  $\{f^n \mid f \in \mathcal{S}\}$ .  $\square$

*Remark 3.1.20.* Given an enveloping system  $V$  consisting of invariant sections, it follows from Proposition 3.1.19, 1 that any basis of  $V$  will give a set of invariants of some positive tensor power of  $L \rightarrow X$  that separates points in  $X^{\mathrm{ss}, \mathrm{fg}}$  (compare with Definition 2.3.10, 1).

We have already seen that when  $X$  is projective,  $L \rightarrow X$  is ample and the ring of invariants  $S^H$  is finitely generated then the enveloping quotient  $X \mathbin{/\!\!/} H = \mathrm{Proj}(S^H)$  is a projective variety. As a first application of enveloping systems, we can prove a sort of converse to this fact for irreducible  $X$ .

**Corollary 3.1.21.** *Suppose  $H$  is a linear algebraic group,  $X$  an irreducible  $H$ -variety and  $L \rightarrow X$  a linearisation. If the enveloping quotient  $X \mathbin{\mathbb{H}} H$  is complete, then  $X \mathbin{\mathbb{H}} H = \text{Proj}(S^H)$ . Furthermore, for suitably divisible integers  $r > 0$  the sheaf  $\mathcal{O}_{X \mathbin{\mathbb{H}} H}(r)$  is an ample line bundle on  $X \mathbin{\mathbb{H}} H$  and the natural structure map*

$$\mathbb{k}[X, L^{\otimes r}]^H = (S^H)^{(r)} \rightarrow \mathbb{k}[X \mathbin{\mathbb{H}} H, \mathcal{O}_{X \mathbin{\mathbb{H}} H}(r)]$$

*is an isomorphism. (In particular,  $\mathbb{k}[X, L^{\otimes r}]^H$  is a finitely generated  $\mathbb{k}$ -algebra for such  $r$  and we have  $X^{\text{NSS}} = X^{\text{ss,fg}}$ .)*

*Proof.* Recall that  $q : X^{\text{NSS}} \rightarrow \text{Proj}(S^H)$  is a dominant morphism, as a result of Lemma 3.1.9. Because  $X$  is irreducible, by 2 of the same lemma  $X^{\text{ss,fg}}$  is a dense open subset of  $X^{\text{NSS}}$ , so the enveloped quotient  $q(X^{\text{ss,fg}})$  is a dense subset of  $\text{Proj}(S^H)$  and hence the enveloping quotient  $X \mathbin{\mathbb{H}} H$  is a dense open subscheme of  $\text{Proj}(S^H)$ . Because  $X \mathbin{\mathbb{H}} H$  is complete, and hence quasi-compact, it is universally closed over  $\text{Spec } \mathbb{k}$ , and since  $\text{Proj}(S^H)$  is separated over  $\text{Spec } \mathbb{k}$  the open immersion  $X \mathbin{\mathbb{H}} H \hookrightarrow \text{Proj}(S^H)$  is a closed morphism [St15, Tag 01W0]. Thus  $X \mathbin{\mathbb{H}} H = \text{Proj}(S^H)$ . Using Proposition 3.1.19, 1 find  $r' > 0$  and an enveloping system  $V \subseteq H^0(X, L^{\otimes r'})^H$  so that the natural map  $V \rightarrow H^0(X \mathbin{\mathbb{H}} H, \mathcal{O}(r')) = H^0(\text{Proj}(S^H), \mathcal{O}(r'))$  defines a closed immersion  $\text{Proj}(S^H) \hookrightarrow \mathbb{P}(V^*)$  (the fact the immersion is closed is implied from the completeness of  $\text{Proj}(S^H) = X \mathbin{\mathbb{H}} H$ ). The line bundle  $\mathcal{O}(r')$  on  $\text{Proj}(S^H)$  is (very) ample, so by Serre vanishing [Har77, Chapter 3, Proposition 5.3] there is  $m_0 > 0$  such that for all  $m \geq m_0$  the restriction map

$$H^0(\mathbb{P}(V^*), \mathcal{O}(m)) = \text{Sym}^m V \rightarrow H^0(\text{Proj}(S^H), \mathcal{O}(mr'))$$

is surjective. Letting  $r$  be any positive multiple of  $m_0 r'$  and  $m = r/r'$ , we see that the restriction map  $\mathbb{k}[\mathbb{P}(V^*), \mathcal{O}(m)] \rightarrow \mathbb{k}[\text{Proj}(S^H), \mathcal{O}(r)]$  is surjective and therefore  $\mathbb{k}[\text{Proj}(S^H), \mathcal{O}(r)]$  is a finitely generated  $\mathbb{k}$ -algebra. The map  $\mathbb{k}[\mathbb{P}(V^*), \mathcal{O}(m)] \rightarrow \mathbb{k}[\text{Proj}(S^H), \mathcal{O}(r)]$  factors through the canonical structure map

$$\mathbb{k}[X, L^{\otimes r}]^H = (S^H)^{(r)} \rightarrow \mathbb{k}[\text{Proj}(S^H), \mathcal{O}(r)],$$

thus this too is a surjective map onto a finitely generated  $\mathbb{k}$ -algebra. On the other hand, the composition of this map with pull-back along the natural map  $q$  from (3.1) agrees with restriction of sections  $\mathbb{k}[X, L^{\otimes r}]^H \rightarrow \mathbb{k}[X^{\text{NSS}}, L^{\otimes r}]^H$ , which is injective because  $X$  is irreducible. It follows that

$$\mathbb{k}[X, L^{\otimes r}]^H = (S^H)^{(r)} \cong \mathbb{k}[\text{Proj}(S^H), \mathcal{O}(r)].$$

In particular,  $\mathbb{k}[X, L^{\otimes r}]^H$  is a finitely generated  $\mathbb{k}$ -algebra. The equality  $X^{\text{NSS}} = X^{\text{ss,fg}}$  now follows from the Definitions 3.1.1 of these sets and Remark 3.1.2.  $\square$

### 3.1.2 Comparison with Mumford's Reductive GIT

The definitions of the naively semistable locus, finitely generated semistable locus and enveloping quotient are direct generalisations of the corresponding notions in [DorK07] for unipotent groups to the context of general varieties with actions of any linear algebraic group. As such they apply



to the situation where  $H = G$  is a reductive group, so we take a moment to compare these notions to those arising in Mumford's GIT [MumFK94] for reductive groups.

Firstly, in the case where  $X$  is affine with a linearisation of the trivial bundle  $L = \mathcal{O}_X \rightarrow X$ , or  $X$  is projective with ample linearisation  $L \rightarrow X$ , then  $X^{\text{nss}} = X^{\text{ss,fg}}$  is equal to Mumford's semistable locus  $X^{\text{ss}}$  for  $G \curvearrowright L \rightarrow X$  [MumFK94, Definition 1.7], and the enveloping quotient is precisely the GIT quotient  $X//G = \text{Proj}(S^G)$  of [MumFK94, Theorem 1.10]. Indeed, we have  $I^{\text{ss,fg}} = \bigcup_{r>0} H^0(X, L^{\otimes r})^G$  in this case: for any invariant section  $f$  of a positive tensor power of  $L$ , the algebra  $(S^G)_f$  is the localisation of a finitely generated algebra  $S^G$  by Nagata's theorem [Na64], and so by Nagata's theorem again  $(S^G)_{(f)}$  is also a finitely generated algebra, being the subalgebra of invariants for the  $\mathbb{G}_m$ -action defining the grading on  $S^G$ . Thus  $X \wr G = \text{Proj}(S^G)$ . Because  $L$  is ample  $X_f$  is affine for each  $f \in I^{\text{ss,fg}}$ , from which it follows that  $X^{\text{ss}} = X^{\text{nss}} = X^{\text{ss,fg}}$ .

However, the similarities with Mumford's GIT when  $G$  is reductive do not extend beyond these cases. For a general variety  $X$  with possibly non-ample linearisation  $L \rightarrow X$  of  $G$ , there may be invariant sections  $f$  whose non-vanishing loci  $X_f$  are not affine. In Mumford's theory only those  $X_f$  that are affine are included in the definition of the semistable locus  $X^{\text{ss}}$ ; see Definition 2.2.2, 1. So for a general linearisation  $G \curvearrowright L \rightarrow X$  with  $G$  reductive, Mumford's semistable locus  $X^{\text{ss}}$  is contained in  $X^{\text{ss,fg}}$  as a (possibly empty) open subset. Given any inner enveloping quotient  $q : X^{\text{ss,fg}} \rightarrow \mathcal{U} \subseteq X \wr G$ , the restriction to Mumford's semistable locus  $X^{\text{ss}}$  coincides with the GIT quotient map, thus  $X//G = q(X^{\text{ss}})$  is an open subvariety of  $\mathcal{U}$ . Hence the GIT quotient  $X//G$  is a (possibly empty) quasi-compact open subscheme of each inner enveloping quotient inside  $X \wr G$ . Finally, as discussed in Remark 3.1.16, the enveloping quotient  $q : X^{\text{ss,fg}} \rightarrow X \wr G$  is not in general a categorical quotient in the category of varieties for the  $G$ -action on  $X^{\text{ss,fg}}$ , whereas Mumford's GIT quotient  $X^{\text{ss}} \rightarrow X//G$  is a categorical quotient for the  $G$ -action on  $X^{\text{ss}}$  [MumFK94, Theorem 1.10].

## 3.2 Natural Properties of Enveloping Quotients with Respect to Induced Group Actions

In this section we will study various natural properties of the enveloping quotient and inner enveloping quotients (adapted to some finite subset) with respect to various natural operations on groups.

### 3.2.1 Restriction and Extension of the Structure Group

We first look at the case of restricting a linearisation under a surjective homomorphism  $\rho : H_1 \rightarrow H_2$  of linear algebraic groups. Suppose  $X$  is an  $H_2$ -variety and  $L \rightarrow X$  line bundle with an  $H_2$ -linearisation. For precision, let us denote this linearisation as  $\mathcal{L}_2 \rightarrow X$ . The homomorphism  $\rho$  induces an  $H_1$ -linearisation on the line bundle  $L \rightarrow X$ , which we denote  $\mathcal{L}_1 \rightarrow X$ . Clearly  $\mathbb{k}[X, \mathcal{L}_1]^{H_1} = \mathbb{k}[X, \mathcal{L}_2]^{H_2}$ , from which it follows that there are canonical identifications

$$X^{\text{nss}(\mathcal{L}_1)} = X^{\text{nss}(\mathcal{L}_2)}, \quad X^{\text{ss,fg}(\mathcal{L}_1)} = X^{\text{ss,fg}(\mathcal{L}_2)}, \quad X \wr_{\mathcal{L}_1} H_1 = X \wr_{\mathcal{L}_2} H_2$$

and the natural maps  $q_1 : X^{\text{nss}(\mathcal{L}_1)} \rightarrow X \wr_{\mathcal{L}_1} H_1$  and  $q_2 : X^{\text{nss}(\mathcal{L}_2)} \rightarrow X \wr_{\mathcal{L}_2} H_2$  of (3.1) coincide under these identifications.

Now let us consider extensions of the structure group. Suppose we have an inclusion  $H_1 \hookrightarrow H_2$  and  $L \rightarrow X$  is an  $H_1$ -linearisation. Recall from Section 2.3.2 that we may consider the fibre space  $H_2 \times^{H_1} X$  associated to the principal  $H_1$ -bundle  $H_2 \rightarrow H_2/H_1$ , which is a variety if  $H_1$  is unipotent or if any finite subset of points in  $X$  is contained in an affine open subset (e.g. if  $X$  is quasi-projective). Then  $H_2 \times^{H_1} L \rightarrow H_2 \times^{H_1} X$  is a line bundle and there is a natural  $H_2$ -linearisation on  $H_2 \times^{H_1} L \rightarrow H_2 \times^{H_1} X$  induced by left multiplication. This extends the  $H_1$ -linearisation  $L \rightarrow X$  under the closed immersion

$$\alpha : X \hookrightarrow H_2 \times^{H_1} X, \quad x \mapsto [e, x].$$

As before, we will usually abuse notation and write  $L \rightarrow H_2 \times^{H_1} X$  for this linearisation instead of  $H_2 \times^{H_1} L$ , unless confusion is likely to arise. Recall also that pullback along  $\alpha$  induces an isomorphism of graded rings

$$\alpha^* : \mathbb{k}[H_2 \times^{H_1} X, L]^{H_2} \xrightarrow{\cong} \mathbb{k}[X, L]^{H_1}.$$

Applying  $\text{Proj}$  gives an isomorphism of schemes

$$\bar{\alpha} : \text{Proj}(\mathbb{k}[X, L]^{H_1}) \xrightarrow{\cong} \text{Proj}(\mathbb{k}[H_2 \times^{H_1} X, L]^{H_2})$$

such that  $\bar{\alpha}^*$  identifies the corresponding twisting sheaves. Let  $q_{H_1} : X^{\text{ss,fg}} \rightarrow X \wr H_1$  be the enveloping quotient map for the linearisation  $H_1 \curvearrowright L \rightarrow X$  and let  $q_{H_2} : (H_2 \times^{H_1} X)^{\text{ss,fg}} \rightarrow (H_2 \times^{H_1} X) \wr H_2$  be the enveloping quotient map for the linearisation  $H_2 \curvearrowright H_2 \times^{H_1} L \rightarrow H_2 \times^{H_1} X$ . Clearly pulling back along  $\alpha$  establishes a bijection  $I^{\text{ss,fg}}(H_2 \times^{H_1} L) \xleftarrow{\cong} I^{\text{ss,fg}}(L)$ . This implies that  $\alpha$  restricts to give a closed immersion of  $X^{\text{ss,fg}(H_1)}$  into  $(H_2 \times^{H_1} X)^{\text{ss,fg}(H_2)}$  and  $\bar{\alpha}$  restricts to an isomorphism of the enveloping quotients, fitting into the following commutative diagram:

$$\begin{array}{ccc} X^{\text{ss,fg}(H_1)} & \xhookrightarrow{\alpha} & (H_2 \times^{H_1} X)^{\text{ss,fg}(H_2)} \\ \downarrow q_{H_1} & & \downarrow q_{H_2} \\ X \wr H_1 & \xrightarrow[\bar{\alpha}]{\cong} & (H_2 \times^{H_1} X) \wr H_2 \end{array}$$

### 3.2.2 Induced Actions of Quotient Groups on Enveloping Quotients

Let us return to the situation where a linear algebraic group  $H$  acts on a variety  $X$  and is equipped with a linearisation  $L \rightarrow X$ , but now also suppose  $H$  has a normal subgroup  $N$ . (We will in particular be interested in the case when  $N = H_u$  is the unipotent radical of  $H$  and  $H_r = H/N$  is reductive). Then we may consider the restricted linearisation  $N \curvearrowright L \rightarrow X$  and form its naively semistable locus  $X^{\text{nss}(N)}$ , semistable finitely generated locus  $X^{\text{ss,fg}(N)}$  and enveloping quotient  $q_N : X^{\text{ss,fg}(N)} \rightarrow X \wr N$ . Because  $N$  is normal in  $H$ , the action of  $H$  on  $S = \mathbb{k}[X, L]$  induces a natural  $H/N$ -action on the ring  $S^N$  of  $N$ -invariant sections. For any  $h \in H$ , the action on  $X$  induces an isomorphism

$$X_f \xrightarrow{\cong} X_{h \cdot f}, \quad x \mapsto hx$$

(with inverse given by acting by  $h^{-1}$ ), so the action of  $H$  on  $X$  restricts to an action on  $X^{\text{ss}(N)}$ . Moreover, for any  $f \in H^0(X, L^{\otimes r})^N$  (with  $r > 0$ ) and  $\bar{h} = hN \in H/N$  the application of  $\bar{h}$  induces an isomorphism

$$\bar{h} \cdot (-) : (S^N)_{(f)} \xrightarrow{\cong} (S^N)_{(h \cdot f)},$$

from which it follows that the action of  $H/N$  on  $S^N$  preserves  $I^{\text{ss}, \text{fg}(N)}$ . Thus  $X^{\text{ss}, \text{fg}(N)}$  is also stable under the  $H$ -action.

**Proposition 3.2.1.** *Retain the notation of the preceding discussion. Then the action of  $H/N$  on  $S^N$  defines a canonical action of  $H/N$  on  $\mathcal{Y} = \text{Proj}(S^N)$  such that the map  $q_N : X^{\text{ss}(N)} \rightarrow \mathcal{Y}$  of (3.1) is equivariant with respect to the quotient  $H \rightarrow H/N$ . In addition, if  $\mathcal{S} \subseteq I^{\text{ss}(N)}$  is any subset that is stable under the canonical  $H/N$ -action on  $S^N$ , then the open subscheme  $\mathcal{U} = \bigcup_{f \in \mathcal{S}} \text{Spec}((S^N)_{(f)})$  of  $\mathcal{Y}$  is preserved under this action. (In particular,  $X \wr N$  is preserved under the action and  $q_N : X^{\text{ss}, \text{fg}(N)} \rightarrow X \wr N$  is equivariant.)*

Before proving Proposition 3.2.1 we need to introduce some notation and prove a lemma. Let

$$\Sigma : H \times X \rightarrow X$$

be the action morphism. A linearisation of  $H$  on  $L$  is equivalent to a choice of line bundle isomorphism

$$\Theta : \Sigma^* L \xrightarrow{\cong} \mathcal{O}_H \boxtimes L = H \times L$$

over  $H \times X$ , satisfying an appropriate cocycle condition (see [MumFK94, Chapter 1, §3]). This naturally extends to isomorphisms of tensor powers of the bundles, giving isomorphisms

$$\theta : H^0(H \times X, \Sigma^*(L^{\otimes r})) \xrightarrow{\cong} H^0(H \times X, H \times L^{\otimes r}) = \mathcal{O}(H) \otimes H^0(X, L^{\otimes r}), \quad r \geq 0,$$

where the last equality comes from the Künneth formula [St15, Tag 02KE]. (Note we abuse notation and suppress mention of  $r$  in the map  $\theta$ .) Composition of  $\theta$  with  $\Sigma$  thus gives us the co-action (or dual action, cf. [MumFK94, Definition 1.2])

$$\Sigma_\theta^* : H^0(X, L^{\otimes r}) \xrightarrow{\theta \circ \Sigma^*} \mathcal{O}(H) \otimes H^0(X, L^{\otimes r}), \quad r \geq 0. \quad (3.2)$$

For any  $h \in H$  and any  $r \geq 0$ , the linearisation  $L^{\otimes r} \rightarrow X$  yields a linear automorphism of  $H^0(X, L^{\otimes r})$  given by the composition

$$H^0(X, L^{\otimes r}) \xrightarrow{\Sigma_\theta^*} \mathcal{O}(H) \otimes H^0(X, L^{\otimes r}) \xrightarrow{\text{ev}_h \otimes \text{id}_X^*} H^0(X, L^{\otimes r}),$$

which satisfies

$$(\text{ev}_h \otimes \text{id}_X^*)(\Sigma_\theta^*(f)) = h^{-1} \cdot f \quad (3.3)$$

for all  $f \in H^0(X, L^{\otimes r})$ .

**Lemma 3.2.2.** *Let  $r \geq 0$  and suppose  $V \subseteq H^0(X, L^{\otimes r})^N$  is an  $H$ -stable subspace of sections. Then the image of  $V$  under  $\Sigma_\theta^*$  lies in  $\mathcal{O}(H)^N \otimes V$ , where  $\mathcal{O}(H)^N$  is the ring of functions invariant under the right multiplication action of  $N$  on  $H$ . (In particular, this holds for  $V = H^0(X, L^{\otimes r})^N$ .)*

*Proof.* This follows from the  $H$ -equivariance of the co-action (3.2) when  $H$  acts on the right-hand side of (3.2) via right multiplication on  $H$ . In more detail, suppose  $f \in V$  is non-zero. Then we may write  $\Sigma_\theta^* f = \sum_{j=1}^m a_j \otimes f_j$ , with  $m > 0$ ,  $q_j \in \mathcal{O}(H)$  and  $f_j \in H^0(X, L^{\otimes r})$  such that the  $a_j$  and the  $f_j$  are linearly independent over  $\mathbb{k}$ . For any  $h \in H$  we have

$$h^{-1} \cdot f = (\text{ev}_h \otimes \text{id}_X^*)(\Sigma_\theta^*(f)) = \sum_j a_j(h) f_j.$$

We can find  $h_1, \dots, h_m \in H$  such that the matrix

$$(a_j(h_i))_{i,j}$$

is invertible:<sup>12</sup> indeed, the morphism  $H \rightarrow \mathbb{k}^m$  defined by the  $a_j$  has image not contained in any proper linear subspace of  $\mathbb{k}^m$ , so there are  $h_1, \dots, h_m$  such that the  $(a_1(h_i), \dots, a_m(h_i)) \in \mathbb{k}^m$  span  $\mathbb{k}^m$ . For such  $h_i$ , the system of linear equations

$$h_i^{-1} \cdot f = \sum_{j=1}^m a_j(h_i) f_j, \quad i = 1, \dots, m$$

tells us that each  $f_j$  is in the span of  $\{h_1^{-1} \cdot f, \dots, h_m^{-1} \cdot f\} \subseteq V$ . So  $f_j \in V$  for each  $j$ . Because  $V \subseteq H^0(X, L^{\otimes r})^N$ , by the associativity property of an action and the fact that  $N$  is normal in  $H$  we have

$$\sum_j a_j(hn) f_j = (n^{-1} h^{-1}) \cdot f = h^{-1} \cdot f = \sum_j a_j(h) f_j.$$

for any  $n \in N$  and  $h \in H$ . Since the  $f_j$  are linearly independent  $a_j(hn) = a_j(h)$  for all  $n \in N$ ,  $h \in H$ , so  $a_j \in \mathcal{O}(H)^N$  for each  $j$ . Hence  $\Sigma_\theta^* f \in \mathcal{O}(H)^N \otimes V$ .  $\square$

*Proof of Proposition 3.2.1.* The proof is divided into two steps. We begin by constructing the morphism  $\bar{\Sigma} : (H/N) \times \mathcal{Y} \rightarrow \mathcal{Y}$  which defines the desired action and show that it maps  $(H/N) \times \mathcal{U}$  to  $\mathcal{U}$ , for  $\mathcal{U}$  as in the statement of the proposition. We then show  $\bar{\Sigma}$  satisfies the axioms for a group action and prove the equivariance of  $q_N$ .

**(Step 1: Definition of  $\bar{\Sigma}$  and restriction to  $\mathcal{U}$ .)** Recall that

$$(H/N) \times \mathcal{Y} = \text{Proj}(\mathcal{O}(H/N) \otimes S^N),$$

where the grading in  $\mathcal{O}(H/N) \otimes S^N$  is induced by  $S^N$ , with  $\mathcal{O}(H/N)$  having degree 0. The corresponding twisting sheaves  $\mathcal{O}(r)$  are given by the exterior tensor product  $\mathcal{O}_{H/N} \boxtimes \mathcal{O}_{\mathcal{Y}}(r)$  for each  $r \geq 0$  [St15, Tag 01MX]. Pullback along the quotient map  $H \rightarrow H/N$  identifies  $\mathcal{O}(H/N)$  with  $\mathcal{O}(H)^N$  and by virtue of Lemma 3.2.2 the diagram

$$\begin{array}{ccc} S^N & \xrightarrow{\Sigma_\theta^*|_{S^N}} & \mathcal{O}(H)^N \otimes S^N \\ \downarrow q_N^* & & \downarrow \\ S & \xrightarrow{\Sigma_\theta^*} & \mathcal{O}(H) \otimes S \end{array} \quad (3.4)$$

<sup>12</sup>A result like this is used in the proof of [Ne78, Lemma 3.1].

of graded rings is well defined and commutes, where  $\Sigma_\theta^*$  is as in (3.2). Applying the Proj functor to the top horizontal map defines a rational map, which we claim is in fact a morphism

$$\bar{\Sigma} := \text{Proj}(\Sigma_\theta^*|_{S^N}) : (H/N) \times \mathcal{Y} \rightarrow \mathcal{Y}.$$

To see this, we need to verify that if  $f \in S^N$  is a homogenous element of positive degree, then there is a homogeneous prime ideal of  $\mathcal{O}(H)^N \otimes S^N$ , different to the irrelevant ideal and not containing  $\Sigma_\theta^*(f)$ . But since  $S^N$  is reduced there is a homogeneous prime  $\mathfrak{p} \in \mathcal{Y} = \text{Proj}(S^N)$  not containing  $f$ , and it follows that

$$(\text{ev}_e \otimes \text{id}_S)^{-1}(\mathfrak{p}) \in \text{Proj}(\mathcal{O}(H)^N \otimes S^N)$$

is a homogeneous prime which does not contain  $\Sigma_\theta^*(f)$  and is different to the irrelevant ideal.

Now let  $\mathcal{S} \subseteq I^{\text{ss}, \text{fg}(N)}$  be a subset that is stable under the  $H$ -action on  $S$ . Notice that this includes the case  $\mathcal{S} = I^{\text{ss}, \text{fg}(N)}$  by virtue of the discussion before the statement of Proposition 3.2.1. Let  $\mathcal{U} = \bigcup_{f \in \mathcal{S}} \text{Spec}((S^N)_{(f)}) \subseteq \mathcal{Y}$  and consider the restriction of  $\bar{\Sigma}$  to  $(H/N) \times \mathcal{U}$ . Given  $y \in \mathcal{U} \subseteq \mathcal{Y}$  and  $f \in \mathcal{S}$  such that  $f(y) \neq 0$ , for any  $h \in H$  the section  $h \cdot f$  is contained in  $\mathcal{S}$  and maps to  $f$  under the composition  $(\text{ev}_h \otimes \text{id}_X^*) \circ \Sigma_\theta^*$  by (3.3). Applying Proj, this says that

$$(h \cdot f)(\bar{\Sigma}(\bar{h}, y)) \neq 0 \iff f(y) \neq 0,$$

where  $\bar{h} = hN$  and we think of  $f$  as a section of some power of  $\mathcal{O}_{\mathcal{Y}}(1)$ . It follows that  $\bar{\Sigma}$  maps  $(\bar{h}, y)$  into  $\text{Spec}((S^N)_{(h \cdot f)}) \subseteq \mathcal{U}$ , hence  $\bar{\Sigma}$  restricts to a map  $(H/N) \times \mathcal{U} \rightarrow \mathcal{U}$ . In the case where  $\mathcal{S} = I^{\text{ss}, \text{fg}(N)}$ , we conclude that  $\bar{\Sigma}$  restricts to a morphism  $(H/N) \times X \curvearrowright N \rightarrow X \curvearrowright N$ .

**(Step 2:**  $\bar{\Sigma}$  is an action of  $(H/N)$  on  $\mathcal{Y}$  and  $q_N$  is equivariant.) Let  $\mu$  (respectively,  $\bar{\mu}$ ) be the morphism defining group multiplication on  $H$  (respectively, on  $(H/N)$ ). By using the Proj functor, the commutative diagrams that  $\bar{\Sigma} : (H/N) \times \mathcal{Y} \rightarrow \mathcal{Y}$  needs to satisfy in order to be an action follow immediately from verifying that the following diagrams of graded rings commute:

$$\begin{array}{ccc} S^N & & \\ \Sigma_\theta^*|_{S^N} \downarrow & \searrow \text{id}_{S^N} & \\ \mathcal{O}(H)^N \otimes S^N & \xrightarrow{\text{ev}_e \otimes \text{id}_{S^N}} & S^N \end{array} \quad (\text{Identity})$$

and

$$\begin{array}{ccc} S^N & \xrightarrow{\Sigma_\theta^*|_{S^N}} & \mathcal{O}(H)^N \otimes S^N \\ \downarrow \Sigma_\theta^*|_{S^N} & & \downarrow \bar{\mu}^* \otimes \text{id}_{S^N} \\ \mathcal{O}(H)^N \otimes S^N & \xrightarrow{\text{id}_{H/N}^* \otimes (\Sigma_\theta^*|_{S^N})} & \mathcal{O}(H)^N \otimes \mathcal{O}(H)^N \otimes S^N \end{array} \quad (\text{Associativity})$$

Note that (Associativity) is well defined by Lemma 3.2.2. The diagram (Identity) is simply (3.3) applied to  $h = e \in H$ . To verify commutativity of diagram (Associativity), note that

$$\bar{\mu}^* : \mathcal{O}(H)^N \rightarrow \mathcal{O}(H)^N \otimes \mathcal{O}(H)^N$$

is just the restriction of  $\mu^* : \mathcal{O}(H) \rightarrow \mathcal{O}(H) \otimes \mathcal{O}(H)$  to the subring  $\mathcal{O}(H)^N$ , so (Associativity) is obtained by restricting the diagram

$$\begin{array}{ccc} S & \xrightarrow{\Sigma_\theta^*} & \mathcal{O}(H) \otimes S \\ \downarrow \Sigma_\theta^* & & \downarrow \mu^* \otimes \text{id}_S \\ \mathcal{O}(H) \otimes S & \xrightarrow{\text{id}_H^* \otimes \Sigma_\theta^*} & \mathcal{O}(H) \otimes \mathcal{O}(H) \otimes S \end{array}$$

to subalgebras of  $N$ -invariants. But this diagram commutes because  $\Sigma : H \times X \rightarrow X$  defines an action.

Finally, let  $\pi : H \rightarrow H/N$  be the canonical quotient map. Applying  $\text{Proj}$  to the commuting diagram (3.4), we see that  $\bar{\Sigma}$  makes the diagram

$$\begin{array}{ccc} H \times X^{\text{nss}(N)} & \xrightarrow{\Sigma} & X^{\text{nss}(N)} \\ \downarrow \pi \times q_N & & \downarrow q_N \\ (H/N) \times \mathcal{Y} & \xrightarrow{\bar{\Sigma}} & \mathcal{Y} \end{array}$$

commute, which is to say that  $q_N$  is equivariant with respect to the projection  $\pi : H \rightarrow H/N$ .  $\square$

A consequence of Proposition 3.2.1 is that there is a canonical action of  $H/N$  on the enveloping quotient  $X \wr N$ . The next result says that, if  $\mathcal{U} \subseteq X \wr N$  is a quasi-compact  $H/N$ -stable open subscheme, then any sufficiently divisible positive tensor power of the twisting sheaf  $\mathcal{O}_{\mathcal{U}}(1)$  has a uniquely defined natural  $H/N$ -linearisation.

**Proposition 3.2.3.** *Retain the notation preceding Proposition 3.2.1. Let  $\mathcal{S} \subseteq I^{\text{ss}, \text{fg}}(N)$  be a finite subset such that  $\mathcal{U} = \bigcup_{f \in \mathcal{S}} \text{Spec}((S^N)_{(f)})$  is stable under the  $H/N$ -action on  $X \wr N$  of Proposition 3.2.1.*

1. *If  $r > 0$  and  $V \subseteq H^0(X, L^{\otimes r})^N$  is an  $H$ -stable enveloping system adapted to  $\mathcal{S}$  for the restricted linearisation  $N \curvearrowright L \rightarrow X$ , then the immersion  $\bar{\phi} : \mathcal{U} \hookrightarrow \mathbb{P}(V^*)$  of Proposition 3.1.19, 2 is equivariant, and pullback of the canonical linearisation  $H/N \curvearrowright \mathcal{O}_{\mathbb{P}(V^*)}(1) \rightarrow \mathbb{P}(V^*)$  along  $\bar{\phi}$  defines a linearisation  $(H/N) \curvearrowright \mathcal{O}_{\mathcal{U}}(r) \rightarrow \mathcal{U}$  such that the natural morphism  $L^{\otimes r}|_{q_N^{-1}(\mathcal{U})} \rightarrow \mathcal{O}_{\mathcal{U}}(r)$  is equivariant with respect to the projection  $H \rightarrow H/N$ .*
2. *Given  $r > 0$  such that  $\mathcal{O}_{\mathcal{U}}(r) \rightarrow \mathcal{U}$  is very ample, there is at most one  $H/N$ -linearisation on  $\mathcal{O}_{\mathcal{U}}(r) \rightarrow \mathcal{U}$  making the natural map  $L^{\otimes r}|_{q_N^{-1}(\mathcal{U})} \rightarrow \mathcal{O}_{\mathcal{U}}(r)$  equivariant with respect to the projection  $H \rightarrow H/N$ .*

*Proof.* (Proof of 1.) The action of  $H$  on  $V$  descends to an action of  $H/N$  on  $V$ , which defines a linearisation  $(H/N) \curvearrowright \mathcal{O}(1) \rightarrow \mathbb{P}(V^*)$ . We show that  $\bar{\phi}$  is equivariant with respect to this action on  $\mathbb{P}(V^*)$ . Let  $\Sigma : H \times X \rightarrow X$  denote the action morphism and  $\bar{\Sigma} : (H/N) \times \mathcal{Y} \rightarrow \mathcal{Y}$  the action morphism on  $\mathcal{Y} = \text{Proj}(S^N)$  constructed in Proposition 3.2.1; note that  $\bar{\Sigma}$  restricts to a morphism  $(H/N) \times \mathcal{U} \rightarrow \mathcal{U}$  by assumption. By Lemma 3.2.2 the linear map  $\Sigma_\theta^*$  of (3.2) restricts

to define a map  $\Sigma_\theta^*|_V : V \rightarrow \mathcal{O}(H)^N \otimes V$ . Applying the  $\text{Sym}^\bullet$  functor, we get a homomorphism of graded rings

$$\text{Sym}^\bullet(\Sigma_\theta^*|_V) : \text{Sym}^\bullet V \rightarrow \mathcal{O}(H)^N \otimes \text{Sym}^\bullet V,$$

where  $\mathcal{O}(H)^N$  is in degree zero in the latter ring, and applying  $\text{Proj}$  to this homomorphism recovers the action of  $H/N$  on  $\mathbb{P}(V^*)$  just described. Furthermore, the following diagram of graded rings commutes (recall  $(S^N)^{(r)}$  is the  $r$ -th Veronese subring of  $S^N$ ):

$$\begin{array}{ccc} \text{Sym}^\bullet V & \xrightarrow{\text{Sym}^\bullet(\Sigma_\theta^*|_V)} & \mathcal{O}(H)^N \otimes \text{Sym}^\bullet V \\ \downarrow \text{mult} & & \downarrow \text{id}_H^* \otimes \text{mult} \\ (S^N)^{(r)} & \xrightarrow{\Sigma_\theta^*|_{S^N}} & \mathcal{O}(H)^N \otimes (S^N)^{(r)} \\ \downarrow & & \downarrow \\ \mathbb{k}[\mathcal{U}, \mathcal{O}_{\mathcal{U}}(r)] & \xrightarrow{\bar{\Sigma}^*} & \mathcal{O}(H)^N \otimes \mathbb{k}[\mathcal{U}, \mathcal{O}_{\mathcal{U}}(r)] \end{array}$$

Under the identification  $\text{Sym}^\bullet V = \mathbb{k}[\mathbb{P}(V^*), \mathcal{O}(1)]$ , the composition of the left-hand vertical arrows corresponds to pull-back along  $\bar{\phi}$  and, by the Künneth isomorphism, the composition of the right-hand vertical arrows corresponds to pulling back along  $\text{id}_{H/N} \times \bar{\phi}$ . Applying  $\text{Proj}$  to this diagram, it follows that  $\bar{\phi} : \mathcal{U} \hookrightarrow \mathbb{P}(V^*)$  is  $H/N$ -equivariant.

