

Adaptive Model Predictive Control with Robust Constraint Satisfaction

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Abstract: Adaptive control for constrained, linear systems is addressed and a solution based on Model Predictive Control (MPC) and set-membership system identification is presented. The paper introduces a computationally tractable solution which uses observations of past state and input trajectories to update the model and improve control performance while maintaining guaranteed constraint satisfaction and recursive feasibility. The developed approach is applied to a stabilizing MPC scheme and practical stability under persistent, additive disturbance is proved. A numerical example and brief comparison with non-adaptive MPC is provided.

Keywords: Model Predictive Control, Adaptive Control, Constraint Satisfaction Problems, Uncertain Linear Systems, System Identification

1. INTRODUCTION

Model Predictive Control (MPC) has become one of the main tools to handle multivariable constrained control problems. The basic idea is to solve an open-loop finite-horizon optimal control problem in each sampling time, given the current state and a model of the process. Since the performance and validity of MPC crucially depends on the model accuracy, Robust and Stochastic MPC has received much attention (Rawlings and Mayne (2009), Kouvaritakis and Cannon (2016)). While these approaches are suitable to handle unmodelled dynamics and fast changing disturbances they are inherently conservative for slowly changing or constant parametric uncertainty. To reduce the cost of manual tuning in MPC and cope with slowly changing dynamics, e.g. due to changing environment and wear, there is a strong interest in self-tuning predictive control formulations but only few solution strategies are available, cf. Qin and Badgwell (2003).

While the main idea is simple, combining closed-loop system identification with Model Predictive Control, the problem is finding a rigorous, yet tractable formulation with provable constraint satisfaction and stability properties. Most system identification methods lack error bounds or convergence guarantees when used for closed-loop systems and in the presence of additive disturbance or measurement noise. Similarly, given a disturbance and uncertainty model, most Robust MPC formulations rely on extensive offline computations and hence are not suitable for a changing uncertainty description.

In Aswani et al. (2013) and Di Cairano (2016) the described problem has been circumvented by using system

identification only to update a nominal prediction model. Recursive feasibility and constraint satisfaction is guaranteed by employing Robust MPC methods based on an a priori given uncertainty set which is not updated. In Kim and Sugie (2008) Adaptive MPC for linear SISO systems based on a modified least squares estimation is introduced but the approach crucially relies on the assumption of noise and additive disturbances being absent. To improve system identification, in Heirung et al. (2012) the MPC cost function explicitly reflects the cost of model uncertainty and balances identification and regulation objectives. Similar to the approach presented in this paper, a combination of set-membership estimation and robust constraint tightening has been proposed in Tanaskovic et al. (2014). But the algorithm relies on uncertain FIR models and is hence restricted to stable systems and requires long prediction horizons which increases the computational complexity since the uncertainty is described as a polytope in a high-dimensional space.

In contrast, we present an Adaptive MPC algorithm for linear systems with model uncertainty and additive disturbances in a more general state-space formulation. In particular, a suitable, computationally tractable combination of set-membership system identification and set-based state prediction to guarantee recursive feasibility and robust constraint satisfaction is introduced. We focus on stabilization, but the proposed combination of system identification and constraint tightening constitutes a solid basis for future work on Dual MPC or tracking problems for systems with uncertain or slowly varying parameters. Our approach builds upon results on Homothetic Tube MPC, cf. Langson et al. (2004), Raković and Cheng (2013).

Remainder: In Section 2 the main assumptions and problem setup are introduced. In Section 3 the system identification and constraint reformulation are presented followed

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by a stabilizing Adaptive MPC algorithm and its properties. A numerical example is presented in Section 4 and an outlook on future research concludes the paper.

Notation: Uppercase letters are used for matrices, lower case for vectors. $[A]_j$ and $[a]_j$ denote the j -th row and entry of the matrix A and vector a , respectively. By $\mathbf{1}$ we denote a column vector of ones of appropriate size. $\mathbb{R}_{\geq 0}$ is the set of the non-negative reals and $\mathbb{N}_a^b = \{n \in \mathbb{N} \mid a \leq n \leq b\}$. Continuous, strictly increasing functions $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ with $\alpha(0) = 0$ (and $\lim_{r \rightarrow \infty} \alpha(r) = \infty$) are denoted \mathcal{K} (\mathcal{K}_∞). $A \oplus B$ denotes the Minkowski set addition.

2. PROBLEM SETUP

Consider the discrete-time, linear system with state $x_k \in \mathbb{R}^n$, input $u_k \in \mathbb{R}^m$, additive disturbance $w_k \in \mathbb{W} \subseteq \mathbb{R}^n$, and unknown but constant parameter $\theta = \theta^* \in \mathbb{R}^p$

$$x_{k+1} = A(\theta)x_k + B(\theta)u_k + w_k. \quad (1)$$

Assumption 1. (Disturbance and Uncertainty). The disturbance set \mathbb{W} is a bounded, convex polytope given by

$$\mathbb{W} = \{w \mid H_w w \leq h_w\}.$$

The system matrices depend affinely on parameters $\theta_i \in \mathbb{R}$

$$(A(\theta), B(\theta)) = (A_0, B_0) + \sum_{i=1}^p (A_i, B_i)\theta_i \quad (2)$$

with the prior parameter set given by

$$\Theta = \{\theta \mid H_\theta \theta \leq h_\theta\} \quad (3)$$

containing the true parameter vector $\theta^* = [\theta_1^*, \dots, \theta_p^*]^\top$.

