Affine Hecke algebras, canonical bases and a derived Deligne-Langlands correspondence

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To my family.

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Abstract

This thesis makes several contributions to the representation theory of affine Hecke algebras and related topics:

- We show that there is an equivalence of triangulated categories relating certain (dg-)modules over affine Hecke algebras with certain categories of constructible sheaves. This can be viewed as a derived version of the celebrated Deligne-Langlands correspondence.

- We provide a new algorithm that computes the canonical basis in any quantum group of finite ADE type or equivalently the stalks of perverse sheaves on a corresponding moduli space of quiver representations. In type A, this also encodes the composition multiplicities of the standard modules for the affine Hecke algebra of GL_n and we show further how this algorithm can be extended to compute the dimensions of the simple modules.

- We construct a geometric realization of any affine Hecke algebra of simple type with two unequal parameters via equivariant algebraic K-theory. This can be viewed as an extension of a classical theorem due to Kazhdan-Lusztig and Ginzburg which provides such a realization in the equal parameter setting.

Contents

Chapter 1

Introduction

Affine Hecke algebras are certain deformations of group algebras associated to affine Weyl groups. These algebras arise in various different places throughout Lie theory, but they are probably most well-known for their role in the representation theory of p-adic groups. In fact, for each split p-adic group $\mathbf{G}$ there is a (specialized) affine Hecke algebra $\mathcal{H}_{q_0}^{\mathrm{aff}}$ such that the category of $\mathcal{H}_{q_0}^{\mathrm{aff}}$-modules is equivalent to the category $\mathrm{Rep}_I(\mathbf{G})$ of smooth $\mathbf{G}$-representations that are generated by their Iwahori-fixed vectors [21]. Thus, the representation theory of affine Hecke algebras is of considerable interest. In their groundbreaking work, Kazhdan and Lusztig were able to classify the irreducible representations of affine Hecke algebras (with equal parameters) by geometric data on the complex reductive group G whose root datum is dual to that of $\mathbf{G}$ [41]. This is often phrased as a finite-to-one map

$$\mathrm{Irr}(\mathcal{H}_{q_0}^{\mathrm{aff}}) \to \{(s,n) \in G \times \mathcal{N} \mid s \text{ semisimple}, s \cdot n = q_0 n\}/G \tag{1.1}$$

known as the Deligne-Langlands correspondence, and it can be understood as a special case of the celebrated local Langlands conjectures. Here $\mathcal{N} \subset \mathfrak{g}$ is the variety of nilpotent elements also known as the nilpotent cone. The tuples (s,n) are also called 'Langlands parameters'.

A derived Deligne-Langlands correspondence

One of the main results of this thesis lifts the Deligne-Langlands correspondence to an equivalence of triangulated categories (Chapter 3). For this, it is useful to rephrase the Deligne-Langlands correspondence in a geometric language involving perverse sheaves (c.f. [24]): We denote by $\mathcal{H}_{\mathbb{C},q}^{\mathrm{aff}}$ the (complexified) affine Hecke algebra with generic paramter q. The semisimple element $s \in G$ in a Langlands parameter corresponds to the choice of a central character

$$\chi_{s,q_0} : Z(\mathcal{H}^{\mathrm{aff}}_{\mathbb{C},q}) \to \mathbb{C}.$$

We can then consider the corresponding variety of Langlands parameters

$$\mathcal{N}^{(s,q_0)} = \{n \in \mathcal{N} \mid s \cdot n = q_0 n\}$$

with fixed semisimple part s. A key result in [24] relates representations of $\mathcal{H}^{\mathrm{aff}}_{\mathbb{C},q}$ with central character $\chi_{(s,q_0)}$ to the category of constructible sheaves $D^b_c(\mathcal{N}^{(s,q_0)})$ and in particular the category of perverse sheaves $\mathrm{Perv}(\mathcal{N}^{(s,q_0)}) \subset D^b_c(\mathcal{N}^{(s,q_0)})$. More specifically, there is a certain semisimple complex $\mathbf{S}^{(s,q_0)} \in D^b_c(\mathcal{N}^{(s,q_0)})$ (which means that $\mathbf{S}^{(s,q_0)}$ is a direct sum of shifts of simple perverse sheaves) such that there is an isomorphism

$$\mathcal{H}^{\mathrm{aff}}_{\mathbb{C},q}/(\ker(\chi_{(s,q_0)})) \cong \mathrm{Hom}^*(\mathbf{S}^{(s,q_0)}, \mathbf{S}^{(s,q_0)}). \tag{1.2}$$

It can be deduced from this that there is a bijection

$$\mathrm{Irr}(\mathcal{H}^{\mathrm{aff}}_{\mathbb{C},q}/(\ker(\chi_{(s,q_0)}))) \xleftrightarrow{1:1} \{X \in \mathrm{Irr}(\mathrm{Perv}(\mathcal{N}^{(s,q_0)})) \mid X \text{ appears in } \mathbf{S}^{(s,q_0)}\}.$$

This can be viewed as a perverse sheaves version of the more combinatorial Deligne-Langlands correspondence from (1.1). In fact, one can associate to any (equivariant) simple perverse sheaf on $\mathcal{N}^{(s,q_0)}$ a $G(s)$-orbit and picking any point n in that orbit yields a pair (s, n) which is the Langlands parameter associated to that simple perverse sheaf. Let

$$D_{\mathrm{Spr}}(\mathcal{N}^{(s,q_0)}) := \langle \mathbf{S}^{(s,q_0)} \rangle_{\Delta,\Subset}$$

be the smallest full triangulated subcategory of $D^b_c(\mathcal{N}^{(s,q_0)})$ which contains $\mathbf{S}^{(s,q_0)}$ and is closed under shifts and direct summands. We will prove the following derived version of the Deligne-Langlands correspondence.

Theorem. *(Theorem 3.5.4)*
There is an equivalence of triangulated categories

$$D_{\mathrm{perf}}(\mathcal{H}^{\mathrm{aff}}_{\mathbb{C},q}/(\ker(\chi_{(s,q_0)})) - \mathrm{dgMod}) \cong D_{\mathrm{Spr}}(\mathcal{N}^{(s,q_0)})^{\mathrm{op}}$$

which identifies $\mathbf{S}^{(s,q_0)}$ with the free dg-module $\mathcal{H}^{\mathrm{aff}}_{\mathbb{C},q}/(\ker(\chi_{(s,q_0)}))$. Here we consider $\mathcal{H}^{\mathrm{aff}}_{\mathbb{C},q}/(\ker(\chi_{(s,q_0)}))$ as a dg-algebra with vanishing differential and grading induced by the Hom^**-grading in* (1.2).

We briefly explain the strategy of the proof of this theorem. By standard dg-techniques, there is an abstract dg-algebra $\mathcal{R}$ such that

$$D_{\mathrm{perf}}(\mathcal{R} - \mathrm{dgMod}) \cong D_{\mathrm{Spr}}(\mathcal{N}^{(s,q_0)})^{\mathrm{op}}$$

and

$$H^*(\mathcal{R}) \cong \mathrm{Hom}^*(\mathbf{S}^{(s,q_0)}, \mathbf{S}^{(s,q_0)}).$$

Thus, the main content of the theorem above is that the dg-algebra $\mathcal{R}$ can be replaced with its cohomology which is known as 'formality'. To prove formality, we first reduce to varieties over a field of positive characteristic using the 'De $\mathbb{F}$ à $\mathbb{C}$' technique from [11]. Then all relevant sheaves can be equipped with a canonical Frobenius action. We show by explicit computation that this action is 'pure' which means that its eigenvalues are of a specific form. It can then be shown that this purity implies formality in a fairly general setting which completes the proof of our derived Delinge-Langlands correspondence.

The idea that purity implies formality goes back to [28, 29]. However, there are several technical problems one has to deal with to obtain a "purity implies formality" result in a general setting as we use it: one difficulty that arises is that the constructible derived category $D^b_c(X) = D^b_c(X, \overline{\mathbb{Q}}_\ell)$ is not a genuine derived category, at least in its standard definition [11, 28]. In particular, $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ is not naturally dg-enhanced in this setting. It is shown in [54] that this problem can be solved by working with a dg-enhancement of $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ arising from an alternative construction of the constructible derived category with integral coefficients $D^b_c(X, \mathcal{O}_E)$ [31,43]. There is yet another construction of $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ coming from the pro-étale topology introduced in [19] which is the most natural setting for dg-purposes. In fact, in the pro-étale topology, the constructible derived category $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ does arise as a full triangulated subcategory of a genuine derived category and thus it is naturally dg-enhanced. For this reason, we choose to work with $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ in its pro-étale realization (which will be reviewed in Section 3.3.3). To get a "purity implies formality" result, one also has to lift the Frobenius action on morphisms of sheaves to the dg-level as in [54, Appendix A]. We explain a variant of this Frobenius lifting argument in Section 3.3.4. Once the Frobenius action is lifted to the dg-level, formality will be a consequence of an algebraic result in [54, Theorem B.1.1]. While it would certainly be possible to directly apply the results from [54] in our setting without referring to the pro-étale topology, we hope that our alternative approach clarifies some of the technical difficulties.

Canonical bases and composition multiplicities

The geometric methods employed in the proof of the Deligne-Langlands correspondence also allow to construct representations of $\mathcal{H}_{q_0}^{\mathrm{aff}}$. In fact, for each Langlands parameter (s, n), there is a corresponding 'standard module' which can be realized geometrically as the Borel-Moore homology of a certain 'Springer fiber'. The irreducible representations of the affine Hecke algebra then appear as quotients of these standard modules. Thus, to study simple modules via standard modules one has to compute the composition multiplicities of the standard modules. It is known that these multiplicities are encoded in the stalks of the (equivariant) simple perverse sheaves on $\mathcal{N}^{(s,q_0)}$.

In the GL_n-case, it turns out that the variety $\mathcal{N}^{(s,q_0)}$ can also be interpreted as a moduli space of quiver representations for a type A quiver (when q_0 is not a root of unity). Such moduli spaces are nice for computations since all irreducible local systems on their orbits are trivial and if the quiver is of finite ADE type, there are only finitely many orbits. We will therefore consider moduli spaces of finite ADE type quivers.

Let I be an indexing set for the vertices of such a finite ADE type quiver. We can then consider the positive part of the corresponding quantum group U^+ with standard generators $\{E_i \mid i \in I\}$. This is a certain quantized version of the positive part of the universal enveloping algebra. The algebra U^+ admits several standard bases of PBW type, but it was observed by Lusztig [48] that U^+ also admits a canonical basis. Moreover, it is known that the change of basis between the standard basis $\mathbf{B}^{\mathrm{std}}$ and the canonical basis $\mathbf{B}^{\mathrm{can}}$ is encoded in the stalks of the (equivariant) simple perverse sheaves on the corresponding moduli space of quiver representations. Thus, in type A, the following problems are equivalent:

- Compute the change of basis between the standard basis $\mathbf{B}^{\mathrm{std}}$ and the canonical basis $\mathbf{B}^{\mathrm{can}}$;
- Compute the stalks of the (equivariant) simple perverse sheaves on $\mathcal{N}^{(s,q_0)}$;
- Compute the composition multiplicities for the standard modules over $\mathcal{H}_{\mathbb{C},q}^{\mathrm{aff}}$ with central character χ_{s,q_0}.

Our main result in Chapter 4 is a new algorithm that computes the change of basis between $\mathbf{B}^{\mathrm{std}}$ and $\mathbf{B}^{\mathrm{can}}$ in U^+ for any quiver of finite ADE type, thus providing a solution to the above problems. The algorithm is inspired by the Lusztig-Shoji algorithm [45, 62] which computes stalks of perverse sheaves on the nilpotent cone. At the heart of the Lusztig-Shoji algorithm lies a Gram-Schmidt type procedure which is applied to a certain Ext-pairing on perverse sheaves. In the quantum groups setting, the replacement for the Ext-pairing is the standard bilinear form

$$(\cdot,\cdot) : U^+ \times U^+ \to \mathbb{Q}(v)$$

defined in [49, 1.2.3]. One of our main results is a new explicit formula for this bilinear form on monomials in the generators $\{E_i \mid i \in I\}$ (Corollary 4.3.11). The formula is combinatorial, but its proof relies on some explicit computations with perverse sheaves. In principle, the bilinear form on U^+ can also be computed from its recursive definition (see also the closed formula in [56]) but our new formula has the advantage that it applies to more general geometric situations (Theorem 4.3.9).

We can now pick a basis $\mathbf{B}^{\text{mon}}$ of U^+ consisting of monomials. Our new formula for the standard bilinear form allows us to explicitly compute the matrix Ψ representing the standard bilinear form with respect to $\mathbf{B}^{\text{mon}}$. The following algorithm explains how to compute the standard and the canonical basis from the matrix Ψ.

Algorithm. *(Algorithm 4.3.12)*

1. *Compute the matrix Ψ using the formula from Corollary 4.3.11;*
2. *Compute the decomposition $\Psi = LDL^T$ using the Gram-Schmidt procedure (where L is lower unitriangular and D is diagonal);*
3. *Compute the unique decomposition $L = QP$ (where Q and P are lower unitriangular matrices with entries below the diagonal in $\mathbb{Z}[v+v^{-1}]$ and $v^{-1}\mathbb{Z}[v^{-1}]$ respectively).*

Output: P, Q (change of basis matrix between $\boldsymbol{B}^{\text{std}}$ and $\boldsymbol{B}^{\text{can}}$ resp. $\boldsymbol{B}^{\text{can}}$ and $\boldsymbol{B}^{\text{mon}}$).

We will execute this algorithm in a few examples to compute canonical bases in type A as well as the corresponding composition multiplicities of standard modules for the affine Hecke algebra of GL_n. We will also extend these methods to compute the dimensions of the corresponding simple modules. In fact, these dimensions can be interpreted geometrically as counting multiplicities of simple perverse sheaves in the semisimple complex $\mathbf{S}^{(s,q_0)} \in D^b_c(\mathcal{N}^{s,q_0})$ [24]. Using the identification of $\mathcal{N}^{(s,q_0)}$ with a moduli space of quiver representations, the sheaves $\mathbf{S}^{(s,q_0)}$ correspond to certain elements in U^+. Decomposing the sheaves $\mathbf{S}^{(s,q_0)}$ into simples can then be reduced to an Ext-computation which we perform using a variation of our formula for the standard bilinear form on monomials. We spell out this extended algorithm for computing dimensions of simple modules in more detail in Algorithm 4.5.10. As an example, we compute the dimensions of certain simple modules for the affine Hecke algebra of GL_4 in Example 4.5.11.

Affine Hecke algebras with unequal parameters

So far, we have only considered affine Hecke algebras with a single parameter q. However, there are also affine Hecke algebras with several unequal parameters which also play an important role in local Langlands. For example, any irreducible unipotent representation of a p-adic group in the sense of Lusztig [50] corresponds to a representation of an associated affine Hecke algebra which may have unequal parameters. Given the success of the geometric methods in the equal parameter setting, one might hope to be able to use similar techniques to study the unequal parameter setting. In the equal parameter setting, the starting point for most geometric considerations is the famous realization of the affine Hecke algebra via the equivariant algebraic K-theory of the so-called 'Steinberg variety' Z

$$K^{G\times\mathbb{G}_m}(Z) \cong \mathcal{H}_q^{\mathrm{aff}}$$

due to Kazhdan-Lusztig [41] and Ginzburg [24]. It is already mentioned in the introduction of [51] that "the K-theoretic approach seems to run into some serious difficulties in the general [unequal parameter] case". Nonetheless, Lusztig was able to classify unipotent representations of p-adic groups by relating irreducible representations of affine Hecke algebras with representations of certain graded Hecke algebras. These graded Hecke algebras do have a geometric interpretation via constructible sheaves. However, apart from being more technical, the graded Hecke algebra approach to p-adic groups has the disadvantage that it seems to be less well-behaved on a categorical level. The K-theory approach on the other hand has received renewed attention in recent years due to its applications to categorical local Langlands [14]. In light of this, the goal of finding a K-theory realization of affine Hecke algebras with unequal parameters remains highly desirable.

In type C, Kato [39] gave a surprising solution to this problem by considering a certain 'exotic Springer resolution' attached to an Sp_{2n}-representation

$$W = V(\epsilon_1) \oplus V(\epsilon_1 + \epsilon_2) \oplus V(\epsilon_1 + \epsilon_2).$$

Note that there is a natural $Sp_{2n} \times \mathbb{G}_m^3$-action on W where each copy of $\mathbb{G}_m$ acts by scaling one of the three summands. One of the main results of [39] then states that the $Sp_{2n} \times \mathbb{G}_m^3$-equivariant K-theory of the associated 'exotic Steinberg variety' realizes a certain affine Hecke algebra with three unequal parameters. This suggests that affine Hecke algebras with unequal parameters should arise more generally from K-theory by considering direct sums of representations and their 'Springer resolutions'. The main difficulty here is to find an appropriate representation and Springer resolution that yield the correct relations in algebraic K-theory. Unfortunately, outside of type C, there are no obvious candidates for complex representations that give rise to relations that resemble those of the affine Hecke algebra.

Surprisingly, the situation improves when passing to small positive characteristic. More specifically, we will show in Chapter 5 that any affine Hecke algebra of simple type with two unequal parameters has a realization via equivariant algebraic K-theory by exploiting the anomalous behavior of the adjoint representation in small characteristic. In [40] Kato already makes the observation that the exotic Springer resolution is related to small characteristic phenomena, but outside of type C, this connection hasn't been explored further in the literature thus far.

The starting point for our construction is a classical result due to Hogeweij and Hiss [35,36] which states that for any simple algebraic group G in characteristic $p = 2$ or $p = 3$ (depending on the type of G) there is a subrepresentation $\mathfrak{g}_s \subset \mathfrak{g}$ whose non-zero weights are precisely the short roots. We then consider a certain 'Springer resolution' $\tilde{V} \to V$ where

$$V := \mathfrak{g}_s \oplus \mathfrak{g}/\mathfrak{g}_s.$$

We equip V (and the associated 'Steinberg variety' Z) with a natural $G \times \mathbb{G}_m \times \mathbb{G}_m$-action where each copy of $\mathbb{G}_m$ acts by scaling one of the summands of V. Note that V has the same weights as $\mathfrak{g}$, namely the roots of G. This will allow us to prove the following geometric realization of the affine Hecke algebra with two unequal parameters $\mathcal{H}^{\mathrm{aff}}_{q_s,q_l}$ associated to G.

Theorem. *(Theorem 5.3.12)*
There is an isomorphism of algebras $K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(Z) \cong \mathcal{H}^{\mathrm{aff}}_{q_s,q_l}$.

We also conjecture that graded Hecke algebras with unequal parameters can be realized in a similar fashion by replacing equivariant K-theory with equivariant Borel-Moore homology (see Conjecture 5.4.2).

Chapter 2

Affine Hecke algebras and their representations

In this chapter, we recall a few algebraic and geometric constructions related to affine Hecke algebras culminating in the statement of the celebrated Deligne-Langlands correspondence. The reader that is familiar with these classical results may skip to the next chapter.

2.1 Affine Hecke algebras and p-adic groups

Fix a root datum $\mathcal{R} = (X^*, X_*, \Phi, \Phi^\vee)$ with Weyl group W and fix a system of positive roots $\Phi^+ \subset \Phi$ with simple roots $\Pi \subset \Phi^+$. Throughout this thesis, q will denote the variable in the ring of Laurent polynomials $\mathbb{Z}[q, q^{-1}]$.

Definition 2.1.1. *The affine Hecke algebra (with equal parameters) $\mathcal{H}_q^{\mathrm{aff}} = \mathcal{H}_q^{\mathrm{aff}}(\mathcal{R})$ associated to $\mathcal{R}$ is the $\mathbb{Z}[q, q^{-1}]$-algebra with basis $\{T_w e^\lambda \mid w \in W, \lambda \in X^*\}$ and multiplication rules*

(i) $T_{s_\alpha}^2 = (q-1)T_{s_\alpha} + q$ $\qquad \alpha \in \Pi$,

(ii) $T_w T_{w'} = T_{ww'}$ $\qquad l(ww') = l(w) + l(w')$,

(iii) $e^\lambda e^\mu = e^{\lambda+\mu}$ $\qquad \lambda, \mu \in X^*$,

(iv) $e^\lambda T_{s_\alpha} - T_{s_\alpha} e^{s_\alpha(\lambda)} = (q-1)\frac{e^\lambda - e^{s_\alpha(\lambda)}}{1-e^{-\alpha}}$ $\qquad \alpha \in \Pi,\ \lambda \in X^*$,

where $T_w := T_w e^0$ and $e^\lambda := T_e e^\lambda$.

Note that the T_w ($w \in W$) span a subalgebra $\mathcal{H}_q \subset \mathcal{H}_q^{\mathrm{aff}}$ which is the (finite) Hecke algebra associated to W. Similarly, the e^λ ($\lambda \in X^*$) span a subalgebra of $\mathcal{H}_q^{\mathrm{aff}}$

isomorphic to the group algebra $\mathbb{Z}[q, q^{-1}][X^*]$. We will also consider the complexified affine Hecke algebra

$$\mathcal{H}^{\text{aff}}_{\mathbb{C},q} := \mathcal{H}^{\text{aff}}_q \otimes_{\mathbb{Z}} \mathbb{C}.$$

Moreover, we can specialize the affine Hecke algebra at any parameter $q_0 \in \mathbb{C}^\times$:

$$\mathcal{H}^{\text{aff}}_{q_0} := \mathcal{H}^{\text{aff}}_{\mathbb{C},q}/(q - q_0).$$

Here are some well-known algebraic properties of the affine Hecke algebra.

Lemma 2.1.2. *[24, 47]*

(i) The center of the affine Hecke algebra is

$$Z(\mathcal{H}^{\text{aff}}_q) = \mathbb{Z}[q, q^{-1}][X^*]^W,$$
$$Z(\mathcal{H}^{\text{aff}}_{\mathbb{C},q}) = \mathbb{C}[q, q^{-1}][X^*]^W.$$

(ii) The affine Hecke algebra $\mathcal{H}^{\text{aff}}_q$ (resp. $\mathcal{H}^{\text{aff}}_{\mathbb{C},q}$) is a free $Z(\mathcal{H}^{\text{aff}}_q)$-module (resp. $Z(\mathcal{H}^{\text{aff}}_{\mathbb{C},q})$-module) of rank $|W|^2$.

The (complex) representation theory of affine Hecke algebras is of considerable interest due to its connection with the representation theory of p-adic groups which we now briefly recall.

Let F be a non-archimedian local field with ring of integers $\mathcal{O}_F$ and residue field κ_F. Let $\mathcal{G}$ be a split reductive group over F and consider the corresponding p-adic group $\mathcal{G}(F)$ which is the group of F-points of $\mathcal{G}$. A central problem is to classify and construct the irreducible smooth representations of $\mathcal{G}(F)$. For this, it is useful to consider fixed-vectors with respect to some interesting compact open subgroups of $\mathcal{G}(F)$. One such subgroup is the so-called Iwahori subgroup $\mathcal{I} \subset \mathcal{G}(F)$ which can be thought of as a p-adic analogue of the Borel subgroup of a reductive algebraic group. In fact, if $\mathcal{G}$ is simply connected, $\mathcal{I}$ is precisely the preimage of the Borel subgroup $B(\kappa_F) \subset G(\kappa_F)$ under the projection $\mathcal{G}(\mathcal{O}_F) \to \mathcal{G}(\kappa_F)$. The following theorem due to Borel and Casselman provides a correspondence between representations of affine Hecke algebras and p-adic groups.

Theorem 2.1.3. *[21] Let $\mathcal{R}$ be the root datum dual to the root datum of $\mathcal{G}$. Then there is a bijection*

$$\{V \in \operatorname{Irr}_{sm}(\mathcal{G}(F)) \mid V^I \textit{ generates } V\} \xleftrightarrow{1:1} \operatorname{Irr}(\mathcal{H}^{\text{aff}}_{|\kappa_F|})$$

where $\mathcal{H}^{\text{aff}}_{|\kappa_F|}$ is the affine Hecke algebra with root datum $\mathcal{R}$ specialized at $|\kappa_F| \in \mathbb{C}^\times$.

Remark 2.1.4. *This bijection can also be upgraded to an equivalence of categories: The category of smooth $\mathcal{G}(F)$-representations $\mathrm{Rep}_{sm}(\mathcal{G}(F))$ breaks into so-called Bernstein blocks and there is a unique 'principal block'*

$$\mathrm{Rep}^0_{sm}(\mathcal{G}(F)) \subset \mathrm{Rep}_{sm}(\mathcal{G}(F))$$

which contains the trivial representation. The bijection from Theorem 2.1.3 then lifts to an equivalence of categories

$$\mathrm{Rep}^0_{sm}(\mathcal{G}(F)) \cong \mathcal{H}^{\mathrm{aff}}_{|\kappa_F|} - \mathrm{Mod}\,.$$

Motivated by the Borel-Casselman equivalence, one would like to parametrize and construct the irreducible representations of affine Hecke algebras. This problem was solved by Kazhdan-Lusztig and Ginzburg using geometric methods and we will review their results in the following sections.

2.2 Equivariant K-theory and the Kazhdan-Lusztig isomorphism

Let H be a complex linear algebraic group and X a quasi-projective H-variety. We denote by $\mathrm{Coh}^H(X)$ the category of equivariant coherent sheaves on X and the equivariant algebraic K-theory of X by

$$K^H(X) := K_0(\mathrm{Coh}^H(X)).$$

Let X be smooth and consider the projections $p_{ij} : X \times X \times X \to X \times X$ onto the i-th and j-th factor ($1 \leq i < j \leq 3$). Then for any H-stable closed subvarieties $Z_{12}, Z_{23} \subset X \times X$, there is a convolution operation (c.f. [24, (5.2.21)])

$$K^H(Z_{12}) \times K^H(Z_{23}) \to K^H(Z_{12} \circ Z_{23})$$

where

$$Z_{12} \circ Z_{23} := p_{13}(p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23})).$$

We will review the construction of this convolution operation and the basic properties of equivariant K-theory in more detail in Chapter 5. For the remainder of this section, we recall how affine Hecke algebras arise from equivariant K-theory.

Let G be a complex reductive group with simply connected derived subgroup. Let $B \subset G$ be a Borel subgroup and let $\mathcal{B} := G/B$ be the flag variety. We denote by $\mathfrak{g}$

(resp. $\mathfrak{b}$) the Lie algebra of G (resp. B). Let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ and let $\mathcal{N} \subset \mathfrak{g}$ the nilpotent cone of G. The Springer resolution is the space

$$\tilde{\mathcal{N}} := G \times^B \mathfrak{n} \cong \{(gB, x) \in \mathcal{B} \times \mathcal{N} \mid g^{-1} \cdot x \in \mathfrak{b}\}$$

This comes with the two projections

$$\begin{tikzcd} & \tilde{\mathcal{N}} \arrow[dl, "\pi"'] \arrow[dr, "\mu"] & \\ \mathcal{B} & & \mathcal{N}. \end{tikzcd} \tag{2.1}$$

The maps π and μ are $G \times \mathbb{G}_m$-equivariant where $t \in \mathbb{G}_m$ acts by scaling with t^{-1} on $\mathcal{N}$ (resp. $\mathfrak{n}$) and trivially on $\mathcal{B}$. Let

$$Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} = \{((g_1B, x_1), (g_2B, x_2)) \in \tilde{\mathcal{N}} \times \tilde{\mathcal{N}} \mid x_1 = x_2\}$$

be the Steinberg variety. It is straightforward to check that

$$Z \circ Z = Z.$$

Hence, convolution equips $K^{G \times \mathbb{G}_m}(Z)$ with the structure of a $\mathbb{Z}[q, q^{-1}]$-algebra where multiplication by q corresponds to tensoring with the irreducible weight 1 representation of $\mathbb{G}_m$. The following theorem is the key result that builds a bridge between affine Hecke algebras and the geometry of the Steinberg variety.

Theorem 2.2.1. *[24, 41] There is an isomorphism of $\mathbb{Z}[q, q^{-1}]$-algebras*

$$\mathcal{H}_q^{\mathrm{aff}} \cong K^{G \times \mathbb{G}_m}(Z).$$

Under this isomorphism, the center $Z(\mathcal{H}_q^{\mathrm{aff}})$ corresponds to $K^{G \times \mathbb{G}_m}(\mathrm{pt})$.

2.3 Constructible and perverse sheaves

For any variety X, we can consider the derived category of constructible sheaves $D_c^b(X)$ with complex coefficients. When X is defined over the complex numbers, this category is often constructed using sheaves on the associated complex analytic space $X(\mathbb{C})$. However, for technical reasons, we will also have to consider varieties (and schemes) which are not defined over the complex numbers. Thus, we will use the ℓ-adic version of the constructible derived category

$$D_c^b(X) := D_c^b(X, \overline{\mathbb{Q}}_\ell)$$

coming from the (pro-)étale topology as defined in [11, 19, 28]. Here ℓ is a prime number which is coprime to the characteristic of X. We will also occasionally consider the constructible derived category with respect to other coefficient systems (e.g. with $\mathbb{F}_\ell$ or $\mathbb{Z}_\ell$ cofficients) but we will reserve the notation $D^b_c(X)$ for $\overline{\mathbb{Q}}_\ell$ coefficients.

Remark 2.3.1. *Note that we can pick an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$, so $D^b_c(X)$ can still be considered as a construcible derived category with complex coefficients.*

The construction of the category $D^b_c(X)$ will be reviewed in more detail in Section 3.3.3. For now, we summarize a few of its basic properties and fix some notation. The constructible derived category comes equipped with the usual 6 (derived) operations

$$(f^*, f_*), (f_!, f^!), (\otimes^L, \mathrm{R}\mathcal{H}om)$$

associated to any morphism of varieties $f : X \to Y$. These are pairs of adjoint functors which are subject to many useful identities (e.g. base change, projection formula, ...). We denote the constant sheaf on X by $\mathbf{1}_X$ and the dualizing complex by $\omega_X := a^! \mathbf{1}_{\mathrm{pt}}$ where $a : X \to \{\mathrm{pt}\}$ is the structure map. The Verdier duality functor will be denoted by $\mathbb{D} := \mathcal{H}\mathrm{om}(-, \omega_X)$. We will write

$$\mathrm{Hom}^i(\mathcal{F}, \mathcal{G}) := \mathrm{Hom}_{D^b_c(X)}(\mathcal{F}, \mathcal{G}[i]).$$

The cohomology and (Borel-Moore) homology of X are defined as

$$H^i(X) := \mathrm{Hom}^i(\mathbf{1}_X, \mathbf{1}_X)$$
$$H_i(X) := \mathrm{Hom}^{-i}(\mathbf{1}_X, \omega_X).$$

The category $D^b_c(X)$ comes equipped with a standard t-structure whose heart is denoted by $\mathrm{Sh}_c(X)$. There is a full subcategory of 'local systems' $\mathrm{Loc}(X) \subset \mathrm{Sh}_c(X)$ and if X is connected, $\mathrm{Loc}(X)$ is equivalent to the category of continuous finite-dimensional $\overline{\mathbb{Q}}_\ell$-representations of the étale fundamental group $\pi_1^{\text{ét}}(X)$.

There is also a non-standard t-structure on $D^b_c(X)$ called the perverse t-structure which is defined by

$$\begin{aligned} \mathcal{F} \in {}^pD^b_c(X)^{\geq 0} &\Leftrightarrow \dim\operatorname{supp} H^{-i}\mathcal{F} \leq i \quad \forall i \in \mathbb{Z} \\ \mathcal{F} \in {}^pD^b_c(X)^{\leq 0} &\Leftrightarrow \dim\operatorname{supp} H^{-i}\mathbb{D}\mathcal{F} \leq i \quad \forall i \in \mathbb{Z}. \end{aligned} \tag{2.2}$$

Its heart will be denoted by $\mathrm{Perv}(X)$. If X is smooth and connected, we have

$$\omega_X = \mathbf{1}_X[2\dim X]$$

and hence $\mathbb{D}(\mathbf{1}_X[\dim X]) = \mathbf{1}_X[\dim X]$. Comparing with (2.2), we see that

$$\mathcal{C}_X := \mathbf{1}_X[\dim X]$$

is a perverse sheaf which we call the constant perverse sheaf. More generally, if X is smooth with connected components $X_1, ..., X_n$, we obtain a constant perverse sheaf

$$\mathcal{C}_X := \bigoplus_{i=1}^{n} \mathbf{1}_{X_i}[\dim X_i]. \tag{2.3}$$

In general, the simple objects of $\mathrm{Perv}(X)$ can be classified as the so-called intersection cohomology complexes on X. More specifically, for any smooth connected locally closed subvariety $Y \subset X$ and any irreducible local system ϕ on Y, there is a unique simple object $\mathrm{IC}(Y, \phi) \in \mathrm{Perv}(X)$ which is supported on $\overline{Y}$ and satisfies

$$H^{-\dim Y}(\mathrm{IC}(Y, \phi))|_Y = \phi.$$

Any simple object in $\mathrm{Perv}(X)$ is of this form. A very powerful tool in the study of perverse sheaves is the decomposition theorem.

Theorem 2.3.2. *[11] Let $f : X \to Y$ be a proper morphism and $Z \subset X$ a smooth locally closed subvariety. Then there is a finite decomposition*

$$f_* \mathrm{IC}(Z, \mathbf{1}_Z) \cong \bigoplus_k \mathrm{IC}(Y_k, \phi_k)[l_k]$$

where each $\mathrm{IC}(Y_k, \phi_k)$ is a simple perverse sheaf on X and $l_k \in \mathbb{Z}$.

The decomposition theorem has many useful applications to representation theory. For example, it can be used to study representation of certain Ext-algebras. To simplify notation, we will write

$$\mathrm{IC}_\phi := \mathrm{IC}(Y, \phi).$$

Let A be a finite dimensional algebra which can be realized as an Ext-algebra

$$A \cong \mathrm{Hom}^*(f_*\mathcal{C}_X, f_*\mathcal{C}_X)$$

for a proper morphism $f : X \to Y$ with X smooth. Using the decomposition theorem, we obtain a decomposition

$$f_*\mathcal{C}_X = f_* \mathrm{IC}(X, \mathbf{1}_X) \cong \bigoplus_\phi \mathrm{IC}_\phi \otimes V_\phi$$

where the V_ϕ are finite-dimensional graded vector spaces which are non-zero for only finitely many ϕ. We now obtain the following (non-grading preserving) isomorphism of algebras

$$\begin{aligned} A &\cong \bigoplus_{k\in\mathbb{Z}} \operatorname{Hom}\left(\bigoplus_{\phi} \mathrm{IC}_\phi \otimes V_\phi, \bigoplus_{\psi} \mathrm{IC}_\psi \otimes V_\psi[k]\right) \\ &\cong \bigoplus_{i,j,k\in\mathbb{Z}} \bigoplus_{\phi,\psi} \operatorname{Hom}\left(\mathrm{IC}_\phi[i], \mathrm{IC}_\psi[j+k]\right) \otimes \operatorname{Hom}_{\mathrm{Vect}}(V_\phi^i, V_\psi^j) \\ &\cong \bigoplus_{k\geq 0} \bigoplus_{\phi,\psi} \operatorname{Hom}^k\left(\mathrm{IC}_\phi, \mathrm{IC}_\psi\right) \otimes \operatorname{Hom}_{\mathrm{Vect}}(V_\phi, V_\psi) \\ &= \bigoplus_{\phi} \operatorname{End}_{\mathrm{Vect}}(V_\phi) \oplus I \end{aligned} \tag{2.4}$$

where

$$I := \bigoplus_{k>0} \bigoplus_{\phi,\psi} \operatorname{Hom}^k\left(\mathrm{IC}_\phi, \mathrm{IC}_\psi\right) \otimes \operatorname{Hom}_{\mathrm{Vect}}(V_\phi, V_\psi).$$

Note that $\bigoplus_{k\geq 0} \bigoplus_{\phi,\psi} \operatorname{Hom}^k\left(\mathrm{IC}_\phi, \mathrm{IC}_\psi\right) \otimes \operatorname{Hom}_{\mathrm{Vect}}(V_\phi, V_\psi)$ comes with a new internal Ext-grading which makes I into a positively graded ideal. In particular, since A is finite-dimensional, we see that I is nilpotent. Hence, the maximal semisimple quotient of A is $A/I \cong \bigoplus \operatorname{End}_{\mathrm{Vect}}(V_\phi)$. Thus, we obtain the following parametrization of simple modules as a consequence of the decomposition theorem.

Corollary 2.3.3. *The non-zero V_ϕ form a complete set of (pairwise distinct) irreducible representation of A.*

The theory of constructible and perverse sheaves can also be formulated in an equivariant setting. Let H be a linear algebraic group acting on a variety X. Following [16], one can define the bounded derived category of constructible equivariant sheaves $D^b_{H,c}(X)$. This also comes with the usual 6 functors where f_*, f^*, $f_!$ and $f^!$ are defined for H-equivariant morphisms $f : X \to Y$. For any $\mathcal{F}, \mathcal{G} \in D^b_{H,c}(X)$ we will write

$$\operatorname{Hom}^i_H(\mathcal{F}, \mathcal{G}) := \operatorname{Hom}_{D^b_{H,c}(X)}(\mathcal{F}, \mathcal{G}[i]).$$

There are equivariant versions of the constant sheaf and the dualizing complex

$$\mathbf{1}_X, \omega_X \in D^b_{H,c}(X)$$

and one can define equivariant cohomology and equivariant (Borel-Moore) homology of X as

$$\begin{aligned} H^i_H(X) &:= \operatorname{Hom}^i_H(\mathbf{1}_X, \mathbf{1}_X) \\ H^H_i(X) &:= \operatorname{Hom}^{-i}_H(\mathbf{1}_X, \omega_X). \end{aligned}$$

As in the non-equivariant setting, there is a perverse t-structure on $D^b_{H,c}(X)$ and we will denote its heart by $\mathrm{Perv}_H(X)$. The simple objects of this category can be classified as equivariant IC-sheaves associated to irreducible equivariant local systems on smooth locally closed H-subvarieties. There also is a forgetful functor $D^b_{H,c}(X) \to D^b_c(X)$ which is exact with respect to the perverse t-structures. If H is connected, the induced functor $\mathrm{Perv}_H(X) \to \mathrm{Perv}(X)$ is fully faithful and closed under taking subquotients (but the whole forgetful functor $D^b_{H,c}(X) \to D^b_c(X)$ is not fully faithful in general).

If X has only finitely many H-orbits, the classification of the simple objects in $\mathrm{Perv}_H(X)$ is particularly nice. In this case, the equivariant local system on an H-orbit $\mathcal{O} \subset X$ correspond to representations of the component group

$$A(\mathcal{O}) = A(x) = H_x/H_x^\circ$$

where $x \in \mathcal{O}$ is any base point. Then any simple object in $\mathrm{Perv}_H(X)$ is of the form $\mathrm{IC}(\mathcal{O}, \rho)$ for some orbit $\mathcal{O}$ and $\rho \in \mathrm{Irr}(A(\mathcal{O}))$. This yields a bijection

$$\mathrm{Irr}(\mathrm{Perv}_H(X)) \xleftrightarrow{1:1} \{(x, \rho) \mid x \in X, \rho \in \mathrm{Irr}(A(x))\}/H. \tag{2.5}$$

Since the component group $A(x)$ is always finite, this implies that there are only finitely many simple objects in $\mathrm{Perv}_H(X)$. For more details on the construction and properties of the equivariant derived category and equivariant perverse sheaves, we refer to [1, Chapter 6].

2.4 Affine Hecke algebras at a central character

In this section, we discuss a geometric incarnation of the affine Hecke algebra at a central character using constructible sheaves. Recall that G is a complex reductive group with simply connected derived subgroup and root datum $\mathcal{R} = (X^*, X_*, \Phi, \Phi^\vee)$. Let $T \subset G$ be a maximal torus. Using Lemma 2.1.2 there is the following geometric interpretation of the center of the affine Hecke algebra with root datum $\mathcal{R}$:

$$Z(\mathcal{H}^{\mathrm{aff}}_{\mathbb{C},q}) = \mathbb{C}[q, q^{-1}][X^*]^W \cong \mathcal{O}(T \times \mathbb{G}_m)^W. \tag{2.6}$$

Hence, the central characters of $\mathcal{H}^{\mathrm{aff}}_{\mathbb{C},q}$ are parametrized by the points of $T/W \times \mathbb{G}_m$ or equivalently by semisimple conjugacy classes in $G \times \mathbb{G}_m$. For $(s, q_0) \in G \times \mathbb{G}_m$ semisimple, we denote the corresponding central character by

$$\chi_{(s,q_0)} : Z(\mathcal{H}^{\mathrm{aff}}_{\mathbb{C},q}) \to \mathbb{C}.$$

By (the countable dimension version of) Schur's lemma [24, Lemma 2.1.3], any simple $\mathcal{H}^{\mathrm{aff}}_{\mathbb{C},q}$-module admits a central character. Together with Lemma 2.1.2 this implies that the simple $\mathcal{H}^{\mathrm{aff}}_{\mathbb{C},q}$-modules are finite-dimensional. Moreover, the simple $\mathcal{H}^{\mathrm{aff}}_{\mathbb{C},q}$-modules with central character $\chi_{(s,q_0)}$ are precisely the simple modules of the finite-dimensional algebra

$$\mathcal{H}^{\mathrm{aff}}_{(s,q_0)} := \mathcal{H}^{\mathrm{aff}}_{\mathbb{C},q}/(\ker(\chi_{(s,q_0)})).$$

Here $(\ker(\chi_{(s,q_0)})) = \mathcal{H}^{\mathrm{aff}}_{\mathbb{C},q} \cdot \ker(\chi_{(s,q_0)})$ is the two-sided ideal in $\mathcal{H}^{\mathrm{aff}}_{\mathbb{C},q}$ generated by $\ker(\chi_{(s,q_0)})$. On the geometric side, fixing a central character is related to taking fixed-points. More specifically, passing to (s,q_0)-fixed points in (2.1), we obtain a diagram

$$\begin{tikzcd} & \tilde{\mathcal{N}}^{(s,q_0)} \arrow[dl, "\pi^{(s,q_0)}"'] \arrow[dr, "\mu^{(s,q_0)}"] & \\ \mathcal{B}^s & & \mathcal{N}^{(s,q_0)}. \end{tikzcd}$$

The variety $\tilde{\mathcal{N}}^{(s,q_0)}$ is smooth (c.f. [24, Lemma 5.11.1]) and thus, we can consider the constant perverse sheaf $\mathcal{C}_{\tilde{\mathcal{N}}^{(s,q_0)}} \in D^b_c(\tilde{\mathcal{N}}^{(s,q_0)})$ and the corresponding 'Springer sheaf'

$$\mathbf{S}^{(s,q_0)} := (\mu^{(s,q_0)})_* \mathcal{C}_{\tilde{\mathcal{N}}^{(s,q_0)}} \in D^b_c(\mathcal{N}^{(s,q_0)}).$$

Then there is the following important geometric incarnation of the affine Hecke algebra at a central character.