Define  $(H/N) \curvearrowright \mathcal{O}_{\mathcal{U}}(r) \rightarrow \mathcal{U}$  to be the linearisation obtained by pulling back  $H/N \curvearrowright \mathcal{O}(1) \rightarrow \mathbb{P}(V^*)$  under  $\bar{\phi}$ . We have  $q_N^* \mathcal{O}_{\mathcal{U}}(r) = L^{\otimes r}|_{q_N^{-1}(\mathcal{U})}$  as line bundles; let  $\psi : L^{\otimes r}|_{q_N^{-1}(\mathcal{U})} \rightarrow \mathcal{O}_{\mathcal{U}}(r)$  be the naturally induced bundle map. To show  $\psi$  is equivariant with respect to  $H \rightarrow H/N$ , argue as follows. The image of  $\bar{\phi}^* : H^0(\mathbb{P}(V^*), \mathcal{O}(1)) \rightarrow H^0(\mathcal{U}, \mathcal{O}(r))$  is an  $H/N$ -stable subspace of  $H^0(\mathcal{U}, \mathcal{O}(r))$  that pulls back under  $q_N$  to the linear system  $V \subseteq H^0(X, L^{\otimes r})^N$ , which is basepoint free on  $q_N^{-1}(\mathcal{U})$ . Let  $x \in q_N^{-1}(\mathcal{U})$ , let  $f \in V$  such that  $f(x) \neq 0$  and let  $F \in H^0(\mathcal{U}, \mathcal{O}(r))$  with  $q_N^* F = f$ . Then because  $q_N^*$  is equivariant with respect to the natural  $H/N$ -actions on  $V$  and  $H^0(\mathcal{U}, \mathcal{O}(r))$ , for any  $h \in H$  we have

$$hf(x) = (h \cdot f)(hx) = (h \cdot (q_N^* F))(hx) = (q_N^*(\bar{h} \cdot F))(hx),$$

(where  $\bar{h} = hN \in H/N$ ), whence

$$\psi(hf(x)) = (\bar{h} \cdot F)(q_N(hx)) = \bar{h}F(q_N(x)) = \bar{h}\psi(f(x)).$$

It follows by linearity that  $\psi(hl) = \bar{h}\psi(l)$  for any  $l \in L^{\otimes r}|_x$ . Hence  $\psi$  is equivariant with respect to  $H \rightarrow H/N$ .

(Proof of 2.) Suppose now  $r > 0$  is such that  $\mathcal{O}_{\mathcal{U}}(r) \rightarrow \mathcal{U}$  is equipped with two  $H/N$ -linearisations  $\mathcal{L}_1, \mathcal{L}_2$  such that the natural maps  $L^{\otimes r}|_{q_N^{-1}(\mathcal{U})} \rightarrow \mathcal{L}_1$  and  $L^{\otimes r}|_{q_N^{-1}(\mathcal{U})} \rightarrow \mathcal{L}_2$  are both equivariant with respect to the projection  $H \rightarrow H/N$ . Then the inclusions

$$\begin{aligned} q_N^* : H^0(\mathcal{U}, \mathcal{L}_1) &\hookrightarrow H^0(q_N^{-1}(\mathcal{U}), L^{\otimes r})^N, \\ q_N^* : H^0(\mathcal{U}, \mathcal{L}_2) &\hookrightarrow H^0(q_N^{-1}(\mathcal{U}), L^{\otimes r})^N \end{aligned}$$

are both  $H/N$ -equivariant linear maps, therefore the  $H/N$ -actions on  $H^0(\mathcal{U}, \mathcal{L}_1)$  and  $H^0(\mathcal{U}, \mathcal{L}_2)$  agree. Because  $\mathcal{O}_{\mathcal{U}}(r) \rightarrow \mathcal{U}$  is very ample, by Lemma 2.1.21 we can find a finite dimensional

rational  $H/N$ -module  $W \subseteq H^0(\mathcal{U}, \mathcal{L}_1) = H^0(\mathcal{U}, \mathcal{L}_2)$ , that is also a complete linear system, with which to equivariantly embed  $\mathcal{U}$  into  $\mathbb{P}(W^*)$ . Then the restriction of  $H/N \curvearrowright \mathcal{O}_{\mathbb{P}(W^*)}(1) \rightarrow \mathbb{P}(W^*)$  to  $\mathcal{U}$  is equal to both the linearisations  $\mathcal{L}_1$  and  $\mathcal{L}_2$ , so that  $\mathcal{L}_1 = \mathcal{L}_2$ .  $\square$

Given an inner enveloping quotient  $q_H : X^{\text{ss}, \text{fg}} \rightarrow \mathcal{U}$  for the  $H$ -linearisation  $L \rightarrow X$ , it is natural and desirable to want to factorise it through an inner enveloping quotient for the  $N$ -linearisation  $N \curvearrowright L \rightarrow X$  obtained by restriction. Unfortunately there is a theoretical obstruction to doing this, in that there may be  $H$ -invariant sections  $f$  over  $X$  where  $(S^H)_{(f)}$  is a finitely generated algebra, but  $(S^N)_{(f)}$  is not.

*Example 3.2.4.* Consider any example where  $N$  is a linear algebraic group acting linearly on a finitely generated graded  $\mathbb{k}$ -algebra  $A = \bigoplus_{d \geq 0} A_d$ , with  $A_0 = \mathbb{k}$ , such that  $A^N$  is not finitely generated over  $\mathbb{k}$  (e.g. Nagata's Example 2.1.25). Let  $X = \text{Spec } A$  and  $L = \mathcal{O}_X$  with the canonical  $N$ -linearisation. Because  $N$  respects the grading on  $A$ , there is a linearisation of  $H = N \times \mathbb{G}_m$  on  $L \rightarrow X$ , where  $\mathbb{G}_m \curvearrowright L \rightarrow X$  is the canonical linearisation defined by the grading on  $A$ . Now consider  $f = 1 \in H^0(X, L)^H = A^H$ . Then  $(S^H)_{(1)} = (\mathbb{k}[X, L]^H)_{(1)} = A^H = \mathbb{k}$  is finitely generated over  $\mathbb{k}$ , because the only  $\mathbb{G}_m$ -invariants in  $A$  are the constant functions. But  $(S^N)_{(1)} = A^N$  is not finitely generated over  $\mathbb{k}$ .

One can remedy this issue by considering a definition of ‘semistability’ that forces finite generation of both the algebras  $N$ -invariants and  $H$ -invariants, as follows. The inclusion  $S^H \hookrightarrow S^N$  induces a rational map

$$q_{H/N} : \text{Proj}(S^N) \dashrightarrow \text{Proj}(S^H) \quad (3.5)$$

that is invariant with respect to the canonical  $H/N$ -action on  $\text{Proj}(S^N)$  of Proposition 3.2.1. For the purpose of this discussion, let  $X^{\text{ss}, N\text{-fg}}$  denote the union of all  $X_f$  such that  $f \in \bigcup_{r \geq 0} H^0(X, L^{\otimes r})^H$  is an  $H$ -invariant section with both of the  $\mathbb{k}$ -algebras  $(S^N)_{(f)}$  and  $(S^H)_{(f)}$  finitely generated. For each such  $f$  we have  $\text{Spec}((S^N)_{(f)}) \subseteq X \curvearrowright N$  and the restriction of  $q_{H/N}$  to  $\text{Spec}((S^N)_{(f)})$  maps into  $\text{Spec}((S^H)_{(f)}) \subseteq X \curvearrowright H$ . Letting  $\mathcal{U}$  be the union of the  $\text{Spec}((S^N)_{(f)})$  defined by such  $f$ , we see that the rational map  $q_{H/N}$  restricts to give a well-defined morphism  $q_{H/N} : \mathcal{U} \rightarrow X \curvearrowright H$ . Furthermore, if  $q_H : X^{\text{ss}, \text{fg}(H)} \rightarrow X \curvearrowright H$  and  $q_N : X^{\text{ss}, \text{fg}(N)} \rightarrow X \curvearrowright N$  are the enveloping quotients for the  $H$ - and  $N$ -linearisations on  $L \rightarrow X$ , respectively, then the diagram

$$\begin{array}{ccccc} X^{\text{ss}, \text{fg}(H)} & \supseteq & X^{\text{ss}, N\text{-fg}} & \subseteq & X^{\text{ss}, \text{fg}(N)} \\ & \searrow q_H & \downarrow q_N & & \downarrow q_N \\ & & \mathcal{U} & \subseteq & X \curvearrowright N \\ & & \downarrow q_{H/N} & & \\ & & X \curvearrowright H & & \end{array}$$

commutes, with all inclusions open. We will take up this theme in the next section when talking about stability.



### 3.3 Stability for Non-Reductive Linearisations

We now turn to the question of defining an open subset of ‘stable’ points of  $X$  for a given linearisation  $L \rightarrow X$  of a linear algebraic group  $H$ , which admits a geometric  $H$ -quotient. A basic requirement we demand of such a definition is that it should extend the definitions of stability in the cases where  $H$  is reductive (Definition 2.2.2, 2) or unipotent (Definition 2.3.5). We will do this in the case where  $X$  is *irreducible*, since it will be helpful to make use of Remark 3.1.10.

#### 3.3.1 An Intrinsic Definition of Stability

Recall from Definition 2.1.22, 2 that for any linear algebraic group  $H$  there is a canonical normal unipotent subgroup  $H_u$  of  $H$ , called the unipotent radical of  $H$ , with the property that the quotient  $H_r = H/H_u$  is a reductive group. According to Proposition 3.2.1, the enveloping quotient  $X \mathbin{\mathbb{H}} H_u$  for the restricted linearisation  $H_u \curvearrowright L \rightarrow X$  has a canonical  $H_r$ -action which makes the enveloping quotient map  $q_{H_u} : X^{\text{ss}, \text{fg}(H_u)} \rightarrow X \mathbin{\mathbb{H}} H_u$  equivariant with respect to  $H \rightarrow H_r$ . This action comes from an action of  $H_r$  on  $\text{Proj}(S^{H_u})$  which has the property that the rational map

$$q_{H_r} : \text{Proj}(S^{H_u}) \dashrightarrow \text{Proj}(S^H) \quad (3.6)$$

defined via (3.5) is  $H_r$ -invariant.

Given an  $H$ -invariant section  $f$  of some positive tensor power of  $L \rightarrow X$  such that  $(S^{H_u})_{(f)}$  is a finitely generated  $\mathbb{k}$ -algebra, the basic open set  $\text{Spec}((S^{H_u})_{(f)})$  is stable under the  $H_r$ -action on  $X \mathbin{\mathbb{H}} H_u$ . The composition of  $q_{H_u} : X_f \rightarrow \text{Spec}((S^{H_u})_{(f)})$  with  $q_{H_r}$  coincides with the restriction of the enveloping quotient map  $q_H : X_f \rightarrow \text{Spec}((S^H)_{(f)})$ . Because  $X$  is irreducible, we have

$$(S^H)_{(f)} = ((S^{H_u})^{H_r})_{(f)} = ((S^{H_u})_{(f)})^{H_r}$$

(see Remark 3.1.10) and since  $H_r$  is reductive we thus have  $(S^H)_{(f)}$  finitely generated over  $\mathbb{k}$  and  $\text{Spec}((S^H)_{(f)}) \subseteq X \mathbin{\mathbb{H}} H$ . Following the ideas of unipotent GIT in Section 2.3.1, we define a notion of stability for the linearisation  $H \curvearrowright L \rightarrow X$  by requiring the restriction of the enveloping quotient map  $q_H : X^{\text{ss}, \text{fg}(H)} \rightarrow X \mathbin{\mathbb{H}} H$  for  $H \curvearrowright L \rightarrow X$  to give a geometric quotient  $X_f \rightarrow \text{Spec}((S^H)_{(f)})$  for the  $H$ -action on  $X_f$ , for suitably chosen sections  $f$ . There are a number of ways one could go about doing this. For example, it is easy to see that if each of  $q_{H_u}$  and  $q_{H_r}$  define geometric quotients for the  $H_u$ - and  $H_r$ -actions on  $X_f$  and  $\text{Spec}((S^{H_u})_{(f)})$ , respectively, then the composition  $q_H$  is a geometric quotient for  $H \curvearrowright X_f$ . But we also want to build on Definition 2.3.5 of stability from [DorK07], where one takes the  $X_f$  which are affine and admit a locally trivial geometric quotient. So it makes sense to further require that  $X_f$  is affine and  $q_{H_u} : X_f \rightarrow \text{Spec}((S^{H_u})_{(f)})$  is a principal bundle for the action of  $H_u$  on  $X_f$ . By [MumFK94, Amplification 1.3], the induced action of the reductive group  $H_r$  on  $\text{Spec}((S^{H_u})_{(f)})$  has a geometric quotient if, and only if, all the orbits are closed in  $\text{Spec}((S^{H_u})_{(f)})$ , and following the ideas of stability in reductive GIT it is also natural to demand that the stabilisers for this action are finite. Because the action of  $H_u$  on  $X_f$  is free and  $H_u$  is normal in  $H$ , these last conditions can be lifted to the action of  $H$  on  $X_f$  using the following lemma.

**Lemma 3.3.1.** *Suppose  $H$  is a linear algebraic group,  $N$  is a normal subgroup of  $H$  and  $X$  is an  $H$ -variety (not necessarily assumed irreducible). Suppose all the stabilisers for the restricted action  $N \curvearrowright X$  are finite and this action has a geometric quotient  $\pi : X \rightarrow X/N$ . Note that  $H/N$  acts canonically on  $X/N$ . Then*

1. *for all the  $H/N$ -orbits in  $X/N$  to be closed, it is necessary and sufficient that all the  $H$ -orbits in  $X$  are closed;*
2. *given  $y \in X/N$ , the stabiliser  $\text{Stab}_{H/N}(y)$  is finite if, and only if,  $\text{Stab}_H(x)$  is finite for some (and hence all)  $x \in \pi^{-1}(y)$ ; and*
3. *if  $H/N$  is reductive and  $X/N$  is affine, then  $X/N$  has a geometric  $H/N$ -quotient if, and only if, all  $H$ -orbits in  $X$  are closed.*

*Proof.* (Proof of 1.) Let  $x \in X$  and  $y = \pi(x)$ . We first show that  $H \cdot x = \pi^{-1}((H/N) \cdot y)$ . Clearly  $H \cdot x \subseteq \pi^{-1}((H/N) \cdot y)$ , because  $\pi$  is equivariant with respect to the projection  $H \rightarrow H/N$ . On the other hand, if  $x' \in \pi^{-1}((H/N) \cdot y)$ , then there is  $h \in H$  such that  $y = \bar{h}\pi(x') = \pi(hx')$ . Since  $\pi^{-1}(y) = N \cdot x$ , there is therefore  $n \in N$  such that  $x' = h^{-1}nx \in H \cdot x$ . Hence  $H \cdot x = \pi^{-1}((H/N) \cdot y)$ . Because  $\pi$  is a submersion,  $H \cdot x$  is closed if, and only if,  $(H/N) \cdot y$  is closed. Since  $\pi$  is surjective, this suffices to prove 1.

(Proof of 2.) Suppose  $y \in X/N$  has finite stabiliser in  $H/N$  and again let  $x \in \pi^{-1}(y)$ . Then

$$\text{Stab}_{H/N}(y) = \{g_1N, \dots, g_mN\}$$

for some finite collection of representatives  $g_1, \dots, g_m \in H$ , which we fix once and for all, such that the cosets  $g_iN$  are pairwise disjoint. If  $h \in \text{Stab}_H(x)$  then  $\bar{h} = hN \in \text{Stab}_{H/N}(y)$ , so  $h$  is contained in a unique coset  $g_{i(h)}N$ , where  $i(h) \in \{1, \dots, m\}$ . In this way we define a function

$$\text{Stab}_H(x) \rightarrow \{g_1, \dots, g_m\}, \quad h \mapsto g_{i(h)}.$$

We claim the fibres of this function are finite. Indeed, let  $h \in \text{Stab}_H(x)$  and suppose  $\tilde{h} \in \text{Stab}_H(x)$  is such that  $g_{i_0} := g_{i(h)} = g_{i(\tilde{h})}$ , with  $i_0 \in \{1, \dots, m\}$ . Then we may find  $n, \tilde{n} \in N$  such that  $h = ng_{i_0}$  and  $\tilde{h} = \tilde{n}g_{i_0}$ , so  $ng_{i_0}x = \tilde{n}g_{i_0}x$ . It follows that, for some  $p \in \text{Stab}_N(g_{i_0}x)$ , we have  $\tilde{n} = pn$  and  $\tilde{h} = ph$ . Since all stabilisers for the  $N$ -action on  $X$  are finite, there are finitely many choices for  $\tilde{h}$  and hence the fibre containing  $h$  is finite, as claimed. We conclude that  $\text{Stab}_H(x)$  is finite for any  $x \in \pi^{-1}(y)$ .

Conversely, suppose  $x \in X$  has finite stabiliser in  $H$  and let  $y = \pi(x) \in X/N$ . Let  $h \in H$  such that  $\bar{h} = hN \in \text{Stab}_{H/N}(y)$ . Then  $\pi(x) = \bar{h}\pi(x) = \pi(hx)$  and, because  $\pi$  is a geometric  $N$ -quotient and  $N$  is normal in  $H$ , there is  $n \in N$  such that  $hnx = x$ . Hence  $hn \in \text{Stab}_H(x)$  and  $\bar{h}$  is in the image of  $\text{Stab}_H(x)$  under the quotient map  $H \rightarrow H/N$ . Thus  $\text{Stab}_{H/N}(y)$  is finite.

(Proof of 3.) If a geometric quotient  $X/N \rightarrow (X/N)/(H/N)$  exists then the composition  $X \xrightarrow{\pi} X/N \rightarrow (X/N)/(H/N)$  is a geometric quotient for the  $H$ -action on  $X$ , which implies that all the  $H$ -orbits in  $X$  are closed. Now suppose all the  $H$ -orbits in  $X$  are closed. Because  $X/N$  is affine and  $H/N$  is reductive the categorical quotient of  $X/N$  by  $H/N$  exists by Theorem 2.2.1. Every  $H/N$ -orbit in  $X/N$  is closed by 1, so the categorical quotient of  $X/N$  by  $H/N$  is a geometric quotient [MumFK94, Amplification 1.3].  $\square$

In light of the above lemma and the preceding discussion, we make the following definition.

**Definition 3.3.2.** Let  $H$  be a linear algebraic group (with as usual unipotent radical  $H_u$  and  $H_r = H/H_u$  reductive) acting on an irreducible variety  $X$  and let  $L \rightarrow X$  be a linearisation for the action. The *stable locus* is the open subset

$$X^s := \bigcup_{f \in I^s} X_f$$

of  $X^{\text{ns}}$ , where  $I^s \subseteq \bigcup_{r>0} H^0(X, L^{\otimes r})^H$  is the subset of  $H$ -invariant sections satisfying the following conditions:

1. the open set  $X_f$  is affine;
2. the action of  $H$  on  $X_f$  is closed with all stabilisers finite groups; and
3. the restriction of the  $H_u$ -enveloping quotient map

$$q_{H_u} : X_f \rightarrow \text{Spec}((S^{H_u})_{(f)})$$

is a principal  $H_u$ -bundle for the action of  $H_u$  on  $X_f$ .

*Remark 3.3.3.* It is clear that this definition of stability extends the definition of stability in [DorK07] for unipotent groups (see Definition 2.3.5). In the case where  $H$  is reductive, then  $H_u$  is trivial and our definition reduces to Mumford's notion of properly stable points [MumFK94] (see Definition 2.2.2, 2).

*Remark 3.3.4.* Observe that requiring 1 and 3 in Definition 3.3.2 is equivalent to demanding that  $X_f$  be an affine open subset of  $X$  that is a *trivial*  $H_u$ -bundle by Proposition 2.1.26.

The significance of assuming that  $X$  is irreducible in Definition 3.3.2 is that it ensures

$$\text{Spec}((S^H)_{(f)}) = \text{Spec}(((S^{H_u})_{(f)})^{H_r}),$$

so that by reductive GIT for affine varieties  $q_{H_r} : \text{Spec}((S^{H_u})_{(f)}) \rightarrow \text{Spec}((S^H)_{(f)})$  is at least a good categorical  $H_r$ -quotient when  $f$  is an  $H$ -invariant (see Theorem 2.2.1) and  $\text{Spec}((S^H)_{(f)}) \subseteq X \wr H$ . Then if  $f \in I^s$ , conditions 1–3 in Definition 3.3.2, combined with Lemma 3.3.1, 3, tell us that

$$q_H : X_f \xrightarrow{q_{H_u}} \text{Spec}((S^{H_u})_{(f)}) \xrightarrow{q_{H_r}} \text{Spec}((S^H)_{(f)})$$

is a composition of geometric quotients, hence a geometric quotient for  $H \curvearrowright X_f$ . Because the property of being a geometric quotient is local on the base, it follows that the enveloping quotient  $q_H : X^{\text{ss}, \text{fg}(H)} \rightarrow X \wr H$  restricts to define a geometric quotient

$$q_H : X^s \rightarrow X^s/H = q_H(X^s).$$

This factorises through the restriction of the enveloping quotient for  $H_u$  in a natural way, and we have the following commutative diagram, with all inclusions open:

$$\begin{array}{ccccc}
X^{\text{ss,fg}}(H) & \supseteq & X^s & \subseteq & X^{\text{ss,fg}}(H_u) \\
\downarrow q_H & & \text{geo} \downarrow q_{H_u} & & \downarrow q_{H_u} \\
& & X^s/H_u & \subseteq & X \wr H_u \\
& & \text{geo} \downarrow q_{H_r} & & \\
X \wr H & \supseteq & X^s/H & & 
\end{array}$$

*Remark 3.3.5.* If  $X$  is irreducible and  $q_H : X^{\text{ss,fg}}(H) \rightarrow \mathcal{U}$  is any inner enveloping quotient for the linearisation  $H \curvearrowright L \rightarrow X$ , then the geometric quotient  $X^s/H$  of  $X^s$  is naturally an open subvariety of  $\mathcal{U}$ .

One of the features of the stable locus defined in Definition 3.3.2 is that it behaves well under affine locally closed immersions, as the next lemma shows.

**Lemma 3.3.6.** *Let  $H$  be a linear algebraic group acting on an irreducible variety  $X$  with linearisation  $L \rightarrow X$ . Suppose  $Y$  is another irreducible variety and  $\phi : Y \hookrightarrow X$  is an  $H$ -equivariant locally closed immersion that is an affine morphism. Then  $\phi^{-1}(X^s)$  is an open subset of  $Y^{\text{ss}(\phi^*L)}$ , the image of  $\phi^{-1}(X^s)$  under the enveloping quotient  $q' : Y^{\text{ss,fg}}(\phi^*L) \rightarrow Y \wr_{\phi^*L} H$  is a geometric quotient for the  $H$ -action on  $\phi^{-1}(X^s)$ , and there is a locally closed immersion  $\bar{\phi} : \phi^{-1}(X^s)/H \hookrightarrow X^s/H$  such that the following diagram commutes (with unmarked inclusions open)*

$$\begin{array}{ccccc}
Y^{\text{ss}(\phi^*L)} & \supseteq & \phi^{-1}(X^s) & \xhookrightarrow{\phi} & X^s \\
\downarrow q' & & \text{geo} \downarrow q' & & \downarrow q \\
Y^{\text{ss}(\phi^*L)}/H & \supseteq & \phi^{-1}(X^s)/H & \xhookrightarrow{\bar{\phi}} & X^s/H
\end{array}$$

*Proof.* Let  $R = \mathbb{k}[Y, \phi^*L]$  and  $S = \mathbb{k}[X, L]$ . The set  $\phi^{-1}(X^s)$  is covered by open subsets of the form  $Y_{\phi^*f}$ , where  $f$  is a section in  $I^s \subseteq \bigcup_{r>0} H^0(X, L^{\otimes r})^H$  of Definition 3.3.2, and by Remark 3.3.4 the map  $q : X_f \rightarrow \text{Spec}((S^H)_{(f)})$  is a trivial  $H_u$ -bundle for such an  $f$ . For each such  $f$  the open subset  $Y_{\phi^*f} = \phi^{-1}(X_f)$  is affine because  $\phi$  is affine and  $X_f$  is affine. It is clear that the action of  $H$  on  $Y_{\phi^*f}$  is closed with all stabilisers finite. By restriction  $Y_{\phi^*f}$  also has the structure of a trivial  $H_u$ -bundle, thus  $Y_{\phi^*f}/H_u$  is affine and isomorphic to  $\text{Spec}((R^{H_u})_{(\phi^*f)})$  (because  $Y$  is irreducible). Thus  $Y_{\phi^*f} \subseteq Y^{\text{ss}(\phi^*L)}$  and the restriction of the enveloping quotient map,

$$q' : Y_{\phi^*f} \rightarrow \text{Spec}((R^H)_{(\phi^*f)}),$$

is a geometric quotient for the  $H$ -action on  $Y_{\phi^*f}$ . On the other hand, by the submersion property of a geometric quotient the image of  $q \circ \phi : Y_{\phi^*f} \rightarrow X_f/H$  is a locally closed subset of  $X_f/H$  that is also a geometric quotient for the  $H$ -action on  $Y_{\phi^*f}$ , hence there is a unique locally closed immersion

$$\text{Spec}((R^H)_{(\phi^*f)}) \hookrightarrow X_f/H$$

factoring  $(q \circ \phi)|_{Y_{\phi f}}$  through  $q'|_{Y_{\phi f}}$ . By varying over suitable  $f$ , we see that  $\phi^{-1}(X^s) \subseteq Y^{s(\phi^*L)}$  and the above morphisms  $\text{Spec}((R^H)_{(\phi^*f)}) \hookrightarrow X_f/H$  glue to give the locally closed immersion

$$\bar{\phi} : q'(\phi^{-1}(X^s)) = \phi^{-1}(X^s)/H \hookrightarrow X^s/H$$

making the required diagram commute.  $\square$

### 3.3.2 Relation to Stability of Reductive Extensions

We next consider how the notion of stability for a linear action of  $H$  proposed in Definition 3.3.2 relates to stability for a reductive group acting on the fibre space  $G \times^{H_u} X$  associated to certain embeddings  $H_u \hookrightarrow G$ , with  $G$  reductive, where  $H_u$  is the unipotent radical of  $H$ . The resulting Proposition 3.3.8 provides an extension of the fact that the Mumford stable locus is equal to the locally trivial stable locus in the unipotent setting (see Proposition 2.3.9). This will be important for the theory of reductive envelopes in the upcoming Section 4, and shows more generally how Mumford's reductive GIT can be used to study non-reductive group actions.

It will be convenient to make the following definition.

**Definition 3.3.7.** Let  $H$  be a linear algebraic group and  $G$  a reductive group. A homomorphism  $H \rightarrow G$  is called  $H_u$ -faithful if its restriction to the unipotent radical  $H_u$  of  $H$  defines a closed embedding  $H_u \hookrightarrow G$ .

Fix a reductive group  $G$  and an  $H_u$ -faithful homomorphism  $\rho : H \rightarrow G$ . Then we can consider the fibre space  $G \times^{H_u} X$  associated to  $X$  and the homomorphism  $\rho|_{H_u} : H_u \hookrightarrow G$ , together with its natural closed immersion  $\alpha : X \hookrightarrow G \times^{H_u} X$ . This space exists as a variety because  $H_u$  is unipotent [EG98, Proposition 23], hence the natural projection

$$G \times X \rightarrow G \times^{H_u} X$$

is a geometric quotient for the diagonal action of  $H_u$  in the category of varieties. Because  $H_u$  is normal in  $H$  the diagonal action of  $H$  on  $G \times X$ , induced by the action of  $H$  on  $X$  and right multiplication on  $G$  through  $\rho$ , descends through this projection to give an action of  $H_r = H/H_u$  on  $G \times^{H_u} X$ . Explicitly, this given by

$$\bar{h} \cdot [g, x] = [g\rho(h)^{-1}, hx]$$

for each  $g \in G$ ,  $x \in X$ , and  $\bar{h} = hH_u \in H_r$ . This action of  $H_r$  commutes with the  $G$ -action on  $G \times^{H_u} X$ , so we can view  $G \times^{H_u} X$  as a  $G \times H_r$ -variety in a natural way. Notice that the inclusion  $\alpha : X \hookrightarrow G \times^{H_u} X$  is equivariant with respect to the diagonal embedding  $H \hookrightarrow G \times H_r$  induced by  $\rho$  and the quotient  $H \rightarrow H_r$ .

As noted in Section 2.3.2, there is a natural  $G$ -linearisation over  $G \times^{H_u} X$  which extends the  $H_u$ -linearisation on  $L$  under the inclusion  $\alpha$ . By abuse of notation, we denote this linearisation  $L = G \times^{H_u} L \rightarrow G \times^{H_u} X$ . The diagonal  $H_r$ -action on  $G \times^{H_u} X$  canonically lifts to the line bundle  $L$  to define an  $H_r$ -linearisation on  $L \rightarrow G \times^{H_u} X$  which commutes with the  $G$ -linearisation, hence there is a natural linearisation

$$G \times H_r \curvearrowright L \rightarrow G \times^{H_u} X.$$

This provides an extension of the  $H$ -linearisation  $H \curvearrowright L \rightarrow X$ , when we let  $H$  act on  $L = G \times^{H_u} L$  via the diagonal homomorphism  $H \rightarrow G \times H_r$ . As such, pulling back sections along  $\alpha$  induces an isomorphism

$$\alpha^* : \mathbb{k}[G \times^{H_u} X, L]^{G \times H_r} \xrightarrow{\cong} \mathbb{k}[X, L]^H.$$

We may now state

**Proposition 3.3.8.** *Let  $H$  be a linear algebraic group (with unipotent radical  $H_u$  and  $H_r = H/H_u$  reductive) acting on an irreducible variety  $X$  with linearisation  $H \curvearrowright L \rightarrow X$ , and let  $G$  be a reductive group with an  $H_u$ -faithful homomorphism  $\rho : H \rightarrow G$ . Let  $\alpha$  be the natural closed immersion of  $X$  into  $G \times^{H_u} X$  and let  $(G \times^{H_u} X)^{s(L)}$  be the stable locus for the  $G \times H_r$ -linearisation  $L \rightarrow G \times^{H_u} X$  in the sense of Definition 2.2.2, 2. Then*

$$X^s = \alpha^{-1}((G \times^{H_u} X)^{s(L)}).$$

*Proof.* (Proof of  $\alpha^{-1}((G \times^{H_u} X)^{s(L)}) \subseteq X^s$ .) Suppose  $x \in X$  and  $\alpha(x) \in (G \times^{H_u} X)_F$ , where  $F$  is a  $G \times H_r$ -invariant section of some positive tensor power of  $L$  over  $G \times^{H_u} X$  such that  $(G \times^{H_u} X)_F$  is affine and the  $G \times H_r$ -action on  $(G \times^{H_u} X)_F$  is closed with finite stabilisers. Let  $f = \alpha^*F$  be the corresponding  $H$ -invariant over  $X$ , so that  $x \in X_f$ . Then  $\alpha$  restricts to an  $H$ -equivariant closed immersion  $X_f \hookrightarrow (G \times^{H_u} X)_F$ , thus  $X_f$  is affine. For any  $y \in X_f$  the orbit  $H \cdot y = \alpha^{-1}((G \times H_r) \cdot \alpha(y))$  is closed in  $X_f$  and  $\text{Stab}_H(y) \subseteq \text{Stab}_{G \times H_r}(\alpha(y))$ , so all  $H$ -orbits in  $X_f$  are closed and all stabilisers for the  $H$ -action on  $X_f$  are finite. It remains to show that the restriction of the  $H_u$ -enveloping quotient

$$q_{H_u} : X_f \rightarrow \text{Spec}((S^{H_u})_{(f)})$$

gives  $X_f$  the structure of a principal  $H_u$ -bundle. The action of  $G$  on  $(G \times^{H_u} X)_F$  is set-theoretically free, because all its stabilisers are conjugate to subgroups of the unipotent group  $H_u$  and, since they are finite, are thus trivial. Furthermore, all  $G$ -orbits in  $(G \times^{H_u} X)_F$  are closed because the action of  $G \times H_r$  on  $(G \times^{H_u} X)_F$  is proper [MumFK94, Corollary 2.5]. Hence  $(G \times^{H_u} X)_F$  is in the stable locus for the restricted linearisation  $G \curvearrowright L \rightarrow G \times^{H_u} X$ . The action of  $G$  on  $(G \times^{H_u} X)_F$  is therefore proper ([MumFK94, Corollary 2.5] again) and so the action of  $G$  on  $(G \times^{H_u} X)_F$  is free by Lemma 2.1.9. The subset  $(G \times^{H_u} X)_F$  has an affine geometric quotient  $(G \times^{H_u} X)_F/G \cong X_f/H_u$  by Theorem 2.2.1, 2, which by [MumFK94, Proposition 0.9] is actually a locally trivial quotient. By descent [Ser58, Proposition 10] this means  $X_f$  has an affine locally trivial geometric quotient, isomorphic to  $\text{Spec}(\mathcal{O}(X_f)^{H_u}) = \text{Spec}((S^{H_u})_{(f)})$ . So  $q_{H_u} : X_f \rightarrow \text{Spec}((S^{H_u})_{(f)})$  is a locally trivial  $H_u$ -quotient.

