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Bordifications of the moduli spaces of tropical curves and abelian varieties, and unstable cohomology of $\mathbf{GL}_g(\mathbb{Z})$ and $\mathbf{SL}_g(\mathbb{Z})$

Francis Brown[1]

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Abstract
We construct bordifications of the moduli spaces of tropical curves and of tropical abelian varieties, and show that the tropical Torelli map extends to their bordifications. We prove that the classical bi-invariant differential forms studied by Cartan and others extend to these bordifications by studying their behaviour at infinity, and consequently deduce infinitely many new non-zero unstable classes in the cohomology of the general and special linear groups $\mathrm{GL}_g(\mathbb{Z})$ and $\mathrm{SL}_g(\mathbb{Z})$. In particular, we obtain a new geometric proof of Borel's theorem on the stable cohomology of these groups. We completely determine the cohomology of the link of the moduli space of tropical abelian varieties within a certain range, and show that it contains the stable cohomology of the general linear group. In addition, we define new transcendental invariants associated to the minimal vectors of quadratic forms, and show that a certain part of the cohomology of the general linear group $\mathrm{GL}_g(\mathbb{Z})$ admits the structure of a motive. In an Appendix, we give an algebraic construction of the Borel-Serre compactification by embedding it in the real points of an iterated blow-up of a projective space along linear subspaces, which may have independent applications.

1 Introduction

The main goal of this paper is an algebro-geometric construction of bordifications of the moduli spaces of tropical curves and abelian varieties. Before describing these in detail, we first present some applications to the cohomology of the special and general linear groups.

1.1 Unstable cohomology of linear groups

Let $\mathcal{P}_g$ denote the space of symmetric positive definite $g \times g$ matrices X with real entries. It is a connected contractible space equipped with a right action of $h \in \mathrm{GL}_g(\mathbb{R})$ via the map $h(X) = h^T X h$.

✉ F. Brown

1 Oxford, UK

g	$H^i(\mathrm{GL}_g(\mathbb{Z}))$					
2	H^0					
3	H^0					
4	H^0					
5	H^0	H^5				
6	H^0	H^5	$\boxed{H^8}$			
7	H^0	H^5		H^9	H^{14}	$\boxed{H^{15}}$

Fig. 1 Known ranges for which the cohomology of $\mathrm{GL}_g(\mathbb{Z})$ has been completely determined [31, 42, 56]. An entry H^i in the table indicates that $H^i(\mathrm{GL}_g(\mathbb{Z}); \mathbb{R})$ is non-zero and of rank 1. All non-zero classes are explained by Theorem 1.1 except for the two boxed entries

g												
2												
3	H_c^6											
4		H_c^7										
5			H_c^{10}					H_c^{15}				
6				H_c^{11}	$\boxed{H_c^{12}}$				H_c^{16}			
7						$\boxed{H_c^{13}}$	H_c^{14}			H_c^{19}	H_c^{23}	H_c^{28}

Fig. 2 Compactly-supported cohomology of $\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})$ in the known ranges [31, 42, 56]. An entry H_c^i in the table indicates that $H_c^i(\mathrm{GL}_g(\mathbb{Z}); \mathbb{R})$ is non-zero and of rank 1. All non-trivial classes are explained by Theorem 1.1 except for the two boxed entries

Since the orbifold $\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})$ is a $K(\pi, 1)$ one has $H^n(\mathrm{GL}_g(\mathbb{Z}); \mathbb{R}) = H^n(\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z}); \mathbb{R})$. Block direct sum of matrices $X \mapsto X \oplus 1$ defines a map $\mathcal{P}_g \to \mathcal{P}_{g+1}$. The stable cohomology is defined to be the limit with respect to these maps:

$$H^n(\mathrm{GL}_\infty(\mathbb{Z}); \mathbb{R}) = \varprojlim_g H^n(\mathrm{GL}_g(\mathbb{Z}); \mathbb{R}) \ .$$

It was famously computed by Borel [11], from which he deduced the ranks of the rational algebraic K-theory of the integers (and, more generally, of all number fields), which is of fundamental importance in the modern theory of motives. Very little is known about the unstable cohomology of the groups $\mathrm{SL}_g(\mathbb{Z})$ and $\mathrm{GL}_g(\mathbb{Z})$. See Figs. 1 and 3 for the range in which their cohomology groups are completely known.

Even less is known about the cohomology with compact supports $H_c^n(\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z}); \mathbb{R})$, which we shall denote by $H_c^n(\mathrm{GL}_g(\mathbb{Z}); \mathbb{R})$. The notation is justified by duality: indeed, when g is odd, the orbifold $\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})$ is orientable and its compactly supported cohomology is Poincaré dual to ordinary cohomology. However, in the case when g is even, the cohomology with and without compact supports are *a priori* unrelated. There is no stability property for cohomology with compact supports, as one may see from Fig. 2.

The ordinary cohomology of the special linear group $\mathrm{SL}_g(\mathbb{Z})$ coincides with that of $\mathrm{GL}_g(\mathbb{Z})$ when g is odd, but is built out of the cohomology of $\mathrm{GL}_g(\mathbb{Z})$ with and without compact supports in the case when g is even.

1.2 Cohomological results

Consider the abstract graded exterior algebra

$$\Omega^{\bullet}_{\mathrm{can}} = \bigwedge \bigoplus_{k \geq 1} \mathbb{Q}\,\omega^{4k+1}$$

with generators ω^{4k+1} in degree $4k+1$. For any $X \in \mathcal{P}_g$ and $k > 1$, let

$$\omega_X^{4k+1} = \mathrm{tr}((X^{-1}dX))^{4k+1} \ , \tag{1.1}$$

which is a well-defined closed differential form on $\mathcal{P}_g$ with a number of remarkable properties, including bi-invariance with respect to left and right multiplication by $\mathrm{GL}_g(\mathbb{R})$. Any $\omega \in \Omega^n_{\mathrm{can}}$ defines a closed differential form of degree n on $\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})$.

Consequently there is a natural map, which is sometimes referred to as either the 'Borel map' or 'Matsushima homomorphism':

$$j : \Omega^n_{\mathrm{can}} \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow H^n(\mathrm{GL}_g(\mathbb{Z}); \mathbb{R}) \ . \tag{1.2}$$

Borel proved that j is injective for $n \leq g/4$. Results of Matsushima [45] and Garland [34] imply that the map j is surjective in the same range. Although this range is narrow, it nonetheless tends to infinity, which allowed Borel to compute the stable limit:

$$\Omega^{\bullet}_{\mathrm{can}} \otimes_{\mathbb{Q}} \mathbb{R} \cong H^{\bullet}(\mathrm{GL}_{\infty}(\mathbb{Z}); \mathbb{R}) \ .$$

Let $g > 1$ be odd, and let $\Omega(g) \subset \Omega_{\mathrm{can}}$ denote the graded subalgebra generated by $\omega^5, \ldots, \omega^{2g-1}$. All forms ω^n of degree $n \geq 2g$ vanish identically on $\mathcal{P}_g$. In order to state our results, we divide $\Omega(g)$ into two subspaces. Let $\Omega_c(g) \subset \Omega(g)$ denote the ideal generated by ω^{2g-1}. We shall call $\Omega_c(g)$ the space of forms of 'compact type' and define $\Omega_{nc}(g) \subset \Omega(g)$ to be the graded subalgebra generated by $1, \omega^5, \ldots, \omega^{2g-5}$. The subscript nc stands for 'non-compact' type. One has $\Omega(g) = \Omega_c(g) \oplus \Omega_{nc}(g)$.

Theorem 1.1 *Let $g > 1$ be odd. There are injective maps*

$$(i).\ \ \Omega^{\bullet}_{nc}(g) \otimes_{\mathbb{Q}} \mathbb{R} \hookrightarrow H^{\bullet}(\mathrm{GL}_h(\mathbb{Z}); \mathbb{R}) \quad \text{for all } h \geq g \ ,$$

$$(ii).\ \ \Omega^{\bullet}_{c}(g) \otimes_{\mathbb{Q}} \mathbb{R} \hookrightarrow H_c^{\bullet+1}(\mathrm{GL}_g(\mathbb{Z}); \mathbb{R}) \ ,$$

$$(iii).\ \Omega^{\bullet}_{c}(g) \otimes_{\mathbb{Q}} \mathbb{R} \hookrightarrow H_c^{\bullet+2}(\mathrm{GL}_{g+1}(\mathbb{Z}); \mathbb{R}) \ .$$

These facts imply the following results about the special linear group:

$$(iv).\ \Omega^{\bullet}_{nc}(g) \otimes_{\mathbb{Q}} \mathbb{R} \hookrightarrow H^{\bullet}(\mathrm{SL}_h(\mathbb{Z}); \mathbb{R}) \quad \text{for all } h \geq g \ ,$$

$$(v).\ \ (\Omega^{\bullet}_{nc}(g) \oplus \Omega_c^{d_{g+1}-\bullet-2}(g)) \otimes_{\mathbb{Q}} \mathbb{R} \hookrightarrow H^{\bullet}(\mathrm{SL}_{g+1}(\mathbb{Z}); \mathbb{R}) \ ,$$

where $d_g = \dim \mathcal{P}_g = \binom{g+1}{2}$.

g	$\dim(\mathcal{P}_g)$	$n=0$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	3	1															
3	6	1															
4	10	1			1												
5	15	1					1										
6	21	1					2			$\boxed{1}$	$\boxed{1}$	1					
7	28	1					1				1					1	$\boxed{1}$

Fig. 3 An entry in row g and column n equals $\dim H^n(\mathrm{SL}_g(\mathbb{Z});\mathbb{R})$; blank entries are zero. All non-zero entries in this table are explained by the previous theorem except for the three boxed entries. These are discussed in §1.11. Some partial computations are known in $g=8$ by [55]

Statement (i) (and implicitly (ii)) is discussed in a research announcement of Ronnie Lee [41], but no proof seems ever to have appeared. The statement also appears in Franke [33, 35], which uses automorphic methods. Our proof bears some similarity to the strategy suggested by Lee and implies that the map (i) for $h=g$ and (ii) are canonically split. The key point in the proof is the construction of representatives for elements in $\Omega_c(g)$ which have compact support.[1] Statement (i) implies, but is much stronger than, Borel's result on the injectivity of (1.2) in small degrees.

Statement (iii) about the unstable cohomology of linear groups of even rank is new. It follows from a much stronger theorem (Theorem 1.3 below) on the existence of cohomology classes in the moduli space of tropical abelian varieties and uses recent results on the acyclicity of the 'inflation complex' established in [15].

To illustrate the content of (iv) and (v), see Fig. 3 for a table of the known cohomology of $\mathrm{SL}_g(\mathbb{Z})$. It is taken from [27] and uses computer calculations of [31].

1.3 Moduli of tropical abelian varieties

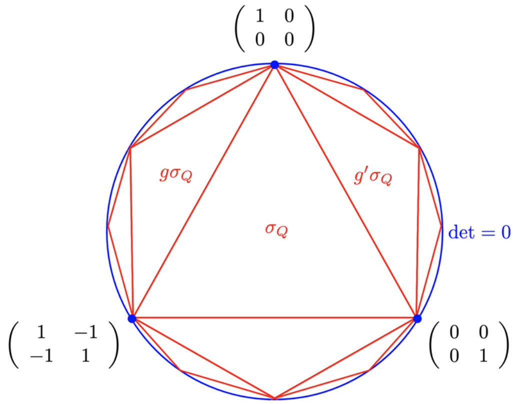
Theorem 1.1 is a consequence of a stronger result concerning cohomology classes on the link $|L\mathcal{A}_g^{\mathrm{trop}}|$ of the moduli space of tropical abelian varieties, which we presently explain. The reason for the vertical bar notation is that $|L\mathcal{A}_g^{\mathrm{trop}}|$ is merely the topological incarnation of a richer kind of hybrid geometric object, denoted by $L\mathcal{A}_g^{\mathrm{trop}}$, which we shall discus later. In any case, as a set, $|L\mathcal{A}_g^{\mathrm{trop}}|$ is the quotient of the rational closure $\mathcal{P}_g^{\mathrm{rt}}$ of $\mathcal{P}_g$ consisting of positive semi-definite matrices with rational kernel, modulo the action of $\mathrm{GL}_g(\mathbb{Z})$, and by $\mathbb{R}^{\times}_{>0}$ acting by scalar multiplication. We consider its decomposition into perfect cones, due to Voronoï [44, 59], which are described as follows. Let Q be a positive definite quadratic form, and let

$$M_Q = \{\lambda \in \mathbb{Z}^g\backslash\{0\} : Q(\lambda) \leq Q(\mu) \quad \text{for all } \mu \in \mathbb{Z}^g\backslash\{0\}\}$$

denote the set of minimal vectors of Q. The set of $\lambda\lambda^T$, for $\lambda \in M_Q$, span a convex polyhedral cone σ_Q in the link $L\mathcal{P}_g^{\mathrm{rt}}$ of $\mathcal{P}_g^{\mathrm{rt}}$, whose points are projective classes of

[1] It is easy to make the mistake of thinking that the canonical forms of compact type vanish along the boundary of the Borel-Serre compactification, but this is insufficient: the difficulty is to prove that they actually extend to the boundary in the first place: one sees from the definition (1.1) that they blow up as one approaches the region where $\det(X) \to 0$. To prove this result, and resolve this apparent contradiction, requires a careful analysis of the geometry of $L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})$ at infinity.

Fig. 4 The decomposition of the link $L\mathcal{P}^{\mathrm{rt}}$ into Voronoï cells viewed inside the projective space $\mathbb{P}^2(\mathbb{R})$ of symmetric 2×2 matrices. The space $|L\mathcal{A}_2^{\mathrm{trop}}|$ is the quotient of the closed simplex σ_Q by the action of Σ_3, and is the one-point compactification of the interior $|L\mathcal{A}^{\circ,\mathrm{trop}}|$ at the cusp. The space $|L\mathcal{A}_2^{\mathrm{trop},\mathcal{B}}|$ has an additional half-interval at the cusp



symmetric matrices. The topological space $|L\mathcal{A}_g^{\mathrm{trop}}|$ is obtained by gluing together $\mathrm{GL}_g(\mathbb{Z})$-equivalence classes of quotients $\sigma_Q/\mathrm{Aut}(\sigma_Q)$ of polyhedral cones by their finite groups of automorphisms.

Examples 1.2 (See Fig. 4). The quadratic form $Q(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$ has minimal vectors $M_Q = \{(\pm 1, 0), (0, \pm 1), \pm(1, -1)\}$, and hence

$$\sigma_Q = \left\{ \begin{pmatrix} \alpha_1 + \alpha_3 & -\alpha_3 \\ -\alpha_3 & \alpha_2 + \alpha_3 \end{pmatrix} : \alpha_1, \alpha_2, \alpha_3 \geq 0 \right\}$$

is the convex hull of the rank 1 matrices $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$ in the space of projective classes of 2×2 symmetric matrices. There is a single $\mathrm{GL}_2(\mathbb{Z})$-orbit of cells of maximal dimension in $|L\mathcal{A}_2^{\mathrm{trop}}|$ generated by σ_Q. The stabiliser of σ_Q is the symmetric group on three elements which permutes the three vertices of σ_Q.

Denote the closed subspace of $|L\mathcal{A}_g^{\mathrm{trop}}|$ corresponding to symmetric matrices with vanishing determinant by $|\partial L\mathcal{A}_g^{\mathrm{trop}}|$. Its open complement:

$$\left| L\mathcal{A}_g^{\circ,\mathrm{trop}} \right| = |L\mathcal{A}_g^{\mathrm{trop}}| \setminus |\partial L\mathcal{A}_g^{\mathrm{trop}}|$$

is nothing other than the space $L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})$ where $L\mathcal{P}_g = \mathcal{P}_g/\mathbb{R}_{>0}^{\times}$.

Theorem 1.3 *Let $g > 1$ be odd. Every form of compact type $\omega \in \Omega_c(g)$ extends to a smooth differential form on $|L\mathcal{A}_g^{\mathrm{trop}}|$. This gives an injective map of graded algebras*:

$$\Omega_c^{\bullet}(g) \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow H_{\mathrm{dR}}^{\bullet}(|L\mathcal{A}_g^{\mathrm{trop}}|) \ . \tag{1.3}$$

Theorem 1.1 follows from Theorem 1.3 using a de Rham theorem for certain kinds of topological spaces, such as $|L\mathcal{A}_g^{\mathrm{trop}}|$, which are defined by gluing quotients of polyhedra together by finite group actions (see below) as well as results from [15].

In fact, Theorem 1.3 enables us to completely determine the cohomology of $|L\mathcal{A}_g^{\mathrm{trop}}|$ in a certain range. If $g > 1$ is odd and $\kappa(g)$ denotes the stable range for the cohomology of the general linear group $\mathrm{GL}_g(\mathbb{Z})$, we show in Corollary 15.14 that

$$\begin{aligned} H^n_{\mathrm{dR}}(|L\mathcal{A}_g^{\mathrm{trop}}|) &\cong \Omega^n_c(g)_{\mathbb{Q}} \otimes \mathbb{R} \\ H^{n-1}_{\mathrm{dR}}(|L\mathcal{A}_{g-1}^{\mathrm{trop}}|) &= 0 \end{aligned} \tag{1.4}$$

for $n \geq d_g - \kappa(g)$. Using the recent results [40, 43], we may take $\kappa(g) = g$.

1.4 Bordification of the moduli space of tropical abelian varieties

In order to prove Theorem 1.3, we construct an explicit bordification

$$\left|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}\right| \longrightarrow \left|L\mathcal{A}_g^{\mathrm{trop}}\right| .$$

It has a boundary $|\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}| \subset |L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|$ with the property that

$$\left|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}\right| \setminus \left|\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}\right| = \left|L\mathcal{A}_g^{\circ,\mathrm{trop}}\right| .$$

The topological space $|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|$ is technically the space obtained by gluing together 'wonderful' compactifications [29] of perfect cones σ_Q by blowing up specific boundary strata which lie at infinity. It may be constructed informally as follows.

Consider the projective space $\mathbb{P}(\mathcal{Q}(V))$ whose points are projective classes of quadratic forms $\mathcal{Q}(V)$ in a vector space V of dimension g over $\mathbb{Q}$. The vanishing of the determinant defines a hypersurface $\mathrm{Det} \subset \mathbb{P}(\mathcal{Q}(V))$ which satisfies:

$$L\mathcal{P}_g \subset (\mathbb{P}(\mathcal{Q}(V)) \setminus \mathrm{Det})(\mathbb{R}) .$$

Now consider the space obtained by blowing up the subspaces $\mathbb{P}(\mathcal{Q}(V/K))$ whose points are quadratic forms with kernel K, for all rational subspaces $0 \neq K \subset V$, in increasing order of dimension. Since $\mathbb{P}(\mathcal{Q}(V/K))$ is contained in the determinant locus Det, every blow-up adds a new boundary component at infinity. Such a boundary component is indexed by a nested sequence $0 < K_1 < K_2 < \cdots < K_n < V$, and is isomorphic to a product of projective spaces. The closure of $L\mathcal{P}_g$ in the iterated blow-up admits an action by $\mathrm{GL}_g(\mathbb{Z})$, whose quotient is the bordification $|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|$. We show in an Appendix that it is homeomorphic to the Borel-Serre compactification [12]. However, from this construction it is not obvious that the quotient $|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|$ embeds into the real points of a finite diagram of algebraic varieties obtained by blowing up projective spaces along *finitely* many linear subspaces. This crucial point is proved in the main body of the text via a different construction, discussed in §1.7

Theorem 1.3 is proven by studying the behaviour of the differential forms (1.1) on the boundary of $|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|$. The existence of this bordification, and the techniques used to construct it, have a variety of other applications, which we discuss below.

1.5 Bordification of the tropical Torelli map

The main thrust of this paper is to provide a general technique for constructing bordifications of spaces built out of quotients of polyhedral cells. For example, we construct a bordification

$$\left|L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}\right| \longrightarrow \left|L\mathcal{M}_g^{\mathrm{trop}}\right|$$

of the link of the moduli space of tropical curves whose existence was alluded to in [19]. It is obtained by gluing together the 'Feynman polytopes' associated to stable graphs and provides a single geometric object whose cells are the spaces underlying the Feynman motives of [10, 17]. This bordification is presumably related to the bordification of Culler and Vogtmann's Outer space which was constructed in [8, 24]. Note that our bordification concerns the quotient of unreduced Outer space by $\mathrm{Out}(F_n)$ and allows graphs with bridges; the latter two papers study reduced Outer space, which only involves bridgeless (core) graphs, and concern the universal covering space, rather than its quotient by $\mathrm{Out}(F_n)$.

The bordifications of $|L\mathcal{A}_g^{\mathrm{trop}}|$ and $|L\mathcal{M}_g^{\mathrm{trop}}|$ are related as follows. The tropical Torelli map $\lambda : |L\mathcal{M}_g^{\mathrm{trop}}| \to |L\mathcal{A}_g^{\mathrm{trop}}|$ was studied in [6, 16, 25, 47, 49]. It is non-degenerate on the moduli space $|L\mathcal{M}_g^{\mathrm{red}}| \subset |L\mathcal{M}_g^{\mathrm{trop}}|$ of 3-edge connected graphs.

Theorem 1.4 *The tropical Torelli map extends to a map $\lambda^{\mathcal{B}}$ of bordifications, giving a commutative diagram where the vertical maps are blow-downs*:

$$\begin{array}{ccc} \left|L\mathcal{M}_g^{\mathrm{red},\mathcal{B}}\right| & \xrightarrow{\lambda^{\mathcal{B}}} & \left|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}\right| \\ \downarrow & & \downarrow \\ \left|L\mathcal{M}_g^{\mathrm{red}}\right| & \xrightarrow{\lambda} & \left|L\mathcal{A}_g^{\mathrm{trop}}\right| \end{array}$$

This diagram gives relations between the cohomology of these four spaces, and the existence of $\lambda^{\mathcal{B}}$ answers a question of Vogtmann. Note that the cohomology of $|L\mathcal{M}_g^{\mathrm{trop}}|$ is related to the cohomology of the commutative graph complex $\mathcal{GC}_0$, and to the top-weight cohomology of the moduli stack $\mathcal{M}_g$ [26]. The cohomology of the bordification $|L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}|$ is described by a new graph complex (see §8.2) involving nested sequences of graphs, and is related to the Hopf algebra structure on graph homology. It would be interesting to study its homology in relation to that of $\mathcal{GC}_0$.

1.6 Canonical integrals of perfect cones

A further consequence of the properties of canonical forms (1.1) on the bordification of $|L\mathcal{A}_g^{\mathrm{trop}}|$ is that we may assign transcendental invariants to polyhedral cones in the Voronoï decomposition.

Theorem 1.5 *Let Q be a positive definite quadratic form of rank g, and let σ_Q be the associated cone. Let $\omega \in \Omega^d_{\mathrm{can}}$ be any canonical form* (1.1) *of degree $d = \dim \sigma_Q$. If*

σ_Q has rank g then the following integral is finite:

$$I_Q(\omega) = \int_{\sigma_Q} \omega < \infty \,. \tag{1.5}$$

The integrals $I_Q(\omega)$ satisfy quadratic relations arising from Stokes' formula (*Theorem* 14.13).

The condition that σ_Q has rank g means that σ_Q meets the interior $L\mathcal{P}_g$ of the link of the space of positive definite matrices (i.e., it does not completely lie at infinity). In particular, every 'perfect' quadratic form of maximal dimension may be assigned a volume, which is non-zero. Theorem 1.5 provides interesting transcendental invariants of quadratic forms which may provide a new perspective on the extensive literature on perfect forms in relation to sphere packing problems, which was one of Voronoï's original goals.

In the case when the cone σ_Q is cographical, i.e., the image under the tropical Torelli map of the cell associated to a graph in $|L\mathcal{M}_g^{\mathrm{red}}|$, the integrals (1.5) are the canonical integrals $I_G(\omega)$ studied in [19]. It was proved in *loc. cit.* that these are generalised Feynman integrals of the sort arising in perturbative quantum field theory. An important slogan, therefore, is that the *volumes of cographical cells in the perfect cone decomposition of the space of symmetric matrices are Feynman integrals*. The Borel regulator, in particular, is a linear combinations of integrals (1.5).

The arithmetic nature of the period integrals (1.5) is not known. However, a computation by Borinsky and Schnetz [13] implies that the volume of the principle cone for $\mathrm{GL}_5(\mathbb{Z})$ is a very special linear combination of non-trivial multiple zeta values of weight 8 which involves $\zeta(3,5)$.

1.7 Polyhedral cell complexes

We finally turn to the main geometric constructions. Theorem 1.3 hinges upon a de Rham theorem for topological spaces obtained by gluing together quotients of polyhedra by finite group actions. In order to make sense of algebraic differential forms such as (1.1) on these spaces, we embed each polyhedron into an ambient algebraic variety, which are glued together according to the same pattern.

The most basic construction along these lines is a category, which we call PLC_k, of polyhedral linear configurations over a field $k \subset \mathbb{R}$. Its objects are triples: $(\mathbb{P}(V), L_\sigma, \sigma)$, where V is a finite-dimensional vector space over k, $\sigma \subset \mathbb{P}(V)(\mathbb{R})$ is a closed convex polyhedron, and $L_\sigma \subset \mathbb{P}(V)$ is the union of linear subspaces defined by the Zariski closure of the boundary $\partial\sigma$. See Figure 5, left. A morphism $\phi : (\mathbb{P}(V), L_\sigma, \sigma) \to (\mathbb{P}(V'), L_{\sigma'}, \sigma')$ in this category is a linear map $\phi : \mathbb{P}(V) \to \mathbb{P}(V')$ such that $\phi(L_\sigma) \subset L_{\sigma'}$ and $\phi(\sigma) \subset \sigma'$. We demand, in addition, that $\phi : \sigma \to \sigma'$ be either the inclusion of a face, or an isomorphism.

We then define a *linear polyhedral complex* to be a space assembled out of such objects (and their quotients by finite linear group actions). More precisely, it is given by a functor

$$F : \mathcal{D} \longrightarrow \mathrm{PLC}_k$$

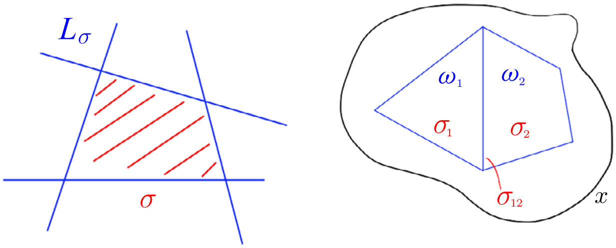


Fig. 5 Left: a polyhedral linear configuration in $\mathbb{P}^2$. Right: two polyhedra σ_1, σ_2 are glued together along the common face $\sigma_{12} = \sigma_1 \cap \sigma_2$. An algebraic differential form ω defines two forms $\omega_i = \omega|_{\sigma_i}$, for $i = 1, 2$, which coincide on σ_{12}. The form ω has poles along a subscheme $\mathcal{X}$, which, as depicted, may not necessarily meet the topological realisation $\sigma_1 \cup_{\sigma_{12}} \sigma_2$

where $\mathcal{D}$ is equivalent to a finite diagram category. It has a topological realisation:

$$|F| = \varinjlim_{x \in \mathcal{D}} \sigma(F(x)) \ ,$$

which is the topological space obtained by gluing the polyhedra $\sigma(F(x))$ along linear maps. Examples of linear polyhedral complexes include the moduli space of tropical curves $L\mathcal{M}_g^{\mathrm{trop}}$ of genus g, where $\mathcal{D}$ is a category of stable graphs, and the moduli space of tropical abelian varieties $L\mathcal{A}_g^{\mathrm{trop}}$, in which case $\mathcal{D}$ is a category of cones associated to quadratic forms. The topological realisation of a polyhedral linear complex is closely related to similar notions in the literature, including stacky fans [16] and generalised CW-complexes [3].

However, a polyhedral linear complex has additional algebraic structure. In particular, there is a functor $\mathsf{S}F : \mathcal{D} \to \mathrm{Sch}_k$ to the category of schemes which picks out the first component in each triple $x \mapsto \mathbb{P}(V_{F(x)})$. We can speak of a subscheme of F: it is simply a subfunctor $\mathcal{X} : \mathcal{D} \to \mathrm{Sch}_k$ of $\mathsf{S}F$. Concretely, it is given by the data of a compatible family of subschemes $\mathcal{X}_x \subset \mathbb{P}(V_{F(x)})$ for every object x in $\mathcal{D}$. A global algebraic differential form on a polyhedral linear complex F with poles along $\mathcal{X}$ (see Figure 5, right) is simply an element of

$$\Omega^\bullet(F \backslash \mathcal{X}) = \varprojlim_{x \in \mathcal{D}} \Omega^\bullet \left(\mathbb{P}(V_{F(x)}) \backslash \mathcal{X}_x \right) \ . \tag{1.6}$$

It is a compatible system of differential forms on $\mathbb{P}(V_{F(x)})$ with poles along $\mathcal{X}_x$ for each x. Our main examples are: the subscheme of $L\mathcal{M}_g^{\mathrm{trop}}$ defined by the graph hypersurface locus, and the subscheme of $L\mathcal{A}_g^{\mathrm{trop}}$ defined by the determinant locus Det.

The above definitions can be generalised. The most important construction, for our purposes, is a category BLC_k whose objects consist of iterated blow-ups of polyhedral linear configurations along linear subspaces. They are 'wonderful' compactifications of linear polyhedra in the sense of [29]. Using this concept, we can efficiently define the bordifications mentioned earlier. For instance, we construct a functor

$$L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}} : \mathcal{D}_g^{\mathrm{perf},\mathcal{B}} \longrightarrow \mathrm{BLC}_{\mathbb{Q}}$$

from a suitable diagram category to $\mathrm{BLC}_{\mathbb{Q}}$. It is defined by blowing up the subspaces of quadratic forms with a non-trivial rational kernel which meet each cone in the Voronoï decomposition. A key point is that this space is a diagram of finitely many algebraic varieties with extra structure; the informal definition of $|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|$ given earlier in this introduction *a priori* involved infinitely many blow-ups.

The next theorem is the key geometric input in the proof of Theorems 1.1 and 1.3.

Theorem 1.6 *There is a commutative diagram*

$$\begin{array}{ccc} \partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}} & \hookrightarrow & L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}} \\ \downarrow \pi_{\mathcal{B}} & & \downarrow \pi_{\mathcal{B}} \\ \partial L\mathcal{A}_g^{\mathrm{trop}} & \hookrightarrow & L\mathcal{A}_g^{\mathrm{trop}} \end{array}$$

where the vertical maps $\pi_{\mathcal{B}}$ are canonical blow-downs. The strict transform of the determinant subscheme $\mathrm{Det} \subset L\mathcal{A}_g^{\mathrm{trop}}$ *defines a subscheme*

$$\widetilde{\mathrm{Det}} \subset L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}$$

which satisfies $\pi_{\mathcal{B}} : \widetilde{\mathrm{Det}} \to \mathrm{Det}$, *and has the property*

$$\widetilde{\mathrm{Det}} \cap \left| L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}} \right| = \emptyset \, . \tag{1.7}$$

On topological realisations, one has a commutative diagram of topological spaces

$$\begin{array}{ccc} & & \left|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}\right| \setminus \left|\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}\right| \\ & \nearrow & \downarrow \\ \left|L\mathcal{A}_g^{\circ,\mathrm{trop}}\right| & \to & \left|L\mathcal{A}_g^{\mathrm{trop}}\right| \setminus \left|\partial L\mathcal{A}_g^{\mathrm{trop}}\right| \end{array}$$

where all arrows are isomorphisms. The open locus $|L\mathcal{A}_g^{\circ,\mathrm{trop}}|$ is homeomorphic to the locally symmetric space $L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z}) = \mathbb{R}^{\times}_{>0} \setminus \mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})$.

A similar theorem holds for the moduli space $L\mathcal{M}_g^{\mathrm{trop}}$ of tropical curves (§8).

1.8 'Motives' of quadratic forms

Hereafter, a motive refers to an object in a suitable Tannakian category $\mathcal{H}_{\mathbb{Q}}$ of realisations, following Deligne [30]. An object M in $\mathcal{H}_{\mathbb{Q}}$ has Betti and de Rham realisations M_{B}, M_{dR} and a period pairing $M_{\mathrm{B}}^{\vee} \otimes_{\mathbb{Q}} M_{\mathrm{dR}} \to \mathbb{C}$. The property (1.7) of the determinant locus enables us to define a canonical object mot_Q associated to a positive definite quadratic form Q such that the integral (1.5) is a period (Definition 16.1). In the special case when $\sigma_{Q_G} = \sigma_G$ is a cographical cone in the image of the tropical Torelli map, mot_{Q_G} is equivalent to the graph motive defined in [10]. This construction provides, in particular, a motivic interpetation of the Borel regulator.

1.9 'Motives' associated to $\mathrm{GL}_g(\mathbb{Z})$

Similarly, we may find a motivic interpretation of the canonical cohomology of $\mathrm{GL}_g(\mathbb{Z})$. For every $g > 1$ and $d \geq 0$ we define a cohomology group $H_c^d(g)$ consisting of closed compatible families of algebraic differential forms with poles along the determinant locus (1.6), modulo exact forms. There is a natural map:

$$H_c^d(g) \longrightarrow H_c^d(L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z});\mathbb{R})$$

which is injective on $\Omega_c^d(g) \subset H_c^d(g)$: in particular, $H_c^d(g)$ contains all the compactly-supported cohomology classes for $\mathrm{GL}_g(\mathbb{Z})$ which are described in Theorem 1.1. Integration of differential forms defines a pairing

$$H_d^{\mathrm{lf}}(\mathrm{GL}_g(\mathbb{Z});\mathbb{Q}) \otimes_{\mathbb{Q}} H_c^d(g) \longrightarrow \mathbb{C}$$

where H^{lf} denotes locally finite (Borel-Moore) homology.

Theorem 1.7 *There is an object M_g^d of $\mathcal{H}_{\mathbb{Q}}$ equipped with a pair of canonical linear maps*

$$H_d^{\mathrm{lf}}(\mathrm{GL}_g(\mathbb{Z});\mathbb{Q}) \longrightarrow (\mathsf{M}_g^d)_{\mathrm{B}}^{\vee}$$
$$H_c^d(g) \longrightarrow (\mathsf{M}_g^d)_{\mathrm{dR}}$$

such that the integration pairing factors through the period pairing: $(\mathsf{M}_g^d)_{\mathrm{B}}^{\vee} \otimes_{\mathbb{Q}} (\mathsf{M}_g^d)_{\mathrm{dR}} \to \mathbb{C}$.

This theorem implies that the part of the locally finite homology $H_d^{\mathrm{lf}}(\mathrm{GL}_g(\mathbb{Z});\mathbb{Q})$ which pairs non-trivially with $H_c^d(g)$ is motivic, since M_g^d lies in the Tannakian subcategory of $\mathcal{H}_{\mathbb{Q}}$ generated by the cohomology of algebraic varieties over $\mathbb{Q}$. In particular, the theorem provides a motivic interpretation of the integrals of canonical differential forms over homology cycles, including the volume integrals over fundamental domains considered by Minkowski. The objects M are interesting: for example, M_3^5 is a non-trivial extension of $\mathbb{Q}(-3)$ by $\mathbb{Q}(0)$ with period a rational multiple of $\zeta(3)$.

1.10 Borel-Serre compactification in algebraic geometry

In an Appendix we study the Borel-Serre compactifiction of $L\mathcal{P}_g/\mathrm{GL}_n(\mathbb{Z})$, which is defined topologically, and prove that it is homeomorphic to $|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|$, which is constructed algebraically.

1.11 Recent progress and discussion of unstable cohomology

It was proven by Bismut and Lott [9] that for $g > 1$ odd, the class of ω^{2g-1} in $H^{2g-1}(\mathrm{GL}_g(\mathbb{Z}))$ vanishes. Conjecture 5.3 in [48] states that the class of ω^{2g-1} also vanishes in $H^{2g-1}(\mathrm{GL}_{g+1}(\mathbb{Z}))$.

Since the first version of this paper became available, a flurry of preprints appeared which have enormously advanced our understanding of the unstable cohomology of the general and special linear groups. Firstly, Ash [5] independently constructed two infinite families of unstable classes in the cohomology of $\mathrm{SL}_n(\mathbb{Z})$, for $n = 3k + 3$ and $n = 3k + 4$. When k is even, they lie in the same degrees as the duals of the classes we denote by $[\omega_c^{2g-1}] \in H_c^{2g}(\mathrm{GL}_g(\mathbb{Z}))$ and their inflations in $H_c^{2g+1}(\mathrm{GL}_{g+1}(\mathbb{Z}))$ (see Theorem 1.1, (ii), (iii)). For k odd, they lie in different degrees from the classes constructed in this paper. Ash suggests that they align with classes announced by Lee [41] (more on which below).

At around the same time, [4] and [21] independently showed that the compactly supported cohomology of the general linear group admits a Hopf algebra structure. By composing the classes constructed in Theorem 1.1, one can not only prove all the claims of Lee, but also deduce the existence of a plethora of new and unexpected classes. See *loc. cit.* for further details. In particular, all boxed classes in our previous figures can now be explained in terms of canonical forms using these new results. For example, the class in $H^9(\mathrm{SL}_6(\mathbb{Z}); \mathbb{R})$ in Fig. 3, which corresponds to a class in $H_c^{12}(\mathrm{GL}_6(\mathbb{Z}); \mathbb{R})$, is denoted by $[\omega^5|\omega^5]$ in Lee's notation. The class in $H^{15}(\mathrm{SL}_7(\mathbb{Z}); \mathbb{R})$ corresponds to a class in $H_c^{13}(\mathrm{GL}_7(\mathbb{Z}); \mathbb{R})$ which is related to it by the inflation map of [15]. The paper [22], which builds on [4] and the methods developed here, establishes infinitely many new classes on $\mathrm{GL}_{2n}(\mathbb{Z})$. In particular, the last remaining boxed class in $H^8(\mathrm{GL}_6(\mathbb{Z}))$ (equivalently $H^8(\mathrm{SL}_6(\mathbb{Z}))$, see Fig. 3) is part of a large infinite family. We refer the reader to the survey [23] for futher context.

In a different direction, cuspidal classes for $\mathrm{GL}_n(\mathbb{Z})$ were constructed recently in [14]. A recent preprint of Vogtmann [58] revisits the bordification of outer space and asks the question of bordifying the tropical Torelli map, which is answered in §12.

1.12 Plan and further comments

Section §2 introduces a general notion of polyhedral cell complexes in algebraic varieties, for which we prove a cellular homology and de Rham cohomology theorem in §3. In Section §4 we study the particular subcategory of *linear* polyhedral cell complexes PLC_k, and their iterated blow-ups in §5.

The first application of this theory is to the moduli space of tropical curves §6, and its bordification §7-8. The discussion of the moduli space of tropical abelian varieties begins in §9 with a study of polyhedra in spaces of quadratic forms and their blow-ups. In Sect. 10 we study $L\mathcal{A}_g^{\mathrm{trop}}$ and define its perfect cone bordification $L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}$. Section 11 studies the properties of the determinant locus and its strict transform. In Sect. 12, we construct the bordification of the tropical Torelli map.

From Sect. 14 onwards, we study the properties of canonical forms and their integrals, and in §15 we prove the main results on the cohomology of $\mathrm{SL}_g(\mathbb{Z})$ and $\mathrm{GL}_g(\mathbb{Z})$ mentioned in this introduction. Finally, §16 discusses the periods and motives associated to canonical integrals and in an Appendix we discuss the relation between the Borel-Serre compactification and the space $|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|$ defined algebraically using blow-ups. We expect that the methods of this paper may be used to define and study the geometric spaces, differential forms, and cohomology classes associated to other types of graph complexes (see, e.g., [7]), as well as the quotients of symmetric spaces by general linear groups over number fields.

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2 Algebraic polyhedral cell complexes

We describe a formalism to construct algebraic models of certain topological spaces defined by gluing together polyhedral cells according to a diagram category. In the first instance, the cells will be objects of a very general category $\mathcal{PC}_k$, but for the applications we shall work with more restrictive subcategories of convex linear polyhedra in projective space (PLC_k), and of their blow-ups along linear spaces (BLC_k).

2.1 A category of polyhedral cells

Let $k \subset \mathbb{C}$ be a field. There are many situations in which one encounters a polyhedron embedded in the real or complex points of an algebraic variety. Since convexity does not make sense in this generality, the concept of a polyhedron in an algebraic variety must be defined using different concepts. To this end, let us denote by $\mathcal{PC}_k$ the category whose objects are triples (P, L, σ) defined recursively as follows:

- P is a smooth scheme over k.
- $\sigma \subset P(\mathbb{C})$ is homeomorphic to a closed real ball $B_n \subset \mathbb{R}^n$ of dimension $n \geq 0$, i.e., there is a continuous map $g : B_n \hookrightarrow P(\mathbb{C})$ such that $g : B_n \cong \sigma$. We call n the dimension of (P, L, σ). It may be strictly smaller than the dimension of P.
- The boundary of σ satisfies $\partial\sigma = \sigma \cap L(\mathbb{C})$, where $L \subset P$ is a subscheme with finitely many distinct smooth irreducible components L_i, $i \in I$.
- Let $n \geq 1$. For every $i \in I$ such that $L_i(\mathbb{C}) \cap \sigma$ is non-empty, the triple

$$\left(L_i, \bigcup_{L_j \neq L_i} L_i \cap L_j, \sigma \cap L_i(\mathbb{C})\right) \tag{2.1}$$

  is required to be an object in $\mathcal{PC}_k$ of dimension $< n$.

The objects in $\mathcal{PC}_k$ of dimension 0 are triples (P, L, σ) where $\sigma \subset L(\mathbb{C})$ is a point. For any object (P, L, σ), one has $\partial\sigma = \bigcup_{i \in I} \sigma_i$, where $\sigma_i = \sigma \cap L_i(\mathbb{C})$ is of smaller dimension. Therefore by repeatedly taking boundaries, one obtains a stratification on

σ giving a structure of a regular CW-complex on the closed ball B_n. Note that the third condition $\partial\sigma = \sigma \cap L(\mathbb{C})$ captures a notion of convexity (it fails, for example, for non-convex Euclidean polyhedra). A morphism:

$$\phi : (P, L, \sigma) \to (P', L', \sigma')$$

is given by a morphism $\phi : P \to P'$ such that $\phi(L) \subset L'$ and $\phi(\sigma) \subset \sigma'$. For any subset $J \subset I$, let $L_J = \bigcap_{j \in J} L_j$. A *face map* is an inclusion of a *face*

$$\iota_J : \left(L_J, \bigcup_{j \in I \setminus J} L_{J \cup \{j\}}, \sigma \cap L_J(\mathbb{C}) \right) \longrightarrow (P, L, \sigma) \tag{2.2}$$

where $\sigma \cap L_J(\mathbb{C}) \neq \emptyset$. A *facet* of (P, L, σ) is a face (2.1) of dimension $\dim \sigma - 1$.

Example 2.1 Let $P = \mathbb{P}^n_{\mathbb{Q}}$ denote projective space of dimension n with homogeneous coordinates $(x_0 : \ldots : x_n)$. The *algebraic simplex* is the triple $(\mathbb{P}^n_{\mathbb{Q}}, L, \sigma)$, where $L = V(x_0 \cdots x_n)$ is the union of coordinate hyperplanes, and $\sigma \subset \mathbb{P}^n_{\mathbb{Q}}(\mathbb{R})$ is the standard coordinate simplex defined by the region $x_i \geq 0$. Its faces are algebraic simplices of smaller dimension.

Notation 2.2 Define the product $\prod_{i=1}^n (P_i, L_i, \sigma_i)$ of objects (P_i, L_i, σ_i) in $\mathcal{PC}_k$ by

$$\left(P_1 \times \cdots \times P_n \,,\, \bigcup_{i=1}^n P_1 \times \cdots \times P_{i-1} \times L_i \times P_{i+1} \times \cdots \times P_n \,,\, \sigma_1 \times \cdots \times \sigma_n \right) .$$

Denote a product of morphisms $f_1, \ldots, f_n$ in $\mathcal{PC}_k$ by $f_1 \times \cdots \times f_n$.

We shall also add to the collection of morphisms in $\mathcal{PC}_k$ the following projection morphisms, defined for any ordered finite sets $I \supset J$:

$$\prod_{i \in I} (P_i, L_i, \sigma_i) \to \prod_{j \in J} (P_j, L_j, \sigma_j) .$$

There are two natural functors, both of which preserve products:

$$\begin{array}{rcl} \sigma : \mathcal{PC}_k & \longrightarrow & \mathcal{T}op \\ \sigma(P, L, \sigma) & = & \sigma \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \mathsf{S} : \mathcal{PC}_k & \longrightarrow & \mathrm{Sch}_k \\ \mathsf{S}(P, L, \sigma) & = & P \end{array} \tag{2.3}$$

to the category of topological spaces, and schemes over k, respectively.

2.2 Algebraic polyhedral cell complexes

Let $\mathcal{C}_k$ denote a subcategory of $\mathcal{PC}_k$. The main cases of interest are $\mathcal{C}_k = \mathrm{PLC}_k$ (§4) and $\mathcal{C}_k = \mathrm{BLC}_k$ (§5).

Definition 2.3 A $\mathcal{C}_k$-*complex* is a functor

$$F : \mathcal{D} \longrightarrow \mathcal{C}_k \tag{2.4}$$

where $\mathcal{D}$ is equivalent to a finite category. We write F_x for $F(x)$ when x is an object of $\mathcal{D}$. A *morphism* (ϕ, Φ) between two functors $F : \mathcal{D} \to \mathcal{C}_k$ and $F' : \mathcal{D}' \to \mathcal{C}_k$ is the data of:

(i) a functor $\phi : \mathcal{D} \to \mathcal{D}'$ and
(ii) a natural transformation $\Phi : F \to F' \circ \phi$.

To spell this out, Φ is the data, for every object x of $\mathcal{D}$, of a morphism

$$\Phi_x : F_x \longrightarrow (F' \circ \phi)_x \qquad \text{in } \mathcal{C}_k$$

such that, for every morphism $f : x \to y$ in $\mathcal{D}$, there is a commutative diagram

$$\begin{array}{ccc} F_x & \xrightarrow{\Phi_x} & (F' \circ \phi)_x \\ \downarrow_{F(f)} & & \downarrow_{(F'\circ\phi)(f)} \\ F_y & \xrightarrow{\Phi_y} & (F' \circ \phi)_y \end{array} .$$

In particular, any functor $\phi : \mathcal{D} \to \mathcal{D}'$ induces a morphism (ϕ, id) between $F = F' \circ \phi$ and F'. If ϕ is an equivalence, then F and F' are isomorphic. In particular, by replacing $\mathcal{D}$ with an equivalent category, we may assume that $\mathcal{D}$ is itself finite.

Definition 2.4 The *topological realisation* of (2.4) is the topological space

$$|F| = \varinjlim_{x \in \mathrm{Ob}(\mathcal{D})} \sigma(F_x) . \tag{2.5}$$

By taking limits, a morphism (Φ, ϕ) from F to F' induces a continuous map between their topological realisations. We denote it by $|\Phi| : |F| \longrightarrow |F'|$.

2.3 Subschemes

Let $F : \mathcal{D} \to \mathcal{PC}_k$ be a functor (2.4) as above.

Definition 2.5 Define a closed (resp. open) *subscheme* of F to be a functor

$$\mathcal{X} : \mathcal{D} \longrightarrow \mathrm{Sch}_k$$

such that $\mathcal{X}_x$ is a closed (resp. open) subscheme of $\mathsf{S}F_x$, for all objects x of $\mathcal{D}$, and such that the canonical embedding $i_x : \mathcal{X}_x \subset \mathsf{S}F_x$ is a natural transformation $i : \mathcal{X} \to \mathsf{S}F$. If $K = \mathbb{R}$ or $\mathbb{C}$, and contains $k \subset K$, then the set of K-points defines a topological space

$$|\mathcal{X}(K)| = \varinjlim_{x \in \mathcal{D}} \mathcal{X}_x(K)$$

with the analytic topology.

Definition 2.5 is analogous to that of a subfunctor. It means that for all morphisms $f : x \to y$ in $\mathcal{D}$ there is a commutative diagram

$$\begin{array}{ccc} \mathsf{S}F_x & \xrightarrow{F(f)} & \mathsf{S}F_y \\ \cup\!\!| & & \cup\!\!| \\ \mathcal{X}_x & \xrightarrow{X(f)} & \mathcal{X}_y \end{array}$$

and hence the morphisms between $\mathcal{X}_x$ are obtained by restricting those from $\mathsf{S}F$.

Definition 2.6 We say that a closed subscheme $\mathcal{Z}$ of F is *at infinity*, which we denote by $|F| \cap \mathcal{Z} = \emptyset$, if its image does not meet any of the polyhedra $\sigma(F_x)$:

$$\sigma(F_x) \cap i_x(\mathcal{Z}_x)(\mathbb{C}) = \emptyset \quad \text{for all} \quad x \in \mathrm{Ob}(\mathcal{D}).$$

Similarly, for an open subscheme U of F we write $|F| \subset U$ if

$$\sigma(F_x) \ \subset \ U_x(\mathbb{C}) \quad \text{for all} \quad x \in \mathrm{Ob}(\mathcal{D}).$$

Definition 2.7 Given an open subscheme $\mathcal{U}$ of F such that $|F| \subset \mathcal{U}$, define

$$\begin{aligned} F \cap \mathcal{U} : \mathcal{D} &\longrightarrow \mathcal{PC}_k \\ x \ &\mapsto \ (F \cap \mathcal{U})_x = (\mathcal{U}_x, L_x \cap U, \sigma) \ , \end{aligned} \tag{2.6}$$

which we may view as an (open) subfunctor of $F : \mathcal{D} \to \mathcal{PC}_k$.

2.4 Algebraic and meromorphic differential forms

We define various notions of differential forms on a $\mathcal{C}_k$-complex. The most important will be the notion of a smooth form (2.8), which we can use to compute cohomology. Since the cohomology classes we construct are in fact algebraic with specific polar locus, we also need to define notions of algebraic forms with poles along a subscheme, as well as meromorphic forms with unspecified poles.

Definition 2.8 Consider a $\mathcal{C}_k$-complex $F : \mathcal{D} \to \mathcal{C}_k$ and a subscheme $\mathcal{X} \subset \mathsf{S}F$. A global differential form of degree d on F with poles along $\mathcal{X}$ is an element of the limit

$$\Omega^d(\mathsf{S}F \backslash \mathcal{X}) = \varprojlim_{x \in \mathrm{Ob}(\mathcal{D})} \Omega^d(\mathsf{S}F_x \backslash \mathcal{X}_x) \ .$$

Equivalently, it is a collection, for every $x \in \mathrm{Ob}(\mathcal{D})$, of regular forms $\omega_x \in \Omega^d(\mathsf{S}F_x \backslash \mathcal{X}_x)$ which are compatible in the sense that

$$F(f)^*(\omega_y) = \omega_x \qquad \text{for every} \qquad f \in \mathrm{Hom}_{\mathcal{D}}(x, y) \ . \tag{2.7}$$

The graded vector space $\Omega^\bullet(\mathsf{S}F \backslash \mathcal{X}) = \bigoplus_d \Omega^d(\mathsf{S}F \backslash \mathcal{X})$ is a differential graded algebra. We may also write it $\Omega^\bullet(F \backslash \mathcal{X})$, bearing in mind that it depends only on the functor $\mathsf{S}F$.

Consider an object (P, L, σ) of $\mathcal{PC}_k$. Let us denote by

$$\Omega^d_{\mathrm{mer}}((P, L, \sigma)) = \varinjlim_{U:\sigma \subset U(\mathbb{C})} \Omega^d(U; k)$$

the space of meromorphic differential forms which are regular on an open affine subset U of P whose complex points contain σ. Such a form may be restricted to the faces of σ, and so $\Omega^\bullet_{\mathrm{mer}} = \bigoplus_{d\geq 0} \Omega^d_{\mathrm{mer}}$ is a contravariant functor from $\mathcal{PC}_k$ to the category of DGA's.

Definition 2.9 Consider a $\mathcal{C}_k$-complex $F : \mathcal{D} \to \mathcal{C}_k$. A meromorphic differential form of degree d on F is an element of the projective limit

$$\Omega^d_{\mathrm{mer}}(F) = \varprojlim_{x\in \mathrm{Ob}(\mathcal{D})} \Omega^d_{\mathrm{mer}}(F_x) .$$

It is a compatible collection of meromorphic forms $\omega_x \in \Omega^d_{\mathrm{mer}}(F_x)$ for $x \in \mathrm{Ob}(\mathcal{D})$. The total space $\Omega^\bullet_{\mathrm{mer}}(F) = \bigoplus_{d\geq 0} \Omega^d_{\mathrm{mer}}(F)$ is a differential graded algebra.

Examples 2.10 The DGA of meromorphic forms $\Omega^d_{\mathrm{mer}}(\mathbb{P}^n_{\mathbb{Q}}, L, \sigma)$ on an algebraic simplex (Example 2.1) contains the polynomial forms on σ in the sense of Sullivan [57].

Definition 2.11 The $\mathbb{C}$-differential graded algebra of smooth forms on $|F|$ is

$$\mathcal{A}^\bullet(|F|) = \varprojlim_{x\in \mathrm{Ob}(\mathcal{D})} \mathcal{A}^\bullet(\sigma(F_x)) , \tag{2.8}$$

where $\mathcal{A}^\bullet(\sigma(F_x))$ denotes the algebra of smooth differential forms over $\mathbb{C}$ which are defined in an open neighbourhood of $\sigma(F_x)$ inside $\mathsf{S}F_x(\mathbb{C})$. When $k \subset \mathbb{R}$, and all polyhedra $\sigma(F_x)$ are in fact contained in $\mathsf{S}F_x(\mathbb{R})$ (which will always be the case in our applications) then $\mathcal{A}^\bullet(|F|)$ has a real structure $\mathcal{A}^\bullet(|F|; \mathbb{R})$ consisting of the $\mathbb{R}$-subalgebra of real forms.

