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A RECONSIDERATION OF THE OPTIMAL INCOME TAX

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ABSTRACT

This paper develops a formula for the optimal nonlinear income tax, the terms of which are familiar from the theory of linear income taxation. The development uses the idea of a perturbation of the optimal schedule and is based upon an assumption of differentiability. It is also shown that the introduction of non-differentiability, implying bunching of taxpayers, may be desirable and, in this case, the optimal schedule may be difficult to determine. The analysis can be applied to nonlinear tax reform:

Keywords: nonlinear taxation, income taxation, bunching, tax reform.

JEL No: H21, H31

[1]This paper has been prepared as a contribution for *Incentives and Organizations: Essays in Honour of Sir James Mirrlees*, edited by P. Hammond and G. Myles. Part of the paper draws on a paper circulated in October 1982 entitled "A New Look at Optimal Income Taxation". The analysis of Section 4 has been completely revised in the present version. I am grateful to Tony Atkinson, Jim Mirrlees and Emmanuel Saez for their suggested comments on versions of this paper.

Introduction

Following on from the seminal work of Mirrlees (1971), an extensive literature on the design of optimal nonlinear tax schedules has developed. Whilst some of this literature (e.g. Sadka (1976)) has dealt with easily understandable properties possessed by such schedules, a major drawback of the "state-of-the-art" is that formulae used to characterize optimal schedules are defined in terms of functions that, whilst open to limited interpretation (Atkinson and Stiglitz (1979)), lack the straightforwardness of direct supply elasticities, etc., that have become a commonplace of linear tax formulae.

There is no guarantee that the particular variational approach to optimization that has been adopted in the literature produces the most appealing tax formulae. Approaches that depend upon different perturbations of the tax schedule will produce formulae that can look very different. One purpose of this paper is to present an approach which is transparent and which produces tax formulae with all of the appeal of the neatest linear tax formulae (Diamond (1975), Dixit and Sandmo (1977)). I consider a particular easily understandable perturbation in the tax schedule which, when broken down by a Taylor expansion, produces the optimal tax formula. As some seem to look upon nonlinear tax theory as an unfathomable area, I hope that the present derivation will be found useful. Here, the aim is to present an approach to nonlinear tax theory that is akin to the derivation of the Keynes-Ramsey rule in growth theory. The appendix shows how the same formula can be derived using a control theoretic approach using the dual approach to nonlinear taxation introduced in Roberts (1979).

There has been little work looking at alternative derivations and formulae for nonlinear tax schedules and the contrast with linear taxation is striking. Saez (1999) has recently looked at nonlinear taxation using the idea of a perturbation of the tax schedule. The perturbation used and presentation of formulae is different to that in this paper though, as in the present analysis, he recognises that one does not need to base formulae on the distribution in the population of some unobservable parameter like ability, as in the original Mirrlees work. Instead, the more natural distribution of income may be used. Saez derives numerical tax formulae using the actual income distribution derived from US tax returns and this offers a great improvement upon the numerical solutions in Mirrlees (1971) and Tuomala (1990). The other work that has investigated alternative formulae is that of Revesz (1989). He uses a calculus of variations approach based upon choosing marginal prices as in the dual approach of Roberts (1979). He derives a formula which incorporates an inverse elasticity rule as in this paper, but in other respects, it retains the structure of the original Mirrlees approach.

All previous derivations of tax formulae depend upon the seemingly innocuous assumption that the tax schedule is differentiable so that taxpayers with different preferences are not 'bunched' together at the same income level. However, the perturbation technique developed in the first part of this paper can be amended to examine the issue of whether optimal schedules are differentiable and it is shown that 'bunching' may be optimal. In earlier work (Ebert (1992), Lollivier and Rochet (1983)), the existence of bunching is related to the failure of taxpayer first-order conditions to define a point of maximum utility along the tax schedule. In this paper, the differentiable tax schedules that are candidates for optimal tax schedules can possess the feature that individual taxpayers are maximizing their own utility. Our analysis uncovers some strange possibilities. For instance, it can be shown that when the income distribution is unbounded, the number of 'bunch' points may be infinite. This casts doubt on the technique of looking at limiting tax rates in such circumstances (Mirrlees (1971), Saez (1999)). A second purpose of this paper is to expose these 'exotic' possibilities.

In the next section, the new formula is derived and interpreted. In Section 3, problems concerned with the marginal rate at the upper-end of the tax schedule are considered. When the highest income is known at the outset, the famous (or infamous) result is obtained that the marginal tax rate should, on efficiency grounds, be zero. When this is not the case, the limiting rate may still be obtainable from a simple formula. The tax formula is derived under the assumption that the optimal tax schedule is twice differentiable. However, in Section 4, a perturbation technique is used to show that 'bunching' may be optimal. The analysis suggests that the structure of optimal tax schedules may be much more complicated than earlier analysis have suggested.

Section 5 considers how various perturbations can be put together to produce a global result like an overall increase in progressivity. When the technique is used to derive the optimal linear income tax, the 'simplified formula' of Dixit and Sandmo (1977) is reproduced. One can also obtain an insight into the structure of optimal polynomial tax schedules.

Concluding remarks are contained in Section 6.