Theorem 2.4.1. *[24] There is an isomorphism of algebras (over $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$)*

$$\mathcal{H}^{\mathrm{aff}}_{(s,q_0)} \cong \mathrm{Hom}^*(\mathbf{S}^{(s,q_0)}, \mathbf{S}^{(s,q_0)}).$$

Proof. This is proved in [24, Proposition 8.1.5, Lemma 8.6.1]. The only difference to our situation is that we work with the constructible derived category of $\overline{\mathbb{Q}}_\ell$-sheaves $D^b_c(X) = D^b_c(X, \overline{\mathbb{Q}}_\ell)$ coming from the (pro-)étale topology whereas *loc.cit.* works with the constructible derived category $D^b_c(X(\mathbb{C}), \mathbb{C})$ coming from the associated complex analytic space $X(\mathbb{C})$. It turns out that the two approaches are essentially equivalent: By [11, p.146] there is a fully faithful functor

$$D^b_c(X, \overline{\mathbb{Q}}_\ell) \to D^b_c(X(\mathbb{C}), \mathbb{C})$$

(which involves choosing an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$). For $X = \mathcal{N}^{(s,q_0)}$ this identifies the $\overline{\mathbb{Q}}_\ell$-Springer sheaf $\mathbf{S}^{(s,q_0)}$ with its analytic version and thus we get an isomorphism of graded algebras

$$\mathrm{Hom}^*_{D^b_c(X,\overline{\mathbb{Q}}_\ell)}(\mathbf{S}^{(s,q_0)}, \mathbf{S}^{(s,q_0)}) \cong \mathrm{Hom}^*_{D^b_c(X(\mathbb{C}),\mathbb{C})}(\mathbf{S}^{(s,q_0)}, \mathbf{S}^{(s,q_0)}).$$

$\square$

2.5 The Deligne-Langlands correspondence

Combining Theorem 2.4.1 with Corollary 2.3.3 we get the following geometric parametrization of simple $\mathcal{H}^{\mathrm{aff}}_{(s,q_0)}$-modules.

Theorem 2.5.1. *[24] There is a canonical injection*

$$\mathrm{Irr}(\mathcal{H}^{\mathrm{aff}}_{(s,q_0)}) \hookrightarrow \mathrm{Irr}(\mathrm{Perv}(\tilde{\mathcal{N}}^{(s,q_0)})).$$

The image of this map consists of all simple perverse sheaves that appear in $\mathbf{S}^{(s,q_0)}$.

This can be viewed as the first instance of the Deligne-Langlands correspondence. Note that the morphism $\mu^{(s,q_0)} : \tilde{\mathcal{N}}^{(s,q_0)} \to \mathcal{N}^{(s,q_0)}$ is equivariant with respect to the centralizer group

$$G(s) = \{g \in G \mid gs = sg\}.$$

This implies that $\mathbf{S}^{(s,q_0)}$ and its simple constituents are also $G(s)$-equivariant. Thus, the injection above is in fact a map

$$\mathrm{Irr}(\mathcal{H}^{\mathrm{aff}}_{(s,q_0)}) \hookrightarrow \mathrm{Irr}(\mathrm{Perv}_{G(s)}(\tilde{\mathcal{N}}^{(s,q_0)})).$$

It can be shown that $\mathcal{N}^{(s,q_0)}$ has only finitely many $G(s)$-orbits (c.f. [24, Proposition 8.1.17]). Thus, the combinatorial description of simple equivariant perverse sheaves from (2.5) yields a map

$$\mathrm{Irr}(\mathcal{H}^{\mathrm{aff}}_{(s,q_0)}) \hookrightarrow \{(n,\rho) \mid n \in \mathcal{N}^{(s,q_0)}, \rho \in \mathrm{Irr}(A(s,n))\}/G(s)$$

where $A(s,n)$ is the component group of the simultaneous centraliser

$$G(s,n) = \{g \in G(s) \mid g \cdot n = n\}.$$

Combining these injections for all central characters we obtain the following more combinatorial version of the Deligne-Langlands correspondence.

Corollary 2.5.2. *[24, 41] There is a canonical injection*

$$\mathrm{Irr}(\mathcal{H}^{\mathrm{aff}}_{q_0}) \hookrightarrow \{(s,n,\rho) \mid s \in G \textit{ semisimple }, n \in \mathcal{N}^{(s,q_0)}, \rho \in \mathrm{Irr}(A(s,n))\}/G. \quad (2.7)$$

If q_0 is not a root of unity, there also is a geometric description of the image of this map. Let

$$\mathcal{B}^s_n := (\mu^{(s,q_0)})^{-1}(n) \cong \{B \in \mathcal{B} \mid s \in B, n \in \mathfrak{b}\}.$$

Then the canonical $G(s,n)$-action on $\mathcal{B}^s_n$ induces an $A(s,n)$-module structure on $H_*(\mathcal{B}^s_n)$.

Theorem 2.5.3. *[24, 41] Let q_0 be not a root of unity. Then, a triple (s, n, ρ) lies in the image of the map* (2.7) *if and only if ρ appears in the $A(s, n)$-module $H_*(\mathcal{B}_n^s)$.*

One can also use geometric techniques to construct modules for the affine Hecke algebra. Note for this that we have a cartesian square

$$\begin{tikzcd} \mathcal{B}_n^s \arrow[r, hook, "\tilde{\iota}"] \arrow[d, "a"] & \tilde{\mathcal{N}}^{(s,q_0)} \arrow[d, "\mu^{(s,q_0)}"] \\ \{n\} \arrow[r, hook, "\iota"] & \mathcal{N}^{(s,q_0)}. \end{tikzcd}$$

Hence, using base change, we can identify

$$\begin{aligned} H_*(\mathcal{B}_n^s) &= \mathrm{Hom}^*(\mathbf{1}_{\mathcal{B}_n^s}, \omega_{\mathcal{B}_n^s}) \\ &= \mathrm{Hom}^*(a^*\mathbf{1}_{\{n\}}, \tilde{\iota}^! \omega_{\tilde{\mathcal{N}}^{(s,q_0)}}) \\ &\cong \mathrm{Hom}^*(\tilde{\iota}_! a^*\mathbf{1}_{\{n\}}, \omega_{\tilde{\mathcal{N}}^{(s,q_0)}}) \\ &\cong \mathrm{Hom}^*((\mu^{(s,q_0)})^* \iota_! \mathbf{1}_{\{n\}}, \omega_{\tilde{\mathcal{N}}^{(s,q_0)}}) \\ &\cong \mathrm{Hom}^*(\iota_! \mathbf{1}_{\{n\}}, (\mu^{(s,q_0)})_* \omega_{\tilde{\mathcal{N}}^{(s,q_0)}}). \end{aligned}$$

Note that the sheaves $\omega_{\tilde{\mathcal{N}}^{(s,q_0)}}$ and $\mathcal{C}_{\tilde{\mathcal{N}}^{(s,q_0)}}$ are the same up to a shift of summands since $\tilde{\mathcal{N}}^{(s,q_0)}$ is smooth. Thus, we get a (non-grading preserving) isomorphism

$$H_*(\mathcal{B}_n^s) \cong \mathrm{Hom}^*(\iota_! \mathbf{1}_{\{n\}}, \mathbf{S}^{(s,q_0)})$$

Hence, by Theorem 2.4.1, there is a canonical $\mathcal{H}^{\mathrm{aff}}_{(s,q_0)}$-module structure on $H_*(\mathcal{B}_n^s)$. It can be shown that this structure is compatible with the $A(s, n)$-module structure. Thus, for any $\rho \in \mathrm{Irr}(A(s, n))$, we obtain an $\mathcal{H}^{\mathrm{aff}}_{(s,q_0)}$-module

$$M_{s,n,\rho} := \mathrm{Hom}_{A(s,n)}(\rho, H_*(\mathcal{B}_n^s))$$

known as the standard module associated to the triple (s, n, ρ). It is straightforward to check that for any $g \in G$, we have

$$M_{(s,n,\rho)} \cong M_{g\cdot(s,n,\rho)}$$

as $\mathcal{H}_q^{\mathrm{aff}}$-modules, so the standard module only depends on the G-orbit of the triple (s, n, ρ). Note that by Theorem 2.5.3 we have $M_{s,n,\rho} \neq 0$ if and only if (s, n, ρ) lies in the image of the map (2.7). The following important result explains how to obtain the simple modules from the standard modules.

Theorem 2.5.4. *[24, 41] Let q_0 be not a root of unity.*

(i) If the $\mathcal{H}^{\mathrm{aff}}_{(s,q_0)}$-module $M_{s,n,\rho}$ is non-zero, it has a unique simple quotient $L_{s,n,\rho}$;

(ii) The $L_{s,n,\rho}$ form a complete set of (distinct) simple $\mathcal{H}^{\mathrm{aff}}_{(s,q_0)}$-modules;

(iii) We have

$$[M_{(s,n,\rho)} : L_{(s,n',\rho')}] = \sum_k \dim \mathrm{Hom}_{A(s,n)}(\rho, H^k(\iota_n^! \, \mathrm{IC}(G(s) \cdot n', \rho')))$$

where $\iota_n : \{n\} \hookrightarrow \mathcal{N}^{(s,q_0)}$ is the inclusion of the point.

Chapter 3

Formality in the Deligne-Langlands correspondence

The goal of this chapter is to lift the Deligne-Langlands correspondence (specifically Theorem 2.5.1) to an equivalence of triangulated categories. These results can also be found in [5].

3.1 Springer geometry

Let $\overline{\mathbb{F}}$ be an algebraically closed field and ℓ a prime number which is invertible in $\overline{\mathbb{F}}$. We will use the notation and definitions from Section 2.3 (e.g. we work with the constructible derived category of $\overline{\mathbb{Q}}_\ell$-complexes $D^b_c(X) := D^b_c(X, \overline{\mathbb{Q}}_\ell)$).

3.1.1 Borel-Moore homology

In this section we recall a few basic facts about Borel-Moore homology. Recall that the i-th Borel-Moore homology is the $\overline{\mathbb{Q}}_\ell$-vector space

$$H_i(X) := \mathrm{Hom}^{-i}(\mathbf{1}_X, \omega_X) = H^{-i}(a_*\omega_X)$$

where $a : X \to \{\mathrm{pt}\}$ is the structure map. It can be shown that $H_i(X)$ is concentrated in degrees $0, 1, ..., 2\dim X$. Moreover, we have the Künneth formula

$$H_*(X \times Y) \cong H_*(X) \otimes H_*(Y).$$

If $i : Y \hookrightarrow X$ is a closed immersion with complement $j : U \hookrightarrow X$ and $\mathcal{F} \in D^b_c(X)$, there is a canonical distinguished triangle

$$i_*i^!\mathcal{F} \to \mathcal{F} \to j_*j^!\mathcal{F} \to [1] \tag{3.1}$$

which is natural in $\mathcal{F}$. For $\mathcal{F} = \omega_X$ this induces a long exact sequence on Borel-Moore homology

$$\cdots \to H_{i+1}(U) \to H_i(Y) \to H_i(X) \to H_i(U) \to H_{i-1}(Y) \to \cdots. \tag{3.2}$$

Let $p : \tilde{X} \to X$ be a smooth morphism of relative dimension d. The adjoint pair (p^*, p_*) gives rise to a canonical morphism

$$\omega_X \to p_* p^* \omega_X = p_* p^! \omega_X[-2d] = p_* \omega_{\tilde{X}}[-2d].$$

This induces a 'smooth pullback' map on Borel-Moore homology

$$H_i(X) \to H_{i+2d}(\tilde{X}).$$

The naturality of the distinguished triangle in (3.1) implies that smooth pullback is compatible with the long exact sequence from (3.2), i.e. if $Y \subset X$ is a closed subvariety with open complement U and $\tilde{Y} := p^{-1}(Y)$, $\tilde{U} := p^{-1}(U)$, the following diagram commutes:

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & H_{i+2d}(\tilde{Y}) & \longrightarrow & H_{i+2d}(\tilde{X}) & \longrightarrow & H_{i+2d}(\tilde{U}) & \longrightarrow & H_{i+2d-1}(\tilde{Y}) & \longrightarrow \cdots \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ \cdots \longrightarrow & H_i(Y) & \longrightarrow & H_i(X) & \longrightarrow & H_i(U) & \longrightarrow & H_{i-1}(Y) & \longrightarrow \cdots . \end{array} \tag{3.3}$$

Lemma 3.1.1. *Let $p : \tilde{X} \to X$ be a Zariski locally trivial fibration with affine fiber $\mathbb{A}^d$. Then the smooth pullback map $H_i(X) \to H_{i+2d}(\tilde{X})$ is an isomorphism for all $i \in \mathbb{Z}$.*

Proof. Using (3.3) and the five lemma, one can reduce to the case where p is trivial. By the Künneth formula, it suffices to consider the case where $p : \mathbb{A}^d \to \mathrm{pt}$. Note that $H_i(\mathbb{A}^d) = 0$ for $i \neq 2d$ and $H_{2d}(\mathbb{A}^d) = \overline{\mathbb{Q}}_\ell$. The claim now follows since the smooth pullback map $H_0(\mathrm{pt}) \to H_{2d}(\mathbb{A}^d)$ is non-zero. $\square$

Recall that if X is smooth and connected, the dualizing complex is given by

$$\omega_X = \mathbf{1}_X[2\dim X].$$

The fundamental class of X is the distinguished element

$$\mathrm{id}_{\mathbf{1}_X} \in \mathrm{Hom}^0(\mathbf{1}_X, \mathbf{1}_X) = \mathrm{Hom}^{-2\dim X}(\mathbf{1}_X, \omega_X) = H_{2\dim X}(X)$$

also denoted by $[X] \in H_{2\dim X}(X)$. More generally, if X is irreducible one can define the fundamental class as follows: Pick a smooth open subset $U \subset X$. Then the long exact sequence (3.2) induces an isomorphism $H_{2\dim X}(U) \cong H_{2\dim X}(X)$. The image of $[U]$ under this isomorphism defines a distinguished element $[X] \in H_{2\dim X}(X)$ called the fundamental class of X. It can be shown that $[X]$ does not depend on the choice of U. If $Y \subset X$ is an irreducible closed subvariety, the image of the fundamental class of Y under $H_{2\dim Y}(Y) \to H_{2\dim Y}(X)$ also defines a fundamental class $[Y] \in H_{2\dim Y}(X)$. If elements of this form span the vector space $H_i(X)$ for each $i \in \mathbb{Z}$, we say that $H_*(X)$ is *spanned by fundamental classes*. Note that being spanned by fundamental classes implies that $H_i(X) = 0$ for i odd. Let $Z_i(X)$ be the free abelian group on the set of i-dimensional irreducible closed subvarieties of X and let $A_i(X) = Z_i(X)/\sim_{Rat}$ be the Chow group (c.f. [33]). The fundamental class construction gives rise to a cycle class map

$$\begin{aligned} cl_X : Z_i(X) &\to H_{2i}(X) \\ [Y] &\mapsto [Y] \end{aligned}$$

which descends to the Chow group

$$cl_X : A_i(X) \to H_{2i}(X) \tag{3.4}$$

by [44, Théorème 6.3]. The open-closed exact sequence and smooth pullback map from Borel-Moore homology have analogues for Chow groups: For any $Y \subset X$ closed with complement $U = X \backslash Y$ there is an exact sequence

$$A_i(Y) \to A_i(X) \to A_i(U) \to 0$$

and for $\tilde{p} : \tilde{X} \to X$ smooth (or more generally flat) of relative dimension d, there is a pullback map

$$A_i(X) \to A_{i+d}(X).$$

The cycle class map cl_X is functorial with respect to these constructions [44, Théoreme 6.1]. We define

$$A_i(X)_{\overline{\mathbb{Q}}_\ell} := A_i(X) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_\ell.$$

Following [26, 1.7], we say that a variety X has property (S) if

- $H_i(X) = 0$ for i odd;
- $cl_X : A_i(X)_{\overline{\mathbb{Q}}_\ell} \to H_{2i}(X)$ is an isomorphism for all $i \in \mathbb{Z}$.

Note that if X has property (S), then $H_*(X)$ is spanned by fundamental classes. The following two lemmas are $\overline{\mathbb{Q}}_\ell$-versions of [26, Lemma 1.8, 1.9].

Lemma 3.1.2. *Let $Y \subset X$ be a closed subvariety with complement $U = X \backslash Y$. If U and Y have property (S) then X also has property (S).*

Proof. The groups $H_i(Y)$ and $H_i(U)$ vanish for i odd. Using the long exact sequence (3.2), we deduce that $H_i(X) = 0$ for i odd. Moreover, for any $i \in \mathbb{Z}$, we get a commutative diagram with exact rows

$$\begin{tikzcd} & A_i(Y)_{\overline{\mathbb{Q}}_\ell} \arrow[r] \arrow[d, "\wr"] & A_i(X)_{\overline{\mathbb{Q}}_\ell} \arrow[r] \arrow[d] & A_i(U)_{\overline{\mathbb{Q}}_\ell} \arrow[r] \arrow[d, "\wr"] & 0 \\ 0 \arrow[r] & H_{2i}(Y) \arrow[r] & H_{2i}(X) \arrow[r] & H_{2i}(U) \arrow[r] & 0. \end{tikzcd}$$

By the five lemma, this implies that the map $A_i(X)_{\overline{\mathbb{Q}}_\ell} \to H_{2i}(X)$ is an isomorphism. $\square$

Lemma 3.1.3. *Let $p : \tilde{X} \to X$ be a Zariski locally trivial fibration with fiber $\mathbb{A}^d$. If X has property (S) then $\tilde{X}$ also has property (S).*

Proof. By Lemma 3.1.1, the pullback map $H_i(X) \to H_{i+2d}(\tilde{X})$ is an isomorphism. In particular, $H_i(X) = 0$ for i odd. Moreover, for any $i \in \mathbb{Z}$ we get a commutative diagram

$$\begin{tikzcd} A_{i+d}(\tilde{X})_{\overline{\mathbb{Q}}_\ell} \arrow[r] & H_{2i+2d}(\tilde{X}) \\ A_i(X)_{\overline{\mathbb{Q}}_\ell} \arrow[u] \arrow[r, "\sim"] & H_{2i}(X). \arrow[u, "\wr"'] \end{tikzcd}$$

As a consequence, the map $A_i(X)_{\overline{\mathbb{Q}}_\ell} \to A_{i+d}(\tilde{X})_{\overline{\mathbb{Q}}_\ell}$ is injective. It is also surjective by [33, Proposition 1.9] and thus an isomorphism. This implies that $A_{i+d}(\tilde{X})_{\overline{\mathbb{Q}}_\ell} \to H_{2i+2d}(\tilde{X})$ is also an isomorphism. $\square$

3.1.2 Morphisms of Springer type

In this section, we introduce a general setting of 'Springer geometry' which provides a framework to study geometric settings such as the one encountered in Chapter 2. Let G be a reductive group defined over $\overline{\mathbb{F}}$. We fix a maximal torus and a Borel subgroup $T \subset B \subset G$ with Weyl group $W = N_G(T)/T$ and flag variety $\mathcal{B} = G/B$. Given a G-representation V and a finite collection $\{V^i \subset V \mid i \in I\}$ of B-stable subspaces, we define for each $i \in I$ the G-variety

$$\tilde{V}^i := G \times^B V^i = \{(gB, x) \in \mathcal{B} \times V \mid g^{-1} \cdot x \in V^i\}.$$

This comes with two G-equivariant projection morphisms

$$\begin{tikzcd} & \tilde{V}^i \arrow[dl, "\pi_i"'] \arrow[dr, "\mu_i"] & \\ \mathcal{B} & & V. \end{tikzcd}$$

Definition 3.1.4. *We call μ_i a morphism of Springer type.*

Example 3.1.5. *Let $V = \mathfrak{g}$ be the Lie algebra of G. Then the morphism of Springer type corresponding to the B-stable subspace $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{g}$ is the Springer resolution*

$$\tilde{\mathcal{N}} = G \times^B \mathfrak{n} \to \mathcal{N} \subset \mathfrak{g}.$$

For $\mathfrak{b} \subset \mathfrak{g}$ we recover the Grothendieck-Springer alteration $G \times^B \mathfrak{b} \to \mathfrak{g}$. Moreover, we will see in Corollary 3.5.2 that (up to taking connected components) the morphism $\tilde{\mathcal{N}}^{(s,q_0)} \to \mathcal{N}^{(s,q_0)}$ considered in Section 2.4 is also of Springer type (see also [24, 41]).

For $i, j \in I$, we consider the Steinberg variety

$$Z^{ij} := \tilde{V}^i \times_V \tilde{V}^j = \{((g_1B, x_1), (g_2B, x_2)) \in \tilde{V}^i \times \tilde{V}^j \mid x_1 = x_2\}.$$

This comes with the projection map $\pi_i \times \pi_j : Z^{ij} \to \mathcal{B} \times \mathcal{B}$. Consider the orbit partition

$$\mathcal{B} \times \mathcal{B} = \bigsqcup_{w \in W} Y_w$$

with

$$Y_w := G \cdot (eB, \dot{w}B) \cong G/(B \cap \dot{w}B\dot{w}^{-1})$$

where $\dot{w} \in N_G(T)$ is a lift of $w \in W$. This induces a partition of Z^{ij} into locally closed subvarieties

$$Z^{ij} = \bigsqcup_{w \in W} Z^{ij}_w$$

where

$$Z^{ij}_w := (\pi_i \times \pi_j)^{-1}(Y_w).$$

Lemma 3.1.6. *(i) The morphism $\mu_i : \tilde{V}^i \to V$ is proper;*

(ii) The morphism $\pi_i : \tilde{V}^i \to \mathcal{B}$ is a Zariski vector bundle with fiber V^i;

(iii) For each $w \in W$ the morphism $\pi_i \times \pi_j : Z^{ij}_w \to Y_w$ is a Zariski vector bundle with fiber $V^i \cap \dot{w}V^j$;

(iv) The first projection $p_1 : Y_w \to \mathcal{B}$ is a Zariski locally trivial fibration with fiber $\mathbb{A}^{l(w)}$.

Proof. The map μ_i can be factored into a closed immersion followed by a projection:

$$\tilde{V}^i \hookrightarrow \mathcal{B} \times V \xrightarrow{p_2} V$$
$$(g, v) \mapsto (gB, gv).$$

Since $\mathcal{B}$ is projective, this implies that μ_i is proper. The local triviality in (ii)-(iv) follows from standard results about quotients (c.f. [37, §I.5.16]). The respective fibers are easily computed. □

Proposition 3.1.7. *The variety Z^{ij} has property (S). In particular, $H_*(Z^{ij})$ is spanned by fundamental classes.*

Proof. By Lemma 3.1.6 the maps $Z_w^{ij} \to Y_w \to \mathcal{B}$ are locally trivial fibrations with affine fibers. It is well known that $\mathcal{B}$ has property (S) (in fact, this follows from Lemma 3.1.2 and the decomposition of $\mathcal{B}$ into Bruhat cells). By Lemma 3.1.3 this implies that Z_w^{ij} has property (S) for all $w \in W$. Pick a total order $\leq$ on W extending the Bruhat order and define

$$Z_{\leq w}^{ij} := \bigsqcup_{y \leq w} Z_y^{ij}$$
$$Z_{< w}^{ij} := \bigsqcup_{y < w} Z_y^{ij}.$$

Note that $Z_{<w}^{ij} = Z_{\leq w'}^{ij}$ where $w' \in W$ is the maximal element with $w' < w$. We show by induction along the total order on W that $Z_{\leq w}^{ij}$ has property (S). We have already proved the claim for $Z_{\leq e}^{ij} = Z_e^{ij}$. Now assume we have shown the claim for each $y < w$. Note that $Z_{<w}^{ij} \subset Z_{\leq w}^{ij}$ is closed with open complement Z_w^{ij}. The varieties $Z_{<w}^{ij}$ and Z_w^{ij} have property (S), so $Z_{\leq w}^{ij}$ also has property (S) by Lemma 3.1.2. This completes the induction. Note that $Z^{ij} = Z_{\leq w_0}^{ij}$ where $w_0 \in W$ is the longest element. Hence, Z^{ij} has property (S). □

Since $\tilde{V}^i$ is smooth, there is the constant perverse sheaf

$$\mathcal{C}_{\tilde{V}^i} = \mathbf{1}_{\tilde{V}^i}[\dim \tilde{V}^i] \in \mathrm{Perv}(\tilde{V}^i).$$

We define the *Springer sheaves*

$$\mathbf{S}^i := (\mu_i)_* \mathcal{C}_{\tilde{V}^i} \in D_c^b(V)$$

and

$$\mathbf{S} := \bigoplus_{i \in I} \mathbf{S}^i\,.$$

The morphism μ_i is proper by Lemma 3.1.6. Hence, the decomposition theorem (Theorem 2.3.2) implies that $\mathbf{S}^i$, and thus also $\mathbf{S}$, is a semisimple complex. In other words, we have

$$\mathbf{S} = \bigoplus_{j=1}^{n} X_j[k_j]$$

for some simple perverse sheaves $X_j \in \mathrm{Perv}(V)$ and integers $k_j \in \mathbb{Z}$. We define the *Springer category*

$$D_{\mathrm{Spr}}(V) := \langle X_1, ..., X_n \rangle_\Delta \tag{3.5}$$

to be the smallest full triangulated subcategory of $D^b_c(V)$ that is closed under isomorphisms and contains the simple perverse sheaves $X_1, ..., X_n$. Our main goal is to give an algebraic description of the Springer category.

3.2 Purity

In this section, we show that the canonical Frobenius action on the space of morphisms between any two Springer sheaves is pure.

3.2.1 The Frobenius action

Let X_0 be a variety defined over a finite field $\mathbb{F}_q$ with structure map $a : X_0 \to \mathrm{Spec}(\mathbb{F}_q)$ and let

$$X := X_0 \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q.$$

Denote by $D^b_c(X_0) := D^b_c(X_0, \overline{\mathbb{Q}}_\ell)$ the constructible derived category of $\overline{\mathbb{Q}}_\ell$-complexes on X_0. For any $\mathcal{F}_0 \in D^b_c(X_0)$ the pullback of $\mathcal{F}_0$ along $X \to X_0$ will be denoted by $\mathcal{F} \in D^b_c(X)$. We also define

$$\underline{\mathrm{Hom}}^i(\mathcal{F}_0, \mathcal{G}_0) := H^i(a_* \,\mathrm{R}\mathcal{H}om(\mathcal{F}_0, \mathcal{G}_0)) \in \mathrm{Sh}_c(\mathrm{Spec}(\mathbb{F}_q)) \tag{3.6}$$

for any $\mathcal{F}_0, \mathcal{G}_0 \in D^b_c(X_0)$. The category $\mathrm{Sh}_c(\mathrm{Spec}(\mathbb{F}_q))$ is equivalent to the category of finite-dimensional continuous $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-representations. Hence, we can consider $\underline{\mathrm{Hom}}^i(\mathcal{F}_0, \mathcal{G}_0)$ as a $\overline{\mathbb{Q}}_\ell$-vector space equipped with a canonical Frobenius action (see Section 3.3.4 for a construction of this action). Forgetting this action recovers the vector space $\mathrm{Hom}^i(\mathcal{F}, \mathcal{G})$. Note that the sheaves $\mathbf{1}_X, \omega_X \in D^b_c(X)$ have canonical

$\mathbb{F}_q$-versions $\mathbf{1}_{X_0}$ and $\omega_{X_0} = a^! \mathbf{1}_{\mathrm{Spec}(\mathbb{F}_q)}$. This induces a canonical Frobenius action on $H_*(X)$ coming from the Frobenius action on

$$\underline{H}_i(X) := \underline{\mathrm{Hom}}^{-i}(\mathbf{1}_{X_0}, \omega_{X_0}) = H^i(a_* \omega_{X_0}).$$

It can be shown that the constructions on Borel-Moore homology from the previous section are compatible with the Frobenius action. For example, if $i : Y_0 \hookrightarrow X_0$ is a closed immersion with open complement $j : U_0 \hookrightarrow X_0$, there is a distinguished triangle

$$i_* i^! \omega_{X_0} \to \omega_{X_0} \to j_* j^! \omega_{X_0} \to [1]$$

inducing a long exact sequence

$$\cdots \to \underline{H}_{i+1}(U) \to \underline{H}_i(Y) \to \underline{H}_i(X) \to \underline{H}_i(U) \to \underline{H}_{i-1}(Y) \to \cdots$$

where all maps commute with the Frobenius action. For any $n \in \mathbb{Z}$, let

$$\mathbf{1}_{\mathrm{Spec}(\mathbb{F}_q)}(n) \in \mathrm{Sh}_c(\mathrm{Spec}(\mathbb{F}_q))$$

be the n-th Tate twist. This corresponds to the 1-dimensional $\overline{\mathbb{Q}}_\ell$-vector space on which the (geometric) Frobenius element acts by multiplication with q^{-n}. Moreover, we define

$$\begin{aligned} \mathbf{1}_{X_0}(n) &:= a^* \mathbf{1}_{\mathrm{Spec}(\mathbb{F}_q)}(n) \in \mathrm{Sh}_c(X_0) \\ \mathcal{F}_0(n) &:= \mathcal{F}_0 \otimes^L \mathbf{1}_{X_0}(n) \in D^b_c(X_0) \end{aligned}$$

for any $\mathcal{F}_0 \in D^b_c(X_0)$ and $n \in \mathbb{Z}$. If X_0 is smooth and connected, the dualizing complex on X_0 is given by

$$\omega_{X_0} = \mathbf{1}_{X_0}[2 \dim X](\dim X).$$

Lemma 3.2.1. *Let X be irreducible. Then Frobenius acts on $[X] \in \underline{H}_{2 \dim X}(X)$ by multiplication with $q^{-\dim X}$.*

Proof. Let $U_0 \subset X_0$ be a smooth open subset. Then we get a canonical element

$$\begin{aligned} \mathrm{id}_{\mathbf{1}_{U_0}} &\in \underline{\mathrm{Hom}}^0(\mathbf{1}_{U_0}, \mathbf{1}_{U_0}) \\ &\cong \underline{\mathrm{Hom}}^0(\mathbf{1}_{U_0}, \omega_{U_0}[-2 \dim U](-\dim U)) \\ &= \underline{\mathrm{Hom}}^{-2 \dim U}(\mathbf{1}_{U_0}, \omega_{U_0})(-\dim U) \end{aligned}$$

which is Frobenius invariant and corresponds to the fundamental class $[U] \in H_{2 \dim U}(U)$ after forgetting the Frobenius action. Note that the Frobenius action on $\underline{H}_{2 \dim U}(U)$

comes from the Frobenius action on $\underline{\mathrm{Hom}}^{-2\dim U}(\mathbf{1}_{U_0}, \omega_{U_0})$. Thus, taking into account the Tate twist, we get that Frobenius acts on $[U]$ by multiplication with $q^{-\dim U}$. The fundamental class $[X]$ is the inverse image of $[U]$ under the Frobenius equivariant restriction isomorphism $\underline{H}_{2\dim X}(X) \xrightarrow{\sim} \underline{H}_{2\dim X}(U)$. Hence, Frobenius also acts on $[X]$ by multiplication with $q^{-\dim U} = q^{-\dim X}$. $\square$

Corollary 3.2.2. *Assume that $H_*(X)$ is spanned by fundamental classes. Then for $\mathbb{F}_q$ large enough, Frobenius acts on $\underline{H}_i(X)$ by multiplication with $q^{-\frac{i}{2}}$.*

Proof. Let $Y_1, ..., Y_n$ be irreducible closed subvarieties of X such that the fundamental classes $[Y_1], ..., [Y_n]$ span $H_*(X)$. For $\mathbb{F}_q$ large enough, we may assume that each of the Y_i can be defined over $\mathbb{F}_q$. Then Lemma 3.2.1 implies that Frobenius acts on $[Y_i] \in \underline{H}_{2\dim Y_i}(X)$ by multiplication with $q^{-\dim Y_i}$. Since the $[Y_i]$ span $H_*(X)$, this proves the claim. $\square$

3.2.2 Frobenius and the Springer sheaf

Let $\mu_i : \tilde{V}^i \to V$ $(i \in I)$ be a finite collection of morphisms of Springer type (Definition 3.1.4) defined over $\overline{\mathbb{F}}_q$. Assume that $\mathbb{F}_q$ is large enough so that there are $\mathbb{F}_q$-forms $\mu_i : \tilde{V}_0^i \to V_0$ for each of the μ_i $(i \in I)$. Then Z^{ij} can also be defined over $\mathbb{F}_q$ by considering $Z_0^{ij} := \tilde{V}_0^i \times_{V_0} \tilde{V}_0^j$.

Lemma 3.2.3. *For $\mathbb{F}_q$ large enough, Frobenius acts on $\underline{H}_k(Z^{ij})$ by multiplication with $q^{-\frac{k}{2}}$.*

Proof. This follows from Proposition 3.1.7 and Corollary 3.2.2. $\square$

We fix a square root $q^{\frac{1}{2}}$ of q in $\overline{\mathbb{Q}}_\ell$. This corresponds to fixing a square root $\mathbf{1}_{\mathrm{Spec}(\mathbb{F}_q)}(\frac{1}{2})$ of the Tate sheaf $\mathbf{1}_{\mathrm{Spec}(\mathbb{F}_q)}(1)$. Then we can form the half integer Tate twists $\mathcal{F}_0(\frac{n}{2})$ for any $\mathcal{F}_0 \in D_c^b(X_0)$ and $n \in \mathbb{Z}$. Consider the constant perverse sheaf of weight 0 on $\tilde{V}_0^i$

$$\mathcal{C}_{\tilde{V}_0^i} := \mathbf{1}_{\tilde{V}_0^i}[d_i](\tfrac{d_i}{2}) \in D_c^b(\tilde{V}_0)$$

where $d_i = \dim \tilde{V}^i$ and let

$$\mathbf{S}_0^i := (\mu_i)_* \mathcal{C}_{\tilde{V}_0^i} \in D_c^b(V_0)$$

be the corresponding $\mathbb{F}_q$-Springer sheaf. We will need the following Frobenius equivariant version of [24, Lemma 8.6.1].

Lemma 3.2.4. *There is a (Frobenius equivariant) isomorphism*

$$\underline{\mathrm{Hom}}^k(\mathbf{S}_0^i, \mathbf{S}_0^j) \cong \underline{H}_{d_i+d_j-k}(Z^{ij})(\tfrac{-d_i-d_j}{2}).$$

Proof. Note that μ_i is proper over $\mathbb{F}_q$ since it is proper over $\overline{\mathbb{F}}_q$ (properness can be checked fpqc locally). Hence, we have $\mathbf{S}_0^i = (\mu_i)_* \mathcal{C}_{\tilde{V}_0^i} = (\mu_i)_! \mathcal{C}_{\tilde{V}_0^i}$. Using base change with respect to the cartesian diagram

$$\begin{tikzcd} Z_0^{ij} \arrow[r, "q"] \arrow[d, "p"'] & \tilde{V}_0^j \arrow[d, "\mu_j"] \\ \tilde{V}_0^i \arrow[r, "\mu_i"] & V_0 \end{tikzcd}$$

we get

$$\begin{aligned}
\mathrm{R}\mathcal{H}om(\mathbf{S}_0^i, \mathbf{S}_0^j) &\cong \mathrm{R}\mathcal{H}om((\mu_i)_! \mathbf{1}_{\tilde{V}_0^i}, (\mu_j)_* \mathbf{1}_{\tilde{V}_0^j})[d_j - d_i](\tfrac{d_j-d_i}{2}) \\
&\cong (\mu_i)_* \,\mathrm{R}\mathcal{H}om(\mathbf{1}_{\tilde{V}_0^i}, \mu_i^!(\mu_j)_* \mathbf{1}_{\tilde{V}_0^j})[d_j - d_i](\tfrac{d_j-d_i}{2}) \\
&\cong (\mu_i)_* \,\mathrm{R}\mathcal{H}om(\mathbf{1}_{\tilde{V}_0^i}, p_* q^! \mathbf{1}_{\tilde{V}_0^j})[d_j - d_i](\tfrac{d_j-d_i}{2}) \\
&\cong (\mu_i)_* p_* \,\mathrm{R}\mathcal{H}om(p^* \mathbf{1}_{\tilde{V}_0^i}, q^! \mathbf{1}_{\tilde{V}_0^j})[d_j - d_i](\tfrac{d_j-d_i}{2}) \\
&\cong (\mu_i)_* p_* \,\mathrm{R}\mathcal{H}om(p^* \mathbf{1}_{\tilde{V}_0^i}, q^! \omega_{\tilde{V}_0^j})[-d_j - d_i](\tfrac{-d_j-d_i}{2}) \\
&\cong (\mu_i)_* p_* \,\mathrm{R}\mathcal{H}om(\mathbf{1}_{Z_0^{ij}}, \omega_{Z_0^{ij}})[-d_j - d_i](\tfrac{-d_j-d_i}{2}).
\end{aligned}$$

Applying $H^k a_*$ to this, we obtain

$$\begin{aligned}
\underline{\mathrm{Hom}}^k(\mathbf{S}_0^i, \mathbf{S}_0^j) &= H^k(a_* \,\mathrm{R}\mathcal{H}om(\mathbf{S}_0^i, \mathbf{S}_0^j)) \\
&\cong H^{k-d_i-d_j}(a_*(\mu_i)_* p_* \,\mathrm{R}\mathcal{H}om(\mathbf{1}_{Z_0^{ij}}, \omega_{Z_0^{ij}}))(\tfrac{-d_i-d_j}{2}) \\
&\cong \underline{H}_{d_i+d_j-k}(Z^{ij})(\tfrac{-d_i-d_j}{2}).
\end{aligned}$$

□

Corollary 3.2.5. *For $\mathbb{F}_q$ large enough, Frobenius acts on $\underline{\mathrm{Hom}}^k(\mathbf{S}_0^i, \mathbf{S}_0^j)$ by multiplication with $q^{\frac{k}{2}}$.*

Proof. By Lemma 3.2.3 Frobenius acts on $\underline{H}_{d_i+d_j-k}(Z^{ij})(\frac{-d_i-d_j}{2})$ by multiplication with

$$q^{-\frac{d_i+d_j-k}{2}} \cdot q^{-\frac{-d_i-d_j}{2}} = q^{\frac{k}{2}}.$$

The claim now follows from Lemma 3.2.4. □

3.3 Formality

3.3.1 Idempotent complete triangulated categories

Recall that an additive category $\mathcal{A}$ is called idempotent complete if any idempotent $e : X \to X$ in $\mathcal{T}$ splits. This is equivalent to the property that all idempotents have kernels (or cokernels). In an idempotent complete category, every idempotent $e : X \to X$ gives rise to a canonical decomposition

$$X = \ker(e) \oplus \operatorname{im}(e).$$

If $\mathcal{A}' \subset \mathcal{A}$ is an additive subcategory that is closed under direct summands and $\mathcal{A}$ is idempotent complete, then $\mathcal{A}'$ is also idempotent complete. Moreover, for any additive category $\mathcal{A}$ one can define its *idempotent completion* $\tilde{\mathcal{A}}$. This is an idempotent complete additive category with a fully faithful additive functor $\iota : \mathcal{A} \to \tilde{\mathcal{A}}$ and the following universal property: Any additive functor $\mathcal{A} \to \mathcal{C}$ with $\mathcal{C}$ idempotent complete factors uniquely (up to isomorphism) through ι. For more details about idempotent completeness we refer to [38, I.6].

A triangulated category is called idempotent complete if its underlying additive category is idempotent complete. All triangulated categories we will encounter are idempotent complete thanks to the following well-known result (c.f. [20, Proposition 3.2], [9, Lemma 2.4], [53, Lemma 1.6.8]).

Lemma 3.3.1. *Any triangulated category with countable coproducts is idempotent complete. Moreover, the bounded below derived category $D^+(\mathcal{A})$ of an abelian category $\mathcal{A}$ is always idempotent complete.*

Let $\mathcal{T}$ be a triangulated category. For any object $X \in \mathcal{T}$, we denote by

$$\langle X \rangle_{\mathcal{T},\Delta} = \langle X \rangle_{\Delta}$$

the smallest full triangulated subcategory of $\mathcal{T}$ that contains X and is closed under isomorphisms. Similarly, we denote by

$$\langle X \rangle_{\mathcal{T},\Delta,\Subset} = \langle X \rangle_{\Delta,\Subset}$$

the smallest full triangulated subcategory of $\mathcal{T}$ that contains X and is closed under isomorphisms and direct summands.

Lemma 3.3.2. *Let $\mathcal{T}$ be an idempotent complete triangulated category and $X \in \mathcal{T}$. Then $\langle X \rangle_{\Delta,\Subset}$ is the idempotent completion of $\langle X \rangle_{\Delta}$.*

Proof. By [38, Theorem I.6.12] the idempotent completion of $\langle X\rangle_\Delta$ can be described as the full additive subcategory $\mathcal{C}\subset\mathcal{T}$ consisting of those objects $Y\in\mathcal{T}$ which are isomorphic to a direct summand of an object in $\langle X\rangle_\Delta$. In particular, we have $\mathcal{C}\subset\langle X\rangle_{\Delta,\Subset}$. The cone of a morphism in $\mathcal{C}$ is a direct summand of the cone of a morphism in $\langle X\rangle_\Delta$. Hence, the category $\mathcal{C}$ is closed under cones and thus $\mathcal{C}=\langle X\rangle_{\Delta,\Subset}$. □

We will also need the following idempotent complete version of Beilinson's Lemma (see [10] and [61, Lemma 6]) .

Lemma 3.3.3. *Let $F:\mathcal{T}\to\mathcal{T}'$ be a triangulated functor between idempotent complete triangulated categories. Let $X\in\mathcal{T}$ such that F induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{T}}(X,X[i])\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{T}'}(F(X),F(X)[i])$$

for all $i\in\mathbb{Z}$. Then F restricts to an equivalence of triangulated categories

$$\langle X\rangle_{\Delta,\Subset}\cong\langle F(X)\rangle_{\Delta,\Subset}.$$

Proof. By a standard dévissage argument, F induces an equivalence $\langle X\rangle_\Delta\cong\langle F(X)\rangle_\Delta$. This extends to an equivalence of the respective idempotent completions and thus to an equivalence $\langle X\rangle_{\Delta,\Subset}\cong\langle F(X)\rangle_{\Delta,\Subset}$ by Lemma 3.3.2. □

3.3.2 Derived categories and dg-algebras

Let k be a commutative ring. For any dg-algebra R over k, we denote by $R-\mathrm{dgMod}$ the category of (left) dg-modules over R. The homotopy category of dg-modules will be denoted by $K(R-\mathrm{dgMod})$ and the derived category by $D(R-\mathrm{dgMod})$. We write

$$D_{\mathrm{perf}}(R-\mathrm{dgMod}):=\langle R\rangle_{D(R-\mathrm{dgMod}),\Delta,\Subset}$$

for the perfect derived category. For more details about dg-algebras and dg-modules we refer to [16, 42].