If $\mathcal{X}$ is at infinity, i.e., $\mathcal{X} \cap |F| = \emptyset$, then there are natural maps of DGA's:

$$\Omega^\bullet(\mathsf{S}F\backslash\mathcal{X}) \subset \Omega^\bullet_{\mathrm{mer}}(F) \longrightarrow \mathcal{A}^\bullet(|F|) . \tag{2.9}$$

Definition 2.12 Define smooth (resp. global algebraic) cohomology groups:

$$H^n_{\mathrm{dR}}(|F|; \mathbb{C}) = H^n(\mathcal{A}^\bullet(|F|)) \quad (\text{resp.} \quad H^n_{\mathrm{dR}}(\mathsf{S}F\backslash\mathcal{X}) = H^n_{\mathrm{dR}}(\Omega^\bullet(F\backslash\mathcal{X}))) . \tag{2.10}$$

The former is a vector space over $\mathbb{C}$ isomorphic to the cohomology of $|F|$ as we shall show; the latter a vector space over k, which does not equal the cohomology of $|F|$ in general.

If $\mathcal{X} \cap |F| = \emptyset$, there is a natural map $H^n_{\mathrm{dR}}(\mathsf{S}F\backslash\mathcal{X}) \to H^n_{\mathrm{dR}}(|F|)$.

3 Homology and cohomology of polyhedral complexes

3.1 Assumptions

Let $\mathcal{C}_k$ be a subcategory of $\mathcal{PC}_k$ such that:

(1) Every face of every object (P, L, σ) in $\mathcal{C}_k$ is also an object of $\mathcal{C}_k$, and the corresponding face map is a morphism in $\mathcal{C}_k$.
(2) All morphisms $f : (P, L, \sigma) \to (P', L', \sigma')$ in $\mathcal{C}_k$ are either face maps, or induce homeomorphisms on topological realisations $f : \sigma \cong \sigma'$.

In particular, the topological realisation $f : \sigma \to \sigma'$ of any morphism in $\mathcal{C}_k$ is necessarily injective. For every (P, L, σ) of dimension $n \geq 1$ one has

$$H_r(\sigma, \partial\sigma; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } r = n \\ 0 & \text{otherwise} \end{cases} . \tag{3.1}$$

An *orientation* on σ is a generator $\varpi \in H_n(\sigma, \partial\sigma; \mathbb{Z})$ if $n \geq 1$ or $\varpi \in H_0(\sigma; \mathbb{Z})$ if $n = 0$. The two main categories of interest, PLC_k and BLC_k, satisfy (1) and (2). An important fact is that an isomorphism between objects of $\mathcal{C}_k$ induces isomorphisms of their faces.

3.2 Cellular homology of a polyhedral $\mathcal{C}_k$-complex

Let $\mathcal{C}_k$ be a category satisfying the assumptions above and let $F : \mathcal{D} \to \mathcal{C}_k$ be a $\mathcal{C}_k$-complex.

Definition 3.1 Define the *face complex* $(\mathfrak{C}_F)_n$ of F to be the graded $\mathbb{Q}$-vector space with generators $[\sigma, \varpi]$, where σ is any face of dimension n of the topological realisation σF_x of an object x of $\mathcal{D}$, and ϖ is an orientation on σ. These symbols are subject to relations:

$$\begin{aligned} &(i) && [\sigma, \lambda\varpi] = \lambda[\sigma, \varpi] && \text{for any } \lambda \in \mathbb{Z} \,, \\ &(ii) && [\sigma, \varpi] = [\sigma', \varpi'] \end{aligned}$$

whenever σ, σ' are n-dimensional faces of $\sigma(F_x), \sigma(F_y)$ respectively, and $g : x \to y$ is a morphism in $\mathcal{D}$ which restricts to an isomorphism $\sigma F(g) : \sigma \xrightarrow{\sim} \sigma'$ which sends ϖ to ϖ'.

Define a differential $d : (\mathfrak{C}_F)_n \to (\mathfrak{C}_F)_{n-1}$ by

$$d[\sigma, \varpi] = \sum_{\tau} [\tau, \varpi\big|_{\tau}] \tag{3.2}$$

where the sum is over all facets τ of σ, where $\dim\tau = \dim\sigma - 1$, and $\varpi\big|_{\tau}$ is the image of ϖ under the boundary map $H_n(\sigma, \partial\sigma) \to H_{n-1}(\tau, \partial\tau)$. One checks that $d^2 = 0$.

Theorem 3.2 *There is a natural isomorphism* $H_n(\mathfrak{C}_F) \xrightarrow{\sim} H_n(|F|; \mathbb{Q})$.

3.3 Differential forms and de Rham complex

To show that the complex $\mathcal{A}^n$ of smooth forms (Definition 2.8) computes cohomology, we only require that:

$$H^\bullet(\mathcal{A}^n(\sigma)) \cong H^n(\sigma;\mathbb{C}) \quad \text{and} \quad H^\bullet(\mathcal{A}^n(\partial\sigma)) = H^n(\partial\sigma;\mathbb{C}) \tag{3.3}$$

and furthermore, that elements of $\mathcal{A}^n$ are *extendable* in the sense of [57, §7]:

$$i^* : \mathcal{A}^\bullet(\sigma) \longrightarrow \mathcal{A}^\bullet(\partial\sigma) \quad \text{is surjective,} \tag{3.4}$$

where $i : \partial\sigma \subset \sigma$ denotes the inclusion map. The following theorem is in fact valid for any complex of differential forms which satisfies these properties.

Theorem 3.3 *Let $F : \mathcal{D} \to \mathcal{C}_k$ be a $\mathcal{C}_k$-complex, and define the de Rham complex of F via* (2.8) (*for which $\mathcal{A}^n$ satisfies* (3.3) *and* (3.4)). *There is an isomorphism*

$$H^n\left(\mathcal{A}^\bullet(|F|)\right) \xrightarrow{\sim} H^n\left(|F|;\mathbb{C}\right) \ ,$$

where $H^n(|F|;\mathbb{C}) = H_n(|F|;\mathbb{C})^\vee$, which is induced by a bilinear pairing

$$\begin{aligned} H^n\left(\mathcal{A}^\bullet(|F|)\right) \otimes H_n(|F|) &\longrightarrow \mathbb{C} \\ \omega \otimes \gamma \ &\mapsto \ \int_\gamma \omega \, . \end{aligned} \tag{3.5}$$

By Theorem 3.2, it may be interpreted as a pairing $H^n(\mathcal{A}^\bullet(|F|)) \otimes H_n(\mathfrak{C}(F)) \to \mathbb{C}$.

Before proceeding with the proof, note that the integral (3.5) makes sense because a differential form $\omega \in \mathcal{A}^\bullet(F)$ gives rise to a well-defined smooth differential form on each simplex of $|F|$, by the compatibility condition (2.7). The integral converges because the simplices in $|F|$ are compact, and have finitely many isomorphism classes.

Remark 3.4 Sullivan defined a differential graded algebra of polynomial differential forms for simplicial complexes [57, §7, (i), p. 297], and proved that it satisfies the extendability condition (3.4). Since they are special cases of meromorphic forms, his argument implies that the induced map $H^n(\Omega^\bullet_{\mathrm{mer}}(F)) \to H^n(\mathcal{A}^\bullet(|F|))$ is surjective.

Even though the complex $\Omega^\bullet_{\mathrm{mer}}(F)$, and hence its cohomology, has a k-structure, the map $H^n(\Omega^\bullet_{\mathrm{mer}}(F)) \otimes_k \mathbb{C} \to H^n(\mathcal{A}^\bullet(|F|))$ is not an isomorphism (the former is infinite-dimensional) and so cannot be used to define a rational structure of finite type on the de Rham cohomology $H^n(\mathcal{A}^\bullet(|F|))$. In other words, the periods one obtains by integration (3.5) depend on the location of the poles of ω.

3.4 Proof of Theorems 3.2 and 3.3

The proof of Theorem 3.2 is standard (compare with [37, Theorem 2.35], [15]). Consider the filtration of $X = |F|$ by subspaces

$$X_p = \mathrm{Im}\left(\coprod_{\dim \sigma_x = p} \sigma_x \longrightarrow X \right)$$

where the disjoint union is over all faces of polyhedral cells of dimension p. It induces a filtration $F_p C_\bullet(X) = C_\bullet(X_p)$ on the singular chain complex, giving rise to a spectral sequence

$$E^1_{p,q} = H_{p+q}(\mathrm{gr}^F_p C_\bullet(X)) \cong H_{p+q}(X_p, X_{p-1})$$

which converges to $\mathrm{gr}^F_\bullet H_{p+q}(X)$. The complex $(E^1_{p,q}, d^1)$ takes the form:

$$\cdots \longrightarrow H_{p+q}(X_p, X_{p-1}) \longrightarrow H_{p+q-1}(X_{p-1}, X_{p-2}) \longrightarrow \cdots \tag{3.6}$$

where the maps are induced on the level of chains by the boundary map. By definition of X_p there is a natural morphism

$$\bigoplus_{\dim \sigma_x = p} \big(H_p(\sigma_x, \partial \sigma_x)\big) / \mathrm{Aut}_{\mathcal{D}}(\sigma_x) \longrightarrow H_p(X_p, X_{p-1}) \tag{3.7}$$

where the sum is over all equivalence classes of faces, where two faces σ_x, σ_y are equivalent if they have the same image in X, and $\mathrm{Aut}_{\mathcal{D}}(\sigma_x)$ is the subgroup of automorphisms on the face σ_x induced by morphisms in the category $\mathcal{D}$. The map (3.7) is surjective since the intersections $\sigma_x \cap \sigma_y$ between all non-isomorphic faces σ_x, σ_y are unions of faces of dimension $\leq p-1$, and contained in X_{p-1}. Since $H_{p+q}(\sigma_x, \partial\sigma_x)$ is concentrated in degree $q = 0$ (3.1), this proves that the spectral sequence degenerates at E^1, and the cohomology of X is isomorphic to the cohomology of the complex (3.6) on setting $q = 0$. In order to identify this complex with the face complex, one may observe that $H_{p+q}(\sigma_x, \partial\sigma_x) \cong H^{\mathrm{lf}}_{p+q}(\mathring{\sigma}_x)$ is isomorphic to the locally finite (Borel-Moore) homology of the interior of σ_x. Since by assumption on $\mathcal{C}_k$ the interiors of faces are either disjoint or isomorphic, we conclude that (3.7) is also injective, and therefore defines an isomorphism

$$\mathfrak{C}(F)_p \cong H_p(X_p, X_{p-1})$$

by Definition 3.1. Furthermore, since the morphisms in (3.6) are induced by the boundary map on relative cohomology, we may identify $(E^1_{p,0}, d^1)$ with the face complex (3.2). This completes the proof of Theorem 3.2, and shows, in passing, that $d^2 = 0$.

Theorem 3.3 is a de Rham version of Theorem 3.2. We shall prove it in a similar way by replacing the singular chain complex with a complex of differential forms and replacing vector spaces and linear maps with their duals. First we need the following lemma.

Lemma 3.5 *Assume* (3.3) *and* (3.4). *Let* $K^\bullet(\sigma) = \ker(i^* : A^\bullet(\sigma) \to A^\bullet(\partial\sigma))$. *Then*

$$H^n(K^\bullet(\sigma)) \xrightarrow{\sim} H^n(\sigma, \partial\sigma)\ . \tag{3.8}$$

Proof By assumption, the relative cohomology group $H^n(\sigma, \partial\sigma)$ is the cohomology of the mapping cone of i^*, which is the complex $\mathcal{A}^n(\sigma) \oplus \mathcal{A}^{n-1}(\partial\sigma)$ with differential $d(\alpha, \beta) = (d\alpha, i^*\alpha - d\beta)$. The map (3.8) is induced by the morphism of complexes:

$$K^\bullet(\sigma) \xrightarrow{\kappa} \mathcal{A}^n(\sigma) \oplus \mathcal{A}^{n-1}(\partial\sigma)$$

where $\kappa(\omega) = (\omega, 0)$. To see that (3.8) is surjective, let $(\omega, \eta) \in \mathcal{A}^n(\sigma) \oplus \mathcal{A}^{n-1}(\partial\sigma)$ be closed, which implies that $d\omega = 0$ and $i^*\omega = d\eta$. By (3.4), there exists $\alpha \in \mathcal{A}^{n-1}(\sigma)$ such that $i^*\alpha = \eta$. The cohomology class of (ω, η) is also represented by $(\omega, \eta) - d(\alpha, 0) = (\omega - d\alpha, 0)$, which equals $\kappa(\omega - d\alpha)$. Note that $\omega - d\alpha \in K^\bullet(\sigma)$ since $i^*(\omega - d\alpha) = i^*(\omega) - d\eta = 0$. To establish the injectivity of (3.8), suppose that $\kappa(\omega) = d(\alpha, \beta)$ is exact. This implies that $d\alpha = \omega$, and $i^*\alpha = d\beta$. By (3.4), there exists $\gamma \in \mathcal{A}^\bullet(\sigma)$ such that $i^*\gamma = \beta$, and hence $\omega = d(\alpha - d\gamma)$ is exact in $K^\bullet(\sigma)$, since $i^*(\alpha - d\gamma) = i^*\alpha - d\beta = 0$. □

The proof of Theorem 3.3 proceeds as for Theorem 3.2. The filtration X_p on X induces a cofiltration on the differential graded algebra $\mathcal{A}^n(|F|)$. It produces a spectral sequence converging to the cohomology of $\mathcal{A}^\bullet(|F|)$. It is enough to show that this spectral sequence is isomorphic, via integration, to the dual of the homology spectral sequence considered above. The integration pairing is well-defined on the level of chain complexes because of Stokes' theorem. The associated graded of the cofiltration on $\mathcal{A}^n(|F|)$ is

$$K_p^\bullet(|F|) = \ker\left(\varprojlim_{\dim\sigma_x \le p} \mathcal{A}^\bullet(\sigma_x) \longrightarrow \varprojlim_{\dim\sigma_x \le p-1} \mathcal{A}^\bullet(\sigma_x)\right)$$

consisting of compatible systems of differential forms on faces of dimension p which vanish on faces of dimension $p-1$. As previously, one has an isomorphism

$$\begin{aligned} H^{p+q}(K_p^\bullet(|F|)) &\cong \bigoplus_{\sigma_x} H^{p+q}\left(K_p^\bullet(\sigma_x)\right)/\mathrm{Aut}_{\mathcal{D}}(\sigma_x) \\ &= \bigoplus_{\sigma_x} H^{p+q}(\sigma_x, \partial\sigma_x)/\mathrm{Aut}_{\mathcal{D}}(\sigma_x) \end{aligned}$$

where the direct sum is over equivalence classes of σ_x, and the second equality follows from (3.8). The relative cohomology $H^{p+q}(\sigma_x, \partial\sigma_x)$ is isomorphic to the compactly supported cohomology $H_c^{p+q}(\mathring{\sigma}_x)$ and is canonically dual to $H_{p+q}(\sigma_x, \partial\sigma_x) \otimes_{\mathbb{Z}} \mathbb{C}$. This implies that $H^{p+q}(K_p^\bullet(|F|))$ vanishes except in degree $q = 0$ and, in this degree, is dual to $\mathfrak{C}(F)_p \otimes \mathbb{C}$ via the integration pairing, which proves Theorem 3.3.

3.5 Relative homology and compact supports

Let F, G be two $\mathcal{C}_k$-complexes and $i : G \to F$ a morphism such that $i : |G| \hookrightarrow |F|$. We may identify $|G|$ with its image $i(|G|)$. Define the DGA of compactly supported forms on the complement by

$$\mathcal{A}_c^\bullet(|F| \setminus |G|) = \varprojlim_{x\in\mathcal{D}} \mathcal{A}_c^\bullet(\sigma(F_x)\backslash(\sigma(F_x)\cap|G|))$$

and write $H^n_{\mathrm{dR},c}(|F| \setminus |G|) = H^n(\mathcal{A}_c^\bullet(|F| \setminus |G|))$. Define the relative de Rham cohomology $H^n_{\mathrm{dR}}(|F|, |G|)$ to be the cohomology of the mapping cone $\mathcal{A}^\bullet(|F|) \oplus \mathcal{A}^{\bullet-1}(|G|)$ with respect to the differential $d(\omega, \eta) = (d\omega, i^*\omega - d\eta)$.

Relative homology and cohomology satisfy the usual long exact sequences.

Theorem 3.6 *There is an isomorphism $H_n(\mathfrak{C}_F/\mathfrak{C}_G) \cong H_n(|F|, |G|; \mathbb{Q})$. Integration defines a canonical isomorphism of $\mathbb{C}$-vector spaces*:

$$H^n_{\mathrm{dR}}(|F|, |G|) \longrightarrow (H_n(|F|, |G|; \mathbb{Q}))^\vee \otimes \mathbb{C}\,.$$

The map defined on complexes by $\omega \mapsto (\omega, 0)$ passes to a canonical isomorphism

$$H^n_{\mathrm{dR},c}(|F| \setminus |G|) \overset{\sim}{\longrightarrow} H^n_{\mathrm{dR}}(|F|, |G|)\ . \tag{3.9}$$

Proof The first part follows formally from Theorems 3.2 and 3.3. To prove (3.9), one may follow the same strategy as Theorem 3.3: the filtration X_p on $|F|$ gives rise to a cofiltration on $\mathcal{A}_c^\bullet(|F| \setminus |G|)$ and a spectral sequence whose E^1_{pq} terms are

$$\bigoplus_{\sigma_x} H^{p+q}_{\mathrm{dR},c}(\sigma_x\backslash(\sigma_x\cap|G|))/\mathrm{Aut}_{\mathcal{D}}(\sigma_x) \tag{3.10}$$

$$\overset{\sim}{\to} \bigoplus_{\sigma_x} H^{p+q}_{\mathrm{dR}}(\sigma_x, \partial\sigma_x \cup (\sigma_x\cap|G|))/\mathrm{Aut}_{\mathcal{D}}(\sigma_x)$$

where the direct sum is over equivalence classes σ_x of cells of dimension p.[2] Since every morphism in $\mathcal{C}_k$ is either a face map or an isomorphism, it follows that either $\sigma_x \cap |G|$ is contained in the boundary of σ_x, in which case $H^{p+q}_{\mathrm{dR}}(\sigma_x, \partial\sigma_x \cup (\sigma_x \cap |G|)) = H^{p+q}_{\mathrm{dR}}(\sigma_x, \partial\sigma_x)$, or $\sigma_x \subset |G|$, in which case this group vanishes. It follows that the complex on the right-hand side of (3.10) is dual, via integration, to $\mathfrak{C}_F/\mathfrak{C}_G \otimes_k \mathbb{C}$. □

4 Linear polyhedral complexes

Let V be a vector space of dimension $n+1$ over a field $k \subset \mathbb{R}$, and let $\mathbb{P}(V)$ denote the associated projective space of dimension n. When V has a preferred basis, we write

[2] One can directly compare this with the associated spectral sequence for the cohomology of the mapping cone $\mathcal{A}^\bullet(|F|) \oplus \mathcal{A}^{\bullet-1}(|G|)$. One replaces $K^\bullet_p(|F|)$ with a complex consisting of forms which vanish on $X_{p-1} \cup |G|$. Its cohomology is $\bigoplus_{\sigma_x} H^{p+q}_{\mathrm{dR}}(\sigma_x, \partial\sigma_x \cup (\sigma_x \cap |G|))/\mathrm{Aut}_{\mathcal{D}}(\sigma_x)$.

$\mathbb{P}^n_k$ instead of $\mathbb{P}(k^{n+1})$. We call a *projective linear configuration* $L \subset \mathbb{P}(V)$ any finite union $L = \bigcup_{i\in I} L_i$ of linear spaces $L_i \subset \mathbb{P}(V)$, all of which have equal dimension. Correspondingly, there are linear subspaces $W_i \subset V$, such that $L_i = \mathbb{P}(W_i)$ for all $i \in I$. For any subset $J \subset I$, we shall write $L_J = \bigcap_{j\in J} L_j$ and $W_J = \bigcap_{j\in J} W_j$.

4.1 Polyhedral linear configurations

Let us write $V_{\mathbb{R}} = V \otimes_k \mathbb{R}$.

Definition 4.1 A *real polyhedral cone* defined over V is the convex hull of a finite set of vectors $v_1, \ldots, v_m \in V$, where $m \geq 1$:

$$\widehat{\sigma} = \mathbb{R}_{\geq 0}\langle v_1, \ldots, v_m\rangle \quad \subset \quad V_{\mathbb{R}} \ . \tag{4.1}$$

Its cone point is the origin. A polyhedral cone is called *strongly convex* if it does not contain any real line $\mathbb{R}w$, for a non-zero vector $w \in V_{\mathbb{R}}$.

A *(projective) polyhedron in* $\mathbb{P}(V)$ is a pair (σ, V), where $\sigma \subset \mathbb{P}(V_{\mathbb{R}})$ is the link of the cone point of a strongly convex polyhedral cone (4.1) defined over V:

$$\sigma = (\widehat{\sigma} \setminus \{0\})\,/\mathbb{R}^{\times}_{>0} \ .$$

Given a polyhedron (σ, V), we write $V_\sigma \subset V$ for the k-linear span of its defining vectors $v_1, \ldots, v_m$ (4.1). The space V_σ only depends on σ, and indeed, the associated projective space $\mathbb{P}(V_\sigma)$ is the Zariski-closure of the k-rational points $\sigma \cap \mathbb{P}(V)(k)$. In particular, (σ, V_σ) is a polyhedron in $\mathbb{P}(V_\sigma)$ and has maximal dimension. In general, the vector space V_σ may be strictly contained in V. We may allow the case when all $v_1, \ldots, v_m = 0$ and σ is empty.

By a well-known theorem of Minkowski and Weyl, a polyhedron may equivalently be described by its facets. There is a unique, finite, minimal set of hyperplanes $(H_j)_{j\in J} \subset V_\sigma \subset V$ where H_j is defined by the vanishing of a non-zero linear form $f_j \in V_\sigma^\vee$ defined over k, such that for all $j \in J$:

$$\sigma \cap (V_\sigma \otimes_k \mathbb{R}) = \{x \in V_\sigma \otimes_k \mathbb{R} : f_j(x) \geq 0 \text{ for all } j \in J\} \ .$$

Note that $\sigma \cap (V_\sigma \otimes_k \mathbb{R})$ is canonically identified with σ via the inclusion $V_\sigma \hookrightarrow V$. A *facet of* σ is a non-empty projective polyhedron of the form $(\sigma \cap H_j, V)$, and has dimension one less than σ. A *face of* σ is any non-empty intersection of facets $(\sigma \cap H_{j_1} \cap \cdots \cap H_{j_k}, V)$, for $k \geq 0$ and a *vertex of* σ is a face of dimension zero. Every vertex of σ thus defines a k-rational point in $\mathbb{P}(V)$ and, one may show, is the image $[v_i] \in \mathbb{P}_k(V)(k)$ of some vector v_i, for $1 \leq i \leq m$, where the v_i are as in (4.1). Not all of the vectors v_i are necessarily vertices and may be redundant in the definition of $\widehat{\sigma}$.

Definition 4.2 A *polyhedral linear configuration* over k is a triple $(\mathbb{P}(V), L_\sigma, \sigma)$ where (σ, V) is a polyhedron, and L_σ is the linear configuration in $\mathbb{P}(V)$ whose components $L_i = V_{\sigma_i}$ are the affine spans of every facet σ_i of σ.

In particular, each component L_i of L_σ satisfies $\dim L_i = \dim\sigma - 1$ and the set of real points $L_\sigma(\mathbb{R})$ is nothing other than the Zariski-closure of the set of points of the boundary $\partial\sigma$ in $\mathbb{P}(V)(\mathbb{R})$. In this manner, a polyhedron (V,σ) uniquely determines a polyhedral linear configuration $(\mathbb{P}(V), L_\sigma, \sigma)$, and vice-versa.

Definition 4.3 A *map of polyhedral linear configurations*, which we denote by

$$\phi : (\mathbb{P}(V), L_\sigma, \sigma) \longrightarrow (\mathbb{P}(V'), L_{\sigma'}, \sigma') \ , \tag{4.2}$$

is given by an injective linear map $\phi : V \hookrightarrow V'$ such that the induced map of projective spaces, also denoted by $\phi : \mathbb{P}(V) \to \mathbb{P}(V')$, satisfies both

$$\phi(L) \subset L' \quad \text{and} \quad \phi(\sigma) \subset \sigma'.$$

In particular, every face of σ maps to a face of σ'.

Remark 4.4 The above definitions are insufficient to express the subdivision of polyhedra into smaller polyhedra. For this one must consider a more general notion where L contains linear subspaces in addition to the Zariski closures of the facets of σ.

Example 4.5 The standard simplex in projective space $\mathbb{P}^n_{\mathbb{Q}}$ with homogeneous coordinates $(x_0 : \ldots : x_n)$ is the polyhedral linear configuration $(\mathbb{P}^n_{\mathbb{Q}}, L, \sigma)$, where $L = V(x_0 \cdots x_n)$, and $\sigma \subset \mathbb{P}^n_{\mathbb{Q}}(\mathbb{R})$ is defined by the region $x_i \geq 0$.

4.2 Faces and their normals

Let $(\mathbb{P}(V), L_\sigma, \sigma)$ be a polyhedral linear configuration, and consider any face σ_F of σ. The polyhedron (σ_F, V_{σ_F}) defines a polyhedral linear configuration $(\mathbb{P}(V_{\sigma_F}), L_{\sigma_F}, \sigma_F)$ and a 'face' map

$$(\mathbb{P}(V_{\sigma_F}), L_{\sigma_F}, \sigma_F) \longrightarrow (\mathbb{P}(V), L, \sigma) \ , \tag{4.3}$$

which is the map (4.2) induced by the inclusion of V_{σ_F} in V, and corresponds to the inclusion of σ_F in σ. Note that (σ_F, V_{σ_F}) has maximal dimension, i.e., $\dim(\sigma_F) = \dim(V_{\sigma_F}) - 1$, but the same is not necessarily true of (σ, V). For the trivial face, when σ_F is equal to σ itself, (4.3) gives a linear map $(\mathbb{P}(V_\sigma), L_\sigma, \sigma) \to (\mathbb{P}(V), L, \sigma)$.

Definition 4.6 Let $W \subset V$ be a vector subspace such that $\sigma_W = \sigma \cap \mathbb{P}(W)$ is a face of σ (here and elsewhere, we write $\sigma \cap \mathbb{P}(W)$ for $\sigma \cap \mathbb{P}(W)(\mathbb{R})$). Consider the polyhedral cone:

$$\widehat{\sigma_{/W}} \subset (V/W)_{\mathbb{R}}$$

which is the image of $\widehat{\sigma}$ (4.1) under the natural map $V_{\mathbb{R}} \to (V/W)_{\mathbb{R}}$. It is defined over V/W since it is the convex hull of the images of a set of defining vectors v_i (4.1) under the natural map $V \to V/W$. Denote by $\sigma_{/W}$ the link of its cone point.

It is important to note that σ_W is not necessarily assumed to be Zariski-dense in $\mathbb{P}(W)$, i.e., W may strictly contain the space V_{σ_W}.

Lemma 4.7 *The pair* $(\sigma_{/W}, V/W)$ *is a polyhedron.*

Proof It suffices to show that $\sigma_{/W}$ is strictly convex. We do this by showing that it is contained in a standard simplex (Example 4.5). Consider any choice of irreducible components L_i of L_σ such that the face σ_W is given by the intersection $\sigma_W = \sigma \cap L_1 \cap \cdots \cap L_m$. We may assume that $L_1, \ldots, L_m$ are normal crossing. Since σ_W is Zariski-dense in $L_1 \cap \cdots \cap L_m$ it follows that $L_1 \cap \cdots \cap L_m \subset \mathbb{P}(W)$, and hence $\dim W \geq \dim V - m$. Since the L_i cross normally, it follows that any subset of $p = \dim(V/W) \leq m$ spaces L_i, and in particular $L_1, \ldots, L_p$, defines a set of coordinate hyperplanes on $\mathbb{P}(V/W)$. Let x_i be a system of coordinates on $\mathbb{P}(V/W)$ whose zero loci are the L_i for $i = 1, \ldots, p$. By replacing x_i with $-x_i$ we may assume that the x_i are non-negative on σ. By construction, the link $\sigma_{/W}$ is contained in the strictly convex region $\{(x_1 : \ldots : x_p) : x_i \geq 0\}$. $\square$

Definition 4.8 Define the *normal* of σ relative to W to be $(\mathbb{P}(V/W), L_{\sigma_{/W}}, \sigma_{/W})$.

A map of polyhedral linear configurations induces maps simultaneously on faces and their normals. More precisely, let ϕ be as in (4.2), and suppose that $W \subset V$ meets σ in a face $\sigma_W = \sigma \cap W$ of σ. Then $\phi(W) = W'$ also meets σ' in the face $\sigma'_{W'} = \sigma' \cap W'$, and we deduce a pair of maps of polyhedral linear configurations:

$$\phi\big|_W : (\mathbb{P}(W), L_{\sigma_W}, \sigma_W) \to (\mathbb{P}(W'), L_{\sigma'_{W'}}, \sigma'_{W'}) \tag{4.4}$$
$$\phi_{/W} : (\mathbb{P}(V/W), L_{\sigma_{/W}}, \sigma_{/W}) \to (\mathbb{P}(V'/W'), L_{\sigma'_{/W'}}, \sigma'_{/W'}) \ .$$

Remark 4.9 The projectivised normal bundle of the linear subspace $\mathbb{P}(W) \subset \mathbb{P}(V)$ is trivial, and is canonically isomorphic to a product of projective spaces:

$$\mathbb{P}(N_{\mathbb{P}(W)|\mathbb{P}(V)}) = \mathbb{P}(W) \times \mathbb{P}(V/W) \ .$$

For any subspace $W \subset V$ meeting σ in a face σ_W, the product of polyhedra

$$\sigma_W \times \sigma_{/W} \quad \subset \quad \mathbb{P}(W)(\mathbb{R}) \times \mathbb{P}(V/W)(\mathbb{R})$$

is contained within it. A map of polyhedral linear configurations ϕ as above induces a map on the products $\sigma_W \times \sigma_{/W} \to \sigma'_{W'} \times \sigma'_{/W'}$. These products encode the infinitesimal structure of σ in the neighbourhood of σ_W. This will be discussed in §5.4.

Lemma 4.10 *Let* (σ, V) *be a polyhedron, and let* $W_1 \subsetneq W_2 \subset V$ *such that* $\sigma \cap \mathbb{P}(W_1)(\mathbb{R}) = \sigma \cap \mathbb{P}(W_2)(\mathbb{R})$ *is a face of* σ. *Then* $\mathbb{P}(W_2/W_1)(\mathbb{R})$ *does not meet* $(\sigma_{/W_1}, V/W_1)$.

Proof If $\mathbb{P}(W_2/W_1)(\mathbb{R})$ were to meet $\sigma_{/W_1}$, then there would exist a $0 \neq v \in W_2/W_1$ whose image in $\mathbb{P}(W_2/W_1)(\mathbb{Q})$ is a vertex of $\sigma_{/W_1}$. This would imply that $\sigma \cap \mathbb{P}(W_2)(\mathbb{R})$ strictly contains $\sigma \cap \mathbb{P}(W_1)(\mathbb{R})$, a contradiction. $\square$

4.3 Category of polyhedral linear configurations

Definition 4.11 Define a category PLC_k whose objects are polyhedral linear configurations over k, and whose morphisms are generated by:

(1) (Linear embeddings) Maps of the form $f : (\mathbb{P}(V), L, \sigma) \to (\mathbb{P}(V'), L', \sigma')$, where f is a linear embedding which satisfies $f(L) = L'$ and $f(\sigma) = \sigma'$,
(2) (Inclusions of faces) For any face σ_F of σ, the face maps (4.3):

$$(\mathbb{P}(V_{\sigma_F}), L_{\sigma_F}, \sigma_F) \longrightarrow (\mathbb{P}(V), L, \sigma) \ .$$

The category PLC_k is a sub-category of $\mathcal{PC}_k$ satisfying the assumptions of §3.1.

Remark 4.12 The above categories are adapted to studying the links of cones. If one is interested in the cones *per se*, one may consider a version in which one replaces projective space $\mathbb{P}^n_k$ with affine space $\mathbb{A}^{n+1}_k$, and σ with $\widehat{\sigma}$, *etc*.

4.4 Linear polyhedral complexes

Definition 2.3 leads to the following:

Definition 4.13 A *linear polyhedral complex* is a PLC_k-complex, i.e., a functor from a finite diagram category to the category of polyhedral linear configurations.

Remark 4.14 The topological realisation of a polyhedral linear complex is obtained by gluing together finitely many quotients of strictly convex polyhedra by finite groups of automorphisms and defines a symmetric CW-complex (see [3]).

We may define subschemes of polyhedral linear complexes, and differential forms upon them in the manner of §2.3, 2.4.

5 Wonderful compactications of linear polyhedral complexes

5.1 Blowing-up linear subspaces

Let $\mathcal{B}$ denote a finite set of linear subspaces $\mathbb{P}(W) \subsetneq \mathbb{P}(V)$ with the property that $\mathcal{B}$ is closed under intersections.

Definition 5.1 The (wonderful) compactification [29] of $\mathbb{P}(V)$ along $\mathcal{B}$ is denoted

$$\pi_{\mathcal{B}} : P^{\mathcal{B}}(V) \longrightarrow \mathbb{P}(V)$$

and is defined to be the iterated blow-up of $\mathbb{P}(V)$ along the strict transforms of the strata $\mathbb{P}(W_j) \in \mathcal{B}$, in increasing order of dimension. It is independent of the order of blow-ups, and hence well-defined. Let $D^{\mathcal{B}} \subset P^{\mathcal{B}}(V)$ denote the exceptional divisor.

Proposition 5.2 *The iterated blow-up $P^{\mathcal{B}}$ has the following properties.*

(i) *Let $f : U \to V$ be an injective linear map, and $\mathcal{B}$ a set of subspaces of $\mathbb{P}(V)$ as above, such that $\mathbb{P}(fU)$ is not contained in any element of $\mathcal{B}$. If $f^{-1}\mathcal{B}$ denotes the set of preimages $f^{-1}\mathbb{P}(W)$ of spaces $\mathbb{P}(W)$ in $\mathcal{B}$, there is a canonical map $f^{\mathcal{B}}$: $P^{f^{-1}\mathcal{B}}(U) \to P^{\mathcal{B}}(V)$ such that*

$$\begin{array}{ccc} P^{f^{-1}\mathcal{B}}(U) & \xrightarrow{f^{\mathcal{B}}} & P^{\mathcal{B}}(V) \\ \downarrow & & \downarrow \\ \mathbb{P}(U) & \xrightarrow{f} & \mathbb{P}(V) \end{array}$$

commutes, where the vertical maps are the blow-downs $\pi_{f^{-1}\mathcal{B}}$, $\pi_{\mathcal{B}}$ respectively.

(ii) *Suppose that $\mathcal{B}$, $\mathcal{B}'$ are two sets of linear subspaces as above which are closed under intersections. Suppose that $\mathcal{B}' \subset \mathcal{B}$. Then there is a canonical morphism*

$$\pi_{\mathcal{B}/\mathcal{B}'} : P^{\mathcal{B}}(V) \longrightarrow P^{\mathcal{B}'}(V) \ .$$

Proof Part (i) follows from repeated application of the universal property of strict transforms [36][II, Corollary 7.15]. For (ii), the proof is by induction. Choose any linear ordering on $\mathcal{B}$ compatible with the order of blowing-up (i.e., in increasing dimension). Let $\mathcal{B}_n \subset \mathcal{B}$ denote the first n elements of $\mathcal{B}$. Suppose by induction on n that we have constructed morphisms $\pi_m : P^{\mathcal{B}_m}(V) \to P^{\mathcal{B}'\cap\mathcal{B}_m}(V)$ for all $m \leq n$ such that the following diagrams commute, where the vertical maps are the natural (blow-down) maps:

$$\begin{array}{ccc} P^{\mathcal{B}_m}(V) & \xrightarrow{\pi_m} & P^{\mathcal{B}'\cap\mathcal{B}_m}(V) \\ \downarrow & & \downarrow \\ P^{\mathcal{B}_{m-1}}(V) & \xrightarrow{\pi_{m-1}} & P^{\mathcal{B}'\cap\mathcal{B}_{m-1}}(V) \end{array} \tag{5.1}$$

and where $\pi_0 : \mathbb{P}(V) \to \mathbb{P}(V)$ is the identity. Let $\mathcal{B}_{n+1} \setminus \mathcal{B}_n = \{\mathbb{P}(W)\}$ where $\mathbb{P}(W) \subset \mathbb{P}(V)$ is a blow-up locus for $P^{\mathcal{B}}(V)$. To define π_{n+1}, there are two cases to consider. Either $\mathbb{P}(W)$ is an element of $\mathcal{B}'$, or it is not. If it is not, then $\mathcal{B}' \cap \mathcal{B}_{n+1} = \mathcal{B}' \cap \mathcal{B}_n$ and the map $\pi_{n+1} : P^{\mathcal{B}_{n+1}}(V) \to P^{\mathcal{B}'\cap\mathcal{B}_{n+1}}(V)$ is simply the composition of the blow-up $P^{\mathcal{B}_{n+1}}(V) \to P^{\mathcal{B}_n}(V)$ with π_n (in this case, the vertical map on the right in (5.1) is the identity when $m = n+1$). In the contrary case, $\mathcal{B}_{n+1} = \mathcal{B}_n \cup \{\mathbb{P}(W)\}$ where $\mathbb{P}(W) \in \mathcal{B}'$. Let Y denote the strict transform of $\mathbb{P}(W)$ in $P^{\mathcal{B}'\cap\mathcal{B}_n}(V)$. By commutativity of the diagrams (5.1) for all $0 \leq m \leq n$, its inverse image $\pi_n^{-1}(Y)$ in $P^{\mathcal{B}_n}(V)$ is a Cartier divisor since it is the union of the strict transform Y' of $\mathbb{P}(W)$ in $P^{\mathcal{B}_n}(V)$, which is Cartier, with exceptional divisors (likewise). By the universal property of blowing up [36][II, Proposition 7.14], there is a unique morphism π : $\mathrm{Bl}_{Y'} P^{\mathcal{B}_n}(V) \to \mathrm{Bl}_Y P^{\mathcal{B}'\cap\mathcal{B}_n}(V)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Bl}_{Y'} P^{\mathcal{B}_n}(V) & \to & \mathrm{Bl}_Y P^{\mathcal{B}'\cap\mathcal{B}_n}(V) \\ \downarrow & & \downarrow \\ P^{\mathcal{B}_n}(V) & \xrightarrow{\pi_n} & P^{\mathcal{B}'\cap\mathcal{B}_n}(V) \end{array}$$

Since $\mathrm{Bl}_{Y'} P^{\mathcal{B}_n}(V) = P^{\mathcal{B}_{n+1}}(V)$ and $\mathrm{Bl}_Y P^{\mathcal{B}'\cap\mathcal{B}_n}(V) = P^{\mathcal{B}'\cap\mathcal{B}_{n+1}}(V)$ we may define π_{n+1} to be the morphism π, which completes the induction step. $\square$

Proposition 5.3 [29]. *The irreducible components $\mathcal{E}_W^{\mathcal{B}}$ of $D^{\mathcal{B}}$ are in one-to-one correspondence with subspaces $\mathbb{P}(W) \in \mathcal{B}$ of codimension ≥ 1, where $\pi_{\mathcal{B}} : \mathcal{E}_W^{\mathcal{B}} \to \mathbb{P}(W)$, and $\mathcal{E}_W^{\mathcal{B}}$ is the Zariski closure of the inverse image with respect to $\pi_{\mathcal{B}}$ of the generic point of $\mathbb{P}(W)$.*

If we define the following sets (which are closed under intersections):

$$\mathcal{B}_W = \{\mathbb{P}(T) \subsetneq \mathbb{P}(W) \ \text{for} \ \mathbb{P}(T) \in \mathcal{B} \ \text{such that} \ T \subseteq W\} \tag{5.2}$$

$$\mathcal{B}_{/W} = \{\mathbb{P}(T/W) \subset \mathbb{P}(V/W) \ \text{for} \ \mathbb{P}(T) \in \mathcal{B} \ \text{such that} \ T \supseteq W\}$$

(recall that $\mathbb{P}(0) = \emptyset$) then there is a commutative diagram

$$\begin{array}{ccc} P^{\mathcal{B}_W} \times P^{\mathcal{B}_{/W}} & \xrightarrow{\sim} & \mathcal{E}_W^{\mathcal{B}} \\ \downarrow \pi_{\mathcal{B}_W} \times \pi_{\mathcal{B}_{/W}} & & \downarrow \pi_{\mathcal{B}} \\ \mathbb{P}(W) \times \mathbb{P}(V/W) & \longrightarrow & \mathbb{P}(W) \end{array} \tag{5.3}$$

where the horizontal map along the top is a canonical isomorphism, and the one along the bottom is projection onto the first factor $\mathbb{P}(W)$. The divisor $D^{\mathcal{B}}$ is simple normal crossing, and two components $\mathcal{E}_W^{\mathcal{B}}$ and $\mathcal{E}_{W'}^{\mathcal{B}}$ have non-empty intersection if and only if one of the two spaces $\mathbb{P}(W)$, $\mathbb{P}(W')$ is contained in the other.

Remark 5.4 The following observation will be useful. By (5.3), $\mathcal{E}_W^{\mathcal{B}}$ is canonically isomorphic to the iterated blow-up of $\mathbb{P}(W) \times \mathbb{P}(V/W)$ relative to $\pi_{\mathcal{B}_W}$ and $\pi_{\mathcal{B}_{/W}}$ on each factor. The product $\mathbb{P}(W) \times \mathbb{P}(V/W)$ is isomorphic to the exceptional divisor of a single blow-up of $\mathbb{P}(V)$ along $\mathbb{P}(W)$. Thus the exceptional divisor $\mathcal{E}_W^{\mathcal{B}}$ may be computed by first blowing-up a single linear space $\mathbb{P}(W)$ inside $\mathbb{P}(V)$, and then computing the iterated blow-ups relative to $\mathcal{B}_W$ and $\mathcal{B}_{/W}$ on each factor $\mathbb{P}(W)$ and $\mathbb{P}(V/W)$ of its exceptional divisor.

By repeated application of Proposition 5.3, one shows that intersections of irreducible components of $D^{\mathcal{B}}$ are in one-to-one correspondence with sequences

$$\mathbb{P}(W_k) \subset \mathbb{P}(W_{k-1}) \subset \cdots \subset \mathbb{P}(W_1) \subset \mathbb{P}(V) \tag{5.4}$$

where $\mathbb{P}(W_i) \in \mathcal{B}$ for $i = 1, \ldots, k$ and all inclusions are strict. The corresponding subscheme of $D^{\mathcal{B}}$ is isomorphic to $\prod_{i=1}^{k+1} P^{\mathcal{B}_i}(W_{i-1}/W_i)$ where $W_{k+1} = 0$ and $W_0 = V$ and where $\mathcal{B}_i = \{\mathbb{P}(T/W_i)$ where $\mathbb{P}(T) \in \mathcal{B}$ such that $W_i \subseteq T \subsetneq W_{i-1}\}$. The previous proposition can be proved using explicit local coordinates, which we describe presently.

5.2 Local coordinates for linear blow ups

Given a sequence (5.4), we may choose projective coordinates $\alpha_1, \ldots, \alpha_n$ on $\mathbb{P}(V)$ such that

$$\mathbb{P}(W_m) = V(\alpha_1, \alpha_2, \ldots, \alpha_{i_m}) \quad \text{for} \ 1 \leq m \leq k \ ,$$

for some increasing sequence $0 < i_1 < \cdots < i_k < n$. A choice of local affine coordinates on $P^{\mathcal{B}}$ lying over the open chart $\alpha_n = 1$ of $\mathbb{P}(V)$ is given by $\beta_1, \ldots, \beta_{n-1}$ where:

$$\beta_1 = \frac{\alpha_1}{\alpha_{i_1}}, \ \ldots, \ \beta_{i_1-1} = \frac{\alpha_{i_1-1}}{\alpha_{i_1}}\ , \ \beta_{i_1} = \frac{\alpha_{i_1}}{\alpha_{i_2}}, \ \ldots, \beta_{i_2-1} = \frac{\alpha_{i_2-1}}{\alpha_{i_2}}\ , \ \ldots$$
$$\beta_{i_{k-1}} = \frac{\alpha_{i_{k-1}}}{\alpha_{i_k}}, \ldots, \ \beta_{i_k-1} = \frac{\alpha_{i_k-1}}{\alpha_{i_k}}, \ \beta_{i_k} = \alpha_{i_k}, \ldots\ , \ \beta_{n-1} = \alpha_{n-1}$$

The equation of the exceptional divisor $\mathcal{E}_{W_m}$ which lies over $\mathbb{P}(W_m)$ is $\beta_{i_m} = 0$.

The isomorphism $\mathcal{E}_W \cong P^{\mathcal{B}_W} \times P^{\mathcal{B}_{/W}}$, in the case $W = W_m$, is represented by the partition of $\beta_1, \ldots, \beta_{n-1}$ into two sets of coordinates $\beta_1, \ldots, \beta_{i_m-1}$ and $\beta_{i_m+1}, \ldots, \beta_{n-1}$ corresponding to the iterated blow-ups of two nested sequences:

$$\mathbb{P}(W_k) \subset \cdots \subset \mathbb{P}(W_{m+2}) \subset \mathbb{P}(W_{m+1}) \subset \mathbb{P}(W_m)$$

$$\mathbb{P}(W_{m-1}/W_m) \subset \cdots \subset \mathbb{P}(W_1/W_m) \subset \mathbb{P}(V/W_m)$$

lying over the affine charts with coordinates $(\alpha_1 : \ldots : \alpha_{i_m-1} : 1)$ for $\mathbb{P}(V/W_m)$, and $(\alpha_{i_m+1} : \ldots : \alpha_{n-1} : 1)$ for $\mathbb{P}(W_m)$. Propositions 5.2, 5.3 may be proven by computing directly in these coordinates.

Lemma 5.5 *Let $H \subset \mathbb{P}(V)$ be a hyperplane and let $\mathbb{P}(W) \in \mathcal{B}$. Denote by $\widetilde{H} \subset P^{\mathcal{B}}$ the strict transform of H under $\pi_{\mathcal{B}}$. Then*

$$\widetilde{H} \cap \mathcal{E}_W = \begin{cases} P^{\mathcal{B}_W} \times \widetilde{H}_{/W} & \text{if} \quad \mathbb{P}(W) \subseteq H \\ \widetilde{H}_W \times P^{\mathcal{B}_{/W}} & \text{if} \quad \mathbb{P}(W) \nsubseteq H\ , \end{cases}$$

where $H_W = H \cap \mathbb{P}(W)$, and, when H corresponds to a subspace $W \subset H_0 \subset V$, we write $H_{/W} = \mathbb{P}(H_0/W)$. Their versions with tildes denote their strict transforms under the iterated blow-ups $\pi_{\mathcal{B}_W}$ and $\pi_{\mathcal{B}_{/W}}$, respectively. If $H \subseteq \mathbb{P}(W)$ then $\widetilde{H} = \emptyset$.

Proof It follows from Remark 5.4 that it is enough to compute the case when $\mathcal{B} = \{\mathbb{P}(W)\}$ reduces to a single blow-up. The strict transform of H is either

$$H_W \times \mathbb{P}(V/W) \quad \text{or} \quad \mathbb{P}(W) \times H_{/W} \quad \text{inside} \quad \mathbb{P}(W) \times \mathbb{P}(V/W)$$

depending on which of the two cases are satisfied, since H meets the normal bundle of $\mathbb{P}(W)$ in the product of a hyperplane and a projective space. Alternatively, one can also verify the lemma by direct computation. Suppose that H has the equation

$$\lambda_1\alpha_1 + \cdots + \lambda_n\alpha_n = 0\ , \tag{5.5}$$

and suppose that $W = W_m$. One has $\mathbb{P}(W_m) \subseteq H$ if and only if $\lambda_{i_m+1} = \cdots = \lambda_n = 0$. After performing the change of variables on the affine chart described above, it becomes

$$\beta_{i_m}\dots\beta_{i_k}\left(\lambda_1\beta_1\beta_{i_1}\dots\beta_{i_{m-1}}+\lambda_2\beta_2\beta_{i_1}\dots\beta_{i_{m-1}}+\dots+\lambda_{i_m-1}\beta_{i_m-1}+\lambda_{i_m}\right)$$
$$+\left(\lambda_{i_m+1}\,\beta_{i_m+1}\beta_{i_{m+1}}\dots\beta_{i_k}+\dots+\lambda_{n-1}\,\beta_{n-1}+\lambda_n\right)\ .$$

In the case when $\mathbb{P}(W_m)\not\subseteq H$, then setting $\beta_{i_m}=0$ annihilates the first line of the previous expression, leaving only the second term in parentheses. It is precisely the equation of the strict transform of H_W in $P^{\mathcal{B}_W}$. In the case when $\mathbb{P}(W_m)\subset H$, the second term in parentheses is identically zero, and therefore the strict transform of H has the equation:

$$\lambda_1\beta_1\beta_{i_1}\dots\beta_{i_{m-1}}+\lambda_2\beta_2\beta_{i_1}\dots\beta_{i_{m-1}}+\dots+\lambda_{i_m-1}\beta_{i_m-1}+\lambda_{i_m}$$

which is the equation of the strict transform of $H_{/W}$ in $P^{\mathcal{B}_{/W}}$. □

5.3 Blow-ups of polyhedral linear configurations

Let $(\mathbb{P}(V),L,\sigma)$ be a polyhedral linear configuration, and let $\mathcal{B}$ be a finite set of linear subspaces $\mathbb{P}(W)\subsetneq\mathbb{P}(V)$ such that:

(B1). $\mathcal{B}$ is stable under intersections, and
(B2). for every $\mathbb{P}(W)\in\mathcal{B}$, the set $\sigma_W=\sigma\cap\mathbb{P}(W)(\mathbb{R})$ is a face of σ, or is empty.

Consider the iterated blow up $\pi_{\mathcal{B}}:P^{\mathcal{B}}\to\mathbb{P}(V)$ defined above. In the usual case when σ is not contained in $\mathbb{P}(W)(\mathbb{R})$ for some $\mathbb{P}(W)\in\mathcal{B}$, we let

$$\sigma^{\mathcal{B}}=\overline{\pi_{\mathcal{B}}^{-1}(\overset{\circ}{\sigma})}\ \subset\ P^{\mathcal{B}}(\mathbb{R}) \tag{5.6}$$

denote the closure, in the analytic topology, of the inverse image of the interior of σ. If σ is contained in a $\mathbb{P}(W)\in\mathcal{B}$, then $\sigma^{\mathcal{B}}$ is defined to be the empty set (alternatively, we may define $\sigma^{\mathcal{B}}$ to be the closure of the inverse image of $\sigma\cap(\mathbb{P}(V)\setminus\bigcup\mathcal{B})(\mathbb{R})$, the intersection of σ with the complement in $\mathbb{P}(V)(\mathbb{R})$ of all blow-up loci in $\mathcal{B}$, which covers both cases). In local coordinates §5.2, the interior of $\sigma^{\mathcal{B}}$ is defined by the positivity of a finite number of polynomials $f_1,\dots,f_k>0$ where the f_i are the local equations of strict transforms of bounding hyperplanes, and exceptional divisors. The semi-algebraic set $\sigma^{\mathcal{B}}$ is then locally defined by $f_1,\dots,f_k\geq 0$.

When $\sigma^{\mathcal{B}}$ is non-empty, we define a *face* of $\sigma^{\mathcal{B}}$ to be a non-empty intersection

$$\sigma^{\mathcal{B}}\cap D(\mathbb{R})$$

where D is any intersection of irreducible components of the total transform $\pi_{\mathcal{B}}^{-1}(L)$. A *facet* is a face of dimension $\dim\sigma-1$. Finally, define $L^{\mathcal{B}}\subset P^{\mathcal{B}}$ to be the union of the Zariski closures of the facets of $\sigma^{\mathcal{B}}$. It depends on σ, but we usually write $L^{\mathcal{B}}$ instead of $L^{\mathcal{B}}_{\sigma}$.

Definition 5.6 The blow-up of $(\mathbb{P}(V),L,\sigma)$ along $\mathcal{B}$ is the triple $(P^{\mathcal{B}},L^{\mathcal{B}},\sigma^{\mathcal{B}})$.

By definition, the facets of $\sigma^{\mathcal{B}}$ are Zariski-dense in the irreducible components L of $L^{\mathcal{B}}$. From the description of the local coordinates on $P^{\mathcal{B}}$ one sees that

$\sigma^{\mathcal{B}}$ is a topological polyhedron inside $P^{\mathcal{B}}(\mathbb{R})$, whose boundary satisfies $\partial\sigma^{\mathcal{B}} = \sigma^{\mathcal{B}} \cap L^{\mathcal{B}}(\mathbb{R})$. Note that $L^{\mathcal{B}}$ contains the strict transform of L, and is contained within its total transform. It will follow from the description of faces in §5.4 that the triple $(P^{\mathcal{B}}, L^{\mathcal{B}}, \sigma^{\mathcal{B}})$ defines an object of $\mathcal{PC}_k$. As in §2, a map of triples $\phi : (P_1^{\mathcal{B}}, L_1^{\mathcal{B}}, \sigma_1^{\mathcal{B}}) \to (P_2^{\mathcal{B}}, L_2^{\mathcal{B}}, \sigma_2^{\mathcal{B}})$ is a morphism of schemes $\phi : P_1^{\mathcal{B}} \to P_2^{\mathcal{B}}$ such that $\phi(L_1^{\mathcal{B}}) \subset L_2^{\mathcal{B}}$ and whose restriction to real points induces $\phi(\sigma_1^{\mathcal{B}}) \subset \sigma_2^{\mathcal{B}}$. In practice, we shall only consider maps of very specific types.

Example 5.7 (i). The blow down map

$$\pi_{\mathcal{B}} : (P^{\mathcal{B}}, L^{\mathcal{B}}, \sigma^{\mathcal{B}}) \to (\mathbb{P}(V), L, \sigma) \tag{5.7}$$

is a morphism in $\mathcal{PC}_k$. The map $\pi_{\mathcal{B}} : \sigma^{\mathcal{B}} \to \sigma$ is not in general a homeomorphism since it may collapse faces.