To capture actuator saturation or restrict the state to desirable operating regions, we assume mixed state and input constraints which can be modelled with linear inequalities. The state and input should be contained in some compact polytopic set

$$\mathbb{Z} = \{x \in \mathbb{R}^n, u \in \mathbb{R}^m \mid Fx + Gu \leq \mathbf{1}\} \quad (4)$$

with given matrices $F \in \mathbb{R}^{c \times n}$ and $G \in \mathbb{R}^{c \times m}$.

This leads us to the following problem formulation: Find a stabilizing controller for system (1) such that the constraints (4) are satisfied robustly and the model uncertainty (3) is taken into account and updated consistently with the state and input history.

To solve the control problem, a Tube MPC algorithm with online set-membership system identification is considered. We introduce the information state Θ_k to explicitly model the knowledge on the parameter at time k and its influence on state prediction, cost and constraint satisfaction. Θ_k is given by the set of parameters consistent with the prior set Θ and the state and input trajectories $\{x_i, u_i\}_{i \in \mathbb{N}_0^k}$ observed up to time k

$$\Theta_k = \{\theta \in \Theta \mid \forall \{x_i, u_i\}_{i \in \mathbb{N}_0^k} \exists w_i \in \mathbb{W} \text{ s.t. (1)}\}. \quad (5)$$

To cope with state predictions under uncertainty, an input parametrization

$$u_{l|k} = Kx_{l|k} + v_{l|k}$$

is introduced with prestabilizing feedback gain $K \in \mathbb{R}^{m \times n}$ and variables $\mathbf{v}_{N|k} = \{v_{l|k}\}_{l \in \mathbb{N}_0^{N-1}}$, $v_{l|k} \in \mathbb{R}^m$. The feedback gain K can be determined by standard robust control methods to satisfy the following assumption.

Assumption 2. The feedback gain K is chosen such that $A_{cl}(\theta) = A(\theta) + B(\theta)K$ is quadratically stable for all $\theta \in \Theta$.

The states $x_{l|k}$, predicted l steps ahead from time k , are modelled by a set-based formulation with dynamics

$$\mathbb{X}_{0|k} \ni x_k, \quad (6a)$$

$$\mathbb{X}_{l+1|k} \supseteq A_{cl}(\theta)\mathbb{X}_{l|k} \oplus B(\theta)v_{l|k} \oplus \mathbb{W} \quad \forall \theta \in \Theta_k \quad (6b)$$

ensuring that $x_{l|k} \in \mathbb{X}_{l|k}$ for all disturbance and uncertainty realizations.

With this the state and input constraints are robustly satisfied if

$$(x_{l|k}, u_{l|k}) \in \mathbb{Z} \quad \forall x_{l|k} \in \mathbb{X}_{l|k}. \quad (7)$$

The MPC cost function to be minimized is given by

$$J_N(x_k, \mathbf{d}_k) = \sum_{l=0}^{N-1} \ell(\mathbb{X}_{l|k}, v_{l|k}) + V(\mathbb{X}_{N|k}) \quad (8)$$

with decision variables $\mathbf{d}_k = \{\mathbb{X}_{l|k}, v_{l|k}\}_{l \in \mathbb{N}_0^N}$, stage cost ℓ , and terminal cost V . For a given state x_k and terminal set \mathbb{X}_T define the optimal value function

$$V_N(x_k) = \min_{\mathbf{d}_k} J_N(x_k, \mathbf{d}_k) \text{ s.t. (6), (7), } \mathbb{X}_{N|k} \subseteq \mathbb{X}_T, \quad (9)$$

and denote its minimizer $\mathbf{d}^* = \{\mathbb{X}_{l|k}^*, v_{l|k}^*\}_{l \in \mathbb{N}_0^N}$. The MPC control law $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is given by

$$u(x_k) = Kx_k + v_{0|k}^*. \quad (10)$$

In the following, we derive a computationally tractable reformulation for the parameter estimation (5) and the optimization (9) with suitable cost functions ℓ , V and sets $\mathbb{X}_{l|k}$ to solve the regulation problem.

3. MAIN RESULTS

In this section we recall set-membership system identification for the state space description (1), (2) followed by a computationally tractable parametrization for the state tube $\{\mathbb{X}_{l|k}\}_{l \in \mathbb{N}_0^N}$ and a derivation of suitable cost and terminal constraints. The section concludes with a summary of the MPC algorithm and an analysis of closed-loop stability.

3.1 System Identification

Let $D(x, u) \in \mathbb{R}^{n \times p}$ be defined by

$$D(x, u) = [A_1x + B_1u, A_2x + B_2u, \dots, A_px + B_pu].$$

At time $k \geq 1$, given x_{k-1} , u_{k-1} , and x_k , let $d_k = A_0x_{k-1} + B_0u_{k-1} - x_k$, then the non-falsified parameter set can be described by

$$\begin{aligned} \Delta_k &= \{\theta \mid x_k - (A(\theta)x_{k-1} + B(\theta)u_{k-1}) \in \mathbb{W}\} \\ &= \{\theta \mid -H_w D(x_{k-1}, u_{k-1})\theta \leq h_w + H_w d_k\}. \end{aligned} \quad (11)$$

With this the information state of the uncertain parameter θ is given by

$$\Theta_k = \Theta_{k-1} \cap \Delta_k \quad (12)$$

with initial condition $\Theta_0 = \Theta$.

