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ABSTRACT

We revisit the paper [*Automorphy lifting for residually reducible l -adic Galois representations*, J. Amer. Math. Soc. **28** (2015), 785–870] by the third author. We prove new automorphy lifting theorems for residually reducible Galois representations of unitary type in which the residual representation is permitted to have an arbitrary number of irreducible constituents.

1. Introduction

In this paper, we prove new automorphy lifting theorems for Galois representations of unitary type. Thus, we are considering representations $\rho : G_F \rightarrow \mathrm{GL}_n(\bar{\mathbb{Q}}_l)$, where G_F is the absolute Galois group of a CM field F and ρ is conjugate self-dual, i.e. there is an isomorphism $\rho^c \cong \rho^\vee \otimes \epsilon^{1-n}$, where $c \in \mathrm{Aut}(F)$ is complex conjugation. We say in this paper that such a representation is automorphic if there exists a regular algebraic, conjugate self-dual, cuspidal (RACSDC) automorphic representation π which is matched with ρ under the Langlands correspondence. (See § 1.1 below for a more precise formulation.)

We revisit the context of the paper [Tho15], proving theorems valid in the case that $\bar{\rho}$ is absolutely reducible, but still satisfies a certain non-degeneracy condition (we say that $\bar{\rho}$ is ‘Schur’). The first theorems of this type were proved in the paper [Tho15], under the assumption that $\bar{\rho}$ has only two irreducible constituents. Our main motivation here is to remove this restriction. Our results are applied to the problem of symmetric power functoriality in [NT19], where they are combined with level-raising theorems to establish automorphy of symmetric powers for certain level 1 Hecke eigenforms congruent to a theta series.

We are also able to weaken some other hypotheses in [Tho15], leading to the following result, which is the main theorem of this paper.

THEOREM 1.1 (Theorem 6.1). *Let F be an imaginary CM number field with maximal totally real subfield F^+ and let $n \geq 2$ be an integer. Let l be a prime and suppose that $\rho : G_F \rightarrow \mathrm{GL}_n(\bar{\mathbb{Q}}_l)$*

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is a continuous semisimple representation satisfying the following hypotheses.

- (i) $\rho^c \cong \rho^\vee \epsilon^{1-n}$.
- (ii) ρ is ramified at only finitely many places.
- (iii) ρ is ordinary of weight λ for some $\lambda \in (\mathbb{Z}_+^n)^{\text{Hom}(F, \bar{\mathbb{Q}}_l)}$.
- (iv) There is an isomorphism $\bar{\rho}^{\text{ss}} \cong \bar{\rho}_1 \oplus \cdots \oplus \bar{\rho}_d$, where each $\bar{\rho}_i$ is absolutely irreducible and satisfies $\bar{\rho}_i^c \cong \bar{\rho}_i^\vee \epsilon^{1-n}$, and $\bar{\rho}_i \not\cong \bar{\rho}_j$ if $i \neq j$.
- (v) There exists a finite place \tilde{v}_0 of F , prime to l , such that $\rho|_{G_{F_{\tilde{v}_0}}}^{\text{ss}} \cong \bigoplus_{i=1}^n \psi \epsilon^{n-i}$ for some unramified character $\psi : G_{F_{\tilde{v}_0}} \rightarrow \bar{\mathbb{Q}}_l^\times$.
- (vi) There exist a RACSDC representation π of $\text{GL}_n(\mathbb{A}_F)$ and $\iota : \bar{\mathbb{Q}}_l \rightarrow \mathbb{C}$ such that:
 - (a) π is ι -ordinary;
 - (b) $r_\iota(\pi)^{\text{ss}} \cong \bar{\rho}^{\text{ss}}$;
 - (c) $\pi_{\tilde{v}_0}$ is an unramified twist of the Steinberg representation.
- (vii) $F(\zeta_l)$ is not contained in $\bar{F}^{\ker \text{ad}(\bar{\rho}^{\text{ss}})}$ and F is not contained in $F^+(\zeta_l)$. For each $1 \leq i, j \leq d$, $\bar{\rho}_i|_{G_{F(\zeta_l)}}$ is absolutely irreducible and $\bar{\rho}_i|_{G_{F(\zeta_l)}} \not\cong \bar{\rho}_j|_{G_{F(\zeta_l)}}$ if $i \neq j$. Moreover, $\bar{\rho}^{\text{ss}}$ is primitive (i.e. not induced from any proper subgroup of G_F) and $\bar{\rho}^{\text{ss}}(G_F)$ has no quotient of order l .
- (viii) $l > 3$ and $l \nmid n$.

Then ρ is automorphic: there exists an ι -ordinary RACSDC automorphic representation Π of $\text{GL}_n(\mathbb{A}_F)$ such that $r_\iota(\Pi) \cong \rho$.

Comparing this with [Tho15, Theorem 7.1], we see that we now allow an arbitrary number of irreducible constituents, while also removing the requirement that the individual constituents are adequate (in the sense of [Tho12]) and potentially automorphic. This assumption of potential automorphy was used in [Tho15], together with the Khare–Wintenberger method, to get a handle on the quotient of the universal deformation ring of $\bar{\rho}$ corresponding to reducible deformations. This made generalizing [Tho15, Theorem 7.1] to the case where more than two irreducible constituents are allowed seem a formidable task: one would want to know that any given direct sum of irreducible constituents of $\bar{\rho}$ was potentially automorphic, and then perhaps use induction on the number of constituents to control the reducible locus.

The first main innovation in this paper that allows us to bypass this is the observation that by fully exploiting the ‘connectedness dimension’ argument to prove that $R = \mathbb{T}$ (which goes back to [SW99] and appears in this paper in the proof of Theorem 5.1), one only needs to control the size of the reducible locus in quotients of the universal deformation ring that are known *a priori* to be finite over the Iwasawa algebra Λ . This can be done easily by hand using the ‘locally Steinberg’ condition (as in § 3.3).

The second main innovation is a finer study of the universal deformation ring R^{univ} of a (reducible but) Schur residual representation. We show that if the residual representation has d absolutely irreducible constituents, then there is an action of a group μ_2^d on R^{univ} and identify the invariant subring $(R^{\text{univ}})^{\mu_2^d}$ with the subring topologically generated by the traces of Frobenius elements (which can also be characterized as the image P of the canonical map to R^{univ} from the universal pseudodeformation ring). This leads to a neat proof that the map $P \rightarrow R^{\text{univ}}$ is étale at prime ideals corresponding to irreducible deformations of $\bar{\rho}$.

We now describe the organization of this paper. Since it is naturally a continuation of [Tho15], we maintain the same notation and use several results and constructions from that paper as black boxes. We begin in §§ 2 and 3 by extending several results from [Tho15] about the relation

between deformations and pseudodeformations to the case where $\bar{\rho}$ is permitted to have more than two irreducible constituents. We also make the above-mentioned study of the dimension of the locus of reducible deformations.

In § 4 we recall from [Tho15] the definition of the unitary group of automorphic forms and Hecke algebras that we use, and state the $\mathbb{T}_{\mathfrak{q}} = R_{\mathfrak{p}}$ type result proved in that paper (here \mathfrak{p} denotes a dimension 1, characteristic l prime of R with good properties, in particular that the associated representation to $\mathrm{GL}_n(\mathrm{Frac} R/\mathfrak{p})$ is absolutely irreducible). In § 5 we carry out the main argument, based on the notion of connectedness dimension, which is described above. Finally, in § 6 we deduce Theorem 1.1, following a simplified version of the argument in [Tho15, § 7] that no longer makes reference to potential automorphy.

1.1 Notation

We use the same notation and normalizations for Galois groups, class field theory, and local Langlands correspondences as in [Tho15, Notation]. Rather than repeat this verbatim here we invite the reader to refer to that paper for more details. We do note the convention that if R is a ring and P is a prime ideal of R , then $R_{(P)}$ denotes the localization of R at P and R_P denotes the completion of the localization.

We recall that $\mathbb{Z}_n^+ \subset \mathbb{Z}^n$ denotes the set of tuples $\lambda = (\lambda_1, \dots, \lambda_n)$ of integers such that $\lambda_1 \geq \dots \geq \lambda_n$. It is identified in a standard way with the set of highest weights of GL_n . If F is a number field and $\lambda = (\lambda_{\tau}) \in (\mathbb{Z}_n^+)^{\mathrm{Hom}(F, \mathbb{C})}$, then we write Ξ_{λ} for the algebraic representation of $\mathrm{GL}_n(F \otimes_{\mathbb{Q}} \mathbb{C}) = \prod_{\tau \in \mathrm{Hom}(F, \mathbb{C})} \mathrm{GL}_n(\mathbb{C})$ of highest weight λ . If π is an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, we say that π is regular algebraic of weight λ if π_{∞} has the same infinitesimal character as the dual Ξ_{λ}^{\vee} .

Let F be a CM field (i.e. a totally imaginary quadratic extension of a totally real field F^+). We always write $c \in \mathrm{Aut}(F)$ for complex conjugation. We say that an automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ is conjugate self-dual if there is an isomorphism $\pi^c \cong \pi^{\vee}$. If π is a RACSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ and $\iota : \bar{\mathbb{Q}}_l \rightarrow \mathbb{C}$ is an isomorphism (for some prime l), then there exists an associated Galois representation $r_{\iota}(\pi) : G_F \rightarrow \mathrm{GL}_n(\bar{\mathbb{Q}}_l)$, characterized up to isomorphism by the requirement of compatibility with the local Langlands correspondence at each finite place of F ; see [Tho15, Theorem 2.2] for a reference. We say that a representation $\rho : G_F \rightarrow \mathrm{GL}_n(\bar{\mathbb{Q}}_l)$ is automorphic if there exists a choice of ι and RACSDC π such that $\rho \cong r_{\iota}(\pi)$.

One can define what it means for a RACSDC automorphic representation π to be ι -ordinary (see [Tho15, Lemma 2.3]; it means that the eigenvalues of certain Hecke operators, *a priori* l -adic integers, are in fact l -adic units). If $\mu \in (\mathbb{Z}_+^n)^{\mathrm{Hom}(F, \bar{\mathbb{Q}}_l)}$, we say (following [Tho15, Definition 2.5]) that a representation $\rho : G_F \rightarrow \mathrm{GL}_n(\bar{\mathbb{Q}}_l)$ is ordinary of weight μ if for each place $v|l$ of F , there is an isomorphism

$$\rho|_{G_{F_v}} \sim \begin{pmatrix} \psi_1 & * & * & * \\ 0 & \psi_2 & * & * \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & \psi_n \end{pmatrix},$$

where $\psi_i : G_{F_v} \rightarrow \bar{\mathbb{Q}}_l^{\times}$ is a continuous character satisfying the identity

$$\psi_i(\sigma) = \prod_{\tau : F_v \hookrightarrow \bar{\mathbb{Q}}_l} \tau(\mathrm{Art}_{F_v}^{-1}(\sigma))^{-(\mu_{\tau, n-i+1+i-1})}$$

for all σ in a suitable open subgroup of I_{F_v} . An important result [Tho15, Theorem 2.4] is that if π is RACSDC of weight λ and ι -ordinary, then $r_i(\pi)$ is ordinary of weight $\iota\lambda$, where by definition $(\iota\lambda)_\tau = \lambda_{\iota\tau}$.

2. Determinants

We first give the definition of a determinant from [Che14]. We recall that if A is a ring and M, N are A -modules, then an A -polynomial law $F : M \rightarrow N$ is a natural transformation $F : h_M \rightarrow h_N$, where $h_M : A\text{-alg} \rightarrow \text{Sets}$ is the functor $h_M(B) = M \otimes_A B$. The A -polynomial law F is called homogeneous of degree $n \geq 1$ if for all $b \in B, x \in M \otimes_A B$, we have $F_B(bx) = b^n F_B(x)$.

DEFINITION 2.1. Let A be a ring and let R be an A -algebra. An A -valued determinant of R of dimension $n \geq 1$ is a multiplicative A -polynomial law $D : R \rightarrow A$ which is homogeneous of degree n .

If D is a determinant, then there are associated polynomial laws $\Lambda_i : R \rightarrow A, i = 0, \dots, n$, given by the formulae

$$D(t - r) = \sum_{i=0}^n (-1)^i \Lambda_i(r) t^{n-i}$$

for all $r \in R \otimes_A B$. We define the characteristic polynomial A -polynomial law $\chi : R \rightarrow R$ by the formula $\chi(r) = \sum_{i=0}^n (-1)^i \Lambda_i(r) r^{n-i}$ ($r \in R \otimes_A B$). We write $\text{CH}(D)$ for the two-sided ideal of R generated by the coefficients of $\chi(r_1 t_1 + \dots + r_m t_m) \in R[t_1, \dots, t_m]$ for all $m \geq 1$ and $r_1, \dots, r_m \in R$. We have $\text{CH}(D) \subseteq \ker(D)$ [Che14, Lemma 1.21]. The determinant D is said to be Cayley–Hamilton if $\text{CH}(D) = 0$, equivalently if $\chi = 0$ (i.e. χ is the zero A -polynomial law).