(ii). More generally, consider two sets $\mathcal{B}, \mathcal{B}'$ which satisfy B1, B2 such that $\mathcal{B}' \subset \mathcal{B}$ (the previous example is the case $\mathcal{B}' = \emptyset$). Then Proposition 5.2 (ii) provides a map

$$\pi_{\mathcal{B}/\mathcal{B}'} : P^{\mathcal{B}} \longrightarrow P^{\mathcal{B}'} .$$

It restricts to a continuous map $\sigma^{\mathcal{B}} \to \sigma^{\mathcal{B}'}$ and defines a morphism

$$\pi_{\mathcal{B}/\mathcal{B}'} : (P^{\mathcal{B}}, L^{\mathcal{B}}, \sigma^{\mathcal{B}}) \to (P^{\mathcal{B}'}, L^{\mathcal{B}'}, \sigma^{\mathcal{B}'}) . \tag{5.8}$$

Definition 5.8 Recall from the proof of Proposition 5.2 (ii) that $\pi_{\mathcal{B}/\mathcal{B}'}$ is constructed inductively by blowing up elements of $\mathcal{B}$. Let $\mathcal{B}_n$ denote the first n elements of $\mathcal{B}$. Suppose that $\mathcal{B}' \cap \mathcal{B}_{n+1} = \mathcal{B}' \cap \mathcal{B}_n$, in which case we have a commutative diagram (5.1):

$$\begin{array}{ccc} P^{\mathcal{B}_{n+1}}(V) & \xrightarrow{\pi_{n+1}} & P^{\mathcal{B}' \cap \mathcal{B}_{n+1}}(V) \\ \downarrow & & || \\ P^{\mathcal{B}_n}(V) & \xrightarrow{\pi_n} & P^{\mathcal{B}' \cap \mathcal{B}_n}(V) \end{array}$$

where the vertical map on the right is the identity. Suppose that $\mathcal{B}_{n+1} = \mathcal{B}_n \cup \{\mathbb{P}(W)\}$, and let Y denote the strict transform of $\mathbb{P}(W)$ in $P^{\mathcal{B}_n}(V)$. Then $P^{\mathcal{B}_{n+1}}(V)$ is the blow-up of $P^{\mathcal{B}_n}(V)$ along Y. We shall call this blow-up *extraneous* (to σ) if

$$Y \cap \sigma^{\mathcal{B}_n} = \emptyset . \tag{5.9}$$

Then, since $P^{\mathcal{B}_{n+1}}(V)$ is isomorphic to $P^{\mathcal{B}_n}(V)$ away from Y, and hence in a Zariski-open neighbourhood of the compact set $\sigma_n^{\mathcal{B}}$, it follows that $\sigma^{\mathcal{B}_{n+1}} \cong \sigma^{\mathcal{B}_n}$. In other words, the extraneous blow up only changes the geometry of $P^{\mathcal{B}_n}(V)$ away from the region $\sigma^{\mathcal{B}_n}$.

More generally, we shall call the map $\pi_{\mathcal{B}/\mathcal{B}'}$ an *extraneous modification* if, every time this situation occurs in the formation of $P^{\mathcal{B}}(V)$, i.e., for all n such that $\mathcal{B}' \cap \mathcal{B}_{n+1} = \mathcal{B}' \cap \mathcal{B}_n$, then the blow-up locus is extraneous (i.e., (5.9) holds). Since every such extraneous blow-up is trivial on a Zariski-open neighbourhood of $\sigma^{\mathcal{B}_n}$, it follows that every π_n is an isomorphism locally near $\sigma^{\mathcal{B}_n}$. We conclude that an extraneous modification satisfies

$$\pi_{\mathcal{B}/\mathcal{B}'} : \sigma^{\mathcal{B}} \xrightarrow{\sim} \sigma^{\mathcal{B}'}$$

since the additional blow-ups in $\mathcal{B} \setminus \mathcal{B}'$ are away from the polyhedron $\sigma^{\mathcal{B}'}$.

Example 5.9 (Linear embeddings). Let $h : V_1 \to V_2$ be an injective linear map and consider the corresponding morphism of polyhedral linear configurations:

$$h : (\mathbb{P}(V_1), L_1, \sigma_1) \to (\mathbb{P}(V_2), L_2, \sigma_2)$$

as in Definition 4.11 (1), where $h(L_1) = L_2$ and $h(\sigma_1) = \sigma_2$. If $\mathcal{B}_2$ is a set of linear subspaces of $\mathbb{P}(V_2)$ satisfying $(B1)$ and $(B2)$ relative to σ_2, such that $\mathbb{P}(V_1)$ is not contained in $\mathbb{P}(W)$ for some $\mathbb{P}(W) \in \mathcal{B}_2$ then let

$$\mathcal{B}_1 = h^{-1}\mathcal{B}_2 = \Big\{\mathbb{P}(h^{-1}W) \subsetneq \mathbb{P}(V_1)\ ,\ \text{for all}\ \mathbb{P}(W) \in \mathcal{B}_2\Big\}\ .$$

The set $\mathcal{B}_1$ satisfies $(B1)$, $(B2)$ relative to σ_1 and Proposition 5.2 (i) gives a morphism

$$(P^{\mathcal{B}_1}(V_1), L_1^{\mathcal{B}_1}, \sigma_1^{\mathcal{B}_1}) \longrightarrow (P^{\mathcal{B}_2}(V_2), L_2^{\mathcal{B}_2}, \sigma_2^{\mathcal{B}_2})\ . \tag{5.10}$$

There is a commutative diagram where the vertical maps are $\pi_{\mathcal{B}_1}$ and $\pi_{\mathcal{B}_2}$:

$$\begin{array}{ccc} (P^{\mathcal{B}_1}(V_1), L_1^{\mathcal{B}_1}, \sigma_1^{\mathcal{B}_1}) & \xrightarrow{\sim} & (P^{\mathcal{B}_2}(V_2), L_2^{\mathcal{B}_2}, \sigma_2^{\mathcal{B}_2}) \\ \downarrow & & \downarrow \\ (\mathbb{P}(V_1), L_1, \sigma_1) & \xrightarrow{\sim} & (\mathbb{P}(V_2), L_2, \sigma_2) \end{array} \tag{5.11}$$

5.4 Faces and their product structure

Proposition 5.10 *Consider a polyhedral linear configuration* $(\mathbb{P}(V), L, \sigma)$ *and* $\mathcal{B}$ *as above. Let* $\mathbb{P}(W) \in \mathcal{B}$ *which meets* σ, *and let* $\mathcal{E}_W^{\mathcal{B}} \subset P^{\mathcal{B}}$ *denote the exceptional divisor lying above* $\mathbb{P}(W)$. *Then, via the isomorphism* $\mathcal{E}_W^{\mathcal{B}} \cong P^{\mathcal{B}_W} \times P^{\mathcal{B}_{/W}}$ *(see* (5.3)*), one has*

$$\sigma^{\mathcal{B}} \cap \mathcal{E}_W^{\mathcal{B}}(\mathbb{R}) = \sigma_W^{\mathcal{B}_W} \times \sigma_{/W}^{\mathcal{B}_{/W}} \tag{5.12}$$

where $\sigma_W = \sigma \cap \mathbb{P}(W)(\mathbb{R})$ *is the face of* σ *cut out by* $\mathbb{P}(W)$, *which exists by assumption* $(B2)$, *and* $\sigma_{/W} \subset \mathbb{P}(V/W)(\mathbb{R})$ *is its normal. If* σ_W *is contained in some* $\mathbb{P}(T) \in \mathcal{B}$, *where* $T \subsetneq W$ *(i.e.,* $\sigma_W = \sigma \cap \mathbb{P}(T)(\mathbb{R})$*), then* $\sigma_W^{\mathcal{B}_W}$ *is the empty set. If not, we have*:

$$L^{\mathcal{B}} \cap \mathcal{E}_W^{\mathcal{B}} = \Big(L_{\sigma_W}^{\mathcal{B}_W} \times P^{\mathcal{B}_{/W}}\Big) \bigcup \Big(P^{\mathcal{B}_W} \times L_{\sigma_{/W}}^{\mathcal{B}_{/W}}\Big)\ .$$

Proof Let $\mathcal{B}' = \{\mathbb{P}(W)\}$. Proposition 5.2 (ii) implies that the blow-down $P^{\mathcal{B}} \to \mathbb{P}(V)$ factorizes through $P^{\mathcal{B}} \to P^{\mathcal{B}'} \to \mathbb{P}(V)$. Consider its restriction to the complement of all exceptional divisors except for $\mathcal{E}_W$. It factors through the maps

$$P^{\mathcal{B}} \setminus \left(\bigcup_{\mathbb{P}(T)\in\mathcal{B}, T\neq W} \mathcal{E}_T \right) \longrightarrow P^{\mathcal{B}'} \setminus D \xrightarrow{\pi_{\mathcal{B}'}} \mathbb{P}(V)$$

where D is the union of the strict transforms of $\mathbb{P}(T) \in \mathcal{B}$ for every T not contained in W, and the total transforms $\pi_{\mathcal{B}'}^{-1}(T)$ for $\mathbb{P}(T) \subsetneq \mathbb{P}(W)$ in $\mathcal{B}$. Since the first map is an isomorphism, the interior of $\sigma^{\mathcal{B}} \cap \mathcal{E}_W^{\mathcal{B}}(\mathbb{R})$ can be computed in the middle space $P^{\mathcal{B}'} \setminus D$, and so it suffices to consider a single-blow up of $\mathbb{P}(V)$ along $\mathbb{P}(W)$. To compute it, note that the interior of the polyhedron σ is the intersection of a finite number of regions $f_i > 0$, where f_i is a linear form such as (5.5) whose zero locus $H_i = V(f_i)$ is a bounding hyperplane of σ. The inverse image of its interior, intersected with the exceptional divisor $\mathcal{E}' \subset P^{\mathcal{B}'}$ of $\pi_{\mathcal{B}'}$, is therefore cut out by the strict transforms of the H_i in $\mathcal{E}' \cong \mathbb{P}(W) \times \mathbb{P}(V/W)$. By Lemma 5.5, the strict transforms are of the form $H \times \mathbb{P}(V/W)$ or $\mathbb{P}(W) \times H$. The former define bounding hyperplanes for $\sigma_W \subset \mathbb{P}(W)(\mathbb{R})$; the latter define bounding hyperplanes for $\sigma_{/W}$, by definition of the normal polyhedron (Definition 4.8). Thus, after a single blow-up of $\mathbb{P}(W)$ in $\mathbb{P}(V)$, the interior of the intersection $\sigma^{\mathcal{B}'} \cap \mathcal{E}'(\mathbb{R})$ is

$$\overset{\circ}{\sigma}_W \times \overset{\circ}{\sigma}_{/W} \;\subset\; \mathbb{P}(W)(\mathbb{R}) \times \mathbb{P}(V/W)(\mathbb{R}) \ ,$$

where a superscript $\circ$ denotes the interior. In the case when there exists $\mathbb{P}(T) \subsetneq \mathbb{P}(W)$ in $\mathcal{B}$ such that $\mathbb{P}(W) \cap \sigma = \mathbb{P}(T) \cap \sigma$, we have $\pi_{\mathcal{B}'}^{-1}(\mathbb{P}(T)) = \mathbb{P}(T) \times \mathbb{P}(V/W)$, and it follows that $\overset{\circ}{\sigma}_W \times \overset{\circ}{\sigma}_{/W}$ is contained in D. We conclude that the interior of $\sigma^{\mathcal{B}} \cap \mathcal{E}_W^{\mathcal{B}}(\mathbb{R})$ is empty. By the earlier description of $\sigma^{\mathcal{B}}$ as a connected semi-algebraic set, the face $\sigma^{\mathcal{B}} \cap \mathcal{E}_W^{\mathcal{B}}(\mathbb{R})$ is equal to the closure of its interior, which in this case is empty. By definition, $\sigma_W^{\mathcal{B}_W}$ is also empty and so formula (5.12) holds.

In the case when there does not exist such a $\mathbb{P}(T)$, every element of $\mathcal{B}_W$ meets $\sigma_W \subset \mathbb{P}(W)(\mathbb{R})$ along a face (and does not strictly contain it). The same holds for $\mathcal{B}_{/W}$ relative to $\sigma_{/W} \subset \mathbb{P}(V/W)(\mathbb{R})$. By Remark 5.4, the restriction of $\pi_{\mathcal{B}/\mathcal{B}'}$ to $\mathcal{E}_W$ is the product $\pi_{\mathcal{B}_W} \times \pi_{\mathcal{B}_{/W}}$. The interior of $\sigma^{\mathcal{B}} \cap \mathcal{E}_W(\mathbb{R})$ is therefore the product

$$\pi_{\mathcal{B}_W}^{-1}(\overset{\circ}{\sigma}_W) \times \pi_{\mathcal{B}_{/W}}^{-1}(\overset{\circ}{\sigma}_{/W}) \ . \tag{5.13}$$

As before, $\sigma^{\mathcal{B}} \cap \mathcal{E}_W(\mathbb{R})$ is equal to the closure of its interior. By (5.6), the topological closure of (5.13) is $\sigma_W^{\mathcal{B}_W} \times \sigma_{/W}^{\mathcal{B}_{/W}}$, which proves (5.12). The final statement concerning $L^{\mathcal{B}} \cap \mathcal{E}_W^{\mathcal{B}}$ follows from the definition of $L^{\mathcal{B}}$ as the union of the Zariski closures of the facets of $\sigma^{\mathcal{B}}$. $\square$

In the particular case when $\mathcal{B} = \{\mathbb{P}(W)\}$ is a singleton and hence $P^{\mathcal{B}}$ is simply the blow-up of $\mathbb{P}(V)$ along $\mathbb{P}(W)$, formula (5.12) states that

$$\sigma^{\mathcal{B}} \cap \mathcal{E}_W(\mathbb{R}) = \sigma_W \times \sigma_{/W} \ ,$$

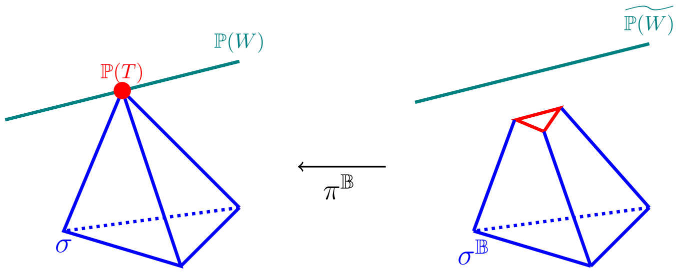


Fig. 6 Two loci $\mathbb{P}(T) \subset \mathbb{P}(W)$ meet σ in the same face, and so $\mathbb{P}(W)$ is not minimal. After performing the blow-up of $\mathcal{B} = \{\mathbb{P}(T)\}$, the strict transform of $\mathbb{P}(W)$ has moved away from the polyhedron $\sigma^{\mathcal{B}}$. The subsequent blow-up of $\mathbb{P}(W)$ is thus extraneous and does not affect $\sigma^{\mathcal{B}}$

which provides an infinitesimal interpretation of the normal polyhedron (Remark 4.9).

The situation when $\sigma_W^{\mathcal{B}_W} = \emptyset$ arises when there exists $\mathbb{P}(T) \subsetneq \mathbb{P}(W)$ in $\mathcal{B}$ such that $\sigma_W = \sigma \cap \mathbb{P}(T)(\mathbb{R}) = \sigma \cap \mathbb{P}(W)(\mathbb{R})$, i.e., W is not minimal amongst the set of spaces in $\mathcal{B}$ which meet σ along σ_W. This phenomenon is discussed in further detail in §5.5. Another way to see that $\sigma_W^{\mathcal{B}_W} = \emptyset$ in this case is as follows. After blowing up the subspace $\mathbb{P}(T)$, it follows from Proposition 5.3 applied to $\mathcal{E}_T$ that the strict transform of $\mathbb{P}(W)$ no longer meets the closure of the inverse image of σ, since by Lemma 4.10, it does not meet $\sigma_{/T}$ (see Fig. 6).

Notation 5.11 Let $(P^{\mathcal{B}}, L^{\mathcal{B}}, \sigma^{\mathcal{B}})$ be as above, and let D be an intersection of irreducible components of $L^{\mathcal{B}} = \bigcup L$. Denote by

$$D \cap (P^{\mathcal{B}}, L^{\mathcal{B}}, \sigma^{\mathcal{B}}) = (D \cap P^{\mathcal{B}}, \bigcup_{L \neq D} (D \cap L), D(\mathbb{R}) \cap \sigma^{\mathcal{B}}) \, .$$

It is a face if $D(\mathbb{R}) \cap \sigma^{\mathcal{B}} \neq \emptyset$.

Corollary 5.12 *A face of the blow-up of a polyhedral linear configuration is isomorphic to a product of blow-ups of polyhedral linear configurations.*

More precisely, for any intersection of irreducible components D of $L^{\mathcal{B}}$ we have

$$D \cap (P^{\mathcal{B}}, L^{\mathcal{B}}, \sigma^{\mathcal{B}}) \cong (P^{\mathcal{B}_1}, L_1^{\mathcal{B}_1}, \sigma_1^{\mathcal{B}_1}) \times \cdots \times (P^{\mathcal{B}_n}, L_n^{\mathcal{B}_n}, \sigma_n^{\mathcal{B}_n}) \tag{5.14}$$

for suitable $\sigma_1, \ldots, \sigma_n$ polyhedra in $\mathbb{P}(V_1), \ldots, \mathbb{P}(V_n)$, and $\mathcal{B}_i$ a subset of linear subspaces in $\mathbb{P}(V_i)$, satisfying $(B1)$ and $(B2)$ relative to σ_i, for all $1 \leq i \leq n$.

Proof Let $(P^{\mathcal{B}}, L^{\mathcal{B}}, \sigma^{\mathcal{B}})$ be the iterated blow up of $(\mathbb{P}(V), L, \sigma)$. Assume for the time being that the face in question is given by intersecting with D, a single irreducible component of $L^{\mathcal{B}}$. Suppose first of all that $D \subset \mathcal{E}_W$ is contained in an exceptional divisor for some $\mathbb{P}(W) \in \mathcal{B}$. Then by Proposition 5.10,

$$\mathcal{E}_W \cap (P^{\mathcal{B}}, L^{\mathcal{B}}, \sigma^{\mathcal{B}}) \cong (P^{\mathcal{B}_W}, L_{\sigma_W}^{\mathcal{B}_W}, \sigma_W^{\mathcal{B}_W}) \times (P^{\mathcal{B}_{/W}}, L_{\sigma_{/W}}^{\mathcal{B}_{/W}}, \sigma_{/W}^{\mathcal{B}_{/W}})$$

and by the final part of the same proposition, $D \subset \mathcal{E}_W \cong P^{\mathcal{B}_W} \times P^{\mathcal{B}_{/W}}$ is of the form $D = D_W \times P^{\mathcal{B}_{/W}}$ or $D = P^{\mathcal{B}_W} \times D_{/W}$.

Now suppose that D is not contained in any exceptional divisor, and is therefore the strict transform of a linear subspace $\mathbb{P}(U) = \pi_{\mathcal{B}}(D)$ for some $U \subset V$, such that $\mathbb{P}(U) \subset L$ is not contained in any $\mathbb{P}(W) \in \mathcal{B}$. It follows from Proposition 5.2 (i) that D is the iterated blow up of $\mathbb{P}(U)$ along the set of linear subspaces $\mathcal{B}_U = \{\mathbb{P}(W \cap U) : \mathbb{P}(W) \in \mathcal{B}\}$. It follows that

$$D \cap (P^{\mathcal{B}}, L^{\mathcal{B}}, \sigma^{\mathcal{B}}) \cong (P^{\mathcal{B}_U}, L_U^{\mathcal{B}_U}, \sigma_U^{\mathcal{B}_U})$$

is the iterated blow up of the face $(\mathbb{P}(U), L_U, \sigma_U)$ of $(\mathbb{P}(V), L, \sigma)$.

In general when D has codimension > 1, we may proceed by induction by repeatedly taking the intersection with irreducible components of D as above. □

5.5 Minimal blow-up

The minimal blow up is obtained from $\mathcal{B}$ by stripping out extraneous modifications (Definition 5.8).

Definition 5.13 Let $(\mathbb{P}(V), L, \sigma)$ be as above, and let $\mathcal{B}$ be a finite set of linear subspaces $\mathbb{P}(W)$ of $\mathbb{P}(V)$ such that (B1) and (B2) hold. Define $\mathcal{B}^{\min,\sigma} \subset \mathcal{B}$ as follows.

First consider the subset $S(\sigma) \subset \mathcal{B}$ of spaces $\mathbb{P}(W)$ such that

(i). $\mathbb{P}(W)(\mathbb{R}) \cap \sigma \neq \emptyset$

(ii). if $\mathbb{P}(W)(\mathbb{R}) \cap \sigma = \mathbb{P}(W')(\mathbb{R}) \cap \sigma$ for some $\mathbb{P}(W') \in \mathcal{B}$ then $W \subset W'$.

The set of U such that $\mathbb{P}(U)(\mathbb{R}) \cap \sigma$ equals the face $\mathbb{P}(W)(\mathbb{R}) \cap \sigma$ is closed under intersections: (ii) requires that W be minimal for this property.

Define $\mathcal{B}^{\min,\sigma} \subseteq \mathcal{B}$ to be the subspace generated by $S(\sigma)$ by taking intersections. In other words, it is the set of intersections of minimal spaces which meet σ along a face.

Note that taking intersections may violate (i): in the situation where σ is not a simplex, there can be spaces $\mathbb{P}(W)$ in $\mathcal{B}^{\min,\sigma}$ which do not meet σ. The definition of $\mathcal{B}^{\min,\sigma}$ makes sense even if $\mathcal{B}$ is infinite: the minimal set $\mathcal{B}^{\min,\sigma}$ is necessarily finite.

Proposition 5.14 *The morphism $\pi : P^{\mathcal{B}} \to P^{\mathcal{B}^{\min,\sigma}}$ constructed in Proposition 5.2 (ii) is an extraneous modification (Definition 5.8). It restricts to an isomorphism*:

$$\pi : \sigma^{\mathcal{B}} \xrightarrow{\sim} \sigma^{\mathcal{B}^{\min,\sigma}} .$$

Proof One must show by Definition 5.8 that every blow-up locus in $\mathcal{B} \setminus \mathcal{B}^{\min,\sigma}$ is extraneous. For those loci in $\mathcal{B}$ which do not meet σ (excluded from $\mathcal{B}^{\min,\sigma}$ in Definition 5.13 (i)), this clear. For non-minimal loci (in the sense of Definition 5.13 (ii)), we argue as follows. Let $\mathbb{P}(T) \subset \mathbb{P}(W)$ be elements of $\mathcal{B}$ such that $\sigma \cap \mathbb{P}(T)(\mathbb{R}) = \sigma \cap \mathbb{P}(W)(\mathbb{R})$. By Proposition 5.3 the exceptional divisor $\mathcal{E}_W \subset P^{\mathcal{B}}$ does not meet $\sigma^{\mathcal{B}}$. Since the iterated blow up is an isomorphism away from exceptional divisors, it follows that, immediately prior to blowing up $\mathbb{P}(W)$ in the formation of $P^{\mathcal{B}}$, the strict transform of $\mathbb{P}(W)$ does not meet the closure of the inverse image of the interior of σ, and hence $\mathbb{P}(W)$ is extraneous. □

We shall call the $P^{\mathcal{B}^{\min,\sigma}}$ the minimal blow-up of $\mathbb{P}(V)$ relative to $\mathcal{B}$, σ.

5.6 Category of blow-ups of polyhedral linear configurations

We may now define a category BLC_k of blow-ups of linear configurations.

Its objects are products (using Notation 2.2):

$$(P^{\mathcal{B}_1}, L^{\mathcal{B}_1}, \sigma^{\mathcal{B}_1}) \times \cdots \times (P^{\mathcal{B}_k}, L^{\mathcal{B}_k}, \sigma^{\mathcal{B}_k})$$

of blow-ups of polyhedral linear configurations.

Morphisms between objects are generated by products of the following:

(1) (Linear Embeddings). The maps induced by linear embeddings (5.10):

$$(P^{\mathcal{B}_1}(V_1), L^{\mathcal{B}_1}, \sigma^{\mathcal{B}_1}) \longrightarrow (P^{\mathcal{B}_2}(V_2), L^{\mathcal{B}_2}, \sigma^{\mathcal{B}_2})$$

They induce isomorphisms $\sigma^{\mathcal{B}_1} \cong \sigma^{\mathcal{B}_2}$.

(2) (Extraneous modifications). With notation as in Definition 5.8, the morphisms

$$\pi_{\mathcal{B}/\mathcal{B}'} : (P^{\mathcal{B}}, L^{\mathcal{B}}, \sigma^{\mathcal{B}}) \longrightarrow (P^{\mathcal{B}'}, L^{\mathcal{B}'}, \sigma^{\mathcal{B}'})$$

where $\mathcal{B}' \subseteq \mathcal{B}$ and $\pi_{\mathcal{B}/\mathcal{B}'}$ is an extraneous modification. Here too, $\sigma^{\mathcal{B}} \cong \sigma^{\mathcal{B}'}$.

(3) (Face maps). A face map is defined to be a composition of inclusions of facets. There are two types of inclusions of facets depending on whether the facet in question is contained in an exceptional divisor or not.

(i). Consider the inclusion of an exceptional divisor $\mathcal{E}_W$ into $P^{\mathcal{B}}$, where $\mathbb{P}(W) \in \mathcal{B}$ such that $\sigma_W = \sigma \cap \mathbb{P}(W)(\mathbb{R})$ is non-empty and hence by $(B2)$ is a face of σ. By identifying $\mathcal{E}_W^{\mathcal{B}} \cong P^{\mathcal{B}_W} \times P^{\mathcal{B}_{/W}}$ we obtain a map via Proposition 5.10:

$$(P^{\mathcal{B}_W}, L^{\mathcal{B}_W}_{\sigma_W}, \sigma^{\mathcal{B}_W}_W) \times (P^{\mathcal{B}_{/W}}, L^{\mathcal{B}_{/W}}_{\sigma_{/W}}, \sigma^{\mathcal{B}_{/W}}_{/W}) \longrightarrow (P^{\mathcal{B}}, L^{\mathcal{B}}_{\sigma}, \sigma^{\mathcal{B}})$$

which defines a morphism in BLC_k.

(ii). Consider the case of an irreducible component D of $L^{\mathcal{B}}$ which is the strict transform of $\mathbb{P}(U) \subset L$ for some $U \subset V$, where $\mathbb{P}(U)$ is not contained in any element of $\mathcal{B}$. By Proposition 5.2 (i) we may identify

$$D \cap (P^{\mathcal{B}}, L^{\mathcal{B}}, \sigma^{\mathcal{B}}) \cong (P^{\mathcal{B}_U}, L^{\mathcal{B}_U}_U, \sigma^{\mathcal{B}_U}_U)$$

to obtain a map: $(P^{\mathcal{B}_U}, L^{\mathcal{B}_U}_U, \sigma^{\mathcal{B}_U}_U) \longrightarrow (P^{\mathcal{B}}, L^{\mathcal{B}}_{\sigma}, \sigma^{\mathcal{B}})$.

The category PLC_k is a subcategory of BLC_k, which in turn is a subcategory of $\mathcal{PC}_k$. Assumptions (1) and (2) of §3.1 follow from the description of iterated blow-ups in local coordinates and the description of the faces of the $\sigma^{\mathcal{B}}$ given in §5.4.

Remark 5.15 Consider a polyhedral linear configuration $(\mathbb{P}(V), L, \sigma)$ and a set $\mathcal{B}$ satisfying $(B1)$, $(B2)$. Let $\mathbb{P}(W) \in \mathcal{B}$. The inclusion of the exceptional divisor $\mathcal{E}_W$

gives rise to the horizontal map along the top of the following commutative diagram in $\mathcal{PC}_k$:

$$\begin{array}{ccccc} (P^{\mathcal{B}_W}, L^{\mathcal{B}_W}_{\sigma_W}, \sigma^{\mathcal{B}_W}_W) & \times & (P^{\mathcal{B}_{/W}}, L^{\mathcal{B}_{/W}}_{\sigma_{/W}}, \sigma^{\mathcal{B}_{/W}}_{/W}) & \longrightarrow & (P^{\mathcal{B}}, L^{\mathcal{B}}_{\sigma}, \sigma^{\mathcal{B}}) \\ & \downarrow & & & \downarrow \\ & (\mathbb{P}(W), L_{\sigma_W}, \sigma_W) & & \longrightarrow & (\mathbb{P}(V), L, \sigma) \end{array}$$

The top row is in the subcategory BLC_k, the bottom row is in PLC_k. The horizontal map along the bottom is the inclusion of the face σ_W of σ. The right-most vertical map is the blow-down $\pi_{\mathcal{B}}$ and the left-hand vertical map is the projection onto the first factor, followed by the blow-down $\pi_{\mathcal{B}_W}$ (on topological realisations, $\sigma_{/W}$ is collapsed to a point.)

5.7 Complexes of blow-ups of polyhedra

Definition 5.16 A complex of blow-ups of polyhedra is a BLC_k-complex, i.e., a functor $F : \mathcal{D} \to \mathrm{BLC}_k$ where $\mathcal{D}$ is equivalent to a finite diagram category.

The definition of a morphism of BLC_k-complexes proceeds in an identical way to Definition 2.3. The definition of subschemes, and differential forms follows §2.3, 2.4. Although blow-downs and projection morphisms exist in the category $\mathcal{PC}_k$, we do not include them in the category BLC_k in order that assumptions §3.1 hold.

6 The moduli space $\mathcal{M}_g^{\mathrm{trop}}$ of tropical curves

We recast the theory of Feynman polytopes in the context of polyhedral linear complexes, before turning to the moduli space of tropical curves.

6.1 Graphs and polyhedral linear complexes

Let G be a finite graph with vertices V_G and edges E_G. We shall call a self-edge (or tadpole) an edge of G with a single endpoint. The loop number (or genus) h_G of G is equal to the number of independent cycles in G, i.e., the rank of the free $\mathbb{Z}$-module $H_1(G;\mathbb{Z})$.

Let $\mathbb{P}^{E_G} = \mathbb{P}(\mathbb{Q}^{E_G})$ denote the projective space with projective coordinates α_e for every edge $e \in E_G$. For the time being its vertices are unweighted (equivalently, have weight zero). For any subgraph γ of G, defined by a subset of edges of E_G, we denote by G/γ the graph obtained by contracting all edges e of E_γ. It does not depend on the ordering of the contractions. The edges of G are labelled in this section.

Definition 6.1 For any G, consider the object in the category $\mathrm{PLC}_{\mathbb{Q}}$ defined by

$$\mathcal{F}(G) = \left(\mathbb{P}^{E_G}, L, \sigma_G\right) \quad \text{where} \quad L = \bigcup_{e \in E_G} L_e,$$

where σ_G is the region in projective space where $\alpha_e \geq 0$, and L_e is the coordinate hyperplane $\alpha_e = 0$. The object $\mathcal{F}(G)$ is none other than a standard simplex (Ex. 4.5).

Remark 6.2 Since σ_G is a simplex, the associated polyhedral configuration has the special property that any non-empty intersection of components of L is the Zariski closure of a face of σ_G. This is not true for general polyhedra.

For every edge $e \in E_G$, there is a canonical face map (Definition 4.11):

$$\mathcal{F}(G/e) \longrightarrow \mathcal{F}(G)$$

corresponding to the inclusion of the face $\sigma_{G/e} \subset \sigma_G$, which identifies $\mathbb{P}^{E_{G/e}}$ with the locus $\alpha_e = 0$ in the projective space $\mathbb{P}^{E_G}$.

The simplex $\mathcal{F}(G)$, together with the data of all its faces, admits a categorical description. For this, all graphs in the following have labelled edges and hence have no non-trivial automorphisms. Consider the category $\mathcal{C}_G$ whose objects are all quotients G/γ of G, for all strict subgraphs $\gamma \subsetneq E_G$, including the case $\gamma = \emptyset$. The morphisms in this category are generated by edge contractions $\Gamma \to \Gamma/e$ for any edge $e \in E_\Gamma$. The map $\mathcal{F}$ is a functor

$$G \mapsto \mathcal{F}(G) \quad : \quad \mathcal{C}_G^{\mathrm{opp}} \longrightarrow \mathrm{PLC}_{\mathbb{Q}}$$

which sends morphisms in $\mathcal{C}_G^{\mathrm{opp}}$ to face maps. Indeed, the faces of the standard simplex $\mathcal{F}(G)$ are in one-to-one correspondence with the elements of the category $\mathcal{C}_G^{\mathrm{opp}}$.

6.2 The moduli space of tropical curves $\mathcal{M}_g^{\mathrm{trop}}$

The moduli space of tropical curves is constructed in a completely analogous manner to the above by gluing together simplices associated to isomorphism classes of stable graphs of a fixed genus.

6.2.1 Weighted and stable graphs

Let G be a finite graph. A *weighting* is a map $w : V(G) \to \mathbb{N}_{\geq 0}$ which assigns a non-negative integer to every vertex.

The *genus* of G is then defined to be:

$$g(G) = h_G + w(G) \quad \text{where} \quad w(G) = \sum_{v \in V_G} w(v)\ . \tag{6.1}$$

A weighted graph (G, w) is called *stable* if every vertex of weight 0 has degree at least 3, and if every vertex of weight 1 has degree ≥ 1. An isomorphism $(G, w) \xrightarrow{\sim} (G', w')$ of weighted graphs is an isomorphism $G \xrightarrow{\sim} G'$ which respects the weightings. For every edge $e \in E_G$, the contraction of e is defined as follows. If e is not a self-edge, then $(G, w)/e$ is the pair $(G/e, w')$ where G/e is the graph in which e is removed and the endpoints v_1, v_2 of G are identified to produce a new vertex v of weight $w'(v) = w(v_1) + w(v_2)$. The weights of all other vertices are unchanged. In the case when e is a self-edge, G/e is defined by simply removing the edge e. In this case, the common endpoint v of e has weight $w'(v) = w(v) + 1$ in the graph G/e; the weights of all other vertices are unchanged.

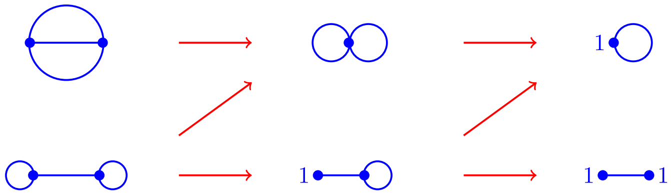
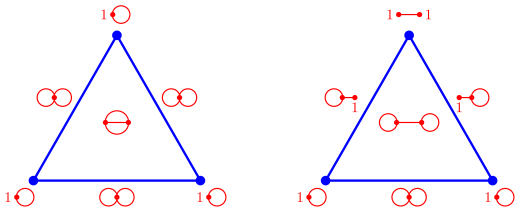


Fig. 7 Illustration of the category I_2. Unlabelled vertices have weight 0. Automorphisms of graphs are not depicted, only edge contractions

Fig. 8 A picture of $|L\mathcal{M}_2^{\mathrm{trop}}|$. Faces with common labels are identified. The simplex on the left is to be quotiented by the action of its symmetry group Σ_3; the one on the right by Σ_2



Definition 6.3 Let $g \geq 2$ and let I_g denote the category of connected, stable, weighted graphs (G, w) of genus g and $|E_G| \geq 1$, whose morphisms are generated by isomorphisms and edge contractions $(G, w) \mapsto (G, w)/e$ (see Figure 7 for the case $g = 2$). Let I_g^{opp} denote the opposite category.

An isomorphism of weighted graphs $(G, w) \cong (G', w')$ is in particular an isomorphism of graphs $G \cong G'$ and induces a bijection between the corresponding sets of edges. This gives rise to a linear isomorphism $\mathbb{P}^{E_G} \cong \mathbb{P}^{E_{G'}}$ and hence a canonical isomorphism $\mathcal{F}(G) \cong \mathcal{F}(G')$ in $\mathrm{PLC}_{\mathbb{Q}}$. Consequently, the restriction of $\mathcal{F}$ to stable, connected graphs defines a functor.

Definition 6.4 The link of the moduli space of tropical curves (in the category $\mathrm{PLC}_{\mathbb{Q}}$) is the polyhedral linear complex defined by the functor

$$L\mathcal{M}_g^{\mathrm{trop}} : I_g^{\mathrm{opp}} \longrightarrow \mathrm{PLC}_{\mathbb{Q}}$$

which to any weighted graph (G, w) assigns the standard simplex (Example 4.5):

$$L\mathcal{M}_g^{\mathrm{trop}}(G, w) = \mathcal{F}(G)\ .$$

Note that $\mathcal{F}(G)$ does not depend on the weighting of G.

Its topological realisation $|L\mathcal{M}_g^{\mathrm{trop}}|$ (2.5) is what is usually called the 'link of the moduli space of tropical curves'. In this paper, we use this phrase to describe $L\mathcal{M}_g^{\mathrm{trop}}$. See Figure 8 for an illustration in the case $g = 2$.

Remark 6.5 The definitions are easily adapted to the moduli space of tropical curves, rather than the link of its cone point. Let $\widetilde{I}_g$ denote the category I_g, with an additional, final, object given by a single vertex of weight g. Consider the functor $\mathcal{M}_g^{\mathrm{trop}}$ from $\widetilde{I}_g^{\mathrm{opp}}$ to $\mathcal{PC}_{\mathbb{Q}}$:

$$(G, w) \mapsto \left(\mathbb{A}^{E_G}, V\left(\prod_{e \in E_G} \alpha_e \right), \widehat{\sigma}_G \right) ,$$

where $\widehat{\sigma}_G$ is the cone $\alpha_e \geq 0$. The final object in $\widetilde{I}_g^{\mathrm{opp}}$ maps to (Spec($\mathbb{Q}$), Spec($\mathbb{Q}$), $\{pt\}$), whose image in $|\mathcal{M}_g^{\mathrm{trop}}|$ is the cone point. The functor $\mathcal{M}_g^{\mathrm{trop}}$ lands in a subcategory of $\mathcal{PC}_{\mathbb{Q}}$ which is an analogue of $\mathrm{PLC}_{\mathbb{Q}}$ defined using affine spaces. Since $|\mathcal{M}_g^{\mathrm{trop}}|$ is topologically trivial, this will not be pursued further in this paper.

6.3 The open moduli space and Outer Space

Let ∂I_g denote the full subcategory of I_g consisting of graphs of total weight $w(G) > 0$. Consider the functor

$$\partial L\mathcal{M}_g^{\mathrm{trop}} : \partial I_g^{\mathrm{opp}} \longrightarrow \mathrm{PLC}_{\mathbb{Q}}$$

obtained by restricting $L\mathcal{M}_g^{\mathrm{trop}}$ to $\partial I_g^{\mathrm{opp}}$. It sends G to $\mathcal{F}(G)$.

Definition 6.6 Let us denote its topological realisation by

$$\left| \partial L\mathcal{M}_g^{\mathrm{trop}} \right| = \varinjlim_{G \in \partial I_g^{\mathrm{opp}}} \sigma_G$$

which canonically embeds in $|L\mathcal{M}_g^{\mathrm{trop}}|$. Let $|L\mathcal{M}_g^{\circ,\mathrm{trop}}| = |L\mathcal{M}_g^{\mathrm{trop}}| \setminus |\partial L\mathcal{M}_g^{\mathrm{trop}}|$ denote its open complement.

The open moduli space is obtained by gluing together the open interiors of simplices σ_G for graphs G which have total weight 0:

$$\left| L\mathcal{M}_g^{\circ,\mathrm{trop}} \right| = \mathrm{Im} \left(\bigcup_{G \in I_g^{\mathrm{opp}}, w(G)=0} \mathring{\sigma}_G \longrightarrow \left| L\mathcal{M}_g^{\mathrm{trop}} \right| \right) ,$$

and is equipped with the subspace topology. Note that the open moduli space $|L\mathcal{M}_g^{\circ,\mathrm{trop}}|$ is not the topological realisation of a polyhedral complex since it is not closed. It is the quotient of Outer space $\mathcal{O}_g$ [28] by the action of $\mathrm{Out}(F_g)$.

6.4 The graph locus

Let G be a graph. The graph polynomial for a connected graph G (with all vertices of weight zero) is defined to be the polynomial

$$\Psi_G = \sum_{T \subset G} \prod_{e \notin T} x_e \quad \in \quad \mathbb{Z}[x_e, e \in E_G]$$

where the sum is over all spanning trees T of G. It is homogeneous of degree h_G. For a graph G with connected components $G_1, \ldots, G_n$, we define

$$\Psi_G = \prod_{i=1}^{n} \Psi_{G_i} \ .$$

If a graph G has a vertex v with $w(v) > 0$, i.e., the total weight $w(G)$ is positive, then we define $\Psi_G = 0$. The graph locus $X_G \subset \mathbb{P}^{E_G}$ is defined to be the zero locus of Ψ_G. It is a hypersurface if G has total vertex weight 0, but equals $\mathbb{P}^{E_G}$ otherwise.

Proposition 6.7 *The map $G \mapsto X_G$ defines a functor*

$$\mathcal{X} : \left(I_g\right)^{\mathrm{opp}} \longrightarrow \mathrm{Sch}_{\mathbb{Q}} \ .$$

It is a subscheme functor of $L\mathcal{M}_g^{\mathrm{trop}}$, i.e., of $\mathsf{S}\mathcal{F} : \left(I_g\right)^{\mathrm{opp}} \to \mathrm{Sch}_{\mathbb{Q}}$ which sends $G \mapsto \mathbb{P}^{E_G}$, (Definition 6.1). Its topological complement is the open moduli space:

$$\left|L\mathcal{M}_g^{\circ,\mathrm{trop}}\right| = \left|L\mathcal{M}_g^{\mathrm{trop}}\right| \setminus \left(\left|L\mathcal{M}_g^{\mathrm{trop}}\right| \cap |\mathcal{X}(\mathbb{R})|\right) = \varinjlim_{x \in I_g^{\mathrm{opp}}} \sigma_G \setminus (\sigma_G \cap X_G(\mathbb{R})) \ .$$

Equivalently, one has $|\mathcal{X}(\mathbb{R})| \cap |L\mathcal{M}_g^{\mathrm{trop}}| = |\partial L\mathcal{M}_g^{\mathrm{trop}}|$.

Proof An isomorphism $(G, w) \cong (G', w')$ of weighted graphs induces a bijection on edge sets, and gives a linear isomorphism $\mathbb{P}^{E_G} \cong \mathbb{P}^{E_{G'}}$ and an isomorphism $X_G \cong X_{G'}$. The image of an edge contraction $G \to G/e$ under the functor $\mathsf{S}\mathcal{F}$ is the inclusion $\mathbb{P}^{E_{G/e}} \to \mathbb{P}^{E_G}$ whose image is the coordinate hyperplane $x_e = 0$. If e is not a self-edge, there is a one-to-one correspondence between the set of spanning trees of G containing e, and the spanning trees of G/e, which implies that $\Psi_G|_{x_e=0} = \Psi_{G/e}$, and therefore $X_{G/e} = X_G \cap V(x_e)$. This formula holds also for e a self-edge (G has no spanning trees containing e), in which case G/e has positive weight, and both $\Psi_G|_{x_e=0}$ and $\Psi_{G/e}$ vanish. This proves that $\mathcal{X}$ indeed defines a subscheme of $\mathsf{S}\mathcal{F}$. For the last part, observe that $\sigma_G \subset X_G(\mathbb{R})$ if $w(G) > 0$ and that

$$\overset{\circ}{\sigma}_G \cap X_G(\mathbb{R}) = \emptyset \quad \text{if} \quad w(G) = 0 \ .$$

This follows from the fact that if $w(G) = 0$ then Ψ_G is a non-trivial sum of positive monomials, and so $\Psi_G > 0$ for $\alpha_e > 0$. The open faces of σ_G are in one-to-one correspondence with the $\overset{\circ}{\sigma}_{G/\gamma}$ for all $\gamma \subset G$. It follows that $\sigma_G \cap X_G(\mathbb{R})$ is the union of the faces $\sigma_{G/\gamma}$ where $h_\gamma > 0$, or equivalently, such that $w(G/\gamma)$ is positive. □

6.5 The graph complex $\mathcal{GC}_0^{\mathrm{stab}}$

Consider the version of the graph complex [39] defined to be the $\mathbb{Q}$-vector space generated by oriented, stable unweighted graphs $[G, \omega]$. It is bigraded by genus g, and the number of edges e_G, which is the homological degree. An orientation ω on G is an element of $\bigwedge(\mathbb{Z}^{E_G})^{\times}$, i.e., an order of the edges up to even permutations. One has relations $[G, -\omega] = -[G, \omega]$ and $[G, \omega] = [G', \omega']$ if $G \cong G'$ is an isomorphism which sends ω to ω'. The differential is:

$$d[G, e_1 \wedge \cdots \wedge e_n] = \sum_e (-1)^i [G//e_i, e_1 \wedge \ldots \widehat{e_i} \wedge \cdots \wedge e_n]$$

where $G//e$ denotes the graph in which the edge e is contracted, and is the empty graph if e is a self-edge. One shows that $\mathcal{GC}_0^{\mathrm{stab}}$ is quasi-isomorphic to the complex usually denoted by $\mathcal{GC}_0$. The following proposition is proven in [26] and also follows from Theorem 3.6.

Proposition 6.8 *The relative face complex* $\mathfrak{C}(L\mathcal{M}_g^{\mathrm{trop}})/\mathfrak{C}(\partial L\mathcal{M}_g^{\mathrm{trop}})$ *is the graph complex* $\mathcal{GC}_0^{\mathrm{stab}}[-1]$ *where the homological degree is given by numbers of edges* -1. *Thus*:

$$H_n(\mathcal{GC}_0) \cong \bigoplus_{g\geq 1} H_{n-1}(|L\mathcal{M}_g^{\mathrm{trop}}|, |\partial L\mathcal{M}_g^{\mathrm{trop}}|; \mathbb{Q}) \ .$$

A more profound statement is the fact, due to [26], that $|\partial L\mathcal{M}_g^{\mathrm{trop}}|$ is contractible. Therefore the graph complex also computes the reduced homology of $|L\mathcal{M}_g^{\mathrm{trop}}|$.

7 The bordification $\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}$ of $\mathcal{M}_g^{\mathrm{trop}}$

We review the blow-ups associated to graphs, and define $\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}$ by gluing together the blow-ups of polyhedral configurations of stable, weighted graphs.

7.1 Blow-ups and Feynman polytopes

A core graph is one which is bridgeless.

Definition 7.1 Let G be a graph. Consider the set of subspaces

$$\mathcal{B}^G = \mathcal{B}^{G,\mathrm{core}} = \{\mathbb{P}(\mathbb{Q}^{E_{G/\gamma}}) : \gamma \subset G \quad \text{core subgraph}\}$$

which, one verifies (see e.g. [10, §7], [17, §5-6]), satisfies $(B1)$, $(B2)$.[3] Define

$$\mathcal{F}^{\mathcal{B}}(G) = \left(P^{\mathcal{B}^G}, L^{\mathcal{B}^G}, \sigma^{\mathcal{B}^G}\right)$$

to be the object in $\mathrm{BLC}_{\mathbb{Q}}$ obtained by blowing up the linear subspaces corresponding to core subgraphs. It is equipped with a canonical blow-down morphism in $\mathcal{PC}_{\mathbb{Q}}$:

$$\pi^{\mathcal{B}} : \mathcal{F}^{\mathcal{B}}(G) \longrightarrow \mathcal{F}(G) \ .$$

[3] If we set $\mathcal{B}^{G,\max} = \{\mathbb{P}(\mathbb{Q}^{E_\gamma}) :$ all $\gamma \subset G\}$, then one may verify that $(\mathcal{B}^{G,\max})^{\min} = \mathcal{B}^{G,\mathrm{core}}$. Therefore the core blow-up is minimal, and Proposition 5.14 defines a morphism $P^{\mathcal{B}^{G,\max}} \to P^{\mathcal{B}^{G,\mathrm{core}}}$.

A key point is that the set $\mathcal{B}^G$ is intrinsic to G, which is why we shall drop the superscript G with impunity. By this we mean the following. Let $\gamma \subset G$ be a core subgraph. Then, employing the notation (5.2), we have canonical identifications

$$\left(\mathcal{B}^G\right)_{/\mathbb{Q}^{E_{G/\gamma}}} = \mathcal{B}^\gamma \qquad \text{and} \qquad \left(\mathcal{B}^G\right)_{\mathbb{Q}^{E_{G/\gamma}}} = \mathcal{B}^{G/\gamma}$$

since, for the first equation, there is a bijection between the set of core subgraphs $g \subset G$ which are contained in γ, and the set of core subgraphs of γ (the property of being a core subgraph is intrinsic, and does not depend on the ambient graph). For the second equation, there is a bijection between core subgraphs of G which contain γ, and core subgraphs of G/γ. For all $e \in E_G$ which is not a self-edge, there is a face morphism

$$\mathcal{F}^{\mathcal{B}}(G/e) \longrightarrow \mathcal{F}^{\mathcal{B}}(G) \tag{7.1}$$

in the category $\mathrm{BLC}_{\mathbb{Q}}$. It is compatible, via blow-down, with the corresponding map $\mathcal{F}(G/e) \to \mathcal{F}(G)$. Exceptional divisors are indexed by core subgraphs $\emptyset \neq \gamma \subset G$. For each such subgraph, there is a canonical face morphism:

$$\mathcal{F}^{\mathcal{B}}(\gamma) \times \mathcal{F}^{\mathcal{B}}(G/\gamma) \longrightarrow \mathcal{F}^{\mathcal{B}}(G)\ . \tag{7.2}$$

Compatibility with the face map $\mathcal{F}(G/\gamma) \to \mathcal{F}(G)$ is expressed by the diagram in $\mathcal{PC}_{\mathbb{Q}}$:

$$\begin{array}{ccccc}
\mathcal{F}^{\mathcal{B}}(\gamma) & \times & \mathcal{F}^{\mathcal{B}}(G/\gamma) & \longrightarrow & \mathcal{F}^{\mathcal{B}}(G) \\
 & \downarrow & & & \downarrow \\
 & \mathcal{F}(G/\gamma) & & \longrightarrow & \mathcal{F}(G)
\end{array}$$

which commutes (see Remark 5.15). The facets of $\mathcal{F}^{\mathcal{B}}(G)$ are in one-to-one correspondence with either non-self-edge edges (7.1), or core subgraphs (7.2).

7.2 Sequences of graphs and categorical formulation of the blow-up

Definition 7.2 Consider a sequence of strictly nested (non-empty) subgraphs

$$(\gamma_1, \gamma_2, \ldots, \gamma_{n-1}, \gamma_n) \ \text{ such that } \ \gamma_i \subsetneq \gamma_{i+1} \text{ for } 1 \leq i \leq n-1\ , \tag{7.3}$$

which are not necessarily connected (even if γ_n is). Such a sequence may equivalently be viewed as the data of a strict filtration $F_\bullet$ on $G = \gamma_n$, where $F_i G = \gamma_i$. The *graded sequence* associated to this filtration is the sequence of graphs:

$$(\gamma_1, \gamma_2/\gamma_1, \ldots, \gamma_n/\gamma_{n-1})\ .$$

Let $e \in E_{\gamma_n}$ be an edge. There is a unique index k such that e is an edge of γ_k/γ_{k-1}. We call e *admissible* if it is neither a self-edge nor the only edge in the quotient γ_k/γ_{k-1}, in which case we define the *edge contraction* with respect to e to be the sequence:

$$(\gamma_1, \ldots, \gamma_n)/e \ = \ ((\gamma_1 \cup e)/e\ ,\ (\gamma_2 \cup e)/e\ ,\ \ldots\ ,\ (\gamma_{n-1} \cup e)/e\ ,\ \gamma_n/e) \tag{7.4}$$

Define a *refinement* of $(\gamma_1, \gamma_2, \ldots, \gamma_{n-1}, \gamma_n)$ to be: $(\gamma, \gamma_1, \ldots, \gamma_n)$ where $\gamma \subsetneq \gamma_1$ or

$$(\gamma_1, \ldots, \gamma_i, \gamma, \gamma_{i+1}, \ldots, \gamma_n) \text{ where } \gamma_i \subsetneq \gamma \subsetneq \gamma_{i+1}$$

for any i. Define the *edge degree* to be $|E_{\gamma_n}| - n + 1$, and the *genus* to be h_{γ_n}.

Given a graph G with labelled edges, define a category $\mathcal{C}_G^{\mathcal{B}}$ whose objects are sequences (7.3) of edge degree > 0 such that $\gamma_n = G/\gamma$ is a quotient of G satisfying $h_{\gamma_n} = h_G$ (i.e., γ does not contain any loops), and such that each graph γ_i, $i < n$ is core. The morphisms are given by contraction of admissible edges, and refinements (insertion of an additional graph in the nested sequence as above): indeed, $\mathcal{C}_G^{\mathcal{B}}$ is simply the category of objects of edge degree ≥ 1 generated by the sequence (G) under these two operations.

Theorem 7.3 *There is a canonical functor* $\left(\mathcal{C}_G^{\mathcal{B}}\right)^{\mathrm{opp}} \to \mathrm{BLC}_{\mathbb{Q}}$ *which sends*

$$(\gamma_1, \gamma_2, \ldots, \gamma_{n-1}, \gamma_n) \mapsto \mathcal{F}^{\mathcal{B}}(\gamma_1) \times \mathcal{F}^{\mathcal{B}}(\gamma_2/\gamma_1) \times \cdots \times \mathcal{F}^{\mathcal{B}}(\gamma_n/\gamma_{n-1})$$

and all morphisms to face morphisms. The objects of $\mathcal{C}_G^{\mathcal{B}}$ *are in one-to-one correspondence with the faces of* $\mathcal{F}^{\mathcal{B}}(G)$; *the morphisms of* $\mathcal{C}_G^{\mathcal{B}}$ *are in one-to-one correspondence with inclusions of faces. The face corresponding to a sequence* $(\gamma_1, \ldots, \gamma_n)$ *has codimension* $e_G - e_{\gamma_n} + n - 1$. *There is a canonical blow-down in* $\mathcal{PC}_{\mathbb{Q}}$ *from the functor*

$$\mathcal{F}^{\mathcal{B}} : \mathcal{C}_G^{\mathcal{B}} \to \mathrm{BLC}_{\mathbb{Q}} \subset \mathcal{PC}_{\mathbb{Q}} \qquad \textit{to} \qquad \mathcal{F} : \mathcal{C}_G \to \mathrm{PLC}_{\mathbb{Q}} \subset \mathcal{PC}_{\mathbb{Q}}$$

which is induced by the pair (ϕ, Φ), *where the functor* $\phi : \mathcal{C}_G^{\mathcal{B}} \to \mathcal{C}_G$ *is defined by*

$$\phi : (\gamma_1, \gamma_2, \ldots, \gamma_n) \mapsto \gamma_n/\gamma_{n-1}$$

and Φ *is the natural transformation defined by the family of morphisms in* $\mathcal{PC}_{\mathbb{Q}}$:

$$\mathcal{F}^{\mathcal{B}}(\gamma_1) \times \mathcal{F}^{\mathcal{B}}(\gamma_2/\gamma_1) \times \cdots \times \mathcal{F}^{\mathcal{B}}(\gamma_n/\gamma_{n-1}) \longrightarrow \mathcal{F}(\gamma_n/\gamma_{n-1})$$

given by projection onto the final component followed by blow-down for $\mathcal{F}^{\mathcal{B}}(\gamma_n/\gamma_{n-1})$.