Let $\mathcal{A}$ be a k-linear abelian category. Then for any two chain complexes $X^\bullet,Y^\bullet\in C(\mathcal{A})$, we can consider the Hom-complex $\mathrm{Hom}^\bullet_{dg}(X^\bullet,Y^\bullet)\in C(k)$. Explicitly, this complex is defined as

$$\mathrm{Hom}^i_{dg}(X^\bullet,Y^\bullet):=\prod_{l+m=i}\mathrm{Hom}_{\mathcal{A}}(X^{-l},Y^m)$$

with differential $d_{\mathrm{Hom}^\bullet_{dg}(X^\bullet,Y^\bullet)}(f)=d_{Y^\bullet}\circ f-(-1)^{|f|}f\circ d_{X^\bullet}$. Taking cohomology of this complex recovers the morphism space in the homotopy category:

$$H^i(\mathrm{Hom}^\bullet_{dg}(X^\bullet,Y^\bullet))=\mathrm{Hom}_{K(\mathcal{A})}(X^\bullet,Y^\bullet[i]).$$

Component-wise composition defines a map

$$\mathrm{Hom}^i_{dg}(Y^\bullet, Z^\bullet) \otimes \mathrm{Hom}^j_{dg}(X^\bullet, Y^\bullet) \to \mathrm{Hom}^{i+j}_{dg}(X^\bullet, Z^\bullet)$$

for any $X^\bullet, Y^\bullet, Z^\bullet \in C(\mathcal{A})$. This equips $\mathrm{Hom}^\bullet_{dg}(Y^\bullet, Y^\bullet)$ with the structure of a dg-algebra and $\mathrm{Hom}^\bullet_{dg}(X^\bullet, Y^\bullet)$ with the structure of a (left) dg-module over $\mathrm{Hom}^\bullet_{dg}(Y^\bullet, Y^\bullet)$. Furthermore, this gives rise to a functor

$$\begin{aligned} \mathrm{Hom}^\bullet_{dg}(-, Y^\bullet) : C(\mathcal{A})^{\mathrm{op}} &\to \mathrm{Hom}^\bullet_{dg}(Y^\bullet, Y^\bullet) - \mathrm{dgMod} \\ X^\bullet &\mapsto \mathrm{Hom}^\bullet_{dg}(X^\bullet, Y^\bullet) \end{aligned} \tag{3.7}$$

which descends to a triangulated functor on the respective homotopy categories

$$\mathrm{Hom}^\bullet_{dg}(-, Y^\bullet) : K(\mathcal{A})^{\mathrm{op}} \to K(\mathrm{Hom}^\bullet_{dg}(Y^\bullet, Y^\bullet) - \mathrm{dgMod}). \tag{3.8}$$

If $Y^\bullet$ is a $\mathcal{K}$-injective complex (e.g. a bounded below complex of injectives), this descends further to a triangulated functor on the corresponding derived categories

$$\mathrm{Hom}^\bullet_{dg}(-, Y^\bullet) : D(\mathcal{A})^{\mathrm{op}} \to D(\mathrm{Hom}^\bullet_{dg}(Y^\bullet, Y^\bullet) - \mathrm{dgMod}). \tag{3.9}$$

Assume now that the abelian category $\mathcal{A}$ has enough injectives. Then for each $X \in D^+(\mathcal{A})$ we can pick a complex of injectives $I^\bullet_X \in C^+(\mathcal{A})$ representing X. Moreover, we can consider the associated dg-algebra

$$R_X := \mathrm{Hom}^\bullet_{dg}(I^\bullet_X, I^\bullet_X) \tag{3.10}$$

and the corresponding functor

$$\mathrm{Hom}^\bullet_{dg}(-, I^\bullet_X) : D(\mathcal{A})^{\mathrm{op}} \to D(R_X - \mathrm{dgMod}). \tag{3.11}$$

The following statement is an idempotent complete version of [61, Proposition 7].

Lemma 3.3.4. *The functor from* (3.11) *induces an equivalence of triangulated categories*

$$\langle X \rangle_{D^+(\mathcal{A})^{\mathrm{op}}, \Delta, \Subset} \cong D_{\mathrm{perf}}(R_X - \mathrm{dgMod})$$

which sends X to the free dg-module R_X.

Proof. By construction, the functor from (3.11) sends X to $\mathrm{Hom}^\bullet_{dg}(I^\bullet_X, I^\bullet_X) = R_X$. Moreover, this functor induces an isomorphism

$$\begin{aligned} \mathrm{Hom}_{D^+(A)}(X, X[i]) &\cong \mathrm{Hom}_{K(\mathcal{A})}(I^\bullet_X, I^\bullet_X[i]) \\ &\cong H^i(R_X) \\ &\cong \mathrm{Hom}_{K(R_X - \mathrm{dgMod})}(R_X, R_X[i]) \\ &\cong \mathrm{Hom}_{D(R_X - \mathrm{dgMod})}(R_X, R_X[i]) \end{aligned}$$

for all $i \in \mathbb{Z}$. Here we have used for the first (resp. last) isomorphism that $I_X^\bullet$ (resp R_X) is a $\mathcal{K}$-injective complex (resp. $\mathcal{K}$-projective dg-module). Note that the categories $D^+(\mathcal{A})^{\mathrm{op}}$ and $D(R_X - \mathrm{dgMod})$ are idempotent complete by Lemma 3.3.1 (using that $D(R_X - \mathrm{dgMod})$ has countable coproducts). By Lemma 3.3.3 this implies that (3.11) restricts to an equivalence of triangulated categories

$$\langle X \rangle_{D^+(\mathcal{A})^{\mathrm{op}},\Delta,\Subset} \cong \langle R_X \rangle_{D(R_X - \mathrm{dgMod}),\Delta,\Subset} = D_{\mathrm{perf}}(R_X - \mathrm{dgMod}).$$

□

The dg-algebra R_X crucially depends on the choice of the injective resolution $I_X^\bullet$ which makes it difficult to compute R_X explicitly. The cohomology of R_X on the other hand can be described more concretely as

$$H^i(R_X) = H^i(\mathrm{Hom}^\bullet_{dg}(I_X^\bullet, I_X^\bullet)) = \mathrm{Hom}^i(X, X).$$

Hence, we would like to replace the dg-algebra R_X in Lemma 3.3.4 with its cohomology. Recall that a dg-algebra R is called *formal* if there is a chain of quasi-isomorphisms of dg-algebras

$$R \leftarrow A_1 \rightarrow A_2 \leftarrow \ldots \rightarrow \ldots \leftarrow A_n \rightarrow H^*(R).$$

Here we consider $H^*(R)$ as a dg-algebra with vanishing differential. Any quasi-isomorphism of dg-algebras $A \to B$ induces an equivalence $D_{\mathrm{perf}}(B) \xrightarrow{\sim} D_{\mathrm{perf}}(A)$ which identifies the free dg-module B with the free dg-module A. Hence, if R is formal, there is an equivalence of triangulated categories

$$D_{\mathrm{perf}}(R - \mathrm{dgMod}) \cong D_{\mathrm{perf}}(H^*(R) - \mathrm{dgMod}).$$

In particular, if R_X is formal Lemma 3.3.4 induces an equivalence

$$\langle X \rangle_{D^+(\mathcal{A})^{\mathrm{op}},\Delta,\Subset} \cong D_{\mathrm{perf}}(\mathrm{Hom}^*(X, X) - \mathrm{dgMod}). \tag{3.12}$$

The following theorem provides a useful criterion for formality.

Theorem 3.3.5. *[54] Let R be a dg-algebra over an algebraically closed field k and $q \in k^\times$ not a root of unity. If R can be equipped with a dg-algebra automorphism $F : R \to R$ such that $H^i(F)$ acts on $H^i(R)$ by multiplication with q^i, then R is formal.*

Proof. This is proved for $k = \mathbb{C}$ in [54, Theorem B.1.1] (with a slightly different condition on the action of $H^i(F)$). The same proof works for general k: By [54, Theorem B.1.2], we may assume that F acts locally finitely on R. Hence, there is a generalized eigenspace decomposition $R^i = \bigoplus_{\alpha \in k^\times} R^i_\alpha$ which satisfies $R^i_\alpha \cdot R^j_\beta \subset R^{i+j}_{\alpha\beta}$. Using this decomposition, we can define a dgg-algebra (i.e. a $\mathbb{Z}^2$-graded algebra with a differential d homogeneous of degree $(1,0)$ satisfying the Leibniz rule) via $\tilde{R} := \bigoplus_{i,j\in\mathbb{Z}} \tilde{R}^i_{q^j}$. Note that the inclusion $\tilde{R} \hookrightarrow R$ is a quasi-isomorphism. Moreover, the cohomology of the dgg-algebra $\tilde{R}$ lives in degrees $\{(i,i) \mid i \in \mathbb{Z}\}$. Such dgg-algebras are known to be formal (c.f. [61, Proposition 4]). □

3.3.3 The pro-étale topology

In this section, we recall the definition of the constructible derived category in the pro-étale topology and a few of its basic properties from [19].

Definition 3.3.6. *A morphism of schemes $f : U \to X$ is called weakly étale if both f and its diagonal $\Delta_f : U \to U \times_X U$ are flat morphisms. Let $X_{\text{proét}}$ be the category of schemes U weakly étale over X. This becomes a site by declaring a family of maps $\{U_i \to U\}$ to be a covering if it is a covering in the fpqc topology.*

The property of being weakly étale is stable under composition and base change. Moreover, any morphism in $X_{\text{proét}}$ is weakly étale. We denote the category of sheaves of abelian groups on $X_{\text{proét}}$ by $\mathrm{Ab}(X_{\text{proét}})$. Let $f : X \to Y$ be a morphism of schemes. Then f induces a morphism of sites $X_{\text{proét}} \to Y_{\text{proét}}$ via the functor

$$\begin{aligned} Y_{\text{proét}} &\to X_{\text{proét}} \\ V &\mapsto V \times_Y X. \end{aligned}$$

There is a corresponding pair of adjoint functors

$$\mathrm{Ab}(Y_{\text{proét}}) \underset{f_*}{\overset{f^{-1}}{\rightleftarrows}} \mathrm{Ab}(X_{\text{proét}}).$$

Let Λ be a topological ring. By [19, Lemma 4.2.12] there is a sheaf of rings Λ_X on $X_{\text{proét}}$ defined by

$$\Lambda_X(U) := \mathrm{Hom}_{\text{cont}}(U, \Lambda).$$

If Λ is totally disconnected and U is quasi-compact, we get

$$\Lambda_X(U) = \mathrm{Hom}_{\text{cont}}(\pi_0(U), \Lambda) \tag{3.13}$$

where $\pi_0(U)$ is the space of connected components. If Λ is discrete, Λ_X is the constant sheaf with values in Λ. For any morphism of schemes $f : X \to Y$, there is a canonical map $f^\# : \Lambda_Y \to f_*\Lambda_X$ and hence a morphism of ringed sites

$$(X_{\text{proét}}, \Lambda_X) \to (Y_{\text{proét}}, \Lambda_Y).$$

Thus we get corresponding pairs of adjoint functors

$$\mathrm{Sh}(Y_{\text{proét}}, \Lambda_Y) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{Sh}(X_{\text{proét}}, \Lambda_X)$$

$$D(Y_{\text{proét}}, \Lambda_Y) \underset{f_*}{\overset{f^*}{\rightleftarrows}} D(X_{\text{proét}}, \Lambda_X).$$

Here we adhere to the derived convention, i.e. the functors f^* and f_* are understood to be derived when applied to complexes. From now on, we fix a prime number ℓ and make the following assumption on the coefficient ring Λ.

Assumption 3.3.7. *The ring Λ is of one of the following forms:*

(i) $\Lambda = \mathbb{F}$ is a finite field of characteristic ℓ;

(ii) $\Lambda = \mathcal{O}_E$ is the ring of integers in an algebraic extension $E/\mathbb{Q}_\ell$;

(iii) $\Lambda = E$ is an algebraic extension $E/\mathbb{Q}_\ell$.

We equip Λ with the ℓ-adic topology.

Note that the assumption above includes the ring $\Lambda = \overline{\mathbb{Q}}_\ell$. For any algebraic extension $E/\mathbb{Q}_\ell$, we denote by $\mathcal{O}_E$ the ring of integers in E. If $E/\mathbb{Q}_\ell$ is finite, we pick a uniformizer $\omega \in \mathcal{O}_E$.

Example 3.3.8. *[19, Lemma 6.8.2] gives an alternative description of Λ_X under Assumption 3.3.7: If Λ is a finite ring (e.g. $\Lambda = \mathbb{F}$ a finite field) then Λ_X is just the constant sheaf on $X_{\text{proét}}$ with values in Λ. In the remaining cases, Λ_X can be constructed as follows:*

$$\begin{aligned}
\mathcal{O}_{E,X} &= \varprojlim (\mathcal{O}_E/\omega^n\mathcal{O}_E)_X && \textit{for } E/\mathbb{Q}_\ell \textit{ finite};\\
\mathcal{O}_{E,X} &= \operatorname*{colim}_{\substack{F/\mathbb{Q}_\ell \textit{ finite},\\ F\subset E}} \mathcal{O}_{F,X} && \textit{for } E/\mathbb{Q}_\ell \textit{ algebraic};\\
E_X &= \mathcal{O}_{E,X}[\ell^{-1}] && \textit{for } E/\mathbb{Q}_\ell \textit{ algebraic}.
\end{aligned}$$

Lemma 3.3.9. *For any morphism of schemes $X \to Y$, the canonical map $f^\# : f^{-1}\Lambda_Y \to \Lambda_X$ is an isomorphism. In particular $f^* = f^{-1}$ is exact.*

Proof. This result can be found in [23, Proposition 4.6]. The proof uses the explicit description of Λ_X in Example 3.3.8. Since pullback commutes with colimits and localization, the main step is to show that $f^\#$ is an isomorphism when $\Lambda = \mathcal{O}_E$ for a finite extension $E/\mathbb{Q}_\ell$. In this case, $\Lambda_Y = \varprojlim(\mathcal{O}_E/\omega^n\mathcal{O}_E)_Y$ is represented by the scheme $\varprojlim(\bigsqcup_{\mathcal{O}_E/\omega^n\mathcal{O}_E} Y)$. Hence, $f^*\Lambda_Y$ is represented by the scheme

$$X \times_Y \varprojlim(\bigsqcup_{\mathcal{O}_E/\omega^n\mathcal{O}_E} Y) = \varprojlim(\bigsqcup_{\mathcal{O}_E/\omega^n\mathcal{O}_E} X \times_Y Y) = \varprojlim(\bigsqcup_{\mathcal{O}_E/\omega^n\mathcal{O}_E} X)$$

and thus $f^*\Lambda_Y = \Lambda_X$. □

Recall that a scheme X is called ℓ-coprime if ℓ is invertible in $\mathcal{O}_X$. Following [19], the constructible derived category can be defined as follows.

Definition 3.3.10. *Let X be a noetherian ℓ-coprime scheme.*

(i) A sheaf $\mathcal{F} \in \mathrm{Sh}(X_{\text{proét}}, \Lambda_X)$ is called locally constant of finite presentation if it is locally isomorphic to $M \otimes_\Lambda \Lambda_X$ for a finitely-presented Λ-module M.

(ii) A sheaf $\mathcal{F} \in \mathrm{Sh}(X_{\text{proét}}, \Lambda_X)$ is called constructible if there exists a finite stratification $\{X_i \to X\}$ such that $\mathcal{F}|_{X_i}$ is locally constant of finite presentation.

(iii) A complex $\mathcal{F} \in D(X_{\text{proét}}, \Lambda_X)$ is called constructible if it is bounded and all its cohomology sheaves are constructible.

We denote by $\mathrm{Sh}_c(X, \Lambda) \subset \mathrm{Sh}(X_{\text{proét}}, \Lambda_X)$ the category of all constructible sheaves. The constructible derived category is the full subcategory $D^b_c(X, \Lambda) \subset D(X_{\text{proét}}, \Lambda_X)$ consisting of constructible complexes.

Remark 3.3.11. *The assumptions on X in the definition above can be weakened to topologically noetherian and qcqs. For our purposes, noetherian is good enough since all schemes we will encounter are noetherian.*

The six functor formalism can also be conveniently described in the pro-étale topology: The functors $\otimes^L, \mathrm{R}\mathcal{H}om$ and f^* preserve constructible complexes and thus they descend to the constructible derived category. The same is true for f_* if $f : X \to Y$ is of finite type and Y satisfies some mild assumptions (e.g. for Y of finite type over a field or more generally for Y quasi-excellent). If $j : U \hookrightarrow X$ is an open immersion, the pullback functor j^* has a left adjoint $j_!$ which preserves constructible complexes. For a general $f : X \to Y$ (of finite type with Y quasi-excellent), we define $f_! := \overline{f}_* \circ j_!$ where we factor f as an open immersion followed by a proper map

$$X \overset{j}{\hookrightarrow} \overline{X} \overset{\overline{f}}{\to} Y. \tag{3.14}$$

When restricted to the constructible derived category, the functor $f_!$ admits a right adjoint denoted by $f^!$. These are the usual six functors $f^*, f_*, f_!, f^!, \otimes^L$ and $\mathrm{R}\mathcal{H}om$.

Remark 3.3.12. *In [19] a slightly different pullback functor f^*_{comp} is used which is defined as f^* followed by a certain completion operation ([19, Lemma 6.5.9]). It turns out that in our situation (i.e. under Assumption 3.3.7) the two functors agree on the constructible derived category. To see this, recall that f^* is exact by Lemma 3.3.9. Using this, it is straightforward to check that f^* preserves constructible complexes. Since constructible complexes are complete ([19, Proposition 6.8.11(3), Definition 6.5.1]), it follows that $f^* = f^*_{comp}$ on $D^b_c(X, \Lambda)$.*

If $\mathbb{F}$ is a finite field of characteristic ℓ, we denote by $D^b_c(X_{\text{ét}}, \mathbb{F})$ the constructible derived category in the étale topology with coefficients in $\mathbb{F}$. The standard compatibility and base change results for the six functors on $D^b_c(X, \Lambda)$ can be deduced from the corresponding results in $D^b_c(X_{\text{ét}}, \mathbb{F})$ using the following reduction steps.

Lemma 3.3.13. *[19] Let X be noetherian and ℓ-coprime.*

(i) Let $\mathbb{F}$ be a finite field of characteristic ℓ. Then the canonical morphism of sites $\nu : X_{\text{proét}} \to X_{\text{ét}}$ induces an equivalence of categories

$$\nu^* : D^b_c(X_{\text{ét}}, \mathbb{F}) \xrightarrow{\sim} D^b_c(X, \mathbb{F});$$

(ii) Let $E/\mathbb{Q}_\ell$ be a finite extension and κ the residue field of $\mathcal{O}_E$. Then the functor

$$\kappa_X \otimes^L_{\mathcal{O}_{E,X}} - : D^b_c(X, \mathcal{O}_E) \to D^b_c(X, \kappa)$$

is well-defined (i.e. preserves constructibility) and conservative (i.e. reflects isomorphisms);

(iii) Let $E/\mathbb{Q}_\ell$ be algebraic. Then the canonical functor

$$\operatorname*{colim}_{\substack{F/\mathbb{Q}_\ell \text{ finite}, \\ F \subset E}} D^b_c(X, \mathcal{O}_F) \to D^b_c(X, \mathcal{O}_E)$$

is an equivalence of categories;

(iv) Let $E/\mathbb{Q}_\ell$ be algebraic. Then the canonical functor

$$D^b_c(X,\mathcal{O}_E)[\ell^{-1}] \to D^b_c(X,E)$$

is an equivalence of categories. Here $D^b_c(X,\mathcal{O}_E)[\ell^{-1}]$ is the category with the same objects as $D^b_c(X,\mathcal{O}_E)$ and

$$\mathrm{Hom}_{D^b_c(X,\mathcal{O}_E)[\ell^{-1}]}(\mathcal{F},\mathcal{G}) := \mathrm{Hom}_{D^b_c(X,\mathcal{O}_E)}(\mathcal{F},\mathcal{G})[\ell^{-1}].$$

Moreover, the functors in (i)-(iv) are compatible with the six functors $f^*, f_*, f_!, f^!, \otimes^L$ and $\mathrm{R}\mathcal{H}om$.

Proof. This follows from various results in [19]: (i) follows from [19, Corollary 5.1.5, 5.1.6]. The well-definedness in (ii) follows from [19, Proposition 6.8.11(3), Definition 6.5.1]. To show that the functor $\kappa_X \otimes^L_{\mathcal{O}_{E,X}} -$ is conservative, it suffices to prove that it reflects zero objects. Let $\mathcal{F} \in D^b_c(X,\mathcal{O}_E)$ with $\kappa_X \otimes^L_{\mathcal{O}_{E,X}} \mathcal{F} = 0$. We need to show that $\mathcal{F} = 0$. Let $\mathfrak{m} \subset \mathcal{O}_E$ be the maximal ideal. Then there is a short exact sequence of $\mathcal{O}_E$-modules

$$0 \to \mathfrak{m}^n/\mathfrak{m}^{n+1} \to \mathcal{O}_E/\mathfrak{m}^{n+1} \to \mathcal{O}_E/\mathfrak{m}^n \to 0$$

with $\mathfrak{m}^n/\mathfrak{m}^{n+1} \cong \kappa$. This induces a distinguished triangle

$$\kappa_X \otimes^L_{\mathcal{O}_{E,X}} \mathcal{F} \to (\mathcal{O}_E/\mathfrak{m}^{n+1})_X \otimes^L_{\mathcal{O}_{E,X}} \mathcal{F} \to (\mathcal{O}_E/\mathfrak{m}^n)_X \otimes^L_{\mathcal{O}_{E,X}} \mathcal{F} \to [1].$$

By an induction argument, this implies that $(\mathcal{O}_E/\mathfrak{m}^n)_X \otimes^L_{\mathcal{O}_{E,X}} \mathcal{F} = 0$ for all $n \geq 1$. Moreover, $\mathcal{F}$ is $\mathfrak{m}$-adically complete by [19, Proposition 6.8.11(3), Definition 6.5.1] which means that

$$\mathcal{F} = \mathrm{Rlim}((\mathcal{O}_E/\mathfrak{m}^n)_X \otimes^L_{\mathcal{O}_{E,X}} \mathcal{F}) = 0.$$

This completes the proof of (ii). The claims in (iii) and (iv) are [19, Proposition 6.8.14]. It remains to prove compatibility with the six functors in (i)-(iv). Note for this that the category $\mathrm{colim}_F\, D^b_c(X,\mathcal{O}_F)$ in (iii) inherits the six functors from the $D^b_c(X,\mathcal{O}_F)$ (a standard argument shows that the transition maps in the colimit are compatible with the six functors using that $\mathcal{O}_{F',X} \cong \mathcal{O}_{F,X}^{\oplus[F':F]}$ as sheaves of $\mathcal{O}_{F,X}$-modules for any finite extension F'/F). Similarly, the category $D^b_c(X,\mathcal{O}_E)[\ell^{-1}]$ in (iv) inherits the six functors from $D^b_c(X,\mathcal{O}_E)$. All functors in (i)-(iv) are induced by pullbacks along morphisms of ringed sites (e.g. the functor in (ii) is the (derived) pullback along the morphism of ringed sites $(X_{\text{proét}},\kappa_X) \to (X_{\text{proét}},\mathcal{O}_{E,X})$). As such, they commute with f^* and $\otimes^L$ by standard results about ringed sites [64, Tag 0D6D, Tag 07A4]. Since the functors in (i),(iii) and (iv) are equivalences, they also commute with the corresponding (right) adjoints f_* and $\mathrm{R}\mathcal{H}om$. Similarly, they commute with $j_!$ for j an open immersion which is left adjoint to j^*. Hence, they also commute with $f_! = \overline{f}_* \circ j_!$ (see (3.14)) and thus also with its right adjoint $f^!$. It remains to show that the functor in (ii) commutes with $f_*, \mathrm{R}\mathcal{H}om, f_!$ and $f^!$. This follows from [19, Lemma 6.5.11(3), 6.7.13, 6.7.14, 6.7.19]. □

We conclude this section by collecting a few useful properties about pro-étale sheaves. In the pro-étale topology, so-called w-contractible affine schemes play a distinguished role.

Definition 3.3.14. *An affine scheme U is called w-contractible if every faithfully-flat weakly-étale map $V \to U$ has a section.*

Lemma 3.3.15. *Let U be a connected w-contractible affine scheme.*

(i) The global sections functor $\Gamma(U, -)$ is exact on $\mathrm{Sh}(U_{\text{proét}}, \Lambda_U)$;

(ii) Any locally constant sheaf of finite presentation $\mathcal{F} \in \mathrm{Sh}_c(U, \Lambda)$ is already constant, i.e. $\mathcal{F} \cong M \otimes_\Lambda \Lambda_U$ for a finitely-presented Λ-module M;

(iii) If R is a strictly Henselian local ring (e.g. an algebraically closed field) then $\mathrm{Spec}(R)$ is w-contractible.

Proof. The exactness in (i) is mentioned in the introduction of [19] (see also [64, Tag 098H, Tag 0946]). Let $\mathcal{F} \in \mathrm{Sh}_c(U, \Lambda)$ be locally constant of finite presentation. Then we can find a weakly-étale cover $V = \bigsqcup V_i \to U$ such that $\mathcal{F}|_{V_i} = M_i \otimes_\Lambda \Lambda_{V_i}$ for some finitely-presented Λ-modules M_i. Since U is weakly contractible, there is a section $s : U \to V$ and by connectedness s lands in some V_i. Thus, we get

$$\mathcal{F} = s^*(\mathcal{F}|_{V_i}) = s^*(M_i \otimes_\Lambda \Lambda_{V_i}) = M_i \otimes_\Lambda \Lambda_U.$$

This proves (ii). (iii) follows from [19, Theorem 1.8]. □

Lemma 3.3.16. *The triangulated category $D^b_c(X, \Lambda) \subset D(X_{\text{proét}}, \Lambda_X)$ is closed under direct summands. In particular, $D^b_c(X, \Lambda)$ is idempotent complete.*

Proof. Since taking cohomology commutes with direct sums, it suffices to prove that $\mathrm{Sh}_c(X, \Lambda) \subset \mathrm{Sh}(X_{\text{proét}}, \Lambda_X)$ is closed under direct summands. In fact, by [19, Lemma 6.8.7, Proposition 6.8.11] the category $\mathrm{Sh}_c(X, \Lambda)$ is abelian, so it is certainly closed under direct summands. □

Lemma 3.3.17. *If $f : X \to Y$ is weakly étale, the pullback functor f^* preserves injectives.*

Proof. Compositions of weakly étale maps are weakly étale and morphisms between weakly étale maps are weakly étale. Together with Lemma 3.3.9 this implies that $(X_{\text{proét}}, \Lambda_X)$ is the localization of the ringed site $(Y_{\text{proét}}, \Lambda_Y)$ at $X \to Y$. By general results on ringed sites [64, Tag 04IX] this implies that f^* has an exact left adjoint $f_!$. In particular, f^* preserves injectives. $\square$

Let $\mathcal{F} \in D^b_c(X, \Lambda)$ and pick a complex of injectives $I^\bullet_{\mathcal{F}} \in C^+(\mathrm{Sh}(X_{\text{proét}}, \Lambda_X))$ representing $\mathcal{F}$. Let $R_{\mathcal{F}} = \mathrm{Hom}^\bullet_{dg}(I^\bullet_{\mathcal{F}}, I^\bullet_{\mathcal{F}})$ be the dg-algebra from (3.10). Thanks to the pro-étale formalism, we obtain the following algebraic description of the category $\langle \mathcal{F} \rangle_{\Delta, \Subset}$.

Corollary 3.3.18. *There is an equivalence of triangulated categories*

$$\langle \mathcal{F} \rangle_{D^b_c(X,\Lambda)^{\text{op}}, \Delta, \Subset} \cong D_{\text{perf}}(R_{\mathcal{F}} - \text{dgMod})$$

which send $\mathcal{F}$ to the free dg-module $R_{\mathcal{F}}$.

Proof. By Lemma 3.3.16 we have $\langle \mathcal{F} \rangle_{D^b_c(X,\Lambda), \Delta, \Subset} = \langle \mathcal{F} \rangle_{D^+(X_{\text{proét}}, \Lambda_X), \Delta, \Subset}$. The result now follows from Lemma 3.3.4. $\square$

3.3.4 The Frobenius action on dg-algebras

In this section we show that the Frobenius action on $\underline{\mathrm{Hom}}$ from (3.6) is compatible with the dg-techniques from Section 3.3.2. These results are similar to the ones in [54, Appendix A] but some of the arguments simplify because the canonical morphism $\mathrm{Spec}(\overline{\mathbb{F}}_q) \to \mathrm{Spec}(\mathbb{F}_q)$ is a weakly-étale (but not étale).

Let X_0 be a variety defined over a finite field $\mathbb{F}_q$. Recall that we define $X := X_0 \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$. Let $\pi : X \to X_0$ be the canonical map and define $\mathcal{F} := \pi^*(\mathcal{F}_0) \in \mathrm{Sh}(X_{\text{proét}}, \Lambda_X)$ for any $\mathcal{F}_0 \in \mathrm{Sh}(X_{0,\text{proét}}, \Lambda_{X_0})$.

Lemma 3.3.19. *The pullback functor $\pi^* : \mathrm{Sh}(X_{0,\text{proét}}, \Lambda_{X_0}) \to \mathrm{Sh}(X_{\text{proét}}, \Lambda_X)$ preserves injectives.*

Proof. The morphism $\mathrm{Spec}(\overline{\mathbb{F}}_q) \to \mathrm{Spec}(\mathbb{F}_q)$ is weakly étale. By base change, $\pi^* : X \to X_0$ is also weakly étale. Hence, π^* preserves injectives by Lemma 3.3.17. $\square$

The geometric q-Frobenius $F \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ defines a morphism $\mathrm{Spec}(\overline{\mathbb{F}}_q) \to \mathrm{Spec}(\overline{\mathbb{F}}_q)$ in the category $\mathrm{Spec}(\mathbb{F}_q)_{\text{proét}}$. Thus, for any $\mathcal{F}_0 \in \mathrm{Sh}(\mathrm{Spec}(\mathbb{F}_q)_{\text{proét}}, \Lambda_{\mathrm{Spec}(\mathbb{F}_q)})$ there is a corresponding restriction map on sections

$$F : \Gamma(\mathrm{Spec}(\overline{\mathbb{F}}_q), \mathcal{F}_0) \to \Gamma(\mathrm{Spec}(\overline{\mathbb{F}}_q), \mathcal{F}_0).$$

This map is an isomorphism with inverse given by restriction along the arithmetic Frobenius $F^{-1} \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. Hence, we can equip $\Gamma(\mathrm{Spec}(\overline{\mathbb{F}}_q), \mathcal{F}_0)$ with the structure of a $\Lambda[F, F^{-1}]$-module. Moreover, for any morphism of sheaves $\mathcal{F}_0 \to \mathcal{G}_0$, we get a commutative diagram

$$\begin{tikzcd}
\Gamma(\mathrm{Spec}(\overline{\mathbb{F}}_q), \mathcal{F}_0) \arrow[r, "F"] \arrow[d] & \Gamma(\mathrm{Spec}(\overline{\mathbb{F}}_q), \mathcal{F}_0) \arrow[d] \\
\Gamma(\mathrm{Spec}(\overline{\mathbb{F}}_q), \mathcal{G}_0) \arrow[r, "F"] & \Gamma(\mathrm{Spec}(\overline{\mathbb{F}}_q), \mathcal{G}_0).
\end{tikzcd}$$

Hence, we obtain a functor

$$\begin{aligned} \Gamma_F : \mathrm{Sh}(\mathrm{Spec}(\mathbb{F}_q)_{\text{proét}}, \Lambda_{\mathrm{Spec}(\mathbb{F}_q)}) &\to \Lambda[F, F^{-1}] - \mathrm{Mod} \\ \mathcal{F}_0 &\mapsto \Gamma(\mathrm{Spec}(\overline{\mathbb{F}}_q), \mathcal{F}_0). \end{aligned}$$

Remark 3.3.20. *Recall from Section 3.2.1 that there is an equivalence between the category of constructible sheaves* $\mathrm{Sh}_c(\mathrm{Spec}(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$ *and the category of finite-dimensional continuous* $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$*-representations over* $\overline{\mathbb{Q}}_\ell$*. On* $\mathrm{Sh}_c(\mathrm{Spec}(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$ *the functor* Γ_F *simply corresponds to restricting the* $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$*-action to the (geometric) Frobenius element* $F \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$*.*

Note that

$$\mathrm{For}_F \circ \Gamma_F(-) = \Gamma(\mathrm{Spec}(\overline{\mathbb{F}}_q), -). \tag{3.15}$$

where

$$\mathrm{For}_F : \Lambda[F, F^{-1}] - \mathrm{Mod} \to \Lambda - \mathrm{Mod}$$

is the forgetful functor.

Lemma 3.3.21. *The functors* For_F, Γ_F *and* $\Gamma(\mathrm{Spec}(\overline{\mathbb{F}}_q), -)$ *are exact.*

Proof. The forgetful functor For_F is exact and reflects exact sequences. Hence, by (3.15) it suffices to prove that $\Gamma(\mathrm{Spec}(\overline{\mathbb{F}}_q), -)$ is exact. This follows from Lemma 3.3.15. $\square$

We define a functor

$$\underline{\mathrm{Hom}}(-,-) : \mathrm{Sh}(X_{0,\text{proét}}, \Lambda_{X_0})^{\mathrm{op}} \times \mathrm{Sh}(X_{0,\text{proét}}, \Lambda_{X_0}) \to \Lambda[F, F^{-1}] - \mathrm{Mod}$$

via

$$(\mathcal{F}_0, \mathcal{G}_0) \mapsto \Gamma_F \circ a_* \circ \mathcal{H}\mathrm{om}(\mathcal{F}_0, \mathcal{G}_0).$$

Note that we have

$$\begin{aligned}
\mathrm{For}_F(\underline{\mathrm{Hom}}(\mathcal{F}_0, \mathcal{G}_0)) &= \mathrm{For}_F \circ \Gamma_F \circ a_* \circ \mathcal{H}\mathrm{om}(\mathcal{F}_0, \mathcal{G}_0) \\
&\overset{(3.15)}{=} \Gamma(\mathrm{Spec}(\overline{\mathbb{F}}_q), a_* \mathcal{H}\mathrm{om}(\mathcal{F}_0, \mathcal{G}_0)) \\
&= \Gamma(X, \mathcal{H}\mathrm{om}(\mathcal{F}_0, \mathcal{G}_0)) \\
&= \mathrm{Hom}(\mathcal{F}, \mathcal{G}).
\end{aligned}$$

Hence, $\underline{\mathrm{Hom}}(\mathcal{F}_0, \mathcal{G}_0)$ is just the vector space $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ together with a canonical Frobenius action. The composition map

$$\mathcal{H}\mathrm{om}(\mathcal{G}_0, \mathcal{K}_0) \otimes \mathcal{H}\mathrm{om}(\mathcal{F}_0, \mathcal{G}_0) \to \mathcal{H}\mathrm{om}(\mathcal{F}_0, \mathcal{K}_0)$$

gives rise to a map

$$a_* \mathcal{H}\mathrm{om}(\mathcal{G}_0, \mathcal{K}_0) \otimes a_* \mathcal{H}\mathrm{om}(\mathcal{F}_0, \mathcal{G}_0) \to a_* \mathcal{H}\mathrm{om}(\mathcal{F}_0, \mathcal{K}_0)$$

and thus to a map of $\Lambda[F, F^{-1}]$-modules

$$\underline{\mathrm{Hom}}(\mathcal{G}_0, \mathcal{K}_0) \otimes_\Lambda \underline{\mathrm{Hom}}(\mathcal{F}_0, \mathcal{G}_0) \to \underline{\mathrm{Hom}}(\mathcal{F}_0, \mathcal{K}_0). \tag{3.16}$$

Similarly, the unit morphism

$$\Lambda_{X_0} \to \mathcal{H}\mathrm{om}(\mathcal{F}_0, \mathcal{F}_0)$$

induces a map

$$\Lambda_{\mathrm{Spec}(\mathbb{F}_q)} \to a_* \mathcal{H}\mathrm{om}(\mathcal{F}_0, \mathcal{F}_0)$$

by adjunction and thus a morphism of $\Lambda[F, F^{-1}]$-modules

$$\Lambda \to \underline{\mathrm{Hom}}(\mathcal{F}_0, \mathcal{F}_0). \tag{3.17}$$

Forgetting the Frobenius action in (3.16) and (3.17) recovers the standard composition map

$$\mathrm{Hom}(\mathcal{G}, \mathcal{K}) \otimes_\Lambda \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\mathcal{F}, \mathcal{K})$$

and unit map

$$\Lambda \to \mathrm{Hom}(\mathcal{F}, \mathcal{F}).$$

Similarly, one can define pre- and post-composition maps

$$\begin{aligned} - \circ \alpha &: \underline{\mathrm{Hom}}(\mathcal{F}'_0, \mathcal{G}_0) \to \underline{\mathrm{Hom}}(\mathcal{F}_0, \mathcal{G}_0) \\ \beta \circ - &: \underline{\mathrm{Hom}}(\mathcal{F}_0, \mathcal{G}_0) \to \underline{\mathrm{Hom}}(\mathcal{F}_0, \mathcal{G}'_0) \end{aligned} \tag{3.18}$$

for any $\alpha : \mathcal{F}_0 \to \mathcal{F}'_0$ and $\beta : \mathcal{G}_0 \to \mathcal{G}'_0$ which recover the maps

$$\begin{aligned} - \circ \pi^*(\alpha) &: \mathrm{Hom}(\mathcal{F}', \mathcal{G}) \to \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \\ \pi^*(\beta) \circ - &: \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\mathcal{F}, \mathcal{G}') \end{aligned}$$

when forgetting the Frobenius action.

We also define a functor on derived categories

$$\mathrm{R}\underline{\mathrm{Hom}}(-,-) : D(X_{0,\text{proét}}, \Lambda_{X_0})^{\mathrm{op}} \times D(X_{0,\text{proét}}, \Lambda_{X_0}) \to D(\Lambda[F, F^{-1}] - \mathrm{Mod})$$

via

$$(\mathcal{F}_0, \mathcal{G}_0) \mapsto \Gamma_F \circ a_* \circ \mathrm{R}\mathcal{H}om(\mathcal{F}_0, \mathcal{G}_0)$$

(recall that we adhere to the derived convention, i.e. Γ_F and a_* are understood to be derived when applied to complexes). Moreover, we set

$$\underline{\mathrm{Hom}}^i(\mathcal{F}_0, \mathcal{G}_0) := H^i(\mathrm{R}\underline{\mathrm{Hom}}(\mathcal{F}_0, \mathcal{G}_0)) = \Gamma_F(H^i(a_* \mathrm{R}\mathcal{H}om(\mathcal{F}_0, \mathcal{G}_0))).$$

Remark 3.3.22. *This definition of* $\underline{\mathrm{Hom}}^i$ *extends our previous definition* (3.6) *for constructible complexes to arbitrary complexes in* $D(X_{0,\text{proét}}, \Lambda_{X_0})$ *(see also Remark 3.3.20).*

For any two chain complexes $\mathcal{F}_0^\bullet, \mathcal{G}_0^\bullet \in C(\mathrm{Sh}(X_{0,\text{proét}}, \Lambda_{X_0}))$ we define a chain complex

$$\underline{\mathrm{Hom}}_{dg}^\bullet(\mathcal{F}_0^\bullet, \mathcal{G}_0^\bullet) \in C(\Lambda[F, F^{-1}] - \mathrm{Mod}).$$

Explicitly, this complex is given by

$$\underline{\mathrm{Hom}}_{dg}^i(\mathcal{F}_0^\bullet, \mathcal{G}_0^\bullet) := \prod_{l+m=i} \underline{\mathrm{Hom}}(\mathcal{F}_0^{-l}, \mathcal{G}_0^m)$$

with differential $d_{\underline{\mathrm{Hom}}_{dg}^\bullet(\mathcal{F}_0^\bullet, \mathcal{G}_0^\bullet)}(f) = d_{\mathcal{G}_0^\bullet} \circ f - (-1)^{|f|} f \circ d_{\mathcal{F}_0^\bullet}$ (where we use (3.18)). The composition map on $\underline{\mathrm{Hom}}(-,-)$ from (3.16) can be applied component-wise to define a composition map

$$\underline{\mathrm{Hom}}_{dg}^i(\mathcal{G}_0^\bullet, \mathcal{K}_0^\bullet) \otimes_\Lambda \underline{\mathrm{Hom}}_{dg}^j(\mathcal{F}_0^\bullet, \mathcal{G}_0^\bullet) \to \underline{\mathrm{Hom}}_{dg}^{i+j}(\mathcal{F}_0^\bullet, \mathcal{K}_0^\bullet).$$

This equips $\underline{\mathrm{Hom}}_{dg}^\bullet(\mathcal{F}_0^\bullet, \mathcal{F}_0^\bullet)$ with the structure of a dg-algebra over Λ together with a dg-algebra automorphism induced by the Frobenius action. Forgetting this Frobenius action recovers the dg-algebra $\mathrm{Hom}_{dg}^\bullet(\mathcal{F}^\bullet, \mathcal{F}^\bullet)$.

Lemma 3.3.23. *Let $\mathcal{F}_0, \mathcal{G}_0 \in D^+(X_{0,\text{proét}}, \Lambda_{X_0})$ and pick bounded below complexes of injectives $I^\bullet_{\mathcal{F}_0}, I^\bullet_{\mathcal{G}_0} \in C^+(\mathrm{Sh}(X_{0,\text{proét}}, \Lambda_{X_0}))$ representing $\mathcal{F}_0$ and $\mathcal{G}_0$. Then*

$$H^i(\underline{\mathrm{Hom}}^\bullet_{dg}(I^\bullet_{\mathcal{F}_0}, I^\bullet_{\mathcal{G}_0})) = \underline{\mathrm{Hom}}^i(\mathcal{F}_0, \mathcal{G}_0)$$

as $\Lambda[F, F^{-1}]$-modules.