We next recall the definition of a generalized matrix algebra [BC09, Definition 1.3.1].

DEFINITION 2.2. Let A be a ring and let R be an A -algebra. We say that R is a *generalized matrix algebra* of type (n_1, \dots, n_d) if it is equipped with the following data:

- (i) a family of orthogonal idempotents e_1, \dots, e_d with $e_1 + \dots + e_d = 1$; and
- (ii) for each $1 \leq i \leq d$, an A -algebra isomorphism $\psi_i : e_i R e_i \rightarrow M_{n_i}(A)$

such that the trace map $T : R \rightarrow A$ defined by $T(x) = \sum_{i=1}^d \text{tr } \psi_i(e_i x e_i)$ satisfies $T(xy) = T(yx)$ for all $x, y \in R$. We refer to the data $\mathcal{E} = \{e_i, \psi_i, 1 \leq i \leq d\}$ as the *data of idempotents* of R .

CONSTRUCTION 2.3. We recall the structure of generalized matrix algebras from [BC09, § 1.3.2]. Let R be a generalized matrix algebra of type (n_1, \dots, n_d) with data of idempotents $\mathcal{E} = \{e_i, \psi_i, 1 \leq i \leq d\}$. For each $1 \leq i \leq d$, let $E_i \in e_i R e_i$ be the unique element such that $\psi_i(E_i)$ is the element of $M_{n_i}(A)$ whose row 1, column 1 entry is 1 and all other entries are 0. We set $\mathcal{A}_{i,j} = E_i R E_j$ for each $1 \leq i, j \leq d$. Note that $\mathcal{A}_{i,j} \mathcal{A}_{j,k} \subseteq \mathcal{A}_{i,k}$ for each $1 \leq i, j, k \leq d$, and the trace map T induces an isomorphism $\mathcal{A}_{i,i} \cong A$ for each $1 \leq i \leq d$. Via this isomorphism, we will tacitly view $\mathcal{A}_{i,j} \mathcal{A}_{j,i}$ as an ideal in A for each $1 \leq i, j \leq d$. With this multiplication, there is an

isomorphism of A -algebras

$$R \cong \begin{pmatrix} M_{n_1}(A) & M_{n_1, n_2}(\mathcal{A}_{1,2}) & \cdots & M_{n_1, n_d}(\mathcal{A}_{1,d}) \\ M_{n_2, n_1}(\mathcal{A}_{2,1}) & M_{n_2}(A) & \cdots & M_{n_2, n_d}(\mathcal{A}_{2,d}) \\ \vdots & \vdots & \ddots & \vdots \\ M_{n_d, n_1}(\mathcal{A}_{d,1}) & M_{n_d, n_2}(\mathcal{A}_{d,2}) & \cdots & M_{n_d}(A) \end{pmatrix}. \tag{1}$$

The following result of Chenevier allows us to use the above structure when studying determinants.

THEOREM 2.4. *Let A be a Henselian local ring with residue field k , let R be an A -algebra, and let $D: R \rightarrow A$ be a Cayley–Hamilton determinant. Suppose that there exist surjective and pairwise non-conjugate k -algebra homomorphisms $\bar{\rho}_i: R \rightarrow M_{n_i}(k)$ such that $\bar{D} = \prod_{i=1}^d (\det \circ \bar{\rho}_i)$, where $\bar{D} = D \otimes_R k$.*

Then there is a datum of idempotents $\mathcal{E} = \{e_i, \psi_i, 1 \leq i \leq d\}$ for which R is a generalized matrix algebra and such that $\psi_i \otimes_A k = \bar{\rho}_i|_{e_i R e_i}$. Any two such data are conjugate by an element of R^\times .

We note that the assumptions of Theorem 2.4 say that D is residually split and multiplicity-free, in the sense of [Che14, Definition 2.19].

Proof. The existence of such a datum of idempotents $\mathcal{E} = \{e_i, \psi_i, 1 \leq i \leq d\}$ is contained in [Che14, Theorem 2.22] and its proof. The statement that two such data are conjugate is exactly as in [BC09, Lemma 1.4.3]. Namely, if $\mathcal{E}' = \{e'_i, \psi'_i, 1 \leq i \leq d\}$ is another such choice, then since $\text{End}_R(R e_i) \cong M_{n_i}(A) \cong \text{End}_R(R e'_i)$ are local rings, the Krull–Schmidt–Azumaya theorem [CR81, Theorem 6.12] (see also [CR81, Remark 6.14 and Chapter 6, Exercise 14]) implies that there is $x \in R^\times$ such that $x e_i x^{-1} = e'_i$ for each $1 \leq i \leq d$. By Skolem–Noether, we can adjust x by an element of $(\oplus_{i=1}^d e_i R e_i)^\times$ so that it further satisfies $x \psi_i x^{-1} = \psi'_i$. \square

We now show that the reducibility ideals of [BC09, Proposition 1.5.1] and their basic properties carry over for determinants (so without having to assume that $n!$ is invertible in A).

PROPOSITION 2.5. *Let A be a Henselian local ring with residue field k , let R be an A -algebra, and let $D: R \rightarrow A$ be a determinant. Assume that $\bar{D} = D \otimes_A k: R \otimes_A k \rightarrow k$ is split and multiplicity free. Write $\bar{D} = \prod_{i=1}^d \bar{D}_i$ with each \bar{D}_i absolutely irreducible of dimension n_i .*

Let $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_s)$ be a partition of $\{1, \dots, d\}$. There is an ideal $I_{\mathcal{P}}$ of A such that an ideal J of A satisfies $I_{\mathcal{P}} \subseteq J$ if and only if there are determinants $D_1, \dots, D_s: R \otimes_A A/J \rightarrow A/J$ such that $D \otimes_A A/J = \prod_{m=1}^s D_m$ and $D_m \otimes_A k = \prod_{i \in \mathcal{P}_m} \bar{D}_i$ for each $1 \leq m \leq s$. If this property holds, then D_1, \dots, D_s are uniquely determined and satisfy $\ker(D \otimes_A A/J) \subseteq \ker(D_m)$.

Moreover, let \mathcal{J} be a two-sided ideal of R with $\text{CH}(D) \subseteq \mathcal{J} \subseteq \ker(D)$ and let $\mathcal{A}_{i,j}$ be the A -modules as in Construction 2.3 for a choice of data of idempotents as in Theorem 2.4 applied to R/\mathcal{J} . Then $I_{\mathcal{P}} = \sum_{i,j} \mathcal{A}_{i,j} \mathcal{A}_{j,i}$, where the sum is over all pairs i, j not belonging to the same $\mathcal{P}_m \in \mathcal{P}$.

Proof. We follow the proof of [BC09, Proposition 1.5.1] closely. Choose a two-sided ideal \mathcal{J} of R with $\text{CH}(D) \subseteq \mathcal{J} \subseteq \ker(D)$, and data of idempotents \mathcal{E} for R/\mathcal{J} as in Theorem 2.4. We let $\mathcal{A}_{i,j}$

be as in Construction 2.3 and define $I_{\mathcal{P}} = \sum_{i,j} \mathcal{A}_{i,j} \mathcal{A}_{j,i}$, where the sum is over all pairs i, j not belonging to the same $\mathcal{P}_m \in \mathcal{P}$. Since another such choice of the data of idempotents is conjugate by an element of $(R/\mathcal{J})^\times$, the ideal $I_{\mathcal{P}}$ does not depend on the choice of \mathcal{E} . To see that it is independent of \mathcal{J} , first note that D further factors through a surjection $\psi: R/\mathcal{J} \rightarrow R/\ker(D)$. Under this surjection, the data of idempotents \mathcal{E} is sent to a data of idempotents for $R/\ker(D)$, and $\text{tr}(\psi(\mathcal{A}_{i,j})\psi(\mathcal{A}_{j,i})) = \text{tr}(\mathcal{A}_{i,j}\mathcal{A}_{j,i})$ since $\text{tr} \circ \psi = \text{tr}$.

We can now replace R with $R/\text{CH}(D)$ and assume that D is Cayley–Hamilton. Since $\text{CH}(D)$ is stable under base change, it suffices to show that $I_{\mathcal{P}} = 0$ if and only if there are determinants $D_1, \dots, D_s: R \rightarrow A$ such that $D = \prod_{m=1}^s D_m$ and $D_m \otimes_A k_A = \prod_{i \in \mathcal{P}_m} \bar{D}_i$ for each $1 \leq m \leq s$ and that, if this happens, then D_1, \dots, D_s are uniquely determined. Fix a datum of idempotents $\mathcal{E} = \{e_i, \psi_i, 1 \leq i \leq d\}$ for R as in Theorem 2.4, and let the notation be as in Construction 2.3. For each $1 \leq m \leq s$, we set $f_m = \sum_{i \in \mathcal{P}_m} e_i$. Then $1 = f_1 + \dots + f_s$ is a decomposition into orthogonal idempotents.

First assume that $I_{\mathcal{P}} = 0$. Let \tilde{D} denote the A -valued determinant on $R/\ker(D)$ arising from D . Fix $x \in R$, an A -algebra B , and $y \in R \otimes_A B$. If $1 \leq i, j \leq d$ do not belong to the same $\mathcal{P}_m \in \mathcal{P}$, then using the algebra structure as in (1) and the fact that $\mathcal{A}_{i,j}\mathcal{A}_{j,i} = 0$, we have $e_i x e_j y = \sum_{l \neq i} e_i x e_j y e_l$, and [Che14, Lemma 1.12(i)] gives

$$D(1 + e_i x e_j y) = D\left(1 + \sum_{l \neq i} e_i x e_j y e_l\right) = D\left(1 + \sum_{l \neq i} x e_j y e_l e_i\right) = D(1) = 1.$$

By [Che14, Lemma 1.19], $e_i x e_j \in \ker(D)$ for all $x \in R$ and all i, j that do not belong to the same $\mathcal{P}_m \in \mathcal{P}$. We then have an isomorphism of A -algebras $R/\ker(D) \cong \prod_{m=1}^s f_m(R/\ker(D))f_m$ and [Che14, Lemma 2.4] gives $D = \prod_{m=1}^s D_m$, where $D_m: R \rightarrow A$ is the composite of the surjection $R \rightarrow f_m(R/\ker(D))f_m$ with the determinant $\tilde{D}_m: f_m(R/\ker(D))f_m \rightarrow A$ given by $x \mapsto \tilde{D}(x + 1 - f_m)$. It is immediate that $D_m \otimes_A k_A = \prod_{i \in \mathcal{P}_m} \bar{D}_i$ for each $1 \leq m \leq s$.

Now assume that there are determinants $D_1, \dots, D_s: R \rightarrow A$ such that $D = \prod_{m=1}^s D_m$ and $D_m \otimes_A k = \prod_{i \in \mathcal{P}_m} \bar{D}_i$ for each $1 \leq m \leq s$. The determinants D_m have dimension $d_m := \sum_{i \in \mathcal{P}_m} n_i$. The trace map yields an equality

$$\sum_{1 \leq m \neq m' \leq s} \text{tr}(f_m R f_{m'} R f_m) = I_{\mathcal{P}}.$$

So, to show that $I_{\mathcal{P}} = 0$, it suffices to show that $\text{tr}(f_m R f_{m'} R f_m) = 0$ for $m \neq m'$. For this, it suffices to show that $f_{m'} \in \ker(D_m)$ for any $m \neq m'$, since this implies that $f_m R f_{m'} \in \ker(D_l)$ for any $1 \leq l \leq s$ and hence

$$D(1 + t f_m R f_{m'} R f_m) = \prod_{l=1}^s D_l(1 + t f_m R f_{m'} R f_m) = 1.$$

For any idempotent f of R , we have the determinant $D_{m,f}: f R f \rightarrow A$ given by $D_{m,f}(x) = D_m(x + 1 - f)$. When $f = f_m$,

$$D_{m,f_m} \otimes_A k = \prod_{i \in \mathcal{P}_m} \bar{D}_{i,f_m} = \prod_{i \in \mathcal{P}_m} \bar{D}_{i,e_i}$$

has dimension d_m . Then [Che14, Lemma 2.4(2)] implies that $D_{m,1-f_m}$ has dimension 0, i.e. is constant and equal to 1. So, for any $m \neq m'$, the characteristic polynomial of $f_{m'}$ with respect

to D_m is

$$D_m(t - f_{m'}) = D_{m,f_m}(t)D_{m,1-f_m}(t - f_{m'}) = t^{d_m}.$$

Then $f_{m'} = f_{m'}^{d_m} \in \text{CH}(D_m) \subseteq \ker(D_m)$, which is what we wanted to prove. This further shows that for each $1 \leq m \leq s$, the determinant D_m is the composite of the surjections

$$R \rightarrow \bigoplus_{l=1}^s f_l R f_l \rightarrow f_m R f_m$$

with the determinant $D_{f_m} : f_m R f_m \rightarrow A$. Since any two choices of the data of idempotents are conjugate under R^\times , each D_m is uniquely determined by D . □

3. Deformations

Galois deformation theory plays an essential role in this paper. The set of results we use is essentially identical to that of [Tho15], with some technical improvements. In this section we recall the notation used in [Tho15], without giving detailed definitions or proofs; we then proceed to prove the new results that we need. Some of the definitions recalled here were first given in [CHT08] or [Ger19], but in order to avoid sending the reader to too many different places we restrict our citations to [Tho15].