Proof To verify that the map in the statement is indeed a functor, we must check that the following diagram commutes, where e is an edge of γ_i but not of γ_{i-1} for $i \geq 2$:

$$\begin{array}{ccc}
(\gamma_1, \gamma_2, \ldots, \gamma_{n-1}, \gamma_n) & \longrightarrow & \mathcal{F}^{\mathcal{B}}(\gamma_1) \times \mathcal{F}^{\mathcal{B}}(\gamma_2/\gamma_1) \times \cdots \times \mathcal{F}^{\mathcal{B}}(\gamma_n/\gamma_{n-1}) \\
\downarrow & & \uparrow \\
((\gamma_1 \cup e)/e, \ldots (\gamma_n \cup e)/e) & \longrightarrow & \mathcal{F}^{\mathcal{B}}(\gamma_1) \times \cdots \times \mathcal{F}^{\mathcal{B}}(\gamma_i/(\gamma_{i-1} \cup e)) \times \cdots \times \mathcal{F}^{\mathcal{B}}(\gamma_n/\gamma_{n-1})
\end{array}$$

This follows from the fact that $\mathcal{F}^{\mathcal{B}}(G/e) \to \mathcal{F}^{\mathcal{B}}(G)$ is a face map (7.1), applied to $G = \gamma_i/\gamma_{i-1}$. The case $i = 1$ is similar. For refinements, we must check that

$$\begin{array}{ccc}
(\gamma_1, \ldots, \gamma_n) & \longrightarrow & \mathcal{F}^{\mathcal{B}}(\gamma_1) \times \mathcal{F}^{\mathcal{B}}(\gamma_2/\gamma_1) \times \cdots \times \mathcal{F}^{\mathcal{B}}(\gamma_n/\gamma_{n-1}) \\
\downarrow & & \uparrow \\
(\gamma_1, \ldots, \gamma_{i-1}, \gamma, \gamma_i, \ldots, \gamma_n) & \longrightarrow & \mathcal{F}^{\mathcal{B}}(\gamma_1) \times \ldots \mathcal{F}^{\mathcal{B}}(\gamma/\gamma_{i-1}) \times \mathcal{F}^{\mathcal{B}}(\gamma_i/\gamma) \ldots \times \mathcal{F}^{\mathcal{B}}(\gamma_n/\gamma_{n-1})
\end{array}$$

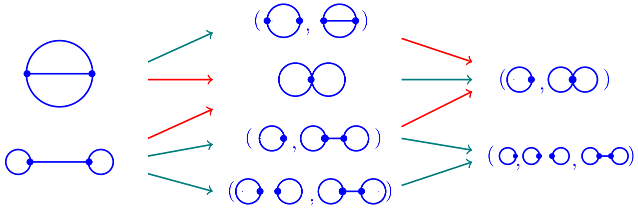


Fig. 9 A picture of $I_2^{\mathcal{B}}$. All vertices have weight 0. Edge contractions are depicted in red, refinements in green, and automorphisms are not shown (colour figure online).

commutes also. This follows from the fact that $\mathcal{F}^{\mathcal{B}}(\gamma/\gamma_{i-1}) \times \mathcal{F}^{\mathcal{B}}(\gamma_i/\gamma) \to \mathcal{F}^{\mathcal{B}}(\gamma_i/\gamma_{i-1})$ is the face map corresponding to the inclusion of an exceptional divisor (7.2). The statement follows from Proposition 5.3: the exceptional divisors of $\mathcal{F}^{\mathcal{B}}(\gamma_n)$ are indexed by strict core subgraphs of γ_i, and so an intersection of exceptional divisors is indexed by nested sequences of core subgraphs. The description of the blow-down map follows from Remark 5.15. $\square$

On topological realisations, $|\Phi|$ induces the blow-down morphism $|\mathcal{F}^{\mathcal{B}}(G)| \to |\mathcal{F}(G)|$ from $|\mathcal{F}^{\mathcal{B}}(G)|$ to the closed simplex. The space $|\mathcal{F}^{\mathcal{B}}(G)|$ is called the Feynman polytope in [17, §6], which provides a similar description of its faces in greater generality.

7.3 Canonical blow-up of $L\mathcal{M}_g^{\mathrm{trop}}$

Consider the category whose objects are the faces of all $\mathcal{F}^{\mathcal{B}}(G)$, where G ranges over stable connected graphs of total weight 0, and whose morphisms are generated by isomorphisms and face morphisms. By construction, it admits a canonical functor to $\mathrm{BLC}_{\mathbb{Q}}$. The complex $L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}$ is defined to be this functor. It may be described more explicitly using nested sequences of graphs.

Definition 7.4 Define a category $I_g^{\mathcal{B}}$ whose objects are nested sequences (7.3) of graphs of genus $h_{\gamma_n} = g$, with the property that each γ_i, for $i < n$, is a core graph (but not necessarily connected), and γ_n is a stable connected graph with all vertices of weight 0. The morphisms in this category are isomorphisms, admissible edge contractions, and refinements. See Fig. 9.

There is a 'collapsing' functor:

$$\begin{aligned} I_g^{\mathcal{B}} &\longrightarrow I_g \\ (\gamma_1, \ldots, \gamma_{n-1}, \gamma_n) &\mapsto \gamma_n/\gamma_{n-1} \end{aligned} \tag{7.5}$$

which sends a filtered graph to its highest graded component, where the quotient γ_n/γ_{n-1} is to be viewed as a weighted graph as follows. First assign weight 0 to every vertex of γ_n, and contract each edge of γ_{n-1}, in any order, keeping track of the induced weights by the process described in §6.2.1. To verify that this is indeed a functor, one only needs to consider refinements where an additional graph γ satisfying $\gamma_{n-1} \subset \gamma \subset \gamma_n$ is inserted between γ_{n-1} and γ_n. It maps to the morphism in I_g given by contracting the edges in $E_\gamma \backslash E_{\gamma_{n-1}}$. We emphasize that the collapsing functor sends *unweighted* nested sequences to *weighted* graphs. It induces a functor between the opposite categories in the usual manner. The functor (7.5) is essentially surjective. Furthermore, $I_g^{\mathcal{B}}$ is generated by singletons (G) for G an unweighted, connected, stable graph of genus g by edge contractions and refinements. Since there are finitely many isomorphism classes, $I_g^{\mathcal{B}}$ is equivalent to a finite category.

Definition 7.5 Consider the map

$$L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}} : \left(I_g^{\mathcal{B}}\right)^{\mathrm{opp}} \longrightarrow \mathrm{BLC}_{\mathbb{Q}} \tag{7.6}$$

$$(\gamma_1, \ldots, \gamma_n) \mapsto \mathcal{F}^{\mathcal{B}}(\gamma_1) \times \mathcal{F}^{\mathcal{B}}(\gamma_2/\gamma_1) \times \cdots \times \mathcal{F}^{\mathcal{B}}(\gamma_n/\gamma_{n-1})$$

sending a nested sequence to the product of the blow-ups of its graded sequence.

Theorem 7.6 *The map $L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}$ is a functor, and sends morphisms in $(I_g^{\mathcal{B}})^{\mathrm{opp}}$ to isomorphisms and face maps. There is a canonical morphism of functors in $\mathcal{PC}_{\mathbb{Q}}$:*

$$L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}} \longrightarrow L\mathcal{M}_g^{\mathrm{trop}}$$

which is induced by the pair $(\phi^{\mathrm{opp}}, \Phi)$ where $\phi : I_g^{\mathcal{B}} \to I_g$ is the collapsing functor, and the natural transformation Φ is defined on the image of $(\gamma_1, \ldots, \gamma_n)$ by

$$\Phi : \mathcal{F}^{\mathcal{B}}(\gamma_1) \times \mathcal{F}^{\mathcal{B}}(\gamma_2/\gamma_1) \times \cdots \times \mathcal{F}^{\mathcal{B}}(\gamma_n/\gamma_{n-1}) \longrightarrow \mathcal{F}(\gamma_n/\gamma_{n-1}) ,$$

namely, projection onto the last component followed by blow down $\mathcal{F}^{\mathcal{B}}(\gamma_n/\gamma_{n-1}) \to \mathcal{F}(\gamma_n/\gamma_{n-1})$.

Proof Any isomorphism of nested sequences $(\gamma_1, \gamma_2, \ldots, \gamma_n) \cong (\gamma_1', \gamma_2', \ldots, \gamma_n')$ induces an isomorphism between the associated graded sequences, and hence a canonical isomorphism in the category $\mathrm{BLC}_{\mathbb{Q}}$ between

$$\mathcal{F}^{\mathcal{B}}(\gamma_1) \times \mathcal{F}^{\mathcal{B}}(\gamma_2/\gamma_1) \times \cdots \times \mathcal{F}^{\mathcal{B}}(\gamma_n/\gamma_{n-1})$$

and its version with each γ_i replaced with γ_i'. Furthermore, this isomorphism is compatible, via blow-downs, with the isomorphism $\mathcal{F}(\gamma_n/\gamma_{n_1}) \cong \mathcal{F}(\gamma_n'/\gamma_{n_1}')$. The fact that $L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}$ is a functor then follows the proof of Theorem 7.3, as does the rest of the statement. □

The topological realisation $|L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}|$ is defined differently, but equivalent, we expect, to the bordification discussed in [24]. See Figure 10 for an illustration in the case $g = 2$. There is a canonical continuous morphism:

$$\left|L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}\right| \longrightarrow \left|L\mathcal{M}_g^{\mathrm{trop}}\right| \tag{7.7}$$

which collapses all exceptional components.

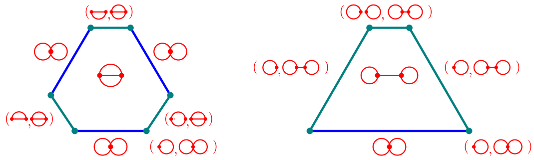


Fig. 10 A picture of $|L\mathcal{M}_2^{\mathrm{trop},\mathcal{B}}|$. Faces with isomorphic labels are to be identified. We have not labelled all faces for reasons of space

8 The strict transform of the graph locus in $L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}$

Let G be a graph with zero weights. Recall that $\pi_{\mathcal{B}} : P^{\mathcal{B}_G} \to \mathbb{P}^{E_G}$ is the corresponding blow-up and $X_G \subset \mathbb{P}^{E_G}$ the graph hypersurface. Denote its strict transform by

$$\widetilde{X}_G \subset P^{\mathcal{B}_G}$$

and let $U_G \subset P^{\mathcal{B}_G}$ be its open complement $P^{\mathcal{B}_G} \backslash \widetilde{X}_G$. Recall from Theorem 7.3 that $\left(\mathcal{C}_G^{\mathcal{B}}\right)^{\mathrm{opp}}$ is equivalent to the category of faces of the blown-up Feynman polytope $\mathcal{F}_G^{\mathcal{B}}$.

Definition 8.1 Define a functor

$$\widetilde{\mathcal{X}} : \left(\mathcal{C}_G^{\mathcal{B}}\right)^{\mathrm{opp}} \longrightarrow \mathrm{Sch}_{\mathbb{Q}}$$

which sends the singleton (G) to the strict transform $\widetilde{X}_G \subset P^{\mathcal{B}_G}$. It is uniquely defined on all other objects (7.3) by restriction to faces (i.e., by intersecting with the corresponding divisors in $P^{\mathcal{B}_G}$), and defines a subscheme functor of $\mathsf{S}\mathcal{F}^{\mathcal{B}}$.

An isomorphism of graphs (or nested sequence of graphs) induces an isomorphism of graph hypersurfaces and hence of their strict transforms by Proposition 5.2 (i). Since $I_g^{\mathcal{B}}$ is generated by stable unweighted graphs of genus g, we deduce the existence of a functor

$$\widetilde{\mathcal{X}} : \left(I_g^{\mathcal{B}}\right)^{\mathrm{opp}} \longrightarrow \mathrm{Sch}_{\mathbb{Q}}$$

which is a subscheme functor of $L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}$.

Proposition 8.2 *The functor* $\widetilde{\mathcal{X}} : \left(I_g^{\mathcal{B}}\right)^{\mathrm{opp}} \to \mathrm{Sch}_{\mathbb{Q}}$ *is given on sequences* (7.3) *by*

$$(\gamma_1, \ldots, \gamma_n) \mapsto \bigcup_{i=1}^{n} P^{\gamma_1} \times \cdots \times P^{\gamma_{i-1}/\gamma_{i-2}} \times \widetilde{X}_{\gamma_i/\gamma_{i-1}} \times P^{\gamma_{i+1}/\gamma_i} \times \cdots \times P^{\gamma_n/\gamma_{n-1}} . \tag{8.1}$$

It is a closed subscheme of $L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}$ *at infinity, i.e., a subscheme of the functor* $\mathsf{S}L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}$*, whose real points do not meet the topological realisation*:

$$\widetilde{\mathcal{X}}(\mathbb{R}) \cap |L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}| = \emptyset . \tag{8.2}$$

The blow-down $(\phi, \Phi) : L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}} \to L\mathcal{M}_g^{\mathrm{trop}}$ *induces a natural transformation* $\widetilde{\mathcal{X}} \to \mathcal{X} \circ \phi$, *where* ϕ *is the collapsing functor.*

Proof If $e \in E_G$ is not a self-edge, there is a canonical inclusion $\widetilde{X}_{G/e} \to \widetilde{X}_G$ induced by the inclusion $P^{\mathcal{B}}_{G/e} \to P^{\mathcal{B}}_G$, since this is true for graph hypersurfaces and strict transforms are functorial (Proposition 5.2 (i)). This proves the formula (8.1) for sequences of length $n = 1$.

The general case follows by taking refinements. Let $\gamma \subset G$ be a core subgraph of G. One has the following identity of graph polynomials [17, (2.2)]

$$\Psi_G = \Psi_\gamma \Psi_{G/\gamma} + R_{G/\gamma} \tag{8.3}$$

where $R_{G/\gamma}$ is a polynomial of homogeneous degree strictly greater than $\deg \Psi_\gamma = h_\gamma$ in the edge variables α_e for $e \in E_\gamma$. The identity implies that if $D \subset P^G$ is the exceptional divisor corresponding to the blow-up of $\mathbb{P}^{E_\gamma} \subset \mathbb{P}^{E_G}$, one has a canonical isomorphism:

$$\widetilde{X}_G \cap D \cong \left(\widetilde{X}_\gamma \times P^{\mathcal{B}}_{G/\gamma}\right) \cup \left(P^{\mathcal{B}}_\gamma \times \widetilde{X}_{G/\gamma}\right) .$$

See also [10, (3.7)]. The right-hand side of (8.1) follows by iterating this isomorphism. The fact that $\widetilde{\mathcal{X}}$ is at infinity follows from the fact that the strict transform locus $\widetilde{X}_G(\mathbb{R})$ does not meet the region $\sigma^{\mathcal{B}}_G$ [10, Proposition 7.3 (iii)], [17, §6.5]. The final statement follows by definition since X_G is the blow-down of $\widetilde{X}_G$. □

The properties of graph hypersurfaces required in the proof will actually be rederived in §11 from a more conceptual viewpoint via properties of the determinant.

The open complement of the subscheme $\widetilde{\mathcal{X}}$ is the functor

$$\begin{aligned} \mathcal{U} : \left(I^{\mathcal{B}}_g\right)^{\mathrm{opp}} &\longrightarrow \mathrm{Sch}_{\mathbb{Q}} \\ (\gamma_1, \ldots, \gamma_n) &\mapsto U_{\gamma_1} \times U_{\gamma_2/\gamma_1} \times \cdots \times U_{\gamma_n/\gamma_{n-1}} \end{aligned}$$

It is an open subscheme of $\mathsf{S}L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}$ and satisfies $|L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}| \subset \mathcal{U}$ (Definition 2.6).

Corollary 8.3 *Via Definition* 2.7 *we deduce a functor*

$$\begin{aligned} L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}} \backslash \widetilde{\mathcal{X}} : \left(I^{\mathcal{B}}_g\right)^{\mathrm{opp}} &\longrightarrow \mathcal{PC}_{\mathbb{Q}} \\ x &\mapsto (\mathcal{U}_x, L^{\mathcal{B}} \cap \mathcal{U}_x, \sigma_x) \end{aligned}$$

Remark 8.4 A functorial system of *linear* subspaces of $\mathsf{S}\mathcal{F}(G)$ was defined in [17, §5.4] whose complement in P^G is affine. It defines a subscheme functor $\mathcal{A} \subset L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}$ such that $|L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}| \subset P^G \backslash \mathcal{A}$. It would be very interesting to construct natural differential forms on $P^G \backslash \mathcal{A}$. They would have linear poles (by contrast with the canonical forms studied here.)

8.1 The boundary and open locus

Definition 8.5 Let $\partial I_g^{\mathcal{B}}$ denote the full subcategory of $I_g^{\mathcal{B}}$ whose objects are sequences of graphs $(\gamma_1, \ldots, \gamma_n)$ where $n \geq 2$. Denote the restriction of $L\mathcal{M}_g^{\mathcal{B}}$ to this category by

$$\partial L\mathcal{M}_g^{\mathcal{B}} : \partial I_g^{\mathcal{B}} \longrightarrow \mathrm{BLC}_{\mathbb{Q}} \ .$$

We shall call it the boundary locus, or exceptional locus, of $L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}$.

The open locus $|L\mathcal{M}_g^{\circ,\mathrm{trop}}|$ embeds canonically into both $|L\mathcal{M}_g^{\mathrm{trop}}|$ and $|L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}|$.

Proposition 8.6 *There is a morphism of $\mathcal{PC}_{\mathbb{Q}}$-complexes*

$$\partial L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}} \longrightarrow \partial L\mathcal{M}_g^{\mathrm{trop}}$$

given by the pair $(\phi^{\mathrm{opp}}, \pi^{\mathcal{B}})$, where $\phi : \partial I_g^{\mathcal{B}} \to \partial I_g$ is the restriction of the collapsing functor, and $\pi^{\mathcal{B}}$ is the canonical blow-down map. In addition, there is a canonical embedding

$$\left|L\mathcal{M}_g^{\circ,\mathrm{trop}}\right| \overset{\sim}{\longrightarrow} \left|L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}\right| \setminus \left|\partial L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}\right| \tag{8.4}$$

whose inverse is the blow-down $|L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}| \setminus |\partial L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}| \overset{\sim}{\longrightarrow} |L\mathcal{M}_g^{\mathrm{trop}}| \setminus |\partial L\mathcal{M}_g^{\mathrm{trop}}|$.

Proof The functor ϕ is given by restricting the collapsing functor (7.5) to nested sequences of length ≥ 2. To see that it lands in the subcategory ∂I_g, note that collapsing a nested sequence of length ≥ 2 results in a graph γ_n/γ_{n-1} of positive weight, since it involves the contraction of a core sugraph γ_{n-1}, which has positive loop number $h_{\gamma_{n-1}} > 0$. The exceptional locus of $\mathcal{F}^{\mathcal{B}}(G) \to \mathcal{F}(G)$ corresponds precisely to sequences (7.3) of length ≥ 2. To prove (8.4), first define the full subcategory $I_{g,w=0}$ of I_g consisting of graphs of weight 0, and consider the functor:

$$\begin{aligned} \sigma \cap \mathcal{U} : I_{g,w=0}^{\mathrm{opp}} &\longrightarrow \mathcal{T}op \\ G \ &\mapsto \ (\sigma \cap \mathcal{U})_G = \sigma_G \setminus (\sigma_G \cap X_G(\mathbb{R})) \ . \end{aligned} \tag{8.5}$$

Note that $(\sigma \cap \mathcal{U})_G$ is contained in σ_G, but contains $\overset{\circ}{\sigma}_G$ since $(\sigma \cap \mathcal{U})_G = \bigcup_{h_\gamma=0} \overset{\circ}{\sigma}_{G/\gamma}$. It follows from Proposition 6.7 and the fact that $(\sigma \cap \mathcal{U})_G$ is the empty set if $w(G) > 0$ that:

$$|\sigma \cap \mathcal{U}| = \varinjlim_{G \in I_{g,w=0}^{\mathrm{opp}}} \sigma_G \setminus (\sigma_G \cap X_G(\mathbb{R})) = \left|L\mathcal{M}_g^{\circ,\mathrm{trop}}\right| \ .$$

Now consider the functor $\sigma\backslash\mathcal{E} : I_g^{\mathcal{B}} \to \mathcal{T}op$ which sends sequences (7.3) of length > 1 to the empty set, and maps singletons (G) to

$$(G) \ \mapsto \ (\sigma\backslash\mathcal{E})_G = \sigma_G^{\mathcal{B}} \setminus \left(\sigma_G^{\mathcal{B}} \cap \mathcal{E}_G(\mathbb{R})\right)$$

where $\mathcal{E}_G \subset P^{\mathcal{B}_G}$ is the exceptional divisor of $\pi_{\mathcal{B}_G} : P^{\mathcal{B}_G} \to \mathbb{P}^{E_G}$. The blow-down morphism defines a functorial isomorphism of algebraic varieties, and of topological spaces:

$$\begin{aligned}\pi_{\mathcal{B}_G} \ : \ P^{\mathcal{B}_G}\backslash\left(\mathcal{E}_G \cup \widetilde{\mathcal{X}}_G\right) &\overset{\sim}{\longrightarrow} \mathbb{P}^{E_G}\backslash X_G \\ \sigma_G^{\mathcal{B}} \setminus \left(\sigma_G^{\mathcal{B}} \cap \mathcal{E}_G(\mathbb{R})\right) &\overset{\sim}{\longrightarrow} \sigma_G\backslash(\sigma_G \cap X_G) = (\sigma \cap \mathcal{U})_G\end{aligned}$$

The second line follows from restriction of the first, since $\widetilde{\mathcal{X}}_G(\mathbb{R})$ does not meet $\sigma_G^{\mathcal{B}}$ by (8.2). Let $j : \sigma_G\backslash(\sigma_G \cap X_G) \overset{\sim}{\to} \sigma_G^{\mathcal{B}} \setminus \left(\sigma_G^{\mathcal{B}} \cap \mathcal{E}_G(\mathbb{R})\right)$ denote its inverse. To define the continuous map (8.4) we define a pair consisting of a functor and natural transformation, and take its limit. The natural transformation is given by j; the functor is given by

$$\iota : I_{g,w=0} \longrightarrow I_g^{\mathcal{B}} \qquad \text{where} \qquad \iota : G \mapsto (G)\ ,$$

and (G) denotes a sequence of graphs of length one. The isomorphism j, which is functorial, thus defines a natural transformation from the functor $\sigma \cap \mathcal{U}$ defined by (8.5) to

$$(\sigma\backslash\mathcal{E}) \circ \iota : I_{g,w=0} \overset{\iota}{\longrightarrow} I_g^{\mathcal{B}} \overset{\sigma\backslash\mathcal{E}}{\longrightarrow} \mathcal{T}op$$

It induces a continuous map

$$|\sigma \cap \mathcal{U}| \longrightarrow |\sigma\backslash\mathcal{E}| = \varinjlim_{G\in(I_g^{\mathcal{B}})^{\mathrm{opp}}} \sigma_G^{\mathcal{B}} \setminus \left(\sigma_G^{\mathcal{B}} \cap \mathcal{E}_G(\mathbb{R})\right)$$

which is exactly (8.4). The last statement follows from the fact that the inverse of the continuous map j is, by definition, the restriction of the blow-down map $\pi^{\mathcal{B}}$. $\square$

The key point in the previous proposition is that simplices which are glued together in the tropical moduli space, and which are not contained in the graph hypersurface locus, continue to be glued together after blowing-up (i.e., if $\mathcal{F}(\gamma)$ is a common face of both $\mathcal{F}(G_1)$ and $\mathcal{F}(G_2)$ and $w(\gamma) = 0$, then $\mathcal{F}^{\mathcal{B}}(\gamma)$ is a common face of both $\mathcal{F}^{\mathcal{B}}(G_1)$ and $\mathcal{F}^{\mathcal{B}}(G_2)$). This is expressed by the fact that ι is a functor, using the fact that core graphs are intrinsic.

8.2 The face complex associated to $L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}$

Using the general Definition 3.1, we may write down the face complex associated to $L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}$.

Definition 8.7 Let $\mathcal{GC}_0^{\mathcal{B}}$ denote the $\mathbb{Q}$-vector space generated by symbols $\left[\underline{\Gamma}, \varpi\right]$

$$\text{where} \quad \underline{\Gamma} = (\gamma_1, \ldots, \gamma_n) \,, \quad \text{and} \quad \varpi = \varpi_{\gamma_1} \wedge \varpi_{\gamma_2/\gamma_1} \wedge \cdots \wedge \varpi_{\gamma_n/\gamma_{n-1}} \,,$$

where $\gamma_1 \subsetneq \ldots \subsetneq \gamma_n$ is a strict nested sequence of graphs where γ_i is core for $i < n$ and γ_n is stable and connected (Definition 7.2), and $\varpi_{\gamma_{i+1}/\gamma_i}$ is an orientation on the quotients γ_{i+1}/γ_i, with relations:

$$[\underline{\Gamma}, -\varpi] = -[\underline{\Gamma}, \varpi] \qquad \text{and} \qquad [\underline{\Gamma}, \varpi] = [\underline{\Gamma}', \varpi']$$

whenever $\underline{\Gamma} \cong \underline{\Gamma}'$ is an isomorphism of nested sequences $f : \gamma_n \cong \gamma_n'$ such that $f(\gamma_i) = \gamma_i'$, and $\varpi' = f(\varpi)$. It is bigraded by genus, and edge number. The differential

$$d : \mathcal{GC}_0^{\mathcal{B}} \longrightarrow \mathcal{GC}_0^{\mathcal{B}}$$

has two components: $d = d^i + d^e$, which we call the internal and exceptional differentials, respectively. The internal differential is defined by edge contraction (7.4)

$$d^i\left[(\gamma_1, \ldots, \gamma_n), \varpi\right] = \sum_{e \in E_{\gamma_j/\gamma_{j-1}}} (-1)^j \left[(\gamma_1, \ldots, \gamma_n)/e, \varpi'\right]$$

where the sum is over all admissible edges, j is the unique index such that $e \in E_{\gamma_j/\gamma_{j-1}}$ and

$$\varpi' = \varepsilon\, \varpi_{\gamma_1} \wedge \cdots \wedge \varpi_{\gamma_{i-1}/\gamma_{i-2}} \wedge \varpi'_{\gamma_i/\gamma_{i-1}} \wedge \cdots \wedge \varpi_{\gamma_n/\gamma_{n-1}}$$

where $\varepsilon = \pm 1$ is defined by $\varpi'_{\gamma_i/\gamma_{i-1}} = \varepsilon\, e \wedge \varpi_{\gamma_i/(\gamma_{i-1} \cup e)}$. The exceptional differential is:

$$d^e\left[(\gamma_1, \ldots, \gamma_n), \varpi\right] = \sum_{\gamma} (-1)^i \left[(\gamma_1, \ldots, \gamma_{i-1}, \gamma, \gamma_i, \ldots, \gamma_n), \varpi'\right]$$

where the sum is over all refinements, i.e., all possible choices of core graphs γ such that $\gamma_i \subsetneq \gamma \subsetneq \gamma_{i+1}$ for some $1 \leq i \leq n$, or $\gamma \subsetneq \gamma_1$. The orientation is given by

$$\varpi' = \varepsilon\, \varpi_{\gamma_1} \wedge \cdots \wedge \varpi_{\gamma_{i-1}/\gamma_{i-2}} \wedge \varpi'_{\gamma/\gamma_{i-1}} \wedge \varpi'_{\gamma_i/\gamma} \wedge \cdots \wedge \varpi_{\gamma_n/\gamma_{n-1}}$$

where $\varepsilon = \pm 1$ is defined by $\varpi_{\gamma_i/\gamma_{i-1}} = \varepsilon \varpi'_{\gamma/\gamma_{i-1}} \wedge \varpi'_{\gamma_i/\gamma}$.

The complex $\mathcal{GC}_0^{\mathcal{B}}$ is filtered by the length n of sequences. Let $F_k\mathcal{GC}_0^{\mathcal{B}}$ denote the subcomplex whose generators $[(\gamma_1, \ldots, \gamma_n), \varpi]$ have $n > k$ components.

Proposition 8.8 *If we write* $\mathcal{GC}_0^{\partial\mathcal{B}} = F_1\mathcal{GC}_0^{\mathcal{B}}$ *then*

$$H_n(\mathcal{GC}_0^{\mathcal{B}}) \cong \bigoplus_g H_n\left(\left|L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}\right|\right) \quad \textit{and} \tag{8.6}$$

$$H_n(\mathcal{GC}_0^{\partial\mathcal{B}}) \cong \bigoplus_g H_n\left(\left|\partial L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}\right|\right).$$

Furthermore, the homology of the usual commutative even graph complex satisfies $H_n(\mathcal{GC}_0) = H_n(\mathcal{GC}_0^{\mathcal{B}}, \mathcal{GC}_0^{\partial\mathcal{B}})$ *and fits into a long exact sequence*

$$\ldots \longrightarrow H_n(\mathcal{GC}_0^{\partial\mathcal{B}}) \longrightarrow H_n(\mathcal{GC}_0^{\mathcal{B}}) \longrightarrow H_n(\mathcal{GC}_0) \xrightarrow{d^e} H_{n-1}(\mathcal{GC}_0^{\partial\mathcal{B}}) \longrightarrow \ldots .$$

This sequence may be identified with the direct sum over all g, of the relative homology sequence of the pair $(|L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}|, |\partial L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}|)$.

Proof The face complex of $|L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}|$ is $(\mathcal{GC}_0^{\mathcal{B}})_g$, the subcomplex of $\mathcal{GC}_0^{\mathcal{B}}$ in genus g. This follows from the categorical description §7.3 of $|\partial L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}|$ and Theorem 7.3: facets of $\sigma_G^{\mathcal{B}}$ are identified either with $\sigma_\gamma^{\mathcal{B}} \times \sigma_{G/\gamma}^{\mathcal{B}}$ for $\gamma \subset G$ a core subgraph corresponding to an exceptional divisor, or $\sigma_{G/e}^{\mathcal{B}}$, where e is a non-tadpole edge corresponding to the intersection of $\sigma_G^{\mathcal{B}}$ with strict transforms of coordinate hyperplanes $x_e = 0$. These two types of facets correspond to the two edge differentials (respectively, exceptional and internal). By iterating one arrives at the description of the face complex given above. Similarly, the face complex of the subspace $|\partial L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}|$ is given by the subcomplex $(\mathcal{GC}_0^{\partial\mathcal{B}})_g$ of $(\mathcal{GC}_0^{\mathcal{B}})_g$ consisting of nested sequences of length ≥ 2, which correspond precisely to the exceptional faces. These facts, together with Theorem 3.2 imply the two equations in (8.6).

Finally, use the relative long exact homology sequence

$$\ldots \longrightarrow H_n(\mathcal{GC}_0^{\partial\mathcal{B}}) \longrightarrow H_n(\mathcal{GC}_0^{\mathcal{B}}) \longrightarrow H_n(\mathcal{GC}_0^{\mathcal{B}}, \mathcal{GC}_0^{\partial\mathcal{B}}) \longrightarrow H_{n-1}(\mathcal{GC}_0^{\partial\mathcal{B}}) \longrightarrow \ldots$$

and use the definition of $\mathcal{GC}_0$ to identify the relative homology group $H_n(\mathcal{GC}_0^{\mathcal{B}}, \mathcal{GC}_0^{\partial\mathcal{B}})$ with $H_n(\mathcal{GC}_0)$. Taking the boundary along $\mathcal{GC}_0^{\partial\mathcal{B}}$ is precisely the exceptional differential d^e.

Note that by exicision, the relative homology $H_n(|L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}|, |\partial L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}|)$ is isomorphic to $H_n(|L\mathcal{M}_g^{\mathrm{trop}}|, |\partial L\mathcal{M}_g^{\mathrm{trop}}|)$, and so these sequences are compatible with Proposition 6.8. $\square$

9 Polyhedra in spaces of quadratic forms

We now turn to the study of polyhedra whose vertices are positive semi-definite quadratic forms with rational null spaces.

9.1 Positive semi-definite matrices

Let $\mathcal{P}_g$ denote the space of positive definite $g \times g$ real symmetric matrices. It may be identified, via the map $X \mapsto X^T X$, with $O_g(\mathbb{R}) \setminus \mathrm{GL}_g(\mathbb{R})$, and admits a right action $M \mapsto h^T M h$ for $h \in \mathrm{GL}_g(\mathbb{R})$. We write $L\mathcal{P}_g = \mathcal{P}_g / \mathbb{R}_{>0}^{\times}$.

The space of real non-zero symmetric matrices up to scalar multiplication may be identified with the real points of the projective space $\mathbb{P}^{d_g-1}$, where $d_g = \binom{g+1}{2}$, and hence $L\mathcal{P}_g \subset \mathbb{P}^{d_g-1}(\mathbb{R})$. A choice of homogeneous coordinates on $\mathbb{P}^{d_g-1}$ are given by the set of matrix entries X_{ij} for $i \leq j$, for X a symmetric matrix. Since the determinant is a homogeneous function of these coordinates, its vanishing locus defines a hypersurface

$$\mathrm{Det} \subset \mathbb{P}^{d_g-1}$$

which satisfies $L\mathcal{P}_g \cap \mathrm{Det}(\mathbb{R}) = \emptyset$. The linear action of GL_g on $\mathbb{P}^{d_g-1}$, which corresponds to the action $M \mapsto h^T M h$ on symmetric matrices, preserves Det and $L\mathcal{P}_g \subset \mathbb{P}^{d_g-1}(\mathbb{R})$.

9.2 Positive polyhedra in the space of symmetric matrices

It is convenient to reformulate the above in a coordinate-free manner. Consider a vector space V of dimension g defined over a field $k \subset \mathbb{R}$. Let us denote by

$$\mathcal{Q}(V) = \left(\mathrm{Sym}^2 V\right)^{\vee}$$

the k-vector space of symmetric bilinear (i.e., quadratic) forms

$$Q : V \otimes_k V \longrightarrow k \ .$$

Such a quadratic form may be viewed as a linear map $Q : V \to V^{\vee}$ which is self-dual, i.e., $Q = Q^T$. Consequently, $\mathcal{Q}$ defines a contravariant functor from the category of vector spaces to itself: for any linear map $V \to W$, there is a natural map $\mathcal{Q}(W) \to \mathcal{Q}(V)$. The functor $\mathcal{Q}$ sends surjective maps to injective maps, and vice-versa. Given an isomorphism $V \cong k^g$ one may identify elements of $\mathcal{Q}(V)$ with symmetric matrices with entries in k.

Recall that the null space $\ker(Q)$ of a quadratic form is defined to be the kernel of $Q : V \to V^{\vee}$. If Q is a positive semi-definite quadratic form, one has

$$Q(v, v) = 0 \quad \Longleftrightarrow \quad v \in \ker(Q) \ . \tag{9.1}$$

Consider the projective space $\mathbb{P}(\mathcal{Q}(V))$. It has distinguished linear subspaces:

$$\mathbb{P}(\mathcal{Q}(V/K)) \hookrightarrow \mathbb{P}(\mathcal{Q}(V)) \qquad \text{for every subspace } K \subset V \ .$$

Such a subspace is contained in the determinant locus $\mathrm{Det}_V \subset \mathbb{P}(\mathcal{Q}(V))$ if and only if $K \neq 0$. Viewing $\mathcal{Q}(V/K)$ as the subspace of $\mathcal{Q}(V)$ of quadratic forms Q satisfying $Q(k, v) = 0$ for all $k \in K$, and $v \in V$, one deduces the formula:

$$\mathcal{Q}\left(V/K_1\right) \cap \mathcal{Q}\left(V/K_2\right) = \mathcal{Q}\left(V/(K_1 + K_2)\right) \tag{9.2}$$

which implies that $\mathbb{P}\left(\mathcal{Q}\left(V/K_1\right)\right) \cap \mathbb{P}\left(\mathcal{Q}\left(V/K_2\right)\right) = \mathbb{P}\left(\mathcal{Q}\left(V/(K_1 + K_2)\right)\right)$.

Consider the convex subsets

$$\mathcal{Q}^{>0}(V) \ \subset \ \mathcal{Q}^{\geq 0}(V) \ \subset \ \mathcal{Q}(V)$$

consisting of positive definite, and positive semi-definite, quadratic forms on V.

Definition 9.1 We shall say that a polyhedron $(\sigma, \mathcal{Q}(V))$ in the space of quadratic forms is *positive*, which we shall denote by $\sigma \geq 0$, if its vertices lie in $\mathcal{Q}^{\geq 0}(V)$, and hence $\sigma \subset \mathcal{Q}^{\geq 0}(V)(\mathbb{R})$ is contained in the set of positive semi-definite quadratic forms.

We shall call a polyhedron *strictly positive*, denoted by $\sigma > 0$, if σ positive, and if in addition it meets the interior $\sigma \cap \mathcal{Q}^{>0}(V) \neq \emptyset$. In other words, σ is strictly positive if it contains at least one positive definite quadratic form.

Positive polyhedra meet the determinant locus in a specific manner.

Lemma 9.2 *Let σ be a positive polyhedron in $\mathcal{Q}(V)$ and let σ_F be a face of σ of dimension* > 0. *The following are equivalent*:

(i) *The face σ_F is contained in the determinant locus* $\mathrm{Det}(\mathbb{R})$.

(ii) *There is a point in the interior of σ_F which lies in* $\mathrm{Det}(\mathbb{R})$, *i.e.*,

$$\overset{\circ}{\sigma}_F \cap \mathrm{Det}(\mathbb{R}) \neq \emptyset$$

(iii) *There is a non-zero linear subspace* $0 \neq K \subset V$ *such that*

$$\sigma_F \ \subset \ \mathbb{P}(\mathcal{Q}(V/K))(\mathbb{R}) \ .$$

Proof Clearly (i) implies (ii) and (iii) implies (i) since $\mathbb{P}(\mathcal{Q}(V/K)) \subset \mathrm{Det}$. It suffices to prove that (ii) implies (iii). Let us write $\widehat{\sigma}_F = \{\sum_{i=1}^n \lambda_i Q_i : \lambda_i \in \mathbb{R}_{\geq 0}\}$ where Q_i are non-zero positive semi-definite quadratic forms on V. If we assume (ii) then there exist $\lambda_i > 0$ such that $\det\left(\sum_{i=1}^n \lambda_i Q_i\right) = 0$ since every interior point of σ admits a (non-unique) representation as a linear combination of Q_i with strictly positive coefficients. Therefore there is a non-zero vector $x \in V$ such that $x \in \ker\left(\sum_{i=1}^n \lambda_i Q_i\right)$. This implies in particular that $\sum_{i=1}^n \lambda_i Q_i(x, x) = 0$. Since $Q_i(x, x) \geq 0$ we deduce that $Q_i(x, x) = 0$ for all $i = 1, \ldots, n$. By (9.1), $x \in \ker Q_i$ for all i. If we set $K_{\sigma_F} = \bigcap_{i=1}^n \ker(Q_i)$, then $x \in K_{\sigma_F}$, and in particular $K_{\sigma_F} \neq 0$. Property (iii) holds on setting $K = K_{\sigma_F}$. □

It follows that if $\sigma \geq 0$ is a positive polyhedron, then it is strictly positive if and only if its interior does not meet the determinant locus:

$$\sigma > 0 \quad \Longleftrightarrow \quad \overset{\circ}{\sigma} \cap \mathrm{Det}(\mathbb{R}) = \emptyset \ . \tag{9.3}$$

Definition 9.3 For any positive polyhedron $\sigma \geq 0$ in $\mathbb{P}(\mathcal{Q}(V))$, we denote by $K_\sigma \subset V$ the intersection of all the null spaces of its vertices.

It follows from (9.2) that K_σ is the unique largest subspace of V such that

$$\sigma \ \subset \ \mathbb{P}(\mathcal{Q}(V/K_\sigma))(\mathbb{R}) \ .$$

In particular, σ meets the strictly positive locus in V/K_σ, i.e., $\sigma \cap \mathcal{Q}^{>0}(V/K_\sigma) \neq \emptyset$ and thus the polyhedron $(\sigma, V/K_\sigma)$ is strictly positive.

Lemma 9.4 *Let $\sigma \geq 0$ be a positive polyhedron. For any subspace $K \subset V$,*

$$\sigma \cap \mathbb{P}(\mathcal{Q}(V/K))(\mathbb{R})$$

is either empty, or is a face of σ.

Proof For every $v \in K$, consider the linear map $f_v : \mathcal{Q}(V) \to k$ defined by $Q \mapsto Q(v, v)$. Its kernel is $\mathcal{Q}(V/kv)$, which contains $\mathcal{Q}(V/K)$ by definition. The linear form f_v is non-negative on the subspace $\mathcal{Q}^{\geq 0}(V) \subset \mathcal{Q}(V)$. It follows from (9.2) that, for any vectors $v_1, \ldots, v_d$ which span K, one has $\mathcal{Q}(V/K) = \bigcap_{i=1}^{d} \mathcal{Q}(V/kv_i) = \bigcap_{i=1}^{d} H_i$, where $H_i = \ker(f_{v_i})$. Since f_{v_i} is non-negative on σ, it follows from the interpretation due to Minkowski and Weyl of a polyhedron as a region bounded by a finite number of hyperplanes, that the set $\sigma \cap \mathbb{P}(H_i)$ is either empty (in the case when $f_{v_i} > 0$ on σ), or a face of σ, and

$$\sigma \cap \mathbb{P}(V/K)(\mathbb{R}) = \bigcap_{i=1}^{d} \sigma \cap H_i(\mathbb{R})$$

is either empty, or a non-empty intersection of faces of σ and hence also a face. $\square$

The previous lemma has the important consequence that in defining $L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}$ we shall only need to blow up linear subspaces of $\mathbb{P}(\mathcal{Q}(V))$ of the special form $\mathbb{P}(\mathcal{Q}(V/K))$.

Definition 9.5 For any face σ_F of σ, define its *essential envelope* to be the face

$$\sigma_F^{\mathrm{ess}} = \sigma \cap \mathbb{P}(\mathcal{Q}(V/K_{\sigma_F}))(\mathbb{R}) \ .$$

It satisfies $\sigma_F \subset \sigma_F^{\mathrm{ess}}$. We call a face σ_F *essential* if $\sigma_F = \sigma_F^{\mathrm{ess}}$.

If $\sigma_F > 0$ is strictly positive, then $K_{\sigma_F} = 0$ and one has $\sigma_F^{\mathrm{ess}} = \sigma$. In general, a face σ_F of σ (whether essential or not) will not be Zariski dense in $\mathbb{P}(\mathcal{Q}(V/K_{\sigma_F}))$. See §13 for some examples of essential faces (we shall show that for the image of Feynman polytopes under the tropical Torelli map, the essential envelope of a face $\sigma_{G/\gamma}$ indexed by a subgraph $\gamma \subset G$ is the face $\sigma_{G/\gamma^{\mathrm{core}}}$ where $\gamma^{\mathrm{core}} \subset \gamma$ is the maximal core subgraph of γ).

Lemma 9.6 *The intersection of an essential face of σ with any other face $\sigma_{Q'}$ of σ, is either empty or an essential face of $\sigma_{Q'}$.*

Proof The set of essential faces of σ are precisely the sets $\sigma \cap \mathbb{P}(\mathcal{Q}(V/K))(\mathbb{R})$ where $K \subset V$ is a linear subspace. The intersection of such a face with $\sigma_{Q'}$ is $\sigma_{Q'} \cap \mathbb{P}(\mathcal{Q}(V/K))(\mathbb{R})$, which is therefore also essential or empty. $\square$

9.3 Collapsing the complement of the determinant in the normal bundle

Restriction of quadratic forms to any subspace $K \subset V$ defines a linear map:

$$\mathcal{Q}(V)/\mathcal{Q}(V/K) \longrightarrow \mathcal{Q}(K)\ . \tag{9.4}$$

It does not induce a map on the corresponding projective spaces since it is not injective in general. To see this, choose a complementary space $V \cong K \oplus C$ where $C \cong V/K$. The space $\mathcal{Q}(V)/\mathcal{Q}(V/K)$ may be represented by symmetric matrices in block matrix form,

$$Q = \left(\begin{array}{c|c} Q_0 & Q_1 \\ \hline Q_1^T & 0 \end{array} \right)\ , \tag{9.5}$$

which implies that $\mathcal{Q}(V)/\mathcal{Q}(V/K)$ is isomorphic to the product $\mathcal{Q}(K) \times \mathrm{Hom}(V/K, K)$. The map $\mathcal{Q}(V)/\mathcal{Q}(V/K) \to \mathcal{Q}(K)$ sends $Q \mapsto Q_0$. The determinant function $\det : \mathcal{Q}(K) \to k$ defines a homogeneous polynomial

$$Q \mapsto \det(Q\big|_K)\ :\ \mathcal{Q}(V)/\mathcal{Q}(V/K) \longrightarrow k\ .$$

In the coordinates (9.5), $\det(Q\big|_K)$ is simply $\det(Q_0)$. Denote its zero locus by

$$\mathrm{Det}\big|_K \subset \mathbb{P}(\mathcal{Q}(V)/\mathcal{Q}(V/K))\ . \tag{9.6}$$

Lemma 9.7 *Restriction of quadratic forms to K gives a well-defined projection*

$$\pi_K\ :\ \mathbb{P}(\mathcal{Q}(V)/\mathcal{Q}(V/K))\ \backslash\ \mathrm{Det}\big|_K \longrightarrow \mathbb{P}(\mathcal{Q}(K))\ \backslash\ \mathrm{Det}_K \tag{9.7}$$

which is a fibration in affine spaces $\mathrm{Hom}(V/K, K) \cong \mathbb{A}^{\dim K \cdot \dim V/K}$. *It is functorial, i.e., for any* $h \in \mathrm{GL}(V)$, *there is a commutative diagram*:

$$\begin{array}{ccc} \mathbb{P}(\mathcal{Q}(V)/\mathcal{Q}(V/K))\ \backslash\ \mathrm{Det}\big|_K & \overset{\pi_K}{\longrightarrow} & \mathbb{P}(\mathcal{Q}(K))\ \backslash\ \mathrm{Det}_K \\ \uparrow & & \uparrow \\ \mathbb{P}\big(\mathcal{Q}(V)/\mathcal{Q}(V/K')\big)\ \backslash\ \mathrm{Det}\big|_{K'} & \overset{\pi_{K'}}{\longrightarrow} & \mathbb{P}\big(\mathcal{Q}(K')\big)\ \backslash\ \mathrm{Det}_{K'} \end{array}$$

where the vertical maps are isomorphisms $Q \mapsto h^T Q h$ and $K' = hK$.

Proof A decomposition $V \cong K \oplus C$ defines projective coordinates on $\mathbb{P}(\mathcal{Q}(V)/\mathcal{Q}(V/K))$. The complement of $\mathrm{Det}\big|_K$ consists of projective classes of matrices (9.5) with $\det(Q_0) \neq 0$. In particular, $Q_0 \neq 0$ and so the map $Q \mapsto Q_0$ is well-defined on projective spaces. Its fiber is isomorphic to the space of matrices Q_1, which is nothing other than the vector space $\mathrm{Hom}(V/K, K)$, viewed as the k points of an affine space of equal dimension. □

Note that the ambiguity in the choice of decomposition $V \cong K \oplus V/K$, is the set of splittings of the exact sequence $0 \to K \to V \to V/K \to 0$, which is a torsor over $\mathrm{Hom}(V/K, K)$.

9.4 Strict positivity of normal faces

For simplicity of notation, let us write

$$\sigma/\sigma_F = \sigma_{/K_{\sigma_F}} \tag{9.8}$$

for the normal polyhedron (Definition 4.8) of σ relative to the null space K_{σ_F} of a face σ_F.

Lemma 9.8 *Let σ be strictly positive and let σ_F be a face of σ. In the product*

$$\sigma_F \times (\sigma/\sigma_F) \subset \mathbb{P}\big(\mathcal{Q}(V/K_{\sigma_F})\big) \times \mathbb{P}\big(\mathcal{Q}(V)/\mathcal{Q}(V/K_{\sigma_F})\big) ,$$

both σ_F and σ/σ_F are strictly positive (Definition 9.1). In the latter case, this means that the interior of the polyhedron σ/σ_F does not meet the determinant locus:

$$(\sigma/\overset{\circ}{\sigma}_F) \subset \mathbb{P}\big(\mathcal{Q}(V)/\mathcal{Q}(V/K_{\sigma_F})\big) \setminus \mathrm{Det}\big|_{K_{\sigma_F}}$$

or equivalently, that the image $\pi_K(\sigma/\sigma_F)$ is a strictly positive polyhedron in $\mathbb{P}(\mathcal{Q}(K_{\sigma_F}))$.

Proof The vertices of σ are projective classes of positive semi-definite quadratic forms $Q_1, \ldots, Q_n \in \mathcal{Q}(V)$. The vertices of the face σ_F correspond to a subset of them: $\{Q_i, i \in I\}$, where $I \subset \{1, \ldots, n\}$. By definition,

$$K_{\sigma_F} = \bigcap_{i \in I} \ker Q_i \ , \tag{9.9}$$

which implies that in any choice of decomposition $V = K_{\sigma_F} \oplus C$, where $C \cong V/K_{\sigma_F}$, the matrices Q_i have the block matrix form

$$Q_i = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \overline{Q}_i \end{array}\right) \quad \text{if} \quad i \in I \ ,$$

where the $\overline{Q}_i$ are the restrictions of Q_i to C, and are positive semi-definite. Suppose that σ_F is not strictly positive. Then σ_F is contained in $\mathrm{Det}_{V/K_{\sigma_F}}(\mathbb{R}) \subset \mathbb{P}(\mathcal{Q}(V/K_{\sigma_F}))(\mathbb{R})$. By Lemma 9.2, there is a subspace $0 \neq K' \subset V/K_{\sigma_F}$ which is contained in $\bigcap_i \ker \overline{Q}_i$. But this implies $\bigcap_i \ker Q_i$ is strictly larger than K_{σ_F}, contradicting (9.9).

The polyhedron σ/σ_F has vertices given by the projective classes of the Q_j, $j \notin I$ which are non-zero in the quotient $\mathcal{Q}(V)/\mathcal{Q}(V/K_{\sigma_F})$ and have block matrix representatives

$$Q_j = \left(\begin{array}{c|c} R_j & P_j \\ \hline P_j^T & 0 \end{array}\right) \quad \text{for all } j \notin I \ .$$

If σ/σ_F were contained in $\mathrm{Det}|_{K_{\sigma_F}}(\mathbb{R})$, then by the argument of Lemma 9.2 applied to the matrices R_j, for $j \notin I$, there would exist $0 \neq x \in K_{\sigma_F}$ such that $x \in \ker(R_j) \subset$

K_{σ_F} for all $j \notin I$. In this case, the image y of x in V satisfies $Q_j(y, y) = R_j(x, x) = 0$ and hence $y \in \ker(Q_j)$ for all $j \notin I$. Since $y \in \ker(Q_i)$ for all $i \in I$ by (9.9), we deduce that y lies in $\ker Q_i$, for all $i = 1, \ldots, n$, contradicting the strict positivity of σ by Lemma 9.2. Note that the image $\pi_K(\sigma/\sigma_F)$ is the link of the convex hull of the quadratic forms R_j, for $j \notin I$. $\square$

Remark 9.9 This lemma easily extends to the case when σ is a strictly positive polyhedron in the space $\mathbb{P}(\mathcal{Q}(V)/\mathcal{Q}(V/K))$. The statement is the following: in the product of polyhedra

$$\sigma_F \times (\sigma/\sigma_F) \subset \mathbb{P}\left(\mathcal{Q}(V/K_{\sigma_F})/\mathcal{Q}(V/K)\right) \times \mathbb{P}\left(\mathcal{Q}(V)/\mathcal{Q}(V/K_{\sigma_F})\right)$$

both σ_F and σ/σ_F are strictly positive. The proof is identical: one only needs to consider symmetric matrices with a number of initial rows and columns which are identically zero.

9.5 Blow-ups and the determinant locus

The determinant locus is well-behaved with respect to blowing up linear subspaces of the form $\mathbb{P}(\mathcal{Q}(V/K))$.

