

# Determinacy analysis in high order dynamic systems: The case of nominal rigidities and limited asset market participation

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## Abstract

We show how to use Hurwitz polynomials to study the stability and uniqueness of Rational Expectation equilibria (REE) in Dynamic General Equilibrium models (DGE). We apply this method to a model characterized by sticky wages and prices and by limited asset market participation (LAMP). We prove analytically in a fourth-order system that, once nominal wage stickiness is taken into account, LAMP does not invalidate the Taylor Principle.

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# 1 Introduction

We show how to use Hurwitz polynomials to study the stability and uniqueness of REE in DGE models. We apply this methodology to a New Keynesian (NK) model featuring sticky wages, prices and LAMP. Under LAMP a portion of agents spend their labor income in each period. Bilbiie (2008) studies determinacy properties of interest rate rules in a NK economy with LAMP and a frictionless labor market. He shows that determinacy of the REE requires an inversion of the Taylor principle, that is, the nominal interest rate needs to react less than one-to-one to inflation. We assess analytically how nominal wage stickiness affects this result. We represent our model as a fourth order dynamic system. Despite this high order, our methodology delivers analytical conditions for the determinacy of the REE. Colciago (2011) shows numerically that wage stickiness helps restoring the Taylor Principle as a necessary condition for determinacy in the presence of LAMP. We prove analytically the generality of this numerical result.

# 2 Methodology

Consider a system of linear difference equations in the form

$$E_t z_{t+1} = A z_t, \quad (1)$$

where  $z_t$  is a  $n \times 1$  vector including  $n_1$  predetermined variables and  $n_2$  non-predetermined variables, where  $n = n_1 + n_2$ .  $A$  is a  $n \times n$  coefficient matrix with characteristic polynomial

$$P_C(\gamma) = \gamma^n + a_1 \gamma^{n-1} + \dots + a_{n-1} \gamma + a_n. \quad (2)$$

The stability and uniqueness of the solution to (1) depend on the location of the roots of  $P_C(\gamma)$  inside the unit circle  $|\gamma| < 1$ . Blanchard and Kahn (BK henceforth, 1980) show that (1) has a stable and unique solution if (2) has  $n_2$  roots larger than one in absolute value and  $n_1$  roots lower than one in absolute value. Verifying if BK conditions are satisfied can be cumbersome, particularly so as  $n$  gets larger and if  $z_t$  contains both predetermined and non-predetermined variables.

Following Felippa and Park (2005) we transform the polynomial  $P_C(\gamma)$  in an Hurwitz polynomial,  $P_H(s)$ , by applying the conformal involuntary transformation

$$\gamma = \frac{1+s}{1-s}. \quad (3)$$

Given (3), it is easy to check that  $|\gamma| \leq 1 \Leftrightarrow s \leq 0$ . Expanding the polynomial, one obtains a quotient of two polynomials:  $\tilde{P}_H(s) = \frac{P_H(s)}{Q_H(s)}$  where the roots of  $\tilde{P}_H(s)$  are the roots of  $P_H(s)$ . The uniqueness and stability properties of  $P_H(s)$  depend on the location of the roots in the left-hand plane  $\Re(s) \leq 0$ . The system (1) has a stable and unique solution if the Hurwitz polynomial associated to (2) has  $n_2$  roots larger than zero and  $n_1$  roots lower than zero. To check how many roots are positive and how many are negative in a high order polynomial is a simpler task than to check how many roots are within or outside the unit circle. Moreover, in microfounded macro-models the sign of the parameters defining functional forms is usually known, while their magnitude is not. Hence, verifying if these conditions are satisfied is more straightforward than verifying if BK conditions are.

### 3 Application: A NK model with nominal rigidities and LAMP

We apply the methodology to a NK model featuring: (i) staggered wage and price contracts; (ii) LAMP, that is, a fraction  $\lambda \in [0, 1]$  of agents do not participate to the financial markets and consume their labor income. The model economy is spelled out in Ascari et al. (2017). Log-linear equilibrium dynamics of the model around the efficient steady state are determined by:

$$\begin{aligned}
(M1) \quad \pi_t &= \beta E_t \pi_{t+1} + \kappa_p \tilde{\omega}_t && \text{NKPC} \\
(M2) \quad \pi_t^w &= \beta E_t \pi_{t+1}^w + \kappa_w [(\sigma + \phi)x_t - \tilde{\omega}_t] && \text{Wage Inflation Curve} \\
(M3) \quad \tilde{\omega}_t &= \tilde{\omega}_{t-1} + \pi_t^w - \pi_t - \Delta \omega_t^{Eff} && \text{Real Wage Gap} \\
(M4) \quad x_t &= E_t x_{t+1} - \frac{1}{\sigma} E_t \left( i_t - \pi_{t+1} - r_t^{Eff} \right) - \frac{\lambda}{(1-\lambda)} E_t \Delta \tilde{\omega}_{t+1} && \text{IS curve}
\end{aligned}$$