(Proof of  $X^s \subseteq \alpha^{-1}((G \times^{H_u} X)^{s(L)})$ .) Let  $x \in X_f$ , where  $f$  is an  $H$ -invariant section of some positive tensor power of  $L \rightarrow X$  such that  $X_f$  is affine, has closed  $H$ -orbits with all stabilisers finite and  $q_{H_u} : X_f \rightarrow \text{Spec}((S^{H_u})_{(f)})$  is a principal  $H_u$ -bundle. Let  $F$  be the  $G \times H_r$ -invariant over  $G \times^{H_u} X$  pulling back to  $f$  under  $\alpha$ . By [Ser58, Proposition 5], the natural morphism  $(G \times^{H_u} X)_F = G \times^{H_u}(X_f) \rightarrow X_f/H_u$  is a principal  $G$ -bundle with affine base. By [MumFK94, Proposition 0.7] this means  $(G \times^{H_u} X)_F \rightarrow X_f/H_u$  is an affine morphism, hence  $(G \times^{H_u} X)_F$  is affine. Now, any  $G \times H_r$ -orbit in  $(G \times^{H_u} X)_F$  is the image  $G \times^{H_u} O$  of a subset of the form

$G \times O \subseteq G \times X_f$  under the geometric quotient  $G \times X_f \rightarrow (G \times^{H_u} X)_F$ , where  $O \subseteq X_f$  is an  $H$ -orbit. Since  $O$  is closed in  $X_f$ , so too is the  $G \times H_r$ -orbit  $G \times^{H_u} O$  inside  $(G \times^{H_u} X)_F$ . Hence all  $G \times H_r$ -orbits in  $(G \times^{H_u} X)_F$  are closed. Moreover, because any point in  $(G \times^{H_u} X)_F$  is in the  $G$ -sweep of a point in  $X_f$  via  $\alpha$ , any stabiliser for the  $G \times H_r$ -action on  $(G \times^{H_u} X)_F$  is conjugate to an  $H$ -stabiliser for a point in  $X_f$  under the inclusion  $H \hookrightarrow G \times H_r$  induced by  $\rho$  and  $H \rightarrow H_r$ . Hence all stabilisers for the  $G \times H_r$ -action on  $(G \times^{H_u} X)_F$  are finite. It follows that  $(G \times^{H_u} X)_F \subseteq (G \times^{H_u} X)^{s(L)}$  and  $x \in \alpha^{-1}((G \times^{H_u} X)^{s(L)})$ .  $\square$

*Remark 3.3.9.* For future reference we note the following fact, which was shown during the proof of Proposition 3.3.8: given an  $H$ -linearisation  $L \rightarrow X$ , an  $H_u$ -faithful homomorphism  $H \rightarrow G$  and an  $H$ -invariant section  $f$  of some positive tensor power of  $L \rightarrow X$  with associated  $G \times H_r$ -invariant section  $F$  over  $G \times^{H_u} X$ , then  $(G \times^{H_u} X)_F = G \times^{H_u} (X_f)$  is affine if, and only if,  $X_f$  is affine and  $X_f \rightarrow \text{Spec}((S^{H_u})_{(f)})$  a principal  $H_u$ -bundle.

It immediately follows from Proposition 3.3.8 that we have an equality

$$(G \times^{H_u} X)^{s(L)} = G \times^{H_u} (X^s).$$

Because  $G$  is a closed reductive subgroup of  $G \times H_r$  it follows that  $(G \times^{H_u} X)^{s(L)}$  is contained in the stable locus for the restricted linearisation  $G \curvearrowright L \rightarrow G \times^{H_u} X$  and hence has a geometric quotient for the  $G$ -action. The inclusion  $\alpha_{H_u} : X^s \hookrightarrow G \times^{H_u} (X^s)$  thus induces an  $H_r$ -equivariant isomorphism

$$X^s/H_u \cong (G \times^{H_u} X)^{s(L)}/G,$$

and since  $(G \times^{H_u} X)^{s(L)}/(G \times H_r) = ((G \times^{H_u} X)^{s(L)}/G)/H_r$ , we conclude that  $\alpha_{H_u}$  descends further to an isomorphism

$$X^s/H \cong (G \times^{H_u} X)^{s(L)}/(G \times H_r).$$

So the significance of Proposition 3.3.8 is that it allows us to describe stability for the linearisation  $H \curvearrowright L \rightarrow X$  and its geometric quotient in terms of stability and the associated quotient for the *reductive* linearisation  $G \times H_r \curvearrowright L \rightarrow G \times^{H_u} X$ .

*Remark 3.3.10.* Even if  $H \curvearrowright L \rightarrow X$  is a linearisation of an ample line bundle over a projective variety, the induced fibre space  $G \times^{H_u} X$  will only be quasi-projective with an ample linearisation, so care needs to be taken when computing stability. Similarly, if  $X$  is affine then  $G \times^{H_u} X$  is not necessarily affine.

*Remark 3.3.11.* One can also consider the fibre space  $G \times^H X$  associated to the  $H_u$ -faithful homomorphism  $\rho : H \rightarrow G$ , together with its natural  $G$ -linearisation  $\tilde{L} := G \times^H L \rightarrow G \times^H X$  and inclusion  $\alpha_H : H \hookrightarrow G \times^H X$  (assuming these spaces exist as varieties). If  $\ker \rho$  is finite, then it can be shown that  $X^s = \alpha_H^{-1}((G \times^H X)^{s(\tilde{L})})$  and the induced embedding  $X^s \hookrightarrow (G \times^H X)^{s(\tilde{L})}$  descends to an isomorphism  $X^s/H \cong ((G \times^H X)^{s(\tilde{L})})/G$ . Since we will not use this in the sequel, we omit the details of the proofs.

### 3.4 Summary of the Intrinsic Picture

We shall shortly draw together the work done so far to give a result that we believe provides a good theoretical basis for doing geometric invariant theory, in the case where  $H$  is any linear algebraic group acting on an irreducible variety  $X$  with linearisation  $L \rightarrow X$ . Before doing so, we make one final observation about the relationship between the notion of stability in Definition 3.3.2 and the various notions of semistability considered before.

As already observed, we have

$$X^s \subseteq X^{\text{ss}, \text{fg}} \subseteq X^{\text{nss}}.$$

This can be further refined using the ideas at the end of Section 3.2.2. The stable locus is patched together with affine open subsets  $X_f$ , for certain  $H$ -invariant sections  $f$  of a positive tensor power of  $L \rightarrow X$  which, among other things, have the property that  $(S^{H_u})_{(f)}$  is a finitely generated  $\mathbb{k}$ -algebra (cf. Definition 3.3.2, 3). Because  $H_r$  is reductive and  $X$  is irreducible then the full invariant algebra  $(S^H)_{(f)}$  is finitely generated over  $\mathbb{k}$ . This idea suggests it is useful to consider another notion of ‘semistability’ that sits inside the finitely generated semistable locus.

**Definition 3.4.1.** Let  $H$  be a linear algebraic group with unipotent radical  $H_u$ , let  $H_r = H/H_u$  and let  $H \curvearrowright L \rightarrow X$  be a linearisation of an irreducible  $H$ -variety  $X$ . We define the  $H_u$ -finitely generated semistable locus to be the open subset

$$X^{\text{ss}, H_u\text{-fg}} := \bigcup_{f \in I^{\text{ss}, H_u\text{-fg}}} X_f,$$

of  $X^{\text{nss}}$ , where

$$I^{\text{ss}, H_u\text{-fg}} = \{f \in \bigcup_{r>0} H^0(X, L^{\otimes r})^H \mid (S^{H_u})_{(f)} \text{ is a finitely generated } \mathbb{k}\text{-algebra}\}.$$

It follows from the discussion above that  $X^s \subseteq X^{\text{ss}, H_u\text{-fg}} \subseteq X^{\text{ss}, \text{fg}}$ . The image of  $X^{\text{ss}, H_u\text{-fg}}$  under the enveloping quotient map  $q_{H_u} : X^{\text{ss}, \text{fg}(H_u)} \rightarrow X \wr H_u$  for the restricted linearisation  $H_u \curvearrowright L \rightarrow X$  is contained in the  $H_r$ -invariant open subscheme

$$\bigcup_{f \in I^{\text{ss}, H_u\text{-fg}}} \text{Spec}((S^{H_u})_{(f)}) \subseteq X \wr H_u.$$

This subscheme is not necessarily quasi-compact, but we can always find a finite subset  $\mathcal{S} \subseteq I^{\text{ss}, H_u\text{-fg}}$  of invariant sections such that the image  $q_{H_u}(X^{\text{ss}, H_u\text{-fg}})$  is contained in the quasi-compact open subscheme

$$\mathcal{U} := \bigcup_{f \in \mathcal{S}} \text{Spec}((S^{H_u})_{(f)}) \subseteq X \wr H_u,$$

which is an inner enveloping quotient of  $X^{\text{ss}, H_u\text{-fg}}$  under the linearisation  $H_u \curvearrowright L \rightarrow X$ . The rational map  $q_{H_r} : \text{Proj}(S^{H_u}) \dashrightarrow \text{Proj}(S^H)$  of (3.6) is defined on  $\mathcal{U}$  and gives a morphism

$$q_{H_r} : \mathcal{U} \rightarrow X \wr H.$$



In fact, the image under  $q_{H_r}$  is precisely the reductive GIT quotient  $\mathcal{U} // H_r$  for the natural  $H_r$ -linearisation  $\mathcal{O}_{\mathcal{U}}(r) \rightarrow \mathcal{U}$  (for  $r > 0$  sufficiently large) defined in Proposition 3.2.3, noting that the semistable set for this linearisation is the whole of  $\mathcal{U}$ . Indeed, we have

$$\mathcal{U} // H_r = \bigcup_{f \in \mathcal{S}} \text{Spec}((S^H)_{(f)}) \subseteq X \wr H.$$

Thus  $q_{H_r} : \mathcal{U} \rightarrow \mathcal{U} // H_r$  is a good categorical quotient by  $H_r$ , and  $q_H : X^{\text{ss}, H_u - \text{fg}} \rightarrow \mathcal{U} // H_r \subseteq X \wr H$  is an inner enveloping quotient of  $X^{\text{ss}, H_u - \text{fg}}$  for the full linearisation  $H \curvearrowright L \rightarrow X$ . It is also clear that the geometric quotient  $X^s / H$  of the stable locus  $X^s$  by  $H$  is naturally an open subvariety of  $\mathcal{U} // H_r$ .

The following theorem summarises the main points of our work so far.

**Theorem 3.4.2.** *Let  $H$  be a linear algebraic group (with unipotent radical  $H_u$ ) acting on an irreducible variety  $X$  and  $L \rightarrow X$  a linearisation for the action. Let  $H_r = H / H_u$ , let  $S = \mathbb{k}[X, L]$  and let*

$$\begin{aligned} q_H &: \text{Proj } S \dashrightarrow \text{Proj}(S^H) \\ q_{H_u} &: \text{Proj } S \dashrightarrow \text{Proj}(S^{H_u}) \\ q_{H_r} &: \text{Proj}(S^{H_u}) \dashrightarrow \text{Proj}(S^H) \end{aligned}$$

be the rational maps defined by the obvious inclusions of graded algebras. Also let

$$\mathcal{U} = \bigcup_{f \in \mathcal{S}} \text{Spec}((S^{H_u})_{(f)}),$$

where  $\mathcal{S}$  is a finite subset of  $I^{\text{ss}, H_u - \text{fg}}$  such that  $X^{\text{ss}, H_u - \text{fg}} = \bigcup_{f \in \mathcal{S}} X_f$ .

1. *There is a commutative diagram*

$$\begin{array}{ccccccc} X^s & \subseteq & X^{\text{ss}, H_u - \text{fg}} & \subseteq & X^{\text{ss}, \text{fg}} & \subseteq & X^{\text{nss}} & \subseteq & X \\ \text{geo} \downarrow q_{H_u} & & \downarrow q_{H_u} & & \downarrow & & \downarrow & & \downarrow q_{H_u} \\ X^s / H_u & \subseteq & \mathcal{U} & & & & & & \text{Proj}(S^{H_u}) \\ \text{geo} \downarrow q_{H_r} & & \text{good} \downarrow q_{H_r} & & \downarrow q_H & & \downarrow q_H & & \downarrow q_{H_r} \\ X^s / H & \subseteq & \mathcal{U} // H_r & \subseteq & X \wr H & \subseteq & \text{Proj}(S^H) & = & \text{Proj}(S^H) \end{array}$$

with good or geometric quotients as indicated and all inclusions open. The induced morphism  $q_H : X^{\text{ss}, H_u - \text{fg}} \rightarrow \mathcal{U} // H_r$  is an inner enveloping quotient of  $X^{\text{ss}, H_u - \text{fg}}$ .

2. *Given any reductive group  $G$  and  $H_u$ -faithful homomorphism  $H \rightarrow G$ , we have*

$$X^s = \alpha^{-1}((G \times^{H_u} X)^{s(L)}),$$

inducing a natural isomorphism

$$X^s / H \cong (G \times^{H_u} X)^{s(L)} / (G \times H_r)$$

where  $\alpha : X \hookrightarrow G \times^{H_u} X$  is the natural inclusion and  $L = G \times^{H_u} L$  is the natural  $G \times H_r$ -linearisation defined in Section 3.3.2.

*Remark 3.4.3.* The spaces involved in the statement of Theorem 3.4.2 are unchanged when we replace the linearisation  $L \rightarrow X$  by any positive tensor power  $L^{\otimes r} \rightarrow X$ . (In the case of  $X^{\text{ss}}$  and  $X^{\text{ss,fg}}$  this was observed in Remark 3.1.2.) It thus makes sense to talk about the notions of stability, finitely generated semistability, enveloping quotients etc. for rational linearisations (see Remark 2.1.20).

Theorem 3.4.2 is a culmination of all the intrinsic notions we have discussed so far and provides what we believe is a good basis for doing geometric invariant theory for non-reductive groups. One reason for this is because, in the case where  $L \rightarrow X$  is an ample linearisation over a projective variety  $X$ , it extends the main geometric invariant theoretic theorems in both the reductive and unipotent settings. However, given a general non-reductive linearisation the question remains of how one can study the stable locus  $X^s$ , finitely generated semistable locus  $X^{\text{ss,fg}}$ , inner enveloping quotients of  $X^{\text{ss,fg}}$  and the geometric quotient  $X^s/H$ , in a more explicit manner. In the next section we consider some techniques for doing this.

## 4 Projective Completions of Enveloped Quotients and Reductive Envelopes

In Section 3 we developed a theoretical framework for identifying open subsets of varieties that admit geometric quotients under a given group action, which seeks to provide an analogue of Mumford's work [MumFK94] to the non-reductive setting. Due to the fact that a non-reductive group does not have such a well behaved invariant theory (notably the possibility of non-finite generation of rings of invariants) there are significant differences with Mumford's theory for reductive groups.

Throughout this section, as before, let  $H$  be a linear algebraic group with unipotent radical  $H_u$ , and let  $H_r = H/H_u$ . Recall that when  $H = H_r = G$  is a reductive group acting on a projective variety  $X$  with an ample linearisation  $L \rightarrow X$ , the quotient of the stable locus (in the sense of Definition 2.2.2, 2) admits a canonical projective completion  $X//G$ , which is a good categorical quotient of the semistable locus. We have  $X//G = \text{Proj}(\mathbb{k}[X, L]^G)$ , where  $\mathbb{k}[X, L]^G$  is a finitely generated  $\mathbb{k}$ -algebra by Nagata's theorem [Na64], and topologically  $X//G$  can be described as the quotient of  $X^{\text{ss}}$  under the S-equivalence relation; see Section 2.2. When  $H$  is not reductive, this picture does not carry over into our non-reductive GIT, due to the fact that the ring of invariants  $\mathbb{k}[X, L]^H$  is not necessarily finitely generated and the image of the enveloping quotient map  $q : X^{\text{ss,fg}} \rightarrow X \wr H$  (within which the quotient  $X^s/H$  of the stable locus  $X^s$  is contained) is not necessarily a variety. To address the first issue, we introduced the notion of an inner enveloping quotient (Definition 3.1.12). Recall this is a choice of quasi-compact open subscheme  $\mathcal{U} \subseteq X \wr H$  which contains the image of the enveloping quotient map  $q : X^{\text{ss,fg}} \rightarrow X \wr H$  as a dense subset. Every inner enveloping quotient is quasi-projective, so we may talk about their projective completions.

A reasonable way to try and recover a picture similar to Mumford's for reductive groups is to therefore consider how to construct projective completions of inner enveloping quotients containing the enveloped quotient as a dense subset. We make

**Definition 4.0.1.** Let  $H$  be a linear algebraic group acting on a variety  $X$  with linearisation  $L \rightarrow X$ . We call a projective variety  $Z$  a *projective completion of the enveloped quotient* if there is an inner enveloping quotient  $\mathcal{U} \subseteq X \curvearrowright H$  and an open immersion  $\mathcal{U} \hookrightarrow Z$ .

The purpose of this section is to describe a method for constructing projective completions of the enveloped quotient, based on extending the work of [DorK07, §5] described in Section 2.3.2. In Section 4.1 we extend the notion of a reductive envelope in Definition 2.3.11 to the more general case where  $H$  is not necessarily unipotent, nor the linearisation  $L \rightarrow X$  ample over a projective variety (see Definition 4.1.4). The idea here is to consider equivariant projective completions  $\beta : G \times^{H_u} X \hookrightarrow \overline{G \times^{H_u} X}$  (where  $G$  is a reductive group and  $G \times^{H_u} X$  is formed with respect to an  $H_u$ -faithful homomorphism  $H \rightarrow G$ ), together with an extension of the linearisation  $L \rightarrow X$  to a  $G \times H_r$ -linearisation  $L' \rightarrow \overline{G \times^{H_u} X}$ , with conditions imposed in order to yield inclusions

$$\begin{aligned} X \cap (\overline{G \times^{H_u} X})^{s(L')} &\subseteq X^s \subseteq X^{\text{ss,fg}} \subseteq X \cap (\overline{G \times^{H_u} X})^{\text{ss}(L')} \\ \text{and } q(X^{\text{ss,fg}}) &\subseteq \mathcal{U} \subseteq \overline{G \times^{H_u} X} //_{L'} (G \times H_r), \end{aligned} \tag{4.1}$$

where  $\mathcal{U}$  is some inner enveloping quotient of  $X^{\text{ss,fg}}$ . The main result concerning reductive envelopes is Theorem 4.1.14. In Section 4.2 we consider certain kinds of reductive envelope, called *strong reductive envelopes*, which yield equalities  $X \cap \overline{G \times^{H_u} X}^{s(L')} = X^s$  and  $X^{\text{ss,fg}} = X \cap \overline{G \times^{H_u} X}^{\text{ss}(L')}$ . These are therefore interesting from the point of view of computing the stable and finitely generated semistable loci for  $H \curvearrowright L \rightarrow X$ . Here we also give an explicit way to construct strong reductive envelopes when some extra simplifying conditions on the linearisation  $H \curvearrowright L \rightarrow X$  and group  $G$  are satisfied.

## 4.1 Reductive Envelopes: General Theory and Main Theorem

Let  $H$  be a linear algebraic group acting on an irreducible variety  $X$  with linearisation  $L \rightarrow X$  and suppose we have an  $H_u$ -faithful homomorphism  $H \rightarrow G$  into a reductive group  $G$ . Consider the fibre space  $G \times^{H_u} X$  associated to this homomorphism, as in Section 3.3.2. As there, we abuse notation and write  $L$  for the natural  $G \times H_r$  linearisation  $G \times^{H_u} L \rightarrow G \times^{H_u} X$ , unless confusion is likely to occur, and also let  $\alpha : X \hookrightarrow G \times^{H_u} X$  be the natural closed immersion. We wish to extend the theory of reductive envelopes from [DorK07] to the more general setting where  $H$  is not necessarily a unipotent group. It is natural to preserve as much of the intrinsic non-reductive GIT picture as possible: we would like the reductive quotients of  $L' \rightarrow \overline{G \times^{H_u} X}$  by  $G$  and  $G \times H_r$  to reflect the ‘quotienting in stages’ aspect of the diagram in Theorem 3.4.2, 1. This requires identifying collections of invariant sections in  $\mathbb{k}[X, L]$  that are large enough to detect the subsets  $X^{\text{nss}}$ ,  $X^{\text{ss,fg}}$ ,  $X^{\text{ss}, H_u\text{-fg}}$  and  $X^s$ , and ensuring these extend to sections over  $\overline{G \times^{H_u} X}$ .

**Definition 4.1.1.** Let  $H$  be a linear algebraic group acting on an irreducible variety  $X$  and  $L \rightarrow X$  a linearisation. Fix a reductive group  $G$  and an  $H_u$ -faithful homomorphism, where  $H_u$  is the unipotent radical of  $H$  and  $H_r = H/H_u$ . We say an enveloping system  $V$  is *fully separating* if it is adapted to a finite subset  $\mathcal{S} \subseteq V$  such that the following properties are satisfied:

1.  $X^{\text{NSS}} = \bigcup_{f \in V^H} X_f$ ;
2. there is a subset  $\mathcal{S}^{\text{ss}, H_u\text{-fg}} \subseteq \mathcal{S}$  such that  $X^{\text{ss}, H_u\text{-fg}} = \bigcup_{f \in \mathcal{S}^{\text{ss}, H_u\text{-fg}}} X_f$  and  $V$  defines an enveloping system adapted to  $\mathcal{S}^{\text{ss}, H_u\text{-fg}}$  for the restricted linearisation  $H_u \curvearrowright L \rightarrow X$ ; and
3. for every  $x \in X^s$  there is  $f \in \mathcal{S}$  with corresponding  $G \times H_r$ -invariant  $F$  over  $G \times^{H_u} X$  such that  $(G \times^{H_u} X)_F$  is affine. (Equivalently, for every  $x \in X^s$  there if  $f \in \mathcal{S}$  such that  $X_f$  is affine and  $X_f \rightarrow \text{Spec}((S^{H_u})_{(f)})$  is a principal  $H_u$ -bundle; see Remark 3.3.9.)

It is not difficult to modify the proof of Proposition 3.1.19, 1 to prove the existence of fully separating enveloping systems, for any given linearisation  $L \rightarrow X$  with  $X$  irreducible, and that such an enveloping system is stable under taking products of sections. This is done in the next lemma.

**Lemma 4.1.2.** *Given any irreducible  $H$ -variety  $X$  with linearisation  $L \rightarrow X$  and any  $H_u$ -faithful homomorphism  $H \rightarrow G$  with  $G$  reductive, for some  $r > 0$  there exists a fully separating enveloping system  $V \subseteq H^0(X, L^{\otimes r})^{H_u}$ . Furthermore, for any fully separating enveloping system  $V \subseteq H^0(X, L^{\otimes m})$  and any  $n > 0$  the image of the natural multiplication map*

$$V^{\otimes n} \rightarrow H^0(X, L^{\otimes mn}),$$

*is again a fully separating enveloping system.*

*Proof.* By Proposition 3.1.19, 1 there is an enveloping system  $V' \subseteq H^0(X, L^{\otimes r})^H$ , for some  $r > 0$ , adapted to a finite subset  $\mathcal{S}$  with  $X^{\text{ss}, \text{fg}} = \bigcup_{f \in \mathcal{S}} X_f$ . We will augment this enveloping system by taking a suitably large multiple of  $r$  and replacing  $V'$  and  $\mathcal{S}$  by their images under the natural multiplication map of sections, as in Proposition 3.1.19, 3. We repeatedly use the fact that  $X_f = X_{f^n}$  for any section  $f$  over  $X$  and integer  $n > 0$ , as well as the equalities  $(S^H)_{(f)} = (S^H)_{(f^n)}$  and  $(S^{H_u})_{(f)} = (S^{H_u})_{(f^n)}$  for invariant sections  $f$ .

Because  $X$  is quasi-compact, by taking a large multiple of  $r$  and replacing  $V'$  and  $\mathcal{S}$  appropriately we may assume there are subsets  $\mathcal{S}^s$  and  $\mathcal{S}^{\text{ss}, H_u\text{-fg}}$  of  $\mathcal{S}$  such that  $X^s$  and  $X^{\text{ss}, H_u\text{-fg}}$  are covered by open subsets of the form  $X_f$  with  $f \in \mathcal{S}^s$  and  $f \in \mathcal{S}^{\text{ss}, H_u\text{-fg}}$ , respectively. We may also assume that  $r$  is chosen so that  $H^0(X, L^{\otimes r})^H$  contains sections  $f_1, \dots, f_n$  with  $X^{\text{NSS}} = \bigcup_{i=1}^n X_{f_i}$ , and furthermore we can choose  $\mathcal{S}^s$  so that each  $f \in \mathcal{S}^s$  extends to  $F$  over  $G \times^{H_u} X$  such that  $(G \times^{H_u} X)_F$  is affine. Following an argument similar to the construction of the subset  $A$  in the proof of Proposition 3.1.19, 1, by taking another suitably large multiple of  $r$  and replacing  $V'$  and the sets  $\mathcal{S}$  and  $\{f_1, \dots, f_n\}$  by their images under the appropriate multiplication map on sections we may assume there is a subset  $A^{\text{ss}, H_u\text{-fg}} \subseteq H^0(X, L^{\otimes r})^{H_u}$ , containing  $\mathcal{S}^{\text{ss}, H_u\text{-fg}}$ , such that  $(S^{H_u})_{(f)}$  is generated by  $\{\tilde{f}/f \mid \tilde{f} \in A^{\text{ss}, H_u\text{-fg}}\}$  for each  $f \in \mathcal{S}^{\text{ss}, H_u\text{-fg}}$ . By Lemma 2.1.21 there is a finite dimensional rational  $H$ -module  $V \subseteq H^0(X, L^{\otimes r})^{H_u}$  for the natural  $H$ -action that contains  $V' \cup A^{\text{ss}, H_u\text{-fg}} \cup \{f_1, \dots, f_n\}$ . Then  $V$  is an enveloping system adapted to  $\mathcal{S}$  such that properties 1–3 of Definition 4.1.1 are satisfied.

The statement about images of fully separating enveloping systems under multiplication maps follows immediately from Proposition 3.1.19, 3 and the equalities  $X_{f^n} = X_f$ ,  $(S^H)_{(f)} = (S^H)_{(f^n)}$  and  $(S^{H_u})_{(f)} = (S^{H_u})_{(f^n)}$  for any  $H$ -invariant  $f$  and  $n > 0$ .  $\square$

*Example 4.1.3.* If  $X$  is an irreducible projective  $H$ -variety and  $L \rightarrow X$  an ample linearisation, then each space of sections  $H^0(X, L^{\otimes r})$ , where  $r > 0$ , is a finite dimensional vector space. Then given an  $H_u$ -faithful homomorphism  $H \rightarrow G$ , an easy consequence of Lemma 4.1.2 is that the space  $V = H^0(X, L^{\otimes r})$  defines a fully separating enveloping system for sufficiently divisible  $r > 0$ .

We now turn to the definition of a reductive envelope. Given an  $H_u$ -faithful homomorphism  $H \rightarrow G$  with  $G$  reductive, the idea is to extend the linearisation  $G \times H_r$  on  $G \times^{H_u} L \rightarrow G \times^{H_u} X$  over a suitable equivariant projective completion  $\overline{G \times^{H_u} X}$  of  $G \times^{H_u} X$ . A key condition for obtaining the diagram (4.1) is to ensure enough invariants over  $X$  (or equivalently over  $G \times^{H_u} X$ ) extend to invariants over  $\overline{G \times^{H_u} X}$ .

**Definition 4.1.4.** Let  $H \curvearrowright L \rightarrow X$  be a linearisation of a non-reductive group  $H$  and suppose  $H \rightarrow G$  is an  $H_u$ -faithful homomorphism into a reductive group  $G$ , where  $H_u$  is the unipotent radical of  $H$  and  $H_r = H/H_u$ . Let  $\overline{G \times^{H_u} X}$  be a projective  $G \times H_r$ -variety with  $G \times H_r$ -equivariant dominant open immersion  $\beta : G \times^{H_u} X \hookrightarrow \overline{G \times^{H_u} X}$  and  $L' \rightarrow \overline{G \times^{H_u} X}$  a  $G \times H_r$ -linearisation that restricts to some positive tensor power of  $L \rightarrow G \times^{H_u} X$  under  $\beta$ . We call  $(\overline{G \times^{H_u} X}, \beta, L')$  a *reductive envelope* for the linearisation  $H \curvearrowright L \rightarrow X$  if there is fully separating enveloping system  $V$  for  $H \curvearrowright L \rightarrow X$  such that

1. each section in  $V^{H_u}$  extends under  $\beta \circ \alpha$  to a  $G$ -invariant section of some tensor power of  $L'$  over  $\overline{G \times^{H_u} X}$ ;
2. each section in  $V^H$  extends under  $\beta \circ \alpha$  to a  $G \times H_r$ -invariant section of some tensor power of  $L'$  over  $\overline{G \times^{H_u} X}$ ; and
3. for  $f \in V^{H_u}$  with extension to  $F$  over  $\overline{G \times^{H_u} X}$ , the open subset  $(\overline{G \times^{H_u} X})_F$  is affine.

If the line bundle  $L'$  is ample, then we call  $(\overline{G \times^{H_u} X}, \beta, L')$  an *ample reductive envelope*.

*Remark 4.1.5.* In the case where  $H$  is unipotent, our notion of reductive envelope corresponds to that of a *fine* reductive envelope in [DorK07]; cf. Definition 2.3.11.

*Remark 4.1.6.* The case where  $L'$  is ample is of most interest to us, because it ensures the GIT quotient  $\overline{G \times^{H_u} X} //_{L'} (G \times H_r)$  is a projective variety. It also means that condition 3 of Definition 4.1.4 is automatically satisfied, so verifying that the data  $(\overline{G \times^{H_u} X}, \beta, L')$  defines a reductive envelope reduces to checking that invariant sections from a fully separating enveloping system extend to sections over  $\overline{G \times^{H_u} X}$ .

The next proposition asserts the existence of an ample reductive envelope for any given ample  $H$ -linearisation  $L \rightarrow X$ , under the standing assumption that stabilisers of general points in  $X$  are finite (see Remark 2.1.3).

**Proposition 4.1.7.** *Let  $H$  be a linear algebraic group acting on an irreducible quasi-projective variety  $X$  such that stabilisers of general points in  $X$  are finite and  $L \rightarrow X$  an ample linearisation for the action. Then  $H \curvearrowright L \rightarrow X$  possesses an ample reductive envelope for some reductive group  $G$  containing  $H$  as a closed subgroup.*

*Proof.* Begin by using Lemma 4.1.2 to find  $r > 0$  with a fully separating enveloping system  $V \subseteq H^0(X, L^{\otimes r})$ . (Note that by Remark 3.3.9 it makes sense to talk about fully separating enveloping systems without reference to any reductive group  $G$  and  $H_u$ -faithful homomorphism  $H \rightarrow G$ .) The line bundle  $L$  is ample, so by taking a sufficiently large multiple of  $r$ , replacing  $V$  by its image under the natural multiplication map of sections and enlarging the resulting  $V$  using Lemma 2.1.21 if necessary, we may assume that  $V$  is a rational  $H$ -module and defines an  $H$ -equivariant locally closed immersion  $X \hookrightarrow \mathbb{P}(V^*)$ . The action of  $H$  on  $V$  defines a homomorphism  $\rho : H \rightarrow G := \mathrm{GL}(V)$  and there is a canonical  $G$ -linearisation on  $\mathcal{O}_{\mathbb{P}(V^*)}(1) \rightarrow \mathbb{P}(V^*)$  extending the  $H$ -linearisation  $L^{\otimes r} \rightarrow X$  via  $\rho$ . Note that any element of the kernel of  $\rho$  must fix every point in  $X$ , so  $\ker \rho$  is a finite group and cannot contain any unipotent elements. Thus  $\rho : H \rightarrow G$  is  $H_u$ -faithful. Consider the fibre bundle  $G \times^{H_u} \mathbb{P}(V^*)$ , together with its  $G \times H_r$ -linearisation  $G \times^{H_u} \mathcal{O}_{\mathbb{P}(V^*)}(1)$ . Then there is an isomorphism of  $G \times H_r$ -varieties

$$\begin{aligned} G \times^{H_u} \mathbb{P}(V^*) &\cong (G/H_u) \times \mathbb{P}(V^*), \\ [g, y] &\mapsto (gH_u, gy), \end{aligned}$$

and the corresponding  $G \times H_r$ -linearisation over  $(G/H_u) \times \mathbb{P}(V^*)$  has underlying line bundle  $\mathcal{O}_{G/H_u} \boxtimes \mathcal{O}_{\mathbb{P}(V^*)}(1)$ .