We will use exactly the same set-up and notation for deformation theory as in [Tho15]. We recall that this means that we fix at the outset the following objects.

- A CM number field F , with its totally real subfield F^+ .
- An odd prime l such that each l -adic place of F^+ splits in F . We write S_l for the set of l -adic places of F^+ .
- A finite set S of finite places of F^+ which split in F . We assume that $S_l \subset S$ and write $F(S)$ for the maximal extension of F which is unramified outside S and set $G_{F,S} = \text{Gal}(F(S)/F)$ and $G_{F^+,S} = \text{Gal}(F(S)/F^+)$. We fix a choice of complex conjugation $c \in G_{F^+,S}$.
- For each $v \in S$, we fix a choice of place \tilde{v} of F such that $\tilde{v}|_{F^+} = v$, and define $\tilde{S} = \{\tilde{v} \mid v \in S\}$.

We also fix the following data.

- A coefficient field $K \subset \bar{\mathbb{Q}}_l$ with ring of integers \mathcal{O} , residue field k , and maximal ideal $\lambda \subset \mathcal{O}$.
- A continuous homomorphism $\chi : G_{F^+,S} \rightarrow \mathcal{O}^\times$. We write $\bar{\chi} = \chi \bmod \lambda$.
- A continuous homomorphism $\bar{r} : G_{F^+,S} \rightarrow \mathcal{G}_n(k)$ such that $\bar{r}^{-1}(\mathcal{G}_n^\circ(k)) = G_{F,S}$. Here \mathcal{G}_n is the algebraic group over \mathbb{Z} defined in [CHT08, §2.1]. We follow the convention that if $R : \Gamma \rightarrow \mathcal{G}_n(A)$ is a homomorphism and $\Delta \subset \Gamma$ is a subgroup such that $R(\Delta) \subset \mathcal{G}_n^0(A)$, then $R|_\Delta$ denotes the composite homomorphism

$$\Delta \rightarrow \mathcal{G}_n^0(A) = \text{GL}_n(A) \times \text{GL}_1(A) \rightarrow \text{GL}_n(A).$$

Thus, $\bar{r}|_{G_{F,S}}$ takes values in $\text{GL}_n(k)$.

If $v \in S_l$, then we write $\Lambda_v = \mathcal{O}[[I_{F_{\tilde{v}}}^{ab}(l)]^n]$, where $I_{F_{\tilde{v}}}^{ab}(l)$ denotes the inertia group in the maximal abelian pro- l extension of $F_{\tilde{v}}$. We set $\Lambda = \hat{\otimes}_v \Lambda_v$, the completed tensor product being over \mathcal{O} . A *global deformation problem*, as defined in [Tho15, §3], then consists of a tuple

$$\mathcal{S} = (F/F^+, S, \tilde{S}, \Lambda, \bar{r}, \chi, \{\mathcal{D}_v\}_{v \in S}).$$

The extra data that we have not defined consists of the choice of a *local deformation problem* \mathcal{D}_v for each $v \in S$. We will not need to define any new local deformation problems in this paper, but we recall that the following have been defined in [Tho15]:

- ‘ordinary deformations’ give rise to a problem \mathcal{D}_v^Δ for each $v \in S_l$ [Tho15, § 3.3.2];
- ‘Steinberg deformations’ give rise to a problem $\mathcal{D}_v^{\text{St}}$ for each place $v \in S$ such that $q_v \equiv 1 \pmod{l}$ and $\bar{r}|_{G_{F_v}}$ is trivial;
- ‘ χ_v -ramified deformations’ give rise to a problem $\mathcal{D}_v^{\chi_v}$ for each place $v \in S$ such that $q_v \equiv 1 \pmod{l}$ and $\bar{r}|_{G_{F_v}}$ is trivial, given the additional data of a tuple $\chi_v = (\chi_{v,1}, \dots, \chi_{v,n})$ of characters $\chi_{v,i} : k(v)^\times(l) \rightarrow k^\times$;
- ‘unrestricted deformations’ give rise to a problem \mathcal{D}_v^\square for any $v \in S$.

If \mathcal{S} is a global deformation problem, then we can define (as in [Tho15]) a functor $\text{Def}_{\mathcal{S}} : \mathcal{C}_\Lambda \rightarrow \text{Sets}$ of ‘deformations of type \mathcal{S} ’. By definition, if $A \in \mathcal{C}_\Lambda$, then $\text{Def}_{\mathcal{S}}(A)$ is the set of $\text{GL}_n(A)$ -conjugacy classes of homomorphisms $r : G_{F^+,S} \rightarrow \mathcal{G}_n(A)$ lifting \bar{r} such that $\nu \circ r = \chi$ and for each $v \in S$, $r|_{G_{F_v}} \in \mathcal{D}_v(A)$. If \bar{r} is *Schur* (see [Tho15, Definition 3.2]), then the functor $\text{Def}_{\mathcal{S}}$ is represented by an object $R_{\mathcal{S}}^{\text{univ}} \in \mathcal{C}_\Lambda$.

3.1 An erratum to [Tho15]

We point out an error in [Tho15]. We thank Lue Pan for bringing this to our attention. In [Tho15, Proposition 3.15] it is asserted that the ring R_v^1 (representing the deformation problem \mathcal{D}_v^1 for $v \in R$, defined under the assumptions $q_v \equiv 1 \pmod{l}$ and $\bar{r}|_{G_{F_v}}$ trivial) has the property that $R_v^1/(\lambda)$ is generically reduced. This is false, even in the case $n = 2$, as can be seen from the statement of [Sho16, Proposition 5.8] (and noting the identification $R_v^1/(\lambda) = R_v^{\chi_v}/(\lambda)$). We offer the following corrected statement.

PROPOSITION 3.1. *Let \bar{R}_v^1 denote the nilreduction of R_v^1 . Then $\bar{R}_v^1/(\lambda)$ is generically reduced.*

Proof. Let \mathcal{M} denote the scheme over \mathcal{O} of pairs of $n \times n$ matrices (Φ, Σ) , where Φ is invertible, the characteristic polynomial of Σ equals $(X - 1)^n$, and we have $\Phi \Sigma \Phi^{-1} = \Sigma^{q_v}$. Then R_v^1 can be identified with the completed local ring of \mathcal{M} at the point $(1_n, 1_n) \in \mathcal{M}(k)$. By [Mat89, Theorem 23.9] (and since \mathcal{M} is excellent), it is enough to show that if $\bar{\mathcal{M}}$ denotes the nilreduction of \mathcal{M} , then $\bar{\mathcal{M}} \otimes_{\mathcal{O}} k$ is generically reduced.

Let $\mathcal{M}_1, \dots, \mathcal{M}_r$ denote the irreducible components of \mathcal{M} with their reduced subscheme structure. According to [Tho12, Lemma 3.15], each $\mathcal{M}_i \otimes_{\mathcal{O}} K$ is non-empty of dimension n^2 , while the $\mathcal{M}_i \otimes_{\mathcal{O}} k$ are the pairwise-distinct irreducible components of $\mathcal{M} \otimes_{\mathcal{O}} k$ and are all generically reduced. Let $\bar{\eta}_i$ denote the generic point of $\mathcal{M}_i \otimes_{\mathcal{O}} k$. Then $\bar{\eta}_i$ admits an open neighbourhood in $\bar{\mathcal{M}}$ not meeting any \mathcal{M}_j ($j \neq i$). Consequently, we have an equality of local rings $\mathcal{O}_{\bar{\mathcal{M}}, \bar{\eta}_i} = \mathcal{O}_{\mathcal{M}_i, \bar{\eta}_i}$, showing that $\mathcal{O}_{\bar{\mathcal{M}}, \bar{\eta}_i}/(\lambda)$ is reduced (in fact, a field). This shows that $\bar{\mathcal{M}} \otimes_{\mathcal{O}} k$ is generically reduced. □

We now need to explain why this error does not affect the proofs of the two results in [Tho15] which rely on the assertion that $R_v^1/(\lambda)$ is generically reduced. The first of these is [Tho15, Proposition 3.17], which states that the Steinberg deformation ring R_v^{St} has the property that $R_v^{\text{St}}/(\lambda)$ is generically reduced. The proof of this result is still valid if one replaces R_v^1 there with \bar{R}_v^1 . Indeed, we need only note that R_v^{St} is \mathcal{O} -flat (by definition) and reduced (since $R_v^{\text{St}}[1/l]$

is regular, by [Tay08, Lemma 3.3]). The map $R_v^1 \rightarrow R_v^{\text{St}}$ therefore factors through a surjection $\bar{R}_v^1 \rightarrow R_v^{\text{St}}$.

The next result is [Tho15, Lemma 3.40(2)], which describes the irreducible components of the localization and completion of a ring R^∞ at a prime ideal P_∞ . The ring R^∞ has R_v^1 as a (completed) tensor factor, and the generic reducedness is used to justify an appeal to [Tho15, Proposition 1.6]. Since passing to nilreduction does not change the underlying topological space, one can argue instead with the quotient of R^∞ , where R_v^1 is replaced by \bar{R}_v^1 . The statement of [Tho15, Lemma 3.40] is therefore still valid.

3.2 Pseudodeformations

In this section, we fix a global deformation problem

$$\mathcal{S} = (F/F^+, S, \tilde{S}, \Lambda, \bar{r}, \chi, \{\mathcal{D}_v\}_{v \in S})$$

such that \bar{r} is Schur. We write $P_S \subset R_S^{\text{univ}}$ for the Λ -subalgebra topologically generated by the coefficients of characteristic polynomials of Frobenius elements $\text{Frob}_w \in G_{F,S}$ (w prime to S). The subring P_S is studied in [Tho15, § 3.4], where it is shown using results of Chenevier that P_S is a complete Noetherian local Λ -algebra and that the inclusion $P_S \subset R_S^{\text{univ}}$ is a finite ring map (see [Tho15, Proposition 3.29]).

In fact, more is true, as we now describe. Let $\bar{B} \in \text{GL}_n(k)$ be the matrix defined by the formula $\bar{r}(c) = (\bar{B}, -\chi(c))_J \in \mathcal{G}_n(k)$. Let $\bar{\rho} = \bar{r}|_{G_{F,S}}$ and suppose that there is a decomposition $\bar{r} = \bigoplus_{i=1}^d \bar{r}_i$ with $\bar{\rho}_i = \bar{r}_i|_{G_{F,S}}$ absolutely irreducible for each i . The representations $\bar{\rho}_i$ are pairwise non-isomorphic, because \bar{r} is Schur (see [Tho15, Lemma 3.3]). We recall [Tho15, Lemma 3.1] that to give a lifting $r : G_{F^+,S} \rightarrow \mathcal{G}_n(R)$ of \bar{r} with $\nu \circ r = \chi$ is equivalent to giving the following data.

- A representation $\rho : G_{F,S} \rightarrow \text{GL}_n(R)$ lifting $\bar{\rho} = \bar{r}|_{G_{F,S}}$.
- A matrix $B \in \text{GL}_n(R)$ lifting \bar{B} with ${}^t B = -\chi(c)B$ and $\chi(\delta)B = \rho(\delta^c)B^t \rho(\delta)$ for all $\delta \in G_{F,S}$.

The equivalence is given by letting $\rho = r|_{G_{F,S}}$ and $r(c) = (B, -\chi(c))_J$. Conjugating r by $M \in \text{GL}_n(R)$ takes B to $MB^t M$. Note that the matrix B defines an isomorphism $\chi \otimes \rho^\vee \xrightarrow{\sim} \rho^c$.

We embed the group μ_2^d in $\text{GL}_n(\mathcal{O})$ as block diagonal matrices, the i th block being of size $\dim_k \bar{\rho}_i$. We assume that the global deformation problem \mathcal{S} has the property that each local deformation problem $\mathcal{D}_v \subset \mathcal{D}_v^\square$ is invariant under conjugation by μ_2^d ; this is the case for each of the local deformation problems recalled above. With this assumption, the group μ_2^d acts on the ring R_S^{univ} by conjugation of the universal deformation and we have the following result.

PROPOSITION 3.2.

- (i) We have an equality $P_S = (R_S^{\text{univ}})^{\mu_2^d}$.
- (ii) Let $\mathfrak{p} \subset R_S^{\text{univ}}$ be a prime ideal and let $\mathfrak{q} = \mathfrak{p} \cap P_S$. Let $E = \text{Frac } R_S^{\text{univ}}/\mathfrak{p}$ and suppose that the associated representation $\rho_{\mathfrak{p}} = r_{\mathfrak{p}}|_{G_{F,S}} \otimes_A E : G_{F,S} \rightarrow \text{GL}_n(E)$ is absolutely irreducible. Then $P_S \rightarrow R_S^{\text{univ}}$ is étale at \mathfrak{q} and μ_2^d acts transitively on the set of primes of R_S^{univ} above \mathfrak{q} .