Fluctuations are caused by shocks to labor productivity,  $a_t$ , and by taste shocks,  $\psi_t$ . (M1) is the NKPC,  $\pi_t$  represents deviations of current inflation from its (zero) steady state;  $\tilde{\omega}_t = \omega_t - \omega_t^{Eff}$  represents the *real wage gap*, defined as the gap between the current and the efficient real wage. The latter is determined by technology,  $\omega_t^{Eff} = a_t$ . The slope of the NKPC is  $\kappa_p = \frac{(1-\beta\xi_p)(1-\xi_p)}{\xi_p}$ ,  $\beta$  is the subjective discount factor and  $\xi_p$  the Calvo-probability for a firm of not changing its price. The variable  $\pi_t^w$  is wage inflation. The wage inflation curve, (M2), has slope  $\kappa_w = \frac{(1-\beta\xi_w)(1-\xi_w)}{\xi_w}$ .  $\xi_w$  is the Calvo-probability for a labor union of not changing its wage. The variable  $x_t = y_t - y_t^{Eff}$  denotes the gap between actual output and the efficient output, which reads as  $y_t^{Eff} = \frac{1+\phi}{\sigma+\phi} a_t + \frac{1}{(\sigma+\phi)} \psi_t$ . The parameters  $\phi$  and  $\sigma$  are the elasticity of intertemporal substitution in labor supply and in consumption, respectively. (M3) defines the real wage gap. (M4) is the IS curve, which differs from a standard IS equation because of the extra term  $\frac{\lambda}{1-\lambda} E_t \Delta \tilde{\omega}_{t+1}$ . The expected growth of the real wage affects aggregate demand relative to the efficient allocation through the consumption of constrained consumers - it will not appear if  $\lambda = 0$ . The efficient real rate of interest is  $r_t^{Eff} = \sigma \left( \frac{1+\phi}{\sigma+\phi} \Delta a_{t+1} - \frac{\phi}{\sigma(\sigma+\phi)} \Delta \psi_{t+1} \right)$ . We consider the same interest rate rule as in Bilbiie (2008)

$$i_t = \phi_\pi \pi_{t+1}. \quad (4)$$

The relevant system to study the determinacy of REE can be represented as in (1) with  $z_t = [\pi_t^w, \pi_t, x_t, \tilde{\omega}_t]'$  and

$$A = \begin{bmatrix} \frac{1}{\beta} & 0 & -\frac{1}{\beta} \kappa_w (\sigma + \phi) & -\frac{1}{\beta} \kappa_w \\ 0 & \frac{1}{\beta} & 0 & -\frac{1}{\beta} \kappa_p \\ \frac{1}{\beta} \chi & \frac{1}{\sigma} \frac{1}{\beta} (\phi_\pi - 1) - \chi \frac{1}{\beta} & 1 - \chi \frac{1}{\beta} \kappa_w (\sigma + \phi) & \chi \frac{1}{\beta} (\kappa_w + \kappa_p) - \frac{1}{\sigma} \frac{1}{\beta} (\phi_\pi - 1) \kappa_p \\ \frac{1}{\beta} & -\frac{1}{\beta} & -\frac{1}{\beta} \kappa_w (\sigma + \phi) & 1 + \frac{1}{\beta} (\kappa_w + \kappa_p) \end{bmatrix}$$

The 4th-order characteristic polynomial reads as

$$P_C(\gamma) = \gamma^4 + a_1 \gamma^3 + a_2 \gamma^2 + a_3 \gamma + a_4.$$

The latter can be transformed into the Hurwitz polynomial using  $\gamma = \frac{1+s}{1-s}$ . In this case

$$\tilde{P}_H(s) = \left( \frac{1+s}{1-s} \right)^4 + a_1 \left( \frac{1+s}{1-s} \right)^3 + a_2 \left( \frac{1+s}{1-s} \right)^2 + a_3 \frac{1+s}{1-s} + a_4. \quad (5)$$