Proof. By definition, $\mathrm{R}\mathcal{H}om(\mathcal{F}_0, \mathcal{G}_0)$ is represented by the complex $\mathcal{H}\mathrm{om}^\bullet_{dg}(I^\bullet_{\mathcal{F}_0}, I^\bullet_{\mathcal{G}_0})$ defined as

$$\mathcal{H}\mathrm{om}^i_{dg}(I^\bullet_{\mathcal{F}_0}, I^\bullet_{\mathcal{G}_0}) := \prod_{l+m=i} \mathcal{H}\mathrm{om}(I^{-l}_{\mathcal{F}_0}, I^m_{\mathcal{G}_0})$$

with differential $d_{\mathcal{H}\mathrm{om}^\bullet_{dg}(I^\bullet_{\mathcal{F}_0}, I^\bullet_{\mathcal{G}_0})}(f) = d_{I^\bullet_{\mathcal{G}_0}} \circ f - (-1)^{|f|} f \circ d_{I^\bullet_{\mathcal{F}_0}}$. Note that $\mathcal{H}\mathrm{om}(I^{-l}_{\mathcal{F}_0}, I^m_{\mathcal{G}_0})$ is acyclic for a_* for all $l, m \in \mathbb{Z}$ (see [6, V-(4.10), V-(5.2)]). Hence, $\mathcal{H}\mathrm{om}^i_{dg}(\mathcal{F}^\bullet_0, \mathcal{G}^\bullet_0)$ is acyclic for a_* for all $i \in \mathbb{Z}$. Since Γ_F is exact, $\mathcal{H}\mathrm{om}^i_{dg}(\mathcal{F}^\bullet_0, \mathcal{G}^\bullet_0)$ is also acyclic for $\Gamma_F \circ a_*$ for all $i \in \mathbb{Z}$. In particular, $\mathrm{R}\underline{\mathrm{Hom}}(\mathcal{F}_0, \mathcal{G}_0) = \Gamma_F \circ a_* \circ \mathrm{R}\mathcal{H}om(\mathcal{F}_0, \mathcal{G}_0)$ is represented by the complex obtained by applying $\Gamma_F \circ a_*$ to each component in $\mathcal{H}\mathrm{om}^\bullet_{dg}(I^\bullet_{\mathcal{F}_0}, I^\bullet_{\mathcal{G}_0})$. This is precisely the complex $\underline{\mathrm{Hom}}^\bullet_{dg}(I^\bullet_{\mathcal{F}_0}, I^\bullet_{\mathcal{G}_0})$. Hence,

$$H^i(\underline{\mathrm{Hom}}^\bullet_{dg}(I^\bullet_{\mathcal{F}_0}, I^\bullet_{\mathcal{G}_0})) = H^i(\mathrm{R}\underline{\mathrm{Hom}}(\mathcal{F}_0, \mathcal{G}_0)) = \underline{\mathrm{Hom}}^i(\mathcal{F}_0, \mathcal{G}_0).$$

□

The constructions of this section can be summarized as follows.

Proposition 3.3.24. *For any $\mathcal{F}_0 \in D^b_c(X_0, \Lambda)$, there is a bounded below complex of injectives $I^\bullet_{\mathcal{F}} \in C^+(\mathrm{Sh}(X_{\text{proét}}, \Lambda_X))$ representing $\mathcal{F}$ and dg-algebra automorphism F of $R_{\mathcal{F}} = \mathrm{Hom}^\bullet_{dg}(I^\bullet_{\mathcal{F}}, I^\bullet_{\mathcal{F}})$ such that the action of $H^i(F)$ on $H^i(R_{\mathcal{F}}) = \mathrm{Hom}^i(\mathcal{F}, \mathcal{F})$ is the Frobenius action coming from $\underline{\mathrm{Hom}}^i(\mathcal{F}_0, \mathcal{F}_0)$.*

Proof. Pick a complex of injectives $I^\bullet_{\mathcal{F}_0} \in C^+(\mathrm{Sh}(X_{0,\text{proét}}, \Lambda_{X_0}))$ representing $\mathcal{F}_0$. Then $I^\bullet_{\mathcal{F}} := \pi^*(I^\bullet_{\mathcal{F}_0})$ is a complex of injectives representing $\mathcal{F}$ by Lemma 3.3.19. The dg-algebra $\underline{\mathrm{Hom}}^\bullet_{dg}(I^\bullet_{\mathcal{F}_0}, I^\bullet_{\mathcal{F}_0})$ is just $R_{\mathcal{F}} = \mathrm{Hom}^\bullet_{dg}(I^\bullet_{\mathcal{F}}, I^\bullet_{\mathcal{F}})$ equipped with a dg-algebra automorphism F. By Lemma 3.3.23, we have $H^i(\underline{\mathrm{Hom}}^\bullet_{dg}(I^\bullet_{\mathcal{F}_0}, I^\bullet_{\mathcal{F}_0})) = \underline{\mathrm{Hom}}^i(\mathcal{F}_0, \mathcal{F}_0)$. Hence, $H^i(F)$ is the canonical Frobenius action on $\underline{\mathrm{Hom}}^i(\mathcal{F}_0, \mathcal{F}_0)$. □

3.3.5 Formality for the Springer category

Let $\mu_i : \tilde{V}^i \to V$ be a finite collection of morphisms of Springer type defined over $\overline{\mathbb{F}}_q$ and let $\mathbf{S} = \bigoplus_{i \in I} \mathbf{S}^i \in D^b_c(V)$ be the associated Springer sheaf (recall that $D^b_c(V) := D^b_c(V, \overline{\mathbb{Q}}_l)$). Recall from (3.5) that the Springer category is defined as

$$D_{\mathrm{Spr}}(V) := \langle X_1, ..., X_n \rangle_\Delta \subset D^b_c(V)$$

where the X_i are the simple perverse sheaves appearing in $\mathbf{S}$.

Lemma 3.3.25. *We have* $D_{\mathrm{Spr}}(V) = \langle \mathbf{S} \rangle_{\Delta,\Subset}$.

Proof. Since $X_1, ..., X_n \in \langle \mathbf{S} \rangle_{\Delta,\Subset}$, we have $D_{\mathrm{Spr}}(V) \subset \langle \mathbf{S} \rangle_{\Delta,\Subset}$. Hence, it suffices to show that $D_{\mathrm{Spr}}(V)$ is closed under direct summands. Let $\mathrm{Perv}_{\mathrm{Spr}}(V) \subset \mathrm{Perv}(V)$ be the Serre subcategory generated by the $X_1, ..., X_n$. Then by a standard dévissage argument, the Springer category can be described as

$$D_{\mathrm{Spr}}(V) = \{\mathcal{F} \in D^b_c(V) \mid {}^pH^i(\mathcal{F}) \in \mathrm{Perv}_{\mathrm{Spr}}(V) \text{ for all } i \in \mathbb{Z}\}.$$

Note that as a Serre subcategory, $\mathrm{Perv}_{\mathrm{Spr}}(V)$ is closed under direct summands in $\mathrm{Perv}(V)$. Hence, $D_{\mathrm{Spr}}(V)$ is also closed under direct summands. □

We obtain the following general formality result for the Springer category.

Theorem 3.3.26. *There is an equivalence of triangulated categories*

$$D_{\mathrm{Spr}}(V)^{\mathrm{op}} \cong D_{\mathrm{perf}}(\mathrm{Hom}^*(\mathbf{S}, \mathbf{S}) - \mathrm{dgMod})$$

which identifies $\mathbf{S}$ *with the free dg-module* $\mathrm{Hom}^*(\mathbf{S}, \mathbf{S})$.

Proof. Combining Corollary 3.3.18 and Lemma 3.3.25 we get an equivalence

$$D_{\mathrm{Spr}}(V)^{\mathrm{op}} = \langle \mathbf{S} \rangle_{\Delta,\Subset} \cong D_{\mathrm{perf}}(R_{\mathbf{S}} - \mathrm{dgMod})$$

which sends $\mathbf{S}$ to the free dg-module $R_{\mathbf{S}}$. By Corollary 3.2.5 and Proposition 3.3.24 the dg-algebra $R_{\mathbf{S}}$ can be equipped with a dg-algebra automorphism $F : R_{\mathbf{S}} \to R_{\mathbf{S}}$ such that $H^i(F)$ acts by multiplication with $q^{\frac{i}{2}}$. This implies that $R_{\mathbf{S}}$ is formal by Theorem 3.3.5. In particular, by (3.12) we get an equivalence

$$D_{\mathrm{Spr}}(V)^{\mathrm{op}} \cong D_{\mathrm{perf}}(R_{\mathbf{S}} - \mathrm{dgMod}) \cong D_{\mathrm{perf}}(\mathrm{Hom}^*(\mathbf{S}, \mathbf{S}) - \mathrm{dgMod})$$

which sends $\mathbf{S}$ to the free dg-module $\mathrm{Hom}^*(\mathbf{S}, \mathbf{S})$. □

Remark 3.3.27. *There are several formality results in the literature similar to the one above. For example, formality of the Springer sheaf in the classical Springer correspondence appears in [30, 54, 59, 60]. There are also similar formality results in the context of flag varieties and Koszul duality [12, 61] where formality has also been studied for modular coefficients [3, 58].*

3.4 De $\mathbb{F}$ à $\mathbb{C}$

In this section we prove a formality result for Springer sheaves on varieties over a field of characteristic 0 by reduction to the positive characteristic case. This is a standard application of the "De $\mathbb{F}$ a $\mathbb{C}$" technique from [11, §6]. We will explain how these arguments work in the pro-étale setting. If X_A is a scheme defined over a ring A and $A \to R$ is a ring homomorphism, we denote by X_R the base change of X_A to R. Similarly, if $S \to \operatorname{Spec}(A)$ is a morphism of schemes, we denote by X_S the corresponding base change to S. Furthermore, for $\mathcal{F} \in D^b_c(X_A, \Lambda)$, we denote by $\mathcal{F}_R \in D^b_c(X_R, \Lambda)$ (resp. $\mathcal{F}_S \in D^b_c(X_S, \Lambda)$) the corresponding pullback to X_R (resp. X_S).

3.4.1 Generic base change

The main tool that we need to compare constructible sheaves on varieties in characteristic p and characteristic 0 is the generic base change theorem. Let us first explain what we mean by generic base change.

Definition 3.4.1. *Let S be a scheme and $f : X \to Y$ a morphism of S-schemes. Let $\mathcal{F}, \mathcal{G} \in D(X_{\text{ét}}, \Lambda)$ for a noetherian ring Λ (resp. $\mathcal{F}, \mathcal{G} \in D(X_{\text{proét}}, \Lambda_X)$ for Λ as in Assumption 3.3.7).*

(i) We say that the formation $f_*\mathcal{F}$ commutes with generic base change if there is a dense open subscheme $U \subset S$ such that for each morphism of schemes $g : S' \to U \subset S$ with corresponding pullback diagram

$$\begin{tikzcd} X_{S'} \arrow[r, "g''"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y_{S'} \arrow[r, "g'"] \arrow[d] & Y \arrow[d] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

the canonical map $(g')^* f_*\mathcal{F} \to f'_*(g'')^*\mathcal{F}$ is an isomorphism.

(ii) We say that the formation $\mathrm{R}\mathcal{H}om(\mathcal{F}, \mathcal{G})$ commutes with generic base change if there is a dense open subscheme $U \subset X$ such that for each morphism of schemes $g : S' \to U \subset S$ the canonical map $g^* \mathrm{R}\mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathrm{R}\mathcal{H}om(g^*\mathcal{F}, g^*\mathcal{G})$ is an isomorphism.

There is the following generic base change theorem for étale sheaves.

Theorem 3.4.2. *[27] Let S be a noetherian scheme, $f : X \to Y$ a morphism of S-schemes of finite type and Λ a noetherian ring annihilated by an integer invertible in $\mathcal{O}_S$. Then for any $\mathcal{F}, \mathcal{G} \in D_c^b(X_{\text{ét}}, \Lambda)$ the formations $f_*\mathcal{F}$ and $\mathrm{R}\mathcal{H}om(\mathcal{F}, \mathcal{G})$ commute with generic base change.*

Proof. This is proved for constructible sheaves $\mathcal{F} \in \mathrm{Sh}_c(X_{\text{ét}}, \Lambda)$ in [27, Th. finitude 1.9, 2.10]. The case of an arbitrary complex $\mathcal{F} \in D_c^b(X_{\text{ét}}, \Lambda)$ can be deduced from this by a dévissage argument. □

We get a similar result for non-torsion coefficients in the pro-étale topology. For the rest of this section, we assume that Λ is as in Assumption 3.3.7.

Lemma 3.4.3. *Let S be a noetherian, quasi-excellent, ℓ-coprime scheme and $f : X \to Y$ a morphism of S-schemes of finite type. Then for any $\mathcal{F}, \mathcal{G} \in D_c^b(X_{\text{proét}}, \Lambda_X)$ the formations $f_*\mathcal{F}$ and $\mathrm{R}\mathcal{H}om(\mathcal{F}, \mathcal{G})$ commute with generic base change.*

Proof. Using the reduction steps for constructible sheaves from Lemma 3.3.13 this can be reduced to the étale generic base change theorem (Theorem 3.4.2). The assumption that S is quasi-excellent ensures that $f_*\mathcal{F}$ is constructible. □

The following lemma can also be deduced from [11, Lemme 6.1.9] or [54, A.5].

Lemma 3.4.4. *Let X_A be a scheme of finite type over a finitely-generated $\mathbb{Z}[\ell^{-1}]$-algebra $A \subset \mathbb{C}$ and $s \in \mathrm{Spec}(A)$ a closed point. For any $\mathcal{F}_A \in D_c^b(X_A, \Lambda)$ there exists a strictly Henselian local ring R such that*

(i) $A \subset R \subset \mathbb{C}$;

(ii) the residue field of R is the algebraic closure $\overline{s}$ of s;

(iii) Pulling back along the morphisms in the diagram $X_{\mathbb{C}} \xrightarrow{u} X_R \xleftarrow{v} X_s$ induces equivalences of categories

$$\langle \mathcal{F}_{\mathbb{C}} \rangle_{D_c^b(X_{\mathbb{C}}, \Lambda), \Delta, \Subset} \cong \langle \mathcal{F}_R \rangle_{D_c^b(X_R, \Lambda), \Delta, \Subset} \cong \langle \mathcal{F}_{\overline{s}} \rangle_{D_c^b(X_{\overline{s}}, \Lambda), \Delta, \Subset}.$$

Proof. By constructibility, we may enlarge A so that the cohomology sheaves of $a_* \mathrm{R}\mathcal{H}om(\mathcal{F}_A, \mathcal{F}_A)$ are locally constant of finite presentation. By Lemma 3.4.3 we may enlarge A further so that $a_* \mathrm{R}\mathcal{H}om(\mathcal{F}_A, \mathcal{G}_A)$ commutes with arbitrary base change, i.e. for any $g : S \to \mathrm{Spec}(A)$, we have

$$g^* a_* \mathrm{R}\mathcal{H}om(\mathcal{F}_A, \mathcal{F}_A) = a_* \mathrm{R}\mathcal{H}om(\mathcal{F}_S, \mathcal{F}_S).$$

Let S be a connected w-contractible affine scheme. Then any locally constant sheaf of finite presentation on $S_{\text{proét}}$ is constant by Lemma 3.3.15 and hence

$$\begin{aligned} H^i(a_* \mathrm{R}\mathcal{H}om(\mathcal{F}_S, \mathcal{F}_S)) &= H^i(g^* a_* \mathrm{R}\mathcal{H}om(\mathcal{F}_A, \mathcal{F}_A)) \\ &= g^* H^i(a_* \mathrm{R}\mathcal{H}om(\mathcal{F}_A, \mathcal{F}_A)) \\ &= M_i \otimes_\Lambda \Lambda_S \end{aligned} \tag{3.19}$$

for some finitely-presented Λ-module M_i. Note that the global sections functor $\Gamma_S = \Gamma(S, -)$ is exact by Lemma 3.3.15. Hence, for S affine, connected and w-contractible, we get

$$\begin{aligned} \mathrm{Hom}^i(\mathcal{F}_S, \mathcal{F}_S) &= H^i(\Gamma_S \circ a_* \circ \mathrm{R}\mathcal{H}om(\mathcal{F}_S, \mathcal{F}_S)) \\ &= \Gamma_S \circ H^i(a_* \mathrm{R}\mathcal{H}om(\mathcal{F}_S, \mathcal{F}_S)) \\ &= \Gamma_S(M_i \otimes_\Lambda \Lambda_S) \\ &= M_i. \end{aligned} \tag{3.20}$$

Here the first equality follows from general results about sites (see [6, V-(4.10), V-(5.2)]). The second uses exactness of Γ_S and the third follows from (3.19). To see the last equality, pick a presentation $\Lambda^n \to \Lambda^m \to M_i \to 0$. This induces a presentation $\Lambda_S^n \to \Lambda_S^m \to M_i \otimes_\Lambda \Lambda_S \to 0$. Note that $\Gamma_S(\Lambda_S) = \Lambda$ by (3.13) and the fact that S is connected. Since Γ_S is exact, this implies $\Gamma_S(M_i \otimes_\Lambda \Lambda_S) = \mathrm{coker}(\Lambda^n \to \Lambda^m) = M_i$. Now let R be any strictly Henselian local ring satisfying (i) and (ii) (it is explained in [11, p.156] that such a ring always exists and one can even assume that R is also a discrete valuation ring). Then the schemes $\mathrm{Spec}(\mathbb{C}), \mathrm{Spec}(R)$, and $\overline{s}$ are all connected w-contractible affine schemes by Lemma 3.3.15. Using (3.20), we get

$$\mathrm{Hom}^i(\mathcal{F}_\mathbb{C}, \mathcal{F}_\mathbb{C}) = \mathrm{Hom}^i(\mathcal{F}_R, \mathcal{F}_R) = \mathrm{Hom}^i(\mathcal{F}_{\overline{s}}, \mathcal{F}_{\overline{s}}) = M_i$$

for all $i \in \mathbb{Z}$. By Lemma 3.3.3 this implies that there are equivalences of triangulated categories

$$\langle \mathcal{F}_\mathbb{C} \rangle_{D_c^b(X_\mathbb{C}, \Lambda), \Delta, \Subset} \cong \langle \mathcal{F}_R \rangle_{D_c^b(X_R, \Lambda), \Delta, \Subset} \cong \langle \mathcal{F}_{\overline{s}} \rangle_{D_c^b(X_{\overline{s}}, \Lambda), \Delta, \Subset}.$$

□

3.4.2 Lifting Springer sheaves

Any variety X over $\mathbb{C}$ can be defined over a finitely-generated $\mathbb{Z}[\ell^{-1}]$-algebra $A \subset \mathbb{C}$, i.e. there exists a scheme X_A (of finite type) over $\mathrm{Spec}(A)$ such that $X = X_A \otimes_A \mathbb{C}$. Choosing a maximal ideal of A gives rise to a ring homomorphism $A \to \mathbb{F}_q$ for a finite field $\mathbb{F}_q$. Hence, it makes sense to consider $X_{\mathbb{F}_q}$ relating X to a scheme over a field of positive characteristic. We first check that this procedure preserves morphisms of Springer type.

Any complex reductive group G can be defined over $\mathbb{Z}$, i.e. there exists a split reductive group scheme $G_{\mathbb{Z}}$ over $\mathbb{Z}$ with $G = G_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$. Fix a Borel subgroup $B_{\mathbb{Z}} \subset G_{\mathbb{Z}}$ and let $B := B_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ be the corresponding Borel subgroup of G. Let V be a G-representation and $\{V^i \subset V \mid i \in I\}$ a finite collection of B-stable subspaces. Recall that a representation of an affine group scheme H is the same as a comodule over the Hopf algebra $\mathcal{O}(H)$.

Lemma 3.4.5. *There is a finitely generated $\mathbb{Z}[\ell^{-1}]$-algebra $A \subset \mathbb{C}$, a G_A-representation V_A and B_A-stable A-submodules $V_A^i \subset V_A$ which recover the $V^i \subset V$ when extending scalars to $\mathbb{C}$.*

Proof. Pick a basis $v_1, ..., v_n \in V$. Then we can write the comultiplication on the v_i as

$$\Delta(v_i) = \sum_j v_j \otimes f_{ij}$$

for some $f_{ij} \in \mathcal{O}(G)$. Since $\mathcal{O}(G) = \mathcal{O}(G_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{C}$, we can write each f_{ij} as

$$f_{ij} = \sum_k g_{ijk} \otimes c_{ijk}$$

for some $g_{ijk} \in \mathcal{O}(G_{\mathbb{Z}})$ and $c_{ijk} \in \mathbb{C}$. Let $A \subset \mathbb{C}$ be the $\mathbb{Z}[\ell^{-1}]$-algebra generated by all the c_{ijk}. Then $V_A := \mathrm{Span}_A\{v_1, ..., v_n\}$ is an $\mathcal{O}(G_A)$-comodule and thus a G_A-representation which recovers V when extending scalars to $\mathbb{C}$. We can use the same argument (potentially enlarging A further) to show that there are B_A-representations V_A^i which are spanned by finitely many vectors $w_j^i \in V^i$ and recover V^i when extending scalars to $\mathbb{C}$. After enlarging A further, we may assume that V_A contains all the w_j^i and thus $V_A^i \subset V_A$. $\square$

Consider the scheme $\tilde{V}_A^i := G_A \times^{B_A} V_A^i$ and the canonical map $\mu_i : \tilde{V}_A^i \to V_A$ induced by the action map $G_A \times V_A^i \to V_A$. For any ring homomorphism $A \to R$, we get a pullback diagram

$$\begin{tikzcd} \tilde{V}_R^i \arrow[r, "u"] \arrow[d, "\mu_i"] & \tilde{V}_A^i \arrow[d, "\mu_i"] \\ V_R \arrow[r, "u"] & V_A \end{tikzcd}$$

where $\tilde{V}_R^i = G_R \times^{B_R} V_R^i$. We define

$$\begin{aligned}
\mathbf{S}_R^i &:= (\mu_i)_! \mathbf{1}_{\tilde{V}_R^i}[\dim \tilde{V}^i] \in D_c^b(V_R) \\
\mathbf{S}_R &:= \bigoplus_{i \in I} \mathbf{S}_R^i \in D_c^b(V_R).
\end{aligned}$$

By proper base change we have $u^*(\mathbf{S}_A^i) = \mathbf{S}_R^i$ and $u^*(\mathbf{S}_A) = \mathbf{S}_R$. Moreover, if $R = \overline{k}$ is an algebraically closed field, the morphism $\mu_i : \tilde{V}_{\overline{k}}^i \to V_{\overline{k}}$ is of Springer type and $\mathbf{S}_{\overline{k}}^i$ is the associated Springer sheaf (c.f. Section 3.1.2). Here we use that

$$\dim \tilde{V}^i = \dim G + \dim V^i - \dim B = \dim G_{\overline{k}} + \dim V_{\overline{k}}^i - \dim B_{\overline{k}} = \dim \tilde{V}_{\overline{k}}^i.$$

Corollary 3.4.6. *Let $\mu_i : \tilde{V}^i \to V$ be a finite collection of morphisms of Springer type over $\mathbb{C}$. Then there is an equivalence of triangulated categories*

$$D_{\mathrm{Spr}}(V)^{\mathrm{op}} \cong D_{\mathrm{perf}}(\mathrm{Hom}^*(\mathbf{S}, \mathbf{S}) - \mathrm{dgMod})$$

which identifies $\mathbf{S}$ with the free dg-module $\mathrm{Hom}^*(\mathbf{S}, \mathbf{S})$.

Proof. Let A, $\tilde{V}_A^i, V_A, \ldots$ be as in Lemma 3.4.5 and pick a closed point $s \to \mathrm{Spec}(A)$ with algebraic closure $\overline{s}$. Note that since A is finitely-generated over $\mathbb{Z}[\ell^{-1}]$, we have $s = \mathrm{Spec}(\mathbb{F}_q)$ for a finite field $\mathbb{F}_q$ with ℓ invertible in $\mathbb{F}_q$. By Lemma 3.4.4 there is an equivalence of triangulated categories $D_{\mathrm{Spr}}(V) \cong D_{\mathrm{Spr}}(V_{\overline{s}})$ which identifies $\mathbf{S}$ with $\mathbf{S}_{\overline{s}}$. In particular, we get

$$\mathrm{Hom}^*(\mathbf{S}, \mathbf{S}) \cong \mathrm{Hom}^*(\mathbf{S}_{\overline{s}}, \mathbf{S}_{\overline{s}}).$$

Hence, by Theorem 3.3.26, we have

$$\begin{aligned}
D_{\mathrm{Spr}}(V)^{\mathrm{op}} &\cong D_{\mathrm{Spr}}(V_{\overline{s}})^{\mathrm{op}} \\
&\cong D_{\mathrm{perf}}(\mathrm{Hom}^*(\mathbf{S}_{\overline{s}}, \mathbf{S}_{\overline{s}}) - \mathrm{dgMod}) \\
&\cong D_{\mathrm{perf}}(\mathrm{Hom}^*(\mathbf{S}, \mathbf{S}) - \mathrm{dgMod}).
\end{aligned}$$

□

3.5 A derived Deligne-Langlands correspondence

Let G be a complex reductive group with simply connected derived subgroup and root datum $\mathcal{R}$ and let $\mathcal{H}_q^{\mathrm{aff}}$ be the associated affine Hecke algebra. Recall from Section 2.4

that the affine Hecke algebra at the central character $\chi_{(s,q_0)}$ can be studied using the maps

$$\begin{tikzcd} & \tilde{\mathcal{N}}^{(s,q_0)} \arrow[dl, "\pi^{(s,q_0)}"'] \arrow[dr, "\mu^{(s,q_0)}"] & \\ \mathcal{B}^s & & \mathcal{N}^{(s,q_0)} \end{tikzcd}$$

and in particular the sheaf

$$\mathbf{S}^{(s,q_0)} = (\mu^{(s,q_0)})_* \mathcal{C}_{\tilde{\mathcal{N}}^{(s,q_0)}} \in D^b_c(\mathcal{N}^{(s,q_0)}).$$

As the notation suggests, the sheaf $\mathbf{S}^{(s,q_0)}$ is a Springer sheaf in the sense of Section 3.1.2. To check this, we need the following standard results about centralizers.

Lemma 3.5.1. *[24, Proposition 8.8.7]*

(i) The centralizer $G(s)$ is connected and reductive;

(ii) Each connected component of the fixed-point variety $\mathcal{B}^s$ is $G(s)$-equivariantly isomorphic to the flag variety of $G(s)$.

We decompose $\tilde{\mathcal{N}}^{(s,q_0)}$ into connected components

$$\tilde{\mathcal{N}}^{(s,q_0)} = \bigsqcup_{i \in I} \tilde{\mathcal{N}}^{(s,q_0),i}.$$

Pick the Borel subgroup $B \subset G$ such that $s \in B$, i.e. $B \in \mathcal{B}^s$. Note that $B(s) = G(s) \cap B$ is the stabilizer of $B \in \mathcal{B}^s$ in $G(s)$. By Lemma 3.5.1(ii) this implies that $B(s) \subset G(s)$ is a Borel subgroup. Let

$$\mathcal{B}^s = \bigsqcup_{j \in J} C_j$$

be the decomposition of $\mathcal{B}^s$ into connected components. Then for each $j \in J$ there is a unique element $B_j \in C_j \cong G(s)/B(s)$ whose stabilizer in $G(s)$ is $B(s)$ (i.e. $B_j(s) = B(s)$). Let $\mathfrak{b}_j$ be the Lie algebra of B_j and $\mathfrak{n}_j = [\mathfrak{b}_j, \mathfrak{b}_j]$. The fiber of B_j under $\pi^{(s,q_0)}$ is given by

$$(\pi^{(s,q_0)})^{-1}(B_j) \cong \mathcal{N}^{(s,q_0)} \cap \pi^{-1}(B_j) = \mathcal{N}^{(s,q_0)} \cap \mathfrak{b}_j = \mathfrak{n}_j^{(s,q_0)}.$$

Hence, we get an isomorphism (c.f. [24, Corollary 8.8.9])

$$(\pi^{(s,q_0)})^{-1}(C_j) \cong G(s) \times^{B(s)} \mathfrak{n}_j^{(s,q_0)}.$$

Note that the variety $G(s) \times^{B(s)} \mathfrak{n}_j^{(s,q_0)}$ is connected, so the $(\pi^{(s,q_0)})^{-1}(C_j)$ are already the connected components $\tilde{\mathcal{N}}^{(s,q_0),i}$ of $\mathcal{N}^{(s,q_0)}$. We have thus shown that the (s, q_0)-Springer resolution is of Springer type (c.f. Definition 3.1.4).

Corollary 3.5.2. *For each $i \in I$, the morphism $\mu^{(s,q_0)} : \tilde{\mathcal{N}}^{(s,q_0),i} \to \mathfrak{g}^{(s,q_0)}$ is of Springer type with Springer sheaf $\mathbf{S}^{(s,q_0),i}$.*

Hence, we can consider the corresponding Springer category

$$D_{\mathrm{Spr}}(\mathcal{N}^{(s,q_0)}) := D_{\mathrm{Spr}}(\mathfrak{g}^{(s,q_0)}) = \langle X_1, ..., X_n \rangle_{D^b_c(\mathfrak{g}^{(s,q_0)}),\Delta}$$

where the X_i are the simple perverse constituents of $\mathbf{S}^{(s,q_0)} \in D^b_c(\mathfrak{g}^{(s,q_0)})$.

Remark 3.5.3. *It might be more natural to consider the Springer category $D_{\mathrm{Spr}}(\mathcal{N}^{(s,q_0)})$ as a subcategory of $D^b_c(\mathcal{N}^{(s,q_0)})$ instead of $D^b_c(\mathfrak{g}^{(s,q_0)})$. Note that this is not much of a difference since the canonical functor $D^b_c(\mathcal{N}^{(s,q_0)}) \to D^b_c(\mathfrak{g}^{(s,q_0)})$ induced by the closed immersion $\mathcal{N}^{(s,q_0)} \hookrightarrow \mathfrak{g}^{(s,q_0)}$ is fully faithful. In particular, we also have*

$$D_{\mathrm{Spr}}(\mathcal{N}^{(s,q_0)}) = \langle X_1, ..., X_n \rangle_{D^b_c(\mathcal{N}^{(s,q_0)}),\Delta}.$$

Moreover, if q_0 is not a root of unity, it can be shown that $\mathcal{N}^{(s,q_0)} \cong \mathfrak{g}^{(s,q_0)}$ (c.f. [24, Proposition 8.8.7]).

By Corollary 3.4.6 and Corollary 3.5.2 there is an equivalence of triangulated categories

$$D_{\mathrm{Spr}}(\mathcal{N}^{(s,q_0)})^{\mathrm{op}} \cong D_{\mathrm{perf}}(\mathrm{Hom}^*(\mathbf{S}^{(s,q_0)}, \mathbf{S}^{(s,q_0)}) - \mathrm{dgMod})$$

which sends $\mathbf{S}^{(s,q_0)}$ to the free dg-module $\mathrm{Hom}^*(\mathbf{S}^{(s,q_0)}, \mathbf{S}^{(s,q_0)})$. Combining this with Theorem 2.4.1, we arrive at the following 'derived Deligne-Langlands correspondence'.

Theorem 3.5.4. *There is an equivalence of triangulated categories*

$$D_{\mathrm{Spr}}(\mathcal{N}^{(s,q_0)})^{\mathrm{op}} \cong D_{\mathrm{perf}}(\mathcal{H}^{\mathrm{aff}}_{(s,q_0)} - \mathrm{dgMod})$$

which sends $\mathbf{S}^{(s,q_0)}$ to the free dg-module $\mathcal{H}^{\mathrm{aff}}_{(s,q_0)}$. Here we consider $\mathcal{H}^{\mathrm{aff}}_{(s,q_0)}$ as a dg-algebra with vanishing differential and grading induced by the Hom^**-grading in Theorem 2.4.1.*

Remark 3.5.5. *It would also be interesting to prove similar formality results for graded Hecke algebras at central characters. These algebras can be used to study a wider range of representations of p-adic groups such as unipotent representations [47, 50]. In terms of geometry, graded Hecke algebras arise as certain Ext-algebras in an equivariant derived category of constructible sheaves [8, 51]. Some formality results in this direction can be found in [63]. There also is a coherent categorical Deligne-Langlands correspondence [14] which works without fixing a central character but replaces constructible sheaves with a certain category of coherent sheaves (see also the conjectures in [34]). The relation between the constructible and the coherent side is discussed in [13].*

Chapter 4

Canonical bases via pairing monomials

In this chapter, we develop a new algorithm to compute canonical bases in quantum groups of finite ADE type and we explore some applications to the representation theory of the affine Hecke algebra of GL_n. We first explain the linear algebra underlying our new algorithm (which is motivated by the Lusztig-Shoji algorithm [45, 62]) before we apply it to quantum groups and their geometric categorification. These results can also be found in [4].

4.1 The linear algebraic skeleton

Let V be a finite-dimensional $\mathbb{Q}(v)$-vector space equipped with the following data:

- a bar involution $\bar{\cdot} : V \to V$ which is $\mathbb{Q}(v)$-antilinear;
- a standard basis $\mathbf{B}^{\mathrm{std}} = \{a_c \mid c \in C\}$ indexed by a partially ordered set $(C, \leq)$ satisfying
$$\overline{a}_c \in a_c + \sum_{c' < c} \mathbb{Z}[v, v^{-1}] \cdot a_{c'}.$$

Then by [49, 24.2.1] there is a unique *canonical basis*

$$\mathbf{B}^{\mathrm{can}} = \{b_c \mid c \in C\}$$

of V satisfying

- $\overline{b}_c = b_c$ for all $c \in C$;
- $b_c = a_c + \sum_{c' < c} p_{cc'}(v) a_{c'}$ for some $p_{cc'}(v) \in v^{-1}\mathbb{Z}[v^{-1}]$.

If we refine the partial order on C to a total order, the last condition implies that the change of basis matrix

$$P := (p_{cc'}(v))_{c,c' \in C}$$

between $\mathbf{B}^{\text{std}}$ and $\mathbf{B}^{\text{can}}$ is lower unitriangular with entries in $v^{-1}\mathbb{Z}[v^{-1}]$ below the diagonal. We want to compute that canonical basis in terms of the standard basis or equivalently the matrix P. For this, we assume that we are given the following additional data: Let

$$(\cdot, \cdot) : V \times V \to \mathbb{Q}(v)$$

be a non-degenerate symmetric bilinear form such that $\mathbf{B}^{\text{std}}$ is orthogonal with respect to $(\cdot, \cdot)$. Consider the matrices

$$D := ((a_c, a_{c'}))_{c,c' \in C}$$
$$\Omega := ((b_c, b_{c'}))_{c,c' \in C}$$

representing $(\cdot, \cdot)$ with respect to $\mathbf{B}^{\text{std}}$ and $\mathbf{B}^{\text{can}}$. By change of basis, we have

$$\Omega = PDP^T \tag{4.1}$$

where Ω is symmetric and invertible, P is lower unitriangular and D is diagonal. In linear algebra, this is called the LDLT decomposition of Ω. The LDLT decomposition of any symmetric invertible matrix is unique (if it exists) and it can be computed using the Gram-Schmidt orthogonalization algorithm.

Corollary 4.1.1. *We can compute $\boldsymbol{B}^{\text{can}}$ from the matrix Ω, i.e. from the values of the bilinear form on $\boldsymbol{B}^{\text{can}}$.*

In applications, computing the bilinear form on $\mathbf{B}^{\text{can}}$ is a non-trivial task. However, there are sometimes certain 'monomial elements' in V for which the bilinear form can be computed explicitly. Assume that we are given the following additional data: Let

$$\mathbf{B}^{\text{mon}} = \{m_c \mid c \in C\}$$

be another ordered basis called the 'monomial basis' satisfying

- $\overline{m}_c = m_c$ for all $c \in C$;
- $m_c = b_c + \sum_{c' < c} q_{cc'} b_{c'}$ for some $q_{cc'} \in \mathbb{Z}[v, v^{-1}]$.

The last condition implies that the change of basis matrix

$$Q := (q_{cc'})_{c,c' \in C}$$

between $\mathbf{B}^{\mathrm{can}}$ and $\mathbf{B}^{\mathrm{mon}}$ is lower unitriangular with entries in $\mathbb{Z}[v, v^{-1}]$. This is equivalent to the change of basis matrix

$$L := QP$$

between $\mathbf{B}^{\mathrm{std}}$ and $\mathbf{B}^{\mathrm{mon}}$ being lower unitriangular with entries in $\mathbb{Z}[v, v^{-1}]$. Since the elements of $\mathbf{B}^{\mathrm{can}}$ and $\mathbf{B}^{\mathrm{mon}}$ are bar invariant, we have

$$\overline{q}_{cc'} = q_{cc'} \tag{4.2}$$

for all $c, c' \in C$. Thus the entries of Q actually lie in $\mathbb{Z}[v + v^{-1}]$. Let

$$\Psi := ((m_c, m_{c'}))_{c,c' \in C}$$

be the matrix representing the bilinear form $(\cdot, \cdot)$ with respect to the basis $\mathbf{B}^{\mathrm{mon}}$. Then by change of basis we get

$$\Psi = LDL^T \tag{4.3}$$

which is the LDLT decomposition of Ψ. Hence, L and D can be computed from Ψ. Note that we also have

$$\Psi = Q\Omega Q^T. \tag{4.4}$$

The following lemma says that P and Q are uniquely determined by L (and hence also by Ψ).

Lemma 4.1.2. *The matrices P and Q are the unique matrices satisfying*

- $L = QP$*;*
- *Q is lower unitriangular with entries in $\mathbb{Z}[v + v^{-1}]$;*
- *P is lower unitriangular with entries in $v^{-1}\mathbb{Z}[v^{-1}]$ below the diagonal;*

Moreover, P and Q can be computed from L by a row elimination algorithm.

Proof. Assume we have $L = Q_1 P_1 = Q_2 P_2$ satisfying the conditions above. Then $P_1 P_2^{-1} = Q_2^{-1} Q_1$ is a lower triangular matrix whose entries below the diagonal lie in $\mathbb{Z}[v + v^{-1}] \cap v^{-1}\mathbb{Z}[v^{-1}] = \{0\}$. Hence, $P_1 = P_2$ and $Q_1 = Q_2$ which proves uniqueness. To compute P and Q, note that we can transform L into a matrix with entries in $v^{-1}\mathbb{Z}[v^{-1}]$ below the diagonal using only elementary row operations with coefficients in $\mathbb{Z}[v + v^{-1}]$. The result of this process is the matrix P and the product of all the row operation is the matrix Q^{-1}. □

We can summarize the results of this section as follows.

Corollary 4.1.3. *The matrix P can be computed from the matrix Ψ by computing the unique matrix decompositions $\Psi = LDL^T$ and $L = QP$.*

Hence, the problem of computing the canonical basis is reduced to finding a nice monomial basis for which the matrix Ψ can be computed explicitly.

4.2 Quantum groups

In this section we recall a few well-known facts about the positive half of quantum groups and its standard, canonical and monomial bases. We denote by $\mathbb{N}$ the set of all natural numbers including zero. For any $k \in \mathbb{N}$, we write $[k]! \in \mathbb{Q}(v)$ for the quantum factorial.

4.2.1 PBW bases

Let Φ be a simply laced root system with Weyl group W. Choose a set of positive roots $\Phi^+ \subset \Phi$ and denote the corresponding simple roots by $\mathbf{e}_i$ $(i \in I)$. The positive half of the quantum group associated to Φ is the $\mathbb{Q}(v)$-algebra $U^+ = U^+(\Phi)$ with generators E_i $(i \in I)$ and relations

$$\begin{aligned} E_iE_j - E_jE_i &= 0 \quad \text{if } \langle \mathbf{e}_i, \check{\mathbf{e}}_j \rangle = 0, \\ E_i^2E_j - (v + v^{-1})E_iE_jE_i + E_jE_i^2 &= 0 \quad \text{if } \langle \mathbf{e}_i, \check{\mathbf{e}}_j \rangle = -1. \end{aligned}$$

This algebra admits a grading

$$U^+ = \bigoplus_{\mathbf{v} \in \mathbb{N}[I]} U^+_{\mathbf{v}}$$

where E_i has degree i. For any $x \in U^+$ we define

$$x^{(k)} := \tfrac{x^k}{[k]!}.$$

The bar involution is the ring homomorphism

$$\bar{\cdot} : U^+ \to U^+$$

with $\overline{v} = v^{-1}$ and $\overline{E}_i = E_i$. This restricts to an involution

$$\bar{\cdot} : U^+_{\mathbf{v}} \to U^+_{\mathbf{v}}$$

on each homogeneous component. Fix a reduced expression

$$w_0 = s_{i_1} s_{i_2} ... s_{i_N}$$

of the longest element of W. This induces a total order on the set of positive roots

$$\Phi^+ = \{\gamma_1 < \gamma_2 < ... < \gamma_N\}$$

where

$$\gamma_j := s_{i_1} s_{i_2} ... s_{i_{j-1}}(\mathbf{e}_{i_j}).$$

Moreover, using braid group actions, one can define for each $\gamma \in \Phi^+$ a distinguished element

$$E_\gamma \in U^+$$

which also depends on the choice of reduced expression for w_0 [49, §40]. For any $\mathbf{v} \in \mathbb{N}[I]$ let

$$\mathrm{KP}(\mathbf{v}) = \{\mathbf{c} : \Phi^+ \to \mathbb{N} \mid \sum_{\gamma \in \Phi^+} \mathbf{c}(\gamma) \cdot \gamma = \sum_{i \in I} \mathbf{v}_i \mathbf{e}_i\}$$

be the set of Kostant partitions of $\mathbf{v}$. For each $\mathbf{c} \in \mathrm{KP}(\mathbf{v})$ the corresponding PBW element is defined as

$$E_{\mathbf{c}} := E_{\gamma_1}^{(c(\gamma_1))} E_{\gamma_2}^{(c(\gamma_2))} ... E_{\gamma_N}^{(c(\gamma_N))} \in U_{\mathbf{v}}^+.$$

These elements form a basis

$$\mathbf{B}_{\mathbf{v}}^{\mathrm{std}} = \{E_{\mathbf{c}} \mid \mathbf{c} \in \mathrm{KP}(\mathbf{v})\}$$

of $U_{\mathbf{v}}^+$ known as the standard basis (or PBW basis) [49, 40.2.2]. There is a coproduct $U^+ \to U^+ \otimes U^+$ which equips U^+ with the structure of a (twisted) bialgebra. Moreover there is a non-degenerate symmetric bilinear form

$$(\cdot,\cdot) : U^+ \times U^+ \to U^+ \tag{4.5}$$

defined in [49, 1.2.3] which is given on generators by

$$(E_i, E_j) = \delta_{i,j}(1 - v^{-2})^{-1}$$

and adjoins the product in U^+ with the coproduct. This bilinear form satisfies

$$(U_{\mathbf{v}}^+, U_{\mathbf{w}}^+) = 0$$

for any $\mathbf{v} \neq \mathbf{w}$ and thus the restricted bilinear form

$$(\cdot,\cdot) : U_{\mathbf{v}}^+ \times U_{\mathbf{v}}^+ \to \mathbb{Q}(v)$$

is still non-degenerate.