We first establish a preliminary lemma, before proving the proposition.

LEMMA 3.3. Let $R = R_S^{\text{univ}}/(\mathfrak{m}_{P_S})$ and let $r : G_{F^+,S} \rightarrow \mathcal{G}_n(R)$ be a representative of the specialization of the universal deformation. Then, after possibly conjugating by an element of

$1 + M_n(\mathfrak{m}_R)$, $r|_{G_{F,S}}$ has (block) diagonal entries given by $\bar{\rho}_1, \dots, \bar{\rho}_d$, and the matrix B defined above is equal to \bar{B} . (Note we are not asserting that the off-diagonal blocks of $r|_{G_{F,S}}$ are zero.)

Proof. We let $\bar{e}_1, \bar{e}_2, \dots, \bar{e}_d \in M_n(k)$ denote the standard idempotents decomposing $\bar{r}|_{G_{F,S}}$ into the block diagonal pieces $\bar{\rho}_1, \dots, \bar{\rho}_d$. We let $\mathcal{A} \subset M_n(R)$ denote the image of $R[G_{F,S}]$ under r . The idempotents \bar{e}_i lift to orthogonal idempotents e_i in \mathcal{A} with $e_1 + \dots + e_d = 1$ and, after conjugating by an element of $1 + M_n(\mathfrak{m}_R)$, we can assume that the e_i are again the standard idempotents on R^n . Moreover, applying the first case of the proof of [BC09, Lemma 1.8.2], we can (and do) choose the e_i so that they are fixed by the anti-involution $\star : \mathcal{A} \rightarrow \mathcal{A}$ given by the formula $M \mapsto B({}^t M)B^{-1}$. This implies that the matrix B is block diagonal. We have $e_i \mathcal{A} e_i = M_{n_i}(R)$ (see [BC09, Lemma 1.4.3] and [Che14, Theorem 2.22]) and, for each $i \neq j$, we have $e_i \mathcal{A} e_j = M_{n_i, n_j}(\mathcal{A}_{i,j})$, where $\mathcal{A}_{i,j} \subset R$ is an ideal [BC09, Proposition 1.3.8].

Since $\det \circ r = \det \circ \bar{r}$, Proposition 2.5 shows that $\sum_{i \neq j} \mathcal{A}_{i,j} \mathcal{A}_{j,i} = 0$. This implies that for each i the map

$$R[G_{F,S}] \rightarrow M_{n_i}(R)$$

given by

$$x \mapsto e_i r(x) e_i$$

is an algebra homomorphism and we get an R -valued lift of $\bar{\rho}_i$. By the uniqueness assertion in Proposition 2.5, the determinant of this lift is equal to $\det \circ \bar{\rho}_i$. Since $\bar{\rho}_i$ is absolutely irreducible, it follows from [Che14, Theorem 2.22] that, after conjugating by a block diagonal matrix in $1 + M_n(\mathfrak{m}_R)$, we can assume that the map

$$x \mapsto e_i r(x) e_i$$

is induced by $\bar{\rho}_i$, which is the desired statement.

Finally, we consider the matrix B . We have already shown that B is block diagonal. For $1 \leq i \leq d$, we denote the corresponding block of B by B_i . It lifts a block \bar{B}_i of \bar{B} . By Schur's lemma, we have $B_i = \beta_i \bar{B}_i$ for some scalars $\beta_i \in 1 + \mathfrak{m}_R$. Since 2 is invertible in R , we can find $\lambda_i \in 1 + \mathfrak{m}_R$ with $\lambda_i^2 = \beta_i^{-1}$. Conjugating r by the diagonal matrix with λ_i in the i th block puts r into the desired form. \square

Proof of Proposition 3.2. We begin by proving the first part. We again let $R = R_S^{\text{univ}}/(\mathfrak{m}_{P_S})$. By Nakayama's lemma, it suffices to show that $R^{\mu_2^d} = k$. Indeed, the natural map $(R_S^{\text{univ}})^{\mu_2^d}/\mathfrak{m}_{P_S}(R_S^{\text{univ}})^{\mu_2^d} \rightarrow R^{\mu_2^d}$ is injective (i.e. $(\mathfrak{m}_{P_S} R_S^{\text{univ}})^{\mu_2^d} = \mathfrak{m}_{P_S}(R_S^{\text{univ}})^{\mu_2^d}$), since if $\sum_i m_i x_i$ is μ_2^d -invariant, with $m_i \in \mathfrak{m}_{P_S}$ and $x_i \in R_S^{\text{univ}}$, we have $\sum_i m_i x_i = (1/2^d) \sum_i m_i \sum_{\sigma \in \mu_2^d} \sigma x_i$, which is an element of $\mathfrak{m}_{P_S}(R_S^{\text{univ}})^{\mu_2^d}$. Let $r : G_{F+,S} \rightarrow \mathcal{G}_n(R)$ be a representative of the specialization of the universal deformation satisfying the conclusion of Lemma 3.3. Then R is a finite k -algebra and is generated as a k -algebra by the matrix entries of r and hence the matrix entries of $\rho = r|_{G_{F,S}}$ (because $B = \bar{B}$). We recall the ideals $\mathcal{A}_{i,j} \subset R$ appearing in the proof of Lemma 3.3, which are generated by the block (i, j) matrix entries of ρ . The conjugate self-duality of ρ is given by ${}^t \rho(\delta) = \chi(\delta) \bar{B}^{-1} \rho((\delta^c)^{-1}) \bar{B}$, $\delta \in G_{F,S}$. Since \bar{B} is block diagonal, we deduce that $\mathcal{A}_{i,j} = \mathcal{A}_{j,i}$. Since $\sum_{i \neq j} \mathcal{A}_{i,j} \mathcal{A}_{j,i} = 0$, we see that $\mathcal{A}_{i,j}^2 = 0$ for $i \neq j$. We deduce that R is generated as a k -module by $1 \in R$ and products of the form

$$a_{\mathcal{P}} = \prod_{(i,j) \in \mathcal{P}} a_{i,j},$$

where $\emptyset \neq \mathcal{P} \subset \{(i, j) : 1 \leq i < j \leq d\}$ and $a_{i,j} \in \mathcal{A}_{i,j}$ has action of μ_2^d given by $((-1)^{\alpha_1}, \dots, (-1)^{\alpha_d})a_{i,j} = (-1)^{\alpha_i + \alpha_j}a_{i,j}$. Suppose that the action of μ_2^d on $a_{\mathcal{P}}$ is trivial. Then, for each $1 \leq i \leq d$, i appears in an even number of elements of \mathcal{P} . A product $a'_{j_1, j_2} = a_{1, j_1} a_{1, j_2}$ lies in \mathcal{A}_{j_1, j_2} and the action of μ_2^d is given by $((-1)^{\alpha_1}, \dots, (-1)^{\alpha_d})a'_{j_1, j_2} = (-1)^{\alpha_{j_1} + \alpha_{j_2}}a'_{j_1, j_2}$. Since 1 appears in an even number of elements of \mathcal{P} , we can ‘pair off’ these elements and rewrite $a_{\mathcal{P}}$ as a product

$$a_{\mathcal{P}'} = \prod_{(i,j) \in \mathcal{P}'} a'_{i,j},$$

where $\mathcal{P}' \subset \{(i, j) : 2 \leq i < j \leq d\}$ and the action of μ_2^d on $a'_{i,j}$ is given by the same formula as for $a_{i,j}$. Continuing in this manner, we deduce that $a_{\mathcal{P}}$ is the product of an even number of elements of $\mathcal{A}_{d-1, d}$ and thus equals 0 since $\mathcal{A}_{d-1, d}^2 = 0$.

The invariant subring $R^{\mu_2^d}$ is equal to the k -submodule of R generated by $\sum_{\sigma \in \mu_2^d} \sigma x$, where x runs over a set of k -module generators of R (since 2 is invertible in k). It follows from the above calculation that $R^{\mu_2^d} = k$.

We now prove the second part. The diagonally embedded subgroup $\mu_2 \subseteq \mu_2^d$ acts trivially on $R_{\mathcal{S}}^{\text{univ}}$, so we have an induced action of μ_2^d/μ_2 . The first part together with [Sta17, Tag 0BRI] implies that μ_2^d/μ_2 acts transitively on the set of primes of $R_{\mathcal{S}}^{\text{univ}}$ above \mathfrak{q} . Let $R = R_{\mathcal{S}}^{\text{univ}}/\mathfrak{p}$ and let $r_{\mathfrak{p}}: G_{F^+, S} \rightarrow \mathcal{G}_n(R)$ be a representative of the specialization of the universal deformation. By [Sta17, Tag 0BST], to finish the proof it will be enough to show that if $\sigma \in \mu_2^d$, $\sigma(\mathfrak{p}) = \mathfrak{p}$, and σ acts as the identity on R , then $\sigma \in \mu_2$.

If $\sigma \in \mu_2^d$ corresponds to the block diagonal matrix $g \in \text{GL}_n(\mathcal{O})$, then these conditions imply that $r_{\mathfrak{p}}$ and $gr_{\mathfrak{p}}g^{-1}$ are conjugate by an element $\gamma \in 1 + M_n(\mathfrak{m}_R)$. Since $r_{\mathfrak{p}}|_{G_{F,S}} \otimes E = \rho_{\mathfrak{p}}$ is absolutely irreducible, this implies that $g\gamma^{-1}$ is scalar and so g must also be scalar as $l > 2$; hence, $g \in \mu_2$. This completes the proof. \square

For each partition $\{1, \dots, d\} = \mathcal{P}_1 \sqcup \mathcal{P}_2$ with $\mathcal{P}_1, \mathcal{P}_2$ both non-empty, Proposition 2.5 gives an ideal $I_{(\mathcal{P}_1, \mathcal{P}_2)} \subset P_{\mathcal{S}}$ cutting out the locus where the determinant $\det r|_{G_{F,S}}$ is $(\mathcal{P}_1, \mathcal{P}_2)$ -reducible. We write $I_{\mathcal{S}}^{\text{red}} = \prod_{(\mathcal{P}_1, \mathcal{P}_2)} I_{(\mathcal{P}_1, \mathcal{P}_2)}$, an ideal of $P_{\mathcal{S}}$.

LEMMA 3.4. *Let $\mathfrak{p} \subset R_{\mathcal{S}}^{\text{univ}}$ be a prime ideal and let $\mathfrak{q} = \mathfrak{p} \cap P_{\mathcal{S}}$. Let $A = R_{\mathcal{S}}^{\text{univ}}/\mathfrak{p}$, $E = \text{Frac } A$. Then $\rho_{\mathfrak{p}} = r_{\mathfrak{p}}|_{G_{F,S}} \otimes_A E$ is absolutely irreducible if and only if $I_{\mathcal{S}}^{\text{red}} \not\subset \mathfrak{q}$.*

Proof. If $I_{\mathcal{S}}^{\text{red}} \subset \mathfrak{q}$, then $I_{(\mathcal{P}_1, \mathcal{P}_2)} \subset \mathfrak{q}$ for some proper partition $(\mathcal{P}_1, \mathcal{P}_2)$. Then Proposition 2.5 implies that $\det r_{\mathfrak{p}}$ admits a decomposition $\det \circ r_{\mathfrak{p}}|_{G_{F,S}} = D_1 D_2$ for two determinants $D_i : A[G_{F,S}] \rightarrow M_{n_i}(A)$. Then [Che14, Corollary 2.13] implies that $\rho_{\mathfrak{p}}$ is not absolutely irreducible.