2. The New Tax Formula

The net effect of an income taxation scheme is that post-tax income, which may be thought of as consumption,[2] is made some function of pre-tax income, the latter being under an individual's control by varying effort, hours of work, etc. In Figure 1, the schedule $c(y)$ gives the consumption-income relationship implied by the *optimal* tax schedule $T^*(y)$, i.e.

$$c(y) = y - T^*(y) \tag{1}$$

For the moment, I will assume that c is twice-differentiable, which follows if and only if T^* is twice differentiable. In Section 4, the relevance of this assumption will be considered in more detail.

Different taxpayers will choose to obtain different pre-tax incomes. An advantage of the present approach is that the reasons for this difference do not need to be considered. But for simplicity, I will assume that those who choose to obtain the same income have identical tastes and are equally valued in society's objective function (a removal of this assumption does not seem to produce any unexpected changes in results). Again, in Figure 1, $C(y)$ shows the indifference curve of taxpayers who, under c, choose to obtain an income y^*. C is also assumed to be twice-differentiable. Taxpayers who choose to obtain different incomes will have different indifference curves that are tangential to the tax schedule. Thus, more generally, we can write $C(y, y^*)$ and this is assumed to be twice differentiable. Its properties will be investigated in more detail in Section 4.

As C is a curve of constant utility, it seems clear that its curvature is related to the compensated elasticity of supply of labour, ε. If w is the rate at which effort (pre-tax income) can be turned into consumption (post-tax income) then, holding utility constant, the taxpayer will choose a supply of effort y where

$$C'(y) = w \tag{2}$$

(where a prime denotes derivatives).

[2] A timeless world is being examined but consumption here could be thought of as that which is available for spending over some defined time period, e.g. a year, and taxation is based upon yearly income. With rational agents, it would be more reasonable to relate the present value of lifetime consumption to the present value of income.

Now,

$$\varepsilon \; = \; \frac{dy}{dw}\,\frac{w}{y} \tag{3}$$

so that, at y^*,

$$C''(y^*) = \frac{(1-t)}{\varepsilon y^*} \tag{4}$$

where t is the marginal tax rate at y^*:

$$t = 1 - c'(y) = 1 - C'(y). \tag{5}$$

It may also be noted that second-order conditions of optimization give

$$C''(y^*) \;\geq\; c''(y^*) \;, \tag{6}$$

which, as

$$c''(y^*) = -t' \;, \tag{7}$$

becomes,

$$\frac{(1-t)}{\varepsilon y^*} \;\geq\; -\,t' \;. \tag{8}$$

This says that if the marginal tax rate falls too rapidly, taxpayers will not choose to be at that part of the income tax schedule.[3]

As c derives from the optimal tax schedule, any perturbation in the schedule must lower welfare. Some perturbations induce inefficiencies and to a first-order, cause a strict reduction in welfare. The trick that permits first-order conditions to be used to characterize optimality depends on finding perturbations that can be operated in a 'reverse' direction: if a perturbation reduces welfare to a first-order then the 'reverse' policy must raise welfare (assuming appropriate differentiability) and, as this is ruled out at the optimum, there can be no welfare change to a first-order from such perturbations.

Consider a perturbation which, in Figure 1, changes the consumption schedule from a, b, d, e to a, b, f, g. Up to b the schedule is unaffected, after d the new schedule is a constant distance above the old one. The effect of such a change is that taxpayers between y and $y + \Delta y$ are encouraged to work harder, so that the disincentive effect of the tax is diminished, but this is achieved at the 'cost' of transferring resources to everybody with an income exceeding $y + \Delta y$.

[3] In the original analyses of Mirrlees (1971) it is shown that there are no 'holes' where no taxpayers choose to locate when the schedule is optimally chosen and there is a desire for redistribution.

[Figure 1 about here]

To evaluate the effects of such a perturbation, it is necessary to define some more terms. First, the number of taxpayers involved is important. Let F be the cumulative distribution of income *when the optimal income tax is in operation* and let f be the associated density function. Second, to evaluate the welfare effect of a change, it is necessary to posit society's trade-off between gains to some and losses to others. For this purpose, let $\alpha(y)$ be the social marginal utility of *consumption* at the optimum for taxpayers who choose to earn income y. Let λ be the marginal value of taxes to the government - this is the lowest social cost which can be incurred to raise revenue. If a pound is given to a taxpayer with income y then the *net* gain to society, what Diamond (1975) has termed the social marginal utility of *income* $\beta(y)$, will depend upon the change in tax revenue from the taxpayer that occurs because of the labour supply response to an increase in lump-sum income. With a nonlinear budget constraint, comparative static effects differ from those under linearity. If $y(w, m)$ is the pre-tax work income that somebody chooses to obtain when w is the rate at which pre-tax income can be converted into post-tax income and m is lump-sum income then if the income y is chosen under $c(\cdot)$ it must satisfy[4]

$$y = y(c'(y),\ c(y) - yc'(y)) \tag{9}$$

The total effect on y of an increase in lump-sum income is given by

$$\frac{dy}{dm} = \frac{y_m}{1 - c'' \cdot \ (y_w \ - yy_m)} \tag{10}$$

Invoking Slutksy's equation allows this to be reduced to

$$\frac{dy}{dm} = \frac{y_m}{1 + \frac{t'y\varepsilon}{1-t}} \tag{11}$$

Using λ to value the change in tax revenue, β is given by

$$\beta(y) = \alpha(y) + \frac{\lambda t y_m}{1 + \frac{t'y\varepsilon}{1-t}} \tag{12}$$

As leisure can be expected to be normal, y_m will be negative and, as the denominator of the second term in (12) is positive (see (8)), β will be less than α.