Lemma 4.2.1. *[49, 38.2.3] The standard basis $\boldsymbol{B}_{\mathbf{v}}^{\mathrm{std}}$ on U_{v}^+ is orthogonal with respect to this bilinear form.*

4.2.2 Canonical and monomial bases

Fix an orientation of the Dynkin diagram of Φ, i.e. a quiver Q whose underlying graph is the Dynkin diagram. The structure of U^+ is closely related to the quiver Q. For any $\mathbf{v} \in \mathbb{N}[I]$ we denote by

$$R_{\mathbf{v}} = R_{\mathbf{v}}(Q) = \bigoplus_{i \to j} \mathrm{Hom}(\mathbb{C}^{\mathbf{v}_i}, \mathbb{C}^{\mathbf{v}_j})$$

the space of $\mathbf{v}$-dimensional representations of Q. This comes with a canonical change of basis action of

$$\mathrm{GL}_{\mathbf{v}} = \prod_{i \in I} \mathrm{GL}_{\mathbf{v}_i}\,.$$

By Gabriel's theorem there is a canonical bijection

$$\begin{aligned} \mathrm{KP}(\mathbf{v}) &\overset{1:1}{\leftrightarrow} \{\mathrm{GL}_{\mathbf{v}}\text{-orbits on } R_{\mathbf{v}}\} \\ \mathbf{c} &\mapsto \mathcal{O}_{\mathbf{c}}. \end{aligned} \tag{4.6}$$

This induces a partial order $\leq$ on $\mathrm{KP}(\mathbf{v})$ where

$$\mathbf{c} < \mathbf{c}' \quad \text{if} \quad \dim \mathcal{O}_{\mathbf{c}} < \dim \mathcal{O}_{\mathbf{c}'}.$$

From now on, we assume that the reduced expression $w_0 = s_{s_1} s_{i_2} \ldots s_{i_N}$ is adapted to Q in the sense of [48, 4.7] (such a reduced expression always exists by [48, 4.12]). Then by [48, 7.9] we have

$$\overline{E}_{\mathbf{c}} \in E_{\mathbf{c}} + \sum_{\mathbf{c}' < \mathbf{c}} \mathbb{Z}[v, v^{-1}] E_{\mathbf{c}'} \tag{4.7}$$

for each $\mathbf{c} \in \mathrm{KP}(\mathbf{v})$. Hence, by [49, 24.2.1] there is a canonical basis of $U^+_{\mathbf{v}}$

$$\mathbf{B}^{\mathrm{can}}_{\mathbf{v}} = \{b_{\mathbf{c}} \mid \mathbf{c} \in \mathrm{KP}(\mathbf{v})\}.$$

It turns out that this basis is independent of the choice reduced expression and orientation [48, 2.3, 3.2] (however the partial order on $\mathrm{KP}(\mathbf{v})$ does depend on the choice of orientation).

Remark 4.2.2. *We can also replace the orbit dimension partial order $\leq$ in* (4.7) *with the orbit closure partial order $\leq_{cl}$ where*

$$\boldsymbol{c} \leq_{cl} \mathbf{c}' \quad \textit{if} \quad \mathcal{O}_{\boldsymbol{c}} \subset \overline{\mathcal{O}}_{\boldsymbol{c}'}.$$

We work with $\leq$ since it is easier to compute in examples.

Next, we discuss the monomial basis. Given a non-negative integer m and a pair of sequences

$$\mathbf{i} = (i_1, ..., i_m) \in I^m$$
$$\mathbf{a} = (a_1, ..., a_m) \in \mathbb{N}^m$$

we say that $(\mathbf{i}, \mathbf{a})$ has weight $\mathbf{v}$ if

$$\mathbf{v}_j = \sum_{i_k = j} a_k$$

for all $j \in I$. For any pair $(\mathbf{i}, \mathbf{a})$ of weight $\mathbf{v}$ we can define a corresponding monomial

$$E_{\mathbf{i},\mathbf{a}} := E_{i_1}^{(a_1)} E_{i_2}^{(a_2)} ... E_{i_m}^{(a_m)} \in U_{\mathbf{v}}^+.$$

Lemma 4.2.3. *[48, 7.8], [55, 4.2] We can find $m \in \mathbb{N}$, $\boldsymbol{i} \in I^m$, and for each $\boldsymbol{c} \in \mathrm{KP}(\boldsymbol{v})$ a sequence $\boldsymbol{a_c} \in \mathbb{N}^m$ such that $(\boldsymbol{i}, \boldsymbol{a_c})$ has weight $\boldsymbol{v}$ and*

$$E_{\boldsymbol{i},\boldsymbol{a_c}} \in E_{\boldsymbol{c}} + \sum_{\boldsymbol{c}' < \boldsymbol{c}} \mathbb{Z}[v, v^{-1}] E_{\boldsymbol{c}'}$$

for all $\boldsymbol{c} \in \mathrm{KP}(\boldsymbol{v})$.

Thus, if we pick $\mathbf{i}$ and for each $\mathbf{c} \in \mathrm{KP}(\mathbf{v})$ a sequence $\mathbf{a_c}$ as above, we obtain a monomial basis

$$\mathbf{B}_{\mathbf{v}}^{\mathrm{mon}} = \{E_{\mathbf{i},\mathbf{a_c}} \mid \mathbf{c} \in \mathrm{KP}(\mathbf{v})\}$$

of $U_{\mathbf{v}}^+$. By Corollary 4.1.3 the canonical basis can then be computed from the matrix

$$\Psi = ((E_{\mathbf{i},\mathbf{a_c}}, E_{\mathbf{i},\mathbf{a_{c'}}}))_{\mathbf{c},\mathbf{c}' \in \mathrm{KP}(\mathbf{v})}. \tag{4.8}$$

In principle, the entries of the matrix Ψ can be computed from the recursive formula in [49, 1.2.3]. We will provide an alternative formula for Ψ in terms of quiver representations in the next section.

Remark 4.2.4. *There is an explicit procedure [55, 4.2] that constructs the sequences $\boldsymbol{a}_c$, depending on the choice of a 'directed partition' on the set of indecomposable representations of Q. We will give explicit examples in type A in Section 4.4.*

4.3 A geometric formula for the bilinear pairing

In this section, we derive a new formula for the standard bilinear form of U^+ evaluated at monomials. The proof relies on geometric techniques.

4.3.1 Notation

Let G be a connected complex algebraic group and X a G-variety. Recall from Section 2.3 that the equivariant cohomology and Borel-Moore homology (with coefficients in $\mathbb{C} \cong \overline{\mathbb{Q}}_l$) are defined as

$$\begin{aligned} H^i_G(X) &:= \mathrm{Hom}^i_G(\mathbf{1}_X, \mathbf{1}_X) \\ H_i^G(X) &:= \mathrm{Hom}^{-i}_G(\mathbf{1}_X, \omega_X). \end{aligned}$$

For $G = \{e\}$ this recovers ordinary cohomology $H^*(X)$ and Borel-Moore homology $H_*(X)$. If X is smooth and connected there is the Poincaré duality isomorphism

$$H^G_k(X) \overset{PD}{\cong} H^{2\dim X - k}_G(X).$$

We will make use of the following standard result about Borel-Moore homology.

Lemma 4.3.1. *Let $X = \bigsqcup_{j \in J} X_j$ be a finite partition into locally closed G-stable subsets such that $H^G_k(X_j)$ vanishes for k odd for all $j \in J$. Assume that we can pick a total order on J such that $X_{\leq j} = \bigsqcup_{j' \leq j} X_{j'}$ is closed in X for all $j \in J$. Then*

$$H^G_k(X) \cong \bigoplus_{j \in J} H^G_k(X_j)$$

for all k.

Proof. This follows by a standard induction argument using the long exact sequence of Borel-Moore homology (see the proof of [32, B.3-Lemma 6] for more details on this type of argument). □

Denote by

$$\begin{aligned} \chi^G_v(X) &:= v^{\dim X} \sum_{k \geq 0} \dim H^k_G(X) v^{-k} \in \mathbb{Z}((v^{-1})) \\ \chi_v(X) &:= v^{\dim X} \sum_{k \geq 0} \dim H^k(X) v^{-k} \in \mathbb{Z}[v, v^{-1}] \end{aligned} \tag{4.9}$$

the (normalized) Poincaré polynomials of X. Denote by $\mathrm{Semis}_G(X) \subset D^b_{G,c}(X)$ the additive category of semisimple complexes, i.e. the full subcategory whose objects are direct sums of shifts is simple perverse sheaves. Then the split Grothendieck group

$$K_\oplus(X) := K_\oplus(\mathrm{Semis}_G(X))$$

is a free $\mathbb{Z}[v, v^{-1}]$-module (where v acts by the homological shift [1]) with a basis given by the classes of simple equivariant perverse sheaves. If X is smooth and connected with structure map $a : X \to \mathrm{pt}$ we have

$$[a_*\mathcal{C}_X] = \chi_v(X) \cdot [\mathcal{C}_{\mathrm{pt}}] \tag{4.10}$$

in $K_\oplus(\mathrm{pt})$. For any Laurent polynomial

$$f(v) = \sum_{k\in\mathbb{Z}} a_k v^k \in \mathbb{N}[v, v^{-1}]$$

and $\mathcal{F} \in \mathrm{Semis}_G(X)$ we define

$$f(v) \cdot \mathcal{F} := \bigoplus_{k\in\mathbb{Z}} \mathcal{F}^{\oplus a_k}[k].$$

4.3.2 Lusztig's geometric categorification of U^+

We retain the notation from Section 4.2: Φ is a simply laced root system with positive roots Φ^+ and simple roots $\{\mathbf{e}_i \mid i \in I\}$, Q is a quiver whose underlying graph is the Dynkin diagram of Φ, $R_{\mathbf{v}}$ is the space of $\mathbf{v}$-dimensional representations of Q and $\mathrm{GL}_{\mathbf{v}}$ is the corresponding change of basis group. The $\mathrm{GL}_{\mathbf{v}}$-stabilizer of any point in $R_{\mathbf{v}}$ is connected. Hence, all equivariant local systems on $\mathrm{GL}_{\mathbf{v}}$-orbits in $R_{\mathbf{v}}$ are trivial and there is a canonical bijection

$$\{\mathrm{GL}_{\mathbf{v}}\text{-orbits on } R_{\mathbf{v}}\} \overset{1:1}{\leftrightarrow} \mathrm{Irr}(\mathrm{Perv}_{\mathrm{GL}_{\mathbf{v}}}(R_{\mathbf{v}})).$$

Combining this with (4.6), we obtain a canonical bijection

$$\begin{aligned} \mathrm{KP}(\mathbf{v}) &\overset{1:1}{\leftrightarrow} \mathrm{Irr}(\mathrm{Perv}_{\mathrm{GL}_{\mathbf{v}}}(R_{\mathbf{v}})) \\ \mathbf{c} &\mapsto \mathrm{IC}_{\mathbf{c}}\,. \end{aligned}$$

This yields a $\mathbb{Z}[v, v^{-1}]$-basis

$$\{[\mathrm{IC}_{\mathbf{c}}] \mid \mathbf{c} \in \mathrm{KP}(\mathbf{v})\}$$

of $K_\oplus(R_{\mathbf{v}})$. This is also a $\mathbb{Q}(v)$-basis of

$$K_\oplus(R_{\mathbf{v}})_{\mathbb{Q}(v)} := \mathbb{Q}(v) \otimes_{\mathbb{Z}[v,v^{-1}]} K_\oplus(R_{\mathbf{v}}).$$

For any $\mathbf{v}, \mathbf{w} \in \mathbb{N}[I]$ there is an induction functor [49, 9.2.5]

$$D^b_{\mathrm{GL}_{\mathbf{v}} \times \mathrm{GL}_{\mathbf{w}},c}(R_{\mathbf{v}} \times R_{\mathbf{w}}) \xrightarrow{\mathrm{Ind}^{\mathbf{v}+\mathbf{w}}_{\mathbf{v},\mathbf{w}}} D^b_{\mathrm{GL}_{\mathbf{v}+\mathbf{w}},c}(R_{\mathbf{v}+\mathbf{w}})$$

which induces a 'convolution map'

$$\begin{aligned} K_\oplus(R_\mathbf{v}) \otimes_{\mathbb{Z}[v,v^{-1}]} K_\oplus(R_\mathbf{w}) &\to K_\oplus(R_{\mathbf{v}+\mathbf{w}}) \\ [\mathcal{F}] \otimes [\mathcal{G}] &\mapsto [\mathrm{Ind}_{\mathbf{v},\mathbf{w}}^{\mathbf{v}+\mathbf{w}}(\mathcal{F} \boxtimes \mathcal{G})]. \end{aligned}$$

This equips

$$K_\oplus(Q) := \bigoplus_{\mathbf{v} \in \mathbb{N}[I]} K_\oplus(R_\mathbf{v})_{\mathbb{Q}(v)}$$

with the structure of a $\mathbb{Q}(v)$-algebra. For any $\mathcal{F}, \mathcal{G} \in \mathrm{Semis}_{\mathrm{GL}_\mathbf{v}}(R_\mathbf{v})$, we define

$$(\mathcal{F}, \mathcal{G}) = \sum_{k \in \mathbb{Z}} \dim \mathrm{Hom}^k_{\mathrm{GL}_\mathbf{v}}(\mathbb{D}\mathcal{F}, \mathcal{G}) \cdot v^{-k} \in \mathbb{Q}(v).$$

A priori, this pairing takes values in $\mathbb{Z}((v^{-1}))$ but it can be shown that on semisimple complexes it actually takes values in $\mathbb{Z}((v^{-1})) \cap \mathbb{Q}(v)$. There is an induced bilinear pairing

$$(\cdot, \cdot) : K_\oplus(R_\mathbf{v}) \times K_\oplus(R_\mathbf{v}) \to \mathbb{Q}(v)$$

which can be extended to a symmetric bilinear form

$$K_\oplus(Q) \times K_\oplus(Q) \to \mathbb{Q}(v) \tag{4.11}$$

by stipulating that $(K_\oplus(R_\mathbf{v}), K_\oplus(R_\mathbf{w})) = 0$ for any $\mathbf{v} \neq \mathbf{w}$. The following theorem is Lusztig's geometric categorification of U^+.

Theorem 4.3.2. *[49, 13.2.11] For each $\mathbf{v} \in \mathbb{N}[I]$ there is an isomorphism of $\mathbb{Q}(v)$-vector spaces*

$$\begin{aligned} K_\oplus(R_\mathbf{v})_{\mathbb{Q}(v)} &\xrightarrow{\sim} U^+_\mathbf{v} \\ [\mathrm{IC}_\mathbf{c}] &\mapsto b_\mathbf{c}. \end{aligned} \tag{4.12}$$

This induces an isomorphism of $\mathbb{N}[I]$-graded algebras

$$K_\oplus(Q) \xrightarrow{\sim} U^+$$

which intertwines the bilinear forms from (4.11) *and* (4.5).

The change of basis matrix between $\mathbf{B}^{\mathrm{std}}_\mathbf{v}$ and $\mathbf{B}^{\mathrm{can}}_\mathbf{v}$ also has a geometric interpretation.

Proposition 4.3.3. *[48, 10.7] The entries of the change of basis matrix P between $\boldsymbol{B}^{\mathrm{std}}_\mathbf{v}$ and $\boldsymbol{B}^{\mathrm{can}}_\mathbf{v}$ are given by*

$$p_{\mathbf{c},\mathbf{c}'}(v) = \sum_{k \in \mathbb{Z}} \dim H^k(\iota_x^* \mathrm{IC}_\mathbf{c}) v^k$$

where $\iota_x : \{x\} \hookrightarrow \mathcal{O}_{\mathbf{c}'}$ is any base point.

Remark 4.3.4. *When Q is the equioriented type A quiver, the polynomials $p_{\mathbf{c},\mathbf{c}'}(v)$ turn out to be certain parabolic Kazhdan-Lusztig polynomials [68].*

Next, we discuss the geometric interpretation of monomials. Let

$$\mathbf{i} = (i_1, ..., i_m) \in I^m$$
$$\mathbf{a} = (a_1, ..., a_m) \in \mathbb{N}^m$$

such that $(\mathbf{i}, \mathbf{a})$ has weight $\mathbf{v}$. Let $B \subset \mathrm{GL}_\mathbf{v}$ be a fixed Borel subgroup. Then there is a unique B-stable flag F^* of I-graded subspaces

$$\mathbb{C}^\mathbf{v} = F^0 \supset F^1 \supset ... \supset F^m = \{0\}$$

of type $(\mathbf{i}, \mathbf{a})$ (i.e. F^{k-1}/F^k is pure of weight i_k and dimension a_k). The stabilizer $P_{\mathbf{i},\mathbf{a}} \subset \mathrm{GL}_\mathbf{v}$ of F^* is a parabolic subgroup containing B. Let $R_{\mathbf{i},\mathbf{a}} \subset R_\mathbf{v}$ be the subspace of all representations x that are compatible with the flag F^*, i.e. $x(F^k) \subset F^k$ for all k. We define

$$\tilde{R}_{\mathbf{i},\mathbf{a}} := \mathrm{GL}_\mathbf{v} \times^{P_{\mathbf{i},\mathbf{a}}} R_{\mathbf{i},\mathbf{a}}.$$

Then the morphism

$$\mu_{\mathbf{i},\mathbf{a}} : \tilde{R}_{\mathbf{i},\mathbf{a}} \to R_\mathbf{v}$$
$$(g, v) \mapsto g \cdot v$$

is proper and we define

$$\mathbf{S}_{\mathbf{i},\mathbf{a}} := (\mu_{\mathbf{i},\mathbf{a}})_* \mathcal{C}_{\tilde{R}_{\mathbf{i},\mathbf{a}}} \in D^b_{\mathrm{GL}_\mathbf{v},c}(R_\mathbf{v}).$$

By the (equivariant) decomposition theorem, we have

$$\mathbf{S}_{\mathbf{i},\mathbf{a}} \in \mathrm{Semis}_{\mathrm{GL}_\mathbf{v}}(R_\mathbf{v}).$$

Moreover,

$$\begin{aligned} \mathbb{D}\,\mathbf{S}_{\mathbf{i},\mathbf{a}} &= \mathbb{D}(\mu_{\mathbf{i},\mathbf{a}})_* \mathcal{C}_{\tilde{R}_{\mathbf{i},\mathbf{a}}} \\ &= (\mu_{\mathbf{i},\mathbf{a}})_* \mathbb{D}\mathcal{C}_{\tilde{R}_{\mathbf{i},\mathbf{a}}} \\ &= (\mu_{\mathbf{i},\mathbf{a}})_* \mathcal{C}_{\tilde{R}_{\mathbf{i},\mathbf{a}}} \\ &= \mathbf{S}_{\mathbf{i},\mathbf{a}}\,. \end{aligned} \tag{4.13}$$

Proposition 4.3.5. *The isomorphism $K_\oplus(R_\mathbf{v})_{\mathbb{Q}(v)} \xrightarrow{\sim} U^+_\mathbf{v}$ from* (4.12) *identifies* $[\mathbf{S}_{\mathbf{i},\mathbf{a}}]$ *with $E_{\mathbf{i},\mathbf{a}}$.*

Proof. See [57, §5] for an argument in the language of Hall algebras. In the language of perverse sheaves this can be explained as follows: It is straightforward to check that $E_{i_j}^{(a_j)}$ is a canonical basis element for all $j = 1, ..., m$. Hence, we have

$$E_{\mathbf{i},\mathbf{a}} = E_{i_1}^{(a_1)} E_{i_2}^{(a_2)} ... E_{i_m}^{(a_m)} \overset{(4.12)}{=} [\mathrm{IC}_{\mathbf{e}_{i_1}^{a_1}}] \cdot [\mathrm{IC}_{\mathbf{e}_{i_2}^{a_2}}] \cdot ... \cdot [\mathrm{IC}_{\mathbf{e}_{i_m}^{a_m}}]$$

Note that $R_{\mathbf{e}_{i_j}^{a_j}} = \{0\}$ is just a point and hence $\mathrm{IC}_{\mathbf{e}_{i_j}^{a_j}}$ is just the constant sheaf $\mathcal{C}_{R_{\mathbf{e}_{i_j}^{a_j}}}$. Spelling out the definition of the corresponding convolution explicitly (see for example [1, (10.3.8)]), we arrive at

$$E_{\mathbf{i},\mathbf{a}} = [\mathcal{C}_{R_{\mathbf{e}_{i_1}^{a_1}}}] \cdot [\mathcal{C}_{R_{\mathbf{e}_{i_2}^{a_2}}}] \cdot ... \cdot [\mathcal{C}_{R_{\mathbf{e}_{i_m}^{a_m}}}] = [(\mu_{\mathbf{i},\mathbf{a}})_* \mathcal{C}_{\tilde{R}_{\mathbf{i},\mathbf{a}}}] = [\mathbf{S}_{\mathbf{i},\mathbf{a}}].$$

□

Combining Theorem 4.3.2 and Proposition 4.3.5 yields the following geometric interpretation of the matrix Ψ from (4.8):

$$\begin{aligned} \Psi_{\mathbf{c},\mathbf{c}'} &= (E_{\mathbf{i},\mathbf{a}_\mathbf{c}}, E_{\mathbf{i},\mathbf{a}_{\mathbf{c}'}}) \\ &= ([\mathbf{S}_{\mathbf{i},\mathbf{a}_\mathbf{c}}], [\mathbf{S}_{\mathbf{i},\mathbf{a}_{\mathbf{c}'}}]) \\ &= \sum_{k\in\mathbb{Z}} \dim \mathrm{Hom}^k_{\mathrm{GL}_\mathbf{v}}(\mathbb{D}(\mathbf{S}_{\mathbf{i},\mathbf{a}_\mathbf{c}}), \mathbf{S}_{\mathbf{i},\mathbf{a}_{\mathbf{c}'}}) \cdot v^{-k} \\ &\overset{(4.13)}{=} \sum_{k\in\mathbb{Z}} \dim \mathrm{Hom}^k_{\mathrm{GL}_\mathbf{v}}(\mathbf{S}_{\mathbf{i},\mathbf{a}_\mathbf{c}}, \mathbf{S}_{\mathbf{i},\mathbf{a}_{\mathbf{c}'}}) \cdot v^{-k}. \end{aligned} \tag{4.14}$$

There also is a geometric version of Lemma 4.2.3.

Lemma 4.3.6. *[57, 2.2] We can find $m \in \mathbb{N}$, $\mathbf{i} \in I^m$, and for each $\mathbf{c} \in \mathrm{KP}(\mathbf{v})$ a sequence $\mathbf{a}_\mathbf{c} \in \mathbb{N}^m$ such that $(\mathbf{i}, \mathbf{a}_\mathbf{c})$ has weight $\mathbf{v}$ and*

(i) $\mu_{\mathbf{i},\mathbf{a}_\mathbf{c}}$ has image $\overline{\mathcal{O}}_\mathbf{c}$,

(ii) $\mu_{\mathbf{i},\mathbf{a}_\mathbf{c}}$ restricts to an isomorphism $\mu^{-1}_{\mathbf{i},\mathbf{a}_\mathbf{c}}(\mathcal{O}_\mathbf{c}) \xrightarrow{\sim} \mathcal{O}_\mathbf{c}$ over $\mathcal{O}_\mathbf{c}$,

for all $\mathbf{c} \in \mathrm{KP}(\mathbf{v})$.

Note that the lemma above (together with the decomposition theorem) implies that

$$\mathbf{S}_{\mathbf{i},\mathbf{a}_\mathbf{c}} \cong \mathrm{IC}_\mathbf{c} \oplus \bigoplus_{k=1}^{n} \mathrm{IC}_{\mathbf{c}_k}[d_k]$$

for some (not necessarily distinct) Kostant partitions $\mathbf{c}_k < \mathbf{c}$ and integers $d_k \in \mathbb{Z}$. Taking the corresponding classes in $K_\oplus(R_\mathbf{v})_{\mathbb{Q}(v)} \cong U_\mathbf{v}^+$ yields

$$E_{\mathbf{i},\mathbf{a}_\mathbf{c}} = [\mathbf{S}_{\mathbf{i},\mathbf{a}_\mathbf{c}}] = [\mathrm{IC}_\mathbf{c}] + \sum_{k=1}^{n} [\mathrm{IC}_{\mathbf{c}_k}[d_k]] = E_\mathbf{c} + \sum_{k=1}^{n} v^{d_k} E_{\mathbf{c}_k}.$$

Hence, Lemma 4.2.3 and the existence of a monomial basis is a consequence of Lemma 4.3.6.

4.3.3 Computation of some Ext groups

In this section we derive a formula for the graded dimensions of Hom-spaces between certain semisimple complexes. Let G be a connected complex reductive group with a fixed Borel subgroup $B \subset G$, torus $T \subset B$ and Weyl group W. Let $P_1, P_2 \subset G$ be two parabolic subgroups containing B with corresponding parabolic subgroups $W_1, W_2 \subset W$. For any $w \in W$, we write ${}^wP_i := \dot{w}P_i\dot{w}^{-1}$ $(i = 1, 2)$ where $\dot{w} \in N_G(T)$ is a lift of w. Consider the orbit partition

$$G/P_1 \times G/P_2 = \bigsqcup_{[w]\in W_1\backslash W/W_2} Y_{[w]} \tag{4.15}$$

where $Y_{[w]} = G \cdot (P_1, {}^wP_2) \cong G/(P_1 \cap {}^wP_2)$. We start by computing the Poincaré polynomial of $Y_{[w]}$.

Lemma 4.3.7. *We have*

$$\chi_v^G(Y_{[w]}) = v^{-r_{[w]}} \frac{\chi_v^G(G/B)}{\chi_v(P_1/B)\chi_v(P_2/B)} \sum_{y\in[w]} v^{2l(y)}$$

where $r_{[w]} = \dim Y_{[w]} + \dim P_1/B + \dim P_2/B - \dim G/B$

Proof. Let

$$q : G/B \times G/B \to G/P_1 \times G/P_2$$

be the projection and

$$X_{[w]} := q^{-1}(Y_{[w]}).$$

We first compute $\chi_v^G(X_{[w]})$. Note that $X_{[w]} = \bigsqcup_{y\in[w]} \mathcal{O}_y$ where

$$G/B \times G/B = \bigsqcup_{y\in W} \mathcal{O}_y$$

is the partition into G-orbits. For any $y \in W$, the map $pr_1 : \mathcal{O}_y \to G/B$ is a G-equivariant locally trivial fibration with fiber $\mathbb{A}^{\dim \mathcal{O}_y - \dim G/B}$. Hence, $H_G^k(\mathcal{O}_y) \cong H_G^k(G/B)$ for all $k \geq 0$ and

$$\chi_v^G(\mathcal{O}_y) = v^{\dim \mathcal{O}_y - \dim G/B}\chi_v^G(G/B). \tag{4.16}$$

Moreover,

$$H_k^G(\mathcal{O}_y) \overset{PD}{\cong} H_G^{2\dim \mathcal{O}_y - k}(\mathcal{O}_y) \cong H_G^{2\dim \mathcal{O}_y - k}(G/B)$$

vanishes for k odd. By Lemma 4.3.1, this implies

$$H_k^G(X_{[w]}) = \bigoplus_{y\in[w]} H_k^G(\mathcal{O}_y). \tag{4.17}$$

Hence, we get

$$\begin{aligned}
\chi_v^G(X_{[w]}) &\overset{(4.9)}{=} v^{\dim X_{[w]}} \sum_{k\in\mathbb{Z}} \dim H_G^k(X_{[w]})v^{-k} \\
&\overset{PD}{=} v^{\dim X_{[w]}} \sum_{k\in\mathbb{Z}} \dim H^G_{2\dim X_{[w]}-k}(X_{[w]})v^{-k} \\
&= v^{-\dim X_{[w]}} \sum_{k\in\mathbb{Z}} \dim H_k^G(X_{[w]})v^{k} \\
&\overset{(4.17)}{=} v^{-\dim X_{[w]}} \sum_{k\in\mathbb{Z}} \sum_{y\in[w]} \dim H_k^G(\mathcal{O}_y)v^{k} \\
&= v^{-\dim X_{[w]}} \sum_{k\in\mathbb{Z}} \sum_{y\in[w]} \dim H^G_{2\dim\mathcal{O}_y-k}(\mathcal{O}_y)v^{2\dim\mathcal{O}_y-k} \\
&\overset{PD}{=} v^{-\dim X_{[w]}} \sum_{k\in\mathbb{Z}} \sum_{y\in[w]} \dim H_G^k(\mathcal{O}_y)v^{2\dim\mathcal{O}_y-k} \\
&= v^{-\dim X_{[w]}} \sum_{y\in[w]} \chi_v^G(\mathcal{O}_y)v^{\dim\mathcal{O}_y} \\
&\overset{(4.16)}{=} v^{-\dim X_{[w]}} \sum_{y\in[w]} \chi_v^G(G/B)v^{2\dim\mathcal{O}_y-\dim G/B}.
\end{aligned} \tag{4.18}$$

Note that $\dim\mathcal{O}_y = \dim G/B + l(y)$, so we can also write (4.18) as

$$\begin{aligned}
\chi_v^G(X_{[w]}) &= v^{-\dim X_{[w]}} \sum_{y\in[w]} \chi_v^G(G/B)v^{2(\dim G/B+l(y))-\dim G/B} \\
&= v^{\dim G/B-\dim X_{[w]}}\chi_v^G(G/B) \sum_{y\in[w]} v^{2l(y)}.
\end{aligned} \tag{4.19}$$

Let $\tilde{q} : X_{[w]} \to Y_{[w]}$ be the restriction of the map q to $Y_{[w]}$. This is a G-equivariant locally trivial fibration with fiber $P_1/B \times P_2/B$. Hence,

$$\dim X_{[w]} - \dim G/B = \dim Y_{[w]} + \dim P_1/B + \dim P_2/B - \dim G/B = r_{[w]}. \tag{4.20}$$

To complete the proof, we claim that

$$\chi_v^G(X_{[w]}) = \chi_v^G(Y_{[w]})\chi_v(P_1/B)\chi_v(P_2/B). \tag{4.21}$$

Note that $\tilde{q}$ is a G-equivariant proper morphisms, so by the decomposition theorem $\tilde{q}_*\mathcal{C}_{X_{[w]}}$ is a semisimple G-equivariant complex on $Y_{[w]}$. The only simple G-equivariant perverse sheaf on $Y_{[w]}$ is $\mathcal{C}_{Y_{[w]}}$. Therefore, we can write

$$\tilde{q}_*\mathcal{C}_{X_{[w]}} = f(v)\cdot\mathcal{C}_{Y_{[w]}} \tag{4.22}$$

for some $f(v) \in \mathbb{N}[v, v^{-1}]$. Pushing (4.22) forward to the point, we get from (4.10) that

$$\chi_v^G(X_{[w]}) = f(v) \cdot \chi_v^G(Y_{[w]}). \tag{4.23}$$

Consider the inclusion maps $\iota_x : \{x\} \hookrightarrow \mathcal{O}_y$ and $\iota_{q^{-1}(x)} : q^{-1}(x) \hookrightarrow X_{[w]}$. Using proper base change (after forgetting the G-equivariance), we get

$$\begin{aligned} f(v) \cdot \mathcal{C}_{\{x\}} &= f(v) \cdot \iota_x^* \mathcal{C}_{Y_{[w]}}[-\dim Y_{[w]}] \\ &\overset{(4.22)}{=} \iota_x^* \tilde{q}_* \mathcal{C}_{X_{[w]}}[-\dim Y_{[w]}] \\ &= \tilde{q}_* \iota_{q^{-1}(x)}^* \mathcal{C}_{X_{[w]}}[-\dim Y_{[w]}] \\ &= \tilde{q}_* \mathcal{C}_{q^{-1}(x)}. \end{aligned}$$

In the Grothendieck group this implies

$$f(v) \cdot [\mathcal{C}_{\{x\}}] = [\tilde{q}_* \mathcal{C}_{q^{-1}(x)}] \overset{(4.10)}{=} \chi_v(q^{-1}(x))[\mathcal{C}_{\{x\}}].$$

Since $q^{-1}(x) \cong P_1/B \times P_2/B$, we get

$$f(v) = \chi_v(q^{-1}(x)) = \chi_v(P_1/B)\chi_v(P_2/B).$$

Thus (4.21) follows from (4.23). Hence, we obtain

$$\begin{aligned} \chi_v^G(Y_{[w]}) &\overset{(4.21)}{=} \frac{1}{\chi_v(P_1/B)\chi_v(P_2/B)} \chi_v^G(X_{[w]}) \\ &\overset{(4.19),(4.20)}{=} v^{-r_{[w]}} \frac{\chi_v^G(G/B)}{\chi_v(P_1/B)\chi_v(P_2/B)} \sum_{y \in [w]} v^{2l(y)}. \end{aligned}$$

□

We now use the lemma above to compute the graded dimensions of certain Hom-spaces which includes the entries of the matrix Ψ from (4.8) as a special case. Let V be a G-representation and $V^i \subset V$ a P_i-stable subspace for $i = 1, 2$. We can then consider the G-space

$$\tilde{V}^i := G \times^{P_i} V^i = \{(gP_i, v) \in G/P_i \times V \mid v \in gV^i\}. \tag{4.24}$$

This comes with the two projection maps

$$\begin{tikzcd} & \tilde{V}^i \arrow[dl, "\pi_i"'] \arrow[dr, "\mu_i"] & \\ G/P_i & & V \end{tikzcd}$$

and the associated 'Springer sheaf'

$$\mathbf{S}^i := (\mu_i)_* \mathcal{C}_{\tilde{V}^i} \in D^b_{G,c}(V).$$

Remark 4.3.8. *This set-up can be viewed as a parabolic version of the Springer morphisms considered in Section 3.1.2.*

For any $w \in W$ we can define the subspace

$$^{w}V^i := \{\dot{w} \cdot v \mid v \in V^i\} \subset V$$

where $\dot{w} \in N_G(T)$ is a lift of $w \in W$.

Theorem 4.3.9. *We have*

$$\sum_{k\in\mathbb{Z}} \dim \mathrm{Hom}^k_G(\mathbf{S}^1, \mathbf{S}^2)v^{-k} = c_v(V^1, V^2) \sum_{w\in W} v^{2(l(w)+\dim V^1 \cap {}^{w}V^2)}$$

where

$$c_v(V^1, V^2) = v^{-\dim G/B - \dim V^1 - \dim V^2} \frac{\chi^G_v(G/B)}{\chi_v(P_1/B)\chi_v(P_2/B)}.$$

Proof. Consider the Steinberg variety

$$Z := \tilde{V}^1 \times_V \tilde{V}^2.$$

By [24, 8.6], there is an isomorphism

$$\mathrm{Hom}^k_G(\mathbf{S}^1, \mathbf{S}^2) \cong H^G_{d_1+d_2-k}(Z) \tag{4.25}$$

where

$$d_i = \dim \tilde{V}^i = \dim G - \dim P_i + \dim V^i.$$

The orbit partition on $G/P_1 \times G/P_2$ from (4.15) induces a partition

$$Z = \bigsqcup_{[w]\in W_1\backslash W/W_2} Z_{[w]}$$

where $Z_{[w]} := (\pi_1 \times \pi_2)^{-1}(Y_{[w]})$. The projection $Z_{[w]} \to Y_{[w]}$ is a G-equivariant vector bundle with fiber $V^1 \cap {}^{w}V^2$. Hence, $Z_{[w]}$ is smooth and

$$H^G_k(Z_{[w]}) \overset{PD}{=} H^{2d_{[w]}-k}_G(Z_{[w]}) \cong H^{2d_{[w]}-k}_G(Y_{[w]}). \tag{4.26}$$

where

$$d_{[w]} := \dim Z_{[w]} = \dim Y_{[w]} + \dim V^1 \cap {}^{w}V^2.$$

It follows from Lemma 4.3.7 that $H^k_G(Y_{[w]})$ vanishes for k odd, so $H^G_k(Z_{[w]})$ also vanishes for k odd. By Lemma 4.3.1, we get

$$H^G_k(Z) = \bigoplus_{[w]\in W_1\backslash W/W_2} H^G_k(Z_{[w]}). \tag{4.27}$$

Hence, we have

$$\begin{aligned}
(\dagger) &:= \sum_{k\in\mathbb{Z}} \dim \operatorname{Hom}^k_G(\mathbf{S}^1, \mathbf{S}^2) v^{-k} \\
&\overset{(4.25)}{=} \sum_{k\in\mathbb{Z}} \dim H^G_{d_1+d_2-k}(Z) v^{-k} \\
&\overset{(4.27)}{=} \sum_{k\in\mathbb{Z}} \sum_{[w]\in W_1\backslash W/W_2} \dim H^G_{d_1+d_2-k}(Z_{[w]}) v^{-k} \\
&\overset{(4.26)}{=} \sum_{k\in\mathbb{Z}} \sum_{[w]\in W_1\backslash W/W_2} \dim H_G^{2d_{[w]}-d_1-d_2+k}(Y_{[w]}) v^{-k} \\
&= \sum_{k\in\mathbb{Z}} \sum_{[w]\in W_1\backslash W/W_2} v^{2d_{[w]}-d_1-d_2} \dim H^k_G(Y_{[w]}) v^{-k} \\
&\overset{(4.9)}{=} \sum_{[w]\in W_1\backslash W/W_2} v^{2d_{[w]}-d_1-d_2-\dim Y_{[w]}} \chi^G_v(Y_{[w]}) \\
&\overset{4.3.7}{=} \sum_{[w]\in W_1\backslash W/W_2} v^{2d_{[w]}-d_1-d_2-\dim Y_{[w]}-r_{[w]}} \frac{\chi^G_v(G/B)}{\chi_v(P_1/B)\chi_v(P_2/B)} \sum_{y\in[w]} v^{2l(y)} \\
&= \frac{\chi^G_v(G/B)}{\chi_v(P_1/B)\chi_v(P_2/B)} \sum_{w\in W} v^{2d_{[w]}-d_1-d_2-\dim Y_{[w]}-r_{[w]}+2l(w)}.
\end{aligned}$$

Recall that

$$\begin{aligned}
d_{[w]} &= \dim Y_{[w]} + \dim V^1 \cap {}^wV^2 \\
r_{[w]} &= \dim Y_{[w]} + \dim P_1/B + \dim P_2/B - \dim G/B \\
d_i &= \dim G - \dim P_i + \dim V^i.
\end{aligned}$$

Hence, we have

$$2d_{[w]} - d_1 - d_2 - \dim Y_{[w]} - r_{[w]} = 2\dim V^1 \cap {}^wV^2 - \dim G/B - \dim V^1 - \dim V^2$$

and thus

$$\begin{aligned}
(\dagger) &= \frac{\chi_v(G/B)}{\chi_v(P_1/B)\chi_v(P_2/B)} \sum_{w\in W} v^{2\dim V^1\cap {}^wV^2 - \dim G/B - \dim V^1 - \dim V^2 + 2l(w)} \\
&= v^{-\dim G/B - \dim V^1 - \dim V^2} \frac{\chi_v(G/B)}{\chi_v(P_1/B)\chi_v(P_2/B)} \sum_{w\in W} v^{2(l(w)+\dim V^1\cap {}^wV^2)}.
\end{aligned}$$

□

Remark 4.3.10. *The theorem above can be used to compute the graded dimension of some interesting Ext-algebras. For example, KLR algebras can be realised as Ext* *algebras of a direct sum of sheaves of the form* $\mathbf{S}^i$ *as above [66]. Similarly, affine Hecke algebras at a central character (away from roots of unity) also have such a realization [24], though the Ext-algebras are non-equivariant. We will make use of this realization in Section 4.5.2 to compute the dimensions of the simple modules for the affine Hecke algebra of* GL_n*. In both of these examples, the parabolic subgroups* P_i *in the definition of the corresponding sheaves* $\mathbf{S}^i$ *are always Borel subgroups. However, to compute the bilinear form on* U^+ *at monomials, we also need to consider parabolic subgroups that are not Borel subgroups.*

We now apply the formula from the theorem above to the quantum groups setting. Recall for this that $S_{\mathbf{v}} = \prod_{i \in I} S_{\mathbf{v}_i}$ is the Weyl group of $\mathrm{GL}_{\mathbf{v}} = \prod_{i \in I} \mathrm{GL}_{\mathbf{v}_i}$, $B \subset \mathrm{GL}_{\mathbf{v}}$ is fixed a Borel subgroup and $R_{\mathbf{i},\mathbf{a}} \subset R_{\mathbf{v}}$ is the subspace that fixes the unique B-stable flag F^* of type $(\mathbf{i}, \mathbf{a})$. For any sequence $\mathbf{a} = (a_1, a_2, ..., a_m) \in \mathbb{N}^m$ we define

$$[\mathbf{a}]! := [a_1]![a_2]! \cdot ... \cdot [a_m]!.$$

Corollary 4.3.11. *Let* $(\boldsymbol{i}, \boldsymbol{a})$ *and* $(\boldsymbol{i}', \boldsymbol{a}')$ *be of weight* $\boldsymbol{v}$*. Then*

$$(E_{\boldsymbol{i},\boldsymbol{a}}, E_{\boldsymbol{i}',\boldsymbol{a}'}) = v^{-\dim R_{\boldsymbol{i},\boldsymbol{a}} - \dim R_{\boldsymbol{i}',\boldsymbol{a}'}} \frac{\prod_{i \in I} (1 - v^{-2})^{-\boldsymbol{v}_i}}{[\boldsymbol{a}]![\boldsymbol{a}']!} \sum_{w \in S_{\boldsymbol{v}}} v^{2(l(w) + \dim R_{\boldsymbol{i},\boldsymbol{a}} \cap^w R_{\boldsymbol{i}',\boldsymbol{a}'})}.$$

Proof. The parabolics $P_{\mathbf{i},\mathbf{a}}, P_{\mathbf{i},\mathbf{a}'}$ contain the common Borel subgroup B. Hence, we can apply Theorem 4.3.9 to obtain

$$\begin{aligned}
(E_{\mathbf{i},\mathbf{a}}, E_{\mathbf{i}',\mathbf{a}'}) &= \sum_{k \in \mathbb{Z}} \mathrm{Hom}_G^k(\mathbf{S}_{\mathbf{i},\mathbf{a}}, \mathbf{S}_{\mathbf{i}',\mathbf{a}'}) v^{-k} \\
&= v^{-s} \frac{\chi_v^{\mathrm{GL}_{\mathbf{v}}}(\mathrm{GL}_{\mathbf{v}}/B)}{\chi_v(P_{\mathbf{i},\mathbf{a}}/B)\chi_v(P_{\mathbf{i}',\mathbf{a}'}/B)} \sum_{w \in S_{\mathbf{v}}} v^{2(l(w) + \dim R_{\mathbf{i},\mathbf{a}} \cap^w R_{\mathbf{i}',\mathbf{a}'})}
\end{aligned}$$

where

$$s = \dim \mathrm{GL}_{\mathbf{v}}/B + \dim R_{\mathbf{i},\mathbf{a}} + \dim R_{\mathbf{i},\mathbf{a}}.$$

Note that the Levi factor of $P_{\mathbf{i},\mathbf{a}}$ is isomorphic to $\mathrm{GL}_{\mathbf{a}_1} \times ... \times \mathrm{GL}_{\mathbf{a}_m}$. The Poincaré polynomials for the flag variety are well-known:

$$\begin{aligned}
\chi_v^{\mathrm{GL}_{\mathbf{v}}}(\mathrm{GL}_{\mathbf{v}}/B) &= v^{\dim \mathrm{GL}_{\mathbf{v}}/B} \chi_v^B(\mathrm{pt}) = v^{\dim \mathrm{GL}_{\mathbf{v}}/B} \prod_{i \in I} (1 - v^{-2})^{-\mathbf{v}_i} \\
\chi_v(P_{\mathbf{i},\mathbf{a}}/B) &= [\mathbf{a}]! \\
\chi_v(P_{\mathbf{i}',\mathbf{a}'}/B) &= [\mathbf{a}']!.
\end{aligned}$$

Thus we have

$$
\begin{aligned}
(E_{\mathbf{i},\mathbf{a}}, E_{\mathbf{i}',\mathbf{a}'}) &= v^{-s} \frac{v^{\dim \mathrm{GL}_{\mathbf{v}}/B} \prod_{i\in I}(1-v^{-2})^{-\mathbf{v}_i}}{[\mathbf{a}]![\mathbf{a}']!} \sum_{w\in S_{\mathbf{v}}} v^{2(l(w)+\dim R_{\mathbf{i},\mathbf{a}} \cap {}^w R_{\mathbf{i}',\mathbf{a}'})} \\
&= v^{-\dim R_{\mathbf{i},\mathbf{a}} - \dim R_{\mathbf{i}',\mathbf{a}'}} \frac{\prod_{i\in I}(1-v^{-2})^{-\mathbf{v}_i}}{[\mathbf{a}]![\mathbf{a}']!} \sum_{w\in S_{\mathbf{v}}} v^{2(l(w)+\dim R_{\mathbf{i},\mathbf{a}} \cap {}^w R_{\mathbf{i}',\mathbf{a}'})}.
\end{aligned}
$$

□

We can now combine Corollary 4.1.3 and Corollary 4.3.11 to obtain the following algorithm that computes the canonical basis.