Lemma 3. The set Θ_k is a convex, polytopic set explicitly given in half-space form

$$\Theta_k = \{\theta \mid H_{\theta_k} \theta \leq h_{\theta_k}\} \quad (13)$$

with $H_{\theta_k} \in \mathbb{R}^{q_k \times p}$ and $h_{\theta_k} \in \mathbb{R}^{q_k}$. For all $k \geq 0$ it holds that $\Theta_{k+1} \subseteq \Theta_k$.

Although not necessary from a theoretical perspective, a minimal representation of Θ_k should be used to decrease the computational load of the MPC algorithm. Redundant constraints in (13) can be removed efficiently by

solving a series of LPs, cf (Blanchini and Miani, 2015, Section 3.3). The number of non-redundant half-spaces q_k might grow without bound, creating the necessity of an approximation under which Lemma 3 remains valid, e.g. by explicitly restricting the number of half-spaces as proposed in Tanaskovic et al. (2014).

3.2 State Tube and Constraint Satisfaction

Restricting the sets $\mathbb{X}_{l|k}$, which contain predicted states, to be translations and dilations of a given polytope \mathbb{X}_0 , the MPC optimization (9) can be recast as a computationally tractable, convex, finite dimensional optimization program. This parametrization for Tube MPC has previously been introduced in Langson et al. (2004). With given $H_x \in \mathbb{R}^{u \times n}$ and free variables $z_{l|k} \in \mathbb{R}^n$, $\alpha_{l|k} \in \mathbb{R}_{\geq 0}$ let

$$\begin{aligned}\mathbb{X}_{l|k} &= \{z_{l|k}\} \oplus \alpha_{l|k} \mathbb{X}_0 \\ &= \{x \mid H_x(x - z_{l|k}) \leq \alpha_{l|k} \mathbf{1}\} \\ &= \{z_{l|k}\} \oplus \alpha_{l|k} \text{co}\{x^1, x^2, \dots, x^v\}.\end{aligned}\quad (14)$$

The explicit description in both, vertex and half-space form of $\mathbb{X}_{l|k}$ can be exploited to obtain linear constraints for the predictions (6) and constraints (7).

Proposition 4. Let $\{\mathbb{X}_{l|k}\}_{l \in \mathbb{N}_0^N}$ be parametrized as in (14) with decision variables $z_{l|k}$, $\alpha_{l|k}$. Define $D_{l|k}^j = D(x_{l|k}^j, u_{l|k}^j)$, $d_{l|k}^j = A_0(x_{l|k}^j) + B_0 u_{l|k}^j - z_{l+1|k}$, with $x_{l|k}^j = z_{l|k} + \alpha_{l|k} x^j$, $u_{l|k}^j = K x_{l|k}^j + v_{l|k}$. Define $[\bar{w}]_i = \max_{w \in \mathbb{W}} [H_x]_i w$ for all $i \in \mathbb{N}_1^u$ and $[\bar{f}]_i = \max_{x \in \mathbb{X}_0} [F + GK]_i x$ for all $i \in \mathbb{N}_1^c$.

Equation (6) and (7) are satisfied if and only if for all $j \in \mathbb{N}_1^v$, $l \in \mathbb{N}_0^{N-1}$ there exists $\Lambda_{l|k}^j \in \mathbb{R}_{\geq 0}^{u \times q_k}$ such that

$$(F + GK)z_{l|k} + Gv_{l|k} + \alpha_{l|k} \bar{f} \leq \mathbf{1} \quad (15a)$$

$$-H_x z_{0|k} - \alpha_{0|k} \mathbf{1} \leq -H_x x_k \quad (15b)$$

$$\Lambda_{l|k}^j h_{\theta_k} + H_x d_{l|k}^j - \alpha_{l+1|k} \mathbf{1} \leq -\bar{w} \quad (15c)$$

$$H_x D_{l|k}^j = \Lambda_{l|k}^j H_{\theta_k}. \quad (15d)$$

Proof. Inequality (7) is equivalent to

$$(F + GK)z_{l|k} + Gv_{l|k} + \alpha_{l|k}(F + GK)x \leq \mathbf{1} \quad \forall x \in \mathbb{X}_0$$

which is equivalent to (15a) by maximising over $x \in \mathbb{X}_0$.

Inequality (15b) is equivalent to (6a) and (15c), (15d) are equivalent to (6b) as shown by the following reformulation.