[4] This technique may be used to neatly follow through the comparative statics exercise for a consumer faced by a nonlinear budget constraint. See Neary and Roberts (1980).

Finally, following Mirrlees (1976), we can define the social marginal utility of *transfer* γ to be the net gain of transferring one pound from the government to an individual:

$$\gamma(y) = \beta(y) - \lambda = \alpha(y) - \lambda\ (1 - \frac{ty_m}{1 + \frac{t'y\varepsilon}{1-t}}) \tag{13}$$

As the tax schedule can be shifted both vertically upwards and vertically downwards, it is clear that with optimal taxation,

$$0 = \int_0^\infty \gamma(y) f(y) dy. \tag{14}$$

Returning to the evaluation of the perturbation, the efficiency effect which reduces the disincentive for a small number must be compared with the distributional effect for a large number. Consider the efficiency effect first. As a particular income is being discussed, let $\widetilde{y}$ be used as the appropriate carrier variable. For small changes, the change in tax revenue is

$$\Delta T = \int_{y}^{y+\Delta y} (y + \Delta y - \widetilde{y} - C(y + \Delta y) + c(\widetilde{y})) f(\widetilde{y})\ d\widetilde{y}\ . \tag{15}$$

For the distributional effect, the distance between d and f in Figure 1 is important. This is given by $C(y+\Delta y) - c(y+ \Delta y)$ and, if this is small, the effect in welfare terms is given by

$$\Delta\ W = \left[\int_{y+\Delta y}^\infty \gamma(\widetilde{y}) f(\widetilde{y}) d(\widetilde{y})\right]\ [C(y + \Delta y) - c(y + \Delta y)] \tag{16}$$

To simplify this expression, assume that Δy is small so that (12) and (13) can be approximated by Taylor expansions. Using (5), (15) becomes

$$\Delta T = \frac{tf(y)(\Delta y)^2}{2} + o(\Delta y^2)\ . \tag{17}$$

and using (4), (5) and (7), (16) becomes

$$\Delta W = \left[\int_y^\infty \gamma(\widetilde{y}) f(\widetilde{y}) d(\widetilde{y})\right] \left[\frac{1-t}{\varepsilon y} + t'\right] \frac{(\Delta y)^2}{2} + o(\Delta y^2) \tag{18}$$

It may be noted that we have ignored the welfare effect of taxpayers between y and $y+ \Delta y$ being made better off. This is a third-order effect and, for small Δt, is dominated by (18).[5]

The net welfare gain from the perturbation is therefore given by $\Delta W + \lambda \Delta T$. To show that this must be zero at the optimum, it is necessary to show that the 'reverse'

[5] See Section 4 below.

perturbation is possible. In a symmetrical fashion, one can consider increasing the progressivity of the tax faced by taxpayers who earn income y and this is accomplished by benefitting all those who earn *lower* incomes. Using (14), one obtains the result that the welfare effect of such a perturbation is the negation of the effect that has been derived. As both changes are feasible, it must be true that the net welfare effect is zero. Thus, letting Δy tend to zero gives (using (14))

$$\lambda t f = \left(\frac{1-t}{\varepsilon y} + t' \right) \Gamma(y) \tag{19}$$

where

$$\Gamma(y) = \int_o^y \gamma(\widetilde{y}) f(\widetilde{y}) \, d\widetilde{y}. \tag{20}$$

Before considering the economic implications of (19) two points may be mentioned. First, (19) has been derived under the assumption that the tax schedule is twice-differentiable and that taxpayers' indifferences curves are smoothly tangential to the curve $c(y)$. An example of where this will not be satisfied occurs when some taxpayers choose not to work and individual optimization leads to a corner solution. The approach taken here can be amended to cope with this situation and the possible optimality of a non-differentiable schedule will be investigated in Section 4.

The second point deals with the content of (19). As it stands, it seems to define a differential equation in the marginal tax rate and its solution will depend upon some initial condition (the *level* of taxation is determined by the government revenue requirement). However, notice that as Γ will be zero at the end-points of the income distribution, if f is bounded away from zero then t will have to be zero (see (23) below). Using this, differentiation of (19) at the upper-end point $\overline{y}$, say, gives

$$t' = \left[\frac{\gamma(\overline{y})}{\lambda - \gamma(\overline{y})} \right] \frac{1}{\varepsilon \overline{y}} \tag{21}$$

and this serves as a further condition to pin down the marginal tax schedule.