Hence one needs to study the following Hurwitz polynomial

$$P_H(s) = \underbrace{\tilde{a}_4}_{\frac{a_1+a_2+a_3+a_4+1}{a_2-a_1-a_3+a_4+1}} + s \underbrace{\tilde{a}_3}_{\frac{2(2+a_1-a_3-2a_4)}{a_2-a_1-a_3+a_4+1}} + s^2 \underbrace{\tilde{a}_2}_{\frac{2(3a_4-a_2+3)}{a_2-a_1-a_3+a_4+1}} + s^3 \underbrace{\tilde{a}_1}_{\frac{2(a_3-a_1-2a_4+2)}{a_2-a_1-a_3+a_4+1}} + s^4 \quad (6)$$

**Proposition 1. Taylor Principle and LAMP.** Under policy rule (4) there exists a locally unique rational expectations equilibrium if and only if:

$$\text{CASE I: } \lambda < \bar{\lambda}^{FR} : \phi_\pi \in \left(1; \bar{\phi}_\pi^{FR}\right);$$

$$\text{CASE II: } \lambda > \bar{\lambda}^{FR} : \phi_\pi \in \left(\bar{\phi}_\pi^{FR}; 1\right);$$

$$\text{where } \bar{\lambda}^{FR} = \frac{1}{1 + \frac{\kappa_w(\sigma+\phi)}{2(1+\beta)+\kappa_w+\kappa_p}} \text{ and } \bar{\phi}_\pi^{FR} = 1 + \frac{2\sigma(1+\beta)[2(1+\beta)+\kappa_p+\kappa_w - \frac{\lambda}{1-\lambda}\kappa_w(\sigma+\phi)]}{\kappa_w\kappa_p(\sigma+\phi)}.$$

Case I generalizes the Taylor principle: as in the full-participation case, the central bank should respond more than one-to-one to increases in inflation. Case II corresponds to the Inverted Taylor Principle. In this case, as in Bilbiee (2008), only a passive policy is consistent with a unique REE. While the proof is in Appendix A.1, here we sketch it to show how to apply the suggested methodology. The key point is to adopt the transformation in (6) to go from the characteristic polynomial of matrix  $A$

$$\begin{aligned} P_C(\gamma) &= \gamma^4 + \left[ \frac{1}{\beta} [-2 - 2\beta - (\kappa_w + \kappa_p) + \chi\kappa_w(\sigma + \phi)] \right] \gamma^3 \\ &\quad + \left[ \frac{1}{\beta} (\kappa_p + \kappa_w + \beta + 1) - \frac{1}{\beta} \kappa_w (\sigma + \phi) \left( \chi \left( 1 + \frac{1}{\beta} \right) + \frac{1}{\sigma\beta} \kappa_p (\phi_\pi - 1) \right) \right. \\ &\quad \left. + \frac{1}{\beta^2} (1 + 3\beta + \kappa_w + \kappa_p) \right] \gamma^2 \\ &\quad + \left[ -\frac{1}{\beta^2} (2 + 2\beta + \kappa_w + \kappa_p - \chi\kappa_w(\sigma + \phi)) \right] \gamma + \frac{1}{\beta^2} \end{aligned}$$

to the associated Hurwitz polynomial

$$\begin{aligned} P_H(s) &= \underbrace{\frac{-\frac{1}{\sigma}\kappa_w\kappa_p(\sigma+\phi)(\phi_\pi-1)}{D}}_{\tilde{a}_4} \\ &\quad + s \underbrace{\frac{2(\beta-1)[- \kappa_w - \kappa_p + \chi\kappa_w(\sigma+\phi)]}{D}}_{\tilde{a}_3} \\ &\quad + s^2 \left[ \underbrace{\frac{4\beta^2 + 4 - 8\beta + 2(1+\beta)[- \kappa_p - \kappa_w + \kappa_w(\sigma+\phi)\chi] + \frac{2}{\sigma}\kappa_w(\sigma+\phi)\kappa_p(\phi_\pi-1)}{D}}_{\tilde{a}_2} \right] \\ &\quad + s^3 \left( \underbrace{\frac{-8 + 8\beta^2 + 2(1-\beta)[- (\kappa_w + \kappa_p) + \chi\kappa_w(\sigma+\phi)]}{D}}_{\tilde{a}_1} \right) + s^4, \end{aligned} \quad (7)$$

where

$$D = 4\beta^2 + 4 + 8\beta + 2[\beta + 1](\kappa_p + \kappa_w) - \frac{1}{\sigma}\kappa_w\kappa_p(\sigma + \phi)(\phi_\pi - 1) - 2(1 + \beta)\chi\kappa_w(\sigma + \phi).$$