Suppose conversely that $\rho_{\mathfrak{p}}$ is not absolutely irreducible. Let $J_{(\mathcal{P}_1, \mathcal{P}_2)}$ denote the image of $I_{(\mathcal{P}_1, \mathcal{P}_2)}$ in A . We must show that some $J_{(\mathcal{P}_1, \mathcal{P}_2)}$ is zero. Let \mathcal{A} denote the image of $A[G_{F,S}]$ in $M_n(A)$ under $r_{\mathfrak{p}}|_{G_{F,S}}$. According to [BC09, Theorem 1.4.4], we can assume that \mathcal{A} has the form

$$\mathcal{A} = \begin{pmatrix} M_{n_1}(A) & M_{n_1, n_2}(\mathcal{A}_{1,2}) & \cdots & M_{n_1, n_d}(\mathcal{A}_{1,d}) \\ M_{n_2, n_1}(\mathcal{A}_{2,1}) & M_{n_2}(A) & \cdots & M_{n_2, n_d}(\mathcal{A}_{2,d}) \\ \vdots & \vdots & \ddots & \vdots \\ M_{n_d, n_1}(\mathcal{A}_{d,1}) & M_{n_d, n_2}(\mathcal{A}_{d,2}) & \cdots & M_{n_d}(A) \end{pmatrix}, \tag{2}$$

where each $\mathcal{A}_{i,j}$ is a fractional ideal of E . Consequently, $\mathcal{A} \otimes_A E$ has the form

$$\mathcal{A} \otimes_A E = \begin{pmatrix} M_{n_1}(E) & M_{n_1,n_2}(\mathcal{E}_{1,2}) & \cdots & M_{n_1,n_d}(\mathcal{E}_{1,d}) \\ M_{n_2,n_1}(\mathcal{E}_{2,1}) & M_{n_2}(E) & \cdots & M_{n_2,n_d}(\mathcal{E}_{2,d}) \\ \vdots & \vdots & \ddots & \vdots \\ M_{n_d,n_1}(\mathcal{E}_{d,1}) & M_{n_d,n_2}(\mathcal{E}_{d,2}) & \cdots & M_{n_d}(E) \end{pmatrix}, \tag{3}$$

where each $\mathcal{E}_{i,j} = \mathcal{A}_{i,j} \otimes_A E$ equals either E or 0 . Let $f_i \in M_n(E)$ denote the matrix with 1 in the (i, i) th entry and 0 everywhere else. If $\rho_{\mathfrak{p}}$ is not absolutely irreducible, then $\mathcal{A} \otimes_A E$ is a proper subspace of $M_n(E)$, so there exists i such that $(\mathcal{A} \otimes_A E)f_i \subset M_n(E)f_i$ is a proper subspace. Since $M_n(E)f_i$ is isomorphic as an $M_n(E)$ -module to the tautological representation E^n , this implies that the $\mathcal{A} \otimes_A E$ -module E^n admits a proper invariant subspace. After permuting the diagonal blocks, we can assume that this subspace is $E^{n_1+\cdots+n_s}$ for some $s < d$ (included as the subspace of E^n where the last $n_{s+1} + \cdots + n_d$ entries are zero). Otherwise said, the spaces $\mathcal{E}_{i,j}$ for $i > s, j \leq s$ are zero. If $\mathcal{P}_1 = \{1, \dots, s\}$ and $\mathcal{P}_2 = \{s + 1, \dots, d\}$, then this implies that $\mathcal{J}_{(\mathcal{P}_1, \mathcal{P}_2)} \otimes_A E = 0$ and hence (as A is a domain) $\mathcal{J}_{(\mathcal{P}_1, \mathcal{P}_2)} = 0$. This completes the proof. \square

LEMMA 3.5. *Let $\mathfrak{p} \subset R_S^{\text{univ}}$ be a prime ideal, $A = R_S^{\text{univ}}/\mathfrak{p}$, $E = \text{Frac } A$. Then $r_{\mathfrak{p}} \otimes_A E$ is Schur and, if $r_{\mathfrak{p}}|_{G_{F,S}} \otimes_A E$ is not absolutely irreducible, then $r_{\mathfrak{p}}$ is equivalent (i.e. conjugate by an element in $1 + M_n(\mathfrak{m}_A)$) to a type- \mathcal{S} lifting of the form $r_{\mathfrak{p}} = r_1 \oplus r_2$, where $r_i : G_{F^+,S} \rightarrow \mathcal{G}_{m_i}(A)$ and $m_1 m_2 \neq 0$.*

Proof. We argue, as in the proof of Lemma 3.4, using the image $\mathcal{A} \subset M_n(A)$ of $A[G_{F,S}]$, which is a generalized matrix algebra. Suppose that we are given $G_{F,S}$ -invariant subspaces $E^n \supset W_1 \supset W_2$ such that W_2 and E^n/W_1 are irreducible. We can assume that \mathcal{A} has the form (2) and that this decomposition is block upper triangular (perhaps with respect to a coarser partition than $n = n_1 + \cdots + n_d$) and moreover that the first block corresponds to W_2 , while the last block corresponds to E^n/W_1 . In particular, W_2 and E^n/W_1 are even absolutely irreducible. Note that there can be no isomorphism $W_2^{c\nu}(\nu \circ r_{\mathfrak{p}}) \cong E^n/W_1$; if there was, then it would imply an identity of A -valued determinants, which we could reduce modulo \mathfrak{m}_A to obtain an identity $\{\rho_i\} = \{\rho_j\}$ of sets of irreducible constituents of $\bar{r}|_{G_{F,S}}$. Since these appear with multiplicity 1 , this is impossible. This all shows that $r_{\mathfrak{p}} \otimes_A E$ is necessarily Schur.

Now suppose that $r_{\mathfrak{p}}|_{G_{F,S}} \otimes_A E$ is not absolutely irreducible. After permuting the diagonal blocks of \bar{r} , we can assume that there is some $1 \leq m \leq d$ such that $\mathcal{A}_{i,j} = 0$ for $i > m, j \leq m$. The existence of the conjugate self-duality of $r_{\mathfrak{p}}$ implies (cf. [BC09, Lemma 1.8.5]) that $\mathcal{A}_{j,i} = 0$ in the same range, giving a decomposition $r_{\mathfrak{p}}|_{G_{F,S}} = \rho_1 \oplus \rho_2$ of representations over A . Since $r_{\mathfrak{p}} \otimes_A E$ is Schur, the conjugate self-duality of $r_{\mathfrak{p}}$ must make ρ_1 and ρ_2 orthogonal, showing that $r_{\mathfrak{p}}$ itself decomposes as $r_{\mathfrak{p}} = r_1 \oplus r_2$. \square

3.3 Dimension bounds

We now suppose that S admits a decomposition $S = S_l \sqcup S(B) \sqcup R \sqcup S_a$, where:

- for each $v \in S(B) \cup R$, $q_v \equiv 1 \pmod{l}$ and $\bar{r}|_{G_{F_{\bar{v}}}}$ is trivial;
- for each $v \in S_a$, $q_v \not\equiv 1 \pmod{l}$, $\bar{r}|_{G_{F_{\bar{v}}}}$ is unramified, and $\bar{r}|_{G_{F_{\bar{v}}}}$ is scalar. (Then any lifting of $\bar{r}|_{G_{F_{\bar{v}}}}$ is unramified.)

We consider the global deformation problem

$$\mathcal{S} = (F/F^+, S, \tilde{S}, \Lambda, \bar{r}, \chi, \{\mathcal{D}_v^\Delta\}_{v \in S_l} \cup \{\mathcal{D}_v^{\text{St}}\}_{v \in S(B)} \cup \{\mathcal{D}_v^1\}_{v \in R} \cup \{\mathcal{D}_v^\square\}_{v \in S_a}),$$

where \bar{r} is assumed to be Schur. We define quantities $d_{F,0} = d_0$ and $d_{F,l} = d_l$ as follows. Let Δ denote the Galois group of the maximal abelian pro- l extension of F which is unramified outside l , and let Δ_0 denote the Galois group of the maximal abelian pro- l extension of F which is unramified outside l and in which each place of $S(B)$ splits completely. We set

$$d_0 = \dim_{\mathbb{Q}_l} \ker(\Delta[1/l] \rightarrow \Delta_0[1/l])^{e=-1}$$

and

$$d_l = \inf_{v \in S_l} [F_v^+ : \mathbb{Q}_l].$$

LEMMA 3.6. *Suppose that $d_l > n(n - 1)/2 + 1$. Let $A \in \mathcal{C}_\Lambda$ be a finite Λ -algebra and let $r : G_{F^+,S} \rightarrow \mathcal{G}_n(A)$ be a lifting of \bar{r} of type \mathcal{S} . Then $\dim A/(I_S^{\text{red}}, \lambda) \leq n[F^+ : \mathbb{Q}] - d_0$.*

Proof. We can assume without loss of generality that $A = A/(I_S^{\text{red}}, \lambda)$ and must show that $\dim A \leq [F^+ : \mathbb{Q}] - d_0$. Since A is Noetherian and we are interested only in dimension, we can assume moreover that A is integral. Let $E = \text{Frac}(A)$. Then (Lemma 3.5) we can find a non-trivial partition $n = n_1 + n_2$ and homomorphisms $r_i : G_{F^+,S} \rightarrow \mathcal{G}_{n_i}(A)$ ($i = 1, 2$) such that $r = r_1 \oplus r_2$.

Let \bar{E} be a choice of algebraic closure of E . Our condition on d_l means that we can appeal to [Tho15, Corollary 3.12] (characterization of A -valued points of \mathcal{D}_v^Δ for each $v \in S_l$). This result implies the existence for each $v \in S_l$ of an increasing filtration

$$0 \subset \text{Fil}_v^1 \subset \text{Fil}_v^2 \subset \dots \subset \text{Fil}_v^n = \bar{E}^n$$

of $r|_{G_{F_v}} \otimes_A \bar{E}$ by G_{F_v} -invariant subspaces such that each $\text{Fil}_v^i / \text{Fil}_v^{i-1}$ is one dimensional, and the character of $I_{F_v}^{ab}(l)$ acting on this space is given by composing the universal character $\psi_v^i : I_{F_v}^{ab}(l) \rightarrow \Lambda_v^\times$ with the homomorphism

$$\Lambda_v \rightarrow \Lambda \rightarrow A \rightarrow \bar{E}.$$

The direct sum decomposition of r leads to a decomposition $r|_{G_{F_v}} = r_1|_{G_{F_v}} \oplus r_2|_{G_{F_v}}$. Let $F_v^i = \text{Fil}_v^i \cap r_1|_{G_{F_v}} \otimes_A \bar{E}$ and $G_v^i = \text{Fil}_v^i \cap r_2|_{G_{F_v}} \otimes_A \bar{E}$. Then F_v^\bullet and G_v^\bullet are increasing filtrations of \bar{E}^{n_1} and \bar{E}^{n_2} , respectively, with graded pieces of dimension at most 1. We write σ_v for the bijection

$$\sigma_v : \{1, \dots, n_1\} \sqcup \{1, \dots, n_2\} \rightarrow \{1, \dots, n\},$$

which is increasing on $\{1, \dots, n_1\}$ and $\{1, \dots, n_2\}$ and which has the property that $\sigma_v(\{1, \dots, n_1\})$ is the set of $i \in \{1, \dots, n\}$ such that the graded piece F_v^i / F_v^{i-1} is non-trivial.

Let $\Lambda_{v,1}$ and $\Lambda_{v,2}$ denote the analogues of the algebra Λ_v in dimensions n_1 and n_2 , respectively. The bijection σ_v determines in an obvious way an isomorphism $\Lambda_{v,1} \hat{\otimes} \Lambda_{v,2} \cong \Lambda_v$. Applying again [Tho15, Corollary 3.12], we see that with this structure on A of $\Lambda_{v,i}$ -algebra, each homomorphism $r_i|_{G_{F_v}} : G_{F_v} \rightarrow \text{GL}_{n_i}(A)$ is of type $\mathcal{D}_v^\Delta(A)$. Similarly if we define $\Lambda_i = \hat{\otimes}_{v \in S_l} \Lambda_{v,i}$ then the collection of bijections $(\sigma_v)_{v \in S_l}$ determines an isomorphism $\Lambda_1 \hat{\otimes} \Lambda_2 \cong \Lambda$.

We also define $\Lambda_{v,0} = \mathcal{O}[I_{F_v}^{ab}(l)]$ and $\Lambda_0 = \hat{\otimes}_{v \in S_l} \Lambda_{v,0}$. Then there are natural maps $\Lambda_0 \rightarrow \Lambda_i$ classifying the characters $\prod_{j=1}^{n_i} \psi_v^j : I_{F_v}^{ab}(l) \rightarrow \Lambda_{v,i}^\times$. Let $\bar{\chi}_i = \det \bar{r}_i|_{G_{F,S}}$. We get a commutative

diagram

$$\begin{CD} \Lambda_0 \hat{\otimes}_{\mathcal{O}} \Lambda_0 @>>> k[[\Delta/(c+1)]] \hat{\otimes}_k k[[\Delta/(c+1)]] \\ @VVV @VVV \\ \Lambda_1 \hat{\otimes}_{\mathcal{O}} \Lambda_2 @>>> A \end{CD}$$

where the map $k[[\Delta/(c+1)]] \hat{\otimes}_k k[[\Delta/(c+1)]] \rightarrow A$ classifies the pair of A -valued characters $(\chi_1, \chi_2) = (\bar{\chi}_1^{-1} \det r_1|_{G_F}, \bar{\chi}_2^{-1} \det r_2|_{G_F})$ of the group $\Delta/(c+1)$. The natural map $\Lambda_0/(\lambda) \rightarrow k[[\Delta/(c+1)]]$ is finite (and dominant). (Note that $\det r_1$ and $\det r_2$ are unramified at places of $S(B) \cup R \cup S_a$ because of our choices of deformation problem.)