Looking at (19), the first point to be made is that, as promised, it provides a formula for the optimal nonlinear income tax which is defined solely by the easily interpretable functions that are familiar from linear tax theory. In particular, echoing Diamond's (1975) remarks, distributional judgements are completely captured by the social marginal utilities of income (or transfer). The next remarkable feature concerns the structure of the formula. If the nonlinearity feature (captured by t') is small then (19) becomes a tax formula which incorporates an inverse elasticity rule - viewing the tax rate as a tax on general consumption, the tax rate should be inversely proportional to the compensated elasticity of labour supply.[6] Furthermore, efficiency considerations (captured by the elasticity) and distributional considerations (captured by the social marginal utilities of transfer) enter into the tax rule multiplicatively. With regard to the general shape of the tax schedule, it has already been mentioned that Γ is zero at the lowest and highest income (recall (14)). With f bounded away from zero at these end-points, (19) says that the marginal tax rate must be zero at these end-points (see Sadka (1976) for the upper-end result, Seade (1977) for the lower-end result). Furthermore, if the government would, on welfare grounds, be in favour of redistributing income from high-income taxpayers so that γ declines with y then Γ will be positive between the end-points and, using (8), the optimal marginal tax rate will be positive. Thus when the optimal schedule is differentiable and there are no corner solutions for taxpayers, the marginal tax rate starts at zero, becomes positive, and falls again to zero at the top of the income distribution. This feature at the upper-end will be investigated in greater detail in the next section.

In simple cases, (19) can be used to obtain an explicit formula for the optimal tax schedule. Assume, following Phelps (1973), that the government is Rawlsian in the sense that α is non-zero only for taxpayers with the lowest income, and that there is a zero income effect of labour supply. For almost everybody γ will then be given by $-\lambda$. If one further assumes that $1-F = k(\overline{y}\text{-}y)^{\sigma}$ and that everybody's compensated labour supply curve possesses the same slope s (this seems to be the simplest assumption possible) then the only feasible solution to (19) is given by

$$t = \frac{1}{(\sigma+1)s}\,(\overline{y} \;- y) \tag{22}$$

which implies that a quadratic tax schedule with a declining marginal tax rate is optimal. But notice that if k is small then low incomes will arise and t will exceed unity at low incomes; this will be incompatible with a smooth tangency with the tax schedule for low income taxpayers. In this case the optimum will involve some non-differentiabilities.

3. The Limiting Tax Rate

[6] If τ is the consumption tax then with income y, consumption c is given by $c = \frac{y}{(1+\tau)}$. For this to be the equivalent to an income tax t so that $c = (1-t)y$, τ must be given by $t/(1-t)$.

Perhaps the strongest result to come out of optimal nonlinear tax theory is that, on efficiency grounds, the taxpayer who earns the highest income should face a zero marginal tax rate (Phelps (1973), Sadka (1976)). Using (19), if $\overline{y}$ is the highest income then, close to $\overline{y}$, Γ is approximated by $\gamma(\overline{y})(y-\overline{y})f(\overline{y})$ and, locally, (19) looks like

$$t' \;=\; \frac{t}{(y - \overline{y})}\,\frac{\lambda}{\varepsilon y \gamma} \tag{23}$$

which solves to give $t(\overline{y}) = 0$.

The problem with this derivation is that it depends upon there being a highest income. Whilst a highest income will always exist, it is not necessarily the case that a government will have knowledge of what it will be in the *ex ante* state when the tax structure is being determined. If the government believes that the income distribution will be $f(y,r)$ with probability $g(r)$ then the maximization of expected welfare is similar to the certainty problem with income distribution h where $h(y) = \int f(y,r)g(r)dr$. Then, with a high probability, the highest income observed will fall short of the upper-bound on the support of h - the simple zero rate at the top result will not usually be applicable.[7]

We are thus interested in tax rates for high incomes but ones below what could, in some circumstances, be considered possible. For simplicity, assume that h has no upper bound and let us investigate the limiting marginal tax rate. If the appropriate functions tend to a limit then (19) gives

$$limt = \frac{1}{1 - \dfrac{\lambda\overline{\epsilon}\ (lim\ \frac{yf}{1-F})}{\overline{\gamma}}} \tag{24}$$

where $\overline{\epsilon}$ and $\overline{\gamma}$ are limiting values of ϵ and γ.

Notice that the income distribution enters through $\lim \frac{yf}{1-F}$. If f is bounded below at the upper bound of its support then the limit will be infinite and this takes us back to the zero marginal tax rate. However, $\frac{yf}{1-F}$ is the constant Pareto parameter of the Pareto distribution so that the widely observed phenomenon that the tail of the income distribution appears to be Pareto in form is equivalent to an observation

[7] Many other arguments against the zero rate at the top can be given. For instance, assume that wages are marginal products plus an error term. Then the highest income earner will almost certainly receive a wage greater than his marginal product and, on efficiency grounds, it will be desirable for him to face a strictly positive marginal tax rate. Of greater importance is the fact that income will be stochastic for fixed effort on the part of a taxpayer. Taxation will then have an insurance role (Varian (1980)) and those with the highest incomes will be those who have had the greatest luck. In this case, the insurance aspect of taxation will be dominant.

that $\mu = lim \frac{yf}{1-F}$ is bounded away from zero and, being more specific, empirical work suggests that it lies between 1.5 and 3.[8] Unfortunately, as an optimal tax formula (24) has a serious defect. To uncover this, it is necessary to look more carefully at the role played by differentiability.