Algorithm 4.3.12. *Input:* $\boldsymbol{v}$ *(weight vector), Output:* P, Q *(change of basis matrix between* $\boldsymbol{B}_{\mathbf{v}}^{\mathrm{std}}$ *and* $\boldsymbol{B}_{\mathbf{v}}^{\mathrm{can}}$ *resp.* $\boldsymbol{B}_{\mathbf{v}}^{\mathrm{can}}$ *and* $\boldsymbol{B}_{\mathbf{v}}^{\mathrm{mon}}$*).*

1. *Compute the matrix* Ψ *using the formula from Corollary 4.3.11;*

2. *Compute the decomposition* $\Psi = LDL^T$ *using the Gram-Schmidt procedure;*

3. *Compute the decomposition* $L = QP$ *using row elimination.*

4.4 Examples in type A

In order to be able to execute Algorithm 4.3.12, we need to explicitly compute:

- the sequences $\mathbf{i}$ and $\mathbf{a}_c$ from Lemma 4.3.6 that give rise to the monomial basis

$$\mathbf{B}^{\mathrm{mon}} = \{E_{\mathbf{i},\mathbf{a}_{\mathbf{c}}} \mid \mathbf{c} \in \mathrm{KP}(\mathbf{v})\};$$

- the partial order $\leq$ on $\mathrm{KP}(\mathbf{v})$, i.e. the dimensions of the orbits $\mathcal{O}_{\mathbf{c}}$.

The orbit dimensions can be computed using Lusztig's formula [48, Proposition 6.6]. Alternatively, they can also be computed from the sequences $\mathbf{i}$ and $\mathbf{a}_c$ using that

$$\dim \mathcal{O}_{\mathbf{c}} = \dim \tilde{R}_{\mathbf{i},\mathbf{a}_{\mathbf{c}}} = \dim G - \dim P_{\mathbf{i},\mathbf{a}_{\mathbf{c}}} + R_{\mathbf{i},\mathbf{a}_{\mathbf{c}}} \tag{4.28}$$

by Lemma 4.3.6. We now recall an explicit description of the sequences $\mathbf{i}, \mathbf{a}_{\mathbf{c}}$ from [57] for the equioriented type A quiver

$$A_n = \bullet \longrightarrow \bullet \longrightarrow \dots \longrightarrow \bullet \tag{4.29}$$

with n vertices and $n-1$ arrows (see also [2, §4] for an alternative combinatorial description using triangular arrays). Let $U^+ = U^+(A_n)$ be the associated positive quantum group. The set of positive roots is given by

$$\Phi^+ = \{\alpha_{ij} = \mathbf{e}_i + \mathbf{e}_{i+1} + ... + \mathbf{e}_j \mid i \leq j\}$$

where $\mathbf{e}_1, ..., \mathbf{e}_n$ are the simple roots labelling the quiver from left to right. The reduced expression

$$w_0 = s_n(s_{n-1}s_n)...(s_i s_{i+1}...s_n)...(s_1 s_2 ... s_n)$$

is adapted to A_n in the sense of [48, 4.7]. Let

$$\mathbf{i} = (n, n-1, n, ..., i, i+1, ..., n, ...1, 2, ..., n).$$

For any Kostant partition $\mathbf{c} = (m_{ij})_{i \leq j} \in \mathrm{KP}(\mathbf{v})$ set

$$\mathbf{a_c} = (m_{nn}, m_{n-1,n-1} + m_{n-1,n}, m_{n-1,n}, ..., m_{1,1} + m_{1,2} + ... + m_{1,n}, ..., m_{1,n}).$$

Then by [57] the $\mathbf{a_c}$ satisfy the conditions from Lemma 4.3.6. Hence,

$$\mathbf{B}_{\mathbf{v}}^{\mathrm{mon}} = \{E_{\mathbf{i},\mathbf{a_c}} \mid \mathbf{c} \in \mathrm{KP}(\mathbf{v})\}$$

is a monomial basis of $U_{\mathbf{v}}^+$. We now compute the canonical basis of $U_{\mathbf{v}}^+$ in two examples.

4.4.1 Example: $\mathbf{v} = (2, 2)$

Let $n = 2$ and $\mathbf{v} = (2, 2)$. The order on the set of positive roots is given by

$$\Phi^+ = \{\mathbf{e}_2 < \mathbf{e}_1 + \mathbf{e}_2 < \mathbf{e}_1\}.$$

There are three Kostant partitions of $\mathbf{v}$:

$$\mathbf{c}_1 = (2, 0, 2), \quad \mathbf{c}_2 = (1, 1, 1), \quad \mathbf{c}_3 = (0, 2, 0).$$

We have

$$\mathbf{i} = (2, 1, 2)$$

and

$$\mathbf{a}_{\mathbf{c}_1} = (2, 2, 0), \quad \mathbf{a}_{\mathbf{c}_2} = (1, 2, 1), \quad \mathbf{a}_{\mathbf{c}_3} = (0, 2, 2).$$

Let $B \subset \mathrm{GL}_2$ be the standard (geometric) Borel subgroup consisting of lower triangular matrices and consider the corresponding Borel $B \times B \subset \mathrm{GL}_2 \times \mathrm{GL}_2 = \mathrm{GL}_{\mathbf{v}}$.

The space of quiver representations $R_{\mathbf{v}}$ can be identified with the set of 2×2 matrices $\left(\begin{smallmatrix} * & * \\ * & * \end{smallmatrix}\right)$. The corresponding spaces $R_{\mathbf{i},\mathbf{a_c}} \subset R_{\mathbf{v}}$ are given by

$$R_{\mathbf{i},\mathbf{a}_{\mathbf{c}_1}} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \quad R_{\mathbf{i},\mathbf{a}_{\mathbf{c}_2}} = \left(\begin{smallmatrix} 0 & 0 \\ * & * \end{smallmatrix}\right), \quad R_{\mathbf{i},\mathbf{a}_{\mathbf{c}_3}} = \left(\begin{smallmatrix} * & * \\ * & * \end{smallmatrix}\right).$$

The corresponding parabolic subgroups of $\mathrm{GL}_{\mathbf{v}}$ are

$$P_{\mathbf{i},\mathbf{a}_{\mathbf{c}_1}} = \mathrm{GL}_{\mathbf{v}}, \quad P_{\mathbf{i},\mathbf{a}_{\mathbf{c}_2}} = \mathrm{GL}_2 \times B, \quad P_{\mathbf{i},\mathbf{a}_{\mathbf{c}_3}} = \mathrm{GL}_{\mathbf{v}}.$$

Using (4.28), we can compute the orbit dimensions

$$\dim \mathcal{O}_{\mathbf{c}_1} = 0, \quad \dim \mathcal{O}_{\mathbf{c}_2} = 3, \quad \dim \mathcal{O}_{\mathbf{c}_3} = 4$$

and hence the order on Kostant partitions is

$$\mathbf{c}_1 < \mathbf{c}_2 < \mathbf{c}_3.$$

We can now evaluate the formula from Corollary 4.3.11 to compute the entries of the matrix Ψ. For example, we have

$$\begin{aligned}
(E_{\mathbf{i},\mathbf{a}_{\mathbf{c}_2}}, E_{\mathbf{i},\mathbf{a}_{\mathbf{c}_2}}) &= v^{-\dim R_{\mathbf{i},\mathbf{a}_{\mathbf{c}_2}} - \dim R_{\mathbf{i},\mathbf{a}_{\mathbf{c}_2}}} \tfrac{\prod_i (1-v^{-2})^{-\mathbf{v}_i}}{[\mathbf{a}_{\mathbf{c}_2}]![\mathbf{a}_{\mathbf{c}_2}]!} \sum_{w \in S_2 \times S_2} v^{2(l(w) + \dim R_{\mathbf{i},\mathbf{a}_{\mathbf{c}_2}} \cap^w R_{\mathbf{i},\mathbf{a}_{\mathbf{c}_2}})} \\
&= v^{-2-2} \tfrac{(1-v^{-2})^{-4}}{(v+v^{-1})(v+v^{-1})} \big(v^{2(0+2)} + v^{2(1+2)} + v^{2(1+0)} + v^{2(2+0)}\big) \\
&= (1-v^{-2})^{-4} \\
&= \tfrac{1+2v^{-2}+v^{-4}}{(1-v^{-2})^2(1-v^{-4})^2}.
\end{aligned} \tag{4.30}$$

More generally, we have

$$\Psi = \tfrac{1}{(1-v^{-2})^2(1-v^{-4})^2} \begin{pmatrix} 1 & v^{-1}+v^{-3} & v^{-4} \\ v^{-1}+v^{-3} & 1+2v^{-2}+v^{-4} & v^{-1}+v^{-3} \\ v^{-4} & v^{-1}+v^{-3} & 1 \end{pmatrix}.$$

The matrices L and D in the corresponding LDLT decomposition $\Psi = LDL^T$ are

$$\begin{aligned}
L &= \begin{pmatrix} 1 & 0 & 0 \\ v^{-1}+v^{-3} & 1 & 0 \\ v^{-4} & v^{-1} & 1 \end{pmatrix} \\
D &= \tfrac{1}{(1-v^{-2})^2(1-v^{-4})^2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+v^{-2}-v^{-4}-v^{-6} & 0 \\ 0 & 0 & 1+v^{-2}-v^{-4}-v^{-6} \end{pmatrix}.
\end{aligned}$$

Note that L already has coefficients in $v^{-1}\mathbb{Z}[v^{-1}]$ below the diagonal. Hence the change of basis matrix between $\mathbf{B}_{\mathbf{v}}^{\mathrm{std}}$ and $\mathbf{B}_{\mathbf{v}}^{\mathrm{can}}$ is

$$P = L = \begin{pmatrix} 1 & 0 & 0 \\ v^{-1}+v^{-3} & 1 & 0 \\ v^{-4} & v^{-1} & 1 \end{pmatrix}.$$

The matrix Q is just the identity matrix which corresponds to the fact that

$$\mathbf{B}_{\mathbf{v}}^{\mathrm{can}} = \mathbf{B}_{\mathbf{v}}^{\mathrm{mon}} = \{E_2^{(2)}E_1^{(2)}, E_1E_2^{(2)}E_1, E_1^{(2)}E_2^{(2)}\}.$$

Remark 4.4.1. *The matrix Ψ can also be computed recursively from [49, 1.2.3] or using the closed formula in [56] derived from this recursion. To illustrate the difference to our formula, we briefly explain how to compute Ψ with the formula from [56]. For any $a \in \mathbb{N}$ let*

$$(\{a\})^{-1} := \prod_{s=1}^{a} (1 - v^{-2s})^{-1}.$$

The formula from [56, Theorem 2.2] is then

$$(E_{\boldsymbol{i},\boldsymbol{a}}, E_{\boldsymbol{i}',\boldsymbol{a}'}) = \sum_{z \in \mathcal{Z}_{\boldsymbol{i}',\boldsymbol{a}'}^{\boldsymbol{i},\boldsymbol{a}}} v^{Q(z)} \prod_{m,p} (\{z_{mp}\})^{-1} \tag{4.31}$$

where $\mathcal{Z}_{\boldsymbol{i}',\boldsymbol{a}'}^{\boldsymbol{i},\boldsymbol{a}}$ is a certain finite set of $|\boldsymbol{i}'| \times |\boldsymbol{i}|$ matrices with entries in $\mathbb{N}$ and $Q(z)$ is a certain integer which can be computed from the entries of z. For example, one can check that

$$\mathcal{Z}_{\boldsymbol{i},\boldsymbol{a}_{c_2}}^{\boldsymbol{i},\boldsymbol{a}_{c_2}} = \left\{ \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right) \right\}$$

and

$$Q\left(\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)\right) = 0, \quad Q\left(\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)\right) = -2.$$

Hence, we get

$$\begin{aligned}(E_{\boldsymbol{i},\boldsymbol{a}_{c_2}}, E_{\boldsymbol{i},\boldsymbol{a}_{c_2}}) &\overset{(4.31)}{=} v^0(\{1\})^{-2}(\{2\})^{-1} + v^{-2}(\{1\})^{-2}(\{2\})^{-1} \\ &= \tfrac{1}{(1-v^{-2})^3(1-v^{-4})} + v^{-2}\tfrac{1}{(1-v^{-2})^3(1-v^{-4})} \\ &= (1-v^{-2})^{-4}\end{aligned}$$

which is the same result as in (4.30).

4.4.2 Example: $\mathbf{v} = (1, 2, 1)$

Let $n = 3$ and $\mathbf{v} = (1, 2, 1)$. The order on the set of positive roots is

$$\Phi^+ = \{\mathbf{e}_3 < \mathbf{e}_2 + \mathbf{e}_3 < \mathbf{e}_2 < \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 < \mathbf{e}_1 + \mathbf{e}_2 < \mathbf{e}_1\}.$$

The Kostant partitions of dimension $\mathbf{v}$ are:

$$(1,0,2,0,0,1), \quad (1,0,1,0,1,0), \quad (0,1,1,0,0,1), \quad (0,1,0,0,1,0), \quad (0,0,1,1,0,0). \tag{4.32}$$

We have

$$\mathbf{i} = (3, 2, 3, 1, 2, 3)$$

and the corresponding sequences $a_{\mathbf{c}}$ are

$$(1,2,0,1,0,0), \quad (1,1,0,1,1,0), \quad (0,2,1,1,0,0), \quad (0,1,1,1,1,0), \quad (0,1,0,1,1,1).$$

Let $B \subset \mathrm{GL}_{\mathbf{v}}$ be the Borel consisting of all lower triangular matrices. Then the corresponding parabolic subgroups $P_{\mathbf{i},\mathbf{a_c}}$ are

$$\mathrm{GL}_{\mathbf{v}}, \quad B, \quad \mathrm{GL}_{\mathbf{v}}, \quad B, \quad B.$$

The corresponding spaces $R_{\mathbf{i},\mathbf{a_c}} \subset R_{\mathbf{v}} = \left(\left(\begin{smallmatrix} * \\ * \end{smallmatrix}\right), \left(\begin{smallmatrix} * & * \end{smallmatrix}\right)\right)$ are

$$\left(\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \end{smallmatrix}\right)\right), \quad \left(\left(\begin{smallmatrix} 0 \\ * \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \end{smallmatrix}\right)\right), \quad \left(\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} * & * \end{smallmatrix}\right)\right), \quad \left(\left(\begin{smallmatrix} 0 \\ * \end{smallmatrix}\right), \left(\begin{smallmatrix} * & 0 \end{smallmatrix}\right)\right), \quad \left(\left(\begin{smallmatrix} * \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} * & * \end{smallmatrix}\right)\right). \tag{4.33}$$

It follows from (4.28) that the corresponding orbit dimensions are

$$0, \quad 2, \quad 2, \quad 3, \quad 4.$$

Thus the Kostant partitions in (4.32) are already ordered with respect to the partial order. The entries of the matrix Ψ can now be computed explicitly. For example, taking

$$\mathbf{c}_3 = (0,1,1,0,0,1), \quad \mathbf{c}_5 = (0,0,1,1,0,0)$$

we get

$$\begin{aligned}
\Psi_{\mathbf{c}_3,\mathbf{c}_5} &= v^{-\dim R_{\mathbf{i},\mathbf{a}_3} - \dim R_{\mathbf{i},\mathbf{a}_5}} \tfrac{\prod_i (1-v^{-2})^{-\mathbf{v}_i}}{[\mathbf{a}_{\mathbf{c}_3}]![\mathbf{a}_{\mathbf{c}_5}]!} \left(v^{2\dim R_{\mathbf{i},\mathbf{a}_{\mathbf{c}_3}} \cap R_{\mathbf{i},\mathbf{a}_{\mathbf{c}_5}}} + v^{2+2\dim R_{\mathbf{i},\mathbf{a}_{\mathbf{c}_3}} \cap^s R_{\mathbf{i},\mathbf{a}_{\mathbf{c}_5}}}\right) \\
&= v^{-5} \tfrac{(1-v^{-2})^{-4}}{[2]!\cdot 1} (v^4 + v^6) \\
&= (1-v^{-2})^{-4} \\
&= \tfrac{1+v^{-2}}{(1-v^{-2})^{-3}(1-v^{-4})}.
\end{aligned}$$

More generally, we have

$$\Psi = \tfrac{1}{(1-v^{-2})^3(1-v^{-4})} \begin{pmatrix} 1 & 1+v^{-2} & v^{-2} & v^{-1}+v^{-3} & v^{-2}+v^{-4} \\ 1+v^{-2} & 2+2v^{-2} & v^{-2}+v^{-4} & 2v^{-1}+2v^{-3} & 2v^{-2}+2v^{-4} \\ v^{-2} & v^{-2}+v^{-4} & 1 & v^{-1}+v^{-3} & 1+v^{-2} \\ v^{-1}+v^{-3} & 2v^{-1}+2v^{-3} & v^{-1}+v^{-3} & 1+2v^{-2}+v^{-4} & 2v^{-1}+2v^{-3} \\ v^{-2}+v^{-4} & 2v^{-2}+2v^{-4} & 1+v^{-2} & 2v^{-1}+2v^{-3} & 2+2v^{-2} \end{pmatrix}. \tag{4.34}$$

The decomposition $\Psi = LDL^T$ is given by

$$\begin{aligned}
L &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1+v^{-2} & 1 & 0 & 0 & 0 \\ v^{-2} & 0 & 1 & 0 & 0 \\ v^{-1}+v^{-3} & v^{-1} & v^{-1} & 1 & 0 \\ v^{-2}+v^{-4} & v^{-2} & 1+v^{-2} & v^{-1} & 1 \end{pmatrix} \\
D &= \tfrac{1}{(1-v^{-2})^3(1-v^{-4})} \begin{pmatrix} 1 & & & & \\ & 1-v^{-4} & & & \\ & & 1-v^{-4} & & \\ & & & 1-v^{2}-v^{-4}+v^{-6} & \\ & & & & 1-v^{-2}-v^{-4}+v^{-6} \end{pmatrix}.
\end{aligned}$$

The matrix L can be decomposed further as

$$L = QP = \begin{pmatrix} 1&0&0&0&0 \\ 1&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&1&0&1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ v^{-2} & 1 & 0 & 0 & 0 \\ v^{-2} & 0 & 1 & 0 & 0 \\ v^{-1}+v^{-3} & v^{-1} & v^{-1} & 1 & 0 \\ v^{-4} & v^{-2} & v^{-2} & v^{-1} & 1 \end{pmatrix}. \tag{4.35}$$

Remark 4.4.2. *The matrix P is also computed in the context of (graded) affine Hecke algebras in [25, 4.2] using the slightly different normalisation*

$$\tilde{P} = (v^{\dim \mathcal{O}_{c'} - \dim \mathcal{O}_c} p_{c',c}(v))_{c,c' \in \mathrm{KP}(v)} = \begin{pmatrix} 1 & 1 & 1 & 1+v^2 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The relation between the quantum groups setting and the affine Hecke algebra setting will be discussed in more detail in the next section.

4.5 Applications to the representation theory of affine Hecke algebras

4.5.1 Affine Hecke algebras and quivers

Let $\mathcal{H}_q^{\mathrm{aff}}$ be the affine Hecke algebra of GL_n (i.e. the affine Hecke algebra associated to the root datum of GL_n). Consider a semisimple element $(s, q_0) \in \mathrm{GL}_n \times \mathbb{G}_m$ where q_0 is not a root of unity. Recall from Section 2.4 that the affine Hecke algebra at the central character $\chi_{(s,q_0)}$ can be studied using the map

$$\mu^{(s,q_0)} : \tilde{\mathcal{N}}^{(s,q_0)} \to \mathcal{N}^{(s,q_0)}.$$

For any $x \in \mathcal{N}^{(s,q_0)}$ let

$$\mathcal{O}_x := \mathrm{GL}_n(s) \cdot x \subset \mathcal{N}^{(s,q_0)}$$

be the corresponding orbit and let

$$M_x := H_*((\mu^{(s,q_0)})^{-1}(x)) = H_*(\mathcal{B}_x^s)$$

be the Borel-Moore homology of the corresponding fiber. It can be shown that the simultaneous centralizer $\mathrm{GL}_n(s,x)$ is always connected. Hence, the component groups $A(s,x)$ are all trivial and thus the simple equivariant perverse sheaves on $\mathcal{N}^{(s,q_0)}$ are of the form

$$\mathrm{IC}_{\mathcal{O}_x} := \mathrm{IC}(\mathcal{O}_x, \mathrm{triv}).$$

In particular, in the GL_n case, the results from Theorem 2.5.4 take the following simpler form.

Theorem 4.5.1. *[17, 24, 41, 67]*

(i) There is a canonical $\mathcal{H}_q^{\mathrm{aff}}$-module structure on M_x with central character $\chi_{(s,q_0)}$.

(ii) If $\mathcal{O}_x = \mathcal{O}_{x'}$, then $M_x \cong M_{x'}$ as $\mathcal{H}_q^{\mathrm{aff}}$-modules.

(iii) The module M_x has a unique simple quotient L_x. This induces a bijection

$$\{\mathrm{GL}_n(s)\text{-}\textit{orbits on } \mathcal{N}^{(s,q_0)}\} \overset{1:1}{\leftrightarrow} \mathrm{Irr}(\mathcal{H}^{\mathrm{aff}}_{(s,q_0)})$$
$$\mathcal{O}_x \mapsto L_x.$$

(iv) The composition multiplicities of the M_x are given by the formula

$$[M_x : L_y] = \sum_{k\in\mathbb{Z}} \dim H^k(\iota_x^! \,\mathrm{IC}_{\mathcal{O}_y}) = \sum_{k\in\mathbb{Z}} \dim H^k(\iota_x^* \,\mathrm{IC}_{\mathcal{O}_y})$$

where $\iota_x : \{x\} \hookrightarrow \mathcal{N}^{(s,q_0)}$ is the inclusion.

It turns out that $\mathcal{N}^{(s,q_0)}$ is just a space of quiver representations in disguise. Assume for simplicity that s is a diagonal matrix of the form

$$s = s_{e,\mathbf{v}} := e \cdot \begin{pmatrix} \mathrm{Id}_{\mathbf{v}_1} & & & \\ & q_0\,\mathrm{Id}_{\mathbf{v}_2} & & \\ & & \ddots & \\ & & & q_0^{k-1}\,\mathrm{Id}_{\mathbf{v}_k} \end{pmatrix} \tag{4.36}$$

for some integers $\mathbf{v}_1, ..., \mathbf{v}_m \in \mathbb{N}$ and $e \in \mathbb{C}^\times$ (see Remark 4.5.5 for the case of a general semisimple s). Then we have

$$\begin{aligned} \mathrm{GL}_n(s) &= \begin{pmatrix} \mathrm{GL}_{\mathbf{v}_1} & & & \\ & \mathrm{GL}_{\mathbf{v}_2} & & \\ & & \ddots & \\ & & & \mathrm{GL}_{\mathbf{v}_m} \end{pmatrix} \\ \mathcal{N}^{(s,q_0)} &= \begin{pmatrix} 0_{\mathbf{v}_1\times\mathbf{v}_1} & & & & \\ \mathrm{Hom}(\mathbb{C}^{\mathbf{v}_1},\mathbb{C}^{\mathbf{v}_2}) & 0_{\mathbf{v}_2\times\mathbf{v}_2} & & & \\ & \mathrm{Hom}(\mathbb{C}^{\mathbf{v}_2},\mathbb{C}^{\mathbf{v}_3}) & 0_{\mathbf{v}_3\times\mathbf{v}_3} & & \\ & & \ddots & \ddots & \\ & & & \mathrm{Hom}(\mathbb{C}^{\mathbf{v}_{m-1}},\mathbb{C}^{\mathbf{v}_m}) & 0_{\mathbf{v}_m\times\mathbf{v}_m} \end{pmatrix}. \end{aligned} \tag{4.37}$$

Thus, we can identify

$$\begin{aligned} \mathrm{GL}_n(s) &\cong \mathrm{GL}_{\mathbf{v}} \\ \mathcal{N}^{(s,q_0)} &\cong R_{\mathbf{v}}(A_m) \end{aligned} \tag{4.38}$$

where A_m is the quiver from (4.29). Hence, the sets of simple and standard modules with central character $\chi_{(s,q_0)}$ are indexed by Kostant partitions:

$$\{L_{\mathbf{c}} \mid \mathbf{c} \in \mathrm{KP}(\mathbf{v})\}, \quad \{M_{\mathbf{c}} \mid \mathbf{c} \in \mathrm{KP}(\mathbf{v})\}.$$

Let U^+ be the positive part of the quantum group associated to A_m.

Lemma 4.5.2. *We have*

$$[M_{\boldsymbol{c}} : L_{\boldsymbol{c}'}] = p_{\boldsymbol{c}',\boldsymbol{c}}(1)$$

where $P = (p_{\boldsymbol{c},\boldsymbol{c}'}(v))_{\boldsymbol{c},\boldsymbol{c}' \in \mathrm{KP}(\boldsymbol{v})}$ is the change of basis matrix P between $\boldsymbol{B}^{\mathrm{can}}_{\mathbf{v}}$ and $\boldsymbol{B}^{\mathrm{std}}_{\mathbf{v}}$ in $U^+_{\boldsymbol{v}}$.

Proof. By Theorem 4.5.1 and Proposition 4.3.3 we have

$$[M_{\mathbf{c}} : L_{\mathbf{c}'}] = \sum_{k \in \mathbb{Z}} \dim H^k(\iota_x^* \operatorname{IC}_{\mathbf{c}'}) = p_{\mathbf{c}',\mathbf{c}}(1)$$

where $\iota_x : \{x\} \hookrightarrow \mathcal{O}_{\mathbf{c}}$ is any base point. $\square$

Corollary 4.5.3. *The multiplicities $[M_{\boldsymbol{c}} : L_{\boldsymbol{c}'}]$ can be computed with Algorithm 4.3.12.*

Example 4.5.4. *Let $\boldsymbol{v} = (1,2,1)$ and consider a semisimple element of the form*

$$s = s_{e,\boldsymbol{v}} = \operatorname{Diag}(e, eq_0, eq_0, eq_0^2) \in \mathrm{GL}_4 .$$

Then

$$\mathrm{GL}_4(s) = \mathrm{GL}_{(1,2,1)}, \quad \mathcal{N}^{(s,q_0)} = R_{(1,2,1)}$$

is the geometry considered in Section 4.4.2. In particular, there are five orbits and hence $\mathcal{H}^{\mathrm{aff}}_q$ has five simple (resp. standard) modules

$$L_1, ..., L_5, \quad M_1, ..., M_5$$

with central character $\chi_{(s,q_0)}$ Recall from (4.35) *that the base change of basis matrix P between $\boldsymbol{B}^{\mathrm{std}}_{\mathbf{v}}$ and $\boldsymbol{B}^{\mathrm{can}}_{\mathbf{v}}$ in $U^+_{\boldsymbol{v}}$ is given by*

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ v^{-2} & 1 & 0 & 0 & 0 \\ v^{-2} & 0 & 1 & 0 & 0 \\ v^{-1}+v^{-3} & v^{-1} & v^{-1} & 1 & 0 \\ v^{-4} & v^{-2} & v^{-2} & v^{-1} & 1 \end{pmatrix}$$

Hence, we get

$$[M_i : L_j] = p_{j,i}(1) = \begin{cases} 0 & i > j \\ 0 & i = 2, j = 3 \\ 2 & i = 1, j = 4 \\ 1 & \textit{otherwise.} \end{cases}$$

Remark 4.5.5. *For a general semisimple pair $(s, q_0) \in \mathrm{GL}_n \times \mathbb{C}^\times$ such that q_0 is not a root of unity, one can still show that there exist integers $m_1, ..., m_k$ and sequences $\boldsymbol{v}^1, ..., \boldsymbol{v}^k$ (where $\boldsymbol{v}^i \in \mathbb{N}^{m_i}$) such that*

$$\begin{aligned} \mathrm{GL}_n(s) &\cong \mathrm{GL}_{\boldsymbol{v}^1} \times ... \times \mathrm{GL}_{\boldsymbol{v}^k} \\ \mathcal{N}^{(s,q_0)} &\cong R_{\boldsymbol{v}^1}(A_{m_1}) \times ... \times R_{\boldsymbol{v}^k}(A_{m_k}). \end{aligned} \tag{4.39}$$

In fact, we may conjugate s into a diagonal matrix of the form

$$s = \mathrm{Diag}(s_{e_1,\boldsymbol{v}^1}, s_{e_2,\boldsymbol{v}^2}, ..., s_{e_k,\boldsymbol{v}^k})$$

for some $e_i \in \mathbb{C}^\times$ and $\boldsymbol{v}^i \in \mathbb{N}^{m_i}$ where $s_{e_i,\boldsymbol{v}^i}$ is as in (4.36)*. Moreover, we may assume that*

$$e_i/e_j \notin \{q_0^{m_i}, q_0^{m_i-1}, ..., q_0^{-m_j}\} \quad \textit{for all } i \neq j$$

since otherwise $s_{e_i,\boldsymbol{v}^i}$ and $s_{e_j,\boldsymbol{v}^j}$ can be combined into a single block. Then one can construct the isomorphisms in (4.39) *by the same argument as in* (4.37)*. When q_0 is a root of unity, the space $\mathcal{N}^{(s,q_0)}$ is related to representations of the cyclic quiver and affine Schubert varieties [48, §11].*

4.5.2 Dimensions of simple modules

We now want to explain how to compute the dimensions of the simple $\mathcal{H}_q^{\mathrm{aff}}$-modules with central character $\chi_{(s,q_0)}$ for $s = s_{e,\mathbf{v}}$ as in (4.36) and $q_0 \in \mathbb{C}^\times$ not a root of unity (the case of a general semisimple element s can be reduced to this case as in Remark 4.5.5). Using the identification from (4.38), we can view $\mathbf{S}^{(s,q_0)}$ as an element of $D^b_{\mathrm{GL}_{\mathbf{v}},c}(R_{\mathbf{v}}(A_m))$. By the decomposition theorem, we have

$$\mathbf{S}^{(s,q_0)} = \sum_{\mathbf{c} \in \mathrm{KP}(\mathbf{v})} f'_{\mathbf{c}}(v) \cdot \mathrm{IC}_{\mathbf{c}}$$

for some $f'_{\mathbf{c}}(v) \in \mathbb{Z}[v, v^{-1}]$. By Corollary 2.3.3 the dimensions of the simple $\mathcal{H}_q^{\mathrm{aff}}$-modules then have the following geometric interpretation.

Lemma 4.5.6. *[24, 8.6.12] We have* $\dim L_{\boldsymbol{c}} = f'_{\boldsymbol{c}}(1)$*.*

Thus, to compute the dimensions of simple modules, it is sufficient to compute the polynomials $f'_{\mathbf{c}}(v)$. For this we need to analyse the geometry of $\tilde{\mathcal{N}}^{(s,q_0)}$ in a bit more detail. Let $B \subset \mathrm{GL}_n$ be a Borel subgroup containing s, $T \subset B$ a maximal torus containing s and $W = S_n$ the corresponding Weyl group. Then $B(s)$ is a Borel subgroup of $\mathrm{GL}_n(s)$ [24, 8.8.7]. The following lemma can be viewed as a refinement of Corollary 3.5.2.

Lemma 4.5.7. *The connected components of $\tilde{\mathcal{N}}^{(s,q_0)}$ are of the form*

$$\tilde{\mathcal{N}}^{(s,q_0),w} := G(s) \times^{B(s)} ({}^w\mathfrak{n} \cap \mathcal{N}^{(s,q_0)})$$

where w runs through the set ${}^{\boldsymbol{v}}S_n$ of minimal (right) coset representatives in $S_{\boldsymbol{v}} \backslash S_n$.

Proof. Let $\mathcal{B} = \mathrm{GL}_n / B$ be the flag variety of GL_n and $\mathcal{B}^s$ the variety of s-fixed points, i.e. the variety of all Borel subgroups containing s. Let

$$\pi^{(s,q_0)} : \tilde{\mathcal{N}}^{(s,q_0)} \to \mathcal{B}^s$$

be the projection. By [24, 8.8.7] the connected components of $\mathcal{B}^s$ are all isomorphic to the flag variety $\mathrm{GL}_n(s)/B(s)$ of $\mathrm{GL}_n(s)$. Thus each connected component C of $\mathcal{B}^s$ contains a unique Borel subgroup B' whose stabilizer in $G(s)$ is $B(s)$, i.e. $B'(s) = B(s)$. By [24, 8.8.9] the connected components of $\tilde{\mathcal{N}}^{(s,q_0)}$ are of the form

$$(\pi^{(s,q_0)})^{-1}(C) \cong \mathrm{GL}_n(s) \times^{B(s)} (\mathfrak{n}' \cap \mathcal{N}^{(s,q_0)}).$$

Thus, to complete the proof, it remains to show that

$$\{B' \in \mathcal{B}^s \mid B'(s) = B(s)\} = \{{}^wB \mid w \in {}^{\mathbf{v}}S_n\}. \tag{4.40}$$

To see this, note that any B' with $B'(s) = B(s)$ contains the maximal torus $T \subset B(s)$ and thus $B' = {}^wB$ for some $w \in S_n$. Let Φ be the root system of GL_n and let

$$\Phi_{\mathbf{v}} := \{\alpha \in \Phi \mid \alpha(s) = 1\}.$$

This a subrootsystem of Φ whose Weyl group is the parabolic subgroup $S_{\mathbf{v}} \subset S_n$. The set of non-zero roots of $B'(s) = {}^wB \cap \mathrm{GL}_n(s)$ is

$$w(\Phi^+) \cap \Phi_{\mathbf{v}}.$$

Thus, we have $B'(s) = B(s)$ if and only if

$$\begin{aligned} &w(\Phi^+) \cap \Phi_{\mathbf{v}} = \Phi^+ \cap \Phi_{\mathbf{v}} \\ \Leftrightarrow &\Phi^+ \cap \Phi_{\mathbf{v}} \subset w(\Phi^+) \\ \Leftrightarrow &w^{-1}(\Phi^+ \cap \Phi_{\mathbf{v}}) \subset \Phi^+. \end{aligned}$$

This is equivalent to w^{-1} being a minimal (left) coset representative in $S_n/S_{\mathbf{v}}$ or to w being a minimal (right) coset representative in $S_{\mathbf{v}} \backslash S_n$. Hence, we have verified (4.40) completing the proof. $\square$

For any $w \in {}^{\mathbf{v}}S_n$ let

$$\mathbf{S}^w := (\mu^{(s,q_0)})_* \mathcal{C}_{\tilde{\mathcal{N}}^{(s,q_0)},w} \in D^b_{\mathrm{GL}_{\mathbf{v}},c}(R_{\mathbf{v}}(A_m))$$

and write

$$\mathbf{S}^w = \sum_{\mathbf{c} \in \mathrm{KP}(\mathbf{v})} f^w_{\mathbf{c}}(v) \cdot \mathrm{IC}_{\mathbf{c}}\,. \tag{4.41}$$

Then we have

$$\begin{aligned}\mathbf{S}^{(s,q_0)} &= \bigoplus_{w\in {}^{\mathbf{v}}S_n} \mathbf{S}^w \\ f'_{\mathbf{c}}(v) &= \sum_{w\in {}^{\mathbf{v}}S_n} f^w_{\mathbf{c}}(v).\end{aligned}\tag{4.42}$$

We also define polynomials

$$h^w_{\mathbf{c}}(v) = \sum_{k\in\mathbb{Z}} \dim \operatorname{Hom}^k(\mathbf{S}_{\mathbf{i},\mathbf{a_c}}, \mathbf{S}^w)v^{-k} \tag{4.43}$$

for any $\mathbf{c} \in \mathrm{KP}(\mathbf{v})$. To get slightly nicer formulas later on, we also define renormalized polynomials:

$$\begin{aligned}f_{\mathbf{c}}(v) &:= \sum_{w\in {}^{\mathbf{v}}S_n} v^{2l(w)+\dim {}^w\mathfrak{n}\cap\mathcal{N}^{(s,q_0)}} f^w_{\mathbf{c}}(v) \\ h_{\mathbf{c}}(v) &:= \sum_{w\in {}^{\mathbf{v}}S_n} v^{2l(w)+\dim {}^w\mathfrak{n}\cap\mathcal{N}^{(s,q_0)}} h^w_{\mathbf{c}}(v).\end{aligned}\tag{4.44}$$

Note that

$$\dim L_{\mathbf{c}} \overset{4.5.6}{=} f'_{\mathbf{c}}(1) \overset{(4.42)}{=} \sum_{w\in {}^{\mathbf{v}}S_n} f^w_{\mathbf{c}}(1) \overset{(4.44)}{=} f_{\mathbf{c}}(1)$$

so for the purpose of determining the dimensions of simple modules it is sufficient to compute the renormalized polynomials $f_{\mathbf{c}}(v)$. For any $w \in S_n$ we can consider the intersection $R_{\mathbf{i},\mathbf{a_c}} \cap {}^w\mathfrak{n}$ by viewing $R_{\mathbf{i},\mathbf{a_c}}$ as a subspace of $\mathfrak{g}$ via

$$R_{\mathbf{i},\mathbf{a_c}} \subset R_{\mathbf{v}} \cong \mathcal{N}^{(s,q_0)} \subset \mathfrak{g}.$$

Here we assume that $R_{\mathbf{i},\mathbf{a_c}}$ is defined with respect to the Borel subgroup $B(s) \subset \mathrm{GL}_n(s) \cong \mathrm{GL}_{\mathbf{v}}$.

Lemma 4.5.8. *We have* $h_{\mathbf{c}}(v) = v^{-\dim R_{i,a_c}} \frac{(1-v^{-2})^{-n}}{[a_c]!} \sum_{w\in S_n} v^{2(l(w)+\dim R_{i,a_c}\cap {}^w\mathfrak{n})}$.