Now because  $H_u$  is unipotent the homogeneous space  $G/H_u$  is quasi-affine [Gros97, Corollary 2.8 and Theorem 2.1], so there is a finite dimensional vector subspace  $W \subseteq \mathcal{O}(G/H_u)$  defining a locally closed immersion of  $G/H_u$  into the affine space  $\mathbb{A} = \mathrm{Spec}(\mathrm{Sym}^\bullet W^*)$ . Right multiplication by  $H$  on  $G$  via  $\rho$  descends to define an  $H_r$ -action on  $G/H_u$ , and  $G$  acts by left multiplication on  $G/H_u$ . By [Bo91, Proposition 1.9] we may assume that  $W$  is invariant under the corresponding actions of  $H_r$  and  $G$  on  $\mathcal{O}(G/H_u)$ . This induces an action of  $G \times H_r$  on  $\mathbb{A}$  together with a linearisation on the trivial line bundle  $\mathcal{O}_{\mathbb{A}} \rightarrow \mathbb{A}$  which restricts to the canonical  $G \times H_r$ -linearisation  $\mathcal{O}_{G/H_u} \rightarrow G/H_u$  under  $G/H_u \hookrightarrow \mathbb{A}$ . Let  $\mathbb{k}$  be a copy of the ground field equipped with the trivial  $G \times H_r$ -representation, set  $\mathbb{P} := \mathbb{P}(W^* \oplus \mathbb{k})$ , and let  $\beta_1 : G/H_u \hookrightarrow \overline{G/H_u}$  be the projective completion of  $G/H_u$  resulting from the embedding  $G/H_u \hookrightarrow \mathbb{A}$  and the standard open immersion  $\mathbb{A} \hookrightarrow \mathbb{P}$ . Then the restriction  $\mathcal{O}_{\overline{G/H_u}}(1) = \mathcal{O}_{\mathbb{P}}(1)|_{\overline{G/H_u}}$  of the canonical  $G \times H_r$ -linearisation  $\mathcal{O}_{\mathbb{P}}(1) \rightarrow \mathbb{P}$  to  $\overline{G/H_u}$  pulls back to the  $G \times H_r$ -linearisation  $\mathcal{O}_{G/H_u} \rightarrow G/H_u$  under  $\beta_1$ . Consider the linearisation

$$G \times H_r \curvearrowright \mathcal{O}_{\overline{G/H_u}}(1) \boxtimes \mathcal{O}_{\mathbb{P}(V^*)}(1) \rightarrow \overline{G/H_u} \times \mathbb{P}(V^*)$$

obtained by taking the product of  $\mathcal{O}_{\overline{G/H_u}}(1)$  with the  $G \times H_r$ -linearisation on  $\mathcal{O}_{\mathbb{P}(V^*)}(1)$  defined by  $G$  and the trivial  $H_r$ -action on  $\mathbb{P}(V^*)$ . Let  $\beta : G \times^{H_u} X \hookrightarrow \overline{G \times^{H_u} X} \subseteq \overline{G/H_u} \times \mathbb{P}(V^*)$  be the projective completion of  $G \times^{H_u} X$  obtained by the composition of the embedding  $G \times^{H_u} X \hookrightarrow G \times^{H_u} \mathbb{P}(V^*) \cong (G/H_u) \times \mathbb{P}(V^*)$  with the open immersion  $\beta_1 \times \mathrm{id}_{\mathbb{P}(V^*)}$ , and let  $L'$  be the restriction of  $\mathcal{O}_{\overline{G/H_u}}(1) \boxtimes \mathcal{O}_{\mathbb{P}(V^*)}(1)$  to  $\overline{G \times^{H_u} X}$ . Then  $\beta^* L' = L^{\otimes r} \rightarrow G \times^{H_u} X$  as  $G \times H_r$ -linearisations.

To conclude we observe that the required extension properties 1–3 of Definition 4.1.4 hold for  $(\overline{G \times^{H_u} X}, \beta, L')$ , as follows. Any invariant section  $f \in V \subseteq H^0(X, L^{\otimes r})$  extends to an invariant (which we also call  $f$ ) of  $\mathcal{O}_{\mathbb{P}(V^*)}(1) \rightarrow \mathbb{P}(V^*)$  by construction, and if  $f$  is  $H_u$ -invariant

(respectively,  $H$ -invariant) over  $\mathbb{P}(V^*)$  then it extends to the  $G$ -invariant (respectively,  $G \times H_r$ -invariant) section

$$1 \otimes f \in \mathcal{O}(G/H_u) \otimes H^0(\mathbb{P}(V^*), \mathcal{O}(1)) = H^0((G/H_u) \times \mathbb{P}(V^*), \mathcal{O}_{G/H_u} \boxtimes \mathcal{O}_{\mathbb{P}(V^*)}(1)).$$

(Here we have used the Künneth formula [St15, Tag 02KE].) But now if  $\epsilon \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$  is the homogeneous coordinate of  $\mathbb{P} = \mathbb{P}(W^* \oplus \mathbb{k})$  corresponding to the trivial  $G \times H_r$ -summand  $\mathbb{k}$  then  $1 \otimes f$  extends to  $\epsilon \otimes f$  under  $\beta$ , which is  $G$  or  $G \times H_r$ -invariant if  $1 \otimes f$  is  $G$  or  $G \times H_r$ -invariant, respectively. Thus  $f \in V^{H_u}$  (respectively,  $f \in V^H$ ) extends to the  $G$ -invariant (respectfully,  $G \times H_r$ -invariant)  $(\epsilon \otimes f)|_{\overline{G \times^{H_u} X}}$  of  $L' \rightarrow \overline{G \times^{H_u} X}$ . This shows properties 1 and 2. Finally, property 3 holds because  $L'$  is (very) ample.  $\square$

*Remark 4.1.8.* In practice, the group  $G = \mathrm{GL}(V)$  containing  $H$  constructed in the proof of Proposition 4.1.7 is too large to be computationally useful. In Section 4.2.1 we will look at a class of reductive envelopes where the reductive group  $G$  contains  $H_u$  as a *Grosshans subgroup*, in which case the geometry of the homogeneous space  $G/H_u$  lends itself to more explicit calculations.

Associated to any reductive envelope  $(\overline{G \times^{H_u} X}, \beta, L')$  are open subsets of  $X$  obtained by pulling back the stable and semistable loci for the  $G \times H_r$ -linearisation  $L' \rightarrow \overline{G \times^{H_u} X}$ . As we will see shortly, one of the key properties of these sets is that they ‘bookend’ the intrinsically defined notions of stability and semistability considered in Section 3. In analogy to [DorK07, Definition 5.2.11] (see the statement of Theorem 2.3.12), we make the following definition.

**Definition 4.1.9.** Let  $H \curvearrowright L \rightarrow X$  be a linearisation of a linear algebraic group  $H$  with unipotent radical  $H_u$ , let  $H_r = H/H_u$  and let  $H \rightarrow G$  be an  $H_u$ -faithful homomorphism into a reductive group  $G$ . Suppose  $(\overline{G \times^{H_u} X}, \beta, L')$  is a reductive envelope. The *completely semistable locus* is the set

$$X^{\overline{\mathrm{ss}}} := (\beta \circ \alpha)^{-1}((\overline{G \times^{H_u} X})^{\mathrm{ss}(L')})$$

and the *completely stable locus* is the set

$$X^{\overline{\mathrm{s}}} := (\beta \circ \alpha)^{-1}((\overline{G \times^{H_u} X})^{\mathrm{s}(L')}),$$

where  $\overline{G \times^{H_u} X}^{\mathrm{ss}(L')}$  and  $\overline{G \times^{H_u} X}^{\mathrm{s}(L')}$  are the semistable and stable loci, respectively, for the reductive linearisation  $G \times H_r \curvearrowright L' \rightarrow \overline{G \times^{H_u} X}$ .

**Proposition 4.1.10.** *Let  $H$  be a linear algebraic group with an  $H_u$ -faithful morphism  $H \rightarrow G$ , with  $G$  reductive, and let  $X$  be an irreducible quasi-projective  $H$ -variety with an ample linearisation  $L \rightarrow X$ . If  $(\overline{G \times^{H_u} X}, \beta, L')$  is a reductive envelope for the linearisation, then  $X^{\overline{\mathrm{ss}}} = X^{\mathrm{ss}}$  and  $X^{\overline{\mathrm{s}}} \subseteq X^{\mathrm{s}}$ .*

*Proof.* Let  $V$  be a fully separating enveloping system adapted to a finite subset  $\mathcal{S} \subseteq V$  satisfying properties 1–3 of Definition 4.1.1, and suppose  $V$  satisfies the extension properties 1–3 of Definition 4.1.4 for the reductive envelope  $(\overline{G \times^{H_u} X}, \beta, L')$ . Then there is a basis  $f_1, \dots, f_n$  of  $V^H$  such that  $X^{\mathrm{ss}} = \bigcup_{i=1}^n X_{f_i}$  and each  $f_i$  extends to a  $G \times H_r$ -invariant  $F_i$  of some positive tensor power of  $L' \rightarrow \overline{G \times^{H_u} X}$  such that  $(\overline{G \times^{H_u} X})_{F_i}$  is affine, so  $X^{\mathrm{ss}} \subseteq X^{\overline{\mathrm{ss}}}$ . On the other hand,

any  $G \times H_r$ -invariant of a tensor power of  $L' \rightarrow \overline{G \times^{H_u} X}$  restricts to a  $G \times H_r$ -invariant over  $G \times^{H_u} X$  under  $\beta$ , which in turn corresponds to an  $H$ -invariant over  $X$  via  $\alpha$ . Hence  $X^{\overline{s}} \subseteq X^{\text{ns}}$  also.

Now suppose  $x \in X^{\overline{s}}$ . Then there is a  $G \times H_r$ -invariant  $F$  of some positive tensor power of  $L' \rightarrow \overline{G \times^{H_u} X}$  such that  $(G \times^{H_u} X)_F$  is an affine open subset containing  $(\beta \circ \alpha)(x)$ , and the  $G \times H_r$ -action on  $(G \times^{H_u} X)_F$  is closed with all stabilisers finite. By abuse of notation, write  $F$  for the section  $\beta^* F$  over  $G \times^{H_u} X$ . Invoking Proposition 3.3.8, to prove  $x \in X^s$  it suffices to show that  $(G \times^{H_u} X)_F \subseteq (G \times^{H_u} X)^{s(L)}$ , where stability is with respect to the canonical  $G \times H_r$ -linearisation  $L \rightarrow G \times^{H_u} X$ . Note that  $(G \times^{H_u} X)_F$  is a  $G \times H_r$ -invariant open subset of  $(\overline{G \times^{H_u} X})_F$ , and  $(\overline{G \times^{H_u} X})_F$  has a geometric  $G \times H_r$ -quotient  $\pi : (\overline{G \times^{H_u} X})_F \rightarrow (\overline{G \times^{H_u} X})_F / (G \times H_r)$  with affine base (Theorem 2.2.1). The image  $\pi((G \times^{H_u} X)_F)$  of  $(G \times^{H_u} X)_F$  is an open subset of  $(\overline{G \times^{H_u} X})_F / (G \times H_r)$ , thus we may cover  $\pi((G \times^{H_u} X)_F)$  with basic affine open subsets of  $\pi((\overline{G \times^{H_u} X})_F)$ . Each of these takes the form  $\pi((G \times^{H_u} X)_{F\tilde{F}}) = (G \times^{H_u} X)_{F\tilde{F}} / (G \times H_r)$ , for a  $G \times H_r$ -invariant section  $\tilde{F}$  over  $\overline{G \times^{H_u} X}$ , by virtue of the canonical isomorphism

$$\mathcal{O}(\pi((G \times^{H_u} X)_F)) \xrightarrow{\pi^\#} (\mathcal{O}((G \times^{H_u} X)_F))^{G \times H_r} = (\mathbb{k}[\overline{G \times^{H_u} X}, L']^{G \times H_r})_{(F)}.$$

(In the final equality we have used the fact that  $\overline{G \times^{H_u} X}$  is irreducible, which is necessarily the case because  $G \times^{H_u} X$  is irreducible and  $\beta$  is dominant.) Thus, for suitable  $G \times H_r$ -invariant sections  $F_i$  over  $\overline{G \times^{H_u} X}$ , we have

$$(G \times^{H_u} X)_F = \bigcup_i \pi^{-1}(\pi((\overline{G \times^{H_u} X})_{F_i})) = \bigcup_i (G \times^{H_u} X)_{F_i}$$

and by affineness of  $\pi$  each  $\pi^{-1}(\pi((\overline{G \times^{H_u} X})_{F_i})) = (G \times^{H_u} X)_{F_i}$  is an affine open subset of  $G \times^{H_u} X$ . By restriction the  $G \times H_r$ -action on each  $(G \times^{H_u} X)_{F_i}$  is closed with finite stabilisers, hence  $(G \times^{H_u} X)_{F_i} \subseteq (G \times^{H_u} X)^{s(L)}$  for each  $i$ . Therefore  $(G \times^{H_u} X)_F \subseteq (G \times^{H_u} X)^{s(L)}$ , as desired.  $\square$

**Corollary 4.1.11.** *For a linearisation  $H \curvearrowright L \rightarrow X$  with  $X$  projective and  $L$  ample, the restriction of the enveloping quotient map to the completely stable locus  $X^{\overline{s}}$  for a reductive envelope  $(\overline{G \times^{H_u} X}, \beta, L')$  defines a geometric quotient  $q_H : X^{\overline{s}} \rightarrow X^{\overline{s}}/H$ , and the composition*

$$X^{\overline{s}} \xrightarrow{\beta \circ \alpha} \overline{G \times^{H_u} X}^{s(L')} \longrightarrow \overline{G \times^{H_u} X}^{s(L')} / (G \times H_r)$$

*induces a natural open immersion  $X^{\overline{s}}/H \hookrightarrow \overline{G \times^{H_u} X}^{s(L')} / (G \times H_r)$ .*

*Proof.* Since  $X^{\overline{s}}$  is an  $H$ -invariant open subset of  $X^s$  the map  $q_H : X^s \rightarrow X^s/H$  restricts to define a geometric quotient  $q_H : X^{\overline{s}} \rightarrow q_H(X^{\overline{s}}) \subseteq X^s/H$ . By definition  $G \times^{H_u}(X^{\overline{s}})$  is an open subset of  $\overline{G \times^{H_u} X}^{s(L')}$  via  $\beta$  and hence

$$X^{\overline{s}}/H = G \times^{H_u}(X^{\overline{s}})/(G \times H_r) \subseteq \overline{G \times^{H_u} X}^{s(L')} / (G \times H_r)$$

with the inclusion open.  $\square$



Suppose we have a reductive envelope  $(\overline{G \times^{H_u} X}, \beta, L')$  for the linearisation  $H \curvearrowright L \rightarrow X$ . Then we may consider the reductive GIT quotients

$$\begin{aligned}\pi_G : \overline{G \times^{H_u} X}^{\text{ss}(G)} &\rightarrow \overline{G \times^{H_u} X} // G, \\ \pi_{G \times H_r} : \overline{G \times^{H_u} X}^{\text{ss}(G \times H_r)} &\rightarrow \overline{G \times^{H_u} X} // (G \times H_r)\end{aligned}$$

for the  $G$  and  $G \times H_r$ -linearisations on  $L'$ , respectively. According to Proposition A.0.1 of the Appendix, there is an induced ample  $H_r$ -linearisation  $M' \rightarrow \overline{G \times^{H_u} X} // G$  such that  $\pi_G$  maps  $\overline{G \times^{H_u} X}^{\text{ss}(G \times H_r)}$  into  $(\overline{G \times^{H_u} X} // G)^{\text{ss}(M')}$ , and if

$$\bar{\pi}_{H_r} : (\overline{G \times^{H_u} X} // G)^{\text{ss}(M')} \rightarrow (\overline{G \times^{H_u} X} // G) //_{M'} H_r$$

is the reductive GIT quotient for the linearisation  $H_r \curvearrowright M' \rightarrow \overline{G \times^{H_u} X} // G$ , then there is a canonical open immersion

$$\psi : \overline{G \times^{H_u} X} // (G \times H_r) \hookrightarrow (\overline{G \times^{H_u} X} // G) //_{M'} H_r$$

such that the diagram

$$\begin{array}{ccc} & \overline{G \times^{H_u} X}^{\text{ss}(G \times H_r)} & \\ & \searrow \pi_{G \times H_r} & \downarrow \pi_G \\ & & (\overline{G \times^{H_u} X} // G)^{\text{ss}(M')} \\ & \swarrow & \downarrow \bar{\pi}_{H_r} \\ \overline{G \times^{H_u} X} // (G \times H_r) & \xrightarrow{\psi} & (\overline{G \times^{H_u} X} // G) //_{M'} H_r \end{array} \quad (4.2)$$

commutes.

**Proposition 4.1.12.** *Let  $H$  be a linear algebraic group acting on an irreducible variety  $X$  with linearisation  $L \rightarrow X$ , let  $H \rightarrow G$  be an  $H_u$ -faithful homomorphism with  $G$  reductive and suppose  $(\overline{G \times^{H_u} X}, \beta, L')$  is a reductive envelope for  $H \curvearrowright L \rightarrow X$ . Retain the notation above.*

1. *There is an inner enveloping quotient  $\mathcal{V} \subseteq X \curvearrowright H$ , together with an open immersion*

$$\theta_H : \mathcal{V} \hookrightarrow \overline{G \times^{H_u} X} // (G \times H_r)$$

*such that  $\theta_H^* \psi^* N' = \mathcal{O}_{\mathcal{V}}(n)$  for some  $n > 0$ , where  $N' \rightarrow (\overline{G \times^{H_u} X} // G) //_{M'} H_r$  is a very ample line bundle pulling back to a positive tensor power of the line bundle  $M' \rightarrow (\overline{G \times^{H_u} X} // G)^{\text{ss}(M')}$ .*

2. *There is an inner enveloping quotient  $\mathcal{U} \subseteq X \curvearrowright H_u$  of  $X^{\text{ss}, H_u - \text{fg}}$  that is stable under the canonical  $H_r$ -action on  $X \curvearrowright H_u$  of Proposition 3.2.1 and an  $H_r$ -equivariant open immersion*

$$\theta_{H_u} : \mathcal{U} \hookrightarrow (\overline{G \times^{H_u} X} // G)^{\text{ss}(M')}$$

*such that  $\theta_{H_u}^* M'$  defines the same linearised polarisation over  $\mathcal{U}$  as the natural one on  $\mathcal{O}_{\mathcal{U}}(n)$ , for  $n$  as in 1. Furthermore,  $\mathcal{U}$  is such that the natural rational map  $q_{H_r} : \text{Proj}(S^{H_u}) \dashrightarrow \text{Proj}(S^H)$  of (3.6) restricts to define a good categorical quotient  $q_{H_r} : \mathcal{U} \rightarrow \mathcal{U} // H_r$  for the  $H_r$ -action on  $\mathcal{U}$ , with  $\mathcal{U} // H_r$  contained in  $\mathcal{V}$  as an open subscheme.*

3. The following diagram commutes (where all unmarked inclusions are natural open immersions):

$$\begin{array}{ccccc}
& X^{\text{ss,fg}} & \hookrightarrow & X^{\text{ss}} & \\
& \swarrow & & \nearrow & \\
X^{\text{ss},H_u\text{-fg}} & \hookrightarrow & X^{\text{ss}} & & \\
\downarrow q_{H_u} & \downarrow q_H & \downarrow \pi_G \circ \beta \circ \alpha & & \downarrow \pi_{G \times H_r} \circ \beta \circ \alpha \\
\mathcal{U} & \xrightarrow{\theta_{H_u}} & (\overline{G \times^{H_u} X} // G)^{\text{ss}(M')} & & \\
\downarrow q_{H_r} & \downarrow \theta_H & \downarrow \bar{\pi}_{H_r} & & \downarrow \psi \\
\mathcal{U} // H_r & \hookrightarrow & (\overline{G \times^{H_u} X} // G) //_{M'} H_r & \hookrightarrow & \overline{G \times^{H_u} X} // (G \times H_r)
\end{array}$$

*Proof.* We begin by fixing some notation. Suppose  $L' \rightarrow \overline{G \times^{H_u} X}$  pulls back to  $L^{\otimes r} \rightarrow X$  under  $\beta \circ \alpha$ , with  $r > 0$ . Let  $V$  be a fully separating enveloping system associated to  $(\overline{G \times^{H_u} X}, \beta, L')$  with  $V \subseteq H^0(X, L^{\otimes r_1})$  for some positive integral multiple  $r_1$  of  $r$ . Let  $\mathcal{S} \subseteq V$  be a finite subset to which  $V$  is adapted and such that properties 1–3 of Definition 4.1.1 are satisfied. By Theorem 2.2.4, 1 there is an ample line bundle

$$N' \rightarrow (\overline{G \times^{H_u} X} // G) //_{M'} H_r$$

that pulls back to  $(M')^{\otimes m} \rightarrow (\overline{G \times^{H_u} X} // G)^{\text{ss}(M')}$  under  $\bar{\pi}_{H_r}$ , for some  $m > 0$ . Similarly,  $M'$  pulls back to  $(L')^{\otimes l} \rightarrow \overline{G \times^{H_u} X}^{\text{ss}(G)}$  under  $\pi_G$ , for some  $l > 0$ . By replacing  $N'$  by a sufficiently positive tensor power of itself, we may assume the following:  $N' \rightarrow (\overline{G \times^{H_u} X} // G) //_{M'} H_r$  and  $(M')^{\otimes m} \rightarrow \overline{G \times^{H_u} X} // G$  are very ample; and there is  $r_2 > 0$  such that  $r_1 r_2 = n := l m r$ . Using this second assumption and Lemma 4.1.2, we may use the multiplication map  $V^{\otimes r_2} \rightarrow H^0(X, L^{\otimes n})$  to further assume that  $\mathcal{S} \subseteq V \subseteq H^0(X, L^{\otimes n})$ .

(Proof of 1.) We now construct the inner enveloping quotient  $\mathcal{V}$  and open immersion  $\theta_H$ . Let

$$\mathcal{V} = \bigcup_{f \in \mathcal{S}} \text{Spec}((S^H)_{(f)}) \subseteq X \wr H.$$

Recall from Definition 3.1.17 of an enveloping system that  $\mathcal{S}$  satisfies  $X^{\text{ss,fg}} = \bigcup_{f \in \mathcal{S}} X_f$  and  $(S^H)_{(f)}$  has generating set  $\{\tilde{f}/f \mid \tilde{f} \in V^H\}$  for each  $f \in \mathcal{S}$ . Given  $f \in \mathcal{S}$ , by Definition 4.1.4, 3 of a reductive envelope, there is an extension  $F \in H^0(\overline{G \times^{H_u} X}, (L')^{\otimes l m})^{G \times H_r}$  of  $f$  under  $\beta \circ \alpha$  such that  $(\overline{G \times^{H_u} X})_F$  is affine. Pulling back along  $\pi_{G \times H_r}$  identifies the ring of regular functions on the affine open subset  $\pi_{G \times H_r}((\overline{G \times^{H_u} X})_F) \subseteq \overline{G \times^{H_u} X} // (G \times H_r)$  with  $\mathcal{O}((\overline{G \times^{H_u} X})_F)^{G \times H_r}$ , and because  $X$  is irreducible  $q_H^\# : (S^H)_{(f)} \hookrightarrow \mathcal{O}(X_f)^H$  is an isomorphism by Lemma 3.1.9, 1. Therefore there is a unique ring homomorphism  $\Theta_f : \mathcal{O}(\pi_{G \times H_r}((\overline{G \times^{H_u} X})_F)) \rightarrow (S^H)_{(f)}$

making the diagram

$$\begin{array}{ccc}
\mathcal{O}(\pi_{G \times H_r}((\overline{G \times^{H_u} X})_F)) & \xrightarrow[\cong]{(\pi_{G \times H_r})^\#} & \mathcal{O}((\overline{G \times^{H_u} X})_F)^{G \times H_r} \\
\downarrow \Theta_f & & \downarrow (\beta \circ \alpha)^\# \\
(S^H)_{(f)} & \xrightarrow[\cong]{q_H^\#} & \mathcal{O}(X_f)^H
\end{array}$$

commute. In fact,  $\Theta_f$  is an isomorphism. Indeed, by Definition 3.1.17, 2 of an enveloping system  $\mathcal{O}(X_f)^H = (S^H)_{(f)}$  is generated by the regular functions  $q_H^\#(\tilde{f}/f)$ , where  $\tilde{f} \in V^H$ . Each such  $\tilde{f}$  extends to some  $\tilde{F} \in H^0(\overline{G \times^{H_u} X}, (L')^{\otimes lm})^{G \times H_r}$  under  $\beta \circ \alpha$  by Definition 4.1.4, 2 of a reductive envelope, and the regular function in  $\mathcal{O}((\overline{G \times^{H_u} X})_F)^{G \times H_r}$  defined by  $\tilde{F}/F$  pulls back to  $q_H^\#(\tilde{f}/f)$  under  $\beta \circ \alpha$ . It follows that  $\Theta_f$  is surjective. On the other hand, because  $\beta : (G \times^{H_u} X)_F \hookrightarrow (\overline{G \times^{H_u} X})_F$  is a dominant morphism and  $\alpha^\#$  identifies  $\mathcal{O}((G \times^{H_u} X)_F)^{G \times H_r}$  with  $\mathcal{O}(X_f)^H$  the map  $(\beta \circ \alpha)^\#$  is injective, hence  $\Theta_f$  is injective also.

It follows that  $\Theta_f$  defines an isomorphism of affine varieties

$$(\theta_H)_f : \text{Spec}((S^H)_{(f)}) \xrightarrow{\cong} \pi_{G \times H_r}((\overline{G \times^{H_u} X})_F)$$

with  $(\theta_H)_f \circ q_H|_{X_f} = \pi_{G \times H_r} \circ \beta \circ \alpha|_{X_f}$ . Because  $\Theta_f$  is defined in terms of compositions of (inverses of) sheaf homomorphisms and taking invariants is natural with respect to equivariant inclusions, it can easily be shown that the maps  $(\theta_H)_f$  glue over the  $\text{Spec}((S^H)_{(f)})$  with  $f \in \mathcal{S}$  to define an open immersion

$$\theta_H : \mathcal{V} \hookrightarrow \overline{G \times^{H_u} X} // (G \times H_r)$$

such that  $\theta_H \circ q_H = \pi_{G \times H_r} \circ \beta \circ \alpha$  on  $X^{\text{ss}, \text{fg}}$ . We can see  $\theta_H^* \psi^* N' = \mathcal{O}_{\mathcal{V}}(n)$  as follows. Because of the extension property 2 of Definition 4.1.4 of a reductive envelope, the space  $V^H$  extends isomorphically under  $\beta \circ \alpha$  to a subspace  $W$  of  $H^0(\overline{G \times^{H_u} X}, (L')^{\otimes lm})^{G \times H_r}$ . Each section in  $W$  descends through the GIT quotient map  $\pi_{G \times H_r}$  to a section in  $H^0(\overline{G \times^{H_u} X} // (G \times H_r), \psi^* N')$ , thus defining a rational map

$$\gamma_H : \overline{G \times^{H_u} X} // (G \times H_r) \dashrightarrow \mathbb{P}(W^*)$$

that defines a morphism on the image of  $\mathcal{V}$  under  $\theta_H$ . There is a natural isomorphism  $\mathbb{P}((V^H)^*) \cong \mathbb{P}(W^*)$  induced by  $(\beta \circ \alpha)^*$  and, by inspection, one sees that the composition  $\gamma_H \circ \theta_H : \mathcal{V} \rightarrow \mathbb{P}(W^*)$  corresponds to the natural locally closed immersion  $\mathcal{V} \hookrightarrow \mathbb{P}((V^H)^*)$  defined by  $V^H$  in Proposition 3.1.19. Hence  $\theta_H^* \psi^* N' = \mathcal{O}_{\mathcal{V}}(n)$ .

(Proof of 2.) The map  $\theta_{H_u}$  is constructed in a similar way to  $\theta_H$ . By Definition 4.1.1, 2 there is  $\mathcal{S}^{\text{ss}, H_u - \text{fg}} \subseteq \mathcal{S}$  such that  $V$  defines an enveloping system adapted to  $\mathcal{S}^{\text{ss}, H_u - \text{fg}}$  for the restricted linearisation  $H_u \curvearrowright L \rightarrow X$ . Letting

$$\mathcal{U} = \bigcup_{f \in \mathcal{S}^{\text{ss}, H_u - \text{fg}}} \text{Spec}((S^{H_u})_{(f)}) \subseteq X \wr H_u,$$

it follows from property 1 of Definition 4.1.4 that, for  $f \in \mathcal{S}^{\text{ss}, H_u\text{-fg}}$  with extension  $F$  over  $\overline{G \times^{H_u} X}$ , there are natural isomorphisms  $\text{Spec}((S^{H_u})_{(f)}) \cong \mathcal{O}(\pi_G(\overline{G \times^{H_u} X}_F))$  which patch to define an open immersion

$$\theta_{H_u} : \mathcal{U} \hookrightarrow (\overline{G \times^{H_u} X} // G)^{\text{ss}(M')}$$

such that  $\theta_{H_u} \circ q_{H_u} = \pi_G \circ \beta \circ \alpha$  on  $X^{\text{ss}, H_u\text{-fg}}$ , and  $\theta_{H_u}^*(M')^{\otimes m} = \mathcal{O}_{\mathcal{U}}(n)$  as line bundles. The arguments are analogous to those for the construction of  $\theta_H : \mathcal{V} \hookrightarrow \overline{G \times^{H_u} X} // (G \times H_r)$ . Notice that each section in  $\mathcal{S}^{\text{ss}, H_u\text{-fg}}$  is fixed by the  $H$ -action on  $H^0(X, L^{\otimes n})^H$ , so by Proposition 3.2.1  $\mathcal{U}$  is stable under the  $H_r$ -action on  $X \wr H_u$ . Furthermore, the equality  $\theta_{H_u} \circ q_{H_u} = \pi_G \circ \beta \circ \alpha$  implies that  $\theta_{H_u}$  is  $H_r$ -equivariant on the image  $q_{H_u}(X^{\text{ss}, H_u\text{-fg}})$ . The interior  $q_{H_u}(X^{\text{ss}, H_u\text{-fg}})^\circ$  inside  $\mathcal{U}$  is therefore a dense open subset of  $\mathcal{U}$  on which  $\theta_{H_u}$  is equivariant, so it follows from the separatedness of  $(\overline{G \times^{H_u} X} // G)^{\text{ss}(M')}$  that  $\theta_{H_u}$  is equivariant on the whole of  $\mathcal{U}$ . Because  $\theta_{H_u} \circ q_{H_u} = \pi_G \circ \beta \circ \alpha$ , there is a naturally induced map  $L^{\otimes n}|_{X^{\text{ss}, H_u\text{-fg}}} \rightarrow \theta_{H_u}^*(M')^{\otimes m}$  which is equivariant with respect to  $H \rightarrow H_r$ , and since  $\theta_{H_u}^*(M')^{\otimes m} \cong \mathcal{O}_{\mathcal{U}}(n)$  as line bundles it follows from Proposition 3.2.3, 2 that the  $H_r$ -linearisation on  $\theta_{H_u}^*(M')^{\otimes m}$  defines the same linearisation as the natural one on  $\mathcal{O}_{\mathcal{U}}(n) \rightarrow \mathcal{U}$ .

Because  $\mathcal{S}^{\text{ss}, H_u\text{-fg}} \subseteq \mathcal{S}$ , the rational map  $q_{H_r} : \text{Proj}(S^{H_u}) \dashrightarrow \text{Proj}(S^H)$  defines an  $H_r$ -invariant morphism  $q_{H_r} : \mathcal{U} \rightarrow \mathcal{V}$ , whose restriction to  $\text{Spec}((S^{H_u})_{(f)})$  for  $f \in \mathcal{S}^{\text{ss}, H_u\text{-fg}}$  is the map

$$\text{Spec}((S^{H_u})_{(f)}) \rightarrow \text{Spec}((S^H)_{(f)}) = \text{Spec}(((S^{H_u})_{(f)})^{H_r})$$

induced by the inclusion  $((S^{H_u})_{(f)})^{H_r} \hookrightarrow (S^{H_u})_{(f)}$ . By reductive GIT for affine varieties (Theorem 2.2.1), each of these restrictions is a good categorical quotient for the  $H_r$ -action on  $\text{Spec}((S^{H_u})_{(f)})$ , and since good categorical quotients are local on the base it follows that  $q_{H_r} : \mathcal{U} \rightarrow \mathcal{U} // H_r = q_{H_r}(\mathcal{U})$  is a good categorical quotient for the  $H_r$ -action on  $\mathcal{U}$ .

(Proof of 3.) It remains to prove the commutativity of the diagram in 3. Most of this follows from the construction of  $\theta_H$  and  $\theta_{H_u}$ —all that is left is to show is the equality

$$\bar{\pi}_{H_r} \circ \theta_{H_u} = \psi \circ \theta_H \circ q_{H_r} : \mathcal{U} \rightarrow (\overline{G \times^{H_u} X} // G) //_{M'} H_r.$$

Note first that both of these morphisms are indeed well defined on  $\mathcal{U}$ . By construction of  $\theta_H$  and diagram (4.2) we have

$$\psi \circ \theta_H \circ q_H = \psi \circ \pi_{G \times H_r} \circ \beta \circ \alpha = \bar{\pi}_{H_r} \circ \pi_G \circ \beta \circ \alpha$$

on  $X^{\text{ss}, \text{fg}}$ . Since  $\pi_G \circ \beta \circ \alpha = \theta_{H_u} \circ q_{H_u}$  and  $q_H = q_{H_r} \circ q_{H_u}$  on  $X^{\text{ss}, H_u\text{-fg}}$ , it follows that

$$\psi \circ \theta_H \circ q_{H_r} \circ q_{H_u} = \bar{\pi}_{H_r} \circ \theta_{H_u} \circ q_{H_u} : X^{\text{ss}, H_u\text{-fg}} \rightarrow (\overline{G \times^{H_u} X} // G) //_{M'} H_r.$$

Applying Proposition 3.1.14, 2 to this morphism, we conclude the desired equality  $\bar{\pi}_{H_r} \circ \theta_{H_u} = \psi \circ \theta_H \circ q_{H_r} : \mathcal{U} \rightarrow (\overline{G \times^{H_u} X} // G) //_{M'} H_r$ .  $\square$

By appealing to Proposition A.0.1, 3 in the Appendix, we obtain a corollary which is particularly relevant for the aims of constructing projective completions of the enveloped quotient.

**Corollary 4.1.13.** *If  $H \curvearrowright L \rightarrow X$  is an ample linearisation and  $(\overline{G \times^{H_u} X}, \beta, L')$  is an ample reductive envelope, then  $\overline{G \times^{H_u} X} // (G \times H_r) = (\overline{G \times^{H_u} X} // G) //_{M'} H_r$  is projective and  $\theta_H : \mathcal{V} \hookrightarrow \overline{G \times^{H_u} X} // (G \times H_r)$  defines a projective completion of the enveloped quotient  $q(X^{\text{ss,fg}})$ , as in Definition 4.0.1. Moreover,  $\theta_{H_u} : \mathcal{U} \hookrightarrow \overline{G \times^{H_u} X} // G$  defines an  $H_r$ -equivariant projective completion of the inner enveloping quotient  $q_{H_u} : X^{\text{ss}, H_u\text{-fg}} \rightarrow \mathcal{U}$  of  $X^{\text{ss}, H_u\text{-fg}}$ .*

We now come to the main theorem of this section, which is the *raison d'être* of reductive envelopes within non-reductive GIT.