$$\begin{aligned}\mathbb{X}_{l+1|k} &\supseteq A_{cl}(\theta)\mathbb{X}_{l|k} \oplus B(\theta)v_{l|k} \oplus \mathbb{W} & \forall \theta \in \Theta_k \\ \Leftrightarrow & H_x(A_{cl}(\theta)x_{l|k} + B(\theta)u_{l|k} + w - z_{l+1|k}) \leq \alpha_{l+1|k} \mathbf{1} \\ & \forall x_{l|k} \in \mathbb{X}_{l|k}, \theta \in \Theta_k, w \in \mathbb{W} \\ \Leftrightarrow & H_x(A_{cl}(\theta)(z_{l|k} + \alpha_{l|k}x^j) + B(\theta)v_{l|k} - z_{l+1|k}) \\ & - \alpha_{l+1|k} \mathbf{1} \leq -\bar{w} \quad \forall j \in \mathbb{N}_1^v, \theta \in \Theta_k \\ \Leftrightarrow & \max_{\theta \in \Theta_k} \{H_x(A_{cl}(\theta)(z_{l|k} + \alpha_{l|k}x^j) + B(\theta)v_{l|k})\} \\ & - H_x z_{l+1|k} - \alpha_{l+1|k} \mathbf{1} \leq -\bar{w} \quad \forall j \in \mathbb{N}_1^v \\ \Leftrightarrow & \max_{\theta \in \Theta_k} \{H_x D_{l|k}^j \theta\} + H_x d_{l|k}^j - \alpha_{l+1|k} \mathbf{1} \leq -\bar{w} \quad \forall j \in \mathbb{N}_1^v \\ \Leftrightarrow & \begin{cases} \Lambda_{l|k}^j h_{\theta_k} + H_x d_{l|k}^j - \alpha_{l+1|k} \mathbf{1} \leq -\bar{w} \\ H_x D_{l|k}^j = \Lambda_{l|k}^j H_{\theta_k} \\ \Lambda_{l|k}^j \in \mathbb{R}_{\geq 0}^{u \times q_k} \end{cases} & \forall j \in \mathbb{N}_1^v\end{aligned}$$

The first equivalence follows from $x \in \mathbb{X}_{l+1|k} \Leftrightarrow H_x(x - z_{l+1|k}) \leq \alpha_{l+1|k} \mathbf{1}$. The second follows from the left hand

side being convex in x for given θ and since $\mathbb{X}_{l|k}$ does not depend on θ , the inequality holds for all $x_{l|k} \in \mathbb{X}_{l|k}$ if it holds for the vertices of $\mathbb{X}_{l|k}$. In the third equivalence the max is to be understood row-wise and the last equivalence follows from strong duality in linear programming. Considering row $i \in \mathbb{N}_1^u$ in the maximisation, which is finite since Θ_k is compact, it holds that

$$\begin{aligned}& \max_{\theta \in \Theta_k} \{[H_x]_i D_{l|k}^j \theta\} \\ &= \min_{\lambda_i \in \mathbb{R}_{\geq 0}^{q_k}} \max_{\theta} \{[H_x]_i D_{l|k}^j \theta + \lambda_i^\top (h_{\theta_k} - H_{\theta_k} \theta)\} \\ &= \min_{\lambda_i \in \mathbb{R}_{\geq 0}^{q_k}} \lambda_i^\top h_{\theta_k} \\ &\text{s.t. } [H_x]_i D_{l|k}^j - \lambda_i^\top H_{\theta_k} = 0.\end{aligned}$$

Since the minimization in the inequality constraint can be removed, this concludes the proof. \square

Remark 5. The equality constraint (15d) can equivalently be written in standard vector form as

$$\begin{bmatrix} H_x A_1 & H_x A_1 x^j & H_x B_1 & -I_u \otimes [H_\theta^\top]_1 \\ H_x A_2 & H_x A_2 x^j & H_x B_2 & -I_u \otimes [H_\theta^\top]_2 \\ & & \ddots & \\ H_x A_p & H_x A_p x^j & H_x B_p & -I_u \otimes [H_\theta^\top]_p \end{bmatrix} \begin{bmatrix} z_{l|k} \\ \alpha_{l|k} \\ u_{l|k} \\ \text{vec}((\Lambda_{l|k}^j)^\top) \end{bmatrix} = 0,$$

with $[\cdot]_l$ being the l -th row of the matrix.

3.3 Terminal Constraint and Min-Max Cost

To derive a stabilizing MPC algorithm we employ a suitable terminal constraint and terminal cost. To take into account the parameter learning for performance, we consider a cost which is positive definite with respect to the desired steady state $x_{ss} = 0$.

The terminal constraint set \mathbb{X}_T is assumed to be robust invariant and λ -contractive for the undisturbed system and state tube $\mathbb{X}_{l|k}$.

Assumption 6. There exists a nonempty terminal set $\mathbb{X}_T = \{(z, \alpha) \in \mathbb{R}^{n+1} \mid H_T z + h_T \alpha \leq \mathbf{1}\}$, $\lambda \in (0, 1)$ such that for all $\theta \in \Theta$ it holds

$$(z, \alpha) \in \mathbb{X}_T \Rightarrow \exists(z^+, \alpha^+) \in \lambda \mathbb{X}_T \text{ s.t.}$$

$$A_{cl}(\theta)(\{z\} \oplus \alpha \mathbb{X}_0) \subseteq \{z^+\} \oplus \alpha^+ \mathbb{X}_0,$$

$$(z, \alpha) \in \mathbb{X}_T \Rightarrow \exists(z^+, \alpha^+) \in \mathbb{X}_T \text{ s.t.}$$

$$A_{cl}(\theta)(\{z\} \oplus \alpha \mathbb{X}_0) \oplus \mathbb{W} \subseteq \{z^+\} \oplus \alpha^+ \mathbb{X}_0,$$

$$(z, \alpha) \in \mathbb{X}_T \Rightarrow (x, Kx) \in \mathbb{Z} \quad \forall x \in \{z\} \oplus \alpha \mathbb{X}_0.$$