This polynomial should exhibit 3 positive roots and 1 negative root for the REE to be unique. This is an easier condition to check than checking whether 3 roots and 1 root of  $P_C(\gamma)$  are outside and inside the unit circle, respectively. The Appendix analyses the signs of the coefficients  $\tilde{a}_i$ , and exploits the Decartes' rule of sign for polynomials.

Besides the methodological aspect, Proposition 1 illustrates the second contribution of the paper. Whenever  $\lambda < \bar{\lambda}^{FR}$ , the Taylor Principle is necessary and sufficient for determinacy. As in Bilbiie (2008), there is a region of the parameter space where the Taylor principle is inverted: when  $\lambda > \bar{\lambda}^{FR}$ ,  $\phi_\pi$  needs to be lower than 1 to yield a unique REE.

The value of  $\bar{\lambda}^{FR}$  is lower than one, it increases monotonically with the degree of wage stickiness, and tends to one when wages are almost fixed. Figure 1 depicts the threshold value  $\bar{\lambda}^{FR}$  as a function of the degree of wage stickiness,  $\xi_w$ . When wages are almost fixed, i.e. when  $\xi_w \rightarrow 1$ , then  $\kappa_w \rightarrow 1$  and  $\bar{\lambda}^{FR} = \frac{1}{1 + \frac{(\sigma + \phi)}{2(1 + \beta) + 1 + \kappa_p}}$ . On the contrary, in the case of flexible wages, i.e.  $\xi_w \rightarrow 0$ , then  $\kappa_w \rightarrow \infty$  as in Bilbiie (2008) and the threshold value becomes  $\bar{\lambda}^{FR, fw} = \frac{1}{1 + (\sigma + \phi)}$ . Under our baseline calibration where wage contracts display an average duration of four quarters, as supported by the empirical evidence,  $\bar{\lambda}^{FR} = 0.9063$ . Thus, as the degree of wage stickiness increases, the parameter space where the Inverted Taylor principle is required for determinacy is restricted.

To gain intuition, consider the following mental experiment. After monetary policy increases in the interest rate, Ricardian agents reduce their demand, while the firms that cannot change their price reduce labor demand. The labor demand curve shifts inward and, under flexible wages, this translates into a reduction in the real wage - the more so the higher  $\phi$  and  $\sigma$ . The decrease in the real wage depresses demand by non-Ricardian agents and reinforces the effects on aggregate demand due to the initial increase in the real interest rate. However, this effect is not monotonic in  $\lambda$ . The decrease in the real wage, hence in marginal costs, together with the small change in hours, hence in output and sales, implies a potential increase in profits. This leads to a positive wealth effect on Ricardian agents. The latter is stronger the larger  $\lambda$ , since Ricardian agents would obtain a higher individual dividend income. If  $\lambda$  is large enough, the income effect may counteract the substitution effect and finally lead to an increase in aggregate demand. In this region of the parameter space an increase in the interest rate leads to an increase in aggregate demand, i.e. to an inversion of the slope of the IS curve. When this happens, then the Taylor principle inverts too. Consider now the case of sticky wages. The inward shift in labor demand due to the reduction in consumption by Ricardian agents after the interest rate increase now results in a modest reduction in the real wage because of wage stickiness. The increase in profits is dampened with respect to the case of flexible wages. Thus, under wage stickiness the inversion of the IS curve requires a larger share of non Ricardian agents to magnify the, eventual, wealth effect at the individual level for the Ricardian agents. Price stickiness has the opposite effect on  $\bar{\lambda}^{FR}$ : the threshold value,  $\bar{\lambda}^{FR}$  decreases with the degree of price stickiness (lower  $\kappa_p$ ) because  $\frac{\partial \bar{\lambda}^{FR}}{\partial \kappa_p} > 0$ . In the case of flexible prices and sticky wages, the model becomes isomorphic to a Ricardian economy,  $\lambda$  would not matter and hence the standard Taylor principle applies: the REE is unique iff

$$\phi_\pi \in \left(1, 1 + \frac{2\sigma(1+\beta)}{\kappa_w(\sigma+\phi)}\right)^1$$