**Theorem 4.1.14.** *Let  $H$  be a linear algebraic group with unipotent radical  $H_u$  and  $H_r = H/H_u$ , let  $L \rightarrow X$  be an ample linearisation of a quasi-projective  $H$ -variety  $X$  and suppose  $(\overline{G \times^{H_u} X}, \beta, L')$  is a reductive envelope for the linearisation, formed with respect to an  $H_u$ -faithful homomorphism  $H \rightarrow G$  with  $G$  reductive. Let*

$$\begin{aligned} \pi : \overline{G \times^{H_u} X}^{\text{ss}(L')} &\rightarrow \overline{G \times^{H_u} X} // (G \times H_r) \\ \pi : \overline{G \times^{H_u} X}^{\text{s}(L')} &\rightarrow \overline{G \times^{H_u} X}^{\text{s}(L')} / (G \times H_r) \end{aligned}$$

be the GIT quotient and geometric quotient of the semistable and stable locus, respectively, for the  $G \times H_r$ -linearisation  $L' \rightarrow \overline{G \times^{H_u} X}$ .

1. *There is an inner enveloping quotient  $q_{H_u} : X^{\text{ss}, H_u\text{-fg}} \rightarrow \mathcal{U}$  of  $X^{\text{ss}, H_u\text{-fg}}$ , with  $\mathcal{U}$  an  $H_r$ -invariant open subset of  $X \wr H_u$  with a good categorical  $H_r$ -quotient  $q_{H_r} : \mathcal{U} \rightarrow \mathcal{U} // H_r$ , and an inner enveloping quotient  $q_H : X^{\text{ss,fg}} \rightarrow \mathcal{V}$  making the diagram*

$$\begin{array}{ccccccccc} X^{\bar{s}} & \subseteq & X^s & \subseteq & X^{\text{ss}, H_u\text{-fg}} & \subseteq & X^{\text{ss,fg}} & \subseteq & X^{\text{ss}} = X^{\text{ns}} \\ \text{geo} \downarrow q_H & & \text{geo} \downarrow q_H & & \downarrow q_H & & \downarrow q_H & & \downarrow \pi \circ \beta \circ \alpha \\ X^{\bar{s}}/H & \subseteq & X^s/H & \subseteq & \mathcal{U} // H_r & \subseteq & \mathcal{V} & \subseteq & \overline{G \times^{H_u} X} // (G \times H_r) \end{array}$$

commute, where all the inclusions are natural open immersions and the two left-most vertical arrows are geometric quotients.

2. *The inclusion  $\beta \circ \alpha : X^{\bar{s}} \hookrightarrow \overline{G \times^{H_u} X}^{\text{s}(L')}$  induces a natural open immersion*

$$X^{\bar{s}}/H \hookrightarrow \overline{G \times^{H_u} X}^{\text{s}(L')} / (G \times H_r).$$

3. *If moreover the reductive envelope  $(\overline{G \times^{H_u} X}, \beta, L')$  is ample, then  $\overline{G \times^{H_u} X} // (G \times H_r)$  is projective and thus gives a projective completion of the enveloped quotient, as in Definition 4.0.1.*

*Proof.* This follows immediately by combining Theorem 3.4.2 with Proposition 4.1.10, Corollary 4.1.11, Proposition 4.1.12 and Corollary 4.1.13.  $\square$

We make a couple of remarks regarding this result.

*Remark 4.1.15.* Notice from Proposition 4.1.12 that the inner enveloping quotient  $\mathcal{U}$  of  $X^{\text{ss}, H_u - \text{fg}}$  embeds naturally into  $(\overline{G \times^{H_u} X} // G)^{\text{ss}(M')}$ , where  $M' \rightarrow \overline{G \times^{H_u} X} // G$  is the naturally induced ample linearisation on the GIT quotient of the linearisation  $G \curvearrowright L' \rightarrow \overline{G \times^{H_u} X}$ . When  $(\overline{G \times^{H_u} X}, \beta, L')$  is an ample reductive envelope then  $\overline{G \times^{H_u} X} // G$  provides an  $H_r$ -equivariant projective completion of  $\mathcal{U}$ , and moreover the composition

$$X^{\text{ss}, H_u - \text{fg}} \xrightarrow{q_{H_u}} \mathcal{U} \xrightarrow{q_{H_r}} \mathcal{U} // H_r$$

can be studied by by doing reductive GIT on  $\overline{G \times^{H_u} X}$  in stages—first by  $G$ , then by  $H_r$  (see Corollary 4.1.13).

*Remark 4.1.16.* If one happens to know that semistability and stability for a reductive envelope  $(\overline{G \times^{H_u} X}, \beta, L')$  coincide, then  $X^{\bar{s}} = X^{\text{ss}}$  and we have a string of equalities

$$X^{\bar{s}} = X^s = X^{\text{ss}, H_u - \text{fg}} = X^{\text{ss}} = X^{\text{nss}} = X^{\text{ss}}.$$

From the point of view of constructing projective completions of enveloped quotients, the most important application of Theorem 4.1.14 is to the case where  $(\overline{G \times^{H_u} X}, \beta, L')$  is an ample reductive envelope, as assumed in statement 3. In this case, the associated completely semistable and stable loci can be computed using the Hilbert-Mumford criteria and the GIT quotient  $\overline{G \times^{H_u} X} // (G \times H_r)$  can be described set-theoretically as the quotient space of  $\overline{G \times^{H_u} X}^{\text{ss}(L')}$  modulo the S-equivalence relation (see Section 2.2).

## 4.2 Strong Reductive Envelopes

We saw in Proposition 4.1.10 that, given a linear algebraic group  $H$  with unipotent radical  $H_u$  acting on an irreducible variety  $X$  with ample linearisation  $L \rightarrow X$ , the completely stable and completely semistable loci associated to a reductive envelope  $(\overline{G \times^{H_u} X}, \beta, L')$  provide approximations of the intrinsically defined stable locus  $X^s$  and the finitely generated semistable locus  $X^{\text{ss}, \text{fg}}$ : one has  $X^{\bar{s}} \subseteq X^s$  and  $X^{\text{ss}, \text{fg}} \subseteq X^{\text{nss}} = X^{\text{ss}}$ . In this section we discuss particular kinds of reductive envelope  $(\overline{G \times^{H_u} X}, \beta, L')$ , which give equalities  $X^{\bar{s}} = X^s$  and  $X^{\text{ss}} = X^{\text{ss}, \text{fg}}$ , thus providing potential ways to compute the finitely generated stable set and stable set for the original linearisation  $H \curvearrowright L \rightarrow X$  using methods from reductive GIT. In light of Theorem 4.1.14 and Proposition 3.3.8, to obtain these equalities we need to make sure that

- no points inside  $(G \times^{H_u} X)^{s(L)}$  suddenly become unstable with respect to  $L'$  as points in  $\overline{G \times^{H_u} X}$ ; and
- any point in  $X \subseteq \overline{G \times^{H_u} X}$  that is semistable for  $L'$  must lie in  $X_f$  for some invariant  $f$  over  $X$  with  $(S^H)_{(f)}$  finitely generated over  $\mathbb{k}$ .

Following the ideas used in [DorK07, §5.2], we adopt the strategy of effectively forcing out any complications arising from the codimension 1 boundary components of  $\overline{G \times^{H_u} X} \setminus (G \times^{H_u} X)$ , by demanding that extensions of appropriate invariants over  $G \times^{H_u} X$  vanish on these components. This, together with a normality assumption on  $G \times^{H_u} X$ , turns out to be enough to get the desired equalities  $X^{\bar{s}} = X^s$  and  $X^{\text{ss}, \text{fg}} = X^{\text{ss}}$ .

**Definition 4.2.1.** Let  $H$  be a linear algebraic group acting on an irreducible variety  $X$  with linearisation  $L \rightarrow X$ . Let  $H \rightarrow G$  be an  $H_u$ -faithful homomorphism with  $G$  a reductive group and let  $(\overline{G \times^{H_u} X}, \beta, L')$  be a reductive envelope. We call  $(\overline{G \times^{H_u} X}, \beta, L')$  a *strong reductive envelope* if there is a fully separating enveloping system  $V$  for  $H \curvearrowright L \rightarrow X$  satisfying the extension properties 1–3 of Definition 4.1.4 and the further property that every  $f \in V^H$  extends to a  $G \times H_r$ -invariant over  $\overline{G \times^{H_u} X}$  that vanishes on each codimension 1 component of the boundary of  $G \times^{H_u} X$  inside  $\overline{G \times^{H_u} X}$ .

**Proposition 4.2.2.** *Let  $H$  be a linear algebraic group acting on an irreducible variety  $X$  with ample linearisation  $L \rightarrow X$  and  $H \rightarrow G$  an  $H_u$ -faithful homomorphism into a reductive group  $G$ . Suppose  $(\overline{G \times^{H_u} X}, \beta, L')$  is a strong reductive envelope with  $\overline{G \times^{H_u} X}$  a normal variety. Then  $X^s = X^{\bar{s}}$  and  $X^{\text{ss}, H_u\text{-fg}} = X^{\text{ss}}$ .*

*Proof.* By Proposition 4.1.10 it suffices to show  $X^s \subseteq X^{\bar{s}}$  and  $X^{\text{ss}} \subseteq X^{\text{ss}, H_u\text{-fg}}$ . Throughout the proof we let  $V$  be a fully separating enveloping system associated to  $(\overline{G \times^{H_u} X}, \beta, L')$  satisfying the conditions in Definition 4.2.1 and let  $D_1, \dots, D_m$  be the codimension 1 components of  $\overline{G \times^{H_u} X} \setminus \beta(G \times^{H_u} X)$ .

(Proof of  $X^s \subseteq X^{\bar{s}}$ .) Let  $x \in X^s$ . By Definition 4.1.1, 3 of a fully separating enveloping system there is  $f \in V^H$  with extension to a  $G \times H_r$ -invariant section  $F$  over  $G \times^{H_u} X$  such that  $f(x) \neq 0$  and  $G \times^{H_u} (X_f) = (G \times^{H_u} X)_F$  is affine. By 3 of Definition 4.1.4 and the definition of a strong reductive envelope, there is a section of some positive tensor power of  $L' \rightarrow \overline{G \times^{H_u} X}$ , which we also call  $F$ , such that  $(\beta \circ \alpha)^* F = f$ , with the open set  $(\overline{G \times^{H_u} X})_F$  affine and  $F$  vanishing on  $\bigcup_i D_i \subseteq \overline{G \times^{H_u} X}$ . Thus the complement of  $G \times^{H_u} (X_f) = \beta^{-1}((\overline{G \times^{H_u} X})_F)$  inside  $(\overline{G \times^{H_u} X})_F$  has codimension at least 2. Because  $(\overline{G \times^{H_u} X})_F$  is normal, pullback along  $\beta$  yields an isomorphism  $\mathcal{O}((\overline{G \times^{H_u} X})_F) \cong \mathcal{O}(G \times^{H_u} (X_f))$  and, since both  $(\overline{G \times^{H_u} X})_F$  and  $G \times^{H_u} (X_f)$  are affine, the open inclusion  $\beta : G \times^{H_u} (X_f) \hookrightarrow (\overline{G \times^{H_u} X})_F$  is therefore an isomorphism. But  $G \times^{H_u} (X_f) = (\overline{G \times^{H_u} X})_F$  is contained in the stable locus for the  $G \times H_r$ -linearisation  $L \rightarrow G \times^{H_u} X$  by Proposition 3.3.8, so the  $G \times H_r$ -action on  $(\overline{G \times^{H_u} X})_F$  is closed with all stabilisers finite. It follows that  $(\overline{G \times^{H_u} X})_F \subseteq \overline{G \times^{H_u} X}^{s(L')}$  and thus  $x \in X^{\bar{s}}$ .

(Proof  $X^{\text{ss}} \subseteq X^{\text{ss}, H_u\text{-fg}}$ .) Now suppose  $x \in X^{\text{ss}}$ . By Definition 4.1.1, 1 of a fully separating enveloping system we have  $X^{\text{ss}} = X^{\text{nss}} = \bigcup_{f \in V^H} X_f$ , so by Definition 4.2.1 there is  $f \in V^H$  with extension to a  $G \times H_r$ -invariant section  $F$  of some positive tensor power of  $L' \rightarrow \overline{G \times^{H_u} X}$ , with  $(\overline{G \times^{H_u} X})_F$  affine, such that  $x \in X_f$  and  $F$  vanishes on  $\bigcup_i D_i$ . As above, the complement of  $G \times^{H_u} (X_f)$  is therefore of codimension at least 2 in  $(\overline{G \times^{H_u} X})_F$ , so from the normality of  $(\overline{G \times^{H_u} X})_F$  it follows that the pullback map  $\beta^\# : \mathcal{O}((\overline{G \times^{H_u} X})_F) \rightarrow \mathcal{O}(G \times^{H_u} (X_f))$  is a  $G \times H_r$ -equivariant isomorphism. Thus  $(\beta \circ \alpha)^\#$  yields isomorphisms

$$\begin{aligned} \mathcal{O}((\overline{G \times^{H_u} X})_F)^G &\xrightarrow{\cong} \mathcal{O}(X_f)^{H_u} = (S^{H_u})_{(f)}, \\ \mathcal{O}((\overline{G \times^{H_u} X})_F)^{G \times H_r} &\xrightarrow{\cong} \mathcal{O}(X_f)^H = (S^H)_{(f)}. \end{aligned}$$

Since  $(\overline{G \times^{H_u} X})_F$  is affine and  $G$  and  $G \times H_r$  are reductive, the  $\mathbb{k}$ -algebras  $(S^{H_u})_{(f)}$  and  $(S^H)_{(f)}$  are therefore finitely generated and thus  $x \in X_f \subseteq X^{\text{ss}, H_u\text{-fg}}$ .  $\square$

*Remark 4.2.3.* Observe that as a corollary to Proposition 4.2.2, for any given linearisation  $H \curvearrowright L \rightarrow X$  a necessary condition for the existence of a strong reductive envelope with normal  $\overline{G \times^{H_u} X}$  is that  $X^{\text{ss}, H_u\text{-fg}} = X^{\text{ss}, \text{fg}} = X^{\text{nss}}$  (cf. Remark 2.3.16).

For the remainder of this section we will consider ways to try to construct strong reductive envelopes. Recall that this means (1) choosing an equivariant completion  $\overline{G \times^{H_u} X}$  of  $G \times^{H_u} X$ , together with (2) an extension  $L' \rightarrow \overline{G \times^{H_u} X}$  of some positive tensor power of the linearisation  $L \rightarrow G \times^{H_u} X$  such that (3) (roughly stated) enough invariant sections over  $G \times^{H_u} X$  extend to sections over  $\overline{G \times^{H_u} X}$  vanishing on the boundary divisors. In general (2) depends heavily on the singularities of the completion in (1). We don't wish to explore in depth here; instead it suffices for us to note that if  $\overline{G \times^{H_u} X}$  is  $\mathbb{Q}$ -factorial then some positive tensor power of  $L \rightarrow G \times^{H_u} X$  extends over the boundary, and if  $\overline{G \times^{H_u} X}$  is even factorial (for example, smooth) then  $L$  itself extends. Moreover, if  $X$  is irreducible and  $H$  and  $G$  are *connected* linear algebraic groups such that the action of  $G \times H_r$  on  $G \times^{H_u} X$  extends to one on  $\overline{G \times^{H_u} X}$ , then any line bundle  $L' \rightarrow \overline{G \times^{H_u} X}$  extending  $L^{\otimes r} \rightarrow G \times^{H_u} X$  (with  $r > 0$ ) has a unique linearisation extending  $L^{\otimes r} \rightarrow G \times^{H_u} X$  (see the proof of [MumFK94, Converse 1.13]).

On the other hand, given any extension  $L' \rightarrow \overline{G \times^{H_u} X}$  of (a power of) the  $G \times H_r$ -linearisation  $L \rightarrow G \times^{H_u} X$ , then as in [DorK07] the question of (3) can be approached by making a fairly mild assumption on the nature of the completion  $\overline{G \times^{H_u} X}$ : namely, that it is a *gentle* completion of  $G \times^{H_u} X$ , in the sense of Definition 2.3.15: recall this means that  $\overline{G \times^{H_u} X}$  is a normal projective variety such that every codimension 1 component of the boundary of  $G \times^{H_u} X$  in  $\overline{G \times^{H_u} X}$  is a  $\mathbb{Q}$ -Cartier divisor. What follows is a generalisation of the constructions in [DorK07, §§5.3.1–5.3.2] (described in Section 2.3.2) that applies to our current setting.

So let us now assume that  $H$  and  $G$  are *connected* linear algebraic groups, with  $G$  reductive, and  $\rho : H \rightarrow G$  is a fixed  $H_u$ -faithful homomorphism. Let  $\overline{G \times^{H_u} X}$  be a gentle completion of  $G \times^{H_u} X$  and  $L' \rightarrow \overline{G \times^{H_u} X}$  a  $G \times H_r$ -linearisation extending  $L^{\otimes r} \rightarrow G \times^{H_u} X$ , for some  $r > 0$ . Also let  $D_1, \dots, D_m \subseteq \overline{G \times^{H_u} X}$  be the codimension 1 irreducible components of the complement of  $G \times^{H_u} X$  in  $\overline{G \times^{H_u} X}$  and define the  $\mathbb{Q}$ -Cartier divisor

$$D := \sum_{i=1}^m D_i.$$

Then for any sufficiently divisible integer  $N > 0$  the divisor  $ND$  is Cartier and defines a line bundle  $\mathcal{O}(ND)$  on  $\overline{G \times^{H_u} X}$  which restricts to the trivial bundle on  $G \times^{H_u} X$ . As in Section 2.3.2, given any line bundle  $M \rightarrow \overline{G \times^{H_u} X}$ , define

$$M_N := M \otimes \mathcal{O}(ND) \rightarrow \overline{G \times^{H_u} X}. \quad (4.3)$$

In the case  $M = L'$  then, because  $G \times H_r$  is connected, the  $G \times H_r$ -linearisation on  $L^{\otimes r} \rightarrow G \times^{H_u} X$  extends uniquely to a  $G \times H_r$ -linearisation on  $L'_N$ .

The next lemma consists of an expanded version of the argument found in [DorK07, Proposition 5.3.10].

**Lemma 4.2.4.** *Let  $\beta : G \times^{H_u} X \hookrightarrow \overline{G \times^{H_u} X}$  be a gentle completion and  $L' \rightarrow \overline{G \times^{H_u} X}$  a line bundle extending  $L$ . Retain the preceding notation.*



1. Let  $f$  be a section of  $L^{\otimes rr'} \rightarrow G \times^{H_u} X$  for some  $r' > 0$ . Then for  $N > 0$  a sufficiently divisible integer,  $f$  extends to a section of  $(L'_N)^{\otimes r'} \rightarrow \overline{G \times^{H_u} X}$  under  $\beta$ .
2. For any  $N > 0$  such that  $ND$  is Cartier and for any integers  $r', m > 0$ , there is a natural inclusion  $H^0(\overline{G \times^{H_u} X}, (L'_N)^{\otimes r'}) \hookrightarrow H^0(\overline{G \times^{H_u} X}, (L'_{mN})^{\otimes r'})$  whose image consists of sections of  $(L'_N)^{\otimes r'}$  that vanish on each of  $D_1, \dots, D_m$ .

*Proof.* Throughout the proof we denote the field of rational functions on an irreducible variety  $Y$  by  $\mathbb{k}(Y)$ .

(Proof of 1.) It suffices to show that for a suitable  $N > 0$  the section  $f$  extends to a section of  $L'_N$  over codimension 1 boundary components—it will canonically extend over components of codimension at least 2 because  $\overline{G \times^{H_u} X}$  is normal. We use the fact that  $L'$  corresponds to a Cartier divisor [Har77, Proposition 6.15]; that is, there is a finite collection of pairs  $(U_j, t_j)$ , with  $U_j \subseteq \overline{G \times^{H_u} X}$  open and  $t_j \in \mathbb{k}(U_j) = \mathbb{k}(G \times^{H_u} X)$ , such that  $L'$  is represented by a Weil divisor whose restriction to  $U_j$  is the principal divisor defined by  $t_j$ . Note that  $(L')^{\otimes r'}$  is then represented by the collection  $(U_j, t_j^{r'})$ . There is a positive integer  $N$  such that the order of the vanishing of  $t_j$  along each  $D_i$  is less than  $N$  for each  $j$ . Thinking of sections of  $L^{\otimes rr'}$  as sections of the constant sheaf of rational functions on  $G \times^{H_u} X$  in the standard way [Har77, Chapter 2, §6], we can write  $f|_{U_j \cap (G \times^{H_u} X)} = b_j/t_j^{r'}$  with  $b_j$  a regular function on  $U_j \cap (G \times^{H_u} X)$ . We can also assume without loss of generality that  $ND$  is a Cartier divisor of  $\overline{G \times^{H_u} X}$ , since the completion  $\overline{G \times^{H_u} X}$  is gentle. Let  $(V_k, s_k)$  represent  $ND$ , with  $V_k \subseteq \overline{G \times^{H_u} X}$  open and  $s_k \in \mathbb{k}(V_k) = \mathbb{k}(G \times^{H_u} X)$ , again with the index set for  $k$  being finite, and set  $a_{jk} = b_j s_k^{r'} \in \mathbb{k}(\overline{G \times^{H_u} X})$ . Now  $b_j$  may have poles along  $U_j \cap V_k \cap (\bigcup_i D_i)$ ; on the other hand each  $s_k^{r'}$  vanishes on  $U_j \cap V_k \cap (\bigcup_i D_i)$  with order  $r'N$ , so by further increasing  $N$  if necessary we may assume each  $a_{jk}$  defines a regular function on  $U_j \cap V_k$ . Thus each  $a_{jk}/(t_j s_k)^{r'}$  is a section of  $(L'_N)^{\otimes r'}$  over  $U_j \cap V_k$ , whose restriction to  $(G \times^{H_u} X) \cap U_j \cap V_k$  is defined by  $b_j/t_j^{r'}$ . One can check that all the  $a_{jk}/(t_j s_k)^{r'}$  (for all  $j$  and  $k$ ) agree on overlaps, so they patch together to give a global section of  $(L'_N)^{\otimes r'}$  which extends  $f$ .

(Proof of 2.) Assume  $N$  is large enough so that  $ND$  is Cartier, and let  $r' > 0$ . Continuing with the notation used above to prove 1, the sheaf  $\mathcal{O}(ND)$  is represented by the collection  $(V_k, s_k)$ , with  $s_k \in \mathcal{O}(V_k)$  such that  $\mathcal{O}(ND)|_{V_k} = \mathcal{O}_{V_k}\langle 1/s_k \rangle$  as sheaves of  $\mathcal{O}_{V_k}$ -modules. Then for each  $m > 0$  there is a well-defined inclusion  $\mathcal{O}(r'ND) \hookrightarrow \mathcal{O}(mr'ND)$ , whose restriction to  $V_k$  corresponds to the multiplication-by- $s_k^{r'(m-1)}$  map  $\mathcal{O}_{V_k} \rightarrow \mathcal{O}_{V_k}$ . Note that sections in the image of this map vanish on each of the  $D_i$ . Because  $(L')^{\otimes r'}$  is locally free the natural map of sheaves

$$(\underline{L}')^{\otimes r'} \otimes \mathcal{O}(r'ND) \rightarrow (\underline{L}')^{\otimes r'} \otimes \mathcal{O}(mr'ND)$$

is again injective [Har77, Chapter 3, Proposition 9.2], and since taking global sections is left-exact we see that this yields an injection

$$H^0(\overline{G \times^{H_u} X}, (L'_N)^{\otimes r'}) \hookrightarrow H^0(\overline{G \times^{H_u} X}, (L'_{mN})^{\otimes r'}).$$

It is immediate that any section in the image of this map vanishes on each of the  $D_i$ .  $\square$

Lemma 4.2.4 says that given any finite collection of sections of some power of  $L^{\otimes r} \rightarrow G \times^{H_u} X$  we can always modify the extension  $L'$  so that these sections extend to sections of the resulting linearisation which vanish on the boundary of  $\overline{G \times^{H_u} X}$ . With additional assumptions on  $L'$  we can use this fact to produce a strong reductive envelope.

**Proposition 4.2.5.** *Let  $H$  be a connected linear algebraic group acting on an irreducible variety  $X$  with linearisation  $L \rightarrow X$  and let  $H \rightarrow G$  be an  $H_u$ -faithful homomorphism into a connected reductive group  $G$  where  $H_u$  is the unipotent radical of  $H$ . Suppose  $\beta : G \times^{H_u} X \hookrightarrow \overline{G \times^{H_u} X}$  is a gentle  $G \times H_r$ -equivariant projective completion and let  $L' \rightarrow \overline{G \times^{H_u} X}$  be any  $G \times H_r$ -linearisation extending some positive tensor power of the  $H$ -linearisation  $L \rightarrow X$ . If either*

- $(\overline{G \times^{H_u} X}, \beta, L')$  defines a reductive envelope for  $H \curvearrowright L \rightarrow X$ ; or
- the line bundle  $L'_n$  of (4.3) is ample, for sufficiently divisible integers  $n > 0$ ;

then for sufficiently divisible integers  $N > 0$  the triple  $(\overline{G \times^{H_u} X}, \beta, L'_N)$  defines a strong reductive envelope.

*Proof.* Suppose  $L' \rightarrow \overline{G \times^{H_u} X}$  pulls back to  $L^{\otimes r} \rightarrow X$  under  $\beta \circ \alpha$  and let  $D_1, \dots, D_m$  be the irreducible codimension 1 components of the boundary of the gentle completion  $\beta : G \times^{H_u} X \hookrightarrow \overline{G \times^{H_u} X}$ . We first show the following: given  $r' > 0$  and an enveloping system  $V \subseteq H^0(X, L^{\otimes rr'})$ , for sufficiently divisible integers  $N > 0$  (depending on  $V$ ) each section  $f \in V^{H_u}$  (respectively,  $f \in V^H$ ) extends to a  $G$ -invariant (respectively,  $G \times H_r$ -invariant) section  $F$  of  $(L'_N)^{\otimes r'} \rightarrow \overline{G \times^{H_u} X}$  under  $\beta \circ \alpha$  which vanishes on each  $D_i$ . To this end, let  $f \in V^{H_u}$  (respectively,  $f \in V^H$ ). By Lemma 4.2.4 there is an integer  $N_f > 0$  such that  $f$  extends to a section  $F$  of  $(L'_{N_f})^{\otimes r'}$  over  $\overline{G \times^{H_u} X}$  which vanishes on the codimension 1 boundary components of  $G \times^{H_u} X$  inside  $\overline{G \times^{H_u} X}$ . Note that  $F$  must be  $G$ -invariant (respectively,  $G \times H_r$ -invariant): by the normality of  $\overline{G \times^{H_u} X}$  the section  $f$  extends canonically to an invariant over the boundary components of codimension at least 2 in  $\overline{G \times^{H_u} X}$ , and since  $F$  vanishes on the remaining boundary components it too must be invariant. Now take bases  $\mathcal{B}_{H_u}$  of  $V^{H_u}$  and  $\mathcal{B}_H$  of  $V^H$  and let  $N > 0$  be any positive integer which is properly divisible by all the  $N_f$  for  $f \in \mathcal{B}_{H_u} \cup \mathcal{B}_H$ . Then by Lemma 4.2.4, 2 any  $f$  in  $V^{H_u}$  or  $V^H$  extends to a section  $F$  of  $(L'_N)^{\otimes r'}$ , invariant in the appropriate sense, which vanishes on the codimension 1 boundary components of  $G \times^{H_u} X$  in  $\overline{G \times^{H_u} X}$ , which was to be shown.

Now suppose  $(\overline{G \times^{H_u} X}, \beta, L')$  defines a reductive envelope for  $H \curvearrowright L \rightarrow X$  and let  $V$  be an associated fully separating enveloping system satisfying 1–3 of Definition 4.1.4 of a reductive envelope. As above, for sufficiently divisible  $N > 0$  each  $f$  in  $V^{H_u}$  or  $V^H$  extends to an invariant  $F$  (in the appropriate sense) of some positive tensor power of  $L'_N \rightarrow \overline{G \times^{H_u} X}$  which vanishes on the codimension 1 complement  $\bigcup_i D_i$  of  $G \times^{H_u} X$  in  $\overline{G \times^{H_u} X}$ . To show that  $(\overline{G \times^{H_u} X}, \beta, L'_N)$  is a strong reductive envelope, we are left to show that each such  $(\overline{G \times^{H_u} X})_F$  is affine (see Definition 4.1.4, 3). But by Definition 4.1.4 of a reductive envelope applied to  $L'$ , the section  $f$  does extend to an invariant  $F'$  of some positive tensor power of  $L' \rightarrow \overline{G \times^{H_u} X}$  with  $(\overline{G \times^{H_u} X})_{F'}$  affine. Observe that the restrictions of  $L'_N$  and  $L'$  to  $\overline{G \times^{H_u} X} \setminus (\bigcup_i D_i)$  are equal, and  $G \times^{H_u} X$  and  $\overline{G \times^{H_u} X} \setminus (\bigcup_i D_i)$  differ only in codimension at least 2, so by normality of  $\overline{G \times^{H_u} X}$  the

sections  $F$  and  $F'$  are equal over  $\overline{G \times^{H_u} X} \setminus (\bigcup_i D_i)$ . It follows that

$$(\overline{G \times^{H_u} X})_F = (\overline{G \times^{H_u} X})_{F'} \setminus (\bigcup_i D_i).$$

Because  $(\overline{G \times^{H_u} X})_{F'}$  is affine and the complement of the support of a Cartier divisor on an affine variety is again affine [St15, Tag 01WQ],  $(\overline{G \times^{H_u} X})_F$  is therefore also affine. Hence,  $(\overline{G \times^{H_u} X}, \beta, L'_N)$  is a strong reductive envelope, for sufficiently divisible  $N > 0$ .

Finally, consider the case where  $L'_n$  is ample for sufficiently divisible  $n > 0$ . Appealing to Lemma 4.1.2 there is an integer  $r' > 0$  such that  $H^0(X, L^{\otimes rr'})$  contains a fully separating enveloping system  $V$ , and for sufficiently divisible  $N > 0$  each  $f$  in  $V^{H_u}$  or  $V^H$  extends to an invariant  $F$  (in the appropriate sense) of some positive tensor power of  $L'_N \rightarrow \overline{G \times^{H_u} X}$  which vanishes on each  $D_i$ . We can choose  $N > 0$  sufficiently divisible so that that  $L'_N$  is ample and thus each  $(\overline{G \times^{H_u} X})_F$  affine. Then  $(\overline{G \times^{H_u} X}, \beta, L'_N)$  is a strong ample reductive envelope.  $\square$

**Corollary 4.2.6.** *In the setting of Proposition 4.2.5, suppose we are in the situation that  $X$  is projective and  $L'_n$  is ample for sufficiently divisible integers  $n > 0$ . If  $S^H = \mathbb{k}[X, L]^H$  is a finitely generated  $\mathbb{k}$ -algebra, then for sufficiently divisible integers  $N > 0$  the inclusion  $\beta \circ \alpha : X^{\text{ss,fg}} \hookrightarrow (\overline{G \times^{H_u} X})^{\text{ss}(L'_N)}$  induces a natural isomorphism*

$$X \wr H = \text{Proj}(S^H) \cong \overline{G \times^{H_u} X} //_{L'_N} (G \times H_r).$$

*Proof.* Because  $X$  is projective and  $L \rightarrow X$  is necessarily ample, the invariant ring  $S^H$  is finitely generated with degree 0 piece equal to the ground field  $\mathbb{k}$ , hence  $X \wr H = \text{Proj}(S^H)$  is a projective variety. Thus  $X \wr H$  is an inner enveloping quotient of  $X^{\text{ss,fg}} = X^{\text{nss}}$ . As in the proof of Proposition 4.2.5, by appealing to Lemma 4.1.2 we may find  $r' > 0$  such that  $H^0(X, L^{\otimes rr'})$  contains a fully separating enveloping system  $V$ . By taking the image of  $V$  under a suitable multiplication map (see Lemma 4.1.2) and enlarging using Lemma 2.1.21 if necessary, we can assume that  $V$  is a fully separating system that also contains a (finite) collection of generators  $f_1, \dots, f_m$  of the ring  $\mathbb{k}[X, L^{\otimes rr'}]^H$ . Then

$$X \wr H \cong \text{Proj}(\mathbb{k}[X, L^{\otimes rr'}]^H) = \bigcup_{j=1}^m \text{Spec}((S^H)_{(f_j)})$$

and each  $(S^H)_{(f_j)}$  is generated by  $\{\tilde{f}/f_j \mid \tilde{f} \in V^H\}$ . Without loss of generality we may therefore view  $V$  as a fully separating enveloping system which is adapted to to a subset  $\mathcal{S}$  containing  $\{f_1, \dots, f_m\}$ . For sufficiently divisible  $N > 0$  we have  $L'_N$  ample and, as in the proof of Proposition 4.2.5, each  $f \in V^H$  extends to a  $G \times H_r$ -invariant  $F$  of some positive tensor power of  $L'_N \rightarrow \overline{G \times^{H_u} X}$ . The inclusion  $\beta \circ \alpha : X^{\text{ss,fg}} \hookrightarrow (\overline{G \times^{H_u} X})^{\text{ss}(L'_N)}$  is therefore well-defined, and defines a dominant open immersion

$$\theta_H : X \wr H \hookrightarrow \overline{G \times^{H_u} X} //_{L'_N} (G \times H_r),$$

as in the proof of Proposition 4.1.12, 1. Since  $X \wr H$  is proper and  $\overline{G \times^{H_u} X} //_{L'_N} (G \times H_r)$  is separated, the image of  $\theta_H$  is also closed in  $\overline{G \times^{H_u} X} //_{L'_N} (G \times H_r)$  [St15, Tag 01W0] and thus is the whole of  $\overline{G \times^{H_u} X} //_{L'_N} (G \times H_r)$ . Therefore  $\theta_H$  defines an isomorphism  $X \wr H \cong \overline{G \times^{H_u} X} //_{L'_N} (G \times H_r)$ .  $\square$

The situation from Proposition 4.2.5 where the linearisation  $L'_N \rightarrow \overline{G \times^{H_u} X}$  is ample for sufficiently divisible  $N$  is potentially the most useful in applications, because it does not require any verification that  $(\overline{G \times^{H_u} X}, \beta, L')$  forms a reductive envelope (in particular, that enough invariants extend to sections  $F$  with  $(\overline{G \times^{H_u} X})_F$  affine). For the proposition to apply however, one needs to know that  $\overline{G \times^{H_u} X}$  provides a gentle completion of  $G \times^{H_u} X$  and this imposes restrictions on the linearisation  $H \curvearrowright L \rightarrow X$  (cf. Remarks 2.3.16 and 4.2.3). For example, if  $X$  is affine and  $L = \mathcal{O}_X \rightarrow X$  with the canonical linearisation, then without loss of generality we may assume that  $N > 0$  is such that  $L'_N \rightarrow \overline{G \times^{H_u} X}$  forms an ample, strong reductive envelope with respect to an enveloping system inside  $H^0(X, L) = \mathcal{O}(X)$  that includes a nonzero constant function  $f$ ; let  $F$  denote the extension of  $f$  to  $L'_N \rightarrow \overline{G \times^{H_u} X}$ . Then  $(\overline{G \times^{H_u} X})_F$  is an affine variety containing  $(G \times^{H_u} X)_F = X_f = X$  as a codimension 2 complement, so if  $\overline{G \times^{H_u} X}$  is gentle then  $\mathcal{O}(X)^H = \mathbb{k}[(\overline{G \times^{H_u} X})_F, L'_N]^G$  is a finitely generated  $\mathbb{k}$ -algebra.