Note that \mathbb{X}_T can be determined recursively, analogous to standard algorithms in nominal linear MPC with the state replaced by the set dynamics in (z, α) . If Assumption 2 is satisfied, \mathbb{X}_0 can be chosen as a robust invariant set and hence Assumption 6 is satisfied if $(x, Kx) \in \mathbb{Z} \forall x \in \mathbb{X}_0$. With Assumption 6, the terminal constraint in the MPC optimization is given by

$$H_T z_{N|k} + h_T \alpha_{N|k} \leq \mathbf{1}. \quad (16)$$

In line with the set-membership approach, we consider a worst case cost for robustly stabilizing the system to a set containing the origin. Define

$$\ell(\mathbb{X}_{l|k}, u_{l|k}) = \max_{x_{l|k} \in \mathbb{X}_{l|k}} \{\|Qx_{l|k}\|_\infty + \|Ru_{l|k}\|_\infty\} \quad (17)$$

with cost matrices Q and R of rank n and m , respectively. A weighted maximum norm instead of a more standard

quadratic cost is used in order to obtain an efficiently solvable linear program. Alternatively, the maximum over a quadratic cost could be employed, but would lead to second order cone constraints.

Let $\bar{t}_r = \max_{x \in \mathbb{X}_T} \{\|Qx\|_\infty + \|RKx\|_\infty\}$ and $t_T > \frac{\bar{t}_r}{1-\lambda}$, we define the terminal cost

$$V(z_{N|k}, \alpha_{N|k}) = \min_{\gamma} \gamma t_T \\ \text{s.t. } z_{N|k} + \alpha_{N|k} \mathbb{X}_0 \subseteq \gamma \mathbb{X}_T.$$

Remark 7. To derive a stabilizing MPC algorithm in the presence of uncertainty and disturbances it is common to consider a cost which is either positive definite with respect to an offline designed (minimal) robust positively invariant target set \mathbb{T} or which penalizes the deviation from the input to some nominal control law κ_0 . In the presented Adaptive MPC this approach would not improve the performance unless the target set \mathbb{T} or nominal control law κ_0 would be recomputed in each iteration, taking the updated parameter set into account.

3.4 Adaptive MPC Algorithm

Having derived tractable reformulations for the parameter estimation (5) and optimal control problem (9), we summarize the Adaptive MPC algorithm and provide a brief analysis of the control theoretic properties.

The algorithm can be divided into two parts: (i) an offline computation of the terminal sets and (ii) the repeated online optimization with decision variables $\mathbf{d}_k = \{z_{l|k}, \alpha_{l|k}, v_{l|k}, \{\Lambda_{l|k}^j\}_{j \in \mathbb{N}_1^N}\}_{l \in \mathbb{N}_0^N}, \gamma\}$ and constraint set

$$\mathbb{D}(x_k, \Theta_k) = \{\mathbf{d}_k \mid (15), (16)\}.$$

Offline: Determine a robustly stabilizing control gain K and terminal set \mathbb{X}_T according to Assumptions 2, 6. Choose cost matrices Q, R and t_T .

Online: For each time step $k = 0, 1, 2, \dots$

- (i) Measure the state x_k .
- (ii) If $k > 0$ update the parameter set Θ_k according to (11), (12).
- (iii) Determine a minimizer of the linear program (9)
$$\mathbf{d}_k^* = \arg \min_{\mathbf{d}_k} J_N(x_k, \mathbf{d}_k) \quad \text{s.t. } \mathbf{d}_k \in \mathbb{D}(x_k, \Theta_k).$$
- (iv) Apply $u_k = Kx_k + v_{0|k}^*$.

The following proposition establishes consistent parameter set estimation, recursive feasibility of the Adaptive MPC algorithm and satisfaction of the constraints (4) if the system (1) is controlled with the MPC control law (10). Consistent parameter set estimation means that the true parameter is always contained in the estimated parameter set if it is contained in the initially given parameter set. Recursive feasibility guarantees that if there exists a solution to the MPC problem for the given initial conditions, a solution to the MPC problem exists for all future states resulting from the application of the proposed MPC control law to system (1).

Proposition 8. Suppose Assumptions 1,2,6 hold. If $\theta^* \in \Theta_{k_0}$, $\mathbb{D}(x_{k_0}, \Theta_{k_0}) \neq \emptyset$, then for all $k \geq k_0$

- (i) $\theta^* \in \Theta_k$
- (ii) $\mathbb{D}(x_k, \Theta_k) \neq \emptyset$
- (iii) $x_k \times u_k \in \mathbb{Z}$.

Proof. We prove claim (i) and (ii) for $k = k_0 + 1$, since k_0 is arbitrary the result follows by induction.

Assume $\theta^* \in \Theta_{k_0}$ and let $u_{k_0} = u(x_{k_0})$ be the applied MPC input. Then

$$x_{k_0+1} = A(\theta^*)x_{k_0} + B(\theta^*)u_{k_0} + w_{k_0}$$

with $w_{k_0} \in \mathbb{W}$ and hence by (11) $\theta^* \in \Delta_{k_0+1}$ which implies $\theta^* \in \Theta_{k_0} \cap \Delta_{k_0+1} = \Theta_{k_0+1}$ as defined in (12).