In an online Appendix we consider alternative interest rate rules. In a model featuring both sticky wages and sticky prices, Galí (2008) considers a policy rule that targets both price and wage inflation, as  $i_t = \phi_\pi \pi_t + \phi_{\pi^w} \pi_t^w$ . He shows numerically that, for  $\phi_\pi, \phi_{\pi^w} \in (0, \infty)$ , the condition  $\phi_\pi + \phi_{\pi^w} > 1$  is necessary and sufficient (see also Flaschel et al., 2008) for the uniqueness of the REE. Applying the proposed methodology, Proposition 2A in the online Appendix shows analytically that such a condition is still crucial in a model with LAMP.

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<sup>1</sup>Notice that  $\lambda^{FR}$  increases with labor elasticity. Compared to Bilbiie (2008) when labor is infinitely elastic the threshold does not become one, but still depends on the degree of wage stickiness. This is so since agents cannot optimally trade hours of work.

# A Appendix

## A.1 Proof of Propositions 1

The system in matrix form ( $\frac{\lambda}{1-\lambda} = \chi$ ):

$$\begin{bmatrix} \pi_{t+1}^w \\ \pi_{t+1} \\ x_{t+1} \\ \tilde{\omega}_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\beta} & 0 & -\frac{1}{\beta}\kappa_w(\sigma + \phi) & -\frac{1}{\beta}\kappa_w \\ 0 & \frac{1}{\beta} & 0 & -\frac{1}{\beta}\kappa_p \\ \frac{1}{\beta}\chi & \frac{1}{\sigma}\frac{1}{\beta}(\phi_\pi - 1) - \chi\frac{1}{\beta} & 1 - \chi\frac{1}{\beta}\kappa_w(\sigma + \phi) & \chi\frac{1}{\beta}(\kappa_w + \kappa_p) - \frac{1}{\sigma}\frac{1}{\beta}(\phi_\pi - 1)\kappa_p \\ \frac{1}{\beta} & -\frac{1}{\beta} & -\frac{1}{\beta}\kappa_w(\sigma + \phi) & 1 + \frac{1}{\beta}(\kappa_w + \kappa_p) \end{bmatrix}}_A \begin{bmatrix} \pi_t^w \\ \pi_t \\ x_t \\ \tilde{\omega}_t \end{bmatrix} \quad (8)$$

The coefficients of the characteristic polynomial:<sup>2</sup>

$a_1 = -\text{trace}(A) = -$  sum of the principal first-order minors of  $A$

$a_2 =$  sum of the principal second-order minors of  $A$

$a_3 = -$  sum of the principal third-order minors of  $A$

$a_4 = \det(A)$  (= principal of fourth-order).

The characteristic polynomial is

$$\begin{aligned} p(\gamma) &= \gamma^4 + \left[ \frac{1}{\beta} [-2 - 2\beta - (\kappa_w + \kappa_p) + \chi\kappa_w(\sigma + \phi)] \right] \gamma^3 \\ &\quad + \left[ \frac{1}{\beta}(\kappa_p + \kappa_w + \beta + 1) - \frac{1}{\beta}\kappa_w(\sigma + \phi) \left( \chi \left( 1 + \frac{1}{\beta} \right) + \frac{1}{\sigma\beta}\kappa_p(\phi_\pi - 1) \right) \right. \\ &\quad \left. + \frac{1}{\beta^2}(1 + 3\beta + \kappa_w + \kappa_p) \right] \gamma^2 \\ &\quad + \left[ -\frac{1}{\beta^2}(2 + 2\beta + \kappa_w + \kappa_p - \chi\kappa_w(\sigma + \phi)) \right] \gamma + \frac{1}{\beta^2} \end{aligned}$$