We now use the existence of the places $S(B)$. For each place $v \in S(B)$, imposing the Steinberg condition on $r_1 \oplus r_2$ determines a relation $\chi_1(\text{Frob}_{\tilde{v}})^{n_2} = \chi_2(\text{Frob}_{\tilde{v}})^{n_1}$ in A . Let \mathcal{E} denote the quotient of the group $\Delta/(c+1) \times \Delta/(c+1)$ by the \mathbb{Z}_l -submodule generated by the elements $(n_2 \text{Frob}_{\tilde{v}}, -n_1 \text{Frob}_{\tilde{v}})$ ($v \in S(B)$). Then $\dim_{\mathbb{Q}_l} \mathcal{E}[1/l] = \dim_{\mathbb{Q}_l}(\Delta/(c+1) \times \Delta/(c+1))[1/l] - d_0$ and we in fact have the following commutative diagram.

$$\begin{CD} \Lambda_0 \hat{\otimes}_{\mathcal{O}} \Lambda_0 @>>> k[[\mathcal{E}]] \\ @VVV @VVV \\ \Lambda_1 \hat{\otimes}_{\mathcal{O}} \Lambda_2 @>>> A \end{CD}$$

We deduce that the map $\Lambda \cong \Lambda_1 \hat{\otimes}_{\mathcal{O}} \Lambda_2 \rightarrow A$ factors through the quotient $\Lambda_1 \hat{\otimes}_{\mathcal{O}} \Lambda_2 \otimes_{\Lambda_0 \hat{\otimes}_{\mathcal{O}} \Lambda_0} k[[\mathcal{E}]]$ of dimension $n[F^+ : \mathbb{Q}] - d_0$. Using finally that A is a finite Λ -algebra, we see that $\dim A$ must satisfy the same estimate. This concludes the proof. \square

DEFINITION 3.7. Let $A \in \mathcal{C}_\Lambda$ and let $r : G_{F^+, S} \rightarrow \mathcal{G}_n(A)$ be a homomorphism of type \mathcal{S} . We say that r is generic at l if it satisfies the following two conditions.

- For each $v \in S_l$, the universal characters $\psi_1^v, \dots, \psi_n^v : I_{F_v}^{ab}(l) \rightarrow A^\times$ are distinct.
- There exist $v \in S_l$ and $\sigma \in I_{F_v}^{ab}(l)$ such that the elements $\psi_1^v(\sigma), \dots, \psi_n^v(\sigma) \in A^\times$ satisfy no non-trivial \mathbb{Z} -linear relation.

We say that r is generic if it is generic at l , A is a domain, and $r|_{G_F} \otimes_A \text{Frac}(A)$ is absolutely irreducible.

LEMMA 3.8. *There exists a countable collection of ideals $I_i \subset \Lambda/(\lambda)$ ($i = 1, 2, \dots$) with the following properties.*

- (i) For each $i = 1, 2, \dots$, we have $\dim \Lambda/I_i \leq n[F^+ : \mathbb{Q}] - d_l$.
- (ii) Suppose that $A \in \mathcal{C}_\Lambda$ and $r : G_{F^+} \rightarrow \mathcal{G}_n(A)$ is a lifting of type \mathcal{S} which is not generic at l . Then there exists $i \geq 1$ such that $I_i A = 0$.

Proof. For each $v \in S_l$, let $d_v = [F_v^+ : \mathbb{Q}_l]$ and let $\sigma_{v,1}, \dots, \sigma_{v,d_v} \in I_{F_v}^{ab}(l)$ be elements which project to a \mathbb{Z}_l -basis of the l -torsion-free quotient of this finitely generated \mathbb{Z}_l -module.

For each $1 \leq i < j \leq n$ and $v \in S_l$, we define an ideal $I(i, j, v) = (\lambda, \psi_v^i(\sigma_{v,k}) - \psi_v^j(\sigma_{v,k}))_{k=1, \dots, d_v}$. Then $\dim \Lambda/I(i, j, v) = n[F^+ : \mathbb{Q}] - d_v$ and, if $\mathfrak{p} \subset \Lambda$ is a prime of characteristic l which does not contain $I(i, j, v)$, then the characters $\psi_v^i \bmod \mathfrak{p}$ and $\psi_v^j \bmod \mathfrak{p}$ are distinct.

Suppose that we are given for each $v \in S_l$ an $n \times d_v$ matrix A_v with integer entries, and with at least one non-zero entry in each column. Then we define an ideal $J((A_v)_{v \in S_l}) \subset \Lambda$ as the ideal generated by λ and all the elements

$$\prod_{i=1}^n \psi_v^i(\sigma_{v,j})^{A_{v,i,j}} - 1 \quad (v \in S_l, j = 1, \dots, d_v).$$

Then $\dim \Lambda/J((A_v)_{v \in S_l}) = (n - 1)[F^+ : \mathbb{Q}]$ and, if $\mathfrak{p} \subset \Lambda$ is a prime of characteristic l not containing any of the ideals $J((A_v)_{v \in S_l})$, then there exist $v \in S_l$ and $1 \leq j \leq d_v$ such that the elements $\psi_v^1(\sigma_{v,j}), \dots, \psi_v^n(\sigma_{v,j}) \in (\Lambda/\mathfrak{p})^\times$ satisfy no non-trivial \mathbb{Z} -linear relation.

The lemma is completed on taking our countable collection of ideals I_i to consist of all the ideals $I(i, j, v)$ and $J((A_v)_{v \in S_l})$ defined above. □

Combining the previous lemmas, we obtain the following result.

LEMMA 3.9. *Suppose that $d_l > n(n - 1)/2 + 1$ and that $A \in \mathcal{C}_\Lambda$ is a finite Λ -algebra such that $\dim A/(\lambda) > \sup(n[F^+ : \mathbb{Q}] - d_0, n[F^+ : \mathbb{Q}] - d_l)$. Let $r : G_{F^+, S} \rightarrow \mathcal{G}_n(A)$ be a homomorphism of type \mathcal{S} . Then we can find a prime ideal $\mathfrak{p} \subset A$ of dimension 1 and characteristic l such that, writing $r_{\mathfrak{p}} = r \bmod \mathfrak{p} : G_{F^+} \rightarrow \mathcal{G}_n(A/\mathfrak{p})$, $r_{\mathfrak{p}}$ is generic.*

Proof. Replacing A by $A/(\lambda)$, we can assume that A is a finite $\Lambda/(\lambda)$ -algebra. Let I_i ($i = 1, 2, \dots$) be the countable collection of ideals of $\Lambda/(\lambda)$ defined in Lemma 3.8. Then $\dim A/I_i A \leq n[F^+ : \mathbb{Q}] - d_l$ for each $i = 1, 2, \dots$. Let $I_0 = I_S^{\text{red}} A$; then Lemma 3.6 shows that $\dim A/I_0 \leq n[F^+ : \mathbb{Q}] - d_0$.

Applying [Tho15, Lemma 1.9], we can find a prime ideal $\mathfrak{p} \subset A$ of dimension 1 (necessarily of characteristic l) such that \mathfrak{p} does not contain any of the ideals $I_0, I_1 A, I_2 A, \dots$. By construction, the homomorphism $r_{\mathfrak{p}}$ is then generic. □

4. Automorphic forms and Hecke algebras on unitary groups

4.1 Hecke algebras

We introduce automorphic forms on unitary groups and related Hecke algebras, using exactly the same notation as in [Tho15, § 4]. This means we start with the following data.

- An integer $n \geq 1$, an odd prime l , and a totally imaginary CM number field L with totally real subfield L^+ . We write S_l for the set of l -adic places of L^+ . We assume that L/L^+ is everywhere unramified. (We note that this implies that $[L^+ : \mathbb{Q}]$ is even. Indeed, the quadratic character of $(L^+)^\times \backslash \mathbb{A}_{L^+}^\times / \hat{\mathcal{O}}_{L^+}^\times$ cutting out L has non-trivial restriction to $(L_v^+)^\times$ for each $v | \infty$ but is trivial on $(-1)_{v | \infty} \in (L_\infty^+)^\times$.)
- A coefficient field $K \subset \bar{\mathbb{Q}}_l$ which contains the image of all embeddings $L \hookrightarrow \bar{\mathbb{Q}}_l$.
- A finite set $S(B)$ of finite, prime-to- l places of L^+ which all split in L . If n is even, then we assume that $n[L^+ : \mathbb{Q}]/2 + |S(B)|$ is also even. We allow the possibility that $S(B)$ may be empty. (Since $[L^+ : \mathbb{Q}]$ is even, we are really just asking that if n is even, then $|S(B)|$ is even.)

We can then find a central simple algebra B over L equipped with an involution \dagger such that $\dim_L B = n^2$, $B^{\text{op}} \cong B \otimes_{L,c} L$, B is split outside $S(B)$, B is a division algebra locally at places $w | S(B)$ of L , and $\dagger|_L = c$. We can moreover assume that the unitary group G over L^+ defined

by the formula (R an L^+ -algebra)

$$G(R) = \{g \in (B \otimes_{L^+} R)^\times \mid gg^{\dagger \otimes 1} = 1\},$$

has the property that $G(L^+ \otimes_{\mathbb{Q}} \mathbb{R})$ is compact and, for each finite place $v \notin S(B)$ of L^+ , G_{L^+} is quasi-split (hence unramified).

We consider automorphic forms on the group G . To define our Hecke algebras, we need to fix the following additional choices.

- A finite set R of finite places of L^+ , disjoint from $S_l \cup S(B)$ and split in L , such that for each $v \in R$, $q_v \equiv 1 \pmod{l}$.
- For each $v \in R$, we fix a choice of n -tuple of characters $\chi_{v,1}, \dots, \chi_{v,n} : k(v)^\times \rightarrow \mathcal{O}^\times$ which are trivial mod ϖ .
- A finite set T of finite places of L^+ split in L , containing $S_l \cup S(B) \cup R$. For each $v \in T$, we fix a choice of place \tilde{v} of L lying above v and set $\tilde{T} = \{\tilde{v} \mid v \in T\}$.

It is also convenient to fix a choice of order $\mathcal{O}_B \subset B$ such that $\mathcal{O}_B^\dagger = \mathcal{O}_B$ and, for each place v of L^+ which splits $v = ww^c$ in L , \mathcal{O}_{B_w} is a maximal order in B_w . We use this maximal order \mathcal{O}_B to extend G to a group scheme over \mathcal{O}_{L^+} . If $v \in T$, then a choice of isomorphism $\mathcal{O}_{B_{\tilde{v}}} \cong M_n(\mathcal{O}_{F_{\tilde{v}}})$ determines an isomorphism $\iota_{\tilde{v}} : G_{\mathcal{O}_{F_{\tilde{v}}^+}} \rightarrow \text{Res}_{\mathcal{O}_{F_{\tilde{v}}}/\mathcal{O}_{F_{\tilde{v}}^+}} \text{GL}_n$ and we fix such a choice.

Now suppose that we are given an open compact subgroup $U = \prod_v U_v \subset G(\mathbb{A}_{L^+}^\infty)$ which satisfies the following conditions.

- There exists a finite place $v \notin S_l$ of L^+ such that U_v contains no non-trivial elements of finite order (in other words, U is sufficiently small in the terminology of [Tho15]).
- If $v \notin T$ is a finite place of L^+ split in L , then $U_v = G(\mathcal{O}_{L_v^+})$.
- If $v \in R$, then $U_v = \iota_{\tilde{v}}^{-1} \text{Iw}(\tilde{v})$, where $\text{Iw}(\tilde{v}) \subset \text{GL}_n(\mathcal{O}_{F_{\tilde{v}}})$ denotes the standard Iwahori subgroup.
- If v is a finite place of L^+ inert in L , then U_v is a hyperspecial maximal compact subgroup of $G(L_v^+)$.

In this case we have defined in [Tho15, Definition 4.2] a Hecke algebra $\mathbb{T}_\chi^T(U(\mathfrak{l}^\infty), \mathcal{O})$. It is a finite faithful Λ -algebra, defined as an inverse limit of Hecke algebras which act on spaces of ordinary automorphic forms on G with coefficients in \mathcal{O} . According to [Tho15, Proposition 4.7], to any maximal ideal $\mathfrak{m} \subset \mathbb{T}_\chi^T(U(\mathfrak{l}^\infty), \mathcal{O})$ of residue field k , one can associate a continuous semisimple representation $\bar{\rho}_\mathfrak{m} : G_{F,T} \rightarrow \text{GL}_n(k)$, uniquely characterized up to $\text{GL}_n(k)$ conjugacy by a formula for the characteristic polynomials of Frobenius elements in terms of Hecke operators.

4.2 Deformation rings

We connect the Hecke algebras defined in the previous section to deformation rings only under the following assumptions (i)–(iii).

- (i) T has the form $T = S_l \sqcup S(B) \sqcup R \sqcup S_a$, where S_a is a non-empty set of places of odd residue characteristic which are absolutely unramified and not split in $L(\zeta_l)$. For every $v \in S_l$, $[F_v^+ : \mathbb{Q}_l] > n(n-1)/2 + 1$.
- (ii) $U = \prod U_v \subset G(\mathbb{A}_{L^+}^\infty)$ is an open compact subgroup such that if $v \in S_a$, then $U_v = \iota_{\tilde{v}}^{-1}(\ker(\text{GL}_n(\mathcal{O}_{L_{\tilde{v}}}) \rightarrow \text{GL}_n(k(\tilde{v}))))$ and, if $v \in S(B)$, then U_v is the unique maximal compact subgroup of $G(F_v^+)$. Since S_a is non-empty, this forces U to be sufficiently small.