At time k_0 , let $\mathbf{u}_{N|k_0}^*, \{\mathbb{X}_{l|k}^*\}_{l \in \mathbb{N}_0^N}$ be a feasible input and state tube trajectory satisfying the MPC constraints (6), (7), (16). For time $k_0 + 1$ and $l \in \mathbb{N}_0^{N-1}$ define the candidate input $\tilde{u}_{l|k_0+1}(x) = Kx + v_{l+1|k_0}$ with $v_{N|k_0} = 0$ and candidate state tube $\tilde{\mathbb{X}}_{l|k_0+1} = \mathbb{X}_{l+1|k_0}^*$. Since $\tilde{\mathbb{X}}_{N-1|k_0+1} \subseteq \mathbb{X}_T$ and $\Theta_{k_0} \subseteq \Theta$, by Assumption 6 there exists $\tilde{\mathbb{X}}_{N|k_0+1} \subseteq \mathbb{X}_T$ satisfying $A(\theta)x \oplus B(\theta)\tilde{u}_{N-1|k_0+1} \oplus \mathbb{W} \subseteq \mathbb{X}_{N|k_0+1}$ for all $x \in \mathbb{X}_{N-1|k_0+1}$, $\theta \in \Theta_{k_0}$. By construction $\{\tilde{u}_{l|k_0+1}, \tilde{\mathbb{X}}_{l|k_0+1}\}_{l \in \mathbb{N}_0^{N-1}}$ satisfy the constraints (7) and since $x_{k_0+1} = A(\theta^*)x_{k_0} + B(\theta^*)u_{k_0} + w_{k_0} \in \mathbb{X}_{0|k_0+1}^* = \tilde{\mathbb{X}}_{0|k_0+1}$ constraints (6) are satisfied. By Proposition 4 this is equivalent to feasibility of (15) and hence $\mathbb{D}(x_{k_0+1}, \Theta_{k_0}) \neq \emptyset$ which implies $\mathbb{D}(x_{k_0+1}, \Theta_{k_0+1}) \neq \emptyset$ as $\Theta_{k_0+1} \subseteq \Theta_{k_0}$.

Claim (iii) is a direct corollary of $\mathbb{D}(x_k, \Theta_k) \neq \emptyset$ and Proposition 4. \square

As highlighted in Remark 7, in Robust MPC for linear systems often a minimal robust positively invariant set (mRPI set) is determined offline and stabilized by choosing a cost which is positive definite with respect to this set. In Min-max MPC it is well known that only practical stability can be guaranteed and convergence to a level set of the Lyapunov function can be proven which is in general less tight than the actual mRPI set which is stabilized (Raimondo et al. (2009)). The following analysis establishes the stability properties of the closed loop system.

Lemma 9. (i) The set $\mathbb{X}_N = \{x \in \mathbb{R}^n \mid \mathbb{D}(x, \Theta) \neq \emptyset\}$ is compact, robust positively invariant for the system (1) with MPC control law (10) and $0 \in \text{int}(\mathbb{X}_N)$.

- (ii) There exists $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, $c_1, c_2 \in \mathbb{R}_{\geq 0}$ such that for all $x \in \mathbb{X}_N$, $w \in \mathbb{W}$, $\theta \in \Theta$ and $x^+ = A(\theta)x + B(\theta)u(x) + w$ it holds

$$\alpha_1(\|x\|) \leq V_N(x) \leq \alpha_2(\|x\|) + c_1 \\ V_N(x^+) - V_N(x) \leq -\alpha_3(\|x\|) + c_2.$$

The proof of Lemma 9 is given in the appendix.

The following technical assumption introduces a set Ω which is robustly stabilized by the Adaptive MPC algorithm. Compared with the simulation results, the bound is quite conservative and future research is devoted to a more detailed stability analysis.

Assumption 10. There exists $\zeta, \rho \in \mathcal{K}_\infty$ such that

$$\Omega = \{x \mid V_N(x) \leq \alpha_4^{-1} \circ \rho^{-1}(\zeta(c_1) + c_2)\} \subseteq \text{int}(\mathbb{X}_N) \quad (18) \\ \text{with } \alpha_4 = \underline{\alpha}_3 \circ \bar{\alpha}_2, \underline{\alpha}_3(s) = \min\{\alpha_3(s/2), \zeta(s/2)\}, \bar{\alpha}_2 = \alpha_2 + Id.$$

Given Lemma 9 and Assumption 10, the following proposition on the stability properties of the closed loop system

is a direct application of Theorem 2 in Raimondo et al. (2009). The origin is practically stable for the closed loop system with the ultimate bound depending on the size of the disturbance set \mathbb{W} .

Proposition 11. Suppose Assumptions 1,2,6,10 hold. Then the origin is practically stable with region of attraction \mathbb{X}_N and $\lim_{k \rightarrow \infty} \|x_k\|_{\Omega} = 0$ for the closed-loop system (1) with MPC control law (10).

Without additive disturbance the constants in Lemma 9 vanish, $c_1 = c_2 = 0$, leading to the following corollary.

Corollary 12. Suppose Assumptions 1,2,6,10 hold and $\mathbb{W} = \{0\}$. Then the origin is asymptotically stable with region of attraction \mathbb{X}_N for the closed-loop system (1) with MPC control law (10).

4. NUMERICAL EXAMPLE

In this section, two examples are presented to illustrate the advantages of the proposed Adaptive MPC scheme. We first demonstrate the online identification and constraint satisfaction in a setup where stabilization of the origin is considered and thereafter the adaptive scheme is compared with a non-adaptive, Robust MPC in an ad-hoc tracking implementation for constant reference signals.