Applying the transformation in (6) to get the Hurwitz polynomial,

$$\begin{aligned} &\underbrace{\frac{-\frac{1}{\sigma}\kappa_w\kappa_p(\sigma + \phi)(\phi_\pi - 1)}{D}}_{\tilde{a}_4} \\ &+ s \underbrace{\frac{2(\beta - 1)[- \kappa_w - \kappa_p + \chi\kappa_w(\sigma + \phi)]}{D}}_{\tilde{a}_3} \\ &+ s^2 \left[ \underbrace{\frac{4\beta^2 + 4 - 8\beta + 2(1 + \beta)[- \kappa_p - \kappa_w + \kappa_w(\sigma + \phi)\chi] + \frac{2}{\sigma}\kappa_w(\sigma + \phi)\kappa_p(\phi_\pi - 1)}{D}}_{\tilde{a}_2} \right] \\ &+ s^3 \left( \underbrace{\frac{-8 + 8\beta^2 + 2(1 - \beta)[- (\kappa_w + \kappa_p) + \chi\kappa_w(\sigma + \phi)]}{D}}_{\tilde{a}_1} \right) + s^4 \end{aligned} \quad (9)$$

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<sup>2</sup>Given an  $n \times n$  matrix  $A$ ,  $k$ th order principal minors are the determinants of the  $k \times k$  submatrices along the diagonal obtained by deleting  $n - k$  columns and the same  $n - k$  rows from  $A$ .

where

$$D = 4\beta^2 + 4 + 8\beta + 2[\beta + 1](\kappa_p + \kappa_w) - \frac{1}{\sigma}\kappa_w\kappa_p(\sigma + \phi)(\phi_\pi - 1) - 2(1 + \beta)\chi\kappa_w(\sigma + \phi).$$

There should be 3 positive roots and 1 negative root for the uniqueness of the REE. A necessary condition must be  $\tilde{a}_4 < 0$ . Proof strategy: look at the signs of the coefficients  $\tilde{a}_i$ , and exploit the Decartes' rule of sign. Look separately at the cases  $\phi_\pi > 1$  and  $\phi_\pi < 1$ .

**Case**  $\phi_\pi > 1$ .

$\tilde{a}_4$ ) The numerator of  $\tilde{a}_4$  (i.e.,  $N_{\tilde{a}_4}$ )<sup>3</sup> is negative, hence the denominator must be positive. For  $D > 0$ , the following restriction must hold:

$$\phi_\pi < 1 + \frac{4\sigma\beta^2 + 4\sigma + 8\sigma\beta + 2\sigma(1+\beta)([\kappa_p + \kappa_w - \chi\kappa_w(\sigma + \phi)])}{\kappa_w\kappa_p(\sigma + \phi)}.$$

$\tilde{a}_3$ ) Since  $D > 0$ , there are two cases:

i)  $N_{\tilde{a}_3} > 0 \Rightarrow \tilde{a}_3 > 0$ , that happens when  $\chi < \frac{\kappa_w + \kappa_p}{\kappa_w(\sigma + \phi)}$ .

In this case  $\frac{4\sigma\beta^2 + 4\sigma + 8\sigma\beta + 2\sigma(1+\beta)([\kappa_p + \kappa_w - \chi\kappa_w(\sigma + \phi)])}{\kappa_w\kappa_p(\sigma + \phi)} + 1 > 1$  and the set is non empty. Moreover  $N_{\tilde{a}_1} = -8 + 8\beta^2 + 2(1 - \beta)[-(\kappa_w + \kappa_p) + \chi\kappa_w(\sigma + \phi)] < 0 \Rightarrow \tilde{a}_1 < 0$ .

Whatever the sign of  $\tilde{a}_2$ , the signs of the coefficients in (9) are: -, +, ?, -, +. By Decartes' rule of sign,  $P_H(s)$  admits then 1 or 3 positive roots. However,  $P_H(-s) = +, -, ?, +, +$ , and hence there can be only one negative root. It follows that under the above conditions

$$\frac{2\sigma(1+\beta)[2(1+\beta) + \kappa_p + \kappa_w - \chi\kappa_w(\sigma + \phi)]}{\kappa_w\kappa_p(\sigma + \phi)} + 1 > \phi_\pi > 1$$

$$\chi < \frac{\kappa_w + \kappa_p}{\kappa_w(\sigma + \phi)}$$

the REE is determinate.