- (iii) $\mathfrak{m} \subset \mathbb{T}_\chi^T(U(\mathfrak{l}^\infty), \mathcal{O})$ is a maximal ideal of residue field k such that $\bar{\rho}_\mathfrak{m} : G_{F,T} \rightarrow \mathrm{GL}_n(k)$ satisfies the following conditions:
 - if $v \in S_a$, then $\bar{\rho}_\mathfrak{m}|_{G_{L_v}}$ is unramified and $\bar{\rho}_\mathfrak{m}(\mathrm{Frob}_v)$ is scalar;
 - if $v \in S_l \cup R \cup S(B)$, then $\bar{\rho}_\mathfrak{m}|_{G_{L_v}}$ is the trivial representation;
 - for each $v \in S(B) \cup R$, $q_v \equiv 1 \pmod{l}$;
 - let $\bar{\rho}_\mathfrak{m} = \bigoplus_{i=1}^d \bar{\rho}_i$ denote the decomposition into simple constituents. Then each $\bar{\rho}_i$ is absolutely irreducible, satisfies $\bar{\rho}_i^c \cong \bar{\rho}_i^\vee \otimes \epsilon^{1-n}$, and, if $i \neq j$, then $\bar{\rho}_i \not\cong \bar{\rho}_j$ (equivalently, the maximal ideal \mathfrak{m} is residually Schur in the sense of [Tho15, Definition 4.8]).

Then [Tho15, Proposition 4.9] $\bar{\rho}_\mathfrak{m}$ extends to a homomorphism $\bar{r}_\mathfrak{m} : G_{L^+,T} \rightarrow \mathcal{G}_n(k)$ such that $\nu \circ \bar{r}_\mathfrak{m} = \epsilon^{1-n} \delta_{L/L^+}^n$ and which is Schur. We can consider the global deformation problem

$$\mathcal{S}_\chi = (L/L^+, T, \tilde{T}, \Lambda, \bar{r}_\mathfrak{m}, \epsilon^{1-n} \delta_{L/L^+}^n, \{R_v^\Delta\}_{v \in S_l} \cup \{R_v^{\chi_v}\}_{v \in R} \cup \{R_v^{\mathrm{St}}\}_{v \in S(B)} \cup \{R_v^\square\}_{v \in S_a}).$$

Then there are Λ -algebra homomorphisms $P_{\mathcal{S}_\chi} \rightarrow R_{\mathcal{S}_\chi}$ and $P_{\mathcal{S}_\chi} \rightarrow \mathbb{T}_\chi^T(U(\mathfrak{l}^\infty), \mathcal{O})_\mathfrak{m}$ (see [Tho15, Proposition 4.13]). We define $J_{\mathcal{S}_\chi} = \ker(P_{\mathcal{S}_\chi} \rightarrow \mathbb{T}_\chi^T(U(\mathfrak{l}^\infty), \mathcal{O})_\mathfrak{m})$.

THEOREM 4.1. *Let $\mathfrak{p} \subset R_{\mathcal{S}_1}^{\mathrm{univ}}$ be a prime ideal of dimension 1 and characteristic l . Let $A = R_{\mathcal{S}_1}^{\mathrm{univ}}/\mathfrak{p}$ and suppose that the following conditions are satisfied.*

- (i) $J_{\mathcal{S}_1} R_{\mathcal{S}_1}^{\mathrm{univ}} \subset \mathfrak{p}$.
- (ii) *The representation $r_\mathfrak{p}$ is generic, in the sense of Definition 3.7.*
- (iii) *For each $v \in R$, $r_\mathfrak{p}|_{G_{L_v}}$ is the trivial representation and, if $l^N \nmid q_v - 1$, then $l^N > n$. For each $v \in S(B)$, $r_\mathfrak{p}|_{G_{L_v}}$ is unramified and $r_\mathfrak{p}(\mathrm{Frob}_v)$ is a scalar matrix.*
- (iv) $\bar{r}_\mathfrak{m}|_{G_{L,S}}$ is primitive.
- (v) $\zeta_l \notin L$, $\bar{r}_\mathfrak{m}|_{G_{L^+(\zeta_l)}}$ is Schur, and $\bar{r}_\mathfrak{m}(G_{L,S})$ does not have a quotient of order l .
- (vi) $l > 3$ and $l \nmid n$.

Let $Q \subset R_{\mathcal{S}_1}^{\mathrm{univ}}$ be a prime such that $Q \subset \mathfrak{p}$. Then $J_{\mathcal{S}_1} R_{\mathcal{S}_1}^{\mathrm{univ}} \subset Q$.

Proof. Let $\mathfrak{q} = \mathfrak{p} \cap P_{\mathcal{S}_1}$. We can assume, after twisting by a character, that $\mathrm{Frac} R_{\mathcal{S}_1}^{\mathrm{univ}}/\mathfrak{p} = \mathrm{Frac} P_{\mathcal{S}_1}/\mathfrak{q}$. (Apply [Tho15, Lemma 3.38] and [Tho15, Corollary 4.14].)

In the case that $\bar{r}_\mathfrak{m}|_{G_{L,S}}$ has two irreducible constituents, the theorem follows on combining [Tho15, Corollary 5.7] (existence of Taylor–Wiles primes under a subset of the hypotheses listed here) and [Tho15, Corollary 4.20] (the assertion $J_{\mathcal{S}_1} R_{\mathcal{S}_1}^{\mathrm{univ}} \subset Q$ assuming existence of Taylor–Wiles primes and, in addition, that $\bar{r}_\mathfrak{m}|_{G_{L,S}}$ has two irreducible constituents).

The proof of [Tho15, Corollary 4.20] can easily be modified to allow $\bar{r}_\mathfrak{m}|_{G_{L,S}}$ to have an arbitrary number d of irreducible constituents: one just needs to replace the μ_2^2 action there by the μ_2^d action on $R_{\mathcal{S}_1}^{\mathrm{univ}}$ described in §3.2 and replace the appeal to [Tho15, Proposition 3.29] with an appeal to Proposition 3.2 of this paper. We omit the details. \square

5. Propagation of potential pro-automorphy

In this section we use Theorem 4.1 (informally, $R = \mathbb{T}$ locally at generic primes) to prove our first automorphy lifting theorem for l -adic Galois representations. The argument follows similar lines to [Tho15, §6]. The main difference is that by making use of Lemma 3.9 we can make do under weaker assumptions. Especially, we do not need an *a priori* bound on the dimension of the locus of reducible deformations inside $R_{\mathcal{S}_1}^{\mathrm{univ}}$.

Let us take up assumptions (1)–(3) of § 4.2. Thus, we have a CM field L , a unitary group G , and a set $S = T = S_l \cup R \cup S(B) \cup S_a$ of finite places of L^+ split in L . We have an open compact subgroup $U \subset G(\mathbb{A}_{L^+}^\infty)$, a maximal ideal \mathfrak{m} of the Hecke algebra $\mathbb{T}^S(U(I^\infty), \mathcal{O})$ of residue field k , a deformation problem

$$\mathcal{S}_1 = (L/L^+, T, \tilde{T}, \Lambda, \bar{r}_m, \epsilon^{1-n} \delta_{L/L^+}^n, \{R_v^\Delta\}_{v \in S_l} \cup \{R_v^1\}_{v \in R} \cup \{R_v^{\text{St}}\}_{v \in S(B)} \cup \{R_v^\square\}_{v \in S_a}),$$

and a diagram of Λ -algebras

$$R_{\mathcal{S}_1}^{\text{univ}} \longleftarrow P_{\mathcal{S}_1} \longrightarrow \mathbb{T}_1^T(U(I^\infty), \mathcal{O})_{\mathfrak{m}}.$$

We can now state the main theorem of this section.

THEOREM 5.1. *Let $r : G_{L^+, S} \rightarrow \mathcal{G}_n(\mathcal{O})$ be a lifting of \bar{r}_m of type \mathcal{S}_1 such that $r|_{G_{L,S}}$ is ordinary of weight λ for some $\lambda \in (\mathbb{Z}_+^n)^{\text{Hom}(L, \bar{\mathbb{Q}}_l)}$. Suppose that the following conditions hold.*

- (i) $\bar{r}_m|_{G_{L,S}}$ is primitive.
- (ii) $\zeta_l \notin L$, $\bar{r}_m|_{G_{L^+(\zeta_l)}}$ is Schur and $\bar{r}_m(G_{L,S})$ does not have a quotient of order l .
- (iii) $l > 3$ and $l \nmid n$.
- (iv) Let $d_{L,0}, d_{L,l}$ be as defined in § 3.3. Then $d_{L,0} > |R|n(n+1) + 2$ and $d_{L,l} > \sup(|R|n(n+1) + 2, n(n-1)/2 + 1)$.

Then $r|_{G_{L,S}}$ is automorphic of weight λ .

We assume the hypotheses of Theorem 5.1 for the rest of § 5. Note that our assumption on $d_{L,0}$ implies that $S(B)$ is non-empty. In particular, the conclusion of Theorem 5.1 implies, by local–global compatibility at a place in $S(B)$, that $r|_{G_{L,S}}$ is absolutely irreducible.

Let us say that a soluble CM extension M/L is good if it is linearly disjoint from the extension of $L(\zeta_l)$ cut out by $\bar{r}_m|_{G_{L(\zeta_l)}}$ and every prime above $S_l \cup S_a \cup R$ splits in M . (Primes of $S(B)$ are not required to split and indeed this possibility plays an important role in the proof; cf. [Tho15, Proposition 6.2].)

LEMMA 5.2. *Let M/L be a good extension. Then $\bar{r}_m|_{G_M}$ is primitive, $\zeta_l \notin M$, $\bar{r}_m|_{G_{M^+(\zeta_l)}}$ is Schur, and $\bar{r}_m(G_M)$ does not have a quotient of order l .*

Proof. We take each property in turn. If M/L is good, then $\bar{r}_m(G_M) = \bar{r}_m(G_L)$, so this image does not have a quotient of order l . Since M/L is linearly disjoint from $L(\zeta_l)$, we have $\zeta_l \notin M$.

To say that $\bar{r}_m|_{G_{M^+(\zeta_l)}}$ is Schur is to say that $L \not\subset M^+(\zeta_l)$ and, if $\bar{\rho}, \bar{\rho}'$ are two Jordan–Hölder factors of $\bar{r}_m|_{G_M}$, then $\bar{\rho}^c \cong \bar{\rho}'^v \otimes \epsilon^{1-n}$ and $\bar{\rho} \not\cong \bar{\rho}'$. Since $\bar{r}_m(G_M) = \bar{r}_m(G_L)$, the latter property is not disturbed. We show that under our assumptions we in fact have $M \not\subset M^+(\zeta_l)$. It suffices to show that the two extensions M/M^+ and $M^+(\zeta_l)/M^+$ are linearly disjoint. Since these extensions arise from the linearly disjoint extensions L/L^+ and $L^+(\zeta_l)/L^+$ by compositum with M^+ , it is enough to check that the extensions M^+/L^+ and $L(\zeta_l)/L^+$ are linearly disjoint or even that $M^+ \cap L(\zeta_l) = L^+$. This follows from the stronger assertion that $M \cap L(\zeta_l) = L$.

Finally, the condition that $\bar{r}_m|_{G_M}$ is primitive depends only on the group $\bar{r}_m(G_M) = \bar{r}_m(G_L)$, so it is inherited from the corresponding condition for \bar{r}_m . \square

If M/L is a good extension and X (respectively \tilde{X}) is a set of places of L^+ (respectively L), then we write X_M (respectively \tilde{X}_M) for the set of places of M^+ lying

above a place of X (respectively places of M lying above a place of \tilde{X}). We write $\Lambda_M = \hat{\otimes}_{v \in S_{l,M}} \mathcal{O}[[I_{M_v}^{ab}(l)^n]]$ for the Iwasawa algebra of M (so $\Lambda_L = \Lambda$). There is a natural surjective homomorphism $\Lambda_M \rightarrow \Lambda$. We can define a deformation problem

$$\mathcal{S}_{1,M} = (M/M^+, T_M, \tilde{T}_M, \Lambda_M, \bar{r}_{\mathfrak{m}}|_{G_{M^+}}, \epsilon^{1-n} \delta_{M/M^+}^n, \{R_v^\Delta\}_{v \in S_{l,M}} \cup \{R_v^1\}_{v \in R_M} \cup \{R_v^{\text{St}}\}_{v \in S(B)_M} \cup \{R_v^\square\}_{v \in S_{a,M}}).$$

On the automorphic side, we can define an open compact subgroup $U_M \subset G(\mathbb{A}_{M^+}^\infty)$ with the property that $U_L = U$; see [Tho15, §4.5] for details. With this choice, it is possible to define a maximal ideal $\mathfrak{m}_M \subset \mathbb{T}^{T_M}(U_M(\mathbb{I}^\infty), \mathcal{O})$ with the property that $\bar{r}_{\mathfrak{m}_M} = \bar{r}_{\mathfrak{m}}|_{G_{M^+}}$ and then there exists a commutative diagram of Λ_M -algebras

$$\begin{array}{ccccc} R_{\mathcal{S}_1}^{\text{univ}} & \longleftarrow & P_{\mathcal{S}_1} & \longrightarrow & \mathbb{T}_1^T(U(\mathbb{I}^\infty), \mathcal{O})_{\mathfrak{m}} \\ \uparrow & & \uparrow & & \uparrow \\ R_{\mathcal{S}_{1,M}}^{\text{univ}} & \longleftarrow & P_{\mathcal{S}_{1,M}} & \longrightarrow & \mathbb{T}_1^{T_M}(U_M(\mathbb{I}^\infty), \mathcal{O})_{\mathfrak{m}_M} \end{array}$$

(see [Tho15, Proposition 4.18]). Let $J_M = J_{\mathcal{S}_{1,M}} P_{\mathcal{S}_1}$. Then we have an inclusion $J_{\mathcal{S}_{1,M}} P_{\mathcal{S}_1} \subset J_{\mathcal{S}_1}$. More generally, if $M_1/M_0/L$ is a tower of good extensions of L , then $J_{M_1} \subset J_{M_0}$. We say that a prime ideal $\mathfrak{p} \subset R_{\mathcal{S}_1}^{\text{univ}}$ is *potentially pro-automorphic* if there exists a good extension M/L such that $J_M R_{\mathcal{S}_1}^{\text{univ}} \subset \mathfrak{p}$.