Proof. The parabolic $P_{\mathbf{i},\mathbf{a_c}}$ contains the Borel $B(s)$. Hence, for any $w \in {}^{\mathbf{v}}S_n$ we can apply Theorem 4.3.9 with $G = \mathrm{GL}_n(s)$, $V = \mathcal{N}^{(s,q_0)} \cong R_{\mathbf{v}}$, $V^1 = R_{\mathbf{i},\mathbf{a_c}}$, $P_1 = P_{\mathbf{i},\mathbf{a_c}}$, $V^2 = {}^w\mathfrak{n} \cap \mathcal{N}^{(s,q_0)}$ and $P_2 = B(s)$ to obtain

$$\begin{aligned}h^w_{\mathbf{c}}(v) &= v^{-\dim \mathrm{GL}_n(s)/B(s) - \dim R_{\mathbf{i},\mathbf{a_c}} - \dim {}^w\mathfrak{n}\cap\mathcal{N}^{(s,q_0)}} \frac{\chi_v^{\mathrm{GL}_n(s)}(\mathrm{GL}_n(s)/B(s))}{\chi_v(P_{\mathbf{i},\mathbf{a_c}}/B(s))} \\ &\quad\cdot \sum_{y\in S_{\mathbf{v}}} v^{2(l(y)+\dim R_{\mathbf{i},\mathbf{a_c}}\cap {}^y({}^w\mathfrak{n}\cap\mathcal{N}^{(s,q_0)}))} \\ &= v^{-\dim R_{\mathbf{i},\mathbf{a_c}} - \dim {}^w\mathfrak{n}\cap\mathcal{N}^{(s,q_0)}} \frac{(1-v^{-2})^{-n}}{[\mathbf{a_c}]!} \sum_{y\in S_{\mathbf{v}}} v^{2(l(y)+\dim R_{\mathbf{i},\mathbf{a_c}}\cap {}^{yw}\mathfrak{n})}.\end{aligned}\tag{4.45}$$

Hence, we get

$$\begin{aligned}
h_{\mathbf{c}}(v) &\overset{(4.44)}{=} \sum_{w \in {}^{\mathbf{v}}S_n} v^{2l(w)+\dim {}^w\mathfrak{n}\cap\mathcal{N}^{(s,q_0)}} h^w_{\mathbf{c}}(v) \\
&\overset{(4.45)}{=} \sum_{w \in {}^{\mathbf{v}}S_n} v^{2l(w)-\dim R_{\mathbf{i},\mathbf{a_c}}} \frac{(1-v^{-2})^{-n}}{[\mathbf{a_c}]!} \sum_{y \in S_{\mathbf{v}}} v^{2(l(y)+\dim R_{\mathbf{i},\mathbf{a_c}}\cap {}^{yw}\mathfrak{n})} \\
&= \sum_{w \in {}^{\mathbf{v}}S_n} v^{-\dim R_{\mathbf{i},\mathbf{a_c}}} \frac{(1-v^{-2})^{-n}}{[\mathbf{a_c}]!} \sum_{y \in S_{\mathbf{v}}} v^{2(l(yw)+\dim R_{\mathbf{i},\mathbf{a_c}}\cap {}^{yw}\mathfrak{n})} \\
&= v^{-\dim R_{\mathbf{i},\mathbf{a_c}}} \frac{(1-v^{-2})^{-n}}{[\mathbf{a_c}]!} \sum_{w \in S_n} v^{2(l(w)+\dim R_{\mathbf{i},\mathbf{a_c}}\cap {}^{w}\mathfrak{n})}.
\end{aligned}$$

□

Consider the column vectors

$$\begin{aligned}
F &:= (f_{\mathbf{c}}(v))_{\mathbf{c}\in\mathrm{KP}(\mathbf{v})} \\
H &:= (h_{\mathbf{c}}(v))_{\mathbf{c}\in\mathrm{KP}(\mathbf{v})}.
\end{aligned}$$

Recall that Ψ is the matrix representing the standard bilinear form on $U^+_{\mathbf{v}}$ with respect to $\mathbf{B}^{\mathrm{mon}}_{\mathbf{v}}$ and Q is the change of basis matrix between $\mathbf{B}^{\mathrm{can}}_{\mathbf{v}}$ and $\mathbf{B}^{\mathrm{mon}}_{\mathbf{v}}$.

Lemma 4.5.9. *We have $H = \Psi Q^{-T} F$.*

Proof. By Theorem 4.3.2 and Proposition 4.3.5 we have

$$\mathbf{S}_{\mathbf{i},\mathbf{a_c}} = \sum_{\mathbf{c}'\in\mathrm{KP}(\mathbf{v})} q_{\mathbf{c}',\mathbf{c}}(v)\cdot \mathrm{IC}_{\mathbf{c}'}\,. \tag{4.46}$$

Hence, we get

$$\begin{aligned}
h^w_{\mathbf{c}}(v) &\overset{(4.43)}{=} \sum_{k\in\mathbb{Z}} \dim \mathrm{Hom}^k(\mathbf{S}_{\mathbf{i},\mathbf{a_c}}, \mathbf{S}^w) v^{-k} \\
&\overset{(4.41),(4.46)}{=} \sum_{k\in\mathbb{Z}} \sum_{\mathbf{c}',\mathbf{c}''} \overline{q}_{\mathbf{c},\mathbf{c}'}(v) f^w_{\mathbf{c}''}(v) \dim \mathrm{Hom}^k(\mathrm{IC}_{\mathbf{c}'}, \mathrm{IC}_{\mathbf{c}''}) v^{-k} \\
&\overset{(4.2)}{=} \sum_{k\in\mathbb{Z}} \sum_{\mathbf{c}',\mathbf{c}''} q_{\mathbf{c},\mathbf{c}'}(v) f^w_{\mathbf{c}''}(v) \dim \mathrm{Hom}^k(\mathrm{IC}_{\mathbf{c}'}, \mathrm{IC}_{\mathbf{c}''}) v^{-k} \\
&\overset{4.3.2}{=} \sum_{\mathbf{c}',\mathbf{c}''} q_{\mathbf{c},\mathbf{c}'}(v) \omega_{\mathbf{c}',\mathbf{c}''}(v) f^w_{\mathbf{c}''}(v)
\end{aligned}$$

where the $\omega_{\mathbf{c}',\mathbf{c}''}(v)$ are the entries of the matrix Ω representing the standard bilinear form on $U_{\mathbf{v}}^+$ with respect to $\mathbf{B}_{\mathbf{v}}^{\mathrm{can}}$. Thus, we get

$$\begin{aligned} h_{\mathbf{c}}(v) &\overset{(4.44)}{=} \sum_{w\in{}^{\mathbf{v}}S_n} v^{2l(w)+\dim {}^w\mathfrak{n}\cap\mathcal{N}^{(s,q_0)}} h_{\mathbf{c}}^w(v) \\ &= \sum_{w\in{}^{\mathbf{v}}S_n} v^{2l(w)+\dim {}^w\mathfrak{n}\cap\mathcal{N}^{(s,q_0)}} \sum_{\mathbf{c}',\mathbf{c}''\in\mathrm{KP}(\mathbf{v})} q_{\mathbf{c},\mathbf{c}'}(v)\omega_{\mathbf{c}',\mathbf{c}''}(v) f_{\mathbf{c}''}^w(v) \\ &\overset{(4.44)}{=} \sum_{\mathbf{c}',\mathbf{c}''\in\mathrm{KP}(\mathbf{v})} q_{\mathbf{c},\mathbf{c}'}(v)\omega_{\mathbf{c}',\mathbf{c}''}(v) f_{\mathbf{c}''}(v). \end{aligned}$$

As a matrix equation, this is precisely $H = Q\Omega F$. Hence, we get

$$H = Q\Omega Q^T Q^{-T} F \overset{(4.4)}{=} \Psi Q^{-T} F.$$

□

Recall that we can compute Q and Ψ using Algorithm 4.3.12. The vector H can be computed explicitly using Lemma 4.5.8. Hence, using the lemma above, we can compute F from Q, Ψ and H. Thus, we arrive at the following algorithm to compute the dimensions of simple $\mathcal{H}_q^{\mathrm{aff}}$-modules with central character $\chi_{(s,q_0)}$ where $s = s_{e,\mathbf{v}}$.

Algorithm 4.5.10. *Input:* $\boldsymbol{v}$, *Output:* $\dim L_{\boldsymbol{c}}$ *for all* $\boldsymbol{c} \in \mathrm{KP}(\boldsymbol{v})$*;*
Step 1: Compute the matrices Ψ *and* Q *using Algorithm 4.3.12;*
Step 2: Compute the vector H *using Lemma 4.5.8;*
Step 3: Compute the vector $F = Q^T\Psi^{-1}H$*.*
Step 4: Compute $\dim L_{\boldsymbol{c}} = f_{\boldsymbol{c}}(1)$*.*

Example 4.5.11. *Let* $\boldsymbol{v} = (1,2,1)$ *and* $s = s_{e,\boldsymbol{v}}$*. We compute the dimensions of the simple* $\mathcal{H}_q^{\mathrm{aff}}(\mathrm{GL}_4)$*-modules with central character* $\chi_{(s,q_0)}$*. This continues the examples from Section 4.4.2 and Example 4.5.4. We use the Borel* $B \subset \mathrm{GL}_4$ *consisting of lower triangular matrices. In Table 4.1, we list for each* $w \in S_4$ *(in cycle notation) the length of the element* w *and the subspace*

$$\mathcal{N}^{(s,q_0)} \cap {}^w\mathfrak{n} = R_{\boldsymbol{v}} \cap {}^w\mathfrak{n} \subset R_{\boldsymbol{v}} = \left(\left(\begin{smallmatrix} * \\ * \end{smallmatrix}\right), \left(\begin{smallmatrix} * & * \end{smallmatrix}\right) \right).$$

Using this table and the explicit description of the $R_{\boldsymbol{i},\boldsymbol{a_c}}$ *in* (4.33)*, we can compute the intersections*

$$R_{\boldsymbol{i},\boldsymbol{a_c}} \cap {}^w\mathfrak{n} = R_{\boldsymbol{i},\boldsymbol{a_c}} \cap (R_{\boldsymbol{v}} \cap {}^w\mathfrak{n}).$$

Table 4.1:

w	$l(w)$	$R_{\mathbf{v}} \cap {}^{w}\mathfrak{n}$
e	0	$\left(\left(\begin{smallmatrix}*\\ *\end{smallmatrix}\right), \left(\begin{smallmatrix}* & *\end{smallmatrix}\right)\right)$
(23)	1	
(12)	1	$\left(\left(\begin{smallmatrix}0\\ *\end{smallmatrix}\right), \left(\begin{smallmatrix}* & *\end{smallmatrix}\right)\right)$
(34)	1	$\left(\left(\begin{smallmatrix}*\\ *\end{smallmatrix}\right), \left(\begin{smallmatrix}* & 0\end{smallmatrix}\right)\right)$
(132)	2	$\left(\left(\begin{smallmatrix}*\\ 0\end{smallmatrix}\right), \left(\begin{smallmatrix}* & *\end{smallmatrix}\right)\right)$
(234)	2	$\left(\left(\begin{smallmatrix}*\\ *\end{smallmatrix}\right), \left(\begin{smallmatrix}0 & *\end{smallmatrix}\right)\right)$
(123)	2	$\left(\left(\begin{smallmatrix}0\\ 0\end{smallmatrix}\right), \left(\begin{smallmatrix}* & *\end{smallmatrix}\right)\right)$
(13)	3	
(1234)	3	
(134)	4	
(243)	2	$\left(\left(\begin{smallmatrix}*\\ *\end{smallmatrix}\right), \left(\begin{smallmatrix}0 & 0\end{smallmatrix}\right)\right)$
(24)	3	
(1432)	3	
(142)	4	
$(12)(34)$	2	$\left(\left(\begin{smallmatrix}0\\ *\end{smallmatrix}\right), \left(\begin{smallmatrix}* & 0\end{smallmatrix}\right)\right)$
(1243)	3	
(1342)	3	$\left(\left(\begin{smallmatrix}*\\ 0\end{smallmatrix}\right), \left(\begin{smallmatrix}0 & *\end{smallmatrix}\right)\right)$
$(13)(24)$	4	
(1324)	5	$\left(\left(\begin{smallmatrix}0\\ 0\end{smallmatrix}\right), \left(\begin{smallmatrix}0 & *\end{smallmatrix}\right)\right)$
(124)	4	$\left(\left(\begin{smallmatrix}0\\ 0\end{smallmatrix}\right), \left(\begin{smallmatrix}* & 0\end{smallmatrix}\right)\right)$
(143)	4	$\left(\left(\begin{smallmatrix}0\\ *\end{smallmatrix}\right), \left(\begin{smallmatrix}0 & 0\end{smallmatrix}\right)\right)$
(1423)	5	$\left(\left(\begin{smallmatrix}*\\ 0\end{smallmatrix}\right), \left(\begin{smallmatrix}0 & 0\end{smallmatrix}\right)\right)$
(14)	5	$\left(\left(\begin{smallmatrix}0\\ 0\end{smallmatrix}\right), \left(\begin{smallmatrix}0 & 0\end{smallmatrix}\right)\right)$
$(14)(23)$	6	

We now have all the ingredients needed to evaluate the formula from Lemma 4.5.8. For example, for

$$\begin{aligned} \boldsymbol{c}_4 &= (0,1,0,0,1,0) \\ \boldsymbol{a}_{\boldsymbol{c}_4} &= (0,1,1,1,1,0) \\ R_{\boldsymbol{i},\boldsymbol{a}_{\boldsymbol{c}_4}} &= \left(\left(\begin{smallmatrix} 0 \\ * \end{smallmatrix}\right), \left(\begin{smallmatrix} * & 0 \end{smallmatrix}\right) \right) \end{aligned}$$

we get

$$\begin{aligned} h_{\boldsymbol{c}_4}(v) = v^{-2}(1-v^{-2})^{-4}(&v^4+v^6+v^6+v^6+v^6+v^6+v^6+v^8+v^8+v^{10}+v^6+v^8+v^8 \\ &+v^{10}+v^8+v^{10}+v^6+v^8+v^{10}+v^{10}+v^{10}+v^{10}+v^{10}+v^{12}) \\ = {}& \tfrac{v^2+8v^4+6v^6+8v^8+v^{10}}{(1-v^{-2})^4}. \end{aligned}$$

More generally, we get

$$H = (1-v^{-2})^{-4} \begin{pmatrix} \scriptstyle v+2v^3+3v^5+3v^7+2v^9+v^{11} \\ \scriptstyle v+5v^3+6v^5+6v^7+5v^9+v^{11} \\ \scriptstyle 3v^3+3v^5+3v^7+3v^9 \\ \scriptstyle v^2+8v^4+6v^6+8v^8+v^{10} \\ \scriptstyle 3v^3+9v^5+9v^7+3v^9 \end{pmatrix}.$$

We have already computed Q and Ψ in (4.34) *and* (4.35) *and hence we can compute*

$$F = Q^T\Psi^{-1}H = \begin{pmatrix} \scriptstyle 3v^9+v^{11} \\ \scriptstyle 2v^5+2v^7+2v^9 \\ \scriptstyle 2v^5+2v^7+2v^9 \\ \scriptstyle v^6+v^8 \\ \scriptstyle v^3+3v^5 \end{pmatrix}.$$

Setting $v=1$, we get that the dimensions of the simple $\mathcal{H}_q^{\mathrm{aff}}$-modules with central character $\chi_{(s,q_0)}$ are

$$\dim L_1 = 4, \quad \dim L_2 = 6, \quad \dim L_3 = 6, \quad \dim L_4 = 2, \quad \dim L_5 = 4.$$

Chapter 5

Geometric realizations of affine Hecke algebras with unequal parameters

In this chapter we provide a new geometric realization of certain affine Hecke algebras with two unequal parameters via equivariant algebraic K-theory. This can be viewed as a generalization of the classical K-theory realization of the affine Hecke algebra with equal parameters (Theorem 2.2.1).

5.1 K-theory and support

We begin by reviewing some elementary facts and constructions in equivariant algebraic K-theory. Let G be a reductive group defined over an algebraically closed field k (not necessarily of characteristic 0) and let X be a quasi-projective G-variety. We denote by $\mathrm{Coh}^G(X)$ the category of G-equivariant coherent sheaves, by $D^b(\mathrm{Coh}^G(X))$ its bounded derived category and by $D^b_{\mathrm{perf}}(\mathrm{Coh}^G(X))$ the category of strictly perfect complexes (i.e. complexes quasi-isomorphic to a bounded complex of G-equivariant locally free sheaves of finite type). Consider the Grothendieck groups

$$K^G(X) := K_0(D^b(\mathrm{Coh}^G(X)))$$
$$K_G(X) := K_0(D^b_{\mathrm{perf}}(\mathrm{Coh}^G(X))).$$

We will write $[\mathcal{F}] \in K^G(X)$ for the class of a coherent sheaf $\mathcal{F} \in \mathrm{Coh}^G(X)$ and $[\mathcal{E}] \in K_G(X)$ for the class of a locally free sheaf $\mathcal{E}$ on X. There are the usual pushforward and pullback maps

$$f_* : K^G(X) \to K^G(Y)$$
$$f^* : K^G(Y) \to K^G(X)$$

which are defined for $f : X \to Y$ proper (resp. flat) and satisfy flat base change. Moreover, there is a pullback map

$$f^* : K_G(Y) \to K_G(X)$$

which is defined without any assumptions of f. The tensor product induces maps

$$- \otimes - : K_G(X) \times K_G(X) \to K_G(X)$$
$$- \otimes - : K^G(X) \times K_G(X) \to K^G(X).$$

This equips $K_G(X)$ with a ring structure and $K^G(X)$ with the structure of a $K_G(X)$-module. For any proper $f : X \to Y$ and $[\mathcal{F}] \in K^G(X), [\mathcal{E}] \in K_G(Y)$, we have the projection formula

$$f_*[\mathcal{F}] \otimes [\mathcal{E}] = f_*([\mathcal{F}] \otimes f^*[\mathcal{E}]). \tag{5.1}$$

There is a canonical map

$$[\mathcal{O}_X] \otimes - : K_G(X) \to K^G(X)$$

which is an isomorphism if X is smooth (this follows from the fact that any G-equivariant coherent sheaf on a smooth variety has a finite resolution by G-equivariant locally-free sheaves [65, Theorem 2.18]). If $H \subset G$ is a closed subgroup and X is an H-variety, then there is an induction isomorphism

$$K^G(G \times^H X) \cong K^H(X)$$

which is compatible with proper pushforward and flat pullback.

For any G-stable closed subvariety $Z \subset X$, we denote by $\iota_Z : Z \hookrightarrow X$ the inclusion. Let $\mathrm{Coh}^G_Z(X) \subset \mathrm{Coh}^G(X)$ be the full subcategory consisting of those coherent sheaves $\mathcal{F}$ on X which are (set-theoretically) supported on Z (i.e. the restriction of $\mathcal{F}$ to $X \backslash Z$ vanishes or equivalently $\mathcal{F}$ is killed by some power of the ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$). Similarly, we denote by $D^b_Z(\mathrm{Coh}^G(X)) \subset D^b(\mathrm{Coh}^G(X))$ the full subcategory consisting of those complexes whose cohomology lies in $\mathrm{Coh}^G_Z(X)$. Then there is the following standard result (see for example [18, §3.1.7]).

Lemma 5.1.1. *The inclusion* $\mathrm{Coh}^G_Z(X) \subset \mathrm{Coh}^G(X)$ *induces an equivalence*

$$D^b(\mathrm{Coh}^G_Z(X)) \xrightarrow{\sim} D^b_Z(\mathrm{Coh}^G(X)).$$

We consider the K-groups with support

$$K_Z^G(X) := K_0(D^b(\mathrm{Coh}_Z^G(X))) \cong K_0(D_Z^b(\mathrm{Coh}^G(X))).$$

Pushforward along the inclusion $Z \hookrightarrow X$ induces a functor

$$D^b(\mathrm{Coh}^G(Z)) \to D^b(\mathrm{Coh}_Z^G(X)).$$

This is not an equivalence, but the induced map on K-groups

$$K^G(Z) \to K_Z^G(X)$$

is an isomorphism with inverse

$$[\mathcal{F}] \mapsto gr_Z[\mathcal{F}] = \sum_{n \geq 0} (-1)^n [\mathcal{I}_Z^n \mathcal{F} / \mathcal{I}_Z^{n+1} \mathcal{F}].$$

From now on we will implicitly identify

$$K^G(Z) = K_Z^G(X)$$

via this isomorphism. If X is smooth, we have the derived tensor product

$$\otimes_X^L : D^b(\mathrm{Coh}^G(X)) \times D^b(\mathrm{Coh}^G(X)) \to D^b(\mathrm{Coh}^G(X)).$$

Passing to K-theory and taking supports into account, this induces a tensor product

$$\otimes_X^L : K^G(Z_1) \times K^G(Z_2) \to K^G(Z_1 \cap Z_2)$$

for any $Z_1, Z_2 \subset X$ closed. For locally free sheaves on X, the tensor product with support reduces to the ordinary tensor product in the following sense.

Lemma 5.1.2. *Let X be smooth and $Z \subset X$ a closed subvariety. Then*

$$[\mathcal{F}] \otimes_X^L [\mathcal{E}] = [\mathcal{F} \otimes \iota_Z^* \mathcal{E}]$$

for any $[\mathcal{F}] \in K^G(Z)$ and $[\mathcal{E}] \in K_G(X) \cong K^G(X)$.

Proof. Since X is smooth, we can pick a finite resolution $E^\bullet$ of $(\iota_Z)_* \mathcal{F}$ by G-equivariant locally free sheaves. Then, by the definition of the tensor product with support, we have

$$[\mathcal{F}] \otimes_X^L [\mathcal{E}] = \sum_i (-1)^i gr_Z[H^i(E^\bullet \otimes \mathcal{E})].$$

Since $-\otimes\mathcal{E}$ is exact, we get $H^i(E^\bullet\otimes\mathcal{E})=0$ for $i\neq 0$ and $H^0(E^\bullet\otimes\mathcal{E})\cong(\iota_Z)_*\mathcal{F}\otimes\mathcal{E}$. Hence, we get

$$\begin{aligned}[\mathcal{F}]\otimes^L_X[\mathcal{E}] &= gr_Z[(\iota_Z)_*\mathcal{F}\otimes\mathcal{E}]\\ &\overset{(5.1)}{=} gr_Z[(\iota_Z)_*(\mathcal{F}\otimes\iota_Z^*\mathcal{E})]\\ &= [\mathcal{F}\otimes\iota_Z^*\mathcal{E}].\end{aligned}$$

□

The following well-known lemma says that the tensor product with support behaves well with respect to transverse intersections (see [22, Lemma 1] for a proof of this result in a more general setting).

Lemma 5.1.3. *Let $Z_1, Z_2\subset X$ be smooth closed subvarieties of a smooth variety X. Assume that Z_1 and Z_2 intersect transversally, i.e. for all $p\in Z_1\cap Z_2$, we have the equality of Zariski tangent spaces*

$$T_pZ_1+T_pZ_2=T_pX.$$

Then we have

$$[\mathcal{O}_{Z_1}]\otimes^L_X[\mathcal{O}_{Z_2}]=[\mathcal{O}_{Z_1\cap Z_2}]$$

where the tensor product with support is taken with respect to $[\mathcal{O}_{Z_1}]\in K^G(Z_1)$ and $[\mathcal{O}_{Z_2}]\in K^G(Z_2)$.

If $f:\tilde{X}\to X$ is a morphism between smooth varieties there is a pullback functor

$$Lf^*:D^b(\mathrm{Coh}^G(X))\to D^b(\mathrm{Coh}^G(\tilde{X})).$$

For any closed subvariety $Z\subset X$ this restricts to a derived functor

$$Lf^*:D^b_Z(\mathrm{Coh}^G(X))\to D^b_{f^{-1}(Z)}(\mathrm{Coh}^G(\tilde{X}))$$

which induces a 'pullback with support' map on K-theory:

$$f^*:K^G(Z)\to K^G(f^{-1}(Z)).$$

The pullback with support construction if functorial (i.e. $f^*\circ g^*=(g\circ f)^*$) and if f is flat, it agrees with the ordinary flat pullback. Moreover, pullback with support essentially commutes with the tensor product with support in the following sense.

Lemma 5.1.4. *Let $f : \tilde{X} \to X$ be a morphism between smooth varieties and let $Z_1, Z_2 \subset X$ be closed. Then for any $[\mathcal{F}_1] \in K^G(Z_1)$ and $[\mathcal{F}_2] \in K^G(Z_2)$ we have*

$$f^*([\mathcal{F}_1] \otimes^L_X [\mathcal{F}_2]) = f^*[\mathcal{F}_1] \otimes^L_{\tilde{X}} f^*[\mathcal{F}_2].$$

Proof. By standard facts about derived categories, we have

$$Lf^*(\mathcal{F}^\bullet \otimes^L \mathcal{G}^\bullet) \cong Lf^*\mathcal{F}^\bullet \otimes^L Lf^*\mathcal{G}^\bullet$$

for any $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in D^b(\mathrm{Coh}^G(X))$. In particular, this holds for any $\mathcal{F}^\bullet \in D^b_{Z_1}(\mathrm{Coh}^G(X))$ and $\mathcal{G}^\bullet \in D^b_{Z_2}(\mathrm{Coh}^G(X))$. The result now follows by taking Grothendieck groups. $\square$

We also have the following relative description of the proper pushforward.

Lemma 5.1.5. *Let $f : \tilde{X} \to X$ be a morphism of varieties and let $\tilde{Z} \subset \tilde{X}$, $Z \subset X$ be G-stable closed subvarieties such that $f(\tilde{Z}) \subset Z$. If the restriction $f : \tilde{Z} \to Z$ is proper, the functor*

$$Rf_* : D^b(\mathrm{QCoh}^G(\tilde{X})) \to D^b(\mathrm{QCoh}^G(X))$$

restricts to a functor

$$Rf_* : D^b_{\tilde{Z}}(\mathrm{Coh}^G(\tilde{X})) \to D^b_Z(\mathrm{Coh}^G(X)).$$

Upon taking Grothendieck groups, this yields the ordinary proper pushforward map

$$f_* : K^G(\tilde{Z}) \to K^G(Z).$$

Proof. By flat base change, $Rf_*\mathcal{F}^\bullet$ is supported on Z for any $\mathcal{F}^\bullet \in D^b_{\tilde{Z}}(\mathrm{Coh}^G(\tilde{X}))$. We need to show that $H^i(Rf_*\mathcal{F}^\bullet)$ is coherent for all i. Since $D^b(\mathrm{Coh}^G_{\tilde{Z}}(X)) \cong D^b_{\tilde{Z}}(\mathrm{Coh}^G(\tilde{X}))$, we may assume $\mathcal{F}^\bullet = \mathcal{F} \in \mathrm{Coh}^G_{\tilde{Z}}(X)$. By dévissage, we may further assume that $\mathcal{I}_{\tilde{Z}}\mathcal{F} = 0$ and thus $\mathcal{F} = (\iota_{\tilde{Z}})_*\mathcal{G}$ for some $\mathcal{G} \in \mathrm{Coh}^G(\tilde{Z})$. Then, by functoriality, we have

$$Rf_*\mathcal{F} = Rf_*(\iota_{\tilde{Z}})_*\mathcal{G} = (\iota_Z)_*Rf_*\mathcal{G} \tag{5.2}$$

Since ι_Z is proper and f is proper on $\tilde{Z}$, this complex has coherent cohomology. The second equality in (5.2) also implies that Rf_* induces the ordinary pushforward on the Grothendieck group. $\square$

There is also the following 'projection formula with support'.

Lemma 5.1.6. *Let $f : \tilde{X} \to X$ be a morphism between smooth varieties and let $Z \subset X$ and $\tilde{Z} \subset \tilde{X}$ be closed subvarieties such that the restriction $f : \tilde{Z} \to f(\tilde{Z})$ (and hence also $f : \tilde{Z} \cap f^{-1}(Z) \to f(\tilde{Z}) \cap Z$) is proper. Then*

$$f_*[\mathcal{F}] \otimes^L_X [\mathcal{G}] = f_*([\mathcal{F}] \otimes^L_{\tilde{X}} f^*[\mathcal{G}])$$

for any $[\mathcal{F}] \in K^G(\tilde{Z})$ and $[\mathcal{G}] \in K^G(Z)$.

Proof. By the Lemma 5.1.5, all relevant functors are computed on full subcategories of $D^b(\mathrm{QCoh}^G(X))$. The claim hence follows from the projection formula in $D^b(\mathrm{QCoh}^G(X))$ (note that any complex $\mathcal{G}^\bullet \in D^b_Z(\mathrm{Coh}^G(X))$ is perfect since X is smooth). $\square$

Let X be a smooth variety, let $Z_{12}, Z_{23} \subset X \times X$ be closed subvarieties such that

$$p_{13} : p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \to X \times X$$

is proper and define

$$Z_{12} \circ Z_{23} := p_{13}(p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23})).$$

Then there is a convolution operation

$$\star : K^G(Z_{12}) \times K^G(Z_{23}) \to K^G(Z_{12} \circ Z_{23}) \tag{5.3}$$

defined via

$$a \star b := (p_{13})_*(p_{12}^* a \otimes^L_{X \times X \times X} p_{23}^* b).$$

One can check using Lemma 5.1.6 that this operation is associative. Moreover, convolution with the diagonal has the following concrete description.

Lemma 5.1.7. *Let $Z_{12}, Z_{23} \subset X \times X$ such that Z_{12} contains the diagonal. Then $Z_{23} \subset Z_{12} \circ Z_{23}$ and for any $[\mathcal{E}] \in K_G(X) \cong K^G(X)$ and $[\mathcal{F}] \in K^G(Z_{23})$ we have*

$$(\Delta_X)_*[\mathcal{E}] \star [\mathcal{F}] = \iota_*([p^*\mathcal{E}] \otimes [\mathcal{F}])$$

where $\Delta_X : X \to Z_{12}$ is the diagonal embedding, ι is the embedding $Z_{23} \hookrightarrow Z_{12} \circ Z_{23}$ and $p : Z_{23} \to X$ is the projection onto the first factor.

Proof. We have $Z_{23} = Z_\Delta \circ Z_{23} \subset Z_{12} \circ Z_{23}$. Let $p_1 : X \times X \to X$ be the first projection. Then, we have

$$\begin{aligned}
(\Delta_X)_*[\mathcal{E}] \star [\mathcal{F}] &\overset{def}{=} (p_{13})_*(p_{12}^*(\Delta_X)_*[\mathcal{E}] \otimes^L_{X\times X\times X} p_{23}^*[\mathcal{F}]) \\
&\overset{base\ change}{=} (p_{13})_*((\Delta_X \times \mathrm{id}_X)_* p_1^*[\mathcal{E}] \otimes^L_{X\times X\times X} p_{23}^*[\mathcal{F}]) \\
&\overset{Lemma\ 5.1.6}{=} (p_{13})_*(\Delta_X \times \mathrm{id}_X)_*(p_1^*[\mathcal{E}] \otimes^L_{X\times X} (\Delta_X \times \mathrm{id}_X)^* p_{23}^*[\mathcal{F}]) \\
&\overset{functoriality}{=} \iota_*(p_1^*[\mathcal{E}] \otimes^L_{X\times X} [\mathcal{F}]) \\
&\overset{Lemma\ 5.1.2}{=} \iota_*(p^*[\mathcal{E}] \otimes [\mathcal{F}]).
\end{aligned}$$

□

5.2 K-theory of vector bundles and Steinberg varieties

In this section we review some elementary properties of the equivariant K-theory of vector bundles and Steinberg varieties. Recall that G is reductive and fix a maximal torus $T \subset G$. Let $(X^*(T), X_*(T), \Phi, \Phi^\vee)$ be the corresponding root datum. Fix a system of positive roots $\Phi^+ \subset \Phi$ with simple roots $\Pi \subset \Phi^+$ and denote the set of negative roots by $\Phi^- = -\Phi^+$. Let B be the corresponding (geometric) Borel subgroup, i.e. the Borel subgroup whose Lie algebra has non-zero weights Φ^-. We denote by $\mathcal{B} = G/B$ the flag variety. For any B-module W there is a corresponding G-equivariant vector bundle $\pi : G \times^B W \to \mathcal{B}$ and there is the Thom isomorphism

$$\pi^* : K^G(\mathcal{B}) \xrightarrow{\sim} K^G(G \times^B W).$$

Combining this with the equivalence

$$K^G(\mathcal{B}) \cong K^B(\mathrm{pt}) \cong K^T(\mathrm{pt}) \cong \mathbb{Z}[X^*(T)]$$

we get an isomorphism

$$\begin{aligned} \mathbb{Z}[X^*(T)] &\cong K^G(G \times^B W) \\ e^\lambda &\mapsto [\pi^*\mathcal{E}_\lambda] \end{aligned} \tag{5.4}$$

where $\mathcal{E}_\lambda$ denotes the line bundle on $\mathcal{B}$ whose fiber over B has weight λ. From now on we will identify $\mathbb{Z}[X^*(T)]$ and $K^G(G \times^B W)$ via this isomorphism. We will need the following two standard results about the equivariant K-theory of vector bundles.

Lemma 5.2.1. *Let W be a B-module, $W' \subset W$ a B-stable subspace and denote by $\iota : G \times^B W' \hookrightarrow G \times^B W$ the inclusion. Then the map*

$$K^G(G \times^B W') \xrightarrow{\iota_*} K^G(G \times^B W)$$

is given by

$$e^\mu \mapsto \lambda^\vee(W/W') \cdot e^\mu$$

where

$$\lambda^\vee(W/W') := \prod_{\lambda \in X^*(T)} (1 - e^{-\lambda})^{\dim(W/W')_\lambda}.$$

Proof. Consider the projections

$$\pi : G \times^B W \to \mathcal{B}$$
$$\pi' : G \times^B W' \to \mathcal{B}.$$

Then, for any $\mu \in X^*(T)$, we have

$$\begin{aligned} \iota_*[(\pi')^*\mathcal{E}_\mu] &\overset{\textit{functoriality}}{=} \iota_*\iota^*[\pi^*\mathcal{E}_\mu] \\ &= \iota_*([\mathcal{O}_{G\times^B W'}] \otimes \iota^*[\pi^*\mathcal{E}_\mu]) \\ &\overset{(5.1)}{=} \iota_*[\mathcal{O}_{G\times^B W'}] \otimes [\pi^*\mathcal{E}_\mu]. \end{aligned}$$

Thus, ι_* acts by multiplication with $\iota_*[\mathcal{O}_{G\times^B W'}]$, so it suffices to show that

$$\iota_*[\mathcal{O}_{G\times^B W'}] = \lambda^\vee(W/W').$$

There is a commutative diagram

$$\begin{tikzcd} K^G(G \times^B W') \arrow[r, equal] \arrow[d, "\iota_*"] & K^B(W') \arrow[r, equal] \arrow[d] & K^T(W') \arrow[d] \\ K^G(G \times^B W) \arrow[r, equal] & K^B(W) \arrow[r, equal] & K^T(W) \end{tikzcd}$$

where the horizontal identifications are induced by induction equivalences and restriction along $T \hookrightarrow B$. Hence, we may assume $G = B = T$. Choosing a splitting $W \cong W' \oplus W/W'$ as T-modules, we can view the inclusion $\iota : W' \hookrightarrow W$ as the zero section of the T-equivariant vector bundle $p_1 : W \to W'$ with fiber W/W'. Let $\mathcal{E}$ be the T-equivariant locally free sheaf on W' corresponding to this vector bundle. In $K^T(W')$, we have

$$[\Lambda^i\mathcal{E}] = \sum_{\lambda \in X^*(T)} \dim(\Lambda^i(W/W'))_\lambda \cdot e^\lambda.$$

It now follows from the Koszul resolution

$$\ldots \to \Lambda^1 p_1^*\mathcal{E}^\vee \to \Lambda^0 p_1^*\mathcal{E}^\vee \to \iota_*\mathcal{O}_{W'} \to 0$$

that

$$\begin{aligned}
\iota_*[\mathcal{O}_{W'}] &= \sum_i (-1)^i [\Lambda^i p_1^* \mathcal{E}^\vee] \\
&= \sum_i \sum_{\lambda \in X^*(T)} (-1)^i \dim(\Lambda^i (W/W'))_\lambda \cdot e^{-\lambda} \\
&= \prod_{\lambda \in X^*(T)} (1 - e^{-\lambda})^{\dim(W/W')_\lambda} \\
&= \lambda^\vee (W/W').
\end{aligned}$$

□

Lemma 5.2.2. *Let $\alpha \in \Pi$ be a simple root and let $P_\alpha = B \sqcup B\dot{s}_\alpha B$ be the corresponding parabolic subgroup. Let W be a P_α-module and consider the projection $\pi_\alpha : G \times^B W \to G \times^{P_\alpha} W$. Then we have*

$$\pi_\alpha^* (\pi_\alpha)_* e^\lambda = \tfrac{e^\lambda - e^{s_\alpha(\lambda) - \alpha}}{1 - e^{-\alpha}}$$

in $K^G(G \times^B W)$.

Proof. There is a commutative diagram

$$\begin{tikzcd}
K^G(G \times^B W) \arrow[r, equal] \arrow[d, "\pi_\alpha"] & K^G(G/B) \arrow[r, equal] \arrow[d] & K^{P_\alpha}(P_\alpha/B) \arrow[r, equal] \arrow[d] & K^{L_\alpha}(P_\alpha/B) \arrow[d] \\
K^G(G \times^{P_\alpha} W) \arrow[r, equal] & K^G(G/P_\alpha) \arrow[r, equal] & K^{P_\alpha}(\mathrm{pt}) \arrow[r, equal] & K^{L_\alpha}(\mathrm{pt})
\end{tikzcd}$$

where the horizontal maps are induced by Thom isomorphisms, induction equivalences and restriction to the standard Levi $L_\alpha \subset P_\alpha$. Note that P_α/B is the flag variety of L_α, so the map $K^{L_\alpha}(P_\alpha/B) \to K^{L_\alpha}(\mathrm{pt})$ is given by the (rank 1) Weyl character formula $e^\lambda \mapsto \frac{e^\lambda - e^{s_\alpha(\lambda)-\alpha}}{1-e^{-\alpha}}$ which holds in arbitrary characteristic (c.f. [37, II.5.10] - this is where we use that the Lie algebra of B has non-zero weights Φ^-). □

Consider a morphism of Springer type (Definition 3.1.4)

$$\tilde{V} \xrightarrow{\mu} V$$

where V is a G-representation, $V^- \subset V$ is a G-stable subspace and

$$\tilde{V} = G \times^B V^- = \{(gB, v) \in \mathcal{B} \times V \mid g^{-1} \cdot v \in V^-\}.$$

The associated 'Steinberg variety'

$$Z := \tilde{V} \times_V \tilde{V} = \{((g_1 B, v_1), (g_2 B, v_2)) \in \tilde{V} \times \tilde{V} \mid v_1 = v_2\}$$

comes with a canonical projection map

$$\pi \times \pi : Z \to \mathcal{B} \times \mathcal{B}.$$

Note that we can view Z as a closed subvariety of the smooth variety $\tilde{V} \times \tilde{V}$. Then convolution with respect to the ambient space $\tilde{V} \times \tilde{V}$ equips $K^G(Z)$ with the structure of a $K^G(\mathrm{pt})$-algebra (see (5.3)). The inclusion of the diagonal

$$Z_\Delta := \tilde{V} \times_{\tilde{V}} \tilde{V} \subset Z$$

induces a map

$$K^G(Z_\Delta) \to K^G(Z)$$

which is an algebra homomorphism by Lemma 5.1.7 (here the algebra structure on $K^G(Z_\Delta)$ is also given by convolution with respect to the ambient space $Z_\Delta \subset \tilde{V} \times \tilde{V}$). Let

$$\mathcal{B} \times \mathcal{B} = \bigsqcup_{w \in W} Y_w$$

be the decomposition into $G-$orbits and let

$$Z_w := (\pi \times \pi)^{-1}(Y_w).$$

We pick a total order on W extending the Bruhat order and define

$$Z_{\leq w} := \bigsqcup_{y \leq w} Z_y$$

$$Z_{< w} := \bigsqcup_{y < w} Z_y.$$

Note that $Z_{<w} = Z_{\leq w'}$ where w' is the maximal element with $w' < w$.

Lemma 5.2.3. *$K^G(Z)$ is a free left $K^G(Z_\Delta)$-module with basis $\{[\mathcal{O}_{\overline{Z}_w}] \mid w \in W\}$.*

Proof. For any G-stable locally closed subvariety $X \subset \tilde{V} \times \tilde{V}$, there is a canonical $K_G(\mathcal{B})$-module structure on $K^G(X)$ via

$$[\mathcal{E}] \cdot [\mathcal{F}] := [p^*\mathcal{E} \otimes \mathcal{F}]$$

where p is the composition

$$X \hookrightarrow \tilde{V} \times \tilde{V} \xrightarrow{p_1} \tilde{V} \to \mathcal{B}.$$

Note that we can identify

$$K^G(Z_\Delta) \cong K^G(\mathcal{B}) \cong K_G(\mathcal{B})$$

via the Thom isomorphism. By Lemma 5.1.7, the $K^G(Z_\Delta)$-module structure on $K^G(X)$ via convolution agrees with the canonical $K_G(\mathcal{B})$-module structure just defined. Thus, it suffices to show that $K^G(Z)$ is a free $K_G(\mathcal{B})$-module with basis $\{[\mathcal{O}_{\overline{Z}_w}] \mid w \in W\}$. We show by induction on the total order that $K^G(Z_{\leq w})$ is a free $K_G(\mathcal{B})$-module with basis $\{[\mathcal{O}_{\overline{Z}_y}] \mid y \leq w\}$. Note that for any $y \in W$, the Thom isomorphism together with Lemma 3.1.6 implies that $K^G(Z_y)$ is a free $K_G(\mathcal{B})$-module of rank 1 with basis $[\mathcal{O}_{Z_y}]$. This proves the claim for $Z_{\leq e} = Z_e$. We have a short exact sequence

$$0 \longrightarrow K^G(Z_{<w}) \longrightarrow K^G(Z_{\leq w}) \longrightarrow K^G(Z_w) \longrightarrow 0 \tag{5.5}$$

where the exactness on the left is proved exactly as in [24, (5.4.4)]. Functoriality of pullback and the projection formula imply that this is a short exact sequence of $K_G(\mathcal{B})$-modules. We know by the induction hypothesis that the outer terms are free $K_G(\mathcal{B})$-modules with bases $\{[\mathcal{O}_{\overline{Z}_y}] \mid y < w\}$ and $\{[\mathcal{O}_{Z_w}]\}$ respectively. Thus the sequence splits and $K^G(Z_{\leq w})$ is also a free $K_G(\mathcal{B})$-module. Note that $[\mathcal{O}_{\overline{Z}_w}] \in K^G(Z_{\leq w})$ lifts the element $[\mathcal{O}_{Z_w}] \in K^G(Z_w)$. Thus, $K^G(Z_{\leq w})$ has the basis $\{[\mathcal{O}_{\overline{Z}_y}] \mid y \leq w\}$ as a $K_G(\mathcal{B})$-module. To complete the proof, note that $Z = Z_{\leq w_0}$ where $w_0 \in W$ is the longest element. $\square$

The result above implies that the algebra homomorphism $K^G(Z_\Delta) \to K^G(Z)$ is injective. Hence, we can view $K^G(Z_\Delta)$ as a subalgebra of $K^G(Z)$.