ii)  $N_{\tilde{a}_3} < 0 \Rightarrow \tilde{a}_3 < 0$ , that happens when  $\chi > \frac{\kappa_w + \kappa_p}{\kappa_w(\sigma + \phi)}$ . In this case the set  $\frac{2\sigma(1+\beta)[2(1+\beta) + \kappa_p + \kappa_w - \chi\kappa_w(\sigma + \phi)]}{\kappa_w\kappa_p(\sigma + \phi)} + 1 > \phi_\pi > 1$  is non empty iff  $\chi < \frac{\kappa_p + \kappa_w}{\kappa_w(\sigma + \phi)} + \frac{2(1+\beta)}{\kappa_w(\sigma + \phi)}$ . Hence now we are looking at values of  $\chi$  such that

$$\frac{\kappa_p + \kappa_w}{\kappa_w(\sigma + \phi)} + \frac{2(1 + \beta)}{\kappa_w(\sigma + \phi)} > \chi > \frac{\kappa_w + \kappa_p}{\kappa_w(\sigma + \phi)} \quad (10)$$

Since the first two coefficients ( $\tilde{a}_4, \tilde{a}_3$ ) are negative and the last is positive, it must be that  $\tilde{a}_2 > 0$  and  $\tilde{a}_1 < 0$  to have three signs inversions. This is true if  $\phi_\pi > 1$  and (10) hold. It follows that:

$$\frac{2\sigma(1+\beta)[2(1+\beta) + \kappa_p + \kappa_w - \chi\kappa_w(\sigma + \phi)]}{\kappa_w\kappa_p(\sigma + \phi)} + 1 > \phi_\pi > 1$$

$$\frac{\kappa_p + \kappa_w}{\kappa_w(\sigma + \phi)} + \frac{2(1+\beta)}{\kappa_w(\sigma + \phi)} > \chi > \frac{\kappa_w + \kappa_p}{\kappa_w(\sigma + \phi)}$$

the REE is determinate.

Joining i) and ii), the equilibrium is determinate iff

$$\frac{2\sigma(1 + \beta)[2(1 + \beta) + \kappa_p + \kappa_w - \chi\kappa_w(\sigma + \phi)]}{\kappa_w\kappa_p(\sigma + \phi)} + 1 > \phi_\pi > 1 \quad (11)$$

and

$$\frac{\kappa_p + \kappa_w}{\kappa_w(\sigma + \phi)} + \frac{2(1 + \beta)}{\kappa_w(\sigma + \phi)} > \chi > \frac{\kappa_w + \kappa_p}{\kappa_w(\sigma + \phi)}. \quad (12)$$

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<sup>3</sup>N stands for numerator, D for denominator. The pedix for correspondent coefficient  $\tilde{a}_i$ .



**Case**  $\phi_\pi < 1$ .

$\tilde{a}_4$ )  $N_{\tilde{a}_4} > 0$ , hence it must be  $D < 0$ , which needs:  $\frac{2\sigma(1+\beta)[2(1+\beta)+\kappa_p+\kappa_w-\chi\kappa_w(\sigma+\phi)]}{\kappa_w\kappa_p(\sigma+\phi)} + 1 < \phi_\pi$ . However, the set

$$1 > \phi_\pi > 1 + \frac{2\sigma(1+\beta)[2(1+\beta)+\kappa_p+\kappa_w-\chi\kappa_w(\sigma+\phi)]}{\kappa_w\kappa_p(\sigma+\phi)} \quad (13)$$

is non empty iff:

$$\chi > \frac{\kappa_p + \kappa_w}{\kappa_w(\sigma + \phi)} + \frac{2(1 + \beta)}{\kappa_w(\sigma + \phi)}. \quad (14)$$

$\tilde{a}_3$ ) Given (14),  $\implies N_{\tilde{a}_3} < 0 \implies \tilde{a}_3 > 0$ , since  $D < 0$ . Since the first two coefficients:  $\tilde{a}_4 < 0, \tilde{a}_3 > 0$ , and the last is positive, the only way to have three signs inversions is that at least one between  $\tilde{a}_2$  and  $\tilde{a}_1$  is negative (they cannot be both positive). Condition for  $\tilde{a}_2 < 0 \implies N_{\tilde{a}_2} > 0 \implies$

$$\phi_\pi > 1 - \frac{\sigma(1+\beta)[\kappa_w(\sigma+\phi)\chi - \kappa_p - \kappa_w]}{\kappa_w(\sigma+\phi)\kappa_p} - \frac{2\sigma(1-\beta)^2}{\kappa_w(\sigma+\phi)\kappa_p}$$

which, if (13) holds, is satisfied iff:

$$\chi < \frac{\kappa_p + \kappa_w}{\kappa_w(\sigma + \phi)} + \frac{4(1 + \beta)}{\kappa_w(\sigma + \phi)} + \frac{2(1 - \beta)^2}{\kappa_w(\sigma + \phi)(1 + \beta)}. \quad (15)$$