Example 1 Consider the second-order discrete-time linear system of the form (1) with

$$\begin{aligned} A_0 &= \begin{bmatrix} 0.5 & 0.2 \\ -0.1 & 0.6 \end{bmatrix}, & B_0 &= \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, \\ A_1 &= \begin{bmatrix} 0.042 & 0 \\ 0.072 & 0.03 \end{bmatrix}, & A_2 &= \begin{bmatrix} 0.015 & 0.019 \\ 0.009 & 0.035 \end{bmatrix}, & A_3 &= 0_{2 \times 2}, \\ \{B_i\}_{i=1,2} &= 0_{2 \times 1}, & B_3 &= \begin{bmatrix} 0.0397 \\ 0.0539 \end{bmatrix} \end{aligned}$$

$$\Theta = \{\theta \mid \|\theta\|_{\infty} \leq 1\} \text{ and } \mathbb{W} = \{w \in \mathbb{R}^2 \mid \|w\|_{\infty} \leq 0.1\}.$$

The MPC parameters were horizon length $N = 9$, cost weights $Q = \text{diag}(1,1)$, $R = 0.001$, and prestabilizing feedback gain $K = [0.017 \quad -0.41]$. Separate state and input constraints $[x_k]_2 \geq -0.3$, $|u_k| \leq 1$ were applied to the system which should be satisfied robustly.

Starting from an initial condition $x_0 = [0, 10]^T$, Figure 1 shows a typical closed loop trajectory, the predicted state tube trajectory at time step $k = 3$ and the state constraint $[x_k]_2 \geq -0.3$. Under the proposed Adaptive MPC scheme, the state constraint is robustly satisfied for all possible predicted states and the state converges to a neighborhood of the origin. The input constraints were satisfied robustly with the input saturated at $u_k = -1$ in the first four steps.

Figure 2 shows the parameter set from time step $k = 0$ to $k = 5$. Given the realized state and input trajectory, falsified parameters are removed and the uncertainty set is non-increasing. We remark that the parameter adaption depends on the initial condition and disturbance realization since the cost does not reflect the advantage of future parameter learning.

The simulation was performed with Matlab and MOSEK. The average optimization time without further tuning or warm-start was 1.5s with a maximum of 1.8s on an Intel Core i7 with 3.4GHz.

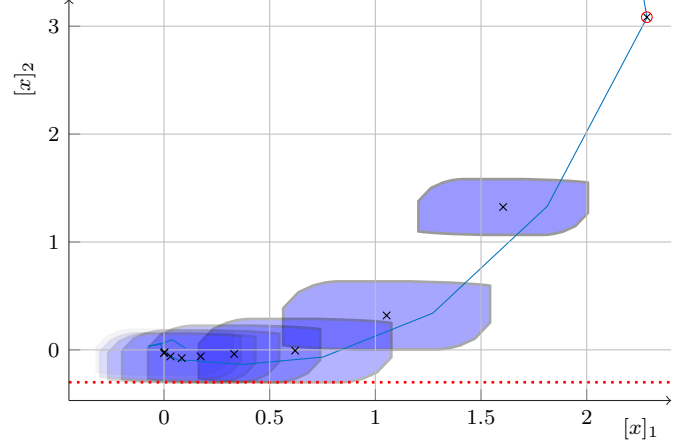


Fig. 1. Realized closed loop trajectory, predicted state tube trajectory and constraint $[x_k]_1 \geq -0.3$.

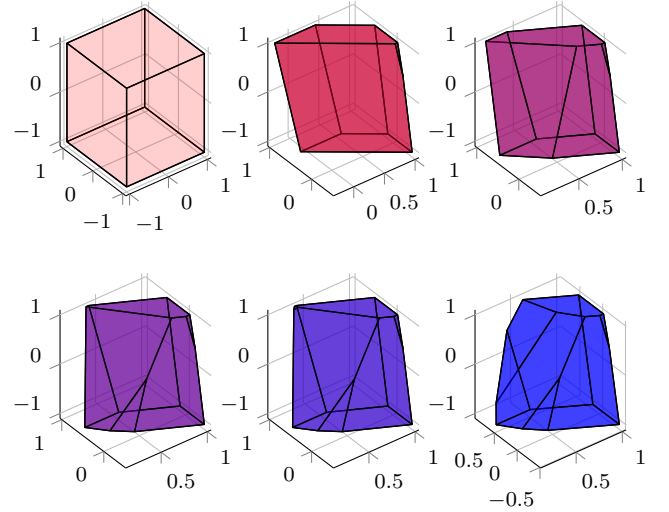


Fig. 2. Evolution of the parameter set from time $k = 0$ to $k = 5$. Note the different axis limits.

Example 2 To highlight the increased performance of the proposed Adaptive MPC scheme we compare it with a non-adaptive robust Tube MPC. Consider a simple mass spring damper system with dynamics described by

$$m\ddot{x} = -c\dot{x} - kx + u + w$$

and nominal parameters mass $m = 1$, damping constant $c = 0.2$, and spring constant $k = 1$. The parameter uncertainty was considered to be $\pm 20\%$, $|w(t)| \leq 0.5$, and input u and state x constrained to $[-5, 5]$ and $[-0.1, 1.1]$ respectively. To apply the MPC algorithm, a first-order discretization with sampling time $T_s = 0.1$ was used. The prediction horizon was set to $N = 14$ and cost weights $Q = \text{diag}(10, 0.001)$, $R = 0.001$.