4. Differentiable Schedules and Optimality

The derivation of the optimal tax schedule assumed implicitly that the schedule was twice-differentiable and that taxpayers' indifference curves were tangential to the schedule at the income that they wished to earn. The possibility of a corner solution at zero income has already been mentioned. Here, the object is to see whether it is desirable for the government to introduce kinks into the tax schedule, the implication of these kinks being that taxpayers with different preferences between leisure and consumption will choose to obtain the same income.

To investigate this problem, we will take the schedule defined by (19), introduce a perturbation that produces a kink, and see what effect is produced. Figure 2 shows the effect of offering the consumption/income point K as well as the schedule $c(y)$. The effect of this is equivalent to introducing a small kink in the schedule at K and all taxpayers who chose to obtain income between $y - x_1$ and $y + x_2$ will now move to K. The change in tax revenue to the government is given by

$$\Delta T = \int_{y-x_1}^{y+x_2} (y - \widetilde{y} - c(y) - d + c(\widetilde{y}))f(\widetilde{y})\, d\widetilde{y} \tag{25}$$

[Figure 2 about here]

To evaluate this, we note first that, if $C(\cdot, \widetilde{y})$ is the indifference curve of a taxpayer which is tangential to $c(\cdot)$ at $\widetilde{y}$, then x_1 and x_2 are defined by

$$C(y, y - x_1) = c(y) + d \tag{26a}$$

$$C(y, y + x_2) = c(y) + d \tag{26b}$$

[8] Using tax return data, Saez (1999) estimates μ to vary between 1.8 and 2.2 over time.

where, for all x,

$$C(y+x,y+x)=c(y+x) \tag{27}$$

and

$$C_1(y+x,y+x)=c'(y+x) \tag{28}$$

Differentiating (27) and applying (28) gives

$$C_2(y+x,y+x)=0 \tag{29}$$

Twice-differentiation of (27) gives

$$C_{11}\ +2C_{12}\ +C_{22}\ =c'' \tag{30}$$

and differentiation of this gives

$$C_{111}\ +3C_{112}\ +3C_{122}+C_{222}=c'''. \tag{31}$$

Multiple differentiation of (29) gives

$$C_{21}\ +C_{22}\ =0 \tag{32}$$

and

$$C_{112}\ +2C_{122}+C_{222}=0 \tag{33}$$

Eliminating d from (26a,b) gives x_2 as a function of $x_{1.}$ Because of the degeneracy induced by (29), the derivative of x_2 with respect to x_1 depends upon the second derivative of C (through l'H$\widehat{o}$pital's rule) and we obtain

$$\begin{aligned} x_2(0)&=0\\ \frac{dx_2}{dx_1}(0)&=1\\ \frac{d^2x_2}{dx_1^2}(0)&=-\frac{2}{3}\frac{C_{222}}{C_{22}} \end{aligned} \tag{34}$$

so that, using (30) and (32) to solve for C_{22}, and (31) and (33) to solve for C_{222}, we obtain

$$\frac{d^2x_2}{dx_1}=-\left(\frac{2}{3}\right)\frac{2C_{111}+3C_{112}-2c'''}{C_{11}-c''}. \tag{35}$$

The denominator is related to the compensated elasticity of labour supply and the curvature of the tax function whilst the numerator is related to how an individual's elasticity charges with income, how elasticity varies across individuals, and changes to the curvature of the tax function.

Turning to (25), a Taylor expansion can be taken around $x_1 = x_2 = 0$. Differentiation with respect to x_1 with d and x_2 viewed as functions of x_1 gives

$$\begin{aligned}\frac{\partial \Delta T}{\partial x_1} &= (-x_2 \; - c(y) - d + c(y + x_2))f(y + x_2)\frac{dx_2}{dx_1} \\ &+ (x_1 - c(y) - d + c(y - x_1))f(y - x_1) \\ &- \int_{y-x_1}^{y+x_2} d'(x_1)f(\widetilde{y})d\widetilde{y}\end{aligned} \tag{36}$$

$$\begin{aligned}\frac{\partial^2 \Delta T}{\partial x_1^2} &= [(-1 + c'(y + x_2))f(y + x_2) + (-x_2 - c(y) - d + c(y + x_2))f'(y + x_2)]\left(\frac{\partial x_2}{\partial x_1}\right)^2 \\ &+ [(-x_2 - c(y) - d + c(y + x_2))f(y + x_2)]\frac{d^2 x_2}{dx_1^2} \\ &+ (1 - c'(y - x_1))f(y - x_1) - (x_1 - c(y) - d + c(y - x_1))f'(y - x_1) \\ &- 2d'(f(y + x_2)\frac{dx_2}{dx_1} + f(y + x_1)) - \int_{y-x_1}^{y+x_2} d''(x_1)f(\widetilde{y})d\widetilde{y}\end{aligned} \tag{37}$$

Differentiation once again and letting x_1 tend to zero gives

$$\frac{\partial^3 \Delta T}{\partial x_1^3} = 2f(y)c''(y) - 6d''(0)f(y) - (1 - c'(y))(4f'(y) + 3f(y)\frac{d^2 x_2}{dx_1^2}) \tag{38}$$

where we have made use of the fact that $d(0) = d'(0) = 0$ (using (26) and (29)). Now,

$$d''(0) = C_{22} = C_{11} - c'' \tag{39}$$

(using (30) and (32)). At $x_1 = 0$, (36) and (37) both become zero so (38) becomes the dominant effect. Using (4), (5), (38) and (39) we have ΔT as a third order Taylor expansion in x_1

$$\Delta T = -\frac{1}{3}[4ft' + \frac{3f(1-t)}{\varepsilon y} + 2tf' + \frac{3}{2}tf\frac{d^2 x_2}{dx_1^2}]x_1^3 + o(x_1^3) \tag{40}$$