Thus (15) guarantees that

$$1 + \frac{2\sigma(1+\beta)[2(1+\beta)+\kappa_p+\kappa_w-\chi\kappa_w(\sigma+\phi)]}{\kappa_w\kappa_p(\sigma+\phi)} > 1 - \frac{\sigma(1+\beta)[\kappa_w(\sigma+\phi)\chi - \kappa_p - \kappa_w]}{\kappa_w(\sigma+\phi)\kappa_p} - \frac{2\sigma(1-\beta)^2}{\kappa_w(\sigma+\phi)\kappa_p}.$$

Condition for  $\tilde{a}_1 < 0 \implies N_{\tilde{a}_1} > 0 \implies$

$$\chi > \frac{4(1 + \beta) + \kappa_w + \kappa_p}{\kappa_w(\sigma + \phi)}. \quad (16)$$

If (14) holds, at least one between (15) and/or (16) is satisfied, since  $\frac{\kappa_p + \kappa_w}{\kappa_w(\sigma + \phi)} + \frac{2(1 + \beta)}{\kappa_w(\sigma + \phi)} < \frac{4(1 + \beta) + \kappa_w + \kappa_p}{\kappa_w(\sigma + \phi)} < \frac{\kappa_p + \kappa_w}{\kappa_w(\sigma + \phi)} + \frac{4(1 + \beta)}{\kappa_w(\sigma + \phi)} + \frac{2(1 - \beta)^2}{\kappa_w(\sigma + \phi)(1 + \beta)}$ . Hence (14) guarantees that at least one between  $\tilde{a}_2$  and  $\tilde{a}_1$  is negative. Decartes' rule of signs then implies 3 positive roots. With  $\phi_\pi < 1$ , the equilibrium is determinate iff

$$1 > \phi_\pi > 1 + \frac{2\sigma(1+\beta)[2(1+\beta)+\kappa_p+\kappa_w-\chi\kappa_w(\sigma+\phi)]}{\kappa_w\kappa_p(\sigma+\phi)} \quad (17)$$

and

$$\chi > \frac{\kappa_p + \kappa_w}{\kappa_w(\sigma + \phi)} + \frac{2(1 + \beta)}{\kappa_w(\sigma + \phi)}. \quad (18)$$

Joining the cases  $\phi_\pi > 1$  and  $\phi_\pi < 1$  yields Proposition 1. QED

## B Figures

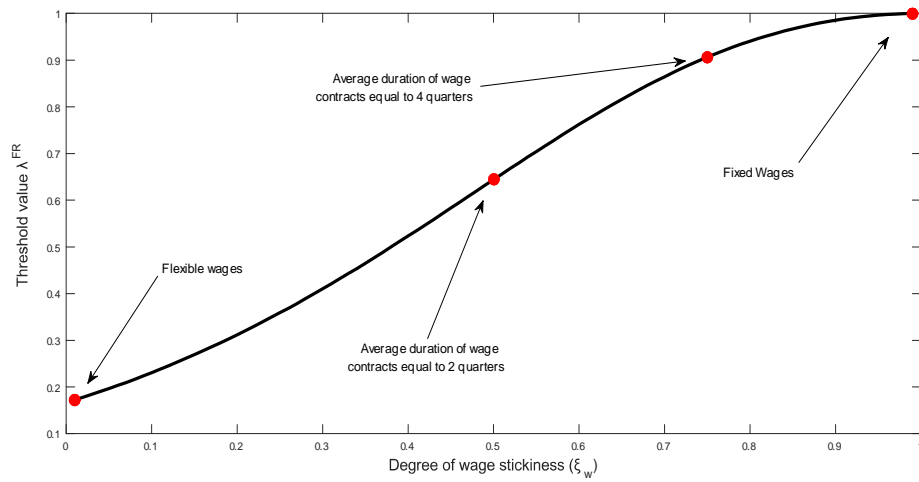


Figure 1: The figure depicts the threshold value  $\lambda^{FR}$ , reported in proposition 1, as a function of the degree of wage stickiness  $\xi_w$ . Remaining parameters are set at their baseline values.

**Supplementary Files**

[Click here to download Supplementary Files: online\\_appendix.pdf](#)