Proposition 9.10 *Let $0 \neq K \subset V$ be a subspace and let*

$$\pi : P^{\mathcal{B}} \longrightarrow \mathbb{P}(\mathcal{Q}(V))$$

denote the blow-up of $\mathbb{P}(\mathcal{Q}(V))$ along $\mathcal{B} = \{\mathbb{P}(\mathcal{Q}(V/K))\}$. The exceptional divisor $\mathcal{E}$ may be canonically identified with the projectivised normal bundle of $\mathbb{P}(\mathcal{Q}(V/K))$:

$$\mathcal{E} \cong \mathbb{P}(\mathcal{Q}(V/K)) \times \mathbb{P}(\mathcal{Q}(V)/\mathcal{Q}(V/K)) \ .$$

If $\widetilde{\mathrm{Det}}_V$ denotes the strict transform of the determinant locus, we have

$$\widetilde{\mathrm{Det}}_V \cap \mathcal{E} \ \cong \ \left(\mathrm{Det}_{V/K} \times \mathbb{P}(\mathcal{Q}(V)/\mathcal{Q}(V/K))\right) \ \cup \ \left(\mathbb{P}(\mathcal{Q}(V/K)) \times \mathrm{Det}\big|_K\right) \ ,$$

where $\mathrm{Det}\big|_K$ was defined in (9.6). In particular one has the product formula:

$$\mathcal{E} \ \setminus \ \left(\mathcal{E} \cap \widetilde{\mathrm{Det}}_V\right) \ \cong \ \left(\mathbb{P}(\mathcal{Q}(V/K)) \ \setminus \ \mathrm{Det}_{V/K}\right) \ \times \ \left(\mathbb{P}(\mathcal{Q}(V)/\mathcal{Q}(V/K)) \ \setminus \ \mathrm{Det}\big|_K\right) \ .$$

Proof Choose a complementary space to K, i.e., $V \cong K \oplus C$ and choose bases of K, C. We can compute the blow up in local affine coordinates (§5.2) as follows. A quadratic form in $Q \in \mathcal{Q}(V)$ may be written in the form

$$Q = \begin{pmatrix} Q_K & P \\ P^T & Q_C \end{pmatrix} \tag{9.10}$$

Let x_{ij} denote the $(i, j)^{\text{th}}$ matrix coefficient. A choice of affine coordinates on $\mathbb{P}(\mathcal{Q}(V))$ is given by the x_{ij} for $i \leq j$, one of which, say $x_{i_0 j_0}$, is set to 1. The locus $\mathbb{P}(\mathcal{Q}(V/K))$ is represented by matrices such that $Q_K = P = 0$, i.e., the vanishing of all other x_{ij} for $i \leq j$ such that $i \leq \dim(K)$. Therefore, suppose that $i_0 \leq \dim(K)$. The blow up may be represented in any such local affine coordinates (see §5.2) by the map:

$$s_K^* : k[x_{ij}, i \leq j] \longrightarrow k[z, x_{ij}, i \leq j]$$
$$\begin{aligned} s_K^*(x_{ij}) &= zx_{ij} \quad \text{if } i \leq \dim(K) \\ s_K^*(x_{ij}) &= x_{ij} \quad \text{otherwise} \end{aligned}$$

The exceptional divisor is defined by the equation $z = 0$. Applying the map s_K^* to the entries of the matrix of a quadratic form (9.10), we obtain

$$s_K^* Q = \begin{pmatrix} zQ_K & zP \\ zP^T & Q_C \end{pmatrix} . \tag{9.11}$$

Taking determinants of the following matrix identity:

$$\begin{pmatrix} zQ_K & zP \\ zP^T & Q_C \end{pmatrix} = \begin{pmatrix} I & 0 \\ P^T Q_K^{-1} & I \end{pmatrix} \begin{pmatrix} zQ_K & 0 \\ 0 & Q_C - zP^T Q_K^{-1} P \end{pmatrix} \begin{pmatrix} I & Q_K^{-1} P \\ 0 & I \end{pmatrix}$$

gives $\det(s_K^* Q) = \det(zQ_K)\det(Q_C - zP^T Q_K^{-1} P)$. This implies that

$$\det(s_K^* Q) = z^{\dim K} \det(Q_K)\det(Q_C) \mod (z^{\dim K+1}) . \tag{9.12}$$

It follows that the intersection of the strict transform of the locus $\det(Q) = 0$ with the exceptional divisor $z = 0$ is given by the equation $\det(Q_K)\det(Q_C) = 0$. The zero loci of $\det(Q_K)$ and $\det(Q_C)$ are $\mathrm{Det}\big|_K$, and $\mathrm{Det}_{V/K}$ respectively. □

This result generalises several asymptotic factorisation theorems for graph polynomials [17, §2] simply by applying the proposition to a suitable graph Laplacian matrix. In particular, (8.3) and other similar formulae can be deduced from the universal fact that the determinant admits an asymptotic factorisation formula (9.12).

10 Perfect cone compactification

Now consider the case when $k = \mathbb{Q}$ and the vector space $V = V_{\mathbb{Z}} \otimes \mathbb{Q}$ has a lattice $V_{\mathbb{Z}}$. It is convenient to identify $V_{\mathbb{Z}} \cong \mathbb{Z}^g$ for the purposes of the ensuing discussion.

10.1 Rational closure of $\mathcal{P}_g$

The rational closure $\mathcal{P}_g^{\mathrm{rt}}$ of the space of positive definite matrices $\mathcal{P}_g$ is defined to be the space of positive semi-definite matrices M whose null space $\ker(M)$ is defined over $\mathbb{Q}$. A positive semi-definite matrix M has rational kernel if and only if there exists an element $h \in \mathrm{GL}_g(\mathbb{Z})$ such that

$$h^T M h = \begin{pmatrix} M_0 & 0 \\ 0 & 0 \end{pmatrix}$$

where M_0 is positive definite, i.e., $M_0 \in \mathcal{P}_{g'}$ for some $g' \leq g$.

Recall from §9.1 that we identify the space of $g \times g$ symmetric matrices with $\mathbb{R}^{d_g}$, where $d_g = \binom{g+1}{2}$. We have inclusions $\mathcal{P}_g \subset \mathcal{P}_g^{\mathrm{rt}} \subset \mathbb{R}^{d_g}$ and

$$L\mathcal{P}_g \subset L\mathcal{P}_g^{\mathrm{rt}} \subset \mathbb{P}^{d_g-1}(\mathbb{R})$$

where the link of $\mathcal{P}_g^{\mathrm{rt}}$ is defined by $L\mathcal{P}_g^{\mathrm{rt}} = (\mathcal{P}_g^{\mathrm{rt}} \setminus \{0\})/\mathbb{R}_{>0}^{\times}$. The group $\mathrm{GL}_g(\mathbb{Z})$ preserves all three spaces. One has $L\mathcal{P}_g^{\mathrm{rt}} \setminus L\mathcal{P}_g \subset \mathrm{Det}(\mathbb{R})$.

Remark 10.1 Lemma 9.2 and the discussion which follows implies that a positive polyhedron $(\sigma, \mathcal{Q}(\mathbb{Q}^g))$ has the property that σ is contained in the space $\mathcal{P}_g^{\mathrm{rt}}$, i.e., every point of σ defines a real symmetric matrix with rational kernel.

10.2 Minimal vectors and polyhedral linear configurations

For any positive definite real quadratic form Q on $\mathbb{R}^g$, denote its set of minimal vectors by

$$M_Q = \{\lambda \in \mathbb{Z}^g \setminus \{0\} : Q(\lambda) \leq Q(\mu) \text{ for all } \mu \in \mathbb{Z}^g \setminus \{0\}\} .$$

The associated polyhedral cone is defined to be

$$\widehat{\sigma}_Q = \mathbb{R}_{\geq 0}\langle \lambda\lambda^T \rangle_{\lambda \in M_Q} \subset \mathcal{P}_g^{\mathrm{rt}}$$

where the $\lambda\lambda^T$ are rank one quadratic forms in $\mathcal{P}_g^{\mathrm{rt}}$. The cone $\widehat{\sigma}_Q$ is strongly convex since this is already the case for $\mathcal{P}_g^{\mathrm{rt}}$ (although $\mathcal{P}_g^{\mathrm{rt}}$ is not itself a polyhedral cone). A positive definite quadratic form Q is called *perfect* if $\widehat{\sigma}_Q$ has maximal dimension d_g. Equivalently, Q is uniquely determined up to a scalar multiple by its set of minimal vectors [44, 59].

Definition 10.2 For any positive definite real quadratic form Q denote by

$$\mathsf{c}_Q = (\mathbb{P}^Q, L_Q, \sigma_Q) \tag{10.1}$$

the polyhedral linear configuration where $\sigma_Q = L\widehat{\sigma}_Q = \left(\widehat{\sigma}_Q \backslash \{0\}\right)/\mathbb{R}_{>0}^{\times}$ is the link of $\widehat{\sigma}_Q$ in $\mathbb{P}^{d_g-1}(\mathbb{R})$, $L_Q = \bigcup_i L_i$ is the union of the Zariski closures of its facets, and $\mathbb{P}^Q \subset \mathbb{P}^{d_g-1} \cong \mathbb{P}(\mathcal{Q}(V))$ is the linear subspace defined by the Zariski-closure of σ_Q.

The configuration c_Q is thus an object of the category $\mathrm{PLC}_{\mathbb{Q}}$. The form Q is perfect if and only if $\mathbb{P}^Q = \mathbb{P}^{d_g-1}$, i.e., σ_Q is Zariski dense in the whole space of quadratic forms.

10.3 Admissible decompositions

See [15, 32, 50, 59].

Definition 10.3 A set $\Sigma = \{\widehat{\sigma}_i\}$ of polyhedral cones $\widehat{\sigma}_i \subset \mathcal{P}_g^{\mathrm{rt}}$ is called *admissible* if it is stable under the action of $\mathrm{GL}_g(\mathbb{Z})$, and such that:

- it covers $\mathcal{P}_g^{\mathrm{rt}}$, i.e., $\bigcup_i \widehat{\sigma}_i = \mathcal{P}_g^{\mathrm{rt}}$,
- if $\widehat{\sigma} \in \Sigma$, then all the faces of $\widehat{\sigma}$ are elements of Σ,
- the intersection of two cones in Σ is a face of both cones,
- the set of $\mathrm{GL}_g(\mathbb{Z})$-orbits of Σ is finite.

Theorem 10.4 (*Voronoï*). *The set of cones $\{\widehat{\sigma}_Q, Q \in \mathcal{P}_g\}$ is an admissible decomposition of $\mathcal{P}_g^{\mathrm{rt}}$. In particular, every face of $\widehat{\sigma}_Q$ is a cone $\widehat{\sigma}_{Q'}$ for some other quadratic form Q'. Furthermore, every cone $\widehat{\sigma}_Q$ is a face of a perfect cone.*

Let us say that two quadratic forms Q, Q' are equivalent if their sets of minimal vectors (and hence their associated cones) coincide: $M_Q = M_{Q'}$. Denote by $[Q]$ the equivalence classes. The group $\mathrm{GL}_g(\mathbb{Z})$ acts upon the classes $[Q]$. Write

$$[Q'] \leq [Q]$$

if the cone $\widehat{\sigma}_{Q'}$ is a face of $\widehat{\sigma}_Q$. Let us write $\dim [Q] = \dim \sigma_Q$.

Definition 10.5 Let $\mathcal{D}_g^{\mathrm{perf}}$ denote the category whose objects are equivalence classes $[Q]$ for $Q \in \mathcal{P}_g$, and whose morphisms are generated by the following two kinds of maps:

(i) face maps $[Q'] \to [Q]$ whenever $[Q'] \leq [Q]$,
(ii) isomorphisms $h : [Q] \to [h^T Q h]$ for any $h \in \mathrm{GL}_g(\mathbb{Z})$.

The category $\mathcal{D}_g^{\mathrm{perf}}$ is equivalent to a finite category [15], since there are only finitely many isomorphism classes of objects, and automorphism groups of cones are finite.

Every $h \in \mathrm{GL}_g(\mathbb{Z})$ defines a linear isomorphism in $\mathrm{PLC}_{\mathbb{Q}}$ we denote by:

$$\mathsf{c}_Q \xrightarrow{h} \mathsf{c}_{h^T Q h} .$$

For every $Q' \leq Q$ one has a face map $\mathsf{c}_{Q'} \to \mathsf{c}_Q$ in the category $\mathrm{PLC}_{\mathbb{Q}}$.

Definition 10.6 Consider the functor

$$L\mathcal{A}_g^{\mathrm{trop}} : \mathcal{D}_g^{\mathrm{perf}} \longrightarrow \mathrm{PLC}_{\mathbb{Q}} \tag{10.2}$$

which sends Q to c_Q. Its topological realisation $|L\mathcal{A}_g^{\mathrm{trop}}|$ is the link of the moduli space of tropical abelian varieties. By construction, it is a generalised cone complex as in [1].

10.4 Blow-ups of cones associated to positive definite quadratic forms

Definition 10.7 Suppose that Q is a positive definite quadratic form on V, such that $\sigma_Q > 0$ is strictly positive (Definition 9.1). Its Zariski closure $\mathbb{P}^Q$ admits a canonical embedding into $\mathbb{P}(\mathcal{Q}(V))$. Consider the finite set of subspaces of $\mathbb{P}^Q$ defined as follows:

$$\mathcal{B}_Q = \bigcup_{n\geq 1} \left\{ \bigcap_{i=1}^{n} \mathbb{P}^Q \cap \mathbb{P}(\mathcal{Q}(V/K_{\sigma_i})) \ : \ \text{where } \sigma_i \text{ are faces of } \sigma_Q \right\} . \qquad (10.3)$$

By definition and (9.2), the spaces which are to be blown up are of the form $\mathbb{P}(\mathcal{Q}(V/K))$, which meet σ_Q precisely along its essential faces.

Alternatively, consider for all $0 \neq K \subset V$, the (infinite) set $\mathcal{B}^{\text{all}}$ of all subspaces

$$\mathbb{P}(\mathcal{Q}(V/K)) \subset \mathbb{P}(\mathcal{Q}(V))$$

Then $\mathcal{B}_Q = (\mathcal{B}^{\text{all}})^{\min,\sigma_Q} \cap \mathbb{P}^Q$, by Definition 5.13 and Lemma 9.4.

Denote the iterated blow-up of c_Q along $\mathcal{B}_Q$ to be:

$$\mathsf{c}_Q^{\mathcal{B}} \quad \in \quad \mathrm{Ob}(\mathrm{BLC}_{\mathbb{Q}}) \ .$$

Let us denote its topological realisation by $\sigma_Q^{\mathcal{B}} = \sigma(\mathsf{c}_Q^{\mathcal{B}})$. There is a natural blow-down morphism $\mathsf{c}_Q^{\mathcal{B}} \to \mathsf{c}_Q$ in the category $\mathcal{PC}_{\mathbb{Q}}$, which induces a continuous map $\sigma_Q^{\mathcal{B}} \to \sigma_Q$.

Remark 10.8 Because σ_Q is not a simplex in general, the set of loci $\mathbb{P}^Q \cap \mathbb{P}(\mathcal{Q}(V/K))$ where $\mathbb{P}(\mathcal{Q}(V/K))$ meets σ_Q is not closed under intersections (the intersection of two essential faces can be empty). Thus the intersection of two blow-up loci may not intersect σ_Q at all, giving rise to an extraneous blow-up. This does not occur in $L\mathcal{M}_g^{\text{trop},\mathcal{B}}$ because σ_G is a simplex. This is a fundamental difference between the constructions for $L\mathcal{M}_g^{\text{trop},\mathcal{B}}$ and $L\mathcal{A}_g^{\text{trop},\mathcal{B}}$. Another difference between these constructions arises for a different reason. Unlike the case of graphs, $\mathcal{B}_Q$ does *not* only consist of the Zariski closures of the faces of σ_Q: we must blow up loci which can be strictly larger. See §13 for some concrete examples.

Consequently, in contrast to the situation with polytopes associated to graphs, the blow-ups are not intrinsic, which means that we do not strictly have face maps. In other words, the blow-up locus is not necessarily stable under passing to faces: if Q' is a face of Q, one has $\mathcal{B}_{Q'} \subseteq \mathbb{P}^{Q'} \cap \mathcal{B}_Q$ but we do not *a priori* have equality. Nevertheless, the following lemma implies that face maps are intrinsic, up to extraneous modifications.

Lemma 10.9 *Let $Q' \leq Q$ be positive definite. Let us denote by $F_{Q'}\mathsf{c}_Q^{\mathcal{B}}$ the face of $\mathsf{c}_Q^{\mathcal{B}}$ corresponding to the face $\sigma_{Q'}$. Then there is a canonical morphism*

$$F_{Q'}\mathsf{c}_Q^{\mathcal{B}} \longrightarrow \mathsf{c}_{Q'}^{\mathcal{B}} \qquad (10.4)$$

in the category $\mathrm{BLC}_{\mathbb{Q}}$ *which is a composition of extraneous modifications. Consequently, it induces an isomorphism on topological realisations*:

$$F_{Q'}\sigma_Q^{\mathcal{B}} \cong \sigma_{Q'}^{\mathcal{B}} \ .$$

Proof By definition and (9.2), $\mathcal{B}_Q$ consists of spaces of the form $\mathbb{P}^Q \cap \mathbb{P}(\mathcal{Q}(V/K))$ for certain $0 \neq K \subset V$. By Lemma 9.4, $\sigma_{Q'} \cap \mathbb{P}(\mathcal{Q}(V/K))(\mathbb{R})$ is either a face of $\sigma_{Q'}$ or the empty set. Since $\mathbb{P}^{Q'}$ is the Zariski-closure of $\sigma_{Q'}$, we deduce that $\mathcal{B}_{Q'} = (\mathcal{B}_Q \cap \mathbb{P}^{Q'})^{\min,\sigma_{Q'}}$ and (10.4) is a consequence of Proposition 5.14 and Proposition 5.2 (i) applied to $\mathbb{P}^{Q'} \subset \mathbb{P}^Q$. □

Face morphisms are thus replaced in this context by *face diagrams* in $\mathrm{BLC}_{\mathbb{Q}}$:

$$\mathsf{c}^{\mathcal{B}}_{Q'} \longleftarrow F_{Q'}\mathsf{c}^{\mathcal{B}}_{Q} \longrightarrow \mathsf{c}^{\mathcal{B}}_{Q} \tag{10.5}$$

where the map on the right is the inclusion of a face, and the map on the left is an extraneous modification (10.4). The blow-down of (10.5) in $\mathrm{PLC}_{\mathbb{Q}}$ is the diagram

$$\mathsf{c}_{Q'} \overset{\sim}{\longleftarrow} \mathsf{c}_{Q'} \longrightarrow \mathsf{c}_{Q}\ . \tag{10.6}$$

10.5 Definition of $L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}$

Every $h \in \mathrm{GL}_g(\mathbb{Z})$ gives rise to a bijection $\mathcal{B}_Q \overset{h}{\to} \mathcal{B}_{h^T Qh}$ and hence induces a linear isomorphism and commutative diagram in $\mathcal{PC}_{\mathbb{Q}}$:

$$\begin{array}{ccc} \mathsf{c}^{\mathcal{B}}_{Q} & \overset{h}{\longrightarrow} & \mathsf{c}^{\mathcal{B}}_{h^T Qh} \\ \downarrow & & \downarrow \\ \mathsf{c}_{Q} & \overset{h}{\longrightarrow} & \mathsf{c}_{h^T Qh} \end{array} \tag{10.7}$$

where the horizontal maps are in $\mathrm{BLC}_{\mathbb{Q}}$. In other words, blow-ups along linear strata are functorial for the action of $\mathrm{GL}_g(\mathbb{Z})$. Consequently, the action of $\mathrm{GL}_g(\mathbb{Z})$ is compatible with inclusions of faces: for every face morphism $F \to \mathsf{c}^{\mathcal{B}}_{Q}$ and $h \in \mathrm{GL}_g(\mathbb{Z})$ there is a unique face $h(F)$ of $\mathsf{c}^{\mathcal{B}}_{h^T Qh}$ such that the following diagram in $\mathrm{BLC}_{\mathbb{Q}}$ commutes:

$$\begin{array}{ccc} F & \overset{h}{\longrightarrow} & h(F) \\ \downarrow & & \downarrow \\ \mathsf{c}^{\mathcal{B}}_{Q} & \overset{h}{\longrightarrow} & \mathsf{c}^{\mathcal{B}}_{h^T Qh} \end{array},$$

where the vertical maps are inclusions of faces. Furthermore, this diagram is compatible with face diagrams (10.5) and their blow-downs (10.6). The reason is that an element $h \in \mathrm{GL}_g(\mathbb{Z})$ induces an isomorphism not only on faces but also on their normals (Remark 4.9).

Definition 10.10 Define a category $\mathcal{D}_g^{\mathrm{perf},\mathcal{B}}$ whose objects are the sets of faces of $\mathsf{c}^{\mathcal{B}}_{Q}$, for all equivalence classes of positive definite quadratic forms $[Q]$, and whose morphisms are generated by face maps, linear isomorphisms h, for all $h \in \mathrm{GL}_g(\mathbb{Z})$, and extraneous modifications of the form (10.4).

Since $\mathsf{c}_Q^{\mathcal{B}}$ has finitely many faces, each of which blows down to a face of c_Q, it follows that $\mathcal{D}_g^{\mathrm{perf},\mathcal{B}}$, like $\mathcal{D}_g^{\mathrm{perf}}$, is equivalent to a finite category. Blow-down defines a canonical functor

$$\phi : \mathcal{D}_g^{\mathrm{perf},\mathcal{B}} \longrightarrow \mathcal{D}_g^{\mathrm{perf}} \ .$$

Definition 10.11 Consider the $\mathrm{BLC}_{\mathbb{Q}}$-complex

$$L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}} : \mathcal{D}_g^{\mathrm{perf},\mathcal{B}} \longrightarrow \mathrm{BLC}_{\mathbb{Q}} \tag{10.8}$$

which to any object of $\mathcal{D}_g^{\mathrm{perf},\mathcal{B}}$ associates the corresponding face of $\mathsf{c}_Q^{\mathcal{B}}$, for suitable Q.

There is a canonical blow-down morphism (in $\mathcal{PC}_{\mathbb{Q}}$)

$$L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}} \longrightarrow L\mathcal{A}_g^{\mathrm{trop}}$$

given by the pair (ϕ, Φ), where $\phi : \mathcal{D}_g^{\mathrm{perf},\mathcal{B}} \to \mathcal{D}_g^{\mathrm{perf}}$ is the functor considered above, and Φ is the natural transformation obtained by restricting the canonical blow-down map $\mathsf{c}_Q^{\mathcal{B}} \to \mathsf{c}_Q$ to faces. It induces a continuous map on topological realisations:

$$\left|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}\right| \longrightarrow \left|L\mathcal{A}_g^{\mathrm{trop}}\right| \ .$$

Remark 10.12 It is instructive to explain how, in the absence of face maps *per se*, the blow ups of cones are glued together along faces. Consider two positive definite quadratic forms Q_1, Q_2 which share a common face $[Q] \leq [Q_1], [Q_2]$. In the perfect cone complex, this is reflected by face maps $\mathsf{c}_Q \to \mathsf{c}_{Q_1}$ and $\mathsf{c}_Q \to \mathsf{c}_{Q_2}$ which we think of as gluing $\mathsf{c}_{Q_1}, \mathsf{c}_{Q_2}$ along the common face c_Q; indeed this is precisely what happens in the topological realisation: the images of the polyhedra σ_{Q_1} and σ_{Q_2} are identified along σ_Q in the limit $|L\mathcal{A}_g^{\mathrm{trop}}|$.

In the complex $L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}$, we have instead a slightly more complicated picture obtained by joining two face diagrams (10.5) in $\mathrm{BLC}_{\mathbb{Q}}$ (below, left):

$$\begin{array}{ccccccc}
F_Q\mathsf{c}_{Q_1}^{\mathcal{B}} & \longrightarrow & \mathsf{c}_{Q_1}^{\mathcal{B}} & & \sigma_Q^{\mathcal{B}} & \longrightarrow & \sigma_{Q_1}^{\mathcal{B}} \\
\downarrow & & & & \downarrow & & \\
\mathsf{c}_Q^{\mathcal{B}} & & & \overset{\sigma}{\mapsto} & \sigma_Q^{\mathcal{B}} & & \\
\uparrow & & & & \uparrow & & \\
F_Q\mathsf{c}_{Q_2}^{\mathcal{B}} & \longrightarrow & \mathsf{c}_{Q_2}^{\mathcal{B}} & & \sigma_Q^{\mathcal{B}} & \longrightarrow & \sigma_{Q_2}^{\mathcal{B}}
\end{array}$$

where the horizontal maps are inclusions of faces, and the vertical maps are extraneous modifications. This diagram means that the blow-ups $\mathsf{c}_{Q_1}^{\mathcal{B}}$ and $\mathsf{c}_{Q_2}^{\mathcal{B}}$ are indeed glued along the common face $\mathsf{c}_Q^{\mathcal{B}}$, but only after collapsing extraneous exceptional divisors. The diagram on the right is the realisation, in the category of topological spaces, of the diagram on the left. The vertical maps are isomorphisms, since they are induced by extraneous modifications. Thus, in the topological realisation $|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|$, this subtlety falls away and one does in effect have canonical face maps, as in the case of the perfect cone complex.

11 The determinant locus in $L\mathcal{A}_g^{\mathrm{trop}}$ and its blow-up

11.1 The determinant locus in $L\mathcal{A}_g^{\mathrm{trop}}$

Lemma 11.1 *The determinant locus defines a subscheme functor of* $L\mathcal{A}_g^{\mathrm{trop}}$:

$$\mathrm{Det} : \mathcal{D}_g^{\mathrm{perf}} \longrightarrow \mathrm{Sch}_{\mathbb{Q}} \,.$$

The topological realisation of its complement is $|L\mathcal{A}_g^{\mathrm{trop}}|\backslash|\mathrm{Det}(\mathbb{R})| \cong L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})$.

Proof The functor $\mathsf{S}L\mathcal{A}_g^{\mathrm{trop}}$ is given on $[Q]$ by the first component $\mathbb{P}^Q$ of $\mathsf{c}_Q = (\mathbb{P}^Q, L_Q, \sigma_Q)$ which is canonically embedded in $\mathbb{P}(\mathcal{Q}(V))$. Define $\mathrm{Det}([Q]) = \mathrm{Det} \cap \mathbb{P}^Q \subseteq \mathbb{P}^Q$. It is compatible with inclusions of faces since they correspond to inclusions $\mathbb{P}^{Q'} \to \mathbb{P}^Q$ of linear subspaces of $\mathbb{P}(\mathcal{Q}(V))$. The compatibility with isomorphisms of Definition 10.5 (ii) follows since the determinant locus is invariant under the action of $\mathrm{GL}_g(\mathbb{Z})$ (i.e., $\det(P) = 0 \Leftrightarrow \det(hPh^T) = 0$ for all $h \in \mathrm{GL}_g(\mathbb{Z})$). The second statement follows from Lemma 9.2. □

The subscheme Det is not at infinity (Definition 2.6) since, in general:

$$\sigma_Q \cap \mathrm{Det}(\mathbb{R}) \neq \emptyset \,.$$

Indeed, any face $\sigma_{Q'}$ where $[Q'] \leq [Q]$ lies at infinity is necessarily contained in $\mathrm{Det}(\mathbb{R})$. Nevertheless, we show below how performing iterated blow-ups separates the strict transform of the determinant locus away from the perfect cones.

11.2 The strict transform of the determinant locus in $L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}$

Lemma 11.2 *The strict transform of the determinant defines a subfunctor of* $\mathsf{S}L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}$:

$$\widetilde{\mathrm{Det}} : \mathcal{D}_g^{\mathrm{perf},\mathcal{B}} \longrightarrow \mathrm{Sch}_{\mathbb{Q}} \,.$$

Proof Recall that $\mathsf{S}L\mathcal{A}_g^{\mathrm{trop}}$ on $[Q]$ is the iterated blow-up $P^{\mathcal{B}_Q}$ of $\mathbb{P}^Q$ along $\mathcal{B}_Q$. Let $\widetilde{\mathrm{Det}}$ denote the strict transform of $\mathrm{Det} \cap \mathbb{P}^Q$. The functoriality with respect to isomorphisms follows from the invariance of the determinant locus under $\mathrm{GL}_g(\mathbb{Z})$. The functoriality with respect to face diagrams follows from Proposition 5.2 (i) in the case of face morphisms, and for extraneous modifications by definition since $\widetilde{\mathrm{Det}}$ is a strict transform. □

The next section is devoted to proving the following theorem.

Theorem 11.3 *The functor* $\widetilde{\mathrm{Det}} : \mathcal{D}_g^{\mathrm{perf},\mathcal{B}} \to \mathrm{Sch}_{\mathbb{Q}}$ *of* $L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}$ *is at infinity, i.e.*

$$\left|L\mathcal{A}_g^{\mathrm{perf},\mathcal{B}}\right| \cap \widetilde{\mathrm{Det}} = \emptyset \,.$$

The proof follows from Corollary 11.5. Consequently we define a functor (Definition 2.7)

$$L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}} \setminus \widetilde{\mathrm{Det}} : \mathcal{D}_g^{\mathrm{perf},\mathcal{B}} \longrightarrow \mathrm{BLC}_{\mathbb{Q}} \,. \tag{11.1}$$

11.3 Structure of faces and proof of Theorem 11.3

Let Q be a positive definite quadratic form such that $\sigma_Q > 0$. Let $\mathcal{B}$ denote the set of subspaces $\mathbb{P}(\mathcal{Q}(V/K))$ of $\mathbb{P}(\mathcal{Q}(V))$ occuring in equation (10.3), such that $\mathcal{B}_Q = \mathcal{B} \cap \mathbb{P}^Q$. By Proposition 5.2 (i), we may form the blow-up $P^{\mathcal{B}}$ either by blowing up $\mathbb{P}^Q$ along $\mathcal{B}_Q$, or by first blowing up $\mathbb{P}(\mathcal{Q}(V))$ along $\mathcal{B}$, and then restricting to the strict transform of $\mathbb{P}^Q$. Here we shall do the latter.

Proposition 11.4 *An intersection $\mathcal{E}$ of irreducible components of the exceptional divisor of the iterated blow-up $P^{\mathcal{B}}$ of $\mathbb{P}(\mathcal{Q}(V))$ along $\mathcal{B}$ is indexed by nested sequences of spaces*

$$0 = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_{n+1} = V \,. \tag{11.2}$$

There is a canonical isomorphism $\mathcal{E} \cong P_1^{\mathcal{B}_1} \times \cdots \times P_n^{\mathcal{B}_n}$ where $P_i^{\mathcal{B}_i}$ is the iterated blow-up of

$$\mathbb{P}\left(\frac{\mathcal{Q}(V/K_i)}{\mathcal{Q}(V/K_{i+1})}\right)$$

along a certain set $\mathcal{B}_i$ of spaces $\mathbb{P}(\mathcal{Q}(V/K)/\mathcal{Q}(V/K_{i+1}))$, where $K_i \subset K \subset K_{i+1}$. The space $\mathcal{B}_i$ is closed under taking intersections.

The intersection of $\mathcal{E}$ with the strict transform $\widetilde{\mathrm{Det}}$ is canonically isomorphic to

$$\mathcal{E} \cap \widetilde{\mathrm{Det}} \;\cong\; \bigcup_{i=1}^{n} P_1^{\mathcal{B}_1} \times \cdots \times P_{i-1}^{\mathcal{B}_{i-1}} \times \widetilde{\mathrm{Det}}_i \times P_{i+1}^{\mathcal{B}_{i+1}} \times \cdots \times P_n^{\mathcal{B}_n}$$

where $\widetilde{\mathrm{Det}}_i \subset P_i^{\mathcal{B}_i}$ is the strict transform of the zero locus of the homogeneous polynomial map obtained by composing the restriction map (9.4) with the determinant:

$$\frac{\mathcal{Q}(V/K_i)}{\mathcal{Q}(V/K_{i+1})} \longrightarrow \mathcal{Q}\left(K_{i+1}/K_i\right) \xrightarrow{\det} \mathbb{Q} \,.$$

Let $\sigma^{\mathcal{B}}$ be the closure of the inverse image of the interior of σ_Q, viewed inside $\mathbb{P}(\mathcal{Q}(V))(\mathbb{R})$. The set $\sigma^{\mathcal{B}} \cap \mathcal{E}(\mathbb{R})$ is either empty or a face which is canonically homeomorphic to a product

$$\sigma_1^{\mathcal{B}_1} \times \sigma_2^{\mathcal{B}_2} \times \cdots \times \sigma_n^{\mathcal{B}_n} \tag{11.3}$$

where $\sigma_i^{\mathcal{B}_i} \subset P_i(\mathbb{R})$ is the blow-up of a polyhedral cone $\sigma_i \subset \mathbb{P}(\mathcal{Q}(V/K_i)/\mathcal{Q}(V/K_{i+1}))(\mathbb{R})$ which is strictly positive.

Proof The first part follows from the description (5.4) of exceptional divisors and their intersections in iterated blow-ups in terms of nested sequences of spaces in $\mathcal{B}$. The rest follows from repeated application of Lemma 9.8, Remark 9.9 and Proposition 9.10. The fact that $\sigma^{\mathcal{B}} \cap \mathcal{E}(\mathbb{R})$ may be empty follows since some blow-ups may be extraneous to σ. □

Corollary 11.5 *Let* $\mathsf{c}_Q^{\mathcal{B}_Q} = (P^{\mathcal{B}_Q}, L^{\mathcal{B}_Q}, \sigma_Q^{\mathcal{B}})$ *be the blow-up of* c_Q *(Definition 10.2). Then*

$$\sigma_Q^{\mathcal{B}} \cap \widetilde{\mathrm{Det}}(\mathbb{C}) = \emptyset\ .$$

Proof One has $\sigma_Q^{\mathcal{B}} \cong \sigma^{\mathcal{B}}$ since $\mathbb{P}^Q$ is the Zariski closure of σ_Q. Since the boundary of $\sigma^{\mathcal{B}}$ is the union of the interiors of its faces, it suffices to show that the interior of each face does not meet the strict transform of the determinant locus. Since each face of σ_Q either lies at infinity (when it is contained in an exceptional divisor), or is a strictly positive polyhedron of the form $\sigma_{Q'}$ for some $[Q'] \leq [Q]$, every face of $\sigma^{\mathcal{B}}$ is a product of blow-ups of strictly positive polyhedra by (11.3) (on applying Proposition 11.4 to a polyhedron $\sigma_{Q'}$). The statement follows from Lemma 9.2 which implies that their interiors do not meet the strict transform of the determinant locus. □

Definition 11.6 With the notations of Proposition 11.4, define a morphism

$$\pi_{\mathrm{red}} : \mathcal{E} \setminus (\mathcal{E} \cap \widetilde{\mathrm{Det}}) \longrightarrow \prod_{i=0}^{n} \left(\mathbb{P}(\mathcal{Q}(K_{i+1}/K_i)) \ \setminus \ \mathrm{Det}_{K_{i+1}/K_i} \right)\ . \tag{11.4}$$

It is obtained from the blow-down map $\mathcal{E} \cong P_1^{\mathcal{B}_1} \times \cdots \times P_n^{\mathcal{B}_n} \to \prod_{i=1}^{n} \mathbb{P}\left(\frac{\mathcal{Q}(V/K_i)}{\mathcal{Q}(V/K_{i+1})}\right)$, by restricting it to the complement of the strict transform of the determinant locus, and composing with the restriction map (9.7) on each component.

11.4 Inverting extraneous modifications

Consider the category $\mathcal{I}_g^{\mathrm{perf},\mathcal{B}}$ which has the same objects as $D_g^{\mathrm{perf},\mathcal{B}}$ but in which all morphisms corresponding to extraneous modifications are inverted. A face diagram in the category $\mathcal{D}_g^{\mathrm{perf},\mathcal{B}}$ which is represented by

$$[Q'] \longleftarrow F_{Q'}[Q] \longrightarrow [Q]$$

is therefore replaced, in the category $\mathcal{I}_g^{\mathrm{perf},\mathcal{B}}$, by morphisms

$$[Q'] \overset{\sim}{\longrightarrow} F_{Q'}[Q] \longrightarrow [Q]$$

giving rise to a genuine face map $[Q'] \to [Q]$ in $\mathcal{I}_g^{\mathrm{perf},\mathcal{B}}$. The functor $L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}$ does not extend to $\mathcal{I}_g^{\mathrm{perf},\mathcal{B}}$, but does if the target category $\mathrm{BLC}_{\mathbb{Q}}$ is replaced by its localisation with respect to extraneous modifications. We shall not pursue this any further in this paper.

However, by the final comments of Remark 10.12, extraneous modifications are already isomorphisms in the topological realisation. As a result, the topological realisation functor $\sigma L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}} : \mathcal{D}_g^{\mathrm{perf},\mathcal{B}} \to \mathcal{T}op$ canonically extends to a functor

$$\sigma L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}} \; : \; \mathcal{I}_g^{\mathrm{perf},\mathcal{B}} \longrightarrow \mathcal{T}op$$

whose associated topological space is precisely

$$\left| L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}} \right| = \varinjlim_{x \in \mathcal{I}_g^{\mathrm{perf},\mathcal{B}}} \sigma_x \; . \tag{11.5}$$

Remark 11.7 The category $\mathcal{I}_g^{\mathrm{perf},\mathcal{B}}$ can be described in terms of nested sequences of objects $([Q_1], \ldots, [Q_r])$ where $[Q_i]$ are objects of the category $\mathcal{D}_g^{\mathrm{perf}}$, in much the same way as Definition 7.2 in the case of graphs. This will not be discussed further.

11.5 The boundary and interior

Let $\partial \mathcal{D}_g^{\mathrm{perf}}$ denote the full subcategory of $\mathcal{D}_g^{\mathrm{perf}}$ generated by classes of positive definite quadratic forms Q such that $\sigma_Q \subset \mathrm{Det}(\mathbb{R})$.

Definition 11.8 Denote the restriction of the functor $L\mathcal{A}_g^{\mathrm{trop}}$ to $\partial \mathcal{D}_g^{\mathrm{perf}} \subset \mathcal{D}_g^{\mathrm{perf}}$ by

$$\partial L\mathcal{A}_g^{\mathrm{trop}} : \partial \mathcal{D}_g^{\mathrm{perf}} \to \mathrm{PLC}_{\mathbb{Q}} \; .$$

Its topological realisation $|\partial L\mathcal{A}_g^{\mathrm{trop}}|$ is the union of all faces of cones in the Voronoï decomposition which lie at infinity, i.e., in $P_g^{\mathrm{rt}} \backslash P_g$. Denote its complement by

$$\left| L\mathcal{A}_g^{\circ,\mathrm{trop}} \right| = \left| L\mathcal{A}_g^{\mathrm{trop}} \right| \backslash \left| \partial L\mathcal{A}_g^{\mathrm{trop}} \right| \; . \tag{11.6}$$

By Lemma 11.1, $|L\mathcal{A}_g^{\mathrm{trop}}| \cap |\mathrm{Det}(\mathbb{R})| = |\partial L\mathcal{A}_g^{\mathrm{trop}}|$ or equivalently,

$$\left| L\mathcal{A}_g^{\circ,\mathrm{trop}} \right| = \left| L\mathcal{A}_g^{\mathrm{trop}} \right| \backslash \left(\left| L\mathcal{A}_g^{\mathrm{trop}} \right| \cap |\mathrm{Det}(\mathbb{R})| \right) \cong L\mathcal{P}_g / \mathrm{GL}_g(\mathbb{Z}) \; .$$

Now let us define $\partial \mathcal{D}_g^{\mathrm{perf},\mathcal{B}}$ to be the full subcategory of $\mathcal{D}_g^{\mathrm{perf},\mathcal{B}}$ consisting of all faces whose image under canonical blow-downs are objects of $\partial \mathcal{D}_g$.

Definition 11.9 Denote the restriction of $L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}$ to $\partial \mathcal{D}_g^{\mathrm{perf},\mathcal{B}}$ by

$$\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}} : \partial \mathcal{D}_g^{\mathrm{perf},\mathcal{B}} \to \mathrm{BLC}_{\mathbb{Q}} \; .$$

We now show the open $|L\mathcal{A}_g^{\circ,\mathrm{trop}}|$ embeds canonically into $|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|$.

Proposition 11.10 *There is a morphism of $\mathcal{PC}_{\mathbb{Q}}$ complexes*

$$\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}} \to \partial L\mathcal{A}_g^{\mathrm{trop}}$$

given by the pair (f, Φ) where the functor $f : \partial\mathcal{D}_g^{\mathrm{perf},\mathcal{B}} \to \partial\mathcal{D}_g^{\mathrm{perf}}$ and the natural transformation Φ is induced by the canonical blow-downs $\mathsf{c}_Q^{\mathcal{B}} \to \mathsf{c}_Q$. There is a canonical embedding

$$\left|L\mathcal{A}_g^{\circ,\mathrm{trop}}\right| \xrightarrow{\sim} \left|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}\right| \setminus \left|\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}\right| \tag{11.7}$$

whose inverse is the blow-down $|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}| \setminus |\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}| \xrightarrow{\sim} |L\mathcal{A}_g^{\mathrm{trop}}| \setminus |\partial L\mathcal{A}_g^{\mathrm{trop}}|$.

Proof Consider the full subcategory $\mathcal{D}_g^{\circ,\mathrm{perf}}$ of $\mathcal{D}_g^{\mathrm{perf}}$ whose objects are classes $[Q]$, where Q is positive definite, such that $\sigma_Q > 0$. Consider the functor $\mathcal{D}_g^{\circ,\mathrm{perf}} \to \mathcal{T}op$ defined by

$$[Q] \mapsto \sigma_Q \setminus \left(\sigma_Q \cap \mathrm{Det}(\mathbb{R})\right) . \tag{11.8}$$

The associated topological space is

$$\left|L\mathcal{A}_g^{\circ,\mathrm{trop}}\right| = \varinjlim_{Q \in \mathcal{D}_g^{\circ,\mathrm{perf}}} \sigma_Q \setminus \left(\sigma_Q \cap \mathrm{Det}(\mathbb{R})\right) .$$

Because $\mathcal{D}_g^{\mathrm{perf},\mathcal{B}}$ does not have face maps, there is no natural functor from $\mathcal{D}_g^{\circ,\mathrm{perf}}$ to $\mathcal{D}_g^{\mathrm{perf},\mathcal{B}}$, but since we are working with topological spaces, for which face maps do exist, we can work in the category $\mathcal{I}_g^{\mathrm{perf},\mathcal{B}}$ (see §11.4) for which $[Q'] \cong F_{Q'}[Q]$. For this category, there is a functor $\iota : \mathcal{D}_g^{\circ,\mathrm{perf}} \to \mathcal{I}_g^{\mathrm{perf},\mathcal{B}}$. From this point onwards, the argument is similar to that of Proposition 8.6 on replacing "$L\mathcal{M}_g$" with "$L\mathcal{A}_g$" and the graph hypersurface locus $\mathcal{X}$ with Det. Indeed, the blow-down induces a canonical isomorphism

$$\sigma_Q^{\mathcal{B}} \setminus \left(\sigma_Q^{\mathcal{B}} \cap \partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}\right) \xrightarrow{\sim} \sigma_Q \setminus (\sigma_Q \cap \mathrm{Det}(\mathbb{R}))$$

whose inverse we shall denote by j. The pair (ι, j) defines a morphism of functors from the functor $\mathcal{D}_g^{\circ,\mathrm{perf}} \to \mathcal{T}op$ defined by (11.8) to the functor $I_g^{\mathrm{perf},\mathcal{B}} \to \mathcal{T}op$ which sends $[Q]$ to $\sigma_Q^{\mathcal{B}} \setminus \left(\sigma_Q^{\mathcal{B}} \cap \partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}\right)$. By taking the limit over objects, we deduce from (11.5) that it induces a continuous map

$$\left|L\mathcal{A}_g^{\circ,\mathrm{trop}}\right| \longrightarrow \left|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}\right| \setminus \left|\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}\right| .$$

The last statement follows since the composition of the previous map with the blow-down map $|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}| \setminus |\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}| \to |L\mathcal{A}_g^{\mathrm{trop}}| \setminus |\partial L\mathcal{A}_g^{\mathrm{trop}}| = |L\mathcal{A}_g^{\circ,\mathrm{trop}}|$ is the identity. $\square$

12 The blow-up of the tropical Torelli map

12.1 The tropical Torelli map

Let G be a connected graph with zero weights. To each edge $e \in E_G$ we assign a variable x_e. Define a bilinear inner product on $\mathbb{Z}^{E_G}$ by setting $\langle e, e' \rangle = \delta_{e,e'} x_e$ for all edges e, e'. The graph Laplacian is defined to be its restriction to $H_1(G;\mathbb{Z}) \subset \mathbb{Z}^{E_G}$ which defines a quadratic form on $H_1(G;\mathbb{Z})$ taking values in $\mathbb{Z}[x_e, e \in E_G]$. See §13 for an example. Equivalently, it is given by a linear map

$$\Lambda_G : H_1(G;\mathbb{Z}) \longrightarrow \mathrm{Hom}\left(H_1(G;\mathbb{Z}), \mathbb{Z}[x_e, e \in e_G]\right) \ .$$

A graph Laplacian matrix is a matrix representative with respect to a basis of $H_1(G;\mathbb{Z})$. It is an $h_G \times h_G$ symmetric matrix whose entries lie in $\mathbb{Z}[x_e, e \in E_G]$. Its determinant, which is well-defined, is equal to the graph polynomial:

$$\det \Lambda_G = \Psi_G \ . \tag{12.1}$$

The definition makes sense for any such graph G, even in the absence of a metric, since the edge parameters x_e are interpreted as abstract variables. In the case when G is a metric graph, with edge lengths $\ell_e \in \mathbb{R}_{\geq 0}$, then the variables x_e are assigned the values ℓ_e and the corresponding graph Laplacian is the real quadratic form

$$\Lambda_G = \Lambda_G\Big|_{x_e=\ell_e} \quad \in \quad \mathcal{Q}(H_1(G;\mathbb{R}))$$

which is denoted, by abuse of notation, by the same symbol. One shows that the graph Laplacian (for a metric graph) is positive semi-definite, with rational kernel.

The tropical Torelli map, defined and studied in [6, 25, 47, 49], is the map

$$\begin{aligned} t_g : \mathcal{M}_g^{\mathrm{trop}} &\longrightarrow \mathcal{A}_g^{\mathrm{trop}} \\ G &\mapsto [\Lambda_G \oplus 0^{w(G)}] \end{aligned} \tag{12.2}$$

which sends a weighted metric graph G to the class of the Laplacian of the associated unweighted metric graph, where $0^{w(G)}$ denotes the $w(G) \times w(G)$ matrix whose entries are all zero. A matrix representative $\Lambda_G \oplus 0^{w(G)}$ has g rows and columns, where g is the genus of G. It was proven in [16] that the tropical Torelli map is a morphism in a category of stacky fans. In this section, we extend it to a map on bordifications. Before doing so, we observe that the tropical Torelli map is degenerate on certain kinds of graphs.

Example 12.1 Consider the sunrise diagram (Fig. 11, top right), with edges oriented to the right, and let $c_1 = e_1 - e_2$, $c_2 = e_2 - e_3$ denote cycles in $\mathbb{Z}^{E_G}$ whose image in $H_1(G;\mathbb{Z})$ is a basis. The graph Laplacian matrix with respect to this basis is

$$\Lambda_G = \begin{pmatrix} x_1 + x_2 & -x_2 \\ -x_2 & x_2 + x_3 \end{pmatrix}$$

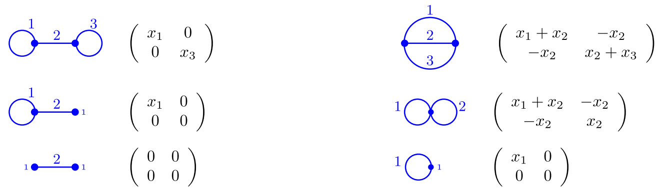


Fig. 11 Representatives for the graph Laplacians of the stable graphs of genus 2

with determinant $\Psi_G = x_1x_2 + x_1x_3 + x_2x_3$. The vanishing of the matrix Λ_G implies that $x_1 = x_2 = x_3 = 0$. The Laplacians of the remaining stable graphs of genus 2 are depicted in the same figure. The tropical Torelli map is not injective on the dumbbell graph (top left), and is even identically zero on the graph depicted on the bottom left. Thus the tropical Torelli map collapses the dumbbell cell into the boundary of the image of the sunrise cell.

Let G be a connected graph. The Laplacian Λ_G defines a linear map

$$\lambda_G : \mathbb{Q}^{E_G} \longrightarrow \mathcal{Q}(H_1(G;\mathbb{Q})) \tag{12.3}$$

which is not injective in general, as the previous example shows. When this is the case, the tropical Torelli map does not extend to a morphism in the category $\mathrm{PLC}_{\mathbb{Q}}$.

Proposition 12.2 *The linear map* (12.3) *is injective if and only if G is* 3*-edge connected* (*the removal of any edge of G is connected and bridgeless*).

Proof Suppose that G is 3-edge connected. Let $e \in E_G$ be any edge of e which is not a self-edge. The graph $G\backslash e$ is 2-edge connected. By Menger's theorem there exist two edge-disjoint paths p_1, p_2 between the endpoints of e which lie in $G\backslash e$. It follows that, by choosing an orientation on the edges of G and adjoining the edge e to the paths p_1, p_2, we may find two cycles $c_1, c_2 \in \mathbb{Z}^{E_G}$ which are independent homology classes in $H_1(G;\mathbb{Z})$, and only overlap in the single edge e. We may complete these to a family of cycles $c_1, c_2, \ldots, c_h$ which represent a basis of $H_1(G;\mathbb{Z})$. It follows from the definition of Λ_G that the entry $(\Lambda_G)_{1,2} = \pm x_e$, where the sign depends on the choice of orientation of edges, and is immaterial. Consequently, the kernel of λ_G is contained in the subspace $x_e = 0$ of $\mathbb{Q}^{E_G}$. Now suppose that e is a self-edge. We may choose the first cycle c_1 in a choice of representatives $c_1, \ldots, c_h$ for $H_1(G;\mathbb{Z})$ to consist of the single edge e. In this case, $(\Lambda_G)_{1,1} = x_e$ and again we conclude that the kernel of (12.3) is contained in $x_e = 0$. Since this holds for all edges $e \in E_G$, λ_G is injective.

Conversely, if G is not 3-edge connected, there exist distinct edges $e_1, e_2 \in E_G$ such that $G' = G\backslash\{e_1, e_2\}$ is disconnected, and G/G' is the graph with two vertices v_1, v_2 and two edges connecting v_1 and v_2. Its Laplacian is represented by the 1×1 matrix with a single entry $x_{e_1} + x_{e_2}$. It follows that $\lambda_{G/G'}$, and a fortiori λ_G (which restricts to $\lambda_{G/G'}$ on the subspace where all $x_e = 0$ for $e \in G'$), are not injective since the linear subspace of $\mathbb{Q}^{E_G}$ given by $x_{e_1} = -x_{e_2}$ and all remaining $x_e = 0$, is contained in their kernel. $\square$

The answer to a more precise question, namely when two tropical curves have equivalent Laplacians, is provided in [25]. Note that for any bridge $e \in E_G$, the variable x_e does not appear at all in the Laplacian Λ_G and one always has $\mathbb{Q}e \subset \ker(\lambda_G)$.

Remark 12.3 For any core (bridgeless) graph G, the intersection of the kernel of λ_G with the region $\widetilde{\sigma}_G = \{x_e \geq 0, e \in E_G\}$ (which is the affine cone over σ_G) is zero. This is because the vanishing locus of λ_G contains the vanishing locus of Ψ_G, which is positive on the interior of $\widetilde{\sigma}_G$ and hence only vanishes along the boundary faces $\widetilde{\sigma}_{G/e} = \widetilde{\sigma}_G \cap V(x_e)$. One concludes by repeating the argument with G/e, which is also core.

12.2 Reduced moduli space of tropical curves

The set of 3-edge connected weighted graphs is stable under edge contractions.

Definition 12.4 Let I_g^{red} be the full subcategory of I_g (Definition 6.3) whose objects are isomorphism classes of weighted connected graphs with edge-connectivity three. Let

$$L\mathcal{M}_g^{\mathrm{red,trop}} : (I_g^{\mathrm{red}})^{\mathrm{opp}} \longrightarrow \mathrm{PLC}_{\mathbb{Q}}$$

denote the restriction of the functor $L\mathcal{M}_g^{\mathrm{trop}}$ to $(I_g^{\mathrm{red}})^{\mathrm{opp}}$. Similarly, let

$$\mathcal{X}^{\mathrm{red}} : (I_g^{\mathrm{red}})^{\mathrm{opp}} \longrightarrow \mathrm{Sch}_{\mathbb{Q}}$$

denote the restriction of the graph locus functor to $(I_g^{\mathrm{red}})^{\mathrm{opp}}$.

There is a natural map $L\mathcal{M}_g^{\mathrm{red,trop}} \rightarrow L\mathcal{M}_g^{\mathrm{trop}}$ and an inclusion $|L\mathcal{M}_g^{\mathrm{red,trop}}| \hookrightarrow |L\mathcal{M}_g^{\mathrm{trop}}|$. In genus 2, for example, the reduced moduli space of tropical curves has only one top-dimensional cell, which is indexed by the sunrise graph (Fig. 11).

12.3 The projective tropical Torelli map

Let G be a 3-edge connected weighted graph. By Proposition 12.2, λ_G is injective and its projectivisation defines a linear morphism:

$$\begin{aligned} \mathbb{P}^{E_G} &\xrightarrow{[\lambda_G]} \mathbb{P}(\mathcal{Q}(H_1(G;\mathbb{Q}))) \\ (x_e)_{e \in E_G} &\mapsto \left[\sum_e x_e Q_e\right], \end{aligned} \tag{12.4}$$

where Q_e is the quadratic form $Q_e(e_1, e_2) = \delta_{e,e_1}\delta_{e,e_2}$. A key theorem [2, Defn. 3.4 and Theorem 6.7] (see also [46, Theorem 4.2.1]), states that for any such G, there exists a quadratic form Q_G whose minimal vectors are exactly the set of Q_e, for $e \in E_G$. Thus, for every 3-edge connected graph G of weight zero, the map $[\lambda_G] : \sigma_G \cong \sigma_{Q_G}$ is an isomorphism. Therefore $[\lambda_G]$ defines a morphism

$$\mathsf{c}_G = (\mathbb{P}^{E_G}, L_G, \sigma_G) \quad \longrightarrow \quad \mathsf{c}_{Q_G} = (\mathbb{P}^{Q_G}, L_{\sigma_{Q_G}}, \sigma_{Q_G}) \tag{12.5}$$

in the category $\mathrm{PLC}_{\mathbb{Q}}$. The second part of the following proposition is a paraphrase of the results [16] for stacky fans, transposed to the setting of polyhedral linear complexes.

Proposition 12.5 *There is a canonical morphism of* $\mathrm{PLC}_{\mathbb{Q}}$*-complexes*:

$$\lambda : L\mathcal{M}_g^{\mathrm{red,trop}} \to L\mathcal{A}_g^{\mathrm{trop}} \tag{12.6}$$

which maps the subscheme $\mathcal{X}^{\mathrm{red}}$ *to* Det. *Its topological incarnation*

$$\left|L\mathcal{M}_g^{\mathrm{red,trop}}\right| \longrightarrow \left|L\mathcal{A}_g^{\mathrm{trop}}\right|$$

is the restriction of the tropical Torelli map (12.2) *to the reduced moduli space.*

Proof By the references quoted above, there exists a functor $t_g : (I_g^{\mathrm{red}})^{\mathrm{opp}} \longrightarrow \mathcal{D}_g^{\mathrm{perf}}$ which sends a 3-edge connected graph G to $[Q_G]$. The pair $(t_g, [\lambda])$ defines the required morphism of $\mathrm{PLC}_{\mathbb{Q}}$-complexes, since $[\lambda]$ is functorial and defines a natural transformation. The compatibility of $\mathcal{X}^{\mathrm{red}}$ and Det follows since $\Psi_G = \det \Lambda_G$, and hence $[\lambda](X_G) = \mathrm{Det}$. □

12.4 The blow-up of the tropical Torelli map

The reader may wish to refer to the example studied in §13.