This expression applies for any differentiable tax schedule. If we abstract from the change in curvature term $\frac{d^2 x_2}{dx_1^2}$, as the second term in square brackets is positive, ΔT will be positive only when the proportional rise in the marginal tax rate is small relative to the proportional fall in the income distribution density. Alternatively, when the tax schedule is approximately linear so that $t' = 0$, ΔT will be positive only when the marginal tax rate is large and the density of the income distribution is falling sufficiently rapidly. This suggests that ΔT will be negative at low income levels but may be positive at high income levels under tax schedules observed in practice.

We now investigate whether the 'optimal' tax schedule derived in Section 2 is in fact optimal by considering the introduction of a kind. Applying (19) to (40) gives the tax effect

$$\Delta T = \frac{1}{3}\left[\frac{f}{\epsilon y} - t(2f' + \frac{4f^2\lambda}{\Gamma} + \frac{f}{\epsilon y} + \frac{3}{2}f\frac{d^2x_2}{dx_1^2})\right] x_1^3 + o(x_1^3) \tag{41}$$

As well as the tax revenue change, another third-order effect will be the welfare improvement brought to those with incomes between $y - x_1$ and $y + x_2$. Even if $\alpha(y) = 0$, however, (41) being positive is possible.

To investigate this further, we concentrate on the optimality of the limiting tax rates derived in the last section. If $\lim \frac{yf}{1-F} = \mu$ then $\lim \frac{yf'}{f} = -(\mu + 1)$. Turning to the change in curvature effect, if this is small then (41) will be positive if

$$t > \frac{1}{1 + \frac{2\varepsilon}{3}(\mu + 1)}. \tag{42}$$

For instance, with a Pareto parameter μ of 2 and an elasticity of $\frac{3}{4}$ then (44) is satisfied if t is above $\frac{2}{5}$. A more reasonable assumption would be to assume that, in the limit, the compensated elasticity of a taxpayer is locally constant, and approaches a limit when viewed across taxpayers. Using (4), C will take the form

$$C_{11}(y, \widetilde{y}) = \frac{C_1(y, \widetilde{y})}{\varepsilon y} \tag{43}$$

Using (43) and (35) gives

$$\frac{d^2x_2}{dx_1^2} = \left(\frac{2}{3}\right)\frac{2 + 1/\varepsilon}{y} \tag{44}$$

Equation (40) will then be positive if

$$t > \frac{3}{1 + 2\mu\varepsilon}. \tag{45}$$

Taking $\mu = 2$ and $\varepsilon = \frac{3}{4}$, this will be satisfied if t is above $\frac{3}{4}$. If the limiting tax rate is set optimally then combining (24) and (45) gives $\Delta T > 0$ if

$$\frac{1}{\mu\varepsilon} < 1 + \frac{3\lambda}{2\gamma}. \tag{46}$$

When high income earners are not given positive welfare weight ($\alpha = 0$), so that there is no direct welfare gain from the introduction of a kink, we have $-\gamma \geq \lambda$. If $-\gamma$ is close to λ then (46) is never satisfied for positive μ and ε. But, recalling (13), if the income effect of labour supply is large, then satisfaction of (46) is made more likely.

If all taxpayers are at kink points at the upper end of the optimal tax function, then it is not difficult to determine something of the structure of these kink points. Assume that there is an upper kink point (A in Figure 3) and that the taxpayer with preferences BAD and everybody with a greater preference for work (less sloped indifference curves) are bunched at A. If B is a point on the tax function that taxpayers will move to if A is withdrawn then it is clear that the government is free to choose A anywhere on the indifference curve to the right of B without changing the composition of the bunched group or affecting everybody else in the population. As it is being assumed that $\alpha = 0$ for the bunched group, the net effect of a movement around the indifference curve is the tax change of the bunched group - clearly, taxes increase by a movement to the right whenever the slope of the indifference curve is less than unity. Therefore, the slope of the indifference curve at the kink point must be at least unity. But now consider offering a new point E with the property that the slope between A and E is, for instance, one-half. E will be chosen in preference to A by those with the greatest preference for work and, as is obvious, the net effect will be to increase tax revenue without producing any welfare effect. Thus we have shown that there can be no upper kink point - in the optimum, there are an infinite number of kink points offered at the tail-end of the tax function.