In the next section we consider a special case which allows us to explicitly construct strong ample reductive envelopes  $(\overline{G \times^{H_u} X}, \beta, L')$  with gentle completions of  $G \times^{H_u} X$ .

#### 4.2.1 Special Case: Extension to a $G$ -Linearisation and Grosshans Subgroups

For this section we suppose that  $X$  is a *projective* irreducible  $H$ -variety. If  $\rho : H \rightarrow G$  is an  $H_u$ -faithful homomorphism into a reductive group  $G$  and the linearisation  $H \curvearrowright L \rightarrow X$  can be partially extended to a  $G$ -linearisation, one can reduce the task of constructing ample strong reductive envelopes to understanding the geometry of the homogeneous space  $G/H_u$  by ‘untwisting’ the  $G$ -action on  $G \times^{H_u} X$ . More precisely, suppose there is a  $G$ -linearisation on the line bundle  $L \rightarrow X$  satisfying the following condition:

- (C1) The linearisation  $H_u \curvearrowright L \rightarrow X$  arising from restricting the  $H$ -linearisation extends to the  $G$ -linearisation through  $\rho|_{H_u}$ .

The extension condition (C1) yields an isomorphism of  $G \times H_r$ -linearisations:

$$\begin{array}{ccc} G \times^{H_u} L & \xrightarrow{\cong} & (G/H_u) \times L & [g, l] \mapsto (gH_u, gl) \\ \downarrow & & \downarrow & \\ G \times^{H_u} X & \xrightarrow{\cong} & (G/H_u) \times X & [g, x] \mapsto (gH_u, gx) \end{array} \quad (4.4)$$

The corresponding  $G$ -linearisation on the right hand side of this diagram is the one given by taking the product of the linearisation on  $L \rightarrow X$  and left multiplication on  $G/H_u$ . The  $H_r$ -linearisation is more complicated and in general cannot be expressed as the product of linearisations over  $G/H_u$  and  $X$ . We make an additional assumption to demand this:

- (C2) There is a linearisation  $G \times H_r \curvearrowright \tilde{L} \rightarrow X$ , with  $\tilde{L} = L$  as line bundles, such that the  $G \times H_r$ -linearisation  $(G/H_u) \times L \rightarrow (G/H_u) \times X$  arising from (4.4) coincides with the product of  $\tilde{L} \rightarrow X$  and the  $G \times H_r$ -action on  $G/H_u$  given by left multiplication by  $G$  and right multiplication by  $H_r$ <sup>13</sup>.

<sup>13</sup>That is,  $(g, \bar{h}) \cdot g_0 H_u = g g_0 \rho(h)^{-1} H_u$  for all  $\bar{h} = h H_u \in H_r = H/H_u$ ,  $g \in G$  and all  $g_0 H_u \in G/H_u$ .

*Example 4.2.7.* Suppose the  $H_u$ -faithful homomorphism  $\rho$  is just a closed embedding  $H \hookrightarrow G$ . Then the  $H_r$ -linearisation on  $(G/H_u) \times L \rightarrow (G/H_u) \times X$  under the isomorphism (4.4) is simply the product of the right multiplication action on  $G/H_u$  together with  $\tilde{L} \rightarrow X$ , where  $\tilde{L} \rightarrow X$  is the line bundle  $L \rightarrow X$  equipped with the trivial  $H_r$ -linearisation.

Assuming  $X$  is projective and conditions (C1) and (C2) are satisfied, then a natural way to complete  $G \times^{H_u} X \cong (G/H_u) \times X$  is to study  $G \times H_r$ -equivariant projective completions  $\overline{G/H_u}$  of  $G/H_u$ . For example, if  $L$  is ample and it is known that  $\overline{G/H_u}$  can be chosen to be normal with  $\mathbb{Q}$ -Cartier prime boundary divisors  $E_1, \dots, E_m$  such that  $\mathcal{O}(N \sum_i E_i) \rightarrow \overline{G/H_u}$  is ample for sufficiently divisible  $N > 0$ , then the codimension 1 components of the boundary  $\overline{G/H_u} \times X$  are precisely the  $E_i \times X$ . These are  $\mathbb{Q}$ -Cartier divisors, so if  $X$  is further assumed to be normal then  $\beta : G \times^{H_u} X \hookrightarrow \overline{G \times^{H_u} X}$  is a gentle completion and Proposition 4.2.5 applies to  $L' = \mathcal{O}(N \sum_i E_i) \boxtimes \tilde{L}$  (with  $\tilde{L} \rightarrow X$  the  $G \times H_r$ -linearisation of (C2)).

This works out particularly well in the case where  $H_u$  is a *Grosshans subgroup* of the reductive group  $G$ . Recall from [Gros97, §4] that this means the pair  $H_u \subseteq G$  satisfies the following equivalent conditions:

- $\mathcal{O}(G/H_u)$  is a finitely generated  $\mathbb{k}$ -algebra;
- there is a finite dimensional  $G$ -module  $W$  and a vector  $w \in W$  such that  $H_u = \text{Stab}_G(w)$  and  $G/H_u$  embeds into the closure  $\overline{G \cdot w} \subseteq W$  with complement having codimension at least 2, via the natural map  $G/H_u \rightarrow G \cdot w$ .

In this case, since  $\mathcal{O}(G/H_u) = \mathcal{O}(G)^{H_u}$  is a normal ring  $\text{Spec}(\mathcal{O}(G/H_u)) = \text{Spec}(\mathcal{O}(G)^{H_u})$  is a normal affine variety upon which  $G \times H_r$  naturally acts, and there is a canonical open immersion

$$G/H_u \hookrightarrow \text{Spec}(\mathcal{O}(G/H_u))$$

that is  $G \times H_r$ -equivariant. It follows from [Gros97, Theorem 4.3] that the boundary of  $G/H_u$  in fact sits inside  $\text{Spec}(\mathcal{O}(G/H_u))$  with a complement of codimension at least 2.

*Remark 4.2.8.* In the situation where (C1) holds we have

$$\mathbb{k}[X, L]^{H_u} = \mathbb{k}[G \times^{H_u} X, L]^G = \mathbb{k}[G/H_u \times X, L]^G \cong (\mathcal{O}(G/H_u) \otimes \mathbb{k}[X, L])^G,$$

where the last isomorphism follows from the Künneth formula [St15, Tag 02KE]. If  $H_u$  is a Grosshans subgroup of  $G$  and  $H \curvearrowright L \rightarrow X$  is an ample linearisation of a projective variety, then by Nagata's theorem [Na64] it follows that  $\mathbb{k}[X, L]^{H_u}$  is finitely generated. Then  $\mathbb{k}[X, L]^H = (\mathbb{k}[X, L]^{H_u})^{H_r}$  is also finitely generated, hence  $X^{\text{ss}, H_u\text{-fg}} = X^{\text{ss}, \text{fg}} = X^{\text{nss}}$  by definition and  $X \curvearrowright H = \text{Proj}(\mathbb{k}[X, L]^H)$  is a projective variety.

Now, any normal affine completion  $\overline{G/H_u}^{\text{aff}}$  of  $G/H_u$  admits an equivariant closed immersion into some  $G \times H_r$ -module  $W$ . As in the proof of Proposition 4.1.7, let  $\mathbb{k}$  be a copy of the ground field equipped with the trivial  $G \times H_r$ -action and consider the closure  $\overline{G/H_u}$  of  $\overline{G/H_u}^{\text{aff}}$  under the natural open immersion  $W \hookrightarrow \mathbb{P} := \mathbb{P}(W \oplus \mathbb{k})$ . The complement  $D_\infty := \overline{G/H_u} \setminus \overline{G/H_u}^{\text{aff}}$  is an effective Cartier divisor corresponding to the hyperplane line bundle  $\mathcal{O}_{\overline{G/H_u}}(1) = \mathcal{O}_{\mathbb{P}}(1)|_{\overline{G/H_u}}$ . Let  $\nu : \widetilde{G/H_u} \rightarrow \overline{G/H_u}$  be the normalisation of  $\overline{G/H_u}$ . Then  $\widetilde{G/H_u}$  naturally contains  $\overline{G/H_u}^{\text{aff}}$

as an open subset with  $\widetilde{G/H_u} \setminus \overline{G/H_u}^{\text{aff}} = \nu^{-1}(D_\infty)$ , and because  $\nu$  is a finite map  $\nu^{-1}(D_\infty)$  is a divisor corresponding to the ample line bundle  $\mathcal{O}_{\widetilde{G/H_u}}(1) = \nu^* \mathcal{O}_{\overline{G/H_u}}(1)$ . The naturally induced  $G \times H_r$ -linearisation on  $\mathcal{O}_{\overline{G/H_u}}(1) \rightarrow \overline{G/H_u}$  canonically defines a linearisation on the normalisation  $\mathcal{O}_{\widetilde{G/H_u}}(1) \rightarrow \widetilde{G/H_u}$  [Ses63, Chapter 1], which pulls back to the canonical  $G \times H_r$ -linearisation on  $\mathcal{O}_{\overline{G/H_u}^{\text{aff}}} \rightarrow \overline{G/H_u}^{\text{aff}}$ .

*Example 4.2.9.* As a simple example, consider the situation where  $H$  is a linear algebraic group with  $H_u \cong \mathbb{G}_a$  and  $\rho : H \rightarrow G = \text{SL}(2, \mathbb{k})$  is an  $H_u$ -faithful homomorphism. Without loss of generality we may assume  $\rho$  maps  $H_u$  onto the subgroup of strictly upper-triangular matrices of  $G$  and, as is well known,  $H_u$  is a Grosshans subgroup of  $G$ . Indeed, consider the defining representation of  $G$  on  $\mathbb{k}^2$ . The orbit of  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  is  $\mathbb{k}^2 \setminus \{0\}$  and has stabiliser equal to  $H_u$ , so that  $G/H_u \cong \mathbb{k}^2 \setminus \{0\}$ . This has a  $G \times H_r$ -equivariant normal affine completion  $\overline{G/H_u}^{\text{aff}} := \mathbb{k}^2$ , containing  $G/H_u$  with codimension 2 complement. By adding a line  $D_\infty$  at infinity, we obtain an equivariant normal projective completion  $\overline{G/H_u} = \mathbb{P}^2$  and then  $\mathcal{O}(D_\infty) = \mathcal{O}_{\mathbb{P}^2}(1)$  has a natural  $G \times H_r$ -linearisation extending the one on  $\mathcal{O} \rightarrow \mathbb{k}^2 \setminus \{0\}$ .

Pulling the previous two observations together, we conclude that when  $H_u$  is a Grosshans subgroup of  $G$  we may find a gentle,  $G \times H_r$ -equivariant projective completion  $\overline{G/H_u}$  of  $G/H_u$  such that the codimension 1 part of the boundary is an effective Cartier divisor  $D_\infty$  corresponding to an ample,  $G \times H_r$ -linearised line bundle  $\mathcal{O}_{\overline{G/H_u}}(1) \rightarrow \overline{G/H_u}$  which restricts to the canonical  $G \times H_r$ -linearisation on  $\mathcal{O}_{G/H_u} \rightarrow G/H_u$ . Given this, and assuming condition (C2), let

$$\beta : G \times^{H_u} X \cong (G/H_u) \times X \hookrightarrow \overline{G/H_u} \times X$$

be the resulting open immersion and let  $L' = \mathcal{O}_{\overline{G/H_u}} \boxtimes \tilde{L} \rightarrow \overline{G/H_u} \times X$  be the  $G \times H_r$ -linearisation required by (C2). Let  $E_1, \dots, E_m$  be prime divisors such that  $D_\infty = \sum_i E_i$ . Then  $\sum_i E_i \times X$  is the Cartier divisor corresponding to the codimension 1 part of the boundary of  $G \times^{H_u} X$  inside  $\overline{G/H_u} \times X$ , so for integers  $N > 0$  (4.3) yields the linearisation  $L'_N = \mathcal{O}(ND_\infty) \boxtimes \tilde{L}$  and we obtain a triple

$$(\overline{G/H_u} \times X, \beta, L'_N = \mathcal{O}(ND_\infty) \boxtimes \tilde{L}) \quad (4.5)$$

such that  $\overline{G/H_u} \times X$  is a gentle completion of  $G \times^{H_u} X$  (assuming  $X$  is normal) and the  $G \times H_r$ -linearisation  $L'_N \rightarrow \overline{G/H_u} \times X$  extends the  $H$ -linearisation  $L \rightarrow X$  under  $\beta \circ \alpha$ . In the case where  $L \rightarrow X$  is ample, we obtain the following corollary to Proposition 4.2.5 and Corollary 4.2.6.

**Corollary 4.2.10.** *Let  $H$  be a connected linear algebraic group acting on a normal, irreducible projective variety  $X$  with ample linearisation  $L \rightarrow X$ . Suppose there is a connected reductive group  $G$  with an  $H_u$ -faithful homomorphism  $H \rightarrow G$  such that the unipotent radical  $H_u$  of  $H$  embeds as a Grosshans subgroup of  $G$  and conditions (C1) and (C2) hold. Then there is a gentle  $G \times H_r$ -equivariant projective completion  $\overline{G/H_u}$  of  $G/H_u$  (where  $H_r = H/H_u$ ) whose boundary divisor  $D_\infty$  is ample, effective and Cartier. Furthermore, given any such completion  $\overline{G/H_u}$  of  $G/H_u$  and associated  $D_\infty$ , for sufficiently divisible  $N > 0$  the triple  $(\overline{G/H_u} \times X, \beta, L'_N =$*

$\mathcal{O}(ND_\infty) \boxtimes \tilde{L}$  of (4.5) is an ample strong reductive envelope for  $H \curvearrowright L \rightarrow X$ , and

$$X \wr H \cong (\overline{G/H_u} \times X) //_{L'_N} (G \times H_r).$$

*Remark 4.2.11.* One can use arguments analogous to those found in the proof of [DorK07, Corollary 5.3.19] to also show that the ring of invariants  $\mathbb{k}[X, L]^H$  is a finitely generated  $\mathbb{k}$ -algebra, and  $X \wr H \cong (\overline{G/H_u} \times X) //_{L'_N} (G \times H_r)$  for sufficiently divisible  $N > 0$ . (Note that [DorK07, Corollary 5.3.19] appears as a corollary to [DorK07, Theorem 5.3.18], whose proof contains an error—see [BeK15, Remark 2.2]. However, since [DorK07, Corollary 5.3.19] includes additional hypotheses of ampleness, its validity is unaffected by this error.)

*Remark 4.2.12.* Because the various intrinsic notions of conventional reductive GIT and non-reductive GIT may be defined for rational linearisations, one can work with rational linearisations in the setting of reductive envelopes. For example, in Corollary 4.2.10 if one assumes  $H \curvearrowright L \rightarrow X$  and  $\tilde{L} \rightarrow X$  are rational linearisations satisfying the natural rational versions of (C1) and (C2), then for  $N \gg 0$  some positive integral multiple of the rational linearisation  $L'_N = \mathcal{O}(ND_\infty) \boxtimes \tilde{L} \rightarrow \overline{G/H_u} \times X$  will define a strong reductive envelope for the corresponding multiple of  $L \rightarrow X$ . The stable locus, finitely generated semistable locus and enveloping quotient for  $H \curvearrowright L \rightarrow X$  can thus still be computed using the rational linearisation  $L'_N$ , which is often more convenient to work with in computations.

Corollary 4.2.10 is a useful result that has the potential to be applied to the study of a number of interesting examples of non-reductive group actions. We will make use of it during the extended example in the upcoming Section 5.

## 5 An Example: $n$ Unordered Points on $\mathbb{P}^1$

In this section we undertake a detailed study of an example to demonstrate the use of strong reductive envelopes for computing the semistable and stable loci and constructing projective completions of the enveloped quotient, in a non-reductive GIT set-up. We will follow this in the final section with an outline of ongoing research applying non-reductive GIT to study moduli spaces occurring naturally in algebraic geometry.

For our detailed study we consider the space of  $n$  unordered points on  $\mathbb{P}^1$  up to compositions of translations and dilations (that is, under the action of the standard Borel subgroup of  $\mathrm{SL}(2, \mathbb{k})$ ). This extends Example 2.3.19 from [DorK07, §6], which only looked at the translation actions. Including the dilations gives a somewhat richer picture, due to the possibility of variation of linearisations and the associated birational transformations on the quotients (as we shall explore in Section 5.3).

We first fix some notation. Let  $G := \mathrm{SL}(2, \mathbb{k})$  and consider its action on  $\mathbb{P}^1$  via Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbb{P}^1 \rightarrow \mathbb{P}^1, \quad [z_0 : z_1] \mapsto [az_0 + bz_1 : cz_0 + dz_1], \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$$

Fix an integer  $n > 0$ . Then  $G$  acts dually on the complete linear system associated to  $\mathcal{O}_{\mathbb{P}^1}(n)$ , which we identify as  $X := \mathbb{P}(V)$  with  $V := \mathbb{k}[x, y]_n$  the vector space of homogeneous polynomials

of degree  $n$  in two indeterminates  $x$  and  $y$ . We write points of  $X$  as  $[\sigma(x, y)]$  with  $\sigma(x, y) \in V$ ; note that  $[\sigma(x, y)]$  defines an effective divisor of zeros on  $\mathbb{P}^1$ , which can be thought of as a collection of  $n$  unordered points on  $\mathbb{P}^1$  with multiplicities.

We consider the action of the subgroup

$$H := \left\{ \begin{pmatrix} t & a \\ 0 & t^{-1} \end{pmatrix} \in G \mid t \in \mathbb{k}^\times, a \in \mathbb{k} \right\}$$

on  $X$ . Geometrically  $H$  corresponds to the Möbius transformations on  $\mathbb{P}^1$  that are compositions of scalings and translations, all of which fix  $\infty = [1 : 0] \in \mathbb{P}^1$ . The unipotent radical of  $H$  is

$$H_u = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{k} \right\} \cong \mathbb{k}^+$$

while the quotient  $H_r = H/H_u$  is isomorphic to the torus

$$T := \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t \in \mathbb{k}^\times \right\} \cong \mathbb{k}^\times$$

via the composition  $T \hookrightarrow H \rightarrow H_r$ . We will typically identify  $H_r$  with  $T$  in this way throughout this section.

We next consider the possible linearisations  $L$  of  $H$  over  $X$ . As line bundles, any linearisation is isomorphic to  $\mathcal{O}_X(m)$  for  $m \in \mathbb{Z}$  [Har77, Corollary 6.17]. The action of  $H$  on  $V$  defines, for each  $m \in \mathbb{Z}$ , a canonical linearisation  $\mathcal{O}_X(m)^{\text{can}} \rightarrow X$  on the line bundle  $\mathcal{O}_X(m)$ . Because  $X$  is irreducible and proper over  $\mathbb{k}$ , any other linearisation on  $\mathcal{O}_X(m)$  is obtained by twisting the canonical linearisation by a character of  $H$  [Dol03, Corollary 7.1]. We take a moment to recall our conventions here: if  $\chi : H \rightarrow \mathbb{k}^\times$  is a character of  $H$ , then let  $\mathcal{O}_X^{(\chi)}$  denote the linearisation on the trivial bundle  $\mathcal{O}_X = X \times \mathbb{k}$  defined by the character  $\chi^{-1}$ :

$$H \times X \times \mathbb{k} \rightarrow X \times \mathbb{k}, \quad (h, x, z) \mapsto (hx, \chi^{-1}(h)z).$$

Then any other linearisation on  $\mathcal{O}_X(m)$  is of the form  $\mathcal{O}_X(m)^{(\chi)} := \mathcal{O}_X(m)^{\text{can}} \otimes \mathcal{O}_X^{(\chi)}$ . Because unipotent groups have no nontrivial characters, the inclusion  $T \hookrightarrow H$  induces an identification between the groups of characters of  $H$  and  $T$ , so that  $\chi$  is of the form

$$\chi : \begin{pmatrix} t & a \\ 0 & t^{-1} \end{pmatrix} \mapsto t^r, \quad \begin{pmatrix} t & a \\ 0 & t^{-1} \end{pmatrix} \in H$$

for some weight  $r \in \mathbb{Z}$ . For each  $m, r \in \mathbb{Z}$  we therefore define linearisations

$$L_{m,r} := \mathcal{O}_X(m)^{(\chi)}, \quad \text{where } \chi \text{ has weight } r \in \mathbb{Z}.$$

of  $H$  over  $X$ . It is these linearisations that we will study in this example.

When  $m \leq 0$  the sets  $X^{s(L_{m,r})}$ ,  $X^{\text{nss}(L_{m,r})}$ ,  $X^{\text{ss,fg}(L_{m,r})}$ ,  $X^{\text{ss},H_u\text{-fg}(L_{m,r})}$  and the enveloping quotient  $X \mathbin{\mathcal{R}}_{L_{m,r}} H$  for the linearisations  $L_{m,r}$  can be easily described by inspection. Indeed, if  $m < 0$  then  $\mathcal{O}_X(m)$  has no nonzero global sections as a line bundle, so that the stable and naively



semistable locus, and enveloping quotient, are all empty. On the other hand, if  $m = 0$ , then the ring of invariants  $\mathbb{k}[X, L_{0,r}]^H$  is isomorphic to  $\text{Sym}^\bullet \mathbb{k}$  if  $r = 0$ , and  $\mathbb{k}$  (in degree 0) otherwise. Thus when  $r = 0$ , we have  $X^{s(L_{0,0})} = \emptyset$  and  $X^{\text{ss}, H_u - \text{fg}(L_{0,0})} = X^{\text{ss}, \text{fg}(L_{0,0})} = X^{\text{nss}(L_{0,0})} = X$ , while  $X \wr_{L_{0,0}} H = \text{pt}$ ; on the other hand, if  $r \neq 0$  then all of  $X^{s(L_{0,r})}$ ,  $X^{\text{ss}, H_u - \text{fg}(L_{0,r})}$ ,  $X^{\text{ss}, \text{fg}(L_{0,r})}$ ,  $X^{\text{nss}(L_{0,r})}$  and  $X \wr_{L_{0,r}} H$  are empty.

In what follows we therefore consider the linearisations  $L_{m,r}$  with  $m > 0$ . In the next section we shall use the methods of Corollary 4.2.10 of Section 4.2.1 to construct strong ample reductive envelopes for these linearisations, which will allow us to compute the stable locus  $X^{s(L_{m,r})}$ , the various semistable loci  $X^{\text{ss}, H_u - \text{fg}(L_{m,r})}$ ,  $X^{\text{ss}, \text{fg}(L_{m,r})}$ ,  $X^{\text{nss}(L_{m,r})}$  (which will all be equal) and a projective completion of the enveloped quotient; cf. Proposition 4.2.2 and Theorem 4.1.14.

## 5.1 The Strong Reductive Envelopes

Fix  $m, r \in \mathbb{Z}$ , with  $m > 0$ , and let  $\chi : H \rightarrow \mathbb{k}^\times$  be the character of  $H$  of weight  $r$ . Consider the inclusion  $H \hookrightarrow G$ , which is clearly an  $H_u$ -faithful homomorphism. By construction, the restricted linearisation of the unipotent radical  $H_u \curvearrowright L_{m,r} \rightarrow X$  extends to a linearisation of  $G = \text{SL}(2, \mathbb{k})$ , and we are in the setting of Example 4.2.7. Furthermore, as we saw in Example 4.2.9,  $H_u$  is a Grosshans subgroup of  $G$ , and  $G/H_u \cong \mathbb{k}^2 \setminus \{0\}$  via the defining representation of  $G$  on  $\mathbb{k}^2$ . This has a  $G \times H_r$ -equivariant normal affine completion  $\overline{G/H_u}^{\text{aff}} := \mathbb{k}^2$ , containing  $G/H_u$  with codimension 2 complement.

*Remark 5.1.1.* Because the restricted linearisation  $H_u \curvearrowright L_{m,r} \rightarrow X$  extends to one of  $G = \text{SL}(2, \mathbb{k})$  and  $H_u$  is a Grosshans subgroup of  $G$ , by Remark 4.2.8 we know that  $\mathbb{k}[X, L_{m,r}]^{H_u}$  and  $\mathbb{k}[X, L_{m,r}]^H$  are both finitely generated  $\mathbb{k}$ -algebras. Therefore  $X \wr_{L_{m,r}} H = \text{Proj}(\mathbb{k}[X, L_{m,r}]^H)$  is a projective variety and  $X^{\text{ss}, H_u - \text{fg}(L_{m,r})} = X^{\text{ss}, \text{fg}(L_{m,r})} = X^{\text{nss}(L_{m,r})}$ .

In what follows we regard elements of  $\mathbb{k}^3$  as column vectors. As in Example 4.2.9, by adding a hyperplane at infinity we obtain a normal (in fact, smooth)  $G \times H_r$ -equivariant projective completion  $\mathbb{P}^2$  of  $\overline{G/H_u}^{\text{aff}}$ . Here we write  $\mathbb{P}^2 = \{[v_0 : v_1 : v_2] \mid 0 \neq (v_0, v_1, v_2)^t \in \mathbb{k}^3\}$  with the hyperplane at infinity defined by  $v_0 = 0$ . The action of  $G \times H_r = G \times T$  on  $\mathbb{P}^2 = \mathbb{P}(\mathbb{k}^3)$  is the one induced by the representation given in block form

$$G \times T \rightarrow \text{GL}(3, \mathbb{k}), \quad (g, \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}) \mapsto \left( \begin{array}{c|c} 1 & 0 \\ \hline 0 & g \begin{pmatrix} t^{-1} & 0 \\ 0 & t^{-1} \end{pmatrix} \end{array} \right)$$

where  $\text{GL}(3, \mathbb{k})$  acts on  $\mathbb{k}^3$  by left multiplication. For each  $N > 0$  this representation canonically defines a  $G \times H_r$ -linearisation  $\mathcal{O}_{\mathbb{P}^2}(N) \rightarrow \mathbb{P}^2$  which restricts to the canonical linearisation on  $\mathcal{O}_{G/H_u} \rightarrow G/H_u$ .

Let  $\beta : G \times^{H_u} X \cong (\mathbb{k}^2 \setminus \{0\}) \times X \hookrightarrow \mathbb{P}^2 \times X$  be the naturally induced open immersion and for integers  $N > 0$  let

$$L'_{m,r,N} := \mathcal{O}_{\mathbb{P}^2}(N) \boxtimes L_{m,r} \rightarrow \mathbb{P}^2 \times X,$$

equipped with its natural  $G \times H_r$ -linearisation. By Corollary 4.2.10, for  $N > 0$  sufficiently divisible depending on  $m$  and  $r$ , the triple

$$(\mathbb{P}^2 \times X, \beta, L'_{m,r,N})$$

defines a strong ample reductive envelope for  $H \curvearrowright L_{m,r} \rightarrow X$ , and

$$X \curvearrowright_{L_{m,r}} H = (\mathbb{P}^2 \times X) //_{L'_{m,r,N}} (G \times H_r).$$

The stable locus  $X^{s(L_{m,r})}$  and finitely generated semistable locus  $X^{\text{ss,fg}(L_{m,r})}$  for the linearisation  $L_{m,r}$  may therefore be computed as the completely stable and completely semistable loci, respectively, associated to the  $G \times H_r$ -linearisation  $L'_{m,r,N}$  by Proposition 4.2.2. We therefore next compute the semistable and stable loci for  $L'_{m,r,N} \rightarrow \mathbb{P}^2 \times X$ .

## 5.2 Semistability and Stability for the $G \times H_r$ -Linearisations $L'_{m,r,N}$

In order to compute semistability and stability for the  $G \times H_r$ -linearisations  $L'_{m,r,N} \rightarrow \mathbb{P}^2 \times X$  we will use the Hilbert-Mumford criteria as stated in Theorem 2.2.6. To do this we use the maximal torus  $T_1 \times T_2 \subseteq G \times H_r$  where  $T_1$  is the maximal torus  $T$  of  $G = \text{SL}(2, \mathbb{k})$  and  $T_2 := H_r$ , also identified with  $T$ . The group of characters of  $T_1 \times T_2$  is then identified with  $\mathbb{Z} \times \mathbb{Z}$  in the natural way.

The set of fixed points for the action  $T_1 \times T_2$ -action on  $\mathbb{P}^2 \times X$  is

$$\{([1 : 0 : 0], [x^{n-i}y^i]), ([0 : 1 : 0], [x^{n-i}y^i]), ([0 : 0 : 1], [x^{n-i}y^i]) \mid i = 0, \dots, n\}.$$

Table 1 gives the weights for each fixed point with respect to the linearisation  $L'_{m,r,N}$  and a general plot of these weights is given in Figure 1.

Fixed point ( $i = 0, \dots, n$ )	Weight in $\text{Hom}(T_1 \times T_2, \mathbb{k}^\times) = \mathbb{Z} \times \mathbb{Z}$
$([1 : 0 : 0], [x^{n-i}y^i])$	$(m(2i - n), r)$
$([0 : 1 : 0], [x^{n-i}y^i])$	$(N + m(2i - n), -N + r)$
$([0 : 0 : 1], [x^{n-i}y^i])$	$(-N + m(2i - n), -N + r)$

Table 1: Weights of the fixed points of  $T_1 \times T_2 \curvearrowright \mathbb{P}^2 \times X$  with respect to the linearisation  $L'_{m,r,N}$ .

By the Hilbert-Mumford criteria Theorem 2.2.6, 2, a point  $p = ([v_0 : v_1 : v_2], [\sigma(x, y)]) \in \mathbb{P}^2 \times X$  is semistable (respectively, stable) for the restricted linearisation  $T_1 \times T_2 \curvearrowright L'_{m,r,N} \rightarrow \mathbb{P}^2 \times X$  if, and only if, the origin of  $\mathbb{R} \otimes_{\mathbb{Z}} (\mathbb{Z} \times \mathbb{Z}) = \mathbb{R} \times \mathbb{R}$  is contained in the weight polytope  $\Delta_p \subseteq \mathbb{R} \times \mathbb{R}$  (respectively, the interior  $\Delta_p^\circ$ ) associated to  $p$ . When  $N$  is taken to be very large with respect to  $m$  and  $r$ , we find that  $p$  can only be  $T_1 \times T_2$ -(semi)stable by satisfying one of the following criteria, split into four cases:

**Case**  $v_0 v_1 v_2 \neq 0$ :

$$\begin{aligned} 0 \in \Delta_p &\iff [1 : 0] \text{ and } [0 : 1] \text{ are both zeros of } \sigma(x, y) \\ &\quad \text{with multiplicity} \leq (n + \frac{r}{m})/2 \\ 0 \in \Delta_p^\circ &\iff [1 : 0] \text{ and } [0 : 1] \text{ are both zeros of } \sigma(x, y) \\ &\quad \text{with multiplicity} < (n + \frac{r}{m})/2 \end{aligned}$$

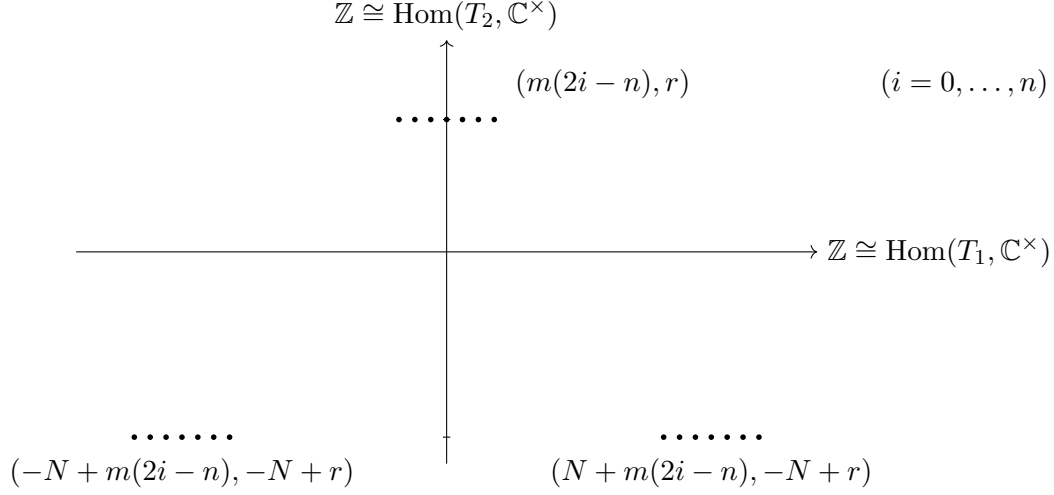


Figure 1: The weight diagram for  $T_1 \times T_2 \hookrightarrow L'_{m,r,N} \rightarrow \mathbb{P}^2 \times X$ .