Proposition 12.6 *Let G be a* 3*-edge connected graph. Then the tropical Torelli map induces a continuous map $\lambda_G^{\mathcal{B}} : \sigma_G^{\mathcal{B}} \to \sigma_{Q_G}^{\mathcal{B}}$ on blown-up polyhedra.*

Proof Because G is 3-edge connected, the map λ_G is injective. Since $\lambda_G : \sigma_G \cong \sigma_{Q_G}$ is an isomorphism and since $\mathbb{P}^{Q_G}$ is the Zariski-closure of σ_{Q_G} it follows that (12.5) is an isomorphism in $\mathrm{PLC}_{\mathbb{Q}}$. It suffices to show that for every face σ_F of σ_{Q_G}, one has $\mathbb{P}(\mathcal{Q}(V/K_{\sigma_F}))(\mathbb{R}) \cap \sigma_{Q_G} = \lambda_G \sigma_{G/\gamma}$ for some core subgraph $\gamma \subset G$. To see this, note that since σ_G is a simplex, its faces $\sigma_G \cap L_\gamma \cong \sigma_{G/\gamma}$ are in one-to-one correspondence with subgraphs $\gamma \subset G$ where L_γ is the linear coordinate space defined by the vanishing of x_e, for $e \in E_\gamma$. Therefore let $\gamma \subset G$ such that $\lambda : \sigma_{G/\gamma} \cong \sigma_F$. We first show that $K_{\sigma_F} = H_1(\gamma; \mathbb{Q})$. For this, choose an isomorphism $H_1(G; \mathbb{Q}) \cong H_1(\gamma; \mathbb{Q}) \oplus H_1(G/\gamma; \mathbb{Q})$. The restriction of the graph Laplacian Λ_G to L_γ takes the block matrix form

$$\Lambda_G\Big|_{L_\gamma} = \begin{pmatrix} 0 & 0 \\ 0 & \Lambda_{G/\gamma} \end{pmatrix} .$$

Since $\det(\Lambda_{G/\gamma}) = \Psi_{G/\gamma}$ is positive on the interior of σ_G, the kernel of this quadratic form is precisely $H_1(\gamma; \mathbb{Q})$ and hence $K_{\sigma_F} = H_1(\gamma; \mathbb{Q})$. Let γ^{core} denote the largest core subgraph of γ. Since $H_1(\gamma; \mathbb{Q}) = H_1(\gamma^{\mathrm{core}}; \mathbb{Q})$, we deduce that $\mathbb{P}(\mathcal{Q}(V/H_1(\gamma; \mathbb{Q}))(\mathbb{R}) \cap \lambda\sigma_G$ contains $\lambda\sigma_{\gamma^{\mathrm{core}}}$. To show the reverse inclusion, note that the restriction of Λ_G to $H^1(\gamma; \mathbb{Q})$ is $\Lambda_{\gamma^{\mathrm{core}}}$. By Remark 12.3, it vanishes on $\widetilde{\sigma}_{\gamma^{\mathrm{core}}}$ only when $x_e = 0$ for all $e \in \gamma^{\mathrm{core}}$ and hence $\mathbb{P}(\mathcal{Q}(V/H_1(\gamma; \mathbb{Q}))(\mathbb{R}) \cap \lambda\sigma_G = \lambda\sigma_{\gamma^{\mathrm{core}}}$.

Consider the smallest set of subspaces $\widetilde{\mathcal{B}}$ of $\mathbb{P}(\mathbb{Q}^{E_G})$ which contains both $\mathcal{B}^G$ and $\lambda^{-1}\mathcal{B}^{Q_G}$ and is closed under intersections. It admits morphisms, by Proposition 5.2 (ii), to both

$$P^{\widetilde{\mathcal{B}}} \longrightarrow P^{\mathcal{B}^G} \quad \text{and} \quad P^{\widetilde{\mathcal{B}}} \longrightarrow P^{\lambda^{-1}\mathcal{B}^{Q_G}} .$$

We have established that the first is an extraneous modification since the subspaces of $\lambda^{-1}\mathcal{B}^{Q_G}$ meet σ_G along $\sigma_{G/\gamma}$ for core γ and therefore contain the Zariski closures of $\sigma_{G/\gamma}$, which are in $\mathcal{B}^G$ by definition. It gives a homeomorphism $\sigma_G^{\mathcal{B}} \cong \sigma_G^{\widetilde{\mathcal{B}}}$. The second can be composed with the map $\lambda : P^{\lambda^{-1}\mathcal{B}^{Q_G}} \to P^{\mathcal{B}^{Q_G}}$ from Proposition 5.2 (i) and leads to a map $P^{\widetilde{\mathcal{B}}} \to P^{\mathcal{B}^{Q_G}}$. It induces a continuous map $\sigma_G^{\widetilde{\mathcal{B}}} \to \sigma_{Q_G}^{\mathcal{B}}$. We deduce the existence of a continous map $\sigma_G^{\mathcal{B}} \to \sigma_{Q_G}^{\mathcal{B}}$ as claimed. □

Define $L\mathcal{M}_g^{\mathrm{red},\mathcal{B}}$ to be the restriction of the functor $L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}$ to the (opposite of the) subcategory $I_g^{\mathrm{red},\mathcal{B}}$ of $I_g^{\mathcal{B}}$ generated by the images of three-edge connected graphs under admissible edge contractions and refinements (Definition 7.4).

Theorem 12.7 *The tropical Torelli map induces a continuous map*

$$\lambda^{\mathcal{B}} : |L\mathcal{M}_g^{\mathrm{red},\mathcal{B}}| \longrightarrow |L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|$$

which maps $|\partial L\mathcal{M}_g^{\mathrm{red},\mathcal{B}}|$ *to* $|\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|$ *and is compatible, via the canonical blow-down maps, with the Torelli map* (12.6).

Proof This follows from Propositions 12.6 and (12.6), which makes essential use of the theorem of [2] on the existence and functoriality of $[Q_G]$, for G a graph. □

13 An example

The following example serves to illustrate the difference in the blow-up constructions for $L\mathcal{M}_3^{\mathrm{trop}}$ and $L\mathcal{A}_3^{\mathrm{trop}}$. Let $G = W_3$ be the wheel with 3 spokes, with inner edges oriented inwards from the center and outer edges oriented counter-clockwise (see below). First we consider the quadratic form in three variables z_1, z_2, z_3 defined by $Q_3 = z_1^2 + z_2^2 + z_3^2 + z_1z_2 + z_2z_3 + z_1z_3$. Its minimal vectors are

$$M = \{\pm(1, 0, 0), \pm(0, 1, 0), \pm(0, 0, 1), \pm(1, -1, 0), \pm(1, 0, -1), \pm(0, 1, -1)\} .$$

The cell σ_{Q_3} is the convex hull of the six matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

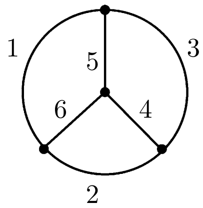
and is given explicitly by the set of projective equivalence classes of matrices

$$\begin{pmatrix} \alpha_1+\alpha_4+\alpha_5 & -\alpha_4 & -\alpha_5 \\ -\alpha_4 & \alpha_2+\alpha_4+\alpha_6 & -\alpha_6 \\ -\alpha_5 & -\alpha_6 & \alpha_3+\alpha_5+\alpha_6 \end{pmatrix}$$

for $\alpha_1,\dots,\alpha_6 \in \mathbb{R}_{\geq 0}$. The essential faces of σ_{Q_3} are those where this matrix has null space exactly K for some subspace $K \subset \mathbb{Q}^3$. For example, if K is the one-dimensional subspace corresponding to z_1, the intersection $\mathcal{Q}(\mathbb{Q}^3/K) \cap \sigma_{Q_3}$ corresponds to when the matrix has vanishing first row and column: $\alpha_1 = \alpha_4 = \alpha_5 = 0$. Such a face will be blown-up. Examples of non-essential faces are given by $\alpha_i = 0$, or even $\alpha_i = \alpha_j = 0$ for any $i,\ j$, since the matrix above continues to have full rank along these loci. They will not be blown-up.

This matrix can be realised as the graph Laplacian for a particular choice of homology basis for the wheel with 3 spokes. However, to illustrate our point, we shall consider a different basis of cycles for $H_1(G;\mathbb{Z})$ consisting of edges $\{1,2,4,5\}$, $\{2,4,6\}$, $\{3,5,4\}$:

$$h_1 = e_1+e_2+e_4-e_5\ ,\ h_2 = e_2+e_4-e_6\ ,\ h_3 = e_3-e_4+e_5$$



With respect to this basis and orientation, the graph Laplacian is represented by the 3×3 matrix

$$\Lambda_G = \mathcal{H}_G^T D_G \mathcal{H}_G = \begin{pmatrix} x_1+x_2+x_4+x_5 & x_2+x_4 & -x_4-x_5 \\ x_2+x_4 & x_2+x_4+x_6 & -x_4 \\ -x_4-x_5 & -x_4 & x_3+x_4+x_5 \end{pmatrix}.$$

The (i,j)th entry is given by $\langle h_i, h_j\rangle$ where $\langle , \rangle$ is the bilinear product on $\mathbb{Z}^{E_G}$ such that $\langle e_i, e_j\rangle = \delta_{ij}x_i$. The matrix Λ_G gives a particular choice of isomorphism from the coordinate simplex σ_{W_3} in $\mathbb{P}^5(\mathbb{R})$ to σ_{Q_3}. The essential faces of σ_{Q_3}, via this isomorphism, correspond precisely to the faces defined where $x_e = 0$, $e \in E(\gamma)$, where γ ranges over core (bridgeless) graphs; all other faces are non-essential.

Consider the subgraph γ spanned by the edges 1, 2, 4, 5. The quotient graph G/γ has a single vertex, and two edges 3, 6. Setting $x_e = 0$ for $e \in E_\gamma$ one obtains

$$\Lambda_G\big|_{L_\gamma} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & x_6 & 0 \\ 0 & 0 & x_3 \end{pmatrix}, \qquad \text{with Zariski closure} \qquad V_{\sigma_{G/\gamma}} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\}.$$

It defines a quadratic form on $V = \bigoplus_{i=1}^3 h_i\mathbb{Q} \cong H_1(G;\mathbb{Q})$ whose kernel is the one-dimensional subspace $K = h_1\mathbb{Q}$. The space of quadratic forms $\mathcal{Q}(V/K)$ with null space K may be identified with the space of symmetric matrices of the form

$$\mathcal{Q}(V/K) \cong \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\} .$$

The image of $\Lambda_G|_{L_\gamma}$ has codimension 1 inside it, and therefore is not Zariski-dense. This example illustrates the difference between $L\mathcal{M}_3^{\mathrm{trop},\mathcal{B}}$ and $L\mathcal{A}_3^{\mathrm{trop},\mathcal{B}}$. The former involves blowing up the subspace $L_\gamma = \{x_e = 0, e \in E_\gamma\}$, corresponding to the core subgraph γ, whose image under the Torelli map λ_G is the projective space $\mathbb{P}(V_{\sigma_{G/\gamma}}) \cong \mathbb{P}^1$. However, in $L\mathcal{A}_3^{\mathrm{trop},\mathcal{B}}$ it is the space $\mathbb{P}(\mathcal{Q}(V/K)) \cong \mathbb{P}^2$ which is blown up, which strictly contains $\mathbb{P}(V_{\sigma_{G/\gamma}})$. Indeed, its preimage under λ_G equals

$$V(x_1 + x_2 + x_4 + x_5, x_2 + x_4, x_4 + x_5) \subset \mathbb{P}(\mathbb{Q}^{E_G})$$

which strictly contains the locus $V(x_1, x_2, x_4, x_5)$ corresponding to γ. It follows that λ_G does not extend to a morphism of blow-ups from $P^{\mathcal{B}_G}$ to $P^{\mathcal{B}^{\min,\sigma_{Q_3}}}$. Nevertheless $\mathbb{P}(V_\sigma)(\mathbb{R})$ and $\mathbb{P}(\mathcal{Q}(V/K))(\mathbb{R})$ both meet σ_G along the same face $\sigma_{G/\gamma}$, and so λ does extend to a morphism between the blown-up polytopes $\sigma_{W_3}^{\mathcal{B}}$ and $\sigma_{Q_3}^{\mathcal{B}}$.

14 Canonical bi-invariant forms and their integrals

14.1 Canonical forms

Employing the same notation as in [19, §4.1], we define

$$\Omega_{\mathrm{can}} = \bigwedge \bigoplus_{k \geq 1} \mathbb{Q}\, \omega^{4k+1} \tag{14.1}$$

to be the graded-commutative Hopf algebra generated by elements ω^{4k+1} in degree $4k+1$, for $k \geq 1$, where the coproduct $\Delta^{\mathrm{can}} : \Omega^{\mathrm{can}} \to \Omega^{\mathrm{can}} \otimes_{\mathbb{Q}} \Omega^{\mathrm{can}}$ has the property that the ω^{4k+1} are primitive, i.e., $\Delta^{\mathrm{can}}(\omega^{4k+1}) = \omega^{4k+1} \otimes 1 + 1 \otimes \omega^{4k+1}$.

Let $R = \bigoplus_{n \geq 0} R^n$ be a differential graded algebra with differential d of degree $+1$, and let $X \in \mathrm{GL}_g(R^0)$ be an invertible $g \times g$ symmetric matrix. Let dX denote the matrix whose entries $(dX)_{ij} = dX_{ij}$ are in R^1. We define

$$\omega_X^{4k+1} = \mathrm{tr}\left((X^{-1} dX)^{4k+1} \right) \quad \in \quad R^{4k+1} .$$

More generally, any $\omega \in \Omega_{\mathrm{can}}$ is a polynomial in the ω^{4k+1}. We write ω_X for the corresponding polynomial in the ω_X^{4k+1}. One can show that the ω_X are:

(1) *Closed*: $d\omega_X = 0$ for any X as above.
(2) *Bi-invariant*: $\omega_{AXB} = \omega_X$ for all $A, B \in \mathrm{GL}_g(R^0)$ such that $dA = dB = 0$.
(3) *Additive*: if we write $\Delta^{\mathrm{can}}\omega = \sum \omega' \otimes \omega''$ and $X_1, X_2 \in \mathrm{GL}_g(R^0)$, then

$$\omega_{X_1 \oplus X_2} = \sum \omega'_{X_1} \wedge \omega''_{X_2} .$$

(4) *Projective*: $\omega_{\lambda X} = \omega_X$ for all $\lambda \in (R^0)^\times$,
(5) *Vanish in low rank*: $\omega_X^{4k+1} = 0$ if $4k+1 > 2g$.

See [19, Lemmas 4.3, 4.4 and Proposition 4.5] for proofs. In *loc. cit.* these forms were studied on the moduli space $|L\mathcal{M}_g^{\mathrm{trop}}|$ and were used to study the cohomology of the even commutative graph complex. In this paper we shall study them on the space $|L\mathcal{A}_g^{\mathrm{trop}}|$, and in particular, reprove and strengthen the results of Borel [11], who showed that

$$\Omega_{\mathrm{can}} \xrightarrow{\sim} \varprojlim_g H^\bullet(\mathrm{GL}_g(\mathbb{Z}); \mathbb{R}) .$$

14.2 Forms of 'compact type'

Let $g > 1$ be odd. By the vanishing property (5), only a finite subspace of Ω_{can} is non-zero on $g \times g$ symmetric matrices.

Definition 14.1 Let $\Omega(g) \subset \Omega_{\mathrm{can}}$ denote the graded exterior algebra generated by 1, $\omega^5, \ldots, \omega^{2g-1}$. It is naturally a quotient of Ω_{can} via the map $\Omega_{\mathrm{can}} \to \Omega(g)$ which sends ω^{4k+1} to zero for all $2k > g$. We shall call a form *simple* if it is a monomial in the generators of $\Omega(g)$, i.e., a product of primitive elements. An element of $\Omega(g)$ is a finite linear combination of simple forms. Call a form $\omega \in \Omega(g)$ of *compact type* in genus g if it is divisible by ω^{2g-1}

$$\omega = \eta \wedge \omega^{2g-1} \quad \text{for some } \eta \in \Omega(g) . \tag{14.2}$$

Otherwise, a simple form ω will be called of *non-compact type*. Denote the space spanned by forms of compact (resp. non-compact) type by $\Omega_c(g)$ (resp. $\Omega_{nc}(g)$). We have

$$\Omega(g) = \Omega_c(g) \oplus \Omega_{nc}(g)$$

The subspace $\Omega_c(g) \subset \Omega(g)$ is an ideal.

Consider the volume form

$$\mathrm{vol}_g = \omega^5 \wedge \cdots \wedge \omega^{2g-1} \quad \in \quad \Omega_c(g) \tag{14.3}$$

which is of degree $d_g - 1$, where $d_g = \frac{g(g+1)}{2}$. It is simple, and of compact type.

Definition 14.2 Let $\omega \in \Omega(g)$ be simple. There is a unique $\star\omega \in \Omega(g)$ such that

$$\omega \wedge \star\omega = \mathrm{vol}_g .$$

The map $\star$ extends by linearity to an endomorphism (the Hodge star operator) on $\Omega(g)$ which interchanges the subspaces of forms of compact and non-compact types:

$$\star : \Omega_c(g) \xrightarrow{\sim} \Omega_{nc}(g) .$$

It is an involution up to sign: $\star\star\,\omega = \pm\omega$.

Example 14.3 Let $g = 7$. Then $\mathrm{vol}_7 = \omega^5 \wedge \omega^9 \wedge \omega^{13}$. The following table lists the simple elements in $\Omega_{nc}(7)$, and below them, their images in $\Omega_c(7)$ under the map $\star$:

$$\begin{array}{cccc} 1 & \omega^5 & \omega^9 & \omega^5 \wedge \omega^9 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \omega^5 \wedge \omega^9 \wedge \omega^{13} & \omega^9 \wedge \omega^{13} & -\,\omega^5 \wedge \omega^{13} & \omega^{13} \end{array}$$

14.3 Canonical forms on $L\mathcal{M}_g^{\mathrm{trop}}$ and $L\mathcal{A}_g^{\mathrm{trop}}$

Lemma 14.4 *Let $\omega \in \Omega^d_{\mathrm{can}}$ be a canonical form of degree d, and let V be a vector space over $k \subset \mathbb{R}$. Then ω defines a closed projective differential form*

$$\omega \quad \in \quad \Omega^d(\mathbb{P}(\mathcal{Q}(V))\backslash \mathrm{Det})$$

which is invariant under the action of $\mathrm{GL}(V)$.

Proof The projectivity follows from §14.1 (4), the invariance from (2). □

Proposition 14.5 *A canonical form $\omega \in \Omega^d(g)$ defines an algebraic differential form (§2.4):*

$$\omega \quad \in \quad \Omega^d(L\mathcal{A}_g^{\mathrm{trop}}\backslash \mathrm{Det})\ . \tag{14.4}$$

It restricts to a smooth differential form, also denoted by ω, on

$$\omega \quad \in \quad \mathcal{A}^d\left(\left|L\mathcal{A}_g^{\circ,\mathrm{trop}}\right|\right) = \mathcal{A}^d\left(L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})\right)\ .$$

Proof By Definition 2.9 a differential form must verify the compatibility relations (2.7). A canonical form ω satisfies the compatibilities for the two types of morphisms in the category $\mathcal{D}_g^{\mathrm{perf}}$ of Definition 10.5: for face morphisms, the compatibility is automatic, and for isomorphisms, this follows from the bi-invariance §14.1 (2). The fact that ω only has poles along the determinant locus follows immediately from its definition since for any square matrix X, the one-form $X^{-1}dX$ is regular outside the locus $\det(X) = 0$. □

In [19], we studied the canonical forms $\omega_G = \omega_{\Lambda_G}$ on the polyhedral linear configuration associated to a graph G.

Proposition 14.6 *A form $\omega \in \Omega^d(g)$ defines an algebraic form $\omega \in \Omega^d(L\mathcal{M}_g^{\mathrm{trop}}\backslash \mathcal{X})$. It restricts to a smooth differential form, also denoted by ω, on*

$$\omega \quad \in \quad \mathcal{A}^d\left(\left|L\mathcal{M}_g^{\mathrm{trop}}\right| \backslash \left|\partial L\mathcal{M}_g^{\mathrm{trop}}\right|\right)\ .$$

The restriction of ω to the reduced moduli space $L\mathcal{M}_g^{\mathrm{red,trop}}$ is the pull-back $\lambda_G^*\omega$ of the corresponding canonical forms (14.4) on $L\mathcal{A}_g^{\mathrm{trop}}$ by the tropical Torelli map.

This follows from Theorem 2.1 in [19]: the first statement is equivalent to the 'compatibility' and 'equivariance' property (2.2) *loc. cit.*; the second on applying Proposition 6.7. That they are pulled back from the tropical Torelli map is automatic from their definition.

14.4 Behaviour at infinity

A key property of canonical forms is their behaviour on blow-ups and in particular, a factorisation property at infinity.

Proposition 14.7 *Let $0 \leq K \subset V$ be a subspace and let $\pi : P^{\mathcal{B}} \to \mathbb{P}(\mathcal{Q}(V))$ denote the blow-up along $\mathcal{B} = \{\mathbb{P}(\mathcal{Q}(V/K))\}$. Let $\omega \in \Omega^d_{\mathrm{can}}$. Denote its coproduct by*

$$\Delta^{\mathrm{can}}\omega = \sum_i \omega_i' \otimes \omega_i'' \ .$$

The pull-back $\pi^*\omega$ to $P^{\mathcal{B}}$ has no poles along the exceptional divisor, and hence defines a regular differential form on the complement of the strict transform $\widetilde{\mathrm{Det}}$ of the determinant:

$$\pi^*(\omega) \quad \in \quad \Omega^d\left(\mathbb{P} \setminus \widetilde{\mathrm{Det}}\right) \ .$$

Its restriction to the exceptional divisor $\mathcal{E}$, which we recall by Proposition 9.10 is canonically isomorphic to the product $\mathcal{E} = \mathbb{P}\left(\mathcal{Q}(V/K)\right) \times \mathbb{P}\left(\mathcal{Q}(V)/\mathcal{Q}(V/K)\right)$, satisfies

$$\pi^*(\omega)\big|_{\mathcal{E}} = \sum_i \omega_i' \wedge \pi_K^*(\omega_i'') \tag{14.5}$$

where $\omega_i' \in \Omega^\bullet\left(\mathbb{P}(\mathcal{Q}(V/K))\backslash \mathrm{Det}_{V/K}\right)$, $\omega_i'' \in \Omega^\bullet\left(\mathbb{P}\left(\mathcal{Q}(K)\right)\backslash \mathrm{Det}_K\right)$, and π_K is the map (9.7).

Proof The proof is almost identical to that of theorem 7.4 in [19] to which we refer for further details. The key point is to choose a complementary subspace $V \cong K \oplus C$ and write Q in block form as in (9.10). Then we may write formula (9.11) in the form:

$$s_K^* Q = \begin{pmatrix} zQ_K & zP \\ zP^T & Q_C \end{pmatrix} = \Lambda U \quad \text{where we set} \quad \Lambda = \begin{pmatrix} zQ_K & 0 \\ 0 & Q_C \end{pmatrix} ,$$

and where

$$U = \Lambda^{-1} s_K^* Q \equiv \begin{pmatrix} 1 & Q_K^{-1} P \\ 0 & 1 \end{pmatrix} \quad (\mathrm{mod}\ z) \ .$$

For simplicity, assume that $\omega = \omega^{4k+1}$ is primitive. Then

$$\omega^{4k+1}_{\Lambda U} = \mathrm{tr}\left((U^{-1}\Lambda^{-1}(d\Lambda)U + U^{-1}dU)^{4k+1}\right) = \mathrm{tr}\left(\left(\Lambda^{-1}d\Lambda + dU.U^{-1}\right)^{4k+1}\right) .$$

The argument follows exactly as in theorem 7.4 of *loc. cit.* to conclude that

$$\omega_{s_K^* Q}\Big|_{z=0} = \omega_{Q_C}^{4k+1} + \omega_{Q_K}^{4k+1}$$

and hence $\pi^*(\omega)\big|_{\mathcal{E}} = \omega_{Q_C}^{4k+1} \wedge 1 + 1 \wedge \omega_{Q_K}^{4k+1}$. The case when ω is a simple form follows by multiplying primitive forms together; the general case follows by linearity. $\square$

Remark 14.8 This implies that the restriction of $\pi^*(\omega)$ to the exceptional divisor is actually just the pre-image of canonical forms under the collapsing map §9.3 of the normal bundle.

It may often happen that the restriction $\pi^*(\omega)$ to the exceptional divisor $\mathcal{E}$ vanishes. There is one important situation in which this always occurs.

Corollary 14.9 *Let $\omega \in \Omega_c^d(g)$ be of compact type, where $g = \dim(V) > 1$ is odd. Then, in the situation of Proposition 14.7, $\pi^*\omega$ vanishes along the exceptional divisor $\mathcal{E}$.*

Proof We may assume that ω is simple. Since ω is of compact type, it follows that in the expression $\Delta^{\mathrm{can}}\omega = \sum_i \omega_i' \otimes \omega_i''$ either ω_i' or ω_i'' is of compact type for each i, since one of these two forms must be divisible by ω^{2g-1}. Note that since $g = 2k+1$ is odd, $\omega^{2g-1} = \omega^{4k+1}$. Property §14.1 (5) implies that ω^{2g-1} vanishes on all matrices of rank h satisfying $2h < 2g-1$, or equivalently, $h < g$. Let us choose a complement $V \cong K \oplus C$ and use the notations of Proposition 14.7. Since $K \neq 0$, both Q_K and Q_C have rank strictly smaller than g, and hence either $(\omega_i')_{Q_C} = 0$ or $(\omega_i'')_{Q_K} = 0$ for every i, since one or other is divisible by ω^{2g-1}. It follows from the formula (14.5) that $\pi^*(\omega)$ vanishes along $\mathcal{E}$. $\square$

The essential part of the proof is the fact that $\Omega_c(g)$ is a Hopf ideal in $\Omega(g)$:

$$\Delta^{\mathrm{can}}(\Omega_c(g)) \subset \big(\Omega_c(g) \otimes_{\mathbb{Q}} \Omega_{nc}(g)\big) + \big(\Omega_{nc}(g) \otimes_{\mathbb{Q}} \Omega_c(g)\big) \ .$$

14.5 Canonical forms on the blow-ups $L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}$ and $L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}$

Theorem 14.10 *Let $\omega \in \Omega_{\mathrm{can}}^d$. It defines a closed algebraic differential form*

$$\widetilde{\omega} \in \Omega^d \left(L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}} \backslash \widetilde{\mathrm{Det}} \right) \tag{14.6}$$

which restricts to a smooth form, also denoted by the same notation,

$$\widetilde{\omega} \in \mathcal{A}^d \left(\left| L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}} \right| \right) \tag{14.7}$$

on the topological realisation. In the case when $g > 1$ is odd and $\omega \in \Omega_c^d(g)$ is of compact type, then the restriction of (14.6) to the boundary $\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}} \setminus (\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}} \cap \widetilde{\mathrm{Det}})$ vanishes. In particular, it follows that the form (14.7) on $|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|$ vanishes on $|\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|$.

Proof Let Q be a positive definite quadratic form such that $\sigma_Q > 0$. Let $\widetilde{\omega} = \pi^*\omega$ denote the pull-back of the form ω along the iterated blow-up $\pi : P^{\mathcal{B}_Q} \to \mathbb{P}^Q$. By repeated application of Proposition 14.7 for each blow-up in the formation of $P^{\mathcal{B}_Q}$, we deduce that $\widetilde{\omega}$ is a meromorphic differential form with no poles along the exceptional divisor. It follows that $\widetilde{\omega} \in \Omega^d(P^{\mathcal{B}_Q} \setminus \widetilde{\mathrm{Det}})$. By restricting it to faces, we obtain a family of forms in $\Omega^d((\mathsf{S}L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}} \setminus \widetilde{\mathrm{Det}})(x))$ for every object x of $\mathcal{D}_g^{\mathrm{perf}}$. By the universal property of strict transforms (Proposition 5.2 (i)) and the fact that restriction of differential forms is compatible with pull-back along blow-ups, the compatibility conditions (2.7) for $\widetilde{\omega}$ follow from those of ω, established in Proposition 14.5. This is because the objects in a face diagram (10.5) are the blow-ups of a face map in $\mathcal{D}_g^{\mathrm{perf}}$, and the form $\widetilde{\omega}$ on each component is the pull-back of ω along a suitable blow-up.

The statement (14.7) is a consequence of the fact that $|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|$ does not meet the strict transform of the determinant locus (Theorem 11.3). When ω is of compact type, $\widetilde{\omega}$ vanishes along exceptional divisors by Corollary 14.9. □

The following statement for $L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}$ may be proved using [19, Theorem 7.4].

Theorem 14.11 *Let $\omega \in \Omega^d_{\mathrm{can}}$. It defines both a holomorphic differential form on $L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}} \setminus \widetilde{\mathcal{X}}$, and restricts to a smooth form on $|L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}|$ which we may denote by*:

$$\widetilde{\omega} \in \Omega^d\left(L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}} \setminus \widetilde{\mathcal{X}}\right) \qquad \textit{resp.} \qquad \widetilde{\omega} \in \mathcal{A}^d\left(\left|L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}\right|\right). \tag{14.8}$$

Note that in the case of $L\mathcal{M}_g^{\mathrm{trop},\mathcal{B}}$ the existence of the dual graph Laplacian implies that most forms of compact type in genus g actually vanish (see [19, Lemma 8.4]), except for the primitive ones. See the discussion in §15.6.

14.6 Integrals of canonical forms

Canonical integrals define new invariants associated to the minimal vectors of quadratic forms.

Corollary 14.12 *Let Q be a positive definite quadratic form such that σ_Q is strictly positive. Then for any canonical form $\omega \in \Omega^d_{\mathrm{can}}$ where $d = \dim \sigma_Q$, the integral*

$$I_Q(\omega) = \int_{\sigma_Q} \omega$$

is finite. In the case where $Q = Q_G$ is associated to a graph, the integral $I_{Q_G}(\omega)$ reduces to the canonical integrals $I_G(\omega)$ studied in [19, §8].

Proof By passing to the blow-up $P^{\mathcal{B}_Q} \to \mathbb{P}^Q$, we may write

$$I_Q(\omega) = \int_{\sigma_Q^{\mathcal{B}}} \widetilde{\omega}$$

since $\overset{\circ}{\sigma}_Q$ is contained in $\sigma_Q^{\mathcal{B}}$, and has complement of Lebesgue measure zero. By Theorem 11.3, $\widetilde{\omega}$ has no poles along the compact region $\sigma_Q^{\mathcal{B}}$ and therefore the integral is finite. □

These integrals satisfy many interesting relations. The analogous relations for graphs were studied in [19, §8.2] and related to the 'waterfall' spectral sequence of [38].

Theorem 14.13 (*Stokes relation*). *Let Q be a positive definite quadratic form such that $\sigma_Q > 0$, and let $\omega \in \Omega_{\mathrm{can}}$ have degree* $\dim \sigma_Q - 1$. *Let $\Delta^{\mathrm{can}}\omega = \sum_i \omega_i' \otimes \omega_i''$. Then*

$$0 = \sum_{Q'} \int_{\sigma_{Q'}} \omega + \sum_i \sum_F \int_{\sigma_F} \omega_i' \int_{\sigma_Q/\sigma_F} \pi^*_{K_{\sigma_F}}(\omega_i'') \tag{14.9}$$

where $Q' \leq Q$ denote the facets of σ_Q which are strictly positive (*not contained in* $\mathrm{Det}(\mathbb{R})$), *and $\sigma_F^{\mathcal{B}} \times (\sigma_Q/\sigma_F)^{\mathcal{B}}$ denote the set of facets of its blow-up $\sigma_Q^{\mathcal{B}}$ which are at infinity* (*in other words, σ_F is an essential face of σ_Q at infinity, and σ_Q/σ_F is defined in* (9.8)). *The map $\pi_{K_{\sigma_F}}$ is defined in Lemma 9.7.*

Proof This follows from an application of Stokes' formula:

$$0 = \int_{\sigma_Q^{\mathcal{B}}} d\widetilde{\omega} = \int_{\partial \sigma_Q^{\mathcal{B}}} \widetilde{\omega} \,,$$

the description of the boundary facets of $\sigma_Q^{\mathcal{B}}$, and the factorisation formula (14.5). □

14.7 Compactly-supported representatives

Every form of compact type has a representative which vanishes identically in a neighbourhood of the boundary.

Corollary 14.14 *Let $g > 1$ be odd. The cohomology class of the differential form ω^{2g-1} in $H^{2g-1}_{\mathrm{dR}}(|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|, |\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|)$ has a compactly supported representative*

$$[\omega_c^{2g-1}] \in H^{2g-1}_{\mathrm{dR},c}\left(\left|L\mathcal{A}_g^{\circ,\mathrm{trop}}\right|\right).$$

Having chosen ω_c^{2g-1}, any other form $\omega = \omega^{2g-1} \wedge \eta \in \Omega_c(g)$ of compact type has a canonical compactly supported representative $\omega_c = \omega_c^{2g-1} \wedge \eta$.

Proof By (3.9), and (11.7) we have $H^{\bullet}_{\mathrm{dR},c}(|L\mathcal{A}_g^{\circ,\mathrm{trop}}|) \cong H^{\bullet}_{\mathrm{dR}}(|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|, |\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|)$. The form ω^{2g-1}, being of compact type, vanishes on $|\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|$ by Theorem 14.10 and therefore defines a class in the latter cohomology group. □

The fact that ω^{2g-1} has a compactly supported representative on $L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})$ was claimed without proof in [41]. Since one also has $|L\mathcal{A}_g^{\circ,\mathrm{trop}}| = |L\mathcal{A}_g^{\mathrm{trop}}| \setminus |\partial L\mathcal{A}_g^{\mathrm{trop}}|$, the blow-down map $\pi : |L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}| \to |L\mathcal{A}_g^{\mathrm{trop}}|$ induces an isomorphism (excision):

$$\pi^* : H^\bullet_{\mathrm{dR}}\left(\left|L\mathcal{A}_g^{\mathrm{trop}}\right|, \left|\partial L\mathcal{A}_g^{\mathrm{trop}}\right|\right) \overset{\sim}{\longrightarrow} H^\bullet_{\mathrm{dR}}\left(\left|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}\right|, \left|\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}\right|\right)$$

since both sides may be identified with the compactly supported cohomology of the interior $|L\mathcal{A}_g^{\circ,\mathrm{trop}}|$. In particular, any compactly supported form ω_c as in the corollary may be viewed as a closed differential form on $|L\mathcal{A}_g^{\mathrm{trop}}|$. Its cohomology class

$$[\omega_c] \in H^\bullet_{\mathrm{dR}}\left(\left|L\mathcal{A}_g^{\mathrm{trop}}\right|\right) \tag{14.10}$$

does not depend on the choice of representative for ω_c^{2g-1} since it is the image of $(\pi^*)^{-1}[\widetilde{\omega}]$ under the natural map $H^\bullet_{\mathrm{dR}}(|L\mathcal{A}_g^{\mathrm{trop}}|, |\partial L\mathcal{A}_g^{\mathrm{trop}}|) \to H^\bullet_{\mathrm{dR}}(|L\mathcal{A}_g^{\mathrm{trop}}|)$.

15 Cohomology classes in $|L\mathcal{A}_g^{\mathrm{trop}}|$

In this section we prove the theorems announced in the introduction.

15.1 Canonical classes in $|L\mathcal{A}_g^{\mathrm{trop}}|$

Let $g > 1$ be odd. We use the notation

$$d_g = \frac{g(g+1)}{2} \quad , \quad \ell_g = \frac{g(g+1)}{2} - 1$$

where $d_g = \dim \mathcal{P}_g$, and $\ell_g = \dim L\mathcal{P}_g$.

Lemma 15.1 *The form* vol_g *defines a non-trivial relative cohomology class*

$$0 \neq [\mathrm{vol}_g] \in H^{\ell_g}_{\mathrm{dR}}\left(\left|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}\right|, \left|\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}\right|\right).$$

Proof Since g is odd, the space $L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})$ is orientable and has a relative fundamental class $[\mathcal{F}_g^{\mathrm{lf}}] \in H^{\mathrm{lf}}_{\ell_g}(|L\mathcal{A}_g^{\circ,\mathrm{trop}}|)$ in locally finite homology, or equivalently,

$$[\mathcal{F}_g^{\mathrm{lf}}] \in H_{\ell_g}\left(\left|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}\right|, \left|\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}\right|\right) .$$

A volume form η_g on $L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})$ is defined on symmetric matrices $M = (M_{ij})_{ij}$ by

$$\eta_g(M) = \partial \left(\frac{1}{\det(M)^{\frac{g+1}{2}}} \bigwedge_{1 \leq i \leq j \leq g} dM_{ij} \right)$$

where $\partial = \sum_{i,j} M_{ij} \frac{\partial}{\partial M_{ij}}$ is the Euler vector field. By H. Cartan [53, Exp. 17, §12] and [11, figure on page 265], the algebra of invariant differential forms on $\mathrm{SL}_g(\mathbb{R})$ is isomorphic to a graded exterior algebra on elements in degree $4k+1$, for $k \geq 0$ (and in degree $4k+1$ for $k \geq 1$ for $\mathrm{GL}_g(\mathbb{R})$). Since the forms ω^{4k+1} are invariant and non-zero they give an explicit choice of algebra generators. It follows that there exists an $\alpha \in \mathbb{Q}^\times$ such that $\eta_g = \alpha \,\mathrm{vol}_g = \alpha\, \omega^5 \wedge \omega^9 \wedge \cdots \wedge \omega^{2g-1}$, since η_g is invariant and vol_g (defined by (14.3)) is the unique generator in $\Omega^\bullet(g)$ of degree ℓ_g. We have

$$\mathrm{vol}(L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})) = \int_{L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})} \eta_g = \alpha \int_{[\mathcal{F}_g^{\mathrm{lf}}]} \mathrm{vol}_g \ .$$

Since $\mathrm{vol}(L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})) \neq 0$ (for a more precise statement see (15.1)), we deduce that the cohomology class of $[\mathrm{vol}_g]$ in $H^{\ell_g}_{dR}\left(|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|, |\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|\right)$ is non-zero. □

Remark 15.2 It follows from a theorem of Minkowski (see [54]) that

$$\int_{L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})} \eta_g \ \in\ \zeta(3)\zeta(5)\ldots\zeta(2g-1)\,\mathbb{Q}^\times \ . \tag{15.1}$$

The point is that the volume of the symmetric space $\mathrm{SL}_g(\mathbb{R})/\mathrm{SL}_g(\mathbb{Z})$ is proportional, by a rational number, to the product of consecutive zeta values $\zeta(2)\ldots\zeta(2g-1)$. One proves [60] that the volume of $L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})$ is obtained by dividing by the volume of $\mathrm{SO}_g(\mathbb{R})$, which is proportional to the product of the even zeta values $\zeta(2)\ldots\zeta(2g-2)$, giving (15.1).

We shall first prove an injectivity theorem for the cohomology of the general linear group, before proving a stronger but more technical result for $|L\mathcal{A}_g^{\mathrm{trop}}|$.

Theorem 15.3 *The map $\omega \mapsto [\omega_c]$ defined in Corollary 14.14 induces an injective map of graded algebras*:

$$\Omega_c^\bullet(g) \hookrightarrow H^\bullet_{\mathrm{dR}}\left(L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})\right) \ .$$

There are canonical injective maps of graded algebras:

$$\Omega_{nc}^\bullet(g) \hookrightarrow H^\bullet_{\mathrm{dR}}\left(|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|\right) \ ,$$

$$\Omega_{nc}^\bullet(g) \hookrightarrow H^\bullet_{\mathrm{dR}}(L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})) \ .$$

There are natural splittings of vector spaces $H^n_{\mathrm{dR}}\left(L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})\right) \to \Omega^n_c(g) \otimes_{\mathbb{Q}} \mathbb{R}$ for each n, and likewise $H^n_{\mathrm{dR}}(L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})) \to \Omega^n_{nc}(g)_{\mathbb{Q}} \otimes \mathbb{R}$.

Proof Recall that $\ell_g = \dim L\mathcal{P}_g$. Let $n \geq 0$ and consider the cup product:

$$H^n(|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|, |\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|) \otimes H^{\ell_g - n}(|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|)$$
$$\longrightarrow H^{\ell_g}(|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|, |\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|)$$

$$[\omega] \otimes [\eta] \mapsto [\omega \wedge \eta].$$

Apply this map to $\omega \in \Omega_c^n(g)$ of compact type, and $\eta = \star\omega$ which is of non-compact type. They define classes in the above cohomology groups by Theorem 14.10. Since the Hodge star operator is non-degenerate, $[\omega \wedge \eta] = [\omega \wedge \star\omega]$ is a non-zero rational multiple of $[\mathrm{vol}_g]$, which is itself non-zero by Lemma 15.1. We conclude that both $[\omega]$ and $[\star\omega]$ are non-vanishing in the respective cohomology groups. This proves the first two equations, since every element of $\Omega^\bullet_{nc}$ is in the image of the star operator.

Recall that $|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}| \setminus |\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}| = |L\mathcal{A}_g^{\circ,\mathrm{trop}}| \cong L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})$. By Theorem 3.6, relative cohomology may be interpreted as compactly supported cohomology. A variant of the above cup product may thus be written as follows:

$$\begin{aligned} &H^n_{c,\mathrm{dR}}\left(L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})\right) \otimes H^{\ell_g-n}_{\mathrm{dR}}\left(L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})\right) \\ &\quad\longrightarrow H^{\ell_g}_{c,\mathrm{dR}}\left(L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})\right). \end{aligned} \tag{15.2}$$

It sends $[\omega_c] \otimes [\star\omega] \mapsto [\omega_c \wedge \star\omega] = [(\mathrm{vol}_g)_c] \neq 0$. It follows that both $[\omega_c]$ and $[\star\omega]$ are non-zero classes in the respective cohomology groups. This proves that $\Omega^\bullet_{nc}(g)$ injects into the ordinary (non compactly-supported) cohomology of $L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})$.

A splitting $H^\bullet_{\mathrm{dR}}(L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})) \to \Omega^\bullet_{nc}(g) \otimes_{\mathbb{Q}} \mathbb{R}$ may be defined as follows. For any $[\alpha] \in H^n_{\mathrm{dR}}(L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z}))$, it follows from Stokes' theorem that the map

$$\omega_c \mapsto \int_{L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})} [\alpha] \wedge \omega_c$$

is a well-defined element of $\mathrm{Hom}(\Omega_c^{\ell_g-n}(g), \mathbb{R})$, which is isomorphic to $\Omega^n_{nc}(g) \otimes_{\mathbb{Q}} \mathbb{R}$ via the star operator since $\star\omega \wedge \omega_c$ pairs non-trivially with the fundamental class $[\mathcal{F}_g^{\mathrm{lf}}]$. This defines a linear map $H^\bullet_{\mathrm{dR}}(L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z}); \mathbb{R}) \to \Omega^\bullet_{nc}(g) \otimes_{\mathbb{Q}} \mathbb{R}$ which, after rescaling, defines the required splitting. A splitting $H^\bullet_{\mathrm{dR}}\left(L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})\right) \to \Omega^\bullet_c(g)$ may be defined similarly. $\square$

We wish to repeat a similar argument in order to prove the more subtle fact that $\Omega^\bullet_c(g)$ injects into the cohomology of $|L\mathcal{A}_g^{\mathrm{trop}}|$. However, replacing the cup product for the *relative* cohomology of $|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|$ with *ordinary* cohomology will not succeed, since the volume form actually vanishes in the cohomology of $|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|$. Replacing $|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|$ with $|L\mathcal{A}_g^{\mathrm{trop}}|$ is also of no avail, since the non-compact canonical forms are not defined on the latter space in the first place. The solution is to work on a space which lies in between $|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}| \to |L\mathcal{A}_g^{\mathrm{trop}}|$. If we interpret the former as the Borel-Serre compactification of $L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})$ (see the Appendix), then an appropriate space is the reductive Borel-Serre compactification. It has the property that its boundary is of sufficiently high codimension that the volume class is non-zero (unlike $|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|$), but is sufficiently regular that the non-compact canonical forms extend to it (unlike $|L\mathcal{A}_g^{\mathrm{trop}}|$). Instead of constructing this space explicitly, we shall imitate its cohomology with an ad-hoc complex whose cohomology will be denoted by H^{red}.

Theorem 15.4 *The map $\omega \mapsto [\omega_c]$ (14.10) induces an injective map of graded algebras*

$$\Omega_c^\bullet(g) \hookrightarrow H_{\mathrm{dR}}^\bullet\left(\left|L\mathcal{A}_g^{\mathrm{trop}}\right|\right) .$$

It factors through the injective map $\Omega_c^\bullet(g) \hookrightarrow H_{\mathrm{dR}}^\bullet(|L\mathcal{A}_g^{\mathrm{trop}}|, |\partial L\mathcal{A}_g^{\mathrm{trop}}|) \cong H_{c,\mathrm{dR}}^\bullet(|L\mathcal{A}_g^{\circ,\mathrm{trop}}|)$ constructed in Theorem 15.3.

Proof We prove a stronger result, namely that the classes $\Omega_c^\bullet(g)$ are non-zero in a larger complex of differential forms with 'reduced boundary'.[4]

$$\Omega^{\bullet,\mathrm{red}}(L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}) \subset \Omega^\bullet(L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}} \setminus \widetilde{\mathrm{Det}})$$
$$\oplus \bigoplus_{\mathcal{E}} \Omega^\bullet\left(\prod_{i=0}^{n} \mathbb{P}\left(\mathcal{Q}(K_{i+1}/K_i)\backslash \mathrm{Det}_{K_{i+1}/K_i}\right)\right)$$

where the direct sum is over all components of the boundary $\partial L\mathcal{A}_g^{\mathcal{B}}$ indexed by nested sequences (11.2). The strategy will be similar to the proof of Theorem 15.3 once we have shown that the volume class is non-zero. The complex $\Omega^{\bullet,\mathrm{red}}(L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}})$ consists of pairs $(\omega, (\eta_{\mathcal{E}})_{\mathcal{E}})$ such that the restriction of ω to a face $\mathcal{E} \cap \partial L\mathcal{A}_g^{\mathcal{B}}$ satisfies

$$\omega\Big|_{\mathcal{E}} = \pi_{\mathrm{red}}^* \eta_{\mathcal{E}} \tag{15.4}$$

where π_{red} was defined in (11.4). There is a corresponding complex $\mathcal{A}^{\bullet,\mathrm{red}}(|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|)$ of smooth differential forms defined in the same way: it consists of pairs $(\omega, (\eta_{\mathcal{E}})_{\mathcal{E}})$ which satisfy (15.4), where $\omega \in \mathcal{A}^\bullet(|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|)$ and $\eta_{\mathcal{E}}$ are smooth forms on $\pi_{\mathrm{red}}(|\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}| \cap \mathcal{E}(\mathbb{R}))$. Let us denote its cohomology by $H^{\bullet,\mathrm{red}}(|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|)$. There is a natural map of complexes $\Omega^{\bullet,\mathrm{red}}(L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}) \to \mathcal{A}^{\bullet,\mathrm{red}}(|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|)$.

[4]To motivate this in the case $n = 2$, consider the following picture of the projectivised normal bundles associated to a single blow-up of $\mathbb{P}(\mathcal{Q}(V/K))$ in $\mathbb{P}(\mathcal{Q}(V))$ where $K \subset V$. We choose, as usual, a complement $V = K \oplus C$. We have the maps

$$\mathbb{P}(\mathcal{Q}(V/K)) \times \mathbb{P}(\mathcal{Q}(V)/\mathcal{Q}(V/K)) \setminus \mathrm{Det}\big|_K \longrightarrow \mathbb{P}(\mathcal{Q}(V/K)) \times \mathbb{P}(\mathcal{Q}(K)) \longrightarrow \mathbb{P}(\mathcal{Q}(V/K)) \tag{15.3}$$

where the first map is $\mathrm{id} \times \pi_K$ (see §9.3, (9.7)) and the second map is projection on to the first factor. Their composition is the blow-down map (see Remark 5.15). The corresponding map on affine spaces, after identifying the middle space as block diagonal matrices, may be viewed on symmetric matrices as the map

$$\begin{pmatrix} Q_K & P \\ P^T & Q_C \end{pmatrix} \longrightarrow \begin{pmatrix} Q_K & 0 \\ 0 & Q_C \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 \\ 0 & Q_C \end{pmatrix}$$

The far left space in (15.3) describes the local structure of an exceptional divisor on $L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}$, the far right space describes the structure of a face at infinity in $L\mathcal{A}_g^{\mathrm{trop}}$. The space in the middle corresponds to the structure at infinity of the reductive Borel-Serre compactification. We consider forms whose restriction to the boundary are constant in the P-direction, i.e., pull-backs from the middle space to the left-most one.

Since the restriction of the blow down to the boundary $|\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}| \to |\partial L\mathcal{A}_g^{\mathrm{trop}}|$ factors through π_{red} (it factors through the map which projects a product (11.3) onto its final component), it follows that the map on cohomology induced by $\pi_{\mathcal{B}}: |L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}| \to |L\mathcal{A}_g^{\mathrm{trop}}|$ factors through the reduced cohomology, i.e., $\pi_{\mathcal{B}}^*$ is the composition of two maps:

$$H^{\bullet}_{\mathrm{dR}}(|L\mathcal{A}_g^{\mathrm{trop}}|) \longrightarrow H^{\bullet,\mathrm{red}}_{\mathrm{dR}}(|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|) \longrightarrow H^{\bullet}_{\mathrm{dR}}(|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|) \tag{15.5}$$

where the first map is $\omega \mapsto (\pi_{\mathcal{B}}^*\omega, 0)$ the second is $(\omega, (\eta)_{\mathcal{E}}) \mapsto \omega$.

We will show that there is an injective map

$$\Omega^{\bullet}(g) \longrightarrow H^{\bullet,\mathrm{red}}_{\mathrm{dR}}\left(\left|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}\right|\right).$$

First of all, observe that there is a well-defined map $\Omega^{\bullet}(g) \to \Omega^{\bullet,\mathrm{red}}(L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}})$ by equation (14.5), which implies that the restriction of a canonical form to an exceptional divisor $\mathcal{E}$ is always of the form $\pi_{\mathrm{red}}^*(\eta_{\mathcal{E}})$. See Remark 14.8. Concretely, the forms $\eta_{\mathcal{E}}$ are products of canonical forms which can be computed explicity from the coproduct of ω.

We next show, as before, that the volume form is non-zero in $H^{\ell_g,\mathrm{red}}(|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|)$. If vol_g were exact, there would exist forms α on $|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|$ and $(\beta_{\mathcal{E}})_{\mathcal{E}}$ of degree $\ell_g - 1$ such that

$$\mathrm{vol}_g = d\alpha \quad \text{ and } \quad \alpha\Big|_{\mathcal{E}(\mathbb{R})} = \pi_{\mathrm{red}}^*\beta\,. \tag{15.6}$$

However, since π_{red} decreases the dimension by at least $g-1$, such a form β is necessarily zero since it has degree $> \ell_g - g$. Equation (15.6) then implies that, since α restricts to zero on every exceptional divisor $\mathcal{E}(\mathbb{R})$, vol_g is zero in relative cohomology $H^{\ell_g}_{\mathrm{dR}}(|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|, |\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|)$, which contradicts Lemma 15.1.

Now let $\omega \in \Omega(g)$. There is a $\lambda \neq 0$ such that $\omega \wedge \star\omega = \lambda \mathrm{vol}_g$. Since (15.4) is compatible with exterior products of differential forms, the complex $\mathcal{A}^{\bullet,\mathrm{red}}(|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|)$ inherits a multiplicative structure giving rise to a product on reduced cohomology:

$$H^{m,\mathrm{red}}_{\mathrm{dR}}(|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|) \otimes H^{n,\mathrm{red}}_{\mathrm{dR}}(|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|) \longrightarrow H^{m+n,\mathrm{red}}_{\mathrm{dR}}(|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|)\,.$$

Since $[\star\omega] \wedge [\omega] = \lambda[\mathrm{vol}_g]$ is non-zero, it follows that $[\omega] \in H^{\bullet,\mathrm{red}}_{\mathrm{dR}}(|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|)$ is non-zero and hence $\Omega(g)$ indeed injects into $H^{\bullet,\mathrm{red}}_{\mathrm{dR}}(|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|)$ as claimed.

By Theorem 14.10 and (15.5), the image of the subspace of compact type $\Omega_c(g)$ in $H^{\bullet,\mathrm{red}}_{\mathrm{dR}}(|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}|)$ factors through its image in $H^{\bullet}_{\mathrm{dR}}(|L\mathcal{A}_g^{\mathrm{trop}}|)$. It follows that the map $\Omega_c(g) \to H^{\bullet}_{\mathrm{dR}}(|L\mathcal{A}_g^{\mathrm{trop}}|)$ given by (14.10) must already be injective.

The last part follows from the fact that the map $\Omega_c(g) \to H^{\bullet}_{\mathrm{dR}}(|L\mathcal{A}_g^{\mathrm{trop}}|)$ factors through $H^{\bullet}_{\mathrm{dR}}(|L\mathcal{A}_g^{\mathrm{trop}}|, |\partial L\mathcal{A}_g^{\mathrm{trop}}|)$ by definition. □

15.2 Cohomology of $\mathrm{GL}_g(\mathbb{Z})$

Theorem 15.3 has the following corollaries. First, some notation. Since $\mathcal{P}_g$ is contractible and an $\mathbb{R}^{\times}_{>0}$-bundle over the space $L\mathcal{P}_g$, we have

$$H^n(\mathrm{GL}_g(\mathbb{Z});\mathbb{R}) = H^n(L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z}))\ , \tag{15.7}$$
$$H^{n+1}_c(\mathrm{GL}_g(\mathbb{Z});\mathbb{R}) = H^n_c(L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z}))\ .$$

The reason for the shift in indices is that $H^n_c(\mathbb{R}^{\times}_{>0}) = 0$ for $n \neq 1$ and $H^1_c(\mathbb{R}^{\times}_{>0}) = \mathbb{R}$.