[Figure 3 about here]

Finally, when α is non-zero, a different picture can emerge. The direct welfare effect of the kink perturbation is approximated by

$$\Delta W = \frac{\lambda t f^2 \alpha}{3}\, x_1^3 \tag{47}$$

and if this is combined with (41) then, as γ must incorporate α, the net effect of the kink perturbation is made more ambiguous. However, (47) may be combined with (41) to give a measure of the net effect of the perturbation. If α is large relative to the weight given to most taxpayers (so that λ is not correspondingly large) then

(47) will dominate (41) and again the introduction of a kink can be desirable. Thus if the government weights low-income taxpayers much more highly than high-income taxpayers then one may expect to observe a kink at the bottom of the tax schedule with a group of low income taxpayers being bunched at the kink. In this case, the result that the optimal tax rate at the lowest income is zero (Seade (1977)) will not hold.[9]

These results point to the complex nature of the nonlinear taxation problem. Given the structure of the problem, one can expect to be able to provide the strongest results about taxation at the two ends of the tax schedule. The analysis of this section suggests that there can be non-differentiabilities at both ends of the schedule and results about marginal tax rates will fail to be meaningful.

5. Global Perturbations and Tax Reform

In Section 2, the fact that perturbations in a tax schedule at a particular point are permissible was used to determine a set of first-order conditions of optimality for nonlinear income tax schedules and, subject to the caveat discussed in the last section, these conditions may be used to determine the optimal schedule. However, the analysis conducted is directly applicable to the situation where the tax schedule is sub-optimal and one wishes to ask whether the marginal tax rate should be raised or lowered at a particular income level. Using λ, (17) and (18) may be combined to give the net welfare effect of a small perturbation. Figure 1 makes clear that the perturbation is, in essence, a reduction in the marginal tax rate from $1-c'(y) = 1-C'(y) to 1-C'(y+\Delta y)$. Therefore, the change in the marginal tax rate is given by

$$\Delta t = -C'' \frac{(\Delta y)^2}{2} + o(\Delta y^2) \tag{48}$$

Thus, for small Δt, the net welfare change is given by (using (17) and (18)):

$$\Delta V = \left[\Gamma(y)(\frac{t'\varepsilon y}{1-t} + 1) - \frac{\lambda t f \varepsilon y}{1-t}\right] \Delta t \tag{49}$$

where it is here assumed that, at the margin, tax revenue is raised in a lump-sum manner.

[9] The possible optimality of a kink should not be too surprising. On welfare grounds it may be desirable to force those who choose to obtain higher incomes to obtain a lower income (work less). The fact that choices are decentralized through a single budget constraint implies that the best that the government can do is to bunch a group of taxpayers at a particular income.

Equation (49) can be used to compute the effect of a small perturbation in the whole schedule. Assume that the government is contemplating a change in the progressivity of the tax structure which involves the marginal tax at $y, t(y)$, changing at rate $p(y)$. The change in welfare will be given by

$$\Delta V = \int_y \left[\Gamma(y)(\frac{t'\varepsilon y}{1-t} + 1) - \frac{\lambda t f \varepsilon y}{1-t} \right] p(y)dy \tag{50}$$

As an example, assume that the tax is initially linear ($t' = 0$) and that the government is contemplating a change to a linear tax with higher rate. Then $p(y)$ would equal some constant p and (50) will become (integrating Γ by parts):

$$\Delta V = \int_y (\gamma - \frac{\lambda t \varepsilon}{1-t}) y p f dy \tag{51}$$

Equating this to zero gives the optimal *linear* tax rate:

$$\frac{t}{1-t} = \frac{\int \gamma y f dy}{\lambda \int \varepsilon y f dy} \tag{52}$$

which is exactly the "simplified" formula of linear taxation derived by Dixit and Sandmo (1977).

Equation (50) can also be used also to derive first-order conditions for optimal polynomial tax schedules, e.g.

$$T = a_o + a_1 y + \frac{a_2 y^2}{2} + ... \frac{a_r y^r}{r} \tag{53}$$

If a change in a_k is contemplated then $p(y)$ is given by $a_k y^{k-1}$ and inserting this in (50) gives, on equating the result to zero, a condition for the optimal a_k.

Two final points concerning (50) are worth mentioning. First, if (50) is used to evaluate a tax reform then the functions in the formula will be evaluated at the initial point. For instance, f will be the present observed distribution of income. Second, a particular example of (50) gives the change in tax revenue as the result of a change in tax structure. For with $\alpha = 0$, (50) becomes, apart from a multiplier λ, the change in tax revenue. Therefore, a change in the tax structure p will lead to a change in tax revenue of

$$\Delta T = -\int \left[\left(\int_y^\infty (1 - \frac{t y_m}{1 + \frac{t'\tilde{y}\varepsilon}{1-t}} f(\tilde{y})) d\tilde{y} \right) (\frac{t'\varepsilon y f}{1-t} + 1) + \frac{t \varepsilon y f}{1-t} \right] p(y)dy \tag{54}$$

Even if p is constant, (54) requires considerable calculation. However, in empirical analysis of budget changes, numerical calculation should be straightforward.

6. Concluding Remarks

The purpose of this paper has been to adopt a non standard approach to the solution of the optimal nonlinear tax problem. This approach seems to have four advantages. First, the approach itself is easily understood, being both direct and simple. Second, the formula produced for the optimal tax rate is easily interpretable as it is defined in terms of functions that are easily interpretable. This formula has close connections to formulae in linear tax theory. Third, an identical approach can be used to evaluate income tax reform and, as was shown in the last section, it is possible to evaluate different types of marginal reform that may be envisaged in a nonlinear framework. Finally, it has been possible to demonstrate some of the inherent complexity of the set of problems to which the optimal income tax problem belongs. In particular, it has been shown that although a well-behaved differentiable tax function that satisfies optimality with respect to well-behaved perturbations can be found, it is sometimes the case that the problem will involve non-differentiabilities at the optimum. The problem raised by this is that it will not always be clear that well-behaved solutions are not optimal - in numerical work, the search for the optimum could be quite difficult.