**Case  $v_0 v_1 \neq 0, v_2 = 0$ :**

$$0 \in \Delta_p \iff \begin{array}{l} [1 : 0] \text{ is a zero of } \sigma(x, y) \text{ with multiplicity } \leq (n - \frac{r}{m})/2 \text{ and} \\ [0 : 1] \text{ is a zero of } \sigma(x, y) \text{ with multiplicity } \leq (n + \frac{r}{m})/2 \end{array}$$

$$0 \in \Delta_p^\circ \iff \begin{array}{l} [1 : 0] \text{ is a zero of } \sigma(x, y) \text{ with multiplicity } < (n - \frac{r}{m})/2 \text{ and} \\ [0 : 1] \text{ is a zero of } \sigma(x, y) \text{ with multiplicity } < (n + \frac{r}{m})/2 \end{array}$$

**Case  $v_0 v_2 \neq 0, v_1 = 0$ :**

$$0 \in \Delta_p \iff \begin{array}{l} [1 : 0] \text{ is a zero of } \sigma(x, y) \text{ with multiplicity } \leq (n + \frac{r}{m})/2 \text{ and} \\ [0 : 1] \text{ is a zero of } \sigma(x, y) \text{ with multiplicity } \leq (n - \frac{r}{m})/2 \end{array}$$

$$0 \in \Delta_p^\circ \iff \begin{array}{l} [1 : 0] \text{ is a zero of } \sigma(x, y) \text{ with multiplicity } < (n + \frac{r}{m})/2 \text{ and} \\ [0 : 1] \text{ is a zero of } \sigma(x, y) \text{ with multiplicity } < (n - \frac{r}{m})/2 \end{array}$$

**Case  $v_0 \neq 0, v_1 = v_2 = 0$  and  $r = 0$ :**

$$0 \in \Delta_p \iff \begin{array}{l} [1 : 0] \text{ and } [0 : 1] \text{ are both zeros of } \sigma(x, y) \\ \text{with multiplicity } \leq n/2 \end{array}$$

$$0 \in \Delta_p^\circ \iff \begin{array}{l} [1 : 0] \text{ and } [0 : 1] \text{ are both zeros of } \sigma(x, y) \\ \text{with multiplicity } < n/2 \end{array}$$

By Theorem 2.2.6, 1, the point  $p$  is (semi)stable for the whole  $G \times H_r$ -linearisation if, and only if,  $(g, \bar{h}) \cdot p$  is  $T_1 \times T_2$ -(semi)stable for each  $(g, \bar{h}) \in G \times H_r$ . Using this, one deduces the following:

**Proposition 5.2.1.** *Let  $m > 0$  and  $r$  be integers. Then for sufficiently large  $N > 0$  (depending on  $m$  and  $r$ ) the semistable and stable loci for the  $G \times H_r$ -linearisations  $L'_{m,r,N} \rightarrow \mathbb{P}^2 \times X$  are as follows:*

**(Case  $r < 0$  or  $\frac{r}{m} > n$ ):** Then  $(\mathbb{P}^2 \times X)^{s(L'_{m,r,N})} = (\mathbb{P}^2 \times X)^{ss(L'_{m,r,N})} = \emptyset$ .

**(Case  $r = 0$ ):** Then  $(\mathbb{P}^2 \times X)^{s(L'_{m,r,N})} = \emptyset$  and

$$(\mathbb{P}^2 \times X)^{ss(L'_{m,r,N})} = \left\{ ([1 : v_1 : v_2], [\sigma(x, y)]) \mid \begin{array}{l} \sigma(x, y) \text{ has no zeros} \\ \text{of multiplicity} > \frac{n}{2} \end{array} \right\}.$$

**(Case  $0 < \frac{r}{m} < n$ ):** Then

$$(\mathbb{P}^2 \times X)^{s(L'_{m,r,N})} = \left\{ ([1 : v_1 : v_2], [\sigma(x, y)]) \mid \begin{array}{l} (v_1, v_2) \neq (0, 0), [v_1 : v_2] \text{ is a zero of} \\ \sigma(x, y) \text{ of multiplicity} < (n - \frac{r}{m})/2 \\ \text{and all zeros of } \sigma(x, y) \text{ have} \\ \text{multiplicity} < (n + \frac{r}{m})/2 \end{array} \right\},$$

$$(\mathbb{P}^2 \times X)^{ss(L'_{m,r,N})} = \left\{ ([1 : v_1 : v_2], [\sigma(x, y)]) \mid \begin{array}{l} (v_1, v_2) \neq (0, 0), [v_1 : v_2] \text{ is a zero of} \\ \sigma(x, y) \text{ of multiplicity} \leq (n - \frac{r}{m})/2 \\ \text{and all zeros of } \sigma(x, y) \text{ have} \\ \text{multiplicity} \leq (n + \frac{r}{m})/2 \end{array} \right\}.$$

**(Case  $\frac{r}{m} = n$ ):** Then  $(\mathbb{P}^2 \times X)^{s(L'_{m,r,N})} = \emptyset$  and

$$(\mathbb{P}^2 \times X)^{ss(L'_{m,r,N})} = \left\{ ([1 : v_1 : v_2], [\sigma(x, y)]) \mid \begin{array}{l} [v_1 : v_2] \text{ is not a} \\ \text{zero of } \sigma(x, y) \end{array} \right\}.$$

For fixed  $m, r, N$  the completely semistable and completely stable loci are by Definition 4.1.9 the intersections of the semistable and stable loci for  $G \times H_r \curvearrowright L'_{m,r,N} \rightarrow \mathbb{P}^2 \times X$  under the inclusion

$$\beta \circ \alpha : X \hookrightarrow \mathbb{P}^2 \times X, \quad [\sigma(x, y)] \mapsto ([1 : 1 : 0], [\sigma(x, y)]).$$

Using Corollary 4.2.10, we therefore deduce

**Corollary 5.2.2.** *For integers  $m > 0$  and  $r$ , the semistable and stable loci for the linearisations  $H \curvearrowright L_{m,r} \rightarrow X$  are as follows:*

**(Case  $r < 0$  or  $\frac{r}{m} > n$ ):** Then  $X^{s(L_{m,r})} = X^{ss,fg(L_{m,r})} = \emptyset$ .

**(Case  $r = 0$ ):** Then  $X^{s(L_{m,r})} = \emptyset$  and

$$X^{ss,fg(L_{m,r})} = \{[\sigma(x, y)] \in X \mid \sigma(x, y) \text{ has no zeros of multiplicity} > n/2\}$$

(Case  $0 < \frac{r}{m} < n$ ): Then

$$X^{s(L_{m,r})} = \left\{ [\sigma(x, y)] \in X \mid \begin{array}{l} [1 : 0] \text{ is a zero of } \sigma(x, y) \text{ with multiplicity} \\ < (n - \frac{r}{m})/2 \text{ and all other zeros} \\ \text{of } \sigma(x, y) \text{ have multiplicity } < (n + \frac{r}{m})/2 \end{array} \right\},$$

$$X^{\text{ss,fg}(L_{m,r})} = \left\{ [\sigma(x, y)] \in X \mid \begin{array}{l} [1 : 0] \text{ is a zero of } \sigma(x, y) \text{ with multiplicity} \\ \leq (n - \frac{r}{m})/2 \text{ and all other zeros} \\ \text{of } \sigma(x, y) \text{ have multiplicity } \leq (n + \frac{r}{m})/2 \end{array} \right\}.$$

(Case  $\frac{r}{m} = n$ ): Then  $X^{s(L_{m,r})} = \emptyset$  and

$$X^{\text{ss,fg}(L_{m,r})} = \{[\sigma(x, y)] \in X \mid [1 : 0] \text{ is not a zero of } \sigma(x, y)\}.$$

*Remark 5.2.3.* Notice that in each case  $x \in X^{s(L_{m,r})}$  (respectively  $x \in X^{\text{ss,fg}(L_{m,r})}$ ) if and only if  $x$  is stable (respectively semistable) for every one-parameter subgroup  $\lambda : \mathbb{G}_m \rightarrow H$ , or equivalently if and only if  $hx$  is stable (respectively semistable) with respect to the action of the standard maximal torus of  $\text{SL}(2, \mathbb{k})$  for every  $h \in H$ . Thus the analogues of the Hilbert–Mumford criteria for reductive GIT hold in these examples for the action of the non-reductive group  $H$ ; they fail, of course, for the action of its unipotent radical  $H_u$ , since there are no one-parameter subgroups  $\lambda : \mathbb{G}_m \rightarrow H_u$ .

When  $r \neq 0$  then, again just as for reductive GIT but in contrast to the unipotent case, the quotient morphism  $q_{m,r} : X^{\text{ss,fg}(L_{m,r})} \rightarrow X \mathbin{\mathbb{H}}_{L_{m,r}} H$  from the semistable locus to the enveloping quotient is surjective, and the projective variety  $X \mathbin{\mathbb{H}}_{L_{m,r}} H$  is a categorical quotient of the semistable locus, with  $q_{m,r}(x) = q_{m,r}(y)$  for  $x, y \in X^{\text{ss,fg}(L_{m,r})}$  if and only if the closures of the  $H$ -orbits of  $x$  and  $y$  meet in  $X^{\text{ss,fg}(L_{m,r})}$ .

### 5.3 Variation of the Enveloping Quotients

We conclude this example by studying how the enveloping quotients  $X^{\text{ss,fg}(L_{m,r})} \rightarrow X \mathbin{\mathbb{H}}_{L_{m,r}} H$  and geometric quotients  $X^{s(L_{m,r})} \rightarrow X^{s(L_{m,r})}/H$  therein change as we range over the different possible ample linearisations  $L_{m,r} \rightarrow X$ . This can be done by examining the variation of the reductive GIT quotients (in the sense of the VGIT of [T96, DolH98, Re00]) of the linearisations  $L'_{m,r,N} \rightarrow \mathbb{P}^2 \times X$ . In order to keep the exposition brief we suppress the details of the VGIT analysis on the reductive envelopes and instead concentrate on the consequences for the  $H$ -linearisations, referencing relevant VGIT results from [T96].

From now on we assume  $N$  is sufficiently divisible with respect to  $m$  and  $r$  so as to satisfy the conclusions of Corollary 4.2.10 and Proposition 5.2.1. Note that two linearisations  $L_{m,r}$  and  $L_{m',r'}$  will have the same stable locus, finitely generated semistable locus and enveloping quotient if  $\frac{r}{m} = \frac{r'}{m'}$  (see Remarks 2.1.20 and 3.4.3). By inspecting Corollary 5.2.2 we see that the changes in stability and finitely generated semistability occur when  $\frac{r}{m} = 0$ , or  $n - \frac{r}{m} \in 2\mathbb{Z}$ , or  $\frac{r}{m} = n$ , and clearly we only need consider the cases where  $\frac{r}{m} \in \mathbb{Q} \cap [0, n]$ . It makes sense therefore to consider four cases: (1) when  $r = 0$ ; (2) when  $0 < \frac{r}{m} < n$  and  $n - \frac{r}{m} \in \mathbb{Q} \setminus 2\mathbb{Z}$ ; (3) when  $0 < \frac{r}{m} < n$  and  $n - \frac{r}{m} \in 2\mathbb{Z}$ ; and (4) when  $\frac{r}{m} = n$ .

### 5.3.1 Case $r = 0$

Observe from Corollary 5.2.2 that  $X^{\text{ss,fg}(L_{m,0})}$  is precisely the semistable locus for the canonical  $G = \text{SL}(2, \mathbb{k})$ -linearisation on  $\mathcal{O}_X(1) \rightarrow X = \mathbb{P}(V)$  [Dol03, §10.2] (which is the classical reductive GIT problem of configurations of  $n$  unordered points on  $\mathbb{P}^1$  up to Möbius transformations). Notice also that there is a  $G$ -equivariant retraction

$$(\mathbb{P}^2 \times X)^{\text{ss}(L'_{m,0,N})} = \mathbb{k}^2 \times X^{\text{ss,fg}(L_{m,0})} \rightarrow \{0\} \times X^{\text{ss,fg}(L_{m,0})} \cong X^{\text{ss,fg}(L_{m,0})}$$

(where  $\mathbb{k}^2 \subseteq \mathbb{P}^2$  corresponds to the gentle affine completion  $\overline{G/H_u}^{\text{aff}}$  of  $G/H_u$ ), defined by taking a limit along the flow of  $t \in T_2 \cong \mathbb{k}^\times$  as  $t \rightarrow \infty$ . Thus two points  $[\sigma(x, y)], [\tilde{\sigma}(x, y)]$  from  $X^{\text{ss,fg}(L_{m,0})}$  get identified in  $X \mathbin{\mathbb{Z}}_{L_{m,0}} H = (\mathbb{P}^2 \times X) //_{L'_{m,0,N}} (G \times H_r)$  if, and only if,  $[\sigma(x, y)]$  and  $[\tilde{\sigma}(x, y)]$  are  $S$ -equivalent for the standard action of  $G$  on  $X$  (see Section 2.2). Writing  $X // G$  for the GIT quotient of the canonical linearisation  $G \curvearrowright \mathcal{O}_X(1) \rightarrow X$ , it follows that the inclusion  $X^{\text{ss,fg}(L_{m,0})} \hookrightarrow (\mathbb{P}^2 \times X)^{\text{ss}(L'_{m,0,N})}$  induces an isomorphism

$$X \mathbin{\mathbb{Z}}_{L_{m,0}} H \cong X // G.$$

In particular, we see that the dimension of  $X \mathbin{\mathbb{Z}}_{L_{m,0}} H$  is one less than the anticipated dimension.

### 5.3.2 Case $0 < \frac{r}{m} < n$ and $n - \frac{r}{m} \in \mathbb{Q} \setminus 2\mathbb{Z}$

In this case we see from Proposition 5.2.1 and Corollary 5.2.2 that  $(\mathbb{P}^2 \times X)^{\text{s}(L'_{m,r,N})} = (\mathbb{P}^2 \times X)^{\text{ss}(L'_{m,r,N})}$  and  $X^{\text{ss,fg}(L_{m,r})} = X^{\text{s}(L_{m,r})}$ . Moreover, by inspection

$$(\mathbb{P}^2 \times X)^{\text{ss}(L'_{m,r,N})} \subseteq (\mathbb{k}^2 \setminus \{0\}) \times X \cong G \times^{H_u} X,$$

thus  $(\mathbb{P}^2 \times X)^{\text{ss}(L'_{m,r,N})} \cong G \times^{H_u} (X^{\text{ss,fg}(L_{m,r})})$  and  $(\mathbb{P}^2 \times X)^{\text{s}(L'_{m,r,N})} \cong G \times^{H_u} (X^{\text{s}(L_{m,r})})$ . From Corollary 4.2.10 we thus have

$$X \mathbin{\mathbb{Z}}_{L_{m,r}} H = (\mathbb{P}^2 \times X) //_{L'_{m,r,N}} (G \times H_r) = (\mathbb{P}^2 \times X)^{\text{s}(L'_{m,r,N})} / (G \times H_r) = X^{\text{s}(L_{m,r})} / H.$$

In particular, the enveloping quotient map  $X^{\text{ss,fg}(L_{m,r})} \rightarrow X \mathbin{\mathbb{Z}}_{L_{m,r}} H$  is a geometric quotient of  $X^{\text{ss,fg}(L_{m,r})}$  and the enveloped quotient is equal to the enveloping quotient, which itself is the canonical choice of inner enveloping quotient (Definition 3.1.12). Indeed, the quotient of  $X^{\text{ss,fg}(L_{m,r})}$  by  $H$  is even projective. So in this case we obtain the best possible geometric picture we could hope for.

### 5.3.3 Case $0 < \frac{r}{m} < n$ and $n - \frac{r}{m} \in 2\mathbb{Z}$

Now  $\frac{r}{m}$  lies on a ‘wall’ (in the sense of Thaddeus [T96, Theorem 2.3]) and  $X^{\text{s}(L_{m,r})}$  is a proper subset of  $X^{\text{ss,fg}(L_{m,r})}$ . As in the above case  $n - \frac{r}{m} \notin 2\mathbb{Z}$  we still have

$$(\mathbb{P}^2 \times X)^{\text{ss}(L'_{m,r,N})} \cong G \times^{H_u} (X^{\text{ss,fg}(L_{m,r})}), \quad (\mathbb{P}^2 \times X)^{\text{s}(L'_{m,r,N})} \cong G \times^{H_u} (X^{\text{s}(L_{m,r})})$$

and  $X \wr_{L_{m,r}} H = (\mathbb{P}^2 \times X) //_{L'_{m,r,N}} (G \times H_r)$ . In particular,  $X^{s(L_{m,r})}/H \cong (\mathbb{P}^2 \times X)^{s(L'_{m,r,N})}/(G \times H_r)$  and the enveloping quotient map  $q : X^{\text{ss,fg}(L_{m,r})} \rightarrow X \wr_{L_{m,r}} H$  is surjective by Theorem 4.1.14 (which means that again the notions of enveloped quotient, inner enveloping quotient and enveloping quotient all coincide in this case). We next compute the complement of  $X^{s(L_{m,r})}/H$  inside  $X \wr_{L_{m,r}} H$ . Let  $[\sigma(x, y)] \in X^{\text{ss,fg}(L_{m,r})} \setminus X^{s(L_{m,r})}$ . We claim that

$$[x^{(n+\frac{r}{m})/2} y^{(n-\frac{r}{m})/2}] \in \overline{H \cdot [\sigma(x, y)]} \cap X^{\text{ss,fg}(L_{m,r})} \quad (\text{closure taken in } X).$$

Indeed, by inspection of Corollary 5.2.2 either (1)  $\sigma(x, y)$  has  $[1 : 0]$  as a root of multiplicity  $(n - \frac{r}{m})/2$ ; or (2)  $\sigma(x, y)$  has a root  $[u_1 : u_2] \neq [1 : 0]$  of multiplicity  $(n + \frac{r}{m})/2$ . In the first case, the limit of any zero of  $\sigma(x, y)$  different to  $[1 : 0]$  under the flow of  $\begin{pmatrix} t_1 & 0 \\ 0 & t_1^{-1} \end{pmatrix} \in T_1 \subseteq G$  as  $t_1 \rightarrow 0$  is equal to  $[0 : 1]$ , so  $[x^{(n+\frac{r}{m})/2} y^{(n-\frac{r}{m})/2}] \in \overline{H \cdot [\sigma(x, y)]} \cap X^{\text{ss,fg}(L_{m,r})}$ . In the second case, there is  $h \in H_u \cong \mathbb{k}^+$  taking  $[u_1 : u_2]$  to  $[0 : 1]$ , and any other zero of  $\sigma(x, y)$  is taken to a point of the form  $[v_1 : v_2]$  with  $v_1 \neq 0$ . Any such  $[v_1 : v_2]$  flows to  $[1 : 0]$  under  $\begin{pmatrix} t_1 & 0 \\ 0 & t_1^{-1} \end{pmatrix}$  as  $t_1 \rightarrow \infty$ , so that  $\overline{H \cdot [\sigma(x, y)]} \cap X^{\text{ss,fg}(L_{m,r})}$  also contains  $[x^{(n+\frac{r}{m})/2} y^{(n-\frac{r}{m})/2}]$ . This proves our claim.

It follows that any two points of  $X^{\text{ss,fg}(L_{m,r})} \setminus X^{s(L_{m,r})}$  are S-equivalent inside  $(\mathbb{P}^2 \times X)^{s(L'_{m,r,N})}$  and so

$$X \wr_{L_{m,r}} H = (X^{s(L_{m,r})}/H) \amalg \text{pt},$$

where pt is the image of  $X^{\text{ss,fg}(L_{m,r})} \setminus X^{s(L_{m,r})}$  under the enveloping quotient  $q : X^{\text{ss,fg}(L_{m,r})} \rightarrow X \wr_{L_{m,r}} H$ . Note that multiple orbits get collapsed to pt, so the enveloping quotient fails to be a geometric quotient.

#### 5.3.4 Case $\frac{r}{m} = n$

In this case  $X^{s(L_{m,r})} = \emptyset$ , while for  $X^{\text{ss,fg}(L_{m,r})}$  a similar analysis to the above case  $0 < \frac{r}{m} < n$  and  $n - \frac{r}{m} \in 2\mathbb{Z}$  holds: again we have  $(\mathbb{P}^2 \times X)^{s(L'_{m,r,N})} \cong G \times^{H_u} (X^{\text{ss,fg}(L_{m,r})})$ , and any  $[\sigma(x, y)] \in X^{\text{ss}(L_{m,r})}$  has limit point equal to  $[x^n]$  inside  $X^{\text{ss}(L_{m,r})}$  under the action of  $T_1$ . So we see that any two points in  $X^{\text{ss,fg}(L_{m,r})}$  are S-equivalent inside  $(\mathbb{P}^2 \times X)^{s(L'_{m,r,N})}$ . Since  $X \wr_{L_{m,r}} H = (\mathbb{P}^2 \times X) //_{L'_{m,r,N}} (G \times H_r)$ , we deduce that

$$X \wr_{L_{m,r}} H = \text{pt}.$$

#### 5.3.5 Birational Transformations of $X^{s(L_{m,r})}/H$

Finally, we examine how the geometric quotients  $X^{s(L_{m,r})}/H$  transform birationally as  $\frac{r}{m}$  crosses the ‘walls’ of integers congruent to  $n$  modulo 2 between 0 and  $n$ , or else equal to 0 or  $n$ .

First consider the case where  $\frac{r}{m} \in \mathbb{Z} \cap (0, n)$  and  $\frac{r}{m} \equiv n \pmod{2}$ , with  $n \geq 3$ . Let  $0 < \epsilon < 1$  be a small rational number and let  $L_{m,r}^+ \rightarrow X$  and  $L_{m,r}^- \rightarrow X$  be the perturbed  $H$ -linearisations, corresponding to the rational numbers  $\frac{r}{m} + \epsilon$  and  $\frac{r}{m} - \epsilon$ , respectively. By inspecting Corollary 5.2.2 we see there are inclusions

$$X^{s(L_{m,r}^-)} \subseteq X^{\text{ss,fg}(L_{m,r})} \supseteq X^{s(L_{m,r}^+)}, \quad X^{s(L_{m,r}^-)} \supseteq X^{s(L_{m,r})} \subseteq X^{s(L_{m,r}^+)},$$

from which we obtain proper birational morphisms

$$\begin{aligned}\psi_- : X \mathcal{Z}_{L_{m,r}^-} H &\rightarrow X \mathcal{Z}_{L_{m,r}} H, \\ \psi_+ : X \mathcal{Z}_{L_{m,r}^+} H &\rightarrow X \mathcal{Z}_{L_{m,r}} H\end{aligned}$$

fitting into the following commutative diagram (with all unmarked inclusions natural open immersions):

$$\begin{array}{ccccc} X^{s(L_{m,r}^-)} & \subseteq & X^{\text{ss,fg}(L_{m,r})} & \supseteq & X^{s(L_{m,r}^+)} \\ \downarrow & & \downarrow q & & \downarrow \\ X \mathcal{Z}_{L_{m,r}^-} H & \xrightarrow{\psi_-} & X \mathcal{Z}_{L_{m,r}} H & \xleftarrow{\psi_+} & X \mathcal{Z}_{L_{m,r}^+} H \\ & \nwarrow & \uparrow & \nearrow & \\ & & X^{s(L_{m,r})}/H & & \end{array}$$

(Cf. [T96, Theorem 3.3].) If  $n = 3$  then in fact

$$X \mathcal{Z}_{L_{m,r}^-} H \xrightarrow{\psi_-} X \mathcal{Z}_{L_{m,r}} H \xleftarrow{\psi_+} X \mathcal{Z}_{L_{m,r}^+} H$$

are all isomorphic to  $\mathbb{P}^1$ .<sup>14</sup> Otherwise  $\psi_-$  and  $\psi_+$  are both small contractions, and the induced birational morphism

$$X \mathcal{Z}_{L_{m,r}^-} H \dashrightarrow X \mathcal{Z}_{L_{m,r}^+} H$$

is a blow-down of  $E_- := \psi_-^{-1}(\text{pt})$  followed by a blow-up of  $E_+ := \psi_+^{-1}(\text{pt})$ , where here  $\text{pt} = (X \mathcal{Z}_{L_{m,r}} H) \setminus (X^{s(L_{m,r})}/H)$  [T96, Theorem 3.5].

In the case where  $n \geq 4$  we claim that  $E_+$  and  $E_-$  are isomorphic to the following weighted projective spaces<sup>15</sup>:  $E_+ \cong \mathbb{P}(1, 2, \dots, s)$  and  $E_- \cong \mathbb{P}(1, 2, \dots, n-s)$ , where  $s = (n - \frac{r}{m})/2$ . Indeed, recall that

$$E_- = (X^{s(L_{m,r}^-)} \setminus X^{s(L_{m,r})})/H, \quad E_+ = (X^{s(L_{m,r}^+)} \setminus X^{s(L_{m,r})})/H.$$

In the case of  $E_+$ , any  $H_u \cong \mathbb{k}^+$ -orbit in  $X^{s(L_{m,r}^+)} \setminus X^{s(L_{m,r})}$  contains a unique point  $[\sigma(x, y)]$  such that  $[0 : 1]$  is a zero of  $\sigma(x, y)$  of multiplicity  $n-s$  and  $[1 : 0]$  is a zero of multiplicity  $0 \leq l < s$ . Thus the locally closed subset

$$Z_+ = \{[a_0 x^n + a_1 x^{n-1} y + \dots + a_s x^{n-s} y^s] \in X \mid a_s \neq 0 \text{ and } a_i \neq 0 \text{ for some } 0 \leq i < s\}$$

of  $X$  provides a slice to the  $H_u$ -action on  $X^{s(L_{m,r}^+)} \setminus X^{s(L_{m,r})}$  which is stable under the  $T_1$ -action. Now  $Z_+ \cong \mathbb{k}^s \setminus \{0\}$  in the obvious way and one can check that the corresponding  $T_1$ -action

<sup>14</sup>To see this, note that  $X \mathcal{Z}_{L_{m,r}} H = (\mathbb{P}^2 \times X) //_{L'_{m,r,N}} (G \times H_r)$  is a one dimensional reductive GIT quotient of a smooth variety, hence is isomorphic to  $\mathbb{P}^1$  [Kem80]; similarly for the linearisations  $L_{m,r}^\pm$ . Alternatively observe that  $X \mathcal{Z}_{L_{m,r}} H$  is a normal projective unirational curve, and hence is isomorphic to  $\mathbb{P}^1$ .

<sup>15</sup>The fact that  $E_+$  and  $E_-$  are weighted projective spaces also follows from [T96, Theorem 5.6].



on  $\mathbb{K}^s \setminus \{0\}$  has weights  $-2s, -2s-2, \dots, -2$ . It follows that  $E_+ = Z_+/T_1$  is isomorphic to  $\mathbb{P}(1, 2, \dots, s)$ . The proof that  $E_- \cong \mathbb{P}(1, 2, \dots, n-s)$  is similar.

In the case where  $r = 0$ , we of course have  $X^{\text{ss,fg}(L_{m,0}^-)} = X^{s(L_{m,0}^-)} = \emptyset$  and  $X^{s(L_{m,0}^+)} \subseteq X^{\text{ss,fg}(L_{m,0})} = X^{\text{ss}(G)}$ , where recall  $X^{\text{ss}(G)}$  is the semistable locus for the canonical linearisation  $G \curvearrowright \mathcal{O}_X(1) \rightarrow X$ . In the case where  $n$  is even, the boundary of the stable locus  $X^{s(G)}$  inside  $X^{\text{ss}(G)}$  is precisely the  $G$ -orbit of the point  $[x^{n/2}y^{n/2}] \in X$  [Dol03, §10.2], so by inspection  $X^{s(L_{m,0}^+)} \subseteq X^{s(G)}$ . Thus there is a commuting diagram

$$\begin{array}{ccccccc} X^{s(L_{m,0}^+)} & \subseteq & X^{s(G)} & \subseteq & X^{\text{ss,fg}(L_{m,0})} & = & X^{\text{ss}(G)} \\ \downarrow q & & \downarrow & & \downarrow q & & \downarrow \text{S-equivalence} \\ X^{s(L_{m,0}^+)}/H & \xrightarrow{\psi} & X^{s(G)}/G & \subseteq & X \wr_{L_{m,0}} H & = & X // G \end{array}$$

where  $\psi : X^{s(L_{m,0}^+)}/H \rightarrow X^{s(G)}/G$  is a fibration. (Indeed, it is a geometric quotient for the unipotent subgroup  $(H_u)^{\text{opp}}$  of strictly lower triangular matrices in  $G$  opposite to  $H_u$ .) When  $n$  is odd, a similar diagram holds, except now  $X^{s(G)} = X^{\text{ss}(G)}$  and  $X^{s(G)}/G = X // G$ .

Lastly, the case where  $\frac{r}{m} = n$  is trivial: we now have  $X^{\text{ss,fg}(L_{m,r}^+)} = X^{s(L_{m,r}^+)} = \emptyset$  and  $X^{s(L_{m,r}^-)} \subseteq X^{\text{ss,fg}(L_{m,r})}$  induces the unique map  $X \wr_{L_{m,r}^-} H \rightarrow X \wr_{L_{m,r}} H = \text{pt}$ .

## 6 Applications

In the example studied in the previous section, where  $H$  is the standard Borel subgroup of  $\text{SL}(2, \mathbb{K})$  and  $X = \mathbb{P}(V)$  with  $V$  an irreducible representation of  $\text{SL}(2, \mathbb{K})$ , the quotient morphism

$$q_{m,r} : X^{\text{ss,fg}(L_{m,r})} \rightarrow X \wr_{L_{m,r}} H$$

from the semistable locus to the enveloping quotient is surjective whenever  $r \neq 0$ , even though the corresponding morphism when  $H$  is replaced with its unipotent radical  $H_u$  is not surjective (see Remark 5.2.3). Furthermore the analogues of the Hilbert–Mumford criteria for reductive GIT hold in these examples for the action of the non-reductive group  $H$ ; that is,  $x \in X^{s(L_{m,r})}$  (respectively  $x \in X^{\text{ss,fg}(L_{m,r})}$ ) if and only if  $x$  is stable (respectively semistable) for every one-parameter subgroup  $\lambda : \mathbb{G}_m \rightarrow H$ . In addition when  $r \neq 0$  the enveloping quotient is a categorical quotient of the semistable locus, with  $q_{m,r}(x) = q_{m,r}(y)$  for  $x, y \in X^{\text{ss,fg}(L_{m,r})}$  if and only if the closures of the  $H$ -orbits of  $x$  and  $y$  meet in  $X^{\text{ss,fg}(L_{m,r})}$ , just as for reductive GIT. Indeed for generic choice of  $r/m \in \mathbb{Q} \setminus \{0\}$  the enveloping quotient is a projective variety which is a geometric quotient of  $X^{\text{ss,fg}(L_{m,r})} = X^{s(L_{m,r})}$ . In this final section we will describe without proof some ongoing research which generalises these observations and has applications to moduli spaces occurring naturally in algebraic geometry.

### 6.1 Graded unipotent group actions

In [BeK15, BeDHK16a, BeDHK16b] the situation is studied when the unipotent radical  $H_u$  of a linear algebraic group  $H$  has a semi-direct product  $\hat{H}_u = H_u \rtimes \mathbb{G}_m$  by the multiplicative

group  $\mathbb{G}_m$  of  $\mathbb{k}$  such that the weights of the action of  $\mathbb{G}_m$  on the Lie algebra of  $H_u$  are all strictly positive; such a unipotent group is called *graded unipotent*. Given any action of  $\hat{H}_u$  on a projective variety  $X$  which is linear with respect to an ample line bundle  $L$  on  $X$ , it is shown in [BeDHK16a, BeDHK16b] that provided two conditions are satisfied:

(i) that we are willing to replace  $L$  with a suitable tensor power and to twist the linearisation of the action of  $\hat{H}_u$  by a suitable (rational) character of  $\hat{H}_u$ , and

(ii) roughly speaking, that ‘semistability coincides with stability’ for the action of  $\hat{H}_u$ , then the  $\hat{H}_u$ -invariants form a finitely generated algebra. Moreover in this situation the natural quotient morphism  $q$  from the semistable locus  $X^{\text{ss,fg},\hat{H}_u}$  to the enveloping quotient  $X \mathbin{\mathbb{G}} \hat{H}_u$  is surjective, and indeed expresses the projective variety  $X \mathbin{\mathbb{G}} \hat{H}_u$  as a geometric quotient of  $X^{\text{ss,fg},\hat{H}_u}$ , and this locus  $X^{\text{ss,fg},\hat{H}_u} = X^{\text{s},\hat{H}_u}$  can be described using Hilbert–Mumford-like criteria.

Suppose that  $H$  acts linearly on a projective variety  $X$ , and that  $H_r = H/H_u$  itself contains a central one-parameter subgroup whose conjugation action on the Lie algebra of  $H_u$  has all weights strictly positive. Then the corresponding semi-direct product  $\hat{H}_u$  is a subgroup of  $H$ , and provided that the condition (ii) that ‘semistability coincides with stability’ for the action of  $\hat{H}_u$  is satisfied,  $X$  can be quotiented first by  $\hat{H}_u$  for a suitably twisted linearisation as above, and then by the induced action of the reductive group  $H/\hat{H}_u \cong H_r/\mathbb{G}_m$ , to obtain a projective variety  $X \mathbin{\mathbb{G}} H$  which is a categorical quotient by  $H$  of  $X^{\text{ss,fg},H}$ . More generally suppose that the linear action of  $H$  on  $X$  extends to a linear action of a semi-direct product  $\hat{H}$  of  $H$  by  $\mathbb{G}_m$  acting by conjugation on the Lie algebra of  $H_u$  with all weights strictly positive, and whose induced conjugation action on  $H_r = H/H_u$  is trivial. Then if the condition (ii) is satisfied, we can quotient first by  $\hat{H}_u$  for a suitably twisted linearisation as above and then by the induced action of the reductive group  $\hat{H}/\hat{H}_u$ , to obtain a projective variety  $X \mathbin{\mathbb{G}} \hat{H}$  which is a categorical quotient by  $\hat{H}$  of the  $\hat{H}$ -invariant open subset  $X^{\text{ss,fg},\hat{H}}$  of  $X$ .