Corollary 15.5 *Let $g > 1$ be odd. The forms of non-compact (resp. compact) type embed into the cohomology (resp. compactly supported cohomology) of $L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})$:*

$$\Omega^{\bullet}_{nc}(g) \otimes_{\mathbb{Q}} \mathbb{R} \hookrightarrow H^{\bullet}(\mathrm{GL}_g(\mathbb{Z});\mathbb{R}) \tag{15.8}$$
$$\Omega^{\bullet}_{c}(g) \otimes_{\mathbb{Q}} \mathbb{R} \hookrightarrow H^{\bullet+1}_c(\mathrm{GL}_g(\mathbb{Z});\mathbb{R})\ .$$

Corollary 15.6 *Let $g > 1$ be odd. Then for all $h \geq g$, we have:*

$$\Omega^{\bullet}_{nc}(g) \quad \hookrightarrow \quad H^{\bullet}(\mathrm{GL}_h(\mathbb{Z});\mathbb{R})\ . \tag{15.9}$$

Proof Let $k \geq 1$ and consider the maps

$$\Omega^{\bullet}_{nc}(g) \to \Omega^{\bullet}_{nc}(g+2k) \to H^{\bullet}(\mathrm{GL}_{g+2k}(\mathbb{Z});\mathbb{R}) \to H^{\bullet}(\mathrm{GL}_{g+2k-1}(\mathbb{Z});\mathbb{R})$$
$$\to H^{\bullet}(\mathrm{GL}_g(\mathbb{Z});\mathbb{R})$$

where the last two maps $H^{\bullet}(\mathrm{GL}_{g+m}(\mathbb{Z});\mathbb{R}) \to H^{\bullet}(\mathrm{GL}_g(\mathbb{Z});\mathbb{R})$ are induced by the stabilisation map $X \mapsto X \oplus I_m$, where I_m is the identity matrix of rank m. The composition of all the maps in the above is the natural map $\Omega^{\bullet}_{nc}(g) \to H^{\bullet}(\mathrm{GL}_g(\mathbb{Z});\mathbb{R})$, since $\omega_{X\oplus I_m} = \omega_X$ for any canonical form ω. We have shown it is injective, by (15.8). Therefore $\Omega^{\bullet}_{nc}(g)$ injects into all intermediate spaces, and hence into $H^{\bullet}(\mathrm{GL}_h(\mathbb{Z});\mathbb{R})$ for all $h \geq g$. $\square$

Remark 15.7 A brief discussion of the history of this result is in the introduction. Note that the previous two corollaries are strictly weaker than Theorem 15.4.

15.3 Compactly supported cohomology

The following proposition is essentially a restatement, in a different form, of results of [15].

Proposition 15.8 *For all $n, g > 1$, there are short exact sequences:*

$$0 \longrightarrow H^{n-1}_{\mathrm{dR}}\left(\left|L\mathcal{A}^{\mathrm{trop}}_{g-1}\right|\right) \longrightarrow H^n_{\mathrm{dR},c}\left(\left|L\mathcal{A}^{\circ,\mathrm{trop}}_g\right|\right) \longrightarrow H^n_{\mathrm{dR}}\left(\left|L\mathcal{A}^{\mathrm{trop}}_g\right|\right) \longrightarrow 0$$

Proof Consider the long exact relative de Rham cohomology sequence:

$$\cdots \longrightarrow H^{n-1}_{\mathrm{dR}}\left(\left|\partial L\mathcal{A}_g^{\mathrm{trop}}\right|\right) \longrightarrow H^n_{\mathrm{dR}}\left(\left|L\mathcal{A}_g^{\mathrm{trop}}\right|, \left|\partial L\mathcal{A}_g^{\mathrm{trop}}\right|\right)$$
$$\longrightarrow H^n_{\mathrm{dR}}\left(\left|L\mathcal{A}_g^{\mathrm{trop}}\right|\right) \longrightarrow \cdots .$$

The statement follows on interpreting the relative cohomology group in the middle as compactly supported cohomology (3.9); from the homeomorphism $|\partial L\mathcal{A}_g^{\mathrm{trop}}| \cong |L\mathcal{A}_{g-1}^{\mathrm{trop}}|$, which follows from [15, Corollary 4.12]; and from the fact that the restriction maps $H^n_{\mathrm{dR}}(|L\mathcal{A}_g^{\mathrm{trop}}|) \to H^n_{\mathrm{dR}}(|\partial L\mathcal{A}_g^{\mathrm{trop}}|)$ are zero. This last fact is dual to the statement that the inclusion of the boundary $|\partial L\mathcal{A}_g^{\mathrm{trop}}| \hookrightarrow |L\mathcal{A}_g^{\mathrm{trop}}|$ induces the zero map in homology $H_n(|\partial L\mathcal{A}_g^{\mathrm{trop}}|; \mathbb{R}) \to H_n(|L\mathcal{A}_g^{\mathrm{trop}}|; \mathbb{R})$. The reason for this is that the image of a chain in the boundary $|\partial L\mathcal{A}_g^{\mathrm{trop}}|$ only involves cones σ_Q which are equivalent to the cone of a positive definite quadratic form of rank strictly less than g, and therefore lies in the inflation subcomplex $I^{(g)}$ of the perfect cone complex $P^{(g)}$. Since the inflation complex is acyclic, by [15, Theorem 5.15], any such chain is trivial in homology. □

Corollary 15.9 *Let $g > 1$ be odd. Then there is a natural injective map*

$$\Omega_c^\bullet(g) \hookrightarrow H_c^{\bullet+2}(\mathrm{GL}_{g+1}(\mathbb{Z}); \mathbb{R}) . \qquad (15.10)$$

Proof By Proposition 15.8, there is an injective map

$$H^\bullet_{\mathrm{dR}}\left(\left|L\mathcal{A}_g^{\mathrm{trop}}\right|\right) \longrightarrow H^{\bullet+1}_{\mathrm{dR},c}(|L\mathcal{A}_{g+1}^{\circ,\mathrm{trop}}|) \cong H^{\bullet+1}_{\mathrm{dR},c}(L\mathcal{P}_{g+1}/\mathrm{GL}_{g+1}(\mathbb{Z}))$$

The result follows from Theorem 15.4 and (15.7). □

15.4 Unstable cohomology of $\mathrm{SL}_g(\mathbb{Z})$

The following lemma is standard [31].

Lemma 15.10 *Let $n \geq 0$. Then*

$$H^n(\mathrm{SL}_g(\mathbb{Z}); \mathbb{R}) \cong \begin{cases} H^n(\mathrm{GL}_g(\mathbb{Z}); \mathbb{R}) & \textit{if } g \textit{ is odd} \\ H^n(\mathrm{GL}_g(\mathbb{Z}); \mathbb{R}) \oplus H_c^{d_g-n}(\mathrm{GL}_g(\mathbb{Z}); \mathbb{R})^\vee & \textit{if } g \textit{ is even} \end{cases}$$

Proof Consider the short exact sequence

$$1 \longrightarrow \mathrm{SL}_g(\mathbb{Z}) \longrightarrow \mathrm{GL}_g(\mathbb{Z}) \xrightarrow{\det} \mathrm{GL}_1(\mathbb{Z}) \longrightarrow 1$$

where $\mathrm{GL}_1(\mathbb{Z}) = \mathbb{Z}^\times \cong \mathbb{Z}/2\mathbb{Z}$. When g is odd, it splits: in fact, since the diagonal matrix $-I_g$ of rank g is central, $\mathrm{GL}_g(\mathbb{Z})$ is isomorphic to the direct product $\mathrm{SL}_g(\mathbb{Z}) \times \mathbb{Z}/2\mathbb{Z}$. Since $\mathbb{Z}/2\mathbb{Z}$ is torsion we deduce that $H^n(\mathrm{SL}_g(\mathbb{Z}); \mathbb{Q}) \cong H^n(\mathrm{GL}_g(\mathbb{Z}); \mathbb{Q})$ when g is odd. Now suppose that g is even, and consider the rank one $\mathbb{R}$-vector space

$\mathbb{R}^{\det}$ equipped with the action of $g \in \mathrm{GL}_g(\mathbb{Z})$ given by $g.v = \det(g)v$. By Shapiro's lemma one has:

$$H^n(\mathrm{SL}_g(\mathbb{Z}); \mathbb{R}) \cong H^n(\mathrm{GL}_g(\mathbb{Z}); \mathrm{Ind}_{\mathrm{SL}_g(\mathbb{Z})}^{\mathrm{GL}_g(\mathbb{Z})} \mathbb{R}) \cong H^n(\mathrm{GL}_g(\mathbb{Z}); \mathbb{R} \oplus \mathbb{R}^{\det}) .$$

Denote by $\mathcal{O}$ the orientation bundle on $L\mathcal{P}_g$, which is non-trivial when g is even. We have

$$H^n(\mathrm{GL}_g(\mathbb{Z}); \mathbb{R}^{\det}) = H^n(\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z}); \mathcal{O}) \cong H_c^{d_g-n}(\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z}); \mathbb{R})^\vee$$

where the first equality follows from the argument of [31, Lemma 7.2] and the second for Poincaré duality for non-orientable homology manifolds. □

The previous lemma and (15.10) imply the following result on the cohomology of $\mathrm{SL}_g(\mathbb{Z})$. Choose a surjective map $s : H_c^{\bullet+2}(\mathrm{GL}_{g+1}(\mathbb{Z}); \mathbb{R}) \to \Omega_c^\bullet(g) \otimes_{\mathbb{Q}} \mathbb{R}$ which is a section of the injective map $\Omega_c^\bullet(g) \otimes_{\mathbb{Q}} \mathbb{R} \hookrightarrow H_c^{\bullet+2}(\mathrm{GL}_{g+1}(\mathbb{Z}); \mathbb{R})$ equivalent to (15.10).

Corollary 15.11 *Let $g > 1$ be odd and $n \geq 0$. Then*

$$\Omega_{nc}^n(g) \otimes_{\mathbb{Q}} \mathbb{R} \hookrightarrow H^n(\mathrm{SL}_g(\mathbb{Z}); \mathbb{R})$$

$$\left(\Omega_{nc}^n(g) \otimes_{\mathbb{Q}} \mathbb{R}\right) \oplus \left(\Omega_c^{d_{g+1}-n-2}(g) \otimes_{\mathbb{Q}} \mathbb{R}\right)^\vee \hookrightarrow H^n(\mathrm{SL}_{g+1}(\mathbb{Z}); \mathbb{R})$$

The first map is natural. The second map depends on the choice of section s: it is given by its dual $s^\vee$ on the second component $(\Omega_c^{d_{g+1}-n-2}(g) \otimes \mathbb{R})^\vee$.

If we identify $(\Omega_c^{d_g-n-1}(g))^\vee$ with $\Omega_{nc}^n(g)$ via the Hodge star operator, the second map in the corollary may be written (non-canonically) in the more convenient form

$$\left(\Omega_{nc}^n(g) \oplus \Omega_{nc}^{n-g}(g)\right) \otimes_{\mathbb{Q}} \mathbb{R} \hookrightarrow H^n(\mathrm{SL}_{g+1}(\mathbb{Z}); \mathbb{R}) .$$

Examples 15.12 For $g = 3$, we have $\Omega_{nc}(3) = \mathbb{Q}$ and $\Omega_c(3) = \omega^5\mathbb{Q}$. The previous corollary produces classes in $H^0(\mathrm{SL}_3(\mathbb{Z}); \mathbb{R})$, as well as $H^0(\mathrm{SL}_4(\mathbb{Z}); \mathbb{R})$, $H^3(\mathrm{SL}_4(\mathbb{Z}); \mathbb{R})$.

For $g = 5$, we have forms of non-compact type $\Omega_{nc}(5) = \mathbb{Q} \oplus \mathbb{Q}\omega^5$. They give rise, via the previous corollary, to classes in $H^0(\mathrm{SL}_5(\mathbb{Z}); \mathbb{R})$ and $H^5(\mathrm{SL}_5(\mathbb{Z}); \mathbb{R})$, and also $H^0(\mathrm{SL}_6(\mathbb{Z}); \mathbb{R})$ and $H^5(\mathrm{SL}_6(\mathbb{Z}); \mathbb{R})$. The forms of compact type $\Omega_c(5) = \omega^9\mathbb{Q} \oplus (\omega^5 \wedge \omega^9)\mathbb{Q}$ give rise to two further classes in $H^5(\mathrm{SL}_6(\mathbb{Z}); \mathbb{R})$ and $H^{10}(\mathrm{SL}_6(\mathbb{Z}); \mathbb{R})$.

Remark 15.13 A natural way to phrase the corollary is to say that there is a complex

$$\Omega_{nc}^n(g) \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow H^n(\mathrm{SL}_{g+1}(\mathbb{Z}); \mathbb{R}) \longrightarrow \left(\Omega_c^{d_{g+1}-n-2}(g) \otimes_{\mathbb{Q}} \mathbb{R}\right)^\vee$$

where the first map is injective, and the second map is surjective.

15.5 Stable range

Let $g > 1$ be odd. By Borel [11], rephrased in terms of forms of non-compact type, there exists a $\kappa(g)$ which grows linearly in g such that

$$\Omega^n_{nc}(g) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} H^n(\mathrm{GL}_g(\mathbb{Z}); \mathbb{R}) \quad \text{for all } n < \kappa(g) \ . \tag{15.11}$$

Borel showed that $\kappa(g)$ is of the order of $g/4$. Using the recent work [40, Theorem A], [43], and (15.9), it may be improved to $\kappa(g) = g$. For n in this range, $\Omega^n_{nc}(g) = \Omega^n(g)$.

Corollary 15.14 *Let $g > 1$ be odd and let $\kappa(g)$ be such that (15.11) holds. Then*

$$H^n_{\mathrm{dR}}\left(\left|L\mathcal{A}^{\mathrm{trop}}_g\right|\right) \cong \Omega^n_c(g) \otimes_{\mathbb{Q}} \mathbb{R}$$
$$H^{n-1}_{\mathrm{dR}}\left(\left|L\mathcal{A}^{\mathrm{trop}}_{g-1}\right|\right) = 0 \ ,$$

for all $n \geq d_g - \kappa(g)$.

Proof Let $g > 1$ be odd. Since $L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})$ is orientable of dimension $\ell_g = d_g - 1$, Poincaré duality implies that $H^n_c(L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z}); \mathbb{R}) \cong H^{\ell_g - n}(L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z}); \mathbb{R})^\vee$ and so we have

$$\dim_{\mathbb{R}} H^n_c(L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z}); \mathbb{R}) = \dim_{\mathbb{R}} H^{\ell_g - n}(L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z}); \mathbb{R}) \overset{(15.11)}{=} \dim_{\mathbb{Q}} \Omega^{\ell_g - n}_{nc}(g)$$

where the last equality holds for $n > \ell_g - \kappa(g)$. In (15.8) we established an injection

$$\Omega^n_c(g) \otimes_{\mathbb{Q}} \mathbb{R} \hookrightarrow H^n_c(L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})) \tag{15.12}$$

for all n. The Hodge star operator formalism implies that $\dim_{\mathbb{Q}} \Omega^n_c(g) = \dim_{\mathbb{Q}} \Omega^{\ell_g - n}_{nc}(g)$ for all n, and hence if $n > \ell_g - \kappa(g)$, we have equality of dimensions in (15.12). In summary,

$$\Omega^n_c(g) \otimes_{\mathbb{Q}} \mathbb{R} \cong H^n_c(L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z}); \mathbb{R}) \quad \text{for } n > \ell_g - \kappa(g) \ .$$

Since $|L\mathcal{A}^{\circ,\mathrm{trop}}_g| \cong L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})$ we deduce that $H^n_{\mathrm{dR},c}(|L\mathcal{A}^{\circ,\mathrm{trop}}_g|) \cong \Omega^n_c(g) \otimes_{\mathbb{Q}} \mathbb{R}$ for n in this range, and the exact sequence of Proposition 15.8 becomes

$$0 \longrightarrow H^{n-1}_{\mathrm{dR}}(|L\mathcal{A}^{\mathrm{trop}}_{g-1}|) \longrightarrow \Omega^n_c(g) \otimes_{\mathbb{Q}} \mathbb{R} \overset{(*)}{\longrightarrow} H^n_{\mathrm{dR}}(|L\mathcal{A}^{\mathrm{trop}}_g|) \longrightarrow 0.$$

By Theorem 15.4, the map $\Omega^n_c(g) \otimes_{\mathbb{Q}} \mathbb{R} \to H^n_{\mathrm{dR}}(|L\mathcal{A}^{\mathrm{trop}}_g|)$ is injective, hence $(*)$ is an isomorphism, giving the first statement. Its kernel vanishes, which gives the second. □

15.6 Canonical forms and the tropical Torelli map

The composition

$$\Omega^n_c(g)\otimes_{\mathbb{Q}}\mathbb{R}\longrightarrow H^n_{\mathrm{dR}}\left(|L\mathcal{A}^{\mathrm{trop}}_g|\right)\xrightarrow{\lambda^*}H^n_{\mathrm{dR}}\left(|L\mathcal{M}^{\mathrm{trop}}_g|\right)$$

sends the majority of forms to zero. Indeed, if $\eta\in\Omega^d(g)$ has degree d then

$$\lambda^*(\omega^{2g-1}\wedge\eta)=0\quad\text{if}\quad d>g-2\,. \tag{15.13}$$

This is because the left-hand side restricts to the form $\omega_G^{2g-1}\wedge\eta_G$ in the notation of [19] on the simplex σ_G associated to a graph G. By [19, Corollary 6.19], it vanishes if its degree is greater than $3g-3$. It might be possible to reduce the degree d in (15.13) by a more detailed analysis of vanishing of graph forms [19, §6.5]. Evidence suggests that $\lambda^*(\omega)$ is non-zero if and only if ω is primitive, or more precisely, that the image of

$$\lambda^*:H^n_{\mathrm{dR}}(L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z}))\longrightarrow H^n_{\mathrm{dR}}\left(\left|L\mathcal{M}^{\mathrm{trop}}_g\right|\right)$$

is one-dimensional: $\mathrm{Im}(\lambda^*)=\mathbb{Q}[\omega^{2g-1}]$. The results of [20] imply that the image of λ^* contains $\mathbb{Q}[\omega^{2g-1}]$ and is dual to the class of the wheel with g spokes.

Remark 15.15 Nevertheless, a conjecture in *loc. cit.* states that the free Lie algebra embeds:

$$\mathrm{Lie}\,(\Omega^{\mathrm{can}})\hookrightarrow\bigoplus_{g\geq 1}H^\bullet\left(|L\mathcal{M}^{\mathrm{trop}}_g|\right)\,.$$

It is corroborated by all known evidence in low degrees, and suggests that all canonical forms do appear in the cohomology of $H^\bullet_{dR}\left(|L\mathcal{M}^{\mathrm{trop}}_g|\right)$, but for a higher than expected genus.

16 Periods

We work in a suitable neutral Tannakian category $\mathcal{H}_{\mathbb{Q}}$ of motivic realisations over $\mathbb{Q}$. There are many possible variants [30]. We assume that it has two fiber functors

$$M\mapsto M_{\mathrm{B}},M_{\mathrm{dR}}\;:\;\mathcal{H}_{\mathbb{Q}}\longrightarrow\mathrm{Vec}_{\mathbb{Q}}$$

and that there is a canonical element $\mathrm{comp}_{\mathrm{B,dR}}\in\mathrm{Isom}^{\otimes}_{\mathcal{H}_{\mathbb{Q}}}(\omega_{\mathrm{dR}},\omega_{\mathrm{B}})(\mathbb{C})$. A convenient option is to let $\mathcal{H}_{\mathbb{Q}}$ be the category of mixed Hodge structures with additional $\mathbb{Q}$-de Rham structure studied in [18]. The element $\mathrm{comp}_{\mathrm{B,dR}}$ defines an isomorphism

$$M_{\mathrm{dR}}\otimes_{\mathbb{Q}}\mathbb{C}\xrightarrow{\sim}M_{\mathrm{B}}\otimes_{\mathbb{Q}}\mathbb{C}\,,$$

which is natural in M, which may be interpreted as a period pairing $M_{\mathrm{dR}} \otimes_{\mathbb{Q}} M_{\mathrm{B}}^{\vee} \to \mathbb{C}$.

Consider the affine group scheme:

$$G_{\mathrm{dR}} = \mathrm{Aut}^{\otimes}_{\mathcal{H}_{\mathbb{Q}}}(\omega_{\mathrm{dR}})$$

given by the automorphisms of the fiber functor $\omega_{\mathrm{dR}} : M \mapsto M_{\mathrm{dR}}$. By the Tannaka theorem [51], the fiber functor ω_{dR} defines an equivalence of categories:

$$\omega_{\mathrm{dR}} : \mathcal{H}_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Rep}\, G_{\mathrm{dR}} \ .$$

16.1 'Motives' of quadratic forms

Definition 16.1 Let Q be a positive definite quadratic form on V such that $\sigma_Q > 0$. Recall that $\mathrm{c}^{\mathcal{B}}_Q = (P^{\mathcal{B}_Q}, L^{\mathcal{B}_Q}, \sigma^{\mathcal{B}}_Q)$. Let $d = \dim \sigma_Q$. Define

$$\mathrm{mot}_Q = H^d\left(P^{\mathcal{B}_Q} \backslash \widetilde{\mathrm{Det}}\ ,\ L^{\mathcal{B}_Q} \backslash (L^{\mathcal{B}_Q} \cap \widetilde{\mathrm{Det}})\right) \ . \tag{16.1}$$

By Corollary 11.5, we have $[\sigma^{\mathcal{B}}_Q] \in (\mathrm{mot}_Q)^{\vee}_{\mathrm{B}}$. Any canonical form $\omega \in \Omega^d_{\mathrm{can}}$ defines a class $[\omega] \in (\mathrm{mot}_Q)_{\mathrm{dR}}$, and so we may define the 'motivic period':

$$I^{\mathfrak{m}}_Q(\omega) = [\mathrm{mot}_Q, [\sigma_Q], [\omega]]^{\mathfrak{m}} \quad \in \quad \mathcal{O}(\mathrm{Isom}^{\otimes}_{\mathcal{H}_{\mathbb{Q}}}(\omega_{\mathrm{dR}}, \omega_{\mathrm{B}})) \ . \tag{16.2}$$

Note that for cones σ_Q which are contained in the determinant locus, [15, Lemma 4.9] implies that they are equivalent to the cone of a quadratic form of smaller rank, so Definition 16.1 in fact covers all cases. The period of $I^{\mathfrak{m}}_Q(\omega)$ is the convergent integral

$$\mathrm{per}\left(I^{\mathfrak{m}}_Q(\omega)\right) = \int_{\sigma_Q} \omega$$

which is finite by Corollary 14.12. Furthermore, the relation of Theorem 14.13 is motivic: it holds verbatim for the objects $I^{\mathfrak{m}}_Q(\omega)$. In the case when $Q = Q_G$ comes from a graph, the object $I^{\mathfrak{m}}_{Q_G}(\omega)$ is equal to the 'canonical motivic Feynman integral' $I^{\mathfrak{m}}_G(\omega)$ defined in [19].

Remark 16.2 Every isomorphism $[Q] \cong [Q']$ in the category $\mathcal{D}^{\mathrm{perf},\mathcal{B}}$ gives rise to an isomorphism $\mathrm{mot}_Q \cong \mathrm{mot}'_Q$. Similarly, face diagrams give rise to a diagram of morphisms in the category $\mathcal{H}_{\mathbb{Q}}$ (see [19] for the case of graphs), but extraneous modifications are *not* necessarily isomorphisms (although they do induce equivalences of motivic periods (16.2)).

Example 16.3 We refer to [19] for many examples of graphs for which $I_G(\omega)$ are known. The most interesting was computed by Borinsky and Schnetz [13]:

$$I_{K_6}(\omega^5 \wedge \omega^9) = 3\,\frac{10!}{16}\,(12\,\zeta(3,5) - 29\,\zeta(8) + 23\,\zeta(3)\zeta(5)) \ . \tag{16.3}$$

It was proven in [19, lemmas 3.7, 6.8] that the canonical integrals are stable under duality, which implies that the volume of the graphic cell $\sigma_{K_6^\vee}$ where $K_6^\vee$ is the matroid dual to K_6, is given by the same integral. It is the 'principal cone' [15, Def. 2.13] in the fundamental domain $\mathcal{F}_5$ for $\mathrm{GL}_5(\mathbb{Z})$. By (15.1) the volume $\int_{\mathcal{F}_5} \omega^5 \wedge \omega^9$ is proportional to a product $\zeta(3)\zeta(5)$. The appearance, therefore, of the non-trivial multiple zeta value $\zeta(3,5)$ in (16.3) suggests that the principal cone does not span the fundamental domain: $\sigma_{K_6^\vee} \subsetneq \mathcal{F}_5$ and that the tropical Torelli map is not surjective. Indeed, it is known (e.g. [31]) that the fundamental domain involves two further cones. We expect from (16.3) that their volumes, which are not yet known, will involve non-trivial multiple zeta values.

As noted in [19, §10.3.1], the particular linear combination $12\,\zeta(3,5) - 29\,\zeta(8)$ is the same one which occurs in quantum field theory as the Feynman residue of $K_{3,4}$. It is a very striking fact that $\zeta(3,5)$ is only ever observed in this particular combination, and never in isolation, amongst the vast array of periods in quantum field theory [52]. This surprising phenomenon is one particular consequence of the 'Cosmic' Galois group. The geometric interpretation (16.3) of this linear combination points to an unexplored and deep connection between quantum fields and the reduction theory of quadratic forms.

16.2 Locally finite homology 'motive' of $\mathrm{GL}_g(\mathbb{Z})$

Let $\mathcal{H}_\mathbb{Q}$, as above, be a Tannakian category of realisations which contains every object mot_Q (16.1).

Consider the cohomology

$$H_c^d(g) = H^d\left(\ker\left(\Omega^\bullet\left(L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}} \setminus \widetilde{\mathrm{Det}}\right)\right.\right.$$
$$\left.\left.\longrightarrow\ \Omega^\bullet\left(\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}} \setminus (\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}} \cap \widetilde{\mathrm{Det}})\right)\right)\right)$$

of the complex of compatible systems (Definition 2.8) of algebraic forms on $L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}} \backslash \widetilde{\mathrm{Det}}$ whose restriction to the boundary $\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}$ vanishes. There is a natural map

$$H_c^d(g) \longrightarrow H_{\mathrm{dR}}^d\left(\left|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}\right|\ ,\ \left|\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}\right|\right) \cong H_c^d(L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z});\mathbb{R})$$

via which elements of $H_c^d(g)$ may be interpreted as smooth compactly supported differential forms on $L\mathcal{P}_g/\mathrm{GL}_g(\mathbb{Z})$. There is a natural map $\Omega_c^d(g) \to H_c^d(g)$ (which is injective).

Theorem 16.4 *For every $g > 1$, and $d \geq 0$, there exists a minimal object M_g^d of $\mathcal{H}_\mathbb{Q}$ which is equipped with a pair of canonical linear maps*

$$H_d^{\mathrm{lf}}(\mathrm{GL}_g(\mathbb{Z});\mathbb{Q}) \longrightarrow (\mathsf{M}_g^d)_\mathrm{B}^\vee$$
$$H_c^d(g) \longrightarrow (\mathsf{M}_g^d)_\mathrm{dR}$$

such that the integration pairing

$$H_d^{\mathrm{lf}}(\mathrm{GL}_g(\mathbb{Z});\mathbb{Q}) \otimes_{\mathbb{Q}} H_c^d(g) \longrightarrow \mathbb{C}$$

factors through the period pairing: $\langle\ ,\ \rangle : (\mathsf{M}_g^d)_{\mathrm{B}}^{\vee} \otimes_{\mathbb{Q}} (\mathsf{M}_g^d)_{\mathrm{dR}} \to \mathbb{C}$.

Proof First consider the object of $\mathcal{H}_{\mathbb{Q}}$ defined by:

$$M_g^d = \bigoplus_{\dim \sigma_Q = d} \mathrm{mot}_Q^{\mathrm{Aut}(\sigma_Q)} ,$$

where the sum is over isomorphism classes of cones σ_Q of dimension d. Consider the subspace $Z_g^d \subset \Omega^d(L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}} \backslash \widetilde{\mathrm{Det}})$ consisting of compatible systems of closed forms whose restriction to $\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}$ vanishes, and let $B_g^d \subset Z_g^d$ be the subspace spanned by $d\omega$, where ω are compatible systems of forms vanishing on $\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}$. There is a natural map $Z_g^d \to (M_g^d)_{\mathrm{dR}}$. By the Tannaka theorem, there are unique subobjects $\langle B_g^d \rangle \subset \langle Z_g^d \rangle \subset M_g^d$ such that

$$\langle Z_g^d \rangle_{\mathrm{dR}} = G_{\mathrm{dR}}\, Z_g^d \qquad \text{and} \qquad \langle B_g^d \rangle_{\mathrm{dR}} = G_{\mathrm{dR}}\, B_g^d\ .$$

Let $N_g^d = \langle Z_g^d \rangle / \langle B_g^d \rangle$. By definition, there is a natural map $H_c^d(g) \to \left(N_g^d\right)_{\mathrm{dR}}$.

By Theorem 11.3, inclusion defines a natural map

$$H_d(\sigma_Q, \partial\sigma_Q) \longrightarrow (\mathrm{mot}_Q)_B^{\vee} = H_d\left((P^{\mathcal{B}_Q} \backslash \widetilde{\mathrm{Det}})(\mathbb{C})\ ,\ L^{\mathcal{B}_Q} \backslash (L^{\mathcal{B}_Q} \cap \widetilde{\mathrm{Det}})(\mathbb{C}); \mathbb{Q}\right)$$

which induces a map

$$\bigoplus_{\dim \sigma_Q = d} H_d(\sigma_Q, \partial\sigma_Q)/\mathrm{Aut}(\sigma_Q) \longrightarrow (M_g^d)_{\mathrm{B}}^{\vee} ,$$

where the direct sum is over all isomorphism classes of cones σ_Q of dimension d. Via the cellular interpretation of homology (Theorem 3.2), we deduce a map from closed cellular chains γ in $L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}$ to $(M_g^d)_{\mathrm{B}}^{\vee}$ and thence to its quotient $\langle Z_g^d \rangle_{\mathrm{B}}^{\vee}$. Since Stokes' theorem (14.9) holds for motivic periods, the period pairing $\langle \gamma, g\omega \rangle$ vanishes for all $g \in G_{\mathrm{dR}}$ if $\omega \in B_g^d$; and for all $g \in G_{\mathrm{dR}}$ and $\omega \in Z_g^d$ if γ is supported on $\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}$, or if $\gamma = \partial\alpha$ is a boundary. This proves, at the same time, that the image of a cellular chain γ lands in the subspace $(\langle Z_g^d \rangle / \langle B_g^d \rangle)_{\mathrm{B}}^{\vee}$ of $\langle Z_g^d \rangle_{\mathrm{B}}^{\vee}$, and that it is zero if γ is a boundary, or supported on $\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}$. Thus by Theorem 3.6, there is a well-defined map

$$H_d\left(\left|L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}\right|\ ,\ \left|\partial L\mathcal{A}_g^{\mathrm{trop},\mathcal{B}}\right|\right) \longrightarrow (N_g^d)_{\mathrm{B}}^{\vee}\ .$$

The left-hand group is isomorphic to the locally finite homology $H_d^{\mathrm{lf}}(\mathrm{GL}_g(\mathbb{Z});\mathbb{Q})$, and so the object N_g^d satisfies the conditions of the theorem. This proves the existence.

For the minimality, one can repeat the argument of [18, §2.4], to show that any object M of $\mathcal{H}_{\mathbb{Q}}$ equipped with maps $V \hookrightarrow M_{\mathrm{B}}^{\vee}$ and $W \hookrightarrow M_{\mathrm{dR}}$, where V, W are $\mathbb{Q}$-vector spaces, has a unique smallest subquotient with the this property (the case when V, and W have dimension 1 is proven in [18, §2.4], but the general case is similar). $\square$

It follows from (15.8) and the relative de Rham Theorem 3.6 that the composition

$$\Omega_c^d(g) \longrightarrow H_c^d(g) \longrightarrow (\mathsf{M}_g^d)_{\mathrm{dR}}$$

is injective. Theorem 16.4 implies that any integral of a canonical form of compact type over a locally finite homology class in $H_d^{\mathrm{lf}}(\mathrm{GL}_g(\mathbb{Z}))$ is motivic. In particular, we deduce a motivic interpretation of Minkowski's volume integrals (15.1).

Remark 16.5 An interesting question is whether the map $H_d^{\mathrm{lf}}(\mathrm{GL}_g(\mathbb{Z}); \mathbb{Q}) \to (\mathsf{M}_g^d)_{\mathrm{B}}^{\vee}$ is injective or not: is it the case that the entire locally finite homology of $\mathrm{GL}_g(\mathbb{Z})$ is motivic in this sense, or only a quotient of it? (One could capture the entire cohomology by enlarging $H_c^d(g)$, for instance by replacing the determinant locus with suitable linear subspaces, since every cohomology class can be represented by a compatible family of algebraic forms by Remark 3.4.) A deeper question is to describe the action of G_{dR} on $(\mathsf{M}_g^d)_{\mathrm{dR}}$. See [19, §9.5] for a related discussion for motives arising from the moduli space of tropical curves.

Example 16.6 In the case $g = 3$, $d = 5$, the map

$$\mathbb{Q} \cong H_5^{\mathrm{lf}}(\mathrm{GL}_3(\mathbb{Z}); \mathbb{Q}) \longrightarrow (M_3^5)_B^{\vee}$$

is injective. In fact, M_3^5 is given by the graph 'motive' [10] of the wheel with 3 spokes W_3, which, one can show, is of rank two, and is a non-trivial extension of $\mathbb{Q}(-3)$ by $\mathbb{Q}(0)$. The space $\mathbb{Q}(-3)_{dR}$ is spanned by the class of the canonical form ω^5; the group $H_5^{\mathrm{lf}}(\mathrm{GL}_3(\mathbb{Z}); \mathbb{Q})$ is spanned by the fundamental class which is the relative homology class of $\sigma_{W_3}^{\mathcal{B}}$. The corresponding period is the 'wheel integral' and equals $60\zeta(3)$.

Appendix: Blow-ups and Borel-Serre for GL_n

In this Appendix, we prove that our blow-up construction can be used to retrieve the Borel-Serre compactification. In view of our applications, we have focused on the case $\mathrm{GL}_n(\mathbb{Z})$, but many of the arguments presented here are quite general. This section is mostly self-contained, although one or two proofs require statements in the main text.

A.1 Statements

Let V be a vector space of dimension n over $\mathbb{Q}$. Let $\mathcal{Q}(V)$ denote the space of quadratic forms on V and $\mathbb{P}(\mathcal{Q}(V))$ the associated projective space. Let

$$X \quad \subset \quad \mathbb{P}(\mathcal{Q}(V))(\mathbb{R}) \tag{A.1}$$

denote the space of positive definite real quadratic forms. It is contained in the complement of the determinant hypersurface $\mathrm{Det} \subset \mathbb{P}(\mathcal{Q}(V))$. For any $0 \neq K \subset V$, the space $\mathbb{P}(\mathcal{Q}(V/K)) \subset \mathrm{Det}$ is the subspace of quadratic forms with null space K.

Definition A.1 Let $\pi_{\mathfrak{B}} : \mathfrak{B} \to \mathbb{P}(\mathcal{Q}(V))$ denote the space[5] obtained by blowing up all (infinitely many) linear subspaces of $\mathbb{P}(\mathcal{Q}(V))$ of the form $\mathbb{P}(\mathcal{Q}(V/K))$ for $0 \neq K \subset V$, in increasing order of dimension. Let $\widetilde{\mathrm{Det}} \subset \mathfrak{B}$ denote the strict transform of Det.

Let X^{BS} denote the Borel-Serre compactification of X. It has a stratification

$$X^{\mathrm{BS}} = X \cup \bigcup_P e(P)$$

where P ranges over rational parabolics (see §A.2). The exceptional divisor $\mathcal{E}$ for $\pi_{\mathfrak{B}}$ defines a stratification on $\mathfrak{B}$ by taking intersections of irreducible components. Since $\pi_{\mathfrak{B}}(\mathcal{E}) \subset \mathrm{Det}$, the inclusion (A.1) induces an injective map:

$$X \longrightarrow (\mathbb{P}(\mathcal{Q}(V)) \setminus \mathrm{Det})\,(\mathbb{R}) \xrightarrow{\pi_{\mathfrak{B}}^{-1}} (\mathfrak{B} \setminus \mathcal{E})\,(\mathbb{R})\,. \tag{A.2}$$

Theorem A.2 *There is a continuous injective map of stratified spaces*

$$f : X^{\mathrm{BS}} \longrightarrow \left(\mathfrak{B} \ \setminus \ \widetilde{\mathrm{Det}}\right)(\mathbb{R}) \tag{A.3}$$

whose restriction to the big open stratum is the embedding (A.2). *The image of* (A.3) *is the closure of X, for the analytic topology, inside $\mathfrak{B}(\mathbb{R})$. The map f is equivariant with respect to the natural actions of* $\mathrm{GL}(V)$ *on both X^{BS} and $\mathfrak{B}$.*

Let P denote the rational parabolic associated to a nested sequence of strict subspaces $0 \subset V_d \subset V_{d-1} \subset \cdots \subset V_1 \subset V$. *The map* (A.3) *restricts to a map*

$$f : e(P) \longrightarrow \mathcal{E}_P \setminus \left(\widetilde{\mathrm{Det}} \cap \mathcal{E}_P\right)(\mathbb{R}) \tag{A.4}$$

where $\mathcal{E}_P \subset \mathfrak{B}$ is the exceptional locus associated to the iterated blow-up of

$$\mathbb{P}(\mathcal{Q}(V/V_1)) \subset \cdots \subset \mathbb{P}(\mathcal{Q}(V/V_d))$$

in increasing order of dimension. In particular, $f(e(P)) = f(X^{\mathrm{BS}}) \cap \mathcal{E}_P(\mathbb{R})$. The complement of the strict transform of Det *is canonically isomorphic to a product of hypersurface complements*:

$$\mathcal{E}_P \setminus \left(\widetilde{\mathrm{Det}} \cap \mathcal{E}_P\right) \cong \prod_{k=0}^{d} \left(\mathbb{P}\left(\frac{\mathcal{Q}(V/V_{k+1})}{\mathcal{Q}(V/V_k)}\right) \ \setminus \ \mathrm{Det}\big|_{V_k/V_{k+1}}\right)$$

where we write $V_0 = V$ and $V_{d+1} = 0$, and $\mathrm{Det}|_{V_k/V_{k+1}}$ is the vanishing locus of the determinant of the restriction of a quadratic form on V/V_{k+1} to the subspace V_k/V_{k+1}.

[5]We shall only consider the set of real points $\mathfrak{B}(\mathbb{R})$, viewed as a topological space with a stratification by exceptional divisors.

In short, the space $\mathfrak{B}$ is an algebraic incarnation of the Borel-Serre compactification with identical combinatorial structure. The space X^{BS} is identified with a semi-algebraic subset of its real points $\mathfrak{B}(\mathbb{R})$. It is defined by algebraic inequalities of the form $u \geq 0$, where u is a homogeneous polynomial given by the determinant of a matrix minor. Note that although $\mathfrak{B}$ is defined by infinitely many blow-ups, the local structure of X^{BS} in terms of spaces $X(P)$ (see §A.2) may be studied by embedding each $X(P)$ into a space obtained by performing only finitely many linear blow-ups of $\mathbb{P}(\mathcal{Q}(V))$. This is how we shall prove Theorem A.2.

Corollary A.3 *Let $|L\mathcal{A}_n|^{\mathrm{trop},\mathfrak{B}}$ be the topological space defined in* (11.5). *Then the there is a canonical homeomorphism*

$$f : X^{\mathrm{BS}}/\mathrm{GL}_n(\mathbb{Z}) \xrightarrow{\sim} |L\mathcal{A}_n|^{\mathrm{trop},\mathfrak{B}} \ .$$

A.2 Reminders on the Borel-Serre compactifcation for $\mathbf{GL}_n$

Let V be a vector space of dimension n over $\mathbb{R}$ equipped with an inner product, or equivalently, an isomorphism $V \cong V^{\vee}$. The group of automorphisms $\mathrm{GL}(V)$ is isomorphic to $\mathrm{GL}_n(\mathbb{R})$, and the subgroup preserving the inner product defines a maximal compact subgroup $K = O(V)$ of $\mathrm{GL}(V)$. It is isomorphic to $O_n(\mathbb{R})$. Given any filtration F of V of length d by subspaces

$$F : \qquad 0 = V_{d+1} \subset V_d \subset \cdots \subset V_1 \subset V_0 = V \ , \tag{A.5}$$

where all inclusions are strict, denote by $P_F \leq \mathrm{GL}(V)$ the parabolic subgroup of automorphisms of V which preserve F. A filtration F'

$$F' : \qquad 0 \subset V_{i_k} \subset \cdots \subset V_{i_1} \subset V$$

where $1 \leq i_1 < \cdots < i_k \leq d$ is obtained by omitting elements from the filtration F, is denoted by $F' \leq F$. The set of filtrations F' such that $F' \leq F$ forms a finite poset. Since for $F' \leq F$, one has $P_F \leq P_{F'}$, the set $\{P_{F'} : F' \leq F\}$ with respect to inclusion of groups is isomorphic to the opposite poset.

For any such $P = P_F$, as above, denote by $K_P = P \cap K$. The natural map

$$P \longrightarrow \prod_{i=0}^{d} \mathrm{GL}(V_i/V_{i+1})$$

induces an isomorphism $K_P \cong \prod_{i=0}^{d} O(V_i/V_{i+1})$, the product of orthogonal groups with respect to the induced inner products on $V_i \to V \cong V^{\vee} \to V_i^{\vee}$.

Let $Z_P \leq P$ denote the subgroup of central elements acting via scalar multiplication by $\mathbb{R}^{\times}_{>0}$ on each quotient V_i/V_{i+1} for $0 \leq i \leq d$. Thus $Z_P \cong (\mathbb{R}^{\times}_{>0})^{d+1}$.

Example A.4 Let $e_1, \ldots, e_n$ denote a basis of V compatible with F. In this basis, automorphisms of V may be identified with $n \times n$ matrices, and the groups P and

Z_P are block-lower triangular, and block diagonal respectively:

$$P = \begin{pmatrix} P_{00} & & & \\ P_{01} & P_{11} & & \\ \vdots & & \ddots & \\ P_{0k} & P_{1k} & \dots & P_{kk} \end{pmatrix} \quad , \qquad Z_P = \begin{pmatrix} \mu_0 I_0 & & & \\ & \mu_1 I_1 & & \\ & & \ddots & \\ & & & \mu_k I_k \end{pmatrix}$$

where $k = d$, $I_0, \dots, I_k$ are block identity matrices and $\mu_0, \dots, \mu_k > 0$. The P_{ij} are likewise block matrices. The group K_P is isomorphic to the group of block diagonal matrices where the matrix O_i in the i^{th} block is orthogonal: $O_i^T O_i = I_i$.

The group Z_P will be made to act upon P by multiplication on the left. Denote the group of central elements by $H = Z_P \cap Z(\mathrm{GL}(V)) \cong \mathbb{R}^\times_{>0}$. In Example A.4, $H \le Z_P$ is the subgroup of diagonal matrices. Let X denote the space of projective equivalence classes of real positive definite quadratic forms on V.

A.2.1 Geodesic action

For any $P = P_F$ as above there is a homeomorphism:

$$\begin{aligned} HK_P \setminus P &\xrightarrow{\sim} X \\ g &\mapsto g^T g \,. \end{aligned} \tag{A.6}$$

Since it commutes with K_P and H, the action of Z_P on P passes to a well-defined action on $HK_P \setminus P$. We write $A_P = H \setminus Z_P$. The action of A_P on X induced by (A.6) is called the geodesic action. Borel and Serre consider the following space of orbits for its action:

$$e(P) = A_P \setminus X \overset{(A.6)}{\cong} Z_P K_P \setminus P \,.$$

It is homeomorphic to a Euclidean space $\mathbb{R}^{\binom{n+1}{2}-d-1}$.

The compactification of X is defined as follows. Consider the isomorphism

$$\begin{aligned} H \setminus Z_P \cong A_P &\cong (\mathbb{R}^\times_{>0})^d \\ (\mu_0, \mu_1, \dots, \mu_d) &\mapsto (\lambda_1, \dots, \lambda_d) \end{aligned} \tag{A.7}$$

where $\lambda_i = \mu_i / \mu_{i-1}$ for $1 \le i \le d$. Consider the multiplicative monoid

$$\overline{A}_P = (\mathbb{R}^\times_{\ge 0})^d$$

with coordinates $\lambda_i \ge 0$. The subgroup $A_P \le \overline{A}_P$ has coordinates $\lambda_i > 0$. Borel and Serre define

$$X(P) = A_P \setminus (\overline{A}_P \times X) \,.$$

The space $\overline{A}_P$ is stratified by the closed subspaces $\lambda_i = 0$ for $i = 1, \ldots, d$. The corresponding open stratification is a union of copies of $(\mathbb{R}^{\times}_{>0})^a$ for $a \leq d$, which may be identified with the subgroups $A_{P_{F'}}$, for all $F' \leq F$. Thus

$$X(P) = X \sqcup \coprod_{P'} A_{P'} \setminus X = X \sqcup \coprod_{P'} e(P')$$

is a disjoint union of spaces $e(P')$, where $P' = P_{F'}$ for all $F' \leq F$.

A.2.2 Borel-Serre compactification

Now suppose that V has a $\mathbb{Q}$-structure and that its inner product is defined over $\mathbb{Q}$. A parabolic $P = P_F$ is called rational if the vector spaces V_i in the corresponding sequence (A.5) are all defined over $\mathbb{Q}$.

The Borel-Serre compactification is defined to be

$$X^{\mathrm{BS}} = \bigcup_{P} X(P) \tag{A.8}$$

where the union is over all rational parabolics P.

Example A.5 In the following example we compare and contrast the Borel-Serre approach with our own.

(1) (Borel-Serre). Let $V = \mathbb{R}^3$, $d = 2$, and $V_i \cong \mathbb{R}^{3-i}$, for $0 \leq i \leq 2$. Write

$$P = \begin{pmatrix} p_{00} & 0 & 0 \\ p_{01} & p_{11} & 0 \\ p_{02} & p_{12} & p_{22} \end{pmatrix} \quad , \quad Z = \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix}$$

where $p_{ij} \in \mathbb{R}$ and $\det(P) = p_{00} p_{11} p_{22} \neq 0$. We set

$$X = P^T P = \begin{pmatrix} p_{00}^2 + p_{01}^2 + p_{02}^2 & p_{02} p_{12} + p_{01} p_{11} & p_{02} p_{22} \\ p_{02} p_{12} + p_{01} p_{11} & p_{12}^2 + p_{11}^2 & p_{12} p_{22} \\ p_{02} p_{22} & p_{12} p_{22} & p_{22}^2 \end{pmatrix}$$

Let $X_{ij} = x_{ij}$. The geodesic action $X \mapsto Z \circ X$ where $Z \circ X = P^T Z^T Z P$, may, as one can check, be written intrinsically as a function of X via

$$Z \circ X = \mu_0^2 \begin{pmatrix} \frac{\det(X)}{\det(X^{11})} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mu_1^2 \begin{pmatrix} \frac{\det(X^{21})^2}{\det(X^{11}) x_{33}} & \frac{\det(X^{21})}{x_{33}} & 0 \\ \frac{\det(X^{21})}{x_{33}} & \frac{\det(X^{11})}{x_{33}} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ \mu_2^2 \begin{pmatrix} \frac{x_{13}^2}{x_{33}} & \frac{x_{13} x_{23}}{x_{33}} & x_{13} \\ \frac{x_{13} x_{23}}{x_{33}} & \frac{x_{23}^2}{x_{33}} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{pmatrix}$$

where $\det(X^{11}) = x_{22}x_{33} - x_{23}^2$ and $\det(X^{21}) = x_{12}x_{33} - x_{13}x_{23}$ are determinants of minors of X. The action $X \mapsto Z \circ X$ is clearly algebraic in the entries of X and is well-defined on the locus where $\det(X)\det(X^{11})x_{33} \neq 0$. This example illustrates that the coordinates given by the geodesic action are not ideally suited to studying the asymptotic behaviour of canonical differential forms at infinity.

(2). (Blow-ups.) With notations as above, the coordinates in the blow-up are, in our view, much simpler and are given by scalar multiplication on matrix blocks (see (9.11)). They correspond to a transformation of the form

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{pmatrix} \mapsto \begin{pmatrix} \nu_0\nu_1\nu_2\, x_{11} & \nu_1\nu_2\, x_{12} & \nu_2\, x_{13} \\ \nu_1\nu_2\, x_{12} & \nu_1\nu_2\, x_{22} & \nu_2\, x_{23} \\ \nu_2\, x_{13} & \nu_2\, x_{23} & \nu_2\, x_{33} \end{pmatrix}$$

for suitable ν_0, ν_1, ν_2.

Although the geometric origin of these two points of view are markedly different, we prove that they are in fact equivalent when restricted to the space of positive definite matrices.

Remark A.6 The following intuition for the structure of $\mathcal{B}(\mathbb{R})$ may be helpful. Consider an analytic family of (projective classes of) symmetric matrices $P_t \subset X \subset \mathcal{B}(\mathbb{R})$, as $t \to 0$, which are positive definite for $t > 0$ and which we think of as quadratic forms on a vector space V. This family is not necessarily geodesic. The limiting matrix $P_0 \in \mathbb{P}(\mathcal{Q}(V))(\mathbb{R})$ may have vanishing determinant, in which case it is positive semi-definite, and has a null space $K \subset V$. By choosing a complementary space $V = K \oplus C$, we may write P_t locally near $t = 0$ in block matrix form

$$P_t = \begin{pmatrix} At & Ut \\ U^T t & P_0 \end{pmatrix} + \text{higher order terms in } t\ .$$

If A is positive definite, the curve $t \mapsto P_t$ in $\mathfrak{B}(\mathbb{R})$ has a limiting point for $t = 0$ which lies on the exceptional divisor associated to the blow-up of $\mathbb{P}(\mathcal{Q}(V/K))$ (if A is not positive definite, then repeat the argument for the restriction of P_t to K, i.e. the top-left block, to define a point on a higher codimension exceptional locus). To describe this limiting point, consider the projective classes of the pair of symmetric matrices:

$$\begin{pmatrix} A & U \\ U^T & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & P_0 \end{pmatrix}$$

which define a point in the exceptional divisor $\mathbb{P}(\mathcal{Q}(V)/\mathcal{Q}(V/K))(\mathbb{R}) \times \mathbb{P}(\mathcal{Q}(V/K))(\mathbb{R})$. The latter is the projectivised normal bundle of $\mathbb{P}(\mathcal{Q}(V/K))$ inside $\mathbb{P}(\mathcal{Q}(V))$ as evidenced by the fact that the limiting point of P_t in $\mathcal{B}(\mathbb{R})$ is the pair: (normal vector, limiting value).

A.3 Linear monoidal actions and blow-ups

The Borel-Serre construction is topological, and relies on the geodesic action which is only defined on the space of positive definite matrices, since it makes crucial use of the surjectivity of (A.6). In order to reformulate it in the language of algebraic geometry, we first need some preliminaries on monoidal actions on vector space schemes and their quotients by multiplicative groups. A possible relation between Borel-Serre's construction and blow-ups is hinted at in [12, pg 437, line 5], but does not seem to have been developed further as far as we are aware.

Let k be a field. All schemes will be over k.

A.3.1 Multiplicative monoid

The multiplicative group is defined by $\mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$, where $\mathcal{O}(\mathbb{G}_m) = k[t, t^{-1}]$ is the Hopf algebra whose coproduct $\Delta : \mathcal{O}(\mathbb{G}_m) \to \mathcal{O}(\mathbb{G}_m) \otimes_k \mathcal{O}(\mathbb{G}_m)$ satisfies $\Delta t = t \otimes t$. The multiplication on $\mathbb{G}_m$ is dual to Δ.

We denote the multiplicative monoid scheme $\mathbb{G}_m \leq \overline{\mathbb{G}}_m$ to be

$$\overline{\mathbb{G}}_m = \operatorname{Spec} k[t]$$

where $k[t]$ is the coalgebra equipped with the coproduct $\Delta : k[t] \to k[t] \otimes_k k[t]$ satisfying the same formula $\Delta t = t \otimes t$. Thus, as a scheme, $\overline{\mathbb{G}}_m$ is simply the affine line $\mathbb{A}^1$, but has a multiplication morphism $\overline{\mathbb{G}}_m \times \overline{\mathbb{G}}_m \to \overline{\mathbb{G}}_m$. It has a distinguished point $0 \in \overline{\mathbb{G}}_m$ such that $\overline{\mathbb{G}}_m \backslash 0 = \mathbb{G}_m$, and has the property that the multiplication satisfies $0 \times \overline{\mathbb{G}}_m \to 0$.