5.3 Affine Hecke algebras with unequal parameters

Let $\mathcal{R} = (X^*, X_*, \Phi, \Phi^\vee)$ be a simple root datum of non-simply laced type (i.e. the $\mathbb{Z}$-span of Φ is of finite index in X^* and Φ is of type B_n, C_n, G_2 or F_4). Then we have

$$\Phi = \Phi_s \sqcup \Phi_l$$

where Φ_s is the set of short roots and Φ_l is the set of long roots. We pick systems of simple and positive roots $\Pi \subset \Phi^+ \subset \Phi$. For any $\alpha \in \Pi$, we denote by $q_\alpha \in \mathbb{Z}[q_s^{\pm 1}, q_l^{\pm 1}]$ the element

$$q_\alpha := \begin{cases} q_s & \alpha \in \Phi_s, \\ q_l & \alpha \in \Phi_l. \end{cases}$$

Definition 5.3.1. *[47] The affine Hecke algebra with two unequal parameters $\mathcal{H}^{\mathrm{aff}}_{q_s,q_l} = \mathcal{H}^{\mathrm{aff}}_{q_s,q_l}(\mathcal{R})$ is the $\mathbb{Z}[q_s^{\pm 1}, q_l^{\pm 1}]$-algebra with basis $\{T_w e^\lambda \mid w \in W, \lambda \in X^*\}$ and multiplication rules* *(i)* $T_{s_\alpha}^2 = (q_\alpha - 1)T_{s_\alpha} + q_\alpha$ $\qquad\qquad$ $\alpha \in \Pi$,

(ii) $T_w T_{w'} = T_{ww'}$ $\qquad\qquad$ $l(ww') = l(w) + l(w')$,

(iii) $e^\lambda e^\mu = e^{\lambda+\mu}$ $\qquad\qquad$ $\lambda, \mu \in X^*$,

(iv) $e^\lambda T_{s_\alpha} - T_{s_\alpha} e^{s_\alpha(\lambda)} = (q_\alpha - 1)\frac{e^\lambda - e^{s_\alpha(\lambda)}}{1-e^{-\alpha}}$ $\qquad\qquad$ $\alpha \in \Pi,\ \lambda \in X^*$,

where $T_w := T_w e^0$ *and* $e^\lambda := T_e e^\lambda$.

Remark 5.3.2. *There are several definitions and conventions in the literature which sometimes differ slightly from our convention. First, the affine Hecke algebra is often defined with respect to the dual root datum, i.e. using the cocharacter lattice instead of the character lattice. Moreover, our definition differs slightly from the one in [47] which only considers parameters which are powers of a single parameter* q*. However, the definitions and basic results also hold for parameters which are independent variables* q_α*. Finally, loc.cit. allows an additional third parameter in adjoint type* B *(i.e. simply connected type* C *in the dual convention) attached to the affine node of the affine type* C *Dynkin diagram. In our definition this additional parameter is taken to be equal to the parameter of the unique simple short root in type* B *(resp. the parameter of the unique simple long root in the dual type* C *convention).*

Here are some elementary algebraic properties of $\mathcal{H}^{\mathrm{aff}}_{q_s,q_l}$ which easily follow from the definition (see also [47]).

Lemma 5.3.3. *(i) The* $\{T_w \mid w \in W\}$ *span a subalgebra* $\mathcal{H}_{q_s,q_l} \subset \mathcal{H}^{\mathrm{aff}}_{q_s,q_l}$ *which is the (finite) Hecke algebra with unequal parameters.*

(ii) The e^λ *span a commutative subalgebra of* $\mathcal{H}^{\mathrm{aff}}_{q_s,q_l}$ *which is isomorphic to the group algebra* $\mathbb{Z}[q_s^{\pm1}, q_l^{\pm1}][X^*]$.

(iii) $\mathcal{H}^{\mathrm{aff}}_{q_s,q_l}$ *is a free right (and also left)* $\mathbb{Z}[q_s^{\pm1}, q_l^{\pm1}][X^*]$*-module with basis* $\{T_w \mid w \in W\}$.

(iv) $\mathcal{H}^{\mathrm{aff}}_{q_s,q_l}$ *is generated as an algebra by* $\mathbb{Z}[q_s^{\pm1}, q_l^{\pm1}][X^*]$ *together with the* T_{s_α} *for* $\alpha \in \Pi$.

Let sgn be the sign representation of $\mathcal{H}_{q_s,q_l}$, i.e. the free $\mathbb{Z}[q_s^{\pm1}, q_l^{\pm1}]$-module of rank 1 on which T_{s_α} acts as -1. The antispherical module is the $\mathcal{H}^{\mathrm{aff}}_{q_s,q_l}$-module

$$M_{asph} := H^{\mathrm{aff}}_{q_s,q_l} \otimes_{\mathcal{H}_{q_s,q_l}} sgn$$

which has the natural basis $\{e^\lambda \otimes 1 \mid \lambda \in X^*\}$.

Lemma 5.3.4. *M_{asph} is the unique $\mathcal{H}^{\mathrm{aff}}_{q_s,q_l}$-module with basis $\{e^\lambda \otimes 1 \mid \lambda \in X^*\}$ and multiplication rules*

(i) $e^\mu \cdot (e^\lambda \otimes 1) = e^{\mu+\lambda} \otimes 1$

(ii) $-(T_{s_\alpha} + q_\alpha e^\alpha) \cdot e^\lambda \otimes 1 = (1 - q_\alpha e^\alpha)\frac{e^\lambda - e^{s_\alpha(\lambda)-\alpha}}{1-e^{-\alpha}} \otimes 1.$

Proof. The uniqueness follows from the fact that the e^λ together with the T_{s_α} generate $\mathcal{H}^{\mathrm{aff}}_{q_s,q_l}$ (Lemma 5.3.3). The equality $e^\mu \cdot (e^\lambda \otimes 1) = e^{\mu+\lambda} \otimes 1$ follows from the definition. Using the relation

$$-T_{s_\alpha} e^\lambda = -e^{s_\alpha(\lambda)} T_{s_\alpha} + (q_\alpha - 1)\frac{e^{s_\alpha(\lambda)} - e^\lambda}{1 - e^{-\alpha}}$$

we get

$$\begin{aligned}
-(T_{s_\alpha} + q_\alpha e^\alpha) \cdot e^\lambda \otimes 1 &= \left(-e^{s_\alpha(\lambda)} T_{s_\alpha} + (q_\alpha - 1)\frac{e^{s_\alpha(\lambda)} - e^\lambda}{1 - e^{-\alpha}} - q_\alpha e^\alpha e^\lambda\right) \otimes 1 \\
&= \left(e^{s_\alpha(\lambda)} + (q_\alpha - 1)\frac{e^{s_\alpha(\lambda)} - e^\lambda}{1 - e^{-\alpha}} - q_\alpha e^{\lambda+\alpha}\right) \otimes 1 \\
&= \left(\frac{e^{s_\alpha(\lambda)} - e^{s_\alpha(\lambda)-\alpha}}{1 - e^{-\alpha}} + (q_\alpha - 1)\frac{e^{s_\alpha(\lambda)} - e^\lambda}{1 - e^{-\alpha}} - q_\alpha \frac{e^{\lambda+\alpha} - e^\lambda}{1 - e^{-\alpha}}\right) \otimes 1 \\
&= (1 - q_\alpha e^\alpha)\frac{e^\lambda - e^{s_\alpha(\lambda)-\alpha}}{1 - e^{-\alpha}} \otimes 1.
\end{aligned}$$

□

Lemma 5.3.5. *[52, (4.3.10)] M_{asph} is a faithful $\mathcal{H}^{\mathrm{aff}}_{q_s,q_l}$-module, i.e. the $\mathcal{H}^{\mathrm{aff}}_{q_s,q_l}$-module structure on M_{asph} induces an injective algebra homomorphism $\mathcal{H}^{\mathrm{aff}}_{q_s,q_l} \hookrightarrow \mathrm{End}_{\mathbb{Z}}(M_{asph})$.*

Let G be the simple algebraic group with root datum $\mathcal{R}$, defined over an algebraically closed field k of characteristic p where

- $p = 2$ if $G \neq G_2$ (i.e. if G is of type B_n, C_n or F_4)
- $p = 3$ if $G = G_2$.

We pick a maximal torus $T \subset G$ and denote the corresponding (geometric) Borel subgroup by B. Let $\mathfrak{g}$ be the adjoint representation of G with the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha.$$

We define

$$\mathfrak{g}_s = \sum_{\alpha\in\Phi_s} \mathfrak{h}_\alpha + \sum_{\alpha\in\Phi_s} \mathfrak{g}_\alpha \subset \mathfrak{g}$$

where $\mathfrak{h}_\alpha := [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$. The following theorem is a consequence of the Chevalley formulas for the adjoint action and our assumption on the characteristic p.

Theorem 5.3.6. *[35, 36] The subspace $\mathfrak{g}_s \subset \mathfrak{g}$ is G-stable.*

We can equip

$$V := \mathfrak{g}_s \oplus \mathfrak{g}/\mathfrak{g}_s$$

with the structure of a $G \times \mathbb{G}_m \times \mathbb{G}_m$-module via

$$(g, t_s, t_l) \cdot (v_s, v_l) = (t_s^{-1} g v_s, t_l^{-1} g v_l).$$

Note that the non-zero weights of V are precisely the roots of G and all non-zero weight spaces are one-dimensional. Consider the B-stable subspace

$$V^- := \bigoplus_{\alpha\in\Phi^-} V_\alpha \subset V.$$

Then we can consider the varieties $\tilde{V} = G \times^B V^-, Z, Z_w, ..$ as in the previous section. For any $w \in W$, we will write ${}^wV^- := \dot{w}V^-$ where $\dot{w} \in N_G(T)$ is any lift of w. For any simple root $\alpha \in \Pi$, we denote the corresponding parabolic subgroup by

$$P_\alpha = B \sqcup B\dot{s}_\alpha B.$$

Lemma 5.3.7. *For each simple root $\alpha \in \Pi$, the subspace $V^- \cap {}^{s_\alpha}V^-$ is P_α-stable.*

Proof. Note that $V^- \cap {}^{s_\alpha}V^- = \bigoplus_{\gamma\in\Phi^-\setminus\{-\alpha\}} V_\gamma$. Since $U_\beta \cdot V_\lambda \subset \bigoplus_{n\geq 0} V_{\lambda+n\beta}$, the subspace $V^- \cap {}^{s_\alpha}V^-$ is U_β stable for all $\beta \in \Phi^-$. It is also T-stable and thus also B-stable. It is also stable under $\dot{s}_\alpha$ and hence also under P_α. □

Define the varieties

$$\begin{aligned}
\tilde{V}_\alpha &:= G \times^B (V^- \cap {}^{s_\alpha}V^-) \\
\bar{V}_\alpha &:= G \times^{P_\alpha} (V^- \cap {}^{s_\alpha}V^-) \\
\Lambda_\alpha &:= \tilde{V}_\alpha \times_{\bar{V}_\alpha} \tilde{V}_\alpha.
\end{aligned}$$

Then $\tilde{V}_\alpha \subset \tilde{V}$ is a subbundle and $\Lambda_\alpha \subset Z$ is closed.

Lemma 5.3.8. *We have $\Lambda_\alpha = \overline{Z}_{s_\alpha}$.*

Proof. The fiber of $(B, \dot{s}_\alpha B) \in \mathcal{B} \times \mathcal{B}$ under the projection $Z \to \mathcal{B} \times \mathcal{B}$ is precisely $V^- \cap {}^{s_\alpha}V^-$ which is also the fiber under $\Lambda_\alpha \to \mathcal{B} \times \mathcal{B}$. By G-equivariance, this implies $Z_{s_\alpha} \subset \Lambda_\alpha$ and hence

$$\overline{Z}_{s_\alpha} \subset \Lambda_\alpha. \tag{5.6}$$

Note that the map $\tilde{V}_\alpha \to \bar{V}_\alpha$ is a locally trivial $\mathbb{P}^1$-fibration. By base change, $\Lambda_\alpha \to \tilde{V}_\alpha$ is also a locally trivial $\mathbb{P}^1$-fibration. In particular, this map is flat and hence open [64, Tag 01UA] which implies that Λ_α is irreducible by [64, Tag 004Z]. Moreover, we have

$$\begin{aligned}
\dim \Lambda_\alpha &= \dim \tilde{V}_\alpha + 1 \\
&= \dim G - \dim B + \dim V^- \cap {}^{s_\alpha}V^- + 1 \\
&= \dim Y_{s_\alpha} + \dim V^- \cap {}^{s_\alpha}V^- \\
&= \dim Z_{s_\alpha} \\
&= \dim \overline{Z}_{s_\alpha}.
\end{aligned}$$

This implies that $\Lambda_\alpha = \overline{Z}_{s_\alpha}$ by the irreducibility of Λ_α and (5.6). $\square$

Given $[\mathcal{F}] \in K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(Z)$ and $[\mathcal{G}] \in K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(\tilde{V})$, we define their convolution as

$$[\mathcal{F}] \star [\mathcal{G}] := (p_1)_*([\mathcal{F}] \otimes^L_{\tilde{V}\times\tilde{V}} p_2^*[\mathcal{G}]) \in K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(\tilde{V}).$$

This equips $K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(\tilde{V})$ with the structure of a $K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(Z)$-module. Recall from (5.4) that we have a canonical identification

$$\mathbb{Z}[q_s^{\pm 1}, q_l^{\pm 1}][X^*] \cong K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(\tilde{V}). \tag{5.7}$$

Proposition 5.3.9. *For any $\lambda \in X^*$, $\alpha \in \Pi$, convolution of $[\mathcal{O}_{\overline{Z}_{s_\alpha}}] \in K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(Z)$ with $e^\lambda \in K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(\tilde{V})$ is given by*

$$[\mathcal{O}_{\overline{Z}_{s_\alpha}}] \star e^\lambda = (1 - q_\alpha e^\alpha)\frac{e^\lambda - e^{s_\alpha(\lambda)-\alpha}}{1 - e^{-\alpha}} \in K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(\tilde{V}).$$

Proof. Recall that by Lemma 5.3.8 we have $\overline{Z}_{s_\alpha} = \Lambda_\alpha$. Hence, we may compute the convolution of $[\mathcal{O}_{\Lambda_\alpha}]$ with e^λ. Let $\tilde{\iota}_\alpha : \Lambda_\alpha \hookrightarrow Z$ be the inclusion. There is a commutative diagram

$$\begin{tikzcd}
\tilde{V} \times \tilde{V} \arrow[rr, "p_2"] & & \tilde{V} \\
Z \arrow[u, hook, "\iota_Z"] \arrow[d, "p_1"'] & \Lambda_\alpha \arrow[l, hook', "\tilde{\iota}_\alpha"'] \arrow[r, "p_2"] \arrow[d, "p_1"] & \tilde{V}_\alpha \arrow[u, hook, "\iota_\alpha"'] \arrow[d, "\pi_\alpha"] \\
\tilde{V} & \tilde{V}_\alpha \arrow[l, hook', "\iota_\alpha"'] \arrow[r, "\pi_\alpha"] & \bar{V}_\alpha
\end{tikzcd}$$

where the bottom right square is cartesian. Using these maps, we have

$$
\begin{aligned}
[\mathcal{O}_{\Lambda_\alpha}] \star e^\lambda &\overset{def}{=} (p_1)_*((\tilde{\iota}_\alpha)_*[\mathcal{O}_{\Lambda_\alpha}] \otimes^L_{\tilde{V}\times\tilde{V}} p_2^* e^\lambda) \\
&\overset{Lemma\ 5.1.2}{=} (p_1)_*((\tilde{\iota}_\alpha)_*[\mathcal{O}_{\Lambda_\alpha}] \otimes \iota_Z^* p_2^* e^\lambda) \\
&\overset{(5.1)}{=} (p_1)_*(\tilde{\iota}_\alpha)_*\tilde{\iota}_\alpha^*\iota_Z^* p_2^* e^\lambda \\
&\overset{functoriality}{=} (\iota_\alpha)_*(p_1)_* p_2^*(\iota_\alpha)^* e^\lambda \\
&\overset{base\ change}{=} (\iota_\alpha)_*(\pi_\alpha)^*(\pi_\alpha)_*(\iota_\alpha)^* e^\lambda.
\end{aligned}
$$

By Lemma 5.2.2 $(\pi_\alpha)^*(\pi_\alpha)_*$ acts as $e^\lambda \mapsto \frac{e^\lambda - e^{s_\alpha(\lambda)-\alpha}}{1-e^{-\alpha}}$. Note that $V^-/(V^- \cap {}^{s_\alpha}V^-) \cong V_{-\alpha}$ is one-dimensional of weight $q_\alpha^{-1}e^{-\alpha}$. Thus, by Lemma 5.2.1 $(\iota_\alpha)_*$ acts as multiplication with $1 - q_\alpha e^\alpha$. Finally, $(\iota_\alpha)^*$ is given by $e^\lambda \mapsto e^\lambda$. Hence, we have

$$
[\mathcal{O}_{\Lambda_\alpha}] \star e^\lambda = (1 - q_\alpha e^\alpha)\frac{e^\lambda - e^{s_\alpha(\lambda)-\alpha}}{1-e^{-\alpha}}.
$$

□

Our next goal is to determine a nice generating set of the algebra $K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(Z)$. For this, we need the following transversality result.

Lemma 5.3.10. *Let $w \in W$ and $\alpha \in \Pi$ such that $l(s_\alpha w) = l(w) + 1$. Then the varieties $Z_{s_\alpha} \times \tilde{V}$ and $\tilde{V} \times Z_w$ intersect transversally in $\tilde{V}^3$. Moreover, the projection onto the first and third factor induces an isomorphism*

$$
p_{13} : (Z_{s_\alpha} \times \tilde{V}) \cap (\tilde{V} \times Z_w) \xrightarrow{\sim} Z_{s_\alpha w}. \tag{5.8}
$$

Proof. Let $(y_1, y_2, y_3) \in \tilde{V}^3$ lie in the intersection of $Z_{s_\alpha} \times \tilde{V}$ and $\tilde{V} \times Z_w$. Note that there is a commutative diagram

$$
\begin{tikzcd}
(Z_{s_\alpha} \times \tilde{V}) \cap (\tilde{V} \times Z_w) \arrow[r, "p_{13}"] \arrow[d, "\pi^3"] & Z_{s_\alpha w} \arrow[d, "\pi^2"] \\
(Y_{s_\alpha} \times \mathcal{B}) \cap (\mathcal{B} \times Y_w) \arrow[r, "p_{13}"] & Y_{s_\alpha w}.
\end{tikzcd} \tag{5.9}
$$

It is well-known that the horizontal map in the bottom row is an isomorphism when $l(s_\alpha w) = l(w) + 1$. Hence, by equivariance, we may assume that (y_1, y_2, y_3) maps to $p_{13}^{-1}(\dot{s}_\alpha B, \dot{w}B) = (\dot{s}_\alpha B, eB, \dot{w}B) \in \mathcal{B}^3$ under the projection $\tilde{V}^3 \xrightarrow{\pi^3} \mathcal{B}^3$. To prove transversality of $Z_{s_\alpha} \times \tilde{V}$ and $\tilde{V} \times Z_w$, it suffices to show that the map

$$
T_{(y_1,y_2)}Z_{s_\alpha} \oplus T_{(y_2,y_3)}Z_w \xrightarrow{(p_2)_*\oplus(p_1)_*} T_{y_2}\tilde{V} \tag{5.10}
$$

is surjective. Note that there are isomorphisms

$$\begin{aligned} G \times^{B \cap {}^{w}B} (V^- \cap {}^{w}V^-) &\xrightarrow{\sim} Z_w \\ (g, v) &\mapsto (gB, gv, g\dot{w}B, gv), \end{aligned}$$

$$\begin{aligned} G \times^{B \cap {}^{s_\alpha}B} (V^- \cap {}^{s_\alpha}V^-) &\xrightarrow{\sim} Z_{s_\alpha} \\ (g, v) &\mapsto (g\dot{s}_\alpha B, gv, gB, gv), \end{aligned}$$

$$\begin{aligned} G \times^{B} (V^-) &\xrightarrow{\sim} \tilde{V} \\ (g, v) &\mapsto (gB, gv). \end{aligned}$$

Let $y_2 = (eB, v)$. Then $v \in {}^{s_\alpha}V^- \cap V^- \cap {}^{w}V^-$ and the isomorphisms above identify the tuple (e, v) with (y_2, y_3), (y_1, y_2) and y_2 respectively. The maps $p_1 : Z_w \to \tilde{V}$ and $p_2 : Z_{s_\alpha} \to \tilde{V}$ then correspond to the maps

$$\begin{aligned} G \times^{B \cap {}^{w}B} (V^- \cap {}^{w}V^-) &\to G \times^{B} V^- \\ (g, v) &\mapsto (g, v) \end{aligned} \tag{5.11}$$

and

$$\begin{aligned} G \times^{B \cap {}^{s_\alpha}B} (V^- \cap {}^{s_\alpha}V^-) &\to G \times^{B} V^- \\ (g, v) &\mapsto (g, v). \end{aligned} \tag{5.12}$$

The map on tangent spaces at (e, v) induced by (5.11) is

$$\begin{aligned} \mathfrak{g}/(\mathfrak{b} \cap {}^{w}\mathfrak{b}) \oplus (V^- \cap {}^{w}V^-) &\to \mathfrak{g}/\mathfrak{b} \oplus V^- \\ (x + \mathfrak{b} \cap {}^{w}\mathfrak{b}, v) &\mapsto (x + \mathfrak{b}, v). \end{aligned}$$

Similarly, (5.12) corresponds to the map

$$\begin{aligned} \mathfrak{g}/(\mathfrak{b} \cap {}^{s_\alpha}\mathfrak{b}) \oplus (V^- \cap {}^{s_\alpha}V^-) &\to \mathfrak{g}/\mathfrak{b} \oplus V^- \\ (x + \mathfrak{b} \cap {}^{w}\mathfrak{b}, v) &\mapsto (x + \mathfrak{b}, v) \end{aligned}$$

on tangent spaces at (e, v). Since $l(s_\alpha w) = l(w) + 1$, we have $\alpha \in {}^{w}\Phi^+$ and thus $V_{-\alpha} \subset V^- \cap {}^{w}V^-$. It now follows that

$$V^- = (V^- \cap {}^{s_\alpha}V^-) + (V^- \cap {}^{w}V^-)$$

Hence, by the explicit description of the maps on tangent spaces above, it follows that (5.10) is surjective. This completes the proof of transversality.

Recall that the map in the bottom row in (5.9) is an isomorphism. Thus, to prove that the map from (5.8) is an isomorphism, it suffices to show that the map in the top row in (5.9) induces an isomorphim on the fibers of the vertical maps. By equivariance, it suffices to do this for the fibers over $(\dot{s}_\alpha B, eB, \dot{w}B) \in \mathcal{B}^3$ and $(\dot{s}_\alpha B, \dot{w}B) \in \mathcal{B}^2$. These fibers are given by ${}^{s_\alpha}V^- \cap V^- \cap {}^{w}V^-$ and ${}^{s_\alpha}V^- \cap {}^{w}V^-$ respectively. Note that

$$ {}^{s_\alpha}\Phi^+ \cap \Phi^+ \cap {}^{w}\Phi^+ = ({}^{s_\alpha}\Phi^+ \backslash \{-\alpha\}) \cap {}^{w}\Phi^+ = {}^{s_\alpha}\Phi^+ \cap {}^{w}\Phi^+ $$

since $-\alpha \notin {}^{w}\Phi^+$. This implies

$$ {}^{s_\alpha}V^- \cap V^- \cap {}^{w}V^- = {}^{s_\alpha}V^- \cap {}^{w}V^- $$

which finishes the proof. $\square$

Corollary 5.3.11. *$K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(Z)$ is generated as an algebra by $K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(Z_\Delta)$ together with the $[\mathcal{O}_{\overline{Z}_{s_\alpha}}]$ for $\alpha \in \Pi$.*

Proof. Let $\mathcal{A}$ be the subalgebra of $K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(Z)$ generated by $K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(Z_\Delta)$ together with the $[\mathcal{O}_{\overline{Z}_{s_\alpha}}]$ for $\alpha \in \Pi$. We prove by induction along the length of w that $[\mathcal{O}_{\overline{Z}_w}] \in \mathcal{A}$ for all $w \in W$. If $l(w) \in \{0,1\}$, the claim is true by construction. Now let $w \in W$ and $\alpha \in \Pi$ such that $l(s_\alpha w) = l(w) + 1$ and assume we have shown the claim for all $y \in W$ with $l(y) \leq l(w)$. Let

$$ \begin{aligned} Z_{s_\alpha,w} &:= (\overline{Z}_{s_\alpha} \times \tilde{V}) \cap (\tilde{V} \times \overline{Z}_w) \\ Z^\circ_{s_\alpha,w} &:= (Z_{s_\alpha} \times \tilde{V}) \cap (\tilde{V} \times Z_w). \end{aligned} $$

The projection onto the first and third factor restricts to a map

$$ p_{13} : Z_{s_\alpha,w} \to Z_{\leq s_\alpha w}. $$

Hence, we have

$$ [\mathcal{O}_{\overline{Z}_{s_\alpha}}] \star [\mathcal{O}_{\overline{Z}_w}] \in K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(Z_{\leq s_\alpha w}). $$

Let

$$ U := \tilde{V}^3 \backslash ((Z_{<s_\alpha} \times \tilde{V}) \cup (\tilde{V} \times Z_{<w})) $$

which is an open subset of $\tilde{V}^3$. Note that

$$ \begin{aligned} U \cap (\overline{Z}_{s_\alpha} \times \tilde{V}) &= U \cap (Z_{s_\alpha} \times \tilde{V}) \\ U \cap (\tilde{V} \times \overline{Z}_w) &= U \cap (\tilde{V} \times Z_w). \end{aligned} \tag{5.13} $$

In particular, we have

$$U \cap Z_{s_\alpha,w} = U \cap Z^\circ_{s_\alpha,w} = Z^\circ_{s_\alpha,w}. \tag{5.14}$$

Thus, $Z^\circ_{s_\alpha,w} \subset Z_{s_\alpha,w}$ is open and the flat pullback

$$\iota^*_{Z^\circ_{s_\alpha,w}} : K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(Z_{s_\alpha,w}) \to K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(Z^\circ_{s_\alpha,w})$$

agrees with the restriction with support map

$$\iota^*_U : K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(Z_{s_\alpha,w}) \to K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(Z^\circ_{s_\alpha,w})$$

coming from the inclusion $\iota_U : U \hookrightarrow \tilde{V}^3$. Moreover, (5.13) together with Lemma 5.3.10 implies that the intersection of $U \cap (\overline{Z}_{s_\alpha} \times \tilde{V})$ and $U \cap (\tilde{V} \times \overline{Z}_w)$ is transverse in U. Using these facts, we can compute the restriction of $[\mathcal{O}_{\overline{Z}_{s_\alpha}}] \star [\mathcal{O}_{\overline{Z}_w}]$ to $Z_{s_\alpha w} \subset Z_{\leq s_\alpha w}$:

$$\begin{aligned}
\iota^*_{Z_{s_\alpha w}}([\mathcal{O}_{\overline{Z}_{s_\alpha}}] \star [\mathcal{O}_{\overline{Z}_w}]) &\overset{def}{=} \iota^*_{Z_{s_\alpha w}}(p_{13})_*(p_{12}^*[\mathcal{O}_{\overline{Z}_{s_\alpha}}] \otimes^L_{\tilde{V}^3} p_{23}^*[\mathcal{O}_{\overline{Z}_w}]) \\
&\overset{base\ change}{=} (p_{13})_*\iota^*_{Z^\circ_{s_\alpha,w}}(p_{12}^*[\mathcal{O}_{\overline{Z}_{s_\alpha}}] \otimes^L_{\tilde{V}^3} p_{23}^*[\mathcal{O}_{\overline{Z}_w}]) \\
&= (p_{13})_*\iota^*_U(p_{12}^*[\mathcal{O}_{\overline{Z}_{s_\alpha}}] \otimes^L_{\tilde{V}^3} p_{23}^*[\mathcal{O}_{\overline{Z}_w}]) \\
&\overset{Lemma\ 5.1.4}{=} (p_{13})_*(\iota^*_U p_{12}^*[\mathcal{O}_{\overline{Z}_{s_\alpha}}] \otimes^L_U \iota^*_U p_{23}^*[\mathcal{O}_{\overline{Z}_w}]) \\
&= (p_{13})_*([\mathcal{O}_{U\cap(\overline{Z}_{s_\alpha}\times\tilde{V})}] \otimes^L_U [\mathcal{O}_{U\cap(\tilde{V}\times\overline{Z}_w)}]) \\
&\overset{Lemma\ 5.1.3}{=} (p_{13})_*[\mathcal{O}_{U\cap Z_{s_\alpha,w}}] \\
&\overset{(5.14)}{=} (p_{13})_*[\mathcal{O}_{Z^\circ_{s_\alpha,w}}] \\
&\overset{Lemma\ 5.3.10}{=} [\mathcal{O}_{Z_{s_\alpha w}}].
\end{aligned}$$

Using the short exact sequence from (5.5), this implies that

$$[\mathcal{O}_{\overline{Z}_{s_\alpha}}] \star [\mathcal{O}_{\overline{Z}_w}] \in [\mathcal{O}_{\overline{Z}_{s_\alpha w}}] + K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(Z_{<s_\alpha w}).$$

The proof of Lemma 5.2.3 shows that $K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(Z_{<s_\alpha w})$ is a free $K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(Z_\Delta)$-module with basis given by the $[\mathcal{O}_{\overline{Z}_y}]$ with $y < s_\alpha w$. Thus, by the induction hypothesis, we get $[\mathcal{O}_{\overline{Z}_{s_\alpha w}}] \in \mathcal{A}$. Hence, $\mathcal{A}$ contains all $[\mathcal{O}_{\overline{Z}_w}]$ for $w \in W$ and thus $\mathcal{A} = K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(Z)$ by Lemma 5.2.3. This completes the proof. □

We are now ready to prove the main result of this chapter.

Theorem 5.3.12. *There is an isomorphism of algebras* $K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(Z) \cong \mathcal{H}^{\mathrm{aff}}_{q_s,q_l}$.

Proof. We can identify

$$M_{asph} \overset{Lemma\ 5.3.4}{\cong} \mathbb{Z}[q_s^{\pm1}, q_l^{\pm1}][X^*] \overset{(5.7)}{\cong} K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(\tilde{V}).$$

Hence, we get algebra homorphisms

$$\begin{aligned}\alpha_1 : K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(Z) &\to \mathrm{End}_{\mathbb{Z}[q_s^\pm,q_l^\pm]}(M_{asph})\\ \alpha_2 : \mathcal{H}^{\mathrm{aff}}_{q_s,q_l} &\to \mathrm{End}_{\mathbb{Z}[q_s^\pm,q_l^\pm]}(M_{asph}).\end{aligned}$$

By Proposition 5.3.9, Lemma 5.1.7 and Lemma 5.3.4 we have

$$\begin{aligned}\alpha_1([\mathcal{O}_{\overline{Z}_{s_\alpha}}]) &= -\alpha_2(T_{s_\alpha} + q_\alpha e^\alpha)\\ \alpha_1((\Delta_{\tilde{V}})_* e^\lambda) &= \alpha_2(e^\lambda)\end{aligned}$$

where $\Delta_{\tilde{V}} : \tilde{V} \hookrightarrow Z$ is the diagonal embedding. Thus, by Corollary 5.3.11 and Lemma 5.3.3, α_1 and α_2 have the same image in $\mathrm{End}_{\mathbb{Z}[q_s^\pm,q_l^\pm]}(M_{asph})$. Moreover, α_2 is injective by Lemma 5.3.5. Hence, we get a surjective homomorphism of algebras

$$\alpha_2^{-1}\circ\alpha_1 : K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(Z) \to \mathcal{H}^{\mathrm{aff}}_{q_s,q_l}$$

which restricts to an isomorphism on polynomial subalgebras

$$\alpha_2^{-1}\circ\alpha_1 : K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(Z_\Delta) \xrightarrow{\sim} \mathbb{Z}[q_s^{\pm1}, q_l^{\pm1}][X^*].$$

Note that $K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(Z)$ and $\mathcal{H}^{\mathrm{aff}}_{q_s,q_l}$ are free modules of rank $|W|$ over this commutative noetherian subalgebra (by Lemmas 5.2.3 and 5.3.3). Surjective maps between free modules of the same (finite) rank over a commutative noetherian ring are always isomorphisms (c.f. [7, Exercise 6.1]), so $\alpha_2^{-1}\circ\alpha_1$ is an isomorphism. □

Remark 5.3.13. *The theorem above provides a geometric realization of all affine Hecke algebras with unequal parameters that arise from simple root data except in adjoint type B (resp. simply connected C if one defines the affine Hecke algebra with respect to the cocharacter lattice) where three unequal parameters are possible. A geometric realization of this affine Hecke algebra with three unequal parameters can be found in [39].*

In the equal parameter setting, the K-theory realization (Theorem 2.2.1) has been used to parametrize the irreducible representations of the affine Hecke algebra and construct standard representations [24, 41] (see also Chapter 2). It would be interesting to obtain similar results for affine Hecke algebras with unequal parameters building on our K-theory realization from Theorem 5.3.12. The author intends to explore this in future work.

5.4 Graded Hecke algebras with unequal parameters

In [47, 51] Lusztig defined certain graded versions of affine Hecke algebras and studied their representations using geometric methods. These graded Hecke algebras can also be defined with respect to several unequal parameters, but the geometric constructions of Lusztig only realize these algebras for certain specific parameters (which is enough for his goal of classifying unipotent representation of p-adic groups [50]). In this section we discuss a conjectural realization of graded Hecke algebras of simple type with arbitrary unequal parameters via constructible sheaves. As before, we assume that $\mathcal{R} = (X^*, X_*, \Phi, \Phi^\vee)$ is a simple root datum of non-simply laced type, $\Pi \subset \Phi$ is a system of simple roots and $\Phi = \Phi_s \sqcup \Phi_l$ is the decomposition into short and long roots. We set

$$\mathfrak{t} := \mathbb{C} \otimes_{\mathbb{Z}} X_*$$

and for any $\alpha \in \Pi$, we denote by $r_\alpha \in \mathbb{C}[r_s, r_l]$ the element

$$r_\alpha := \begin{cases} r_s & \alpha \in \Phi_s, \\ r_l & \alpha \in \Phi_l. \end{cases}$$

Definition 5.4.1. *[47] The graded Hecke algebra with two unequal parameters $\mathbb{H}_{r_s,r_l} = \mathbb{H}_{r_s,r_l}(\mathcal{R})$ is the $\mathbb{C}$-vector space*

$$\mathbb{H}_{r_s,r_l} = \mathbb{C}[r_s, r_l] \otimes_{\mathbb{C}} \mathbb{C}[W] \otimes_{\mathbb{C}} S(\mathfrak{t}^*)$$

with algebra structure uniquely determined by

(i) $\mathbb{C}[r_s, r_l] \otimes_{\mathbb{C}} \mathbb{C}[W] \subset \mathbb{H}_{r_s,r_l}$ *is a subalgebra,*

(ii) $\mathbb{C}[r_s, r_l] \otimes_{\mathbb{C}} S(\mathfrak{t}^*) \subset \mathbb{H}_{r_s,r_l}$ *is a subalgebra,*

(iii) $\xi \cdot s_\alpha - s_\alpha \cdot s_\alpha(\xi) = r_\alpha \frac{\xi - s_\alpha(\xi)}{\alpha}$ *for any* $\alpha \in \Pi$, $\xi \in S(\mathfrak{t}^*)$.

This becomes a graded $\mathbb{C}$-algebra by setting

$$\begin{aligned} \deg(w) &= 0 && w \in W, \\ \deg(r_\alpha) &= 2 && \alpha \in \Pi, \\ \deg(\xi) &= 2 && \xi \in \mathfrak{t}^* \backslash \{0\}. \end{aligned}$$

Let $G, V = \mathfrak{g}_s \oplus \mathfrak{g}/\mathfrak{g}_s, \tilde{V}, Z, \ldots$ be as in the previous section. Then we can consider the (equivariant) constructible derived category as defined in Section 2.3. In particular, there is the 'Springer sheaf'

$$\mathbf{S} := \mu_* \mathcal{C}_{\tilde{V}} \in D^b_{G\times\mathbb{G}_m\times\mathbb{G}_m,c}(V).$$

We conjecture that there is the following geometric realization of the graded Hecke algebra with unequal parameters.

Conjecture 5.4.2. *There is an isomorphism of graded $\mathbb{C}$-algebras*

$$\mathrm{Hom}^*_{G\times\mathbb{G}_m\times\mathbb{G}_m}(\mathbf{S}, \mathbf{S}) \cong \mathbb{H}_{r_s, r_l}.$$

Remark 5.4.3. *An interesting consequence of this conjecture would be that*

$$\mathrm{Hom}_G(\mathbf{S}, \mathbf{S}) \cong \mathbb{C}[W].$$

This would allow to parametrize the irreducible representations of W by the simple perverse constituents of $\mathbf{S}$ similar to Springer theory. Note that $\mathbf{S}$ is supported on the analogue of the nilpotent cone

$$\mathfrak{N} := \mathrm{im}(\mu).$$

We expect that $\mathfrak{N}$ has only finitely many G-orbits (the author has verified this by explicit computation in the G_2 case, but we do not know if this is true in general). Using (2.5)*, this would yield a combinatorial parametrization of the irreducible W-representations via pairs (n, ρ) similar to the Springer correspondence where $n \in \mathfrak{N}$ and ρ is an irreducible representation of the component group $A(n)$.*

It seems reasonable to expect that Conjecture 5.4.2 can be proved by techniques similar to the proof of Theorem 5.3.12. We briefly sketch how one might attempt to execute such a proof. For simplicity, we will ignore any grading shifts/flips. Recall that the proof of Theorem 5.3.12 relies on the observation that the antispherical module arises via the convolution action of $K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(Z)$ on $K^{G\times\mathbb{G}_m\times\mathbb{G}_m}(\tilde{V})$. Note that there is an isomorphism

$$\mathrm{Hom}^*_{G\times\mathbb{G}_m\times\mathbb{G}_m}(\mathbf{S}, \mathbf{S}) \cong H_*^{G\times\mathbb{G}_m\times\mathbb{G}_m}(Z)$$

and $H_*^{G\times\mathbb{G}_m\times\mathbb{G}_m}(Z)$ naturally acts on $H_*^{G\times\mathbb{G}_m\times\mathbb{G}_m}(\tilde{V})$ via an analogously defined convolution operation (see [24, §2.7] for more details on this in the characteristic 0 setting).

Moreover, there is a canonical isomorphism

$$\begin{aligned} H_*^{G\times\mathbb{G}_m\times\mathbb{G}_m}(\tilde{V}) &\cong H^*_{G\times\mathbb{G}_m\times\mathbb{G}_m}(\tilde{V}) \\ &\cong H^*_{G\times\mathbb{G}_m\times\mathbb{G}_m}(\mathcal{B}) \\ &\cong H^*_{B\times\mathbb{G}_m\times\mathbb{G}_m}(\mathrm{pt}) \\ &\cong \mathbb{C}[r_s, r_l]\otimes_{\mathbb{C}} S(\mathfrak{t}^*). \end{aligned}$$

Hence, as a $\mathbb{C}[r_s, r_l]$-module, $H_*^{G\times\mathbb{G}_m\times\mathbb{G}_m}(\tilde{V})$ can be identified with the antispherical module

$$\tilde{M}_{asph} := \mathbb{H}_{r_s,r_l}\otimes_{\mathbb{C}[W]} sgn$$

where sgn is the sign representation of W. The action of $-(s_\alpha+1)$ on the antispherical module is given by

$$\begin{aligned} -(s_\alpha+1)\cdot\xi\otimes 1 &= -(s_\alpha(\xi)\cdot s_\alpha + r_\alpha\frac{\xi - s_\alpha(\xi)}{\alpha} + \xi)\otimes 1 \\ &= -(-s_\alpha(\xi) + r_\alpha\frac{\xi - s_\alpha(\xi)}{\alpha} + \xi)\otimes 1 \\ &= -(r_\alpha+\alpha)\frac{\xi - s_\alpha(\xi)}{\alpha}\otimes 1. \end{aligned}$$

This operation arises geometrically as follows: The map

$$\xi\mapsto\frac{\xi - s_\alpha(\xi)}{\alpha}$$

is the Demazure operator which is well-known to arise geometrically via

$$(p_\alpha)^*(p_\alpha)_* : H_*^{G\times\mathbb{G}_m\times\mathbb{G}_m}(\mathcal{B})\to H_*^{G\times\mathbb{G}_m\times\mathbb{G}_m}(\mathcal{B})$$

where $p_\alpha : G/B\to G/P_\alpha$ is the projection [15]. One can deduce from this (by an argument similar to Lemma 5.2.2) that the Demazure operator also corresponds to the map

$$(\pi_\alpha)^*(\pi_\alpha)_* : H_*^{G\times\mathbb{G}_m\times\mathbb{G}_m}(\tilde{V}_\alpha)\to H_*^{G\times\mathbb{G}_m\times\mathbb{G}_m}(\tilde{V}_\alpha)$$

where π_α is the map from Lemma 5.2.2. The map

$$\xi\mapsto -(r_\alpha+\alpha)\xi \tag{5.15}$$

arises geometrically as the map (c.f. [46, 1.10])

$$\iota_* : H_*^{T\times\mathbb{G}_m\times\mathbb{G}_m}(\{0\})\to H_*^{T\times\mathbb{G}_m\times\mathbb{G}_m}(k_{-\alpha})$$

where $\iota : \{0\} \hookrightarrow k_{-\alpha}$ is the inclusion and $k_{-\alpha}$ is the one-dimensional $T \times \mathbb{G}_m \times \mathbb{G}_m$-module with

$$(t, c_s, c_l) \cdot x = \begin{cases} \alpha(t)^{-1} c_s^{-1} x & \alpha \text{ short}, \\ \alpha(t)^{-1} c_l^{-1} x & \alpha \text{ long}. \end{cases}$$

Note that $k_{-\alpha} \cong V^-/(V^- \cap {}^{s_\alpha}V^-)$. It follows from this (by an argument similar to Lemma 5.2.1) that

$$(\iota_\alpha)_* : H_*^{G\times\mathbb{G}_m\times\mathbb{G}_m}(\tilde{V}_\alpha) \to H_*^{G\times\mathbb{G}_m\times\mathbb{G}_m}(\tilde{V})$$

is also given by the map in (5.15). Following the proof of Proposition 5.3.9, we expect that convolution with $[\overline{Z}_{s_\alpha}]$ on $H_*^{G\times\mathbb{G}_m\times\mathbb{G}_m}(\tilde{V})$ is given by $(\iota_\alpha)_*(\pi_\alpha)^*(\pi_\alpha)_*(\iota_\alpha)^*$ which is the map

$$\xi \mapsto -(r_\alpha + \alpha)\frac{\xi - s_\alpha(\xi)}{\alpha}.$$

This is precisely the action of $-(s_\alpha + 1)$ on the antispherical module. Hence, we expect that the convolution action of $H_*^{G\times\mathbb{G}_m\times\mathbb{G}_m}(Z)$ on $H_*^{G\times\mathbb{G}_m\times\mathbb{G}_m}(\tilde{V})$ realizes the antispherical module $\tilde{M}_{asph}$. The rest of the proof of Theorem 5.3.12 is purely formal, so it seems reasonable to expect that a similar argument should work in the equivariant Bore-Moore homology setting. This would prove Conjecture 5.4.2.

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