In order to describe the condition (ii), that ‘semistability coincides with stability’ for the action of  $\hat{H}_u$ , more precisely, let  $L \rightarrow X$  be a very ample linearisation of the action of  $\hat{H}$  on an irreducible projective variety  $X$ . Let  $\chi : \hat{H} \rightarrow \mathbb{G}_m$  be a character of  $\hat{H}$  with kernel containing  $H$ ; such characters  $\chi$  can be identified with integers so that the integer 1 corresponds to the character which fits into the exact sequence  $H \rightarrow \hat{H} \rightarrow \mathbb{G}_m$ . Let  $\omega_{\min}$  be the minimal weight for the  $\mathbb{G}_m$ -action on  $V := H^0(X, L)^*$  and let  $V_{\min}$  be the weight space of weight  $\omega_{\min}$  in  $V$ . Suppose that  $\omega_{\min} < \omega_{\min+1} < \dots < \omega_{\max}$  are the weights with which the one-parameter subgroup  $\mathbb{G}_m \leq \hat{H}_u \leq \hat{H}$  acts on the fibres of the tautological line bundle  $\mathcal{O}_{\mathbb{P}((H^0(X, L)^*)}(-1)$  over points of the connected components of the fixed point set  $\mathbb{P}((H^0(X, L)^*)^{\mathbb{G}_m})$  for the action of  $\mathbb{G}_m$  on  $\mathbb{P}((H^0(X, L)^*)$ ; since  $L$  is very ample  $X$  embeds in  $\mathbb{P}((H^0(X, L)^*)$  and the line bundle  $L$  extends to the dual  $\mathcal{O}_{\mathbb{P}((H^0(X, L)^*)}(1)$  of the tautological line bundle  $\mathcal{O}_{\mathbb{P}((H^0(X, L)^*)}(-1)$ . Without loss of generality we may assume that there exist at least two distinct such weights, since otherwise the action of the unipotent radical  $H_u$  of  $H$  on  $X$  is trivial, and so the action of  $H$  is via an action of the reductive group  $H_r = H/H_u$  and reductive GIT can be applied. Let  $c$  be a positive integer such that

$$\frac{\chi}{c} = \omega_{\min} + \epsilon$$

where  $\epsilon > 0$  is sufficiently small; we will call rational characters  $\chi/c$  with this property *well adapted* to the linear action of  $\hat{H}$ , and we will call the linearisation *well adapted* if  $\omega_{\min} < 0 \leq$

$\omega_{\min} + \epsilon$  for sufficiently small  $\epsilon > 0$ . The linearisation of the action of  $\hat{H}$  on  $X$  with respect to the ample line bundle  $L^{\otimes c}$  can be twisted by the character  $\chi$  so that the weights  $\omega_j$  are replaced with  $\omega_j c - \chi$ ; let  $L_\chi^{\otimes c}$  denote this twisted linearisation. Let  $X_{\min+}^{s, \mathbb{G}_m}$  denote the stable subset of  $X$  for the linear action of  $\mathbb{G}_m$  with respect to the linearisation  $L_\chi^{\otimes c}$ ; by the theory of variation of (classical) GIT [DolH98, T96], if  $L$  is very ample then  $X_{\min+}^{s, \mathbb{G}_m}$  is the stable set for the action of  $\mathbb{G}_m$  with respect to any rational character  $\chi/c$  such that  $\omega_{\min} < \chi/c < \omega_{\min+1}$ . Let

$$X_{\min+}^{s, \hat{H}_u} = X \setminus \hat{H}_u(X \setminus X_{\min+}^{s, \mathbb{G}_m}) = \bigcap_{u \in H_u} u X_{\min+}^{s, \mathbb{G}_m}$$

be the complement of the  $\hat{H}_u$ -sweep (or equivalently the  $H_u$ -sweep) of the complement of  $X_{\min+}^{s, \mathbb{G}_m}$ , let

$$Z_{\min} := X \cap \mathbb{P}(V_{\min}) = \left\{ x \in X \mid \begin{array}{l} x \text{ is a } \mathbb{G}_m\text{-fixed point and} \\ \mathbb{G}_m \text{ acts on } L^*|_x \text{ with weight } \omega_{\min} \end{array} \right\}$$

and

$$X_{\min}^0 := \{x \in X \mid \lim_{t \rightarrow 0, t \in \mathbb{G}_m} t \cdot x \in Z_{\min}\}.$$

Then  $X_{\min}^0$  is  $\hat{H}_u$ -invariant and  $X_{\min+}^{s, \hat{H}_u} = X_{\min}^0 \setminus H_u Z_{\min}$ .

The condition that ‘semistability coincides with stability’ for the linear action of  $\hat{H}_u$  required in [BeDHK16a] is slightly stronger than that required in [BeDHK16b], where the hypothesis needed for the  $\hat{H}_u$ -linearisation  $L \rightarrow X$  is that

$$\text{Stab}_{H_u}(z) = \{e\} \text{ for every } z \in Z_{\min} \quad (\mathfrak{C}^*)$$

(note that this condition is satisfied in the examples studied in §5) and the following result is proved.

**Theorem 6.1.1.** [BeDHK16b] *Let  $H$  be a linear algebraic group over  $\mathbb{k}$  with unipotent radical  $H_u$ . Let  $\hat{H} = H \rtimes \mathbb{G}_m$  be a semidirect product of  $H$  by  $\mathbb{G}_m$  with subgroup  $\hat{H}_u = H_u \rtimes \mathbb{G}_m$ , where the conjugation action of  $\mathbb{G}_m$  on  $H_u$  is such that all the weights of the induced  $\mathbb{G}_m$ -action on the Lie algebra of  $H_u$  are strictly positive, while the induced conjugation action of  $\mathbb{G}_m$  on  $H_r = H/H_u$  is trivial. Suppose that  $\hat{H}$  acts linearly on an irreducible projective variety  $X$  with respect to an ample line bundle  $L$ , and that  $c$  is a sufficiently divisible positive integer and  $\chi : \hat{H} \rightarrow \mathbb{G}_m$  is a character of  $\hat{H}$  with kernel containing  $H$  such that the rational character  $\chi/c$  is well adapted for the linear action of  $\hat{H}_u$ . Suppose also that the linear action of  $\hat{H}_u$  on  $X$  satisfies the condition  $(\mathfrak{C}^*)$  above. Then the algebras of invariants  $\oplus_{m=0}^{\infty} H^0(X, L_{m\chi}^{\otimes cm})^{\hat{H}_u}$  and*

$$\oplus_{m=0}^{\infty} H^0(X, L_{m\chi}^{\otimes cm})^{\hat{H}} = (\oplus_{m=0}^{\infty} H^0(X, L_{m\chi}^{\otimes cm})^{\hat{H}_u})^{H_r}$$

*are finitely generated. Moreover the enveloping quotient  $X \mathbin{\mathbb{P}} \hat{H}_u$  is the projective variety associated to the algebra of invariants  $\oplus_{m=0}^{\infty} H^0(X, L_{m\chi}^{\otimes cm})^{\hat{H}_u}$  and is a geometric quotient of the open subset  $X_{\min+}^{s, \hat{H}_u}$  of  $X$  by  $\hat{H}_u$ , while the enveloping quotient  $X \mathbin{\mathbb{P}} \hat{H}$  is the projective variety associated to the algebra of invariants  $\oplus_{m=0}^{\infty} H^0(X, L_{m\chi}^{\otimes cm})^{\hat{H}}$  and is the reductive GIT quotient of  $X \mathbin{\mathbb{P}} \hat{H}_u$  by the induced action of the reductive group  $\hat{H}/\hat{H}_u \cong H_r$  with respect to the linearisation induced by a sufficiently divisible tensor power of  $L$ .*

Applying this result with  $X$  replaced by  $X \times \mathbb{P}^1$ , with respect to the tensor power of the linearisation  $L$  (over  $X$ ) with  $\mathcal{O}_{\mathbb{P}^1}(M)$  (over  $\mathbb{P}^1$ ) for  $M \gg 1$ , gives us a projective variety  $(X \times \mathbb{P}^1) \wr \hat{H}$  which is a categorical quotient by  $\hat{H}$  of an  $\hat{H}$ -invariant open subset of  $X \times \mathbb{k}$ . This open subset is the inverse image in  $(X \times \mathbb{P}^1)_{\min+}^{s, \hat{H}_u}$  of the  $H_r$ -semistable subset  $((X \times \mathbb{P}^1) \wr \hat{H}_u)^{ss, H_r}$  of  $(X \times \mathbb{P}^1) \wr \hat{H}_u = (X \times \mathbb{P}^1)_{\min+}^{s, \hat{H}_u} / \hat{H}_u$ , and contains as an open subset a geometric quotient by  $H$  of an  $H$ -invariant open subset  $X^{\hat{s}, H}$  of  $X$ . Here  $X^{\hat{s}, H}$  can be identified in the obvious way with  $X^{\hat{s}, H} \times \{[1 : 1]\}$  which is the intersection with  $X \times \{[1 : 1]\}$  of the inverse image in  $(X \times \mathbb{P}^1)_{\min+}^{s, \hat{H}_u} = (X \times \mathbb{P}^1)_{\min+}^{ss, \text{fg}, \hat{H}_u}$  of the  $H_r$ -stable subset  $((X \times \mathbb{P}^1) \wr \hat{H}_u)^{s, H_r}$  of

$$(X \times \mathbb{P}^1) \wr \hat{H}_u = ((X_{\min}^0 \times \mathbb{k}^*) \sqcup (X_{\min+}^{s, \hat{H}_u} \times \{0\})) / \hat{H}_u \cong (X_{\min}^0 / H_u) \sqcup (X_{\min+}^{s, \hat{H}_u} / \hat{H}_u).$$

Furthermore the geometric quotient  $X^{\hat{s}, H} / H$  and its projective completion  $(X \times \mathbb{P}^1) \wr \hat{H}$  can be described using Hilbert–Mumford-like criteria, by combining the description of  $(X \times \mathbb{P}^1) \wr \hat{H}_u$  as the geometric quotient  $(X \times \mathbb{P}^1)_{\min+}^{s, \hat{H}_u} / \hat{H}_u$  with reductive GIT for the induced linear action of the reductive group  $H_r = H / H_u$  on  $(X \times \mathbb{P}^1) \wr \hat{U}$ .

In [BeDHK16b] it is also shown that when the condition  $(\mathfrak{C}^*)$  is not satisfied, but is replaced with the much weaker condition

$$\min_{x \in X} \dim(\text{Stab}_{H_u}(x)) = 0, \quad (\mathfrak{C}^{**})$$

then there is a sequence of blow-ups of  $X$  along  $\hat{H}$ -invariant subvarieties (analogous to that of [Ki85] in the reductive case) resulting in a projective variety  $\hat{X}$  with an induced linear action of  $\hat{H}$  satisfying the condition  $(\mathfrak{C}^*)$ . In this way we obtain a projective variety  $\widehat{X \times \mathbb{P}^1} \wr \hat{H}$  which is a categorical quotient by  $\hat{H}$  of a  $\hat{H}$ -invariant open subset of a blow-up of  $X \times \mathbb{k}$  and contains as an open subset a geometric quotient of an  $H$ -invariant open subset  $X^{\hat{s}, H}$  of  $X$  by  $H$ , where the geometric quotient  $X^{\hat{s}, H} / H$  and its projective completion  $\widehat{X \times \mathbb{P}^1} \wr \hat{H}$  have descriptions in terms of Hilbert–Mumford-like criteria and the explicit blow-up construction.

*Remark 6.1.2.* It is observed in [BeDHK16b] that, at least when  $H_u$  is abelian, most of these conclusions hold even when the condition  $(\mathfrak{C}^{**})$  is not satisfied. The case when  $\min_{x \in X} \dim(\text{Stab}_{H_u}(x)) > 0$  is studied in [BeJKon, BeHJKon].

## 6.2 Automorphism groups of complete simplicial toric varieties

The automorphism group of the weighted projective plane  $\mathbb{P}(1, 1, 2) = (\mathbb{k}^3 \setminus \{0\}) / \mathbb{G}_m$ , for  $\mathbb{G}_m$  acting linearly on  $\mathbb{k}^3$  with weights 1, 1, 2, is given by

$$\text{Aut}(\mathbb{P}(1, 1, 2)) \cong R \ltimes U$$

where  $R \cong \text{GL}(2, \mathbb{k})$  is reductive and  $U \cong (\mathbb{k})^3$  is unipotent. Here  $(\lambda, \mu, \nu) \in (\mathbb{k})^3$  acts on  $\mathbb{P}(1, 1, 2)$  as

$$[x, y, z] \mapsto [x, y, z + \lambda x^2 + \mu xy + \nu y^2].$$

The central one-parameter subgroup  $\mathbb{G}_m$  of  $R \cong \mathrm{GL}(2, \mathbb{k})$  acts on the Lie algebra of  $H_u$  with all positive weights, and the associated semi-direct product

$$\hat{U} = U \rtimes \mathbb{G}_m$$

can be identified with a subgroup of  $\mathrm{Aut}(\mathbb{P}(1, 1, 2))$ . Thus the results discussed in §6.1 have an immediate application to linear actions of  $\mathrm{Aut}(\mathbb{P}(1, 1, 2))$ .

**Corollary 6.2.1.** *Suppose that  $H = \mathrm{Aut}(\mathbb{P}(1, 1, 2))$  acts linearly on a projective variety  $X$ . If the linearisation is replaced with a suitable positive tensor power and twisted by an appropriate character of  $H$ , then when condition  $(\mathfrak{C}^*)$  holds the enveloping quotient  $X \mathbin{\mathbb{G}} H$  is the projective variety associated to the  $H$ -invariants on  $X$ , and is a categorical quotient by  $H$  of an  $H$ -invariant open subset of  $X$  which can be described using Hilbert–Mumford-like criteria. Even if  $(\mathfrak{C}^*)$  fails, provided that the weaker condition  $(\mathfrak{C}^{**})$  holds there is a geometric quotient by  $H$  of an open subset of  $X$  described by Hilbert–Mumford-like criteria, with a projective completion which is a categorical quotient of an open subset of an  $H$ -equivariant blow-up  $\tilde{X}$  of  $X$  and coincides with  $X \mathbin{\mathbb{G}} H$  when  $(\mathfrak{C}^*)$  holds.*

In fact the same is true for the automorphism group of any complete simplicial toric variety. For it was observed in [BeDHK16a] using the description in [C95] that the automorphism group  $H$  of any complete simplicial toric variety is a linear algebraic group with a graded unipotent radical  $U$  such that the grading is defined by a one parameter subgroup  $\mathbb{G}_m$  of  $H$  acting by conjugation on the Lie algebra of  $U$  with all weights strictly positive, and inducing a central one-parameter subgroup of  $R = H/U$ . Thus the results of §6.1 can be applied to any linear action of  $H$  on an irreducible projective variety with respect to an ample linearisation.

### 6.3 Groups of $k$ -jets of holomorphic reparametrisations of $(\mathbb{C}^p, 0)$

Suppose now that  $\mathbb{k} = \mathbb{C}$  and consider  $k$ -jets at 0 of holomorphic maps from  $\mathbb{C}^p$  to a complex manifold  $Y$  for any  $k, p \geq 1$ . It was observed in [BeK15] that the group  $\mathbb{G}_{k,p}$  of  $k$ -jets of holomorphic reparametrisations of  $(\mathbb{C}^p, 0)$  has a graded unipotent radical  $\mathbb{U}_{k,p}$  such that the grading is defined by a one-parameter subgroup of  $\mathbb{G}_{k,p}$  acting by conjugation on the Lie algebra of  $\mathbb{U}_{k,p}$  with all weights strictly positive, and inducing a central one-parameter subgroup of the reductive group  $\mathbb{G}_{k,p}/\mathbb{U}_{k,p}$ . So the results discussed in §6.1 apply to any linear action of the reparametrisation group  $\mathbb{G}_{k,p}$ .

Here  $\mathbb{G}_k = \mathbb{G}_{k,1}$  is the group of  $k$ -jets of germs of biholomorphisms of  $(\mathbb{C}, 0)$  given by

$$t \mapsto \phi(t) = a_1 t + a_2 t^2 + \dots + a_k t^k,$$

for  $a_1, \dots, a_k \in \mathbb{C}$  and  $a_1 \neq 0$ , under composition modulo  $t^{k+1}$ . It is isomorphic to the group of matrices

$$\mathbb{G}_k \cong \left\{ \begin{pmatrix} a_1 & a_2 & \dots & a_k \\ 0 & a_1^2 & \dots & \\ & & \dots & \\ 0 & 0 & \dots & a_1^k \end{pmatrix} : a_1 \in \mathbb{C}^*, a_2, \dots, a_k \in \mathbb{C} \right\}$$

and hence is a linear algebraic group.  $\mathbb{G}_k$  has a subgroup  $\mathbb{C}^*$  (represented by  $\phi(t) = a_1 t$ ) and a unipotent subgroup  $\mathbb{U}_k$  (represented by  $\phi(t) = t + a_2 t^2 + \dots + a_k t^k$ ) which is its unipotent radical, with

$$\mathbb{G}_k \cong \mathbb{U}_k \rtimes \mathbb{C}^*.$$

If  $Y$  is a complex manifold then  $\mathbb{G}_k$  acts fibrewise on the bundle  $J_k \rightarrow Y$  of  $k$ -jets at 0 of holomorphic curves  $f : \mathbb{C} \rightarrow Y$  by reparametrising  $k$ -jets. Similarly the group  $\mathbb{G}_{k,p}$  of  $k$ -jets of germs of biholomorphisms of  $(\mathbb{C}^p, 0)$  acts fibrewise on the bundle  $J_{k,p} \rightarrow Y$  of  $k$ -jets at the origin of holomorphic maps  $f : \mathbb{C}^p \rightarrow X$ , and

$$\mathbb{G}_{k,p} \cong \mathbb{U}_{k,p} \rtimes \mathrm{GL}(p, \mathbb{C})$$

where  $\mathbb{U}_{k,p}$  is the unipotent radical of  $\mathbb{G}_{k,p}$ , and the central one-parameter subgroup  $\mathbb{C}^*$  of  $\mathrm{GL}(p, \mathbb{C})$  acts on the Lie algebra of  $\mathbb{U}_{k,p}$  with all weights strictly positive. Thus  $\mathbb{G}_{k,p}$  has the structure required in §6.1.

## 6.4 Unstable strata for linear actions of reductive groups

Now let  $G$  be a reductive group over an algebraically closed field  $\mathbb{k}$  of characteristic zero, acting linearly on a projective variety  $X$  with respect to an ample line bundle  $L$ . Associated to this linear  $G$ -action and an invariant inner product on the Lie algebra of  $G$ , there is a stratification

$$X = \bigsqcup_{\beta \in \mathcal{B}} S_\beta$$

of  $X$  by locally closed subvarieties  $S_\beta$ , indexed by a partially ordered finite subset  $\mathcal{B}$  of a positive Weyl chamber for the reductive group  $G$ , such that

(i)  $S_0 = X^{ss}$ ,

and for each  $\beta \in \mathcal{B}$

(ii) the closure of  $S_\beta$  is contained in  $\bigcup_{\gamma \geq \beta} S_\gamma$ , and

(iii)  $S_\beta \cong G \times_{P_\beta} Y_\beta^{ss}$

where  $P_\beta$  is a parabolic subgroup of  $G$  acting on a projective subvariety  $\bar{Y}_\beta$  of  $X$  with an open subset  $Y_\beta^{ss}$  which is determined by the action of the Levi subgroup of  $P_\beta$  with respect to a suitably twisted linearisation [Ki84].

Here the original linearisation for the action of  $G$  on  $L \rightarrow X$  is restricted to the action of the parabolic subgroup  $P_\beta$  over  $\bar{Y}_\beta$ , and then twisted by a rational character of  $P_\beta$  which is well adapted in the sense of §6.1 for a central one-parameter subgroup of the Levi subgroup of  $P_\beta$  acting with all weights strictly positive on the Lie algebra of the unipotent radical of  $P_\beta$ . Thus to construct a quotient by  $G$  of (an open subset of) an unstable stratum  $S_\beta$ , we can study the linear action on  $\bar{Y}_\beta$  of the parabolic subgroup  $P_\beta$ , twisted appropriately, and apply the results discussed in §6.1.

In particular we can consider moduli spaces of sheaves of fixed Harder–Narasimhan type over a nonsingular projective variety  $W$  (cf. [HoK12]). There are well known constructions going back to Simpson [Si94] of the moduli spaces of semistable pure sheaves on  $W$  of fixed Hilbert polynomial as GIT quotients of linear actions of suitable special linear groups  $G$  on schemes

$Q$  (closely related to quot-schemes) which are  $G$ -equivariantly embedded in projective spaces. These constructions can be chosen so that elements of  $Q$  which parametrise sheaves of a fixed Harder–Narasimhan type form a stratum in the stratification of  $Q$  associated to the linear action of  $G$  (at least modulo taking connected components of strata) [HoK12]. §6.1 can be applied to the associated linear actions of parabolic subgroups of these special linear groups  $G$ , appropriately twisted, to construct and study moduli spaces of sheaves of fixed Harder–Narasimhan type over  $W$  [BeHJKon]. The simplest non-trivial case is that of unstable vector bundles of rank 2 and fixed Harder–Narasimhan type over a nonsingular projective curve  $W$  (cf. [BraPMN09]).

## A Appendix: Linearisations of Products of Reductive Groups

We discuss GIT quotients of direct products of reductive groups. For this section, suppose  $G_1$  and  $G_2$  are reductive groups and  $X$  is a  $G_1 \times G_2$ -variety equipped with a  $G_1 \times G_2$ -linearisation  $L \rightarrow X$ . Via the natural embeddings  $G_i \hookrightarrow G_1 \times G_2$ ,  $i = 1, 2$ , this data is equivalent to saying that the variety  $X$  and the line bundle  $L$  are equipped with two commuting linearisations  $G_i \curvearrowright L \rightarrow X$ . In particular, it makes sense to consider the semistable loci  $X^{\text{ss}(G_1)}$  and  $X^{\text{ss}(G_1 \times G_2)}$  with respect to the linearisations  $G_1 \curvearrowright L \rightarrow X$  and  $G_1 \times G_2 \curvearrowright L \rightarrow X$  respectively, together with their reductive GIT quotients

$$\begin{aligned}\pi_{G_1} : X^{\text{ss}(G_1)} &\rightarrow X//G_1, \\ \pi_{G_1 \times G_2} : X^{\text{ss}(G_1 \times G_2)} &\rightarrow X//(G_1 \times G_2).\end{aligned}$$

In the case where  $L \rightarrow X$  is ample and  $X$  is projective over an affine variety the following result is well known (cf. [OST99] and [Sc08, Section 1.5.3] for the case  $X = \mathbb{P}^n$  and also [T96]), though proofs in the general case are hard to come by. For the reader's convenience, we include here a proof for the more general case of when  $X$  is any variety and  $L \rightarrow X$  is any linearisation.

**Proposition A.0.1.** *Retain the notation above.*

1. *The set  $X^{\text{ss}(G_1)}$  is stable under the  $G_2$ -action on  $X$  and there is a canonical action of  $G_2$  on  $X//G_1$  such that  $\pi_{G_1}$  is  $G_2$ -equivariant.*
2. *There is a natural ample  $G_2$ -linearisation  $M \rightarrow X//G_1$  such that, for some  $n > 0$ , we have  $\pi_{G_1}^* M = L^{\otimes n}|_{X^{\text{ss}(G_1)}}$  as  $G_2$ -linearisations and  $X^{\text{ss}(G_1 \times G_2)} \subseteq \pi_{G_1}^{-1}((X//G_1)^{\text{ss}(M)})$ . Letting*

$$\bar{\pi}_{G_2} : (X//G_1)^{\text{ss}(M)} \rightarrow (X//G_1)//_M G_2$$

*denote the reductive GIT quotient with respect to this linearisation, there is a canonical open immersion  $\psi : X//(G_1 \times G_2) \hookrightarrow (X//G_1)//_M G_2$  such that the following diagram commutes:*

$$\begin{array}{ccc} & X^{\text{ss}(G_1 \times G_2)} & \\ & \searrow \pi_{G_1 \times G_2} & \downarrow \pi_{G_1} \\ & & (X//G_1)^{\text{ss}(M)} \\ & \swarrow & \downarrow \bar{\pi}_{G_2} \\ X//(G_1 \times G_2) & \xrightarrow{\psi} & (X//G_1)//_M G_2\end{array}$$

3. If  $X$  is further assumed to be projective, then  $X^{\text{ss}(G_1 \times G_2)} = \pi^{-1}((X//G_1)^{\text{ss}(M)})$  and  $\psi$  is an isomorphism.

*Proof.* (Proof of 1.) Suppose  $f \in H^0(X, L^{\otimes r})^{G_1}$ , for some  $r > 0$ , such that  $X_f$  is affine. For any  $g_2 \in G_2$  the section  $g_2 \cdot f$  is again  $G_1$ -invariant, and acting on  $X$  by  $g_2$  induces an isomorphism (with inverse given by  $g_2^{-1}$ )  $X_f \rightarrow X_{g_2 \cdot f}$ , so that  $X_{g_2 \cdot f}$  is also affine. Hence the  $G_2$ -action on  $X$  restricts to define an action  $\sigma : G_2 \times X^{\text{ss}(G_1)} \rightarrow X^{\text{ss}(G_1)}$ . Recall that the GIT quotient  $\pi_{G_1} : X^{\text{ss}(G_1)} \rightarrow X//G_1$  is a categorical quotient for the action of  $G_1$  on  $X^{\text{ss}(G_1)}$ . Let  $G_1$  act on  $G_2 \times X^{\text{ss}(G_1)}$  by demanding that  $G_1$  acts trivially on  $G_2$ . Then the composition

$$G_2 \times X^{\text{ss}(G_1)} \xrightarrow{\sigma} X^{\text{ss}(G_1)} \xrightarrow{\pi_{G_1}} X//G_1$$

is  $G_1$ -invariant by virtue of the fact that  $G_1$  is normal in  $G_1 \times G_2$ , so there is a canonical map  $\bar{\sigma} : G_2 \times X//G_1 \rightarrow X//G_1$  such that the diagram

$$\begin{array}{ccc} G_2 \times X^{\text{ss}(G_1)} & \xrightarrow{\sigma} & X^{\text{ss}(G_1)} \\ \downarrow \text{id}_{G_2} \times \pi_{G_1} & & \downarrow \pi_{G_1} \\ G_2 \times (X//G_1) & \xrightarrow{\bar{\sigma}} & X//G_1 \end{array}$$

commutes. Using the universal property of categorical quotients it is easy to verify that  $\bar{\sigma}$  defines an action of  $G_2$  on  $X//G_1$ —we omit the details.

(Proof of 2.) The construction of the GIT quotient  $X//G_1$  comes with an ample line bundle  $M \rightarrow X//G_1$  such that  $\pi_{G_1}^* M = L^{\otimes n}|_{X^{\text{ss}(G_1)}}$ , for some  $n > 0$  [MumFK94, Theorem 1.10]. In fact, the natural map  $L^{\otimes n}|_{X^{\text{ss}(G_1)}} \rightarrow M$  thus arising is a good categorical quotient of the action of  $G_1$  on  $L^{\otimes n}|_{X^{\text{ss}(G_1)}}$ . (This can be shown by following through the proof of the following more general statement [Ne78, Proposition 3.12]: if  $G$  is a reductive group acting on varieties  $X$  and  $Y$ , if  $X \rightarrow Y$  is an affine  $G$ -equivariant morphism and  $Y$  possesses a good categorical quotient by  $G$ , then so too does  $X$ .) Following an argument similar to that in the proof of 1, one sees that there is a canonical  $G_2$ -action on  $M$  such that  $L^{\otimes n}|_{X^{\text{ss}(G_1)}} \rightarrow M$  is  $G_2$ -equivariant and the line bundle projection  $M \rightarrow X//G_1$  is equivariant.

We next show that  $X^{\text{ss}(G_1 \times G_2)} \subseteq \pi_{G_1}^{-1}((X//G_1)^{\text{ss}(M)})$ . Let  $x \in X^{\text{ss}(G_1 \times G_2)}$ . Then without loss of generality there is an invariant section  $f \in H^0(X, L^{\otimes mn})^{G_1 \times G_2}$  with  $m > 0$  such that  $x \in X_f$  and  $X_f$  is affine. Clearly  $\pi_{G_1}$  is defined at  $x$ . Because both  $\pi_{G_1} : X^{\text{ss}(G_1)} \rightarrow X//G_1$  and  $L^{\otimes mn}|_{X^{\text{ss}(G_1)}} \rightarrow M^{\otimes m}$  are  $G_2$ -equivariant maps that are categorical quotients for the  $G_1$ -actions, pulling back along  $\pi_{G_1}$  defines a canonical  $G_2$ -equivariant isomorphism

$$\pi_{G_1}^* : H^0(X//G_1, M^{\otimes m}) \xrightarrow{\cong} H^0(X^{\text{ss}(G_1)}, L^{\otimes mn})^{G_1}.$$

Hence there is  $F \in H^0(X//G_1, M^{\otimes m})^{G_2}$  such that  $\pi_{G_1}^{-1}((X//G_1)_F) = X_f$ . The map  $\pi_{G_1}$  restricts to a good categorical quotient  $\pi_{G_1} : X_f \rightarrow (X//G_1)_F$  for the  $G_1$ -action on  $X_f$ , and since  $X_f$  is affine so too is  $(X//G_1)_F$  by Theorem 2.2.1, 1. Thus  $(X//G_1)_F \subseteq (X//G_1)^{\text{ss}(M)}$  and  $\pi_{G_1}(x) \in (X//G_1)^{\text{ss}(M)}$ .

The composition  $\bar{\pi}_{G_2} \circ \pi_{G_1} : X^{\text{ss}(G_1 \times G_2)} \rightarrow (X//G_1)//_M G_2$  is  $G_1 \times G_2$ -invariant, so induces a unique morphism  $\psi : X//(G_1 \times G_2) \rightarrow (X//G_1)//_M G_2$  making the required diagram commute.



Recall from the construction of the GIT quotient that  $X//(G_1 \times G_2)$  is covered by affine open subsets  $\pi_{G_1 \times G_2}(X_f) = \text{Spec}(\mathcal{O}(X_f)^{G_1 \times G_2})$ , for  $f \in H^0(X, L^{\otimes mn})^{G_1 \times G_2}$  with  $m > 0$ . The morphism  $\psi$  maps  $\pi_{G_1 \times G_2}(X_f)$  to the affine open subset  $\bar{\pi}_{G_2}((X//G_1)_F)$  of  $(X//G_1)//_M G_2$ , where as above  $F$  is a  $G_2$ -invariant section such that  $\pi_{G_1}^* F = f|_{X^{\text{ss}(G_1)}}$ ; this map corresponds to the isomorphism of rings

$$\mathcal{O}(\bar{\pi}_{G_2}((X//G_1)_F)) \xrightarrow{\bar{\pi}_{G_2}^*} \mathcal{O}((X//G_1)_F)^{G_2} \xrightarrow{\pi_{G_1}^*} \mathcal{O}(X_f)^{G_1 \times G_2}.$$

Hence  $\psi$  restricts to an isomorphism  $\pi_{G_1 \times G_2}(X_f) \rightarrow \bar{\pi}_{G_2}((X//G_1)_F)$ . Patching over all such  $\pi_{G_1 \times G_2}(X_f)$  shows that  $\psi$  is an open immersion.

(Proof of 3.) Suppose now that  $X$  is projective and  $L$  is ample. Then the GIT quotient  $X//G_1$  is canonically isomorphic to  $\text{Proj}(\mathbb{k}[X, L^{\otimes n}]^{G_1})$ , with  $\mathbb{k}[X, L^{\otimes n}]^{G_1}$  finitely generated and  $M \rightarrow X//G_1$  corresponding to the twisting sheaf  $\mathcal{O}(1)$  on  $\text{Proj}(\mathbb{k}[X, L^{\otimes n}]^{G_1})$  [MumFK94, Page 40]. The GIT quotient  $\pi_{G_1} : X^{\text{ss}(G_1)} \rightarrow X//G_1$  is the morphism defined by the inclusion  $\mathbb{k}[X, L^{\otimes n}]^{G_1} \hookrightarrow \mathbb{k}[X, L^{\otimes n}]$ . Moreover, by Serre vanishing [Har77, Chapter 3, Proposition 5.3], for sufficiently large  $m > 0$  the natural map  $H^0(X, L^{\otimes mn})^{G_1} \rightarrow H^0(X//G_1, M^{\otimes m})$  is surjective. Now suppose  $x \in X^{\text{ss}(G_1)}$  maps to  $(X//G_1)^{\text{ss}(M)}$  under  $\pi_{G_1}$ . Then there is  $F \in H^0(X//G_1, M^{\otimes m})^{G_2}$  such that  $F(\pi_{G_1}(x)) \neq 0$ , with  $m$  sufficiently large so that  $\pi_{G_1}^* F = f|_{X^{\text{ss}(G_1)}}$  for some global invariant section  $f \in H^0(X, L^{\otimes mn})^{G_1 \times G_2}$ , so that  $x \in X_f \subseteq X^{\text{ss}(G_1 \times G_2)}$ . Thus  $X^{\text{ss}(G_1 \times G_2)} = \pi_{G_1}^{-1}((X//G_1)^{\text{ss}(M)})$ . The induced map  $\pi_{G_1} : X^{\text{ss}(G_1 \times G_2)} \rightarrow (X//G_1)^{\text{ss}(M)}$  is therefore a categorical quotient for the  $G_1$ -action on  $X^{\text{ss}(G_1 \times G_2)}$ , and so its composition with the categorical  $G_2$ -quotient  $\bar{\pi}_{G_2} : (X//G_1)^{\text{ss}(M)} \rightarrow (X//G_1)//_M G_2$  is a categorical quotient for the full  $G_1 \times G_2$ -action on  $X^{\text{ss}(G_1 \times G_2)}$ . It follows that the canonically induced map  $\psi : X//(G_1 \times G_2) \rightarrow (X//G_1)//_M G_2$  is an isomorphism.  $\square$

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