APPENDIX

The purpose of this appendix is to show how the optimal tax rule (19) can be derived using standard control methods. To do this, assume that individuals are indexed by a parameter z which determines tastes and social deservingness. The parameter z will be defined so that, *at the optimum*, the supply of effort, i.e. pre-tax income y, will be z for a $z-$type individual.

If $v(m, wz)$ is the indirect utility function of a z type individual where m is (lump-sum) income and w is the rate at which pre-tax income can be converted into post-tax income, i.e. the wage, then Roy's Identity gives

$$v_w(m, w, z) = y(m, w, z)v_m(m, w, z) \tag{A.1}$$

where $y(m, w, z)$ is the (uncompensated) supply equation ((9) in the text). When an individual chooses to be at a point on a nonlinear tax schedule, he makes the same choice as he would under a linear tax schedule passing through that point with the same slope as the nonlinear schedule at that point (see Figure 4).[10] Thus a nonlinear schedule is equivalent to a situation where individuals face linear tax schedules though the implied m and w will depend upon the tastes of the individual z. For a nonlinear schedule $c(\cdot)$, $m(z)$ and $w(z)$ will be defined by

$$m(z) = c(y(z)) - c'(y(z))y(z) \tag{A.2}$$

$$w(z) = c'(y(z)) \tag{A.3}$$

where $y(z)$ is the pre-tax income choice of an individual. Differentiating (A.2) and (A.3) gives

$$\frac{dm}{dz} + y(z)\,\frac{dw}{dz} = 0 \tag{A.4}$$

which is the constraint imposed by the fact that $m(\cdot)$ and $w(\cdot)$ are formed from a nonlinear tax schedule.

Locally around the optimum, the government's objective can be viewed as being linear in utilities:

$$W = \int_0^\infty \delta(z)v(m(z), w(z), z)f(z)dz \tag{A.5}$$

[10] This is discussed further in Roberts (1979).

where $f(z)$ is the density of z - this will, by assumption, describe the income distribution at the optimum.

Finally, there is a resource constraint faced by the government - aggregate income must be at least as great as aggregate consumption plus required government revenue R:

$$\int_0^\infty [y(z) - c(y(z))]\; f(z)dz \geq\; R \tag{A.6}$$

which, using (A.2) and (A.3), becomes

$$\int_0^\infty [m(z) + (w(z) - 1)y(z)]\; f(z)dz + R \leq\; 0 \tag{A.7}$$

Instead of choosing a tax schedule, the government may be viewed as choosing functions $m(\cdot)$ and $w(\cdot)$ to maximize (A.5) subject to (A.4), (A.7) and (A.8):

$$y(z) = y(m(z), w(z), z). \tag{A.8}$$

As λ is the value of a change in tax revenue, λ will be the multiplier associated with (A.7) in the optimization problem. Letting μ_1 and μ_2 be the multipliers associated with (A.4) and (A.8), standard control theory gives first-order conditions:

$$\delta v_m f - \;\lambda f - \frac{d\mu_1}{dz} \;-\; \mu_2\; y_m \;= 0 \tag{A.9}$$

$$\delta v_w f - \lambda y f - y\,\frac{d\mu_1}{dz} \text{ - } \mu_1\,\frac{dy}{dz} \;-\; \mu_2 y_w \;= 0 \tag{A.10}$$

$$-\lambda(w-1)f + \mu_1\,\frac{dw}{dz} + \mu_2 \;= 0 \tag{A.11}$$

$$\mu_1(o) = \;\mu_1(\infty) \;= 0(\text{transversality}) \tag{A.12}$$

Now, by definition, $\delta v_m = \alpha$ and (A.1) gives $\delta v_w = \alpha y$. Combining (A.9) and (A.10) gives

$$\mu_1\,\frac{dy}{dz} \;= -\mu_2(y_w \;- y y_m) = \;-\;\frac{\mu_2 y \varepsilon}{w}\;, \tag{A.13}$$

the second equality following from Slutsky's equation (see (12) in the text). Now, at the optimum$\frac{dy}{dz} = 1$ so that solving for μ_2 from (A.11) and (A.13) and replacing μ_2 in (A.9) gives

$$\frac{d\mu_1}{dy} \;=\; \left[\alpha - \lambda \;+\; \frac{\lambda t y_m}{1 + \;\frac{t' y \varepsilon}{(1-t)}}\right] f(y) = \;\gamma(y) f(y) \tag{A.14}$$

from (13) in the text ($y(z) = z, t = 1 - w, t' = -\frac{dw}{dz}$). Equation (20) therefore gives (using (A.12)):

$$\Gamma(y) = \mu_1(y). \tag{A.15}$$

Finally, (A.11), (A.13) and (A.15) give, on rearrangement

$$\lambda t f = \left(\frac{1-t}{\varepsilon y} + t' \right) \Gamma(y) \tag{A.16}$$

which is the optimal tax formula in the text.

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