A.3.2 Linear monoidal action

A vector space of dimension n over a field k will be viewed as the k-points of an affine scheme $\mathbb{A}^n$ in the usual manner. Direct sums correspond to products of vector-space schemes. Consider a finite-dimensional graded vector space W over a field k

$$W = W_n \oplus W_{n-1} \oplus \cdots \oplus W_1 \oplus W_0$$

where $\dim W_i \geq 1$ for all i. The multiplicative group $\mathbb{G}_m \times W_i \to W_i$ acts by scalar multiplication on each component, denoted by $(\lambda, w) \mapsto \lambda w$. It extends to an action $\overline{\mathbb{G}}_m \times W_i \to W_i$ which sends $(0, w)$ to the origin of W_i. Consider the monoid action

$$m : \overline{\mathbb{G}}_m^n \times W \longrightarrow W \ , \tag{A.9}$$

where the i^{th} component of $\overline{\mathbb{G}}_m$ for $i = 1, \ldots, n$, acts via

$$\begin{aligned} \overline{\mathbb{G}}_m \times W &\longrightarrow W \\ (\lambda_i, (w_n, \ldots, w_0)) &\mapsto (\lambda_i w_n, \ldots, \lambda_i w_i, w_{i-1}, \ldots, w_0) \ . \end{aligned} \tag{A.10}$$

The action (A.9) restricts to an action of $\mathbb{G}_m^n$ on W. Furthermore, let $\mathbb{G}_m^n$ act on $\overline{\mathbb{G}}_m^n$ componentwise via the map $\mathbb{G}_m \times \overline{\mathbb{G}}_m \to \overline{\mathbb{G}}_m$ given by $(\mu, \lambda) \mapsto \lambda\mu^{-1}$, where μ^{-1} is the inverse in the group $\mathbb{G}_m$. We deduce that (A.9) is equivariant for the diagonal action of $\mathbb{G}_m^n$ on $\overline{\mathbb{G}}_m^n \times W$, in other words, the following diagram commutes

$$\begin{array}{ccc} \mathbb{G}_m^n \times (\overline{\mathbb{G}}_m^n \times W) & \longrightarrow & \overline{\mathbb{G}}_m^n \times W \\ \downarrow & & \downarrow m \\ \overline{\mathbb{G}}_m^n \times W & \overset{m}{\longrightarrow} & W \end{array} \tag{A.11}$$

where the vertical map on the left is projection onto $\overline{\mathbb{G}}_m^n \times W$, and the horizontal map is the diagonal action of $\mathbb{G}_m^n$ on both $\overline{\mathbb{G}}_m^n$ and W. In the case $n = 1$ this diagram merely expresses the trivial fact that $m(\mu.(\lambda, w)) = m(\lambda\mu^{-1}, \mu w) = \lambda w = m(\lambda, w)$.

A.3.3 Quotients and blowups

For any finite dimensional vector space W over k, let $W^\times = W\backslash\{0\}$. It is an open subscheme of W stable under the action of $\mathbb{G}_m$, but not under the extended action of $\overline{\mathbb{G}}_m$, since $\overline{\mathbb{G}}_m \times W^\times \to W$. Let us denote by:

$$W^\star = W_n^\times \times W_{n-1}^\times \times \cdots \times W_1^\times \times W_0 \ ,$$

(component W_0 *sic*). The morphism (A.9) restricts to a morphism $m : \overline{\mathbb{G}}_m^n \times W^\star \to W$, which, by (A.11), is equivariant with respect to the action of $\mathbb{G}_m^n$.

We wish to define its quotient by $\mathbb{G}_m^n$ in the category of schemes.

Example A.7 We warm up with the construction of projective space as a quotient. Let $x_i^1, \ldots, x_i^n$ denote coordinates on the k-vector space $W_i \cong \mathbb{A}^n$, where $n = \dim W_i$, and write

$$W_i^\times = \bigcup_{j=1}^n U_i^j$$

where $U_i^j \subset \mathbb{A}^n$ is the open complement of $x_i^j = 0$. Let $U_i^{jk} = U_i^j \cap U_i^k$. The opens U_i^j, U_i^{jk} are stable under the action of $\mathbb{G}_m$ on W, and the inclusions $U_i^{jk} \hookrightarrow U_i^j$ are $\mathbb{G}_m$-equivariant. One may define the quotient by $\mathbb{G}_m$:

$$\mathbb{G}_m \backslash\backslash W_i^\times := \bigcup_{i=1}^n \mathbb{G}_m \backslash\backslash U_i^j$$

to be the union of schemes $\mathbb{G}_m \backslash\backslash U_i^i$ glued along the $\mathbb{G}_m \backslash\backslash U_i^{jk}$, where $\mathbb{G}_m \backslash\backslash U_i^j = \mathrm{Spec}\,(\mathcal{O}(U_i^j)^{\mathbb{G}_m})$ (and similarly $\mathbb{G}_m \backslash\backslash U_i^{jk}$) is the affine scheme whose coordinate ring consists of $\mathbb{G}_m$-invariant polynomials. Since $\mathcal{O}(U_i^j)^{\mathbb{G}_m} = k[\frac{x_i^1}{x_i^j}, \ldots, \frac{x_i^n}{x_i^j}]$, we recover the standard affine covering of projective space and thus

$$\mathbb{G}_m \backslash\backslash W_i^\times \cong \mathbb{P}(W_i)$$

is canonically isomorphic to the projective space of W_i.

Definition A.8 Consider a $\mathbb{G}_m$-equivariant covering $W_i^\times = \bigcup_{1 \le j \le \dim W_i} U_i^j$ of W_i for every $i = 1, \ldots, n$ as in the previous example, and define a scheme:

$$\mathbb{G}_m^n \backslash\backslash(\overline{\mathbb{G}}_m^n \times W^\star) := \bigcup_{i_1,\ldots,i_n} \mathbb{G}_m^n \backslash\backslash \left(\overline{\mathbb{G}}_m^n \times U_n^{i_n} \times \cdots \times U_1^{i_1} \times W_0\right) \ , \qquad \text{(A.12)}$$

where each $1 \le i_k \le \dim W_k$. It does not depend on the choice of coordinates (since $\mathbb{G}_m^n$ is central in the product of general linear groups $\mathrm{GL}(W_i)$).

Since it is $\mathbb{G}_m^n$-equivariant, the map m (A.9) defines a morphism of schemes also denoted by

$$m \, : \, \mathbb{G}_m^n \backslash\backslash (\overline{\mathbb{G}}_m^n \times W^\star) \longrightarrow W \, . \tag{A.13}$$

Proposition A.9 *The scheme* (A.12) *canonically embeds as an open subscheme of the iterated blow-up $B \to W$ of the affine space W along the linear subspaces*

$$W_0 \subset W_0 \oplus W_1 \subset \cdots \subset W_0 \oplus W_1 \oplus \cdots \oplus W_{n-1}$$

in increasing order of dimension. It is Zariski-dense in B.

Proof The most direct proof is via explicit coordinates generalising Example A.7. Consider the coordinate ring of a particular open in (A.12)

$$\mathcal{O}\left(\overline{\mathbb{G}}_m^n \times U_n^{i_n} \times \cdots \times U_1^{i_1} \times W_0\right)$$

where $1 \leq i_r \leq \dim W_r$. With the labelling of coordinates of Example A.7, it is

$$k\left[\lambda_1, \ldots, \lambda_n, \left(x_j^i\right)_{\substack{0 \leq j \leq n \\ 1 \leq i \leq \dim W_i}}, \frac{1}{x_1^{i_1}}, \ldots, \frac{1}{x_n^{i_n}}\right]$$

where λ_i is the coordinate on the ith component of $\overline{\mathbb{G}}_m$ (and the x_0^j are coordinates on W_0). The action of the ith component of $\mathbb{G}_m$ is equivalent to the action

$$\lambda_j \mapsto \begin{cases} \lambda_j \mu_i^{-1} & \text{if } i = j \, , \\ \lambda_j & \text{if } i \neq j \end{cases} \qquad \text{and} \qquad x_k^j \mapsto \begin{cases} \mu_i x_k^j & \text{if } k \geq i \\ x_k^j & \text{if } k < i \end{cases}$$

for all $j \geq 1$. The ring of $\mathbb{G}_m^n$-invariants of $\mathcal{O}\left(\overline{\mathbb{G}}_m^n \times U_n^{i_n} \times \cdots \times U_1^{i_1} \times W_0\right)$ is

$$\mathcal{O}(W_0)\left[\left(\frac{x_1^i}{x_1^{i_1}}\right)_{1 \leq i \leq \dim W_1}, \left(\frac{x_2^i}{x_2^{i_2}}\right)_{1 \leq i \leq \dim W_2}, \ldots, \left(\frac{x_n^i}{x_n^{i_n}}\right)_{1 \leq i \leq \dim W_n}, \right.$$
$$\left. \lambda_1 x_1^{i_1}, \lambda_2 \frac{x_2^{i_2}}{x_1^{i_1}}, \ldots, \lambda_n \frac{x_n^{i_n}}{x_{n-1}^{i_{n-1}}}\right],$$

since the former is generated over the latter (which is clearly $\mathbb{G}_m^n$-invariant) by $(x_1^{i_1})^{\pm 1}, \ldots, (x_n^{i_n})^{\pm 1}$. By introducing $\mathbb{G}_m^n$-invariant variables

$$\alpha_j^i = \lambda_1 \lambda_2 \ldots \lambda_j x_j^i$$

we can rewrite the ring of $\mathbb{G}_m^n$-invariants in the slightly different form

$$\mathcal{O}(W_0)\left[\alpha_1^{i_1}, \left(\frac{\alpha_1^i}{\alpha_1^{i_1}}\right)_{1 \leq i \leq \dim W_1}, \frac{\alpha_2^{i_2}}{\alpha_1^{i_1}}, \left(\frac{\alpha_2^i}{\alpha_2^{i_2}}\right)_{1 \leq i \leq \dim W_2}, \ldots, \right.$$

$$\frac{\alpha_n^{i_n}}{\alpha_{n-1}^{i_{n-1}}}, \left(\frac{\alpha_n^i}{\alpha_n^{i_n}}\right)_{1 \leq i \leq \dim W_n} \Bigg].$$

These are identical to a system of local coordinates on the iterated blow-up described in §5.2 (after reversing the indexation of the subscripts of the α's) along

$$V(\alpha_j^i : j \geq 1) \quad \subset \quad V(\alpha_j^i : j \geq 2) \quad \subset \quad \dots \quad \subset \quad V(\alpha_j^i : j \geq n) \, .$$

These coordinates define a canonical open immersion of this affine chart of (A.12) into B. Since these coordinate charts are compatible they glue together to form an explicit open immersion of schemes from (A.12) to B. □

Inspection of the proof shows that the scheme (A.12) is canonically isomorphic to the complement in B of strict transforms of some linear subspaces of W which may be written in terms of the W_i, although we will not need such an explicit description.

Remark A.10 The recursive structure of the iterated blow-up may be interpreted using the monoidal action. Consider the morphism

$$\pi_r : \overline{\mathbb{G}}_m^r \times W_n \times \cdots \times W_{r+1} \times W_r^\times \times \cdots \times W_1^\times \times W_0 \longrightarrow \\ \overline{\mathbb{G}}_m^{r-1} \times W_n \times \cdots \times W_r \times W_{r-1}^\times \times \cdots \times W_1^\times \times W_0 \quad \text{(A.14)}$$

which sends $((\lambda_1, \dots, \lambda_r), (w_n, \dots, w_0))$ to $((\lambda_1, \dots, \lambda_{r-1}), (\lambda_r w_n, \dots, \lambda_r w_r, w_{r-1}, \dots, w_0))$.

Lemma A.11 *The morphism π_r induces a map*

$$\mathbb{G}_m^r \mathbin{\backslash\backslash} \left(\overline{\mathbb{G}}_m^r \times W_n \times \cdots \times W_{r+1} \times W_r^\times \times \cdots \times W_1^\times \times W_0\right) \longrightarrow \\ \mathbb{G}_m^{r-1} \mathbin{\backslash\backslash} \left(\overline{\mathbb{G}}_m^{r-1} \times W_n \times \cdots \times W_r \times W_{r-1}^\times \times \cdots \times W_1^\times \times W_0\right) \quad \text{(A.15)}$$

which sends $V(\lambda_r)$, the vanishing locus of λ_r, to

$$Z_r = 0 \times \cdots \times 0 \times \mathbb{G}_m^{r-1} \mathbin{\backslash\backslash} \left(\overline{\mathbb{G}}_m^{r-1} \times (W_{r-1}^\times \times \cdots \times W_1^\times \times W_0)\right) \, .$$

Its restriction to the complement of $V(\lambda_r)$ is an isomorphism onto the complement of Z_r in its image. (*It is not surjective, since the locus $w_r = 0$ is not in its image*).

Proof The map π_r is $\mathbb{G}_m$-invariant for the action of the last component of $\mathbb{G}_m^r = \mathbb{G}_m^{r-1} \times \mathbb{G}_m$ (with coordinate μ_r) and equivariant for the action of $\mathbb{G}_m^{r-1}$ (with coordinates $(\mu_1, \dots, \mu_{r-1})$). Since $w_r \neq 0$, the equation $\lambda_r w_r \neq 0$ is equivalent to $\lambda_r \neq 0$. Hence $\pi_r^{-1}(Z_r) = V(\lambda_r)$. Recall that the last copy of $\mathbb{G}_m$ in $\mathbb{G}_m^r = \mathbb{G}_m^{r-1} \times \mathbb{G}_m$, with coordinate μ_r, acts via

$$\mu_r \times (\lambda_1, \dots, \lambda_r, w_n, \dots, w_0)$$

$$\mapsto (\lambda_1, \ldots, \lambda_{r-1}, \lambda_r \mu_r^{-1}, \mu_r w_n, \ldots, \mu_r w_r, w_{r-1}, \ldots, w_0)$$

By passing to the quotient for this action, every point with $\lambda_r \neq 0$ has a unique representative with $\lambda_r = 1$. On such points, the map in question is the identity and hence π_r is an isomorphism onto its image outside $V(\lambda_r)$. □

Now observe that the map $\mathbb{G}_m^n \backslash\!\backslash (\overline{\mathbb{G}}_m^n \times W^\star) \to W$ is a composition of morphisms

$$\begin{aligned}
\mathbb{G}_m^n \backslash\!\backslash \left(\overline{\mathbb{G}}_m^n \times W^\star\right) \xrightarrow{\pi_n} \; & W_n \times \mathbb{G}_m^{n-1} \backslash\!\backslash \left(\overline{\mathbb{G}}_m^{n-1} \times (W_{n-1}^\times \times \cdots \times W_1^\times \times W_0)\right) \\
\xrightarrow{\mathrm{id}\times\pi_{n-1}} \; & W_n \times W_{n-1} \\
& \times \mathbb{G}_m^{n-2} \backslash\!\backslash \left(\overline{\mathbb{G}}_m^{n-2} \times (W_{n-2}^\times \times \cdots \times W_1^\times \times W_0)\right) \\
\vdots \; & \\
\longrightarrow \; & W_n \times W_{n-1} \times \ldots W_1 \times W_0 \,.
\end{aligned}$$

The divisors $V(\lambda_r)$ can be described by a computation given in §A.3.5. It follows from this and Lemma A.11, that each map is a blow-up along Z_r, and one deduces that $\mathbb{G}_m^n \backslash\!\backslash (\overline{\mathbb{G}}_m^n \times W^\star)$ is an iterated blow-up along the strict transforms of $0 \times \cdots \times 0 \times W_{r-1} \times \cdots \times W_0$.

A.3.4 Variant: projective version

With notation as above, let

$$W^\circ = W_n^\times \times W_{n-1}^\times \times \cdots \times W_1^\times \times W_0^\times$$

with an action of $\mathbb{G}_m^{n+1} = \mathbb{G}_m^n \times \mathbb{G}_m$ where $\mathbb{G}_m^n$ acts as above, but with an additional copy of $\mathbb{G}_m$ (with coordinate μ_0) acting by scalar multiplication on W° (i.e. diagonally on each copy $W_i^\times$). Then we may define the quotient $\mathbb{G}_m^{n+1} \backslash\!\backslash (\overline{\mathbb{G}}_m^n \times W^\circ)$ as before. By (A.10), all components of $\mathbb{G}_m^n$ act trivially on W_0 and the morphism (A.13) now corresponds to:

$$\mathbb{G}_m^{n+1} \backslash\!\backslash (\overline{\mathbb{G}}_m^n \times W^\circ) \longrightarrow \mathbb{G}_m \backslash\!\backslash (W \backslash \{0\}) = \mathbb{P}(W) \,.$$

Note that its image is the open complement of $\mathbb{P}(W_n \oplus \cdots \oplus W_1 \oplus \{0\})$ in $\mathbb{P}(W)$. The projective version of Proposition A.9 is:

Proposition A.12 *The scheme $\mathbb{G}_m^{n+1} \backslash\!\backslash (\overline{\mathbb{G}}_m^n \times W^\circ)$, relative to the morphism*:

$$\mathbb{G}_m^{n+1} \backslash\!\backslash (\overline{\mathbb{G}}_m^n \times W^\circ) \longrightarrow \mathbb{P}(W)$$

canonically embeds into the iterated blow-up $B_W \to \mathbb{P}(W)$ along the linear subspaces

$$\mathbb{P}(W_0) \subset \mathbb{P}(W_0 \oplus W_1) \subset \ldots \subset \mathbb{P}(W_0 \oplus W_1 \oplus \cdots \oplus W_{n-1})$$

in increasing order of dimension. It is Zariski-dense in B_W.

A.3.5 Exceptional divisors

The exceptional divisors of $\mathbb{G}_m^{n+1} \backslash\backslash (\overline{\mathbb{G}}_m^n \times W^\circ)$ are in one-to-one correspondence with the divisors $V(\lambda_i)$ defined by $\lambda_i = 0$ for $i = 1, \ldots, n$.

Lemma A.13 *The divisor $V(\lambda_i)$ is canonically isomorphic to a product*

$$V(\lambda_i) \cong \mathbb{G}_m^{n-i+1} \backslash\backslash (\overline{\mathbb{G}}_m^{n-i} \times W_n^\times \times \cdots \times W_i^\times) \times \mathbb{G}_m^i \backslash\backslash (\overline{\mathbb{G}}_m^{i-1} \times W_{i-1}^\times \times \cdots \times W_0^\times) \, .$$

Proof There is a canonical isomorphism from $\overline{\mathbb{G}}_m^{i-1} \times 0 \times \overline{\mathbb{G}}_m^{n-i} \times W_n^\times \times \cdots \times W_0^\times$ to

$$\overline{\mathbb{G}}_m^{i-1} \times W_{i-1}^\times \times \cdots \times W_0^\times \times \overline{\mathbb{G}}_m^{n-i} \times W_n^\times \times \cdots \times W_i^\times$$

which maps an element $(\lambda_1, \ldots, \lambda_{i-1}, 0, \lambda_{i+1}, \ldots, \lambda_n, w_n, \ldots, w_0)$ to

$$(\lambda_1, \ldots, \lambda_{i-1}, w_{i-1}, \ldots, w_0) \times (\lambda_{i+1}, \ldots, \lambda_n, w_n, \ldots, w_{i+1}, w_i) \, .$$

One checks that it is equivariant with respect to the respective actions of

$$\begin{aligned} &\mathbb{G}_m^i \times \mathbb{G}_m^{n-i+1} \xrightarrow{\sim} \mathbb{G}_m^{n+1} \\ &(\mu_0, \ldots, \mu_{i-1}) \times (\mu_i, \ldots, \mu_n) \\ &\quad \mapsto (\mu_0, \ldots, \mu_{i-1}, (\mu_0 \ldots \mu_{i-1})^{-1} \mu_i, \mu_{i+1}, \ldots, \mu_n) \, . \end{aligned}$$

The statement follows by passing to the quotient. □

If we write $W' = W_{i-1} \oplus \cdots \oplus W_0$, it follows that $V(\lambda_i)$ embeds into the product

$$V(\lambda_i) \hookrightarrow B_{W/W'} \mathbb{P}(W/W') \times B_{W'} \mathbb{P}(W') \tag{A.16}$$

where the blow-ups are with respect to the flags of subspaces defined by the filtration $F_m = \bigoplus_{k=0}^m W_k$ on W and the filtrations it induces on W' and W/W'.

The irreducible components $\mathcal{S}_i$ of the exceptional divisor $\mathcal{S}$ define a stratification on the blow-up B_W of $\mathbb{P}(W)$. Consider the corresponding quasi-projective stratification

$$B_W = \bigcup_I \mathcal{S}_I^\circ \tag{A.17}$$

where the union is over all subsets $I \subset \{1, \ldots, n\}$, and where we denote $\mathcal{S}_I = \bigcap_{i \in I} \mathcal{S}_i$, and $\mathcal{S}_\emptyset = \mathbb{P}(W)$, and define

$$\mathcal{S}_I^\circ = \mathcal{S}_I \setminus \left(\bigcup_{j \notin I} \mathcal{S}_I \cap \mathcal{S}_j \right) \, .$$

It follows from the description in Proposition 5.3 that each stratum of (A.17) (and each stratum of $\mathbb{G}_m^{n+1} \backslash\backslash (\overline{\mathbb{G}}_m^n \times W^\circ)$ via (A.16)), is embedded in a product of projective spaces:

$$\mathcal{S}_I^\circ \quad \subset \quad \prod_{k=1}^{|I|} \mathbb{P}\left(\bigoplus_{i_{k-1} \leq j < i_k} W_j \right) = \prod_{k=1}^{|I|} \mathbb{P}\left(\mathrm{gr}_k^{F'} W \right) \ , \tag{A.18}$$

where $F'W$ is the increasing filtration on W whose k^{th} graded quotient is $\bigoplus_{i_{k-1} \leq j < i_k} W_j$ where we set $i_0 = 0$, $i_{n+1} = n+1$. The strata of (A.17) are in one-to-one correspondence with filtrations $F' \leq F$.

The restriction of the embedding (A.18) to the locus in $\mathbb{G}_m^{n+1} \backslash\backslash (\overline{\mathbb{G}}_m^n \times W^\circ)$ where $\lambda_i = 0$ for all $i \in I$ and $\lambda_j \neq 0$ for all $j \neq 0$, is given explicitly by the map

$$\begin{aligned}(\lambda_1, \ldots, \lambda_n, w_n, \ldots, w_0) \mapsto (w_0, \ldots, w_{i_1 - 1}) \times (w_{i_1}, \ldots, w_{i_2 - 1}) \\ \times \cdots \times (w_{i_{n-1}}, \ldots, w_n)\end{aligned}$$

where the right-hand side is in the space $\prod_{k=1}^{n+1} \mathbb{G}_m \backslash\backslash (W_{i_{k-1}}^\times \times \cdots \times W_{i_k - 1}^\times)$ which embeds in a product of projective spaces by Example A.7. One may verify that it is an embedding because, for every $\lambda_j \neq 0$, we may apply a unique element of the jth copy of $\mathbb{G}_m$ in $\mathbb{G}_m^{n+1}$ to make the coordinate λ_j equal to 1.

A.4 Spaces of parabolic and symmetric matrices

Using the previous construction, we may define an algebraic version of $A_P \setminus (\overline{A}_P \times P)$ associated to a rational parabolic P and embed it in an iterated blow-up. We then provide an algebraic interpretation of the space $X(P)$ by embedding it into the real points of a finite iterated blow-up of projective space. We shall denote algebraic groups with caligraphic letters to distinguish them from the topological groups considered in §A.2.

Let V be a finite-dimensional vector space over $\mathbb{Q}$ and consider a flag of subspaces

$$F \ : \ 0 \subset V_d \subset V_{d-1} \subset \cdots \subset V_1 \subset V_0 = V$$

which has a splitting: $V \cong \bigoplus_{i \geq 0} V_i / V_{i+1}$. Let $\mathcal{P}_F \subset \mathrm{End}_F(V)$ denote the affine variety over $\mathbb{Q}$ of endomorphisms of V which preserve F. Using the splitting of V we may write

$$\mathrm{End}_F(V) = \mathrm{Hom}_F(V \ , \ \bigoplus_i V_i / V_{i+1}) = \bigoplus_i \mathrm{Hom}_F(V, V_i / V_{i+1})$$

which, upon setting $\mathcal{P}_i = \mathrm{Hom}_F(V, V_i / V_{i+1})$, gives a decomposition (of schemes, corresponding to the direct sum in the category of vector spaces)

$$\mathcal{P}_F = \mathcal{P}_d \times \mathcal{P}_{d-1} \times \cdots \times \mathcal{P}_0 \ . \tag{A.19}$$

Define a subscheme $\mathcal{P}_F^\circ = \mathcal{P}_d^\times \times \mathcal{P}_{d-1}^\times \times \cdots \times \mathcal{P}_0^\times$ of $\mathcal{P}_F$ as in §A.3.4. Let

$$\mathcal{Z}_P = \mathbb{G}_m^{d+1} \hookrightarrow \mathrm{GL}\left(\bigoplus_i V_i/V_{i+1}\right) = \prod_i \mathrm{GL}(V_i/V_{i+1})$$

be the product of multiplicative groups whose k^{th} component, where $0 \leq k \leq d$, acts by scalar multiplication on V_k/V_{k+1}. Denote its coordinate ring $\mathcal{O}(\mathcal{Z}_P) = k[\mu_0, \mu_0^{-1}, \ldots, \mu_d, \mu_d^{-1}]$.

Consider the affine group scheme $\mathcal{A}_P = \mathbb{G}_m^d$, with coordinates $\lambda_1, \ldots, \lambda_d$. There is a morphism of affine group schemes $\mathcal{A}_P \to \mathcal{Z}_P$, which on coordinate rings is the map

$$(\mu_0, \ldots, \mu_d) \mapsto (1,\ \lambda_1,\ \lambda_1\lambda_2,\ \ldots,\ \lambda_1 \cdots \lambda_d) : \mathcal{O}(\mathcal{Z}_P) \longrightarrow \mathcal{O}(\mathcal{A}_P)\ .$$

This is consistent with (A.7). The previous morphism, via the action of $\mathcal{Z}_P$ on $\bigoplus_i V_i/V_{i+1}$, defines an action of $\mathcal{A}_P$ on $\mathcal{P}_F^\circ$. This action extends to an action of the multiplicative monoid $\overline{\mathcal{A}}_P = \overline{\mathbb{G}}_m^d$ whose k^{th} component has coordinate λ_k where $1 \leq k \leq d$. It acts by scalar multiplication on V_k as in §A.3, i.e., for $v_i \in V_i/V_{i+1}$, for $0 \leq i \leq d$ we have:

$$(\lambda_1, \ldots, \lambda_d) \times (v_d, \ldots, v_1, v_0) \mapsto (\lambda_d \cdots \lambda_1 v_d,\ \ldots,\ \lambda_2\lambda_1 v_2,\ \lambda_1 v_1,\ v_0)\ .$$

Since the components of $\mathcal{Z}_P$ act centrally on V_i/V_{i+1}, this defines an action $\overline{\mathcal{A}}_P \times \mathcal{P}_F^\circ \to \mathcal{P}_F$ by scalar multiplication on each component. It is given by the identical formula to the above, upon replacing v_i with p_i, for $p_i \in \mathcal{P}_i^\times$. On points, an element $a \in \mathcal{A}_P$ acts upon $(\lambda, p) \in \overline{\mathcal{A}}_P \times \mathcal{P}_F^\circ$ by $a.(\lambda, p) = (\lambda.a^{-1}, a.p)$. Finally let $\mathcal{H} = \mathbb{G}_m$ denote the subgroup of $\mathcal{Z}_P$ with coordinates $\mu_0 = \cdots = \mu_d$, corresponding to diagonal matrices. It acts on $\overline{\mathcal{A}}_P \times \mathcal{P}_F^\circ$ by acting trivially on $\overline{\mathcal{A}}_P$, and acting via scalar multiplication on $\mathcal{P}_F^\circ$.

Theorem A.14 *There is a canonical Zariski-dense embedding*:

$$(\mathcal{A}_P \times \mathcal{H}) \setminus\!\setminus \left(\overline{\mathcal{A}}_P \times \mathcal{P}_F^\circ\right) \longrightarrow B_F\,\mathbb{P}(\mathcal{P}_F)\ , \tag{A.20}$$

where $B_F\mathbb{P}(\mathcal{P}_F)$ is the iterated blow-up of $\mathbb{P}(\mathcal{P}_F)$ along the subspaces

$$\mathbb{P}(\mathcal{P}_0)\ \subset\ \mathbb{P}(\mathcal{P}_0 \oplus \mathcal{P}_1)\ \subset\ \cdots\ \subset\ \mathbb{P}(\mathcal{P}_0 \oplus \cdots \oplus \mathcal{P}_{d-1})\ .$$

Let $I = \{i_1, \ldots, i_k\}$ and $1 \leq i_1 < \cdots < i_k \leq d$. The image of the locus $V(\lambda_i, i \in I)$ under the morphism (A.20) *is contained in the codimension k stratum $\mathcal{S}_I$ of the exceptional divisor of $B_F\mathbb{P}(\mathcal{P}_F)$ which corresponds to the flag $F' \leq F$*:

$$F'\ :\ 0 = V_{i_{k+1}} \subset V_{i_k} \subset V_{i_{k-1}} \subset \cdots \subset V_{i_1} \subset V_{i_0} = V\ . \tag{A.21}$$

Both $V(\lambda_i, i \in I)$ and its Zariski-closure $\mathcal{S}_I$ are canonically isomorphic to products, and the morphism (A.20) *restricts to the embedding*:

$$V(\lambda_i, i \in I) \cong \prod_{m=0}^{k} (\mathcal{A}_{P_{F_m}} \times \mathcal{H}) \setminus\!\setminus \left(\overline{\mathcal{A}}_{P_{F_m}} \times \mathcal{P}_{F_m}^\circ\right) \longrightarrow \mathcal{S}_I \tag{A.22}$$

$$\cong \prod_{m=0}^{k} B_{F_m}\mathbb{P}(P_{F_m})$$

where $\mathcal{P}_{F_m} = \prod_{i_m \leq j < i_{m+1}} \mathcal{P}_j$, $\mathcal{P}^{\circ}_{F_m} = \prod_{i_m \leq j < i_{m+1}} \mathcal{P}^{\times}_j$ *and* F_m *are the filtrations induced by* F *on the quotients* $V_{i_m}/V_{i_{m+1}}$.

The restriction of (A.22) *to the open locus* $\{\lambda_j \neq 0 : j \notin I\}$ *gives an embedding*

$$(\mathcal{A}_{P_{F'}} \times \mathcal{H}) \mathbin{\backslash\backslash} \mathcal{P}^{\circ}_F \quad \hookrightarrow \quad \mathcal{S}^{\circ}_I \subseteq \prod_{m=0}^{k} \mathbb{P}(P_{F_m})\,. \tag{A.23}$$

Proof Apply Proposition A.12 to $\mathcal{P}_F$. The second part follows by iterating the description of a codimension one exceptional divisor given in §A.3.5. The only novelty is (A.23). It follows from the fact that the locus $\{\lambda_j \neq 0 : j \notin I\}$ is isomorphic to

$$\left(\prod_{m=0}^{k} (\mathcal{A}_{P_{F_m}} \times \mathcal{H}) \mathbin{\backslash\backslash} \left(\overline{\mathcal{A}}_{P_{F_m}} \times \mathcal{P}^{\circ}_{F_m}\right)\right) \setminus \bigcup_{j \notin I} V(\lambda_j) \cong (\mathcal{A}_{P_{F'}} \times \mathcal{H}) \mathbin{\backslash\backslash} \mathcal{P}^{\circ}_F\,.$$

This follows from the isomorphism $\mathcal{P}^{\circ}_F = \mathcal{P}^{\circ}_{F_0} \times \cdots \times \mathcal{P}^{\circ}_{F_m}$, and the fact that the locus where $\lambda_j \neq 0$ is a torsor over the jth copy of $\mathbb{G}_m$ in $\mathcal{A}_{P_{F_m}}$, and so any point with coordinate $\lambda_j \neq 0$ has a unique representative in its $\mathbb{G}_m$-orbit with $\lambda_j = 1$. □

Since the determinant is a homogeneous polynomial in matrix entries, we may define the determinant hypersurface $\mathrm{Det} \subset \mathbb{P}(\mathcal{P}_F)$ to be its vanishing locus. Let

$$\widetilde{\mathrm{Det}} \quad \subset \quad B_F\mathbb{P}(\mathcal{P}_F)$$

denote its strict transform. Let $P = \mathrm{GL}_F(V \otimes_{\mathbb{Q}} \mathbb{R})$ denote the (topological) group of automorphisms considered in §A.2, and likewise $A_P \leq P$. Then we may interpret

$$P \;\cong\; (\mathcal{P}_F \backslash \mathrm{Det})\,(\mathbb{R})$$

and it follows from Theorem A.14 that there are injections:

$$\begin{aligned}(A_P \times H) \setminus \left(\overline{A}_P \times P\right) &\longrightarrow \left((\mathcal{A}_P \times \mathcal{H}) \mathbin{\backslash\backslash} \left(\overline{\mathcal{A}}_P \times (\mathcal{P}^{\circ}_F \backslash \mathrm{Det}\right)\right)(\mathbb{R}) \\ &\longrightarrow \left(B_F\mathbb{P}(\mathcal{P}_F) \setminus \widetilde{\mathrm{Det}}\right)(\mathbb{R})\,.\end{aligned} \tag{A.24}$$

On the left is a topological group; in the middle, the real points of a scheme-theoretic group quotient; on the right, the real points of an iterated blow up.

Example A.15 Consider the case when $d = 2$ and F is $0 \subset V_2 \subset V_1 \subset V$. By choosing a basis of V adapted to this filtration, the points of $\mathcal{P}_F$ are given by block lower triangular matrices, and $\overline{\mathcal{A}}_P \cong \overline{\mathbb{G}}^2_m$ acts by

$$(\lambda_1, \lambda_2) \times \begin{pmatrix} P_0 & 0 & 0 \\ P_{01} & P_1 & 0 \\ P_{02} & P_{12} & P_2 \end{pmatrix} \mapsto \begin{pmatrix} P_0 & 0 & 0 \\ \lambda_1 P_{01} & \lambda_1 P_1 & 0 \\ \lambda_1\lambda_2 P_{02} & \lambda_1\lambda_2 P_{12} & \lambda_1\lambda_2 P_2 \end{pmatrix}$$

where λ_1, λ_2 are points on $\overline{\mathbb{G}}_m^2$. The points of $\mathcal{P}_i$, for $i = 0, 1, 2$ are given by the subsets of matrices which vanish everywhere except in (block) row $i+1$, and the subspaces $\mathcal{P}_0 \subset \mathcal{P}_0 \oplus \mathcal{P}_1$ which are to be blown up are the spaces of matrices of the form:

$$\begin{pmatrix} P_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \subseteq \begin{pmatrix} P_0 & 0 & 0 \\ P_{01} & P_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

The points of the subspace $\mathcal{P}_F^\circ$ satisfy $P_0 \neq 0$, $P_1 \neq 0$ and $P_2 \neq 0$. This strictly contains the complement of the locus Det defined by $\det(P_0)\det(P_1)\det(P_2) = 0$.

The stratification of $\mathbb{G}_m \backslash\backslash(\overline{\mathbb{G}}_m^2 \times \mathcal{P}_F^\circ)$ is generated by the equations $\lambda_1 = 0$ and $\lambda_2 = 0$ (below left). The strata themselves are described as quotients (below, middle), which embed into the following products of blow-ups (below, right):

$$\begin{array}{rcccl} V(\lambda_1) & : & (\mathbb{G}_m \backslash\backslash \mathcal{P}_0^\circ) \times \mathbb{G}_m^2 \backslash\backslash \left(\overline{\mathbb{G}}_m \times \mathcal{P}_1^\circ \times \mathcal{P}_2^\circ\right) & \subseteq & \mathbb{P}(\mathcal{P}_0) \times B_{\mathbb{P}(\mathcal{P}_1)}\mathbb{P}(\mathcal{P}_2 \oplus \mathcal{P}_1) \\ V(\lambda_2) & : & \mathbb{G}_m^2 \backslash\backslash \left(\overline{\mathbb{G}}_m \times \mathcal{P}_0^\circ \times \mathcal{P}_1^\circ\right) \times (\mathbb{G}_m \backslash\backslash \mathcal{P}_2^\circ) & \subseteq & B_{\mathbb{P}(\mathcal{P}_0)}\mathbb{P}(\mathcal{P}_0 \oplus \mathcal{P}_1) \times \mathbb{P}(\mathcal{P}_2) \cdot \\ V(\lambda_1, \lambda_2) & : & (\mathbb{G}_m \backslash\backslash \mathcal{P}_0^\circ) \times (\mathbb{G}_m \backslash\backslash \mathcal{P}_1^\circ) \times (\mathbb{G}_m \backslash\backslash \mathcal{P}_2^\circ) & \subseteq & \mathbb{P}(\mathcal{P}_0) \times \mathbb{P}(\mathcal{P}_1) \times \mathbb{P}(\mathcal{P}_2) . \end{array}$$

These strata, respectively, correspond to the flags: $0 \subset V_1 \subset V$ for $\lambda_1 = 0$; $0 \subset V_2 \subset V$ for $\lambda_2 = 0$; and the full flag $0 \subset V_2 \subset V_1 \subset V$ for $\lambda_1 = \lambda_2 = 0$.

See Example A.17 for the continuation of this example.

A.5 Spaces of quadratic forms, blow-ups and $X(P)$

Let F, P be as in §A.4. There is a natural morphism of algebraic varieties

$$\begin{aligned} \mathbb{P}(\mathcal{P}_F) &\longrightarrow \mathbb{P}(\mathcal{Q}(V)) \\ M &\mapsto M^T M \end{aligned} \tag{A.25}$$

which sends the subspace $\mathbb{P}(\mathcal{P}_0 \oplus \cdots \oplus \mathcal{P}_i)$ to $\mathbb{P}(\mathcal{Q}(V/V_{i+1})) \subset \mathbb{P}(\mathcal{Q}(V))$, the projective space of the subspace of quadratic forms which vanish on V_{i+1}.

Consequently, by the universal property of blow-ups, (A.25) induces a morphism

$$B_F\mathbb{P}(\mathcal{P}_F) \longrightarrow B_F\mathbb{P}(\mathcal{Q}(V)) \tag{A.26}$$

where $B_F\mathbb{P}(\mathcal{Q}(V))$ is the iterated blow-up of $\mathbb{P}(\mathcal{Q}(V))$ along the linear subspaces

$$\mathbb{P}(\mathcal{Q}(V/V_1)) \subset \mathbb{P}(\mathcal{Q}(V/V_2)) \subset \cdots \subset \mathbb{P}(\mathcal{Q}(V/V_d))$$

associated to the flag F. Combining with Theorem A.14 we deduce a natural map:

$$(\mathcal{A}_P \times \mathcal{H}) \backslash\backslash \left(\overline{\mathcal{A}}_P \times \mathcal{P}_F^\circ\right) \longrightarrow B_F\mathbb{P}(\mathcal{P}_F) \longrightarrow B_F\mathbb{P}(\mathcal{Q}(V)) . \tag{A.27}$$

Recall from §9.5 that the exceptional divisor $\mathcal{E}$ of $B_F\mathbb{P}(\mathcal{Q}(V))$ defines a stratification whose strata $\mathcal{E}_{F'}$ are in one-to-one correspondence with flags $F' \leq F$ (A.21), and are canonically isomorphic to a product of iterated blow-ups

$$\mathcal{E}_{F'} \cong \prod_{m=0}^{k} B_{F_i} \mathbb{P}\left(\frac{\mathcal{Q}(V/V_{i_{m+1}})}{\mathcal{Q}(V/V_{i_m})}\right) ,$$

where F_i is the induced filtration on the successive quotients V_i/V_{i+1} of (A.21). Let $\mathrm{Det} \subset \mathbb{P}(\mathcal{Q}(V))$ denote the vanishing locus of the determinant, and $\widetilde{\mathrm{Det}}$ its strict transform in $B_F\mathbb{P}(\mathcal{Q}(V))$. It was proven in Proposition 11.4 that

$$\mathcal{E}_{F'} \setminus \left(\mathcal{E}_{F'} \cap \widetilde{\mathrm{Det}}\right) \cong \prod_{m=0}^{k} \left(\mathbb{P}\left(\frac{\mathcal{Q}(V/V_{i_{m+1}})}{\mathcal{Q}(V/V_{i_m})}\right) \setminus \mathrm{Det}\big|_{V_{i_m}/V_{i_{m+1}}}\right) \tag{A.28}$$

where $\mathrm{Det}\big|_{V_{i_m}/V_{i_{m+1}}}$ is the zero locus of the homogeneous map

$$\frac{\mathcal{Q}(V/V_{i_{m+1}})}{\mathcal{Q}(V/V_{i_m})} \longrightarrow \mathcal{Q}(V_{i_m}/V_{i_{m+1}}) \xrightarrow{\det} \mathbb{Q},$$

where the first map is restriction of quadratic forms (compare §9.3).

Theorem A.16 *The map* (A.27) *induces an injective continuous map*

$$X(P) \longrightarrow \left(B_F\, \mathbb{P}(\mathcal{Q}(V)) \ \setminus \ \widetilde{\mathrm{Det}}\right)(\mathbb{R}) \ . \tag{A.29}$$

For every flag F' of the form (A.21) *the map* (A.29) *restricts to a map*

$$e(P_{F'}) \longrightarrow \mathcal{E}_{F'} \setminus \left(\mathcal{E}_{F'} \cap \widetilde{\mathrm{Det}}\right)(\mathbb{R}) \tag{A.30}$$

to the corresponding stratum in the exceptional locus of $B_F\, \mathbb{P}(\mathcal{Q}(V))$. The image of $e(P_{F'})$, via the identification (A.28), *is the product $e(P_{F'}) \cong \prod_m X_m$ where*

$$X_m \ \subset \ \mathbb{P}\left(\frac{\mathcal{Q}(V/V_{i_{m+1}})}{\mathcal{Q}(V/V_{i_m})}\right)(\mathbb{R})$$

is the set of projective classes of matrices whose image in $\mathbb{P}(\mathcal{Q}(V_{i_m}/V_{i_{m+1}}))(\mathbb{R})$ is positive definite.

Proof The equation $\det(M^T M) = \det(M)^2$ implies that (A.25) preserves the determinant loci Det in $\mathbb{P}(\mathcal{P}_F)$ and $\mathbb{P}(\mathcal{Q}(V))$, and so the morphism (A.26) preserves their strict transforms. It follows that (A.27) defines a natural map

$$(\mathcal{A}_P \times \mathcal{H}) \setminus\!\!\setminus \left(\overline{\mathcal{A}}_P \times \mathcal{P}_F^{\circ}\right) \longrightarrow B_F\mathbb{P}(\mathcal{Q}(V)) \setminus \widetilde{\mathrm{Det}} \tag{A.31}$$

whose restriction to the open locus $\{\lambda_i \neq 0, \ \text{for } 1 \leq i \leq d\}$ is the morphism

$$\mathbb{G}_m \setminus\!\!\setminus \mathcal{P}_F^{\circ} \longrightarrow \mathbb{P}(\mathcal{Q}(V)) \setminus \mathrm{Det}$$

which, in coordinates given by a basis of V, sends a matrix M to the projective class of $M^T M$. This map is invariant by left-multiplication by the subscheme $\mathcal{K}_{P_F} \leq \mathcal{P}_F^{\circ}$

whose points are matrices O such that $O^T O = 1$, since one has $(OM)^T OM = M$. Since this equation is algebraic, it remains true on the Zariski closure of $\mathcal{P}_F^\circ$, and hence (A.31) is also invariant under left multiplication by $\mathcal{K}_{P_F}$, since the action of $\mathcal{K}_{P_F}$ on $\mathcal{P}_F^\circ$ commutes with that of $\mathcal{A}_P$ and $\mathcal{H}$. We deduce that the map of topological spaces

$$(A_P \times H) \setminus \left(\overline{A}_P \times P\right) \longrightarrow \left(B_F \mathbb{P}(\mathcal{Q}(V)) \setminus \widetilde{\mathrm{Det}}\right)(\mathbb{R})$$

which is induced by taking real points of the map (A.31) and composing with the first map in (A.24), is $K_P \leq \mathcal{K}_{P_F}(\mathbb{R})$ invariant and so passes to the quotient

$$X(P) = Z_P K_P \setminus \left(\overline{A}_P \times P\right) \longrightarrow \left(B_F \mathbb{P}(\mathcal{Q}(V)) \setminus \widetilde{\mathrm{Det}}\right)(\mathbb{R}) .$$

It gives a well-defined continuous map (A.29) such that the following diagram commutes:

$$\begin{array}{ccc} Z_P \setminus \left(\overline{A}_P \times P\right) & \longrightarrow & \left(B_F \mathbb{P}(P_F) \setminus \widetilde{\mathrm{Det}}\right)(\mathbb{R}) \\ \downarrow & & \downarrow \\ X(P) = A_P \setminus \left(\overline{A}_P \times X\right) & \longrightarrow & \left(B_F \mathbb{P}(\mathcal{Q}(V)) \setminus \widetilde{\mathrm{Det}}\right)(\mathbb{R}) \end{array} ,$$

where the horizontal map along the top is the morphism (A.20) and the vertical maps are $M \mapsto M^T M$ on the left, and (A.26) on the right.

Now compute the restriction of the map (A.29) on each stratum indexed by a flag F' of the form (A.21). The map (A.31) restricts to a map on exceptional divisors

$$(\mathcal{A}_{P_{F'}} \times \mathcal{H}) \backslash\backslash \mathcal{P}_F^\circ \overset{(A.23)}{\longrightarrow} \mathcal{S}_{F'}^\circ \overset{(A.26)}{\longrightarrow} \mathcal{E}_{F'} .$$

Since it is left-invariant by $\mathcal{K}_{P_F}$, it descends, after taking real points, to the quotient

$$e(P_{F'}) = A_{P_{F'}} \setminus X \cong Z_{P_{F'}} K_{P_{F'}} \setminus P_{F'} \longrightarrow \mathcal{E}_{F'} \setminus \left(\mathcal{E}_{F'} \setminus \widetilde{\mathrm{Det}}\right)(\mathbb{R}) .$$

This morphism is injective on each stratum $e(P_{F'})$ since $M^T M = N^T N$ holds for two invertible real matrices M, N if and only if $M = ON$, for O an orthogonal matrix. The final statement follows from the fact that (A.29) identifies the open $X \subset X(P)$ with the subspace of projective classes of positive-definite matrices inside $\mathbb{P}(\mathcal{Q}(V))(\mathbb{R})$. This proves the result for the trivial flag. In the case of a flag $0 \subset V_1 \subset V$ of length $d = 1$, the map from $e(P)$ to the corresponding exceptional locus is given on the level of matrices by

$$\begin{pmatrix} A & 0 \\ U & B \end{pmatrix} \mapsto \begin{pmatrix} A^T A & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & U^T B \\ B^T U & B^T B \end{pmatrix} , \tag{A.32}$$

where the matrices are on the right denote projective equivalence classes. We can choose A, B such that $A^T A$, and $B^T B$ are arbitrary (projective classes of) positive definite real symmetric matrices, and finally choose U such that $U^T B$ is any other matrix, since B is invertible. This indeed proves that $e(P)$ surjects onto the space described in the statement of the theorem. The general case, for a flag of arbitrary length, is similar (or follows from the case of length one by induction). $\square$

Examples A.17 As in Example A.15, use the splitting $V \cong V/V_1 \oplus V_1/V_2 \oplus V_2$ to write endomorphisms of V in block matrix form. Consider the deepest stratum which is associated to the full flag F, which corresponds to $\mathcal{E}_{12}$ (denoted $V(\lambda_1, \lambda_2)$ in Example A.15)).

The morphism (A.30)

$$e(P) \longrightarrow \mathbb{P}(\mathcal{Q}(V/V_1))(\mathbb{R}) \times \mathbb{P}\left(\frac{\mathcal{Q}(V/V_2)}{\mathcal{Q}(V/V_1)}\right)(\mathbb{R}) \times \mathbb{P}\left(\frac{\mathcal{Q}(V)}{\mathcal{Q}(V/V_2)}\right)(\mathbb{R})$$

is given explicitly on matrix representatives in P of classes in $e(P) = A_P K_P \setminus P$ by:

$$\begin{pmatrix} P_0 & & \\ P_{01} & P_1 & \\ P_{02} & P_{12} & P_2 \end{pmatrix} \mapsto \begin{pmatrix} P_0^T P_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & P_{01}^T P_1 & 0 \\ P_1^T P_{01} & P_1^T P_1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 & P_{02}^T P_2 \\ 0 & 0 & P_{12}^T P_2 \\ P_2^T P_{02} & P_2^T P_{12} & P_2^T P_2 \end{pmatrix}$$

Now consider the stratum associated to the flag $F' : 0 \subset V_1 \subset V$, which corresponds to the exceptional divisor $\mathcal{E}_1$ (corresponding to $V(\lambda_1)$ in Example A.15). The morphism (A.30)

$$e(P') \longrightarrow \mathbb{P}(\mathcal{Q}(V/V_1))(\mathbb{R}) \times \mathbb{P}\left(\frac{\mathcal{Q}(V)}{\mathcal{Q}(V/V_1)}\right)(\mathbb{R})$$

is given on matrix representatives in P of classes in $e(P') = A_{P'} K_{P'} \setminus P' = A_{P'} K_P \setminus P$ by setting $A = P_0$, $U = \begin{pmatrix} P_{01} \\ P_{02} \end{pmatrix}$, and $B = \begin{pmatrix} P_1 & 0 \\ P_{12} & P_2 \end{pmatrix}$ in (A.32). This gives the map

$$\begin{pmatrix} P_0 & 0 & 0 \\ P_{01} & P_1 & 0 \\ P_{02} & P_{12} & P_2 \end{pmatrix} \mapsto \begin{pmatrix} P_0^T P_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & P_{01}^T P_1 + P_{02}^T P_{12} & P_{02}^T P_2 \\ P_1^T P_{01} + P_{12}^T P_{02} & P_1^T P_1 + P_{12}^T P_{12} & P_{12}^T P_2 \\ P_2^T P_{02} & P_2^T P_{12} & P_2^T P_2 \end{pmatrix}$$

A.6 Proof of the statements in §A.1

A.6.1 Proof of Theorem A.2

Theorem A.16 provides an injective map of stratified spaces from $X(P)$ to the iterated blow-up $B_F \mathbb{P}(Q(V))$. By the argument of Proposition 5.2 (ii), there is a canonical map $\mathfrak{B}(\mathbb{R}) \to B_F\mathbb{P}(Q(V))$ which is an isomorphism away from exceptional divisors. Since the image of the exceptional divisors is contained in the (strict transform) of the determinant locus, which does not meet the image of $X(P)$, it follows that the continuous map $X(P) \to B_F\mathbb{P}(Q(V)) \setminus \widetilde{\mathrm{Det}}$ canonically lifts to a map $X(P) \to (\mathfrak{B} \setminus \widetilde{\mathrm{Det}})(\mathbb{R})$. Via the description (A.8) of X^{BS} as a union of $X(P)$, for P rational parabolics, we deduce the canonical continous map $X^{\mathrm{BS}} \to (\mathfrak{B} \setminus \widetilde{\mathrm{Det}})(\mathbb{R})$. It is injective since, by Theorem A.16, the image of each open stratum $e(P)$ of X^{BS} is contained in the complement of the determinant locus in a unique exceptional stratum. They are disjoint, since the intersections of exceptional strata are contained in the determinant loci. The statement abut the closure of X will follow from the argument in the next paragraph below. The description of the image of $e(P)$ follows from Theorem A.16 and (A.28). Finally, the group of rational points $\mathrm{GL}(V)(\mathbb{Q})$ acts on X via $P \mapsto Pg$. Correspondingly, the algebraic group $\mathrm{GL}(V)$ acts upon $\mathbb{P}(Q(V))$ via $(g, M) \mapsto g^T Mg$. Since it permutes the set of flags (A.5) and hence the centers of all the blow-ups which define $\mathfrak{B}$, it extends to a natural action of $\mathrm{GL}(V)$ upon $\mathfrak{B}$. Its group of $\mathbb{Q}$-points $\mathrm{GL}(V)(\mathbb{Q})$ acts algebraically upon $\mathfrak{B}$. This action is by compatible, via $P \mapsto P^T P$, with the action of $\mathrm{GL}(V)(\mathbb{Q})$ upon X.

A.6.2 Proof of Corollary A.3

Consider a $\mathrm{GL}_g(\mathbb{Z})$-equivariant admissible decomposition $X = \bigcup_\sigma \sigma$, obtained, for example, as the intersection of an admissible decomposition §10.3 with the set of positive definite symmetric matrices. Let us identify X^{BS} with its image in $\mathfrak{B}(\mathbb{R})$ under the map f. Thus we may write

$$X \subset X^{\mathrm{BS}} \subset \mathfrak{B}(\mathbb{R}) \ .$$

Consider a polyhedron $\sigma \subset X$, and denote its closure in X^{BS} by $\overline{\sigma}$. Its closure in $\mathfrak{B}(\mathbb{R})$ is the compact polyhedron $\sigma^{\mathcal{B}} = \sigma^{\mathcal{B}^{\min,\sigma}}$ in the minimal blow-up of σ, since, by definition, the latter is the topological closure of the inverse image of the interior of σ in its minimal blow up. The equality $\sigma^{\mathcal{B}} = \sigma^{\mathcal{B}^{\min,\sigma}}$ follows from invariance under extraneous modifications (Proposition 5.14). By Proposition 11.4, each open face of $\sigma^{\mathcal{B}}$ is a product of strictly positive polyhedra, and is therefore contained in the image of an $e(P)$, for a certain P, by the last part of Theorem A.16. It follows that $\sigma^{\mathcal{B}} \subset X^{\mathrm{BS}}$. Thus

$$X \subset \bigcup_{\sigma} \sigma^{\mathcal{B}} \subset X^{\mathrm{BS}}$$

and in particular $\overline{\sigma} = \sigma^{\mathcal{B}}$. By construction, $X(P) = A_P \setminus (\overline{A}_P \times X)$ is contained in the closure of $X = A_P \setminus (A_P \times X)$ (let the coordinates λ_i on A_P tend to zero) and so X^{BS} is contained in the closure of X. The previous equation then implies that

$$X^{\mathrm{BS}} = \bigcup_{\sigma} \sigma^{\mathcal{B}} \tag{A.33}$$

It follows that X^{BS} equals the closure of X in $\mathfrak{B}(\mathbb{R})$, which was the remaining statement to prove in Theorem A.2. The decomposition (A.33) has similar properties to an admissible decomposition as in §10.3 (finitely many $\mathrm{GL}_g(\mathbb{Z})$ orbits, intersections are closures of faces, etc). Conclude using the fact that, by definition (11.5), the topological space $|L\mathcal{A}_g^{\mathrm{trop},\mathfrak{B}}|$ is the quotient of $\bigcup_\sigma \sigma^{\mathcal{B}}$ by the action of $\mathrm{GL}_g(\mathbb{Z})$, since the gluing relations in the diagram category $\mathcal{D}_g^{\mathrm{trop},\mathcal{B}}$ are those induced by $\mathrm{GL}_g(\mathbb{Z})$.

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