Figure 3 shows the closed loop response under the proposed Adaptive MPC and robust Tube MPC algorithm. After time steps $k = 20, 40, 60$ the desired set point was switched between 0 and 1. Due to the model uncertainty the desired steady state $x_{ss} = 1$ is only stabilized with an offset which, due to the parameter adaption, is decreased in the Adaptive MPC scheme but not in the Robust MPC. Similarly, while each transient between the steady states is similar in the Robust MPC, the Adaptive MPC shows a faster, improved convergence in each transient. The closed

loop cost of the Robust MPC was 199 compared to 146 for the Adaptive MPC, i.e. 47% higher.

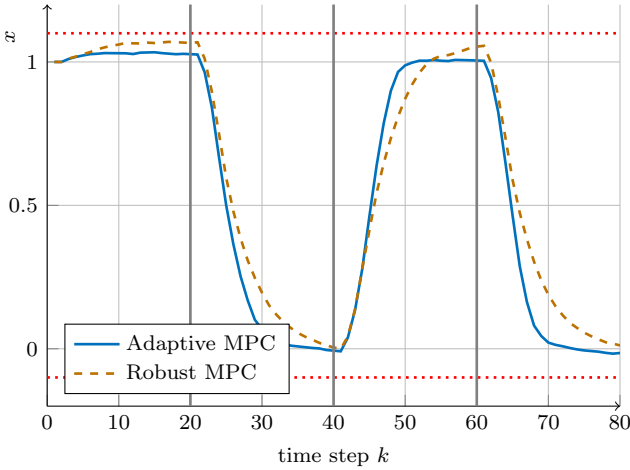


Fig. 3. Comparison of closed loop trajectories with setpoint changing between 0 and 1 after $k = 20, 40, 60$ for the proposed Adaptive MPC and a non-adaptive Robust MPC.

5. CONCLUSIONS AND FURTHER WORK

The presented combination of parameter estimation and constraint tightening provides a rigorous and computationally efficient basis for robust Adaptive MPC algorithms based on state-space models.

In this work, practical stability of the origin was established through combining the presented ansatz with a terminal constraint and worst-case cost. Two examples were presented to illustrate the advantages of the proposed Adaptive MPC scheme compared to a non-adaptive Robust MPC.

The presented approach is only a first step towards combining modern state-space MPC algorithms with a suitable parameter adaption. Based thereon, there are several future research directions worth to be considered. Instead of a worst-case cost which does not encourage learning, a ‘dual-cost’ which introduces learning based on the regulation objective similar to the observations made in Feldbaum (2001) would be of interest. Since a state-space formulation is employed, recent results on output feedback and offset-free tracking in MPC can be combined with the presented approach to obtain a zero steady-state offset for tracking constant reference signals. The presented results hold only for constant uncertainties. Adapting the parameter estimation to allow for slowly time varying parameters, e.g. due to changing environment or wear in mechanical devices, would be of interest to increase the applicability.

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Appendix A. TECHNICAL PROOFS

Proof. [Lemma 9] The set \mathbb{X}_N is the projection of the polytopic set given by the intersection of (15), (16) and hence closed. It is compact since it is contained in the bounded set $\text{Proj}_x(\mathbb{Z})$ and robust positively invariant by Proposition 8.

The MPC value function $V_N(x) \geq \ell(x, u(x)) \geq \|Qx\|_\infty \geq \frac{\|Q\|_\infty}{\sqrt{n}} \|x\| = \alpha_1(\|x\|)$ for all $x \in \mathbb{X}_N$. For the upper bound let $x \in \mathbb{X}_T$ and $V_0(x) = V(x) = t_T \Psi_T(x) \leq \tilde{\alpha}_2(\|x\|)$ with Ψ_T being the Minkowski function for the terminal set \mathbb{X}_T and $\tilde{\alpha}_2 \in \mathcal{K}_\infty$. Let $\mathbb{X} = z + \alpha \mathbb{X}_0 \subseteq \gamma \mathbb{X}_T$ with $\gamma \in (0, 1]$ and \mathbb{X}^+ according to Assumption 6. Note that it exists $d \in \mathbb{R}$ s.t.

$$\begin{aligned} V(\mathbb{X}^+) - V(\mathbb{X}) + \ell(\mathbb{X}, \kappa(x)) \\ \leq (\lambda\gamma + d)t_T - \gamma t_T + \gamma \bar{t}_r \\ = \gamma(\bar{t}_r - (1 - \lambda)t_T) + dt_T \leq dt_T \end{aligned} \quad (\text{A.1})$$

and hence for $N \geq 1$

$$\begin{aligned} V_N(x) &\leq V_{N-1}(x) + dt_T \leq V_0(x) + Ndt_T \\ &\leq \tilde{\alpha}_2(\|x\|) + c_1. \end{aligned}$$

The optimal value function, being the solution of a parametric LP is continuous (Rawlings and Mayne, 2009, Theorem C.34) and hence $\tilde{\alpha}_2$ can be extended to α_2 on \mathbb{X}_N . Using the candidate solution introduced in the proof of Proposition 8 and equation (A.1) one obtains

$$V_N(x^+) - V_N(x) \leq dt_T - \|Qx\|_\infty \leq -\alpha_3(\|x\|) + c_2$$

which concludes the proof. \square