PROPOSITION 5.3. *Let $\mathfrak{p} \subset R_{\mathcal{S}_1}^{\text{univ}}$ be a prime of dimension 1 and characteristic l which is potentially pro-automorphic and generic. Suppose further that for each $v \in R$, the restriction $r_{\mathfrak{p}}|_{G_{L_v}}$ is trivial. Then every minimal prime $Q \subset \mathfrak{p}$ is potentially pro-automorphic.*

Proof. The proof is the same as the proof of [Tho15, Proposition 6.2]. We need only replace the reference there to [Tho15, Corollary 4.20] to Theorem 4.1 here. □

Proof of Theorem 5.1. For any good extension M/L , the ring $R_{\mathcal{S}_1}^{\text{univ}}/(\lambda, J_M)$ is a finite Λ -algebra. Indeed, $R_{\mathcal{S}_{1,M}}^{\text{univ}}/(\lambda, J_{\mathcal{S}_{1,M}})$ is a finite Λ_M -algebra and we can appeal to [Tho15, Lemma 4.16] and [Tho15, Proposition 4.17]. It follows from Lemma 3.9 that any quotient of $R_{\mathcal{S}_1}^{\text{univ}}/(\lambda, J_M)$ of dimension at least $1 + \sup(n[L^+ : \mathbb{Q}] - d_{L,0}, n[L^+ : \mathbb{Q}] - d_{L,l})$ contains a generic, potentially pro-automorphic prime \mathfrak{p} of dimension 1 and characteristic l .

Fix a choice of lifting $r_{\mathcal{S}_1}^{\text{univ}} : G_{L^+,S} \rightarrow \mathcal{G}_n(R_{\mathcal{S}_1}^{\text{univ}})$ representing the universal deformation. This induces for each $v \in R$ a homomorphism $R_v^1 \rightarrow R_{\mathcal{S}_1}^{\text{univ}}$ and we let J_R denote the ideal generated by the images of $\mathfrak{m}_{R_v^1}, v \in R$. The ideal J_R is independent of the choice of lifting and, for any quotient $R_{\mathcal{S}_1}^{\text{univ}}/I$ of characteristic l , we have $\dim R_{\mathcal{S}_1}^{\text{univ}}/(J_R, I) \geq \dim R_{\mathcal{S}_1}^{\text{univ}}/I - |R|n^2$ by [Mat89, Theorem 15.1] (note that $R_v^1/(\lambda)$ has dimension n^2 [Tho15, Proposition 3.15]). It follows that there exists a generic prime $\mathfrak{p} \subset R_{\mathcal{S}_1}^{\text{univ}}/(J_R, J_L)$ of dimension 1 and characteristic l , since $\dim R_{\mathcal{S}_1}^{\text{univ}}/(J_R, J_L) \geq n[L^+ : \mathbb{Q}] - n^2|R| > \sup(n[L^+ : \mathbb{Q}] - d_{L,0}, n[L^+ : \mathbb{Q}] - d_{L,l})$ (here we are using assumption (4) in the statement of Theorem 5.1). By Proposition 5.3, any minimal prime $Q \subset \mathfrak{p}$ of $R_{\mathcal{S}_1}^{\text{univ}}$ is potentially pro-automorphic.

We now consider the partition of the set of minimal primes of $R_{\mathcal{S}_1}^{\text{univ}}$ into two sets $\mathcal{C}_1, \mathcal{C}_2$, consisting of those primes which respectively are and are not potentially pro-automorphic. We have shown that \mathcal{C}_1 is non-empty. We claim that \mathcal{C}_2 is empty. Otherwise, [Tho15, Lemma 3.21]

implies the existence of primes $Q_1 \in \mathcal{C}_1, Q_2 \in \mathcal{C}_2$ such that

$$\dim R_{\mathcal{S}_1}^{\text{univ}}/(Q_1, Q_2) \geq n[L^+ : \mathbb{Q}] - |R|n - 2$$

and hence

$$\dim R_{\mathcal{S}_1}^{\text{univ}}/(Q_1, Q_2, J_R) \geq n[L^+ : \mathbb{Q}] - |R|n - |R|n^2 - 2 = n[L^+ : \mathbb{Q}] - |R|n(n + 1) - 2.$$

The ring $R_{\mathcal{S}_1}^{\text{univ}}/Q_1$ (and hence each of its quotients) is finite over Λ_L . Applying Lemma 3.9 and assumption (4) once again, we see that $R_{\mathcal{S}_1}^{\text{univ}}/(Q_1, Q_2, J_R)$ contains a generic, potentially pro-automorphic prime \mathfrak{p}' of dimension 1 and characteristic l . Applying Proposition 5.3 to \mathfrak{p}' , we deduce that Q_2 is potentially pro-automorphic, which is a contradiction.

Now let $r : G_{L^+, S} \rightarrow \mathcal{G}_n(\mathcal{O})$ be a lifting of \bar{r}_m which is ordinary of weight λ and of type \mathcal{S}_1 , as in the statement of the theorem. This induces a homomorphism $R_{\mathcal{S}_1}^{\text{univ}} \rightarrow \mathcal{O}$. Let Q be a minimal prime contained inside the kernel of this homomorphism. Then there is a good extension M/L such that $J_M \subset Q$ and so the induced homomorphism $R_{\mathcal{S}_1, M}^{\text{univ}} \rightarrow \mathcal{O}$ kills $J_{\mathcal{S}_1, M}$, and the map $P_{\mathcal{S}_1, M} \rightarrow \mathcal{O}$ induced by $r|_{G_{M, S_M}}$ factors through $\mathbb{T}_1^{TM}(U_M(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}_M}$. Using [Ger19, Lemma 2.6.4], [CHT08, Proposition 3.3.2], and [Tho15, Lemma 2.7] (respectively a classicality statement in Hida theory, base change for the unitary group G_{M^+} , and soluble descent for $\text{GL}_n(\mathbb{A}_M)$), we deduce that the representation $r|_{G_L}$ is automorphic of weight λ . \square

The following corollary was established in the course of the above proof.

COROLLARY 5.4. *With hypotheses on \bar{r}_m as in Theorem 5.1, $R_{\mathcal{S}_1}$ is a finite Λ -algebra.*

6. The end

We are now in a position to state and prove the main theorem of this paper.

THEOREM 6.1. *Let F be an imaginary CM number field with maximal totally real subfield F^+ and let $n \geq 2$ be an integer. Let l be a prime and suppose that $\rho : G_F \rightarrow \text{GL}_n(\bar{\mathbb{Q}}_l)$ is a continuous semisimple representation satisfying the following hypotheses.*

- (i) $\rho^c \cong \rho^\vee \epsilon^{1-n}$.
- (ii) ρ is ramified at only finitely many places.
- (iii) ρ is ordinary of weight λ for some $\lambda \in (\mathbb{Z}_+^n)^{\text{Hom}(F, \bar{\mathbb{Q}}_l)}$.
- (iv) There is an isomorphism $\bar{\rho}^{\text{ss}} \cong \bar{\rho}_1 \oplus \cdots \oplus \bar{\rho}_d$, where each $\bar{\rho}_i$ is absolutely irreducible and satisfies $\bar{\rho}_i^c \cong \bar{\rho}_i^\vee \epsilon^{1-n}$, and $\bar{\rho}_i \not\cong \bar{\rho}_j$ if $i \neq j$.
- (v) There exists a finite place \tilde{v}_0 of F , prime to l , such that $\rho|_{G_{F_{\tilde{v}_0}}}^{\text{ss}} \cong \bigoplus_{i=1}^n \psi \epsilon^{n-i}$ for some unramified character $\psi : G_{F_{\tilde{v}_0}} \rightarrow \bar{\mathbb{Q}}_l^\times$.
- (vi) There exist a RACSDC representation π of $\text{GL}_n(\mathbb{A}_F)$ and $\iota : \bar{\mathbb{Q}}_l \rightarrow \mathbb{C}$ such that:
 - (a) π is ι -ordinary;
 - (b) $r_\iota(\pi)^{\text{ss}} \cong \bar{\rho}^{\text{ss}}$;
 - (c) $\pi_{\tilde{v}_0}$ is an unramified twist of the Steinberg representation.
- (vii) $F(\zeta_l)$ is not contained in $\bar{F}^{\ker \text{ad}(\bar{\rho}^{\text{ss}})}$ and F is not contained in $F^+(\zeta_l)$. For each $1 \leq i, j \leq d$, $\bar{\rho}_i|_{G_{F(\zeta_l)}}$ is absolutely irreducible and $\bar{\rho}_i|_{G_{F(\zeta_l)}} \not\cong \bar{\rho}_j|_{G_{F(\zeta_l)}}$ if $i \neq j$. Moreover, $\bar{\rho}^{\text{ss}}$ is primitive and $\bar{\rho}^{\text{ss}}(G_F)$ has no quotient of order l .
- (viii) $l > 3$ and $l \nmid n$.

Then ρ is automorphic: there exists an ι -ordinary RACSDC automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ such that $r_\iota(\Pi) \cong \rho$.

Proof. The proof is similar to, and even simpler than, the proof of [Tho15, Theorem 7.1]. We can find a coefficient field $K \subset \bar{\mathbb{Q}}_l$ and, after replacing ρ by a conjugate, assume that $\rho = r|_{G_F}$, where $r : G_{F^+} \rightarrow \mathcal{G}_n(\mathcal{O})$ is a homomorphism such that \bar{r} is Schur and $\nu \circ r = \epsilon^{1-n} \delta_{F/F^+}^n$. Similarly, we can assume the existence of a model $\rho' : G_F \rightarrow \mathrm{GL}_n(\mathcal{O})$ for $r_\iota(\pi)$ which extends to a homomorphism $r' : G_{F^+} \rightarrow \mathcal{G}_n(\mathcal{O})$ such that $\nu \circ r' = \epsilon^{1-n} \delta_{F/F^+}^n$ and such that $\bar{r}' = \bar{r}$. In fact, $\bar{r}|_{G_{F^+(\zeta_l)}}$ is Schur. Indeed, making use of assumption 6.1, we just need to check that $\bar{r}(G_{F^+(\zeta_l)})$ meets both components of \mathcal{G}_n . This is our assumption that $F \not\subset F^+(\zeta_l)$.

After making a preliminary soluble base change, we can assume that the following further conditions are satisfied.

- (i) F/F^+ is everywhere unramified and each place of F^+ dividing l or above which ρ or π is ramified splits in F .
- (ii) The place \tilde{v}_0 is split over F^+ . We write $v_0 = \tilde{v}_0|_{F^+}$.
- (iii) For each place w of F at which ρ or π is ramified, or which divides l , the representation $\bar{r}|_{G_{F_w}}$ is trivial.
- (iv) For each prime-to- l place w of F at which ρ or π is ramified, we have $q_w \equiv 1 \pmod{l}$ and, if $l^N \mid (q_w - 1)$, then $l^N > n$. Moreover, $\rho|_{G_{F_w}}$ and $\rho'|_{G_{F_w}}$ are unipotently ramified.

We can find a finite set \tilde{X}_0 of finite places of F satisfying the following conditions.

- \tilde{X}_0 does not contain any place at which ρ or π is ramified, or any place dividing l .
- Let $E/F(\zeta_l)$ denote the extension cut out by $\bar{\rho}|_{G_{F(\zeta_l)}}$. Then, for any Galois subextension $E/E'/F$ with $\mathrm{Gal}(E'/F)$ simple, there exists a place $w \in \tilde{X}_0$ which does not split in E' .
- \tilde{X}_0 contains an absolutely unramified place \tilde{v}_1 such that $\bar{\rho}(\mathrm{Frob}_{\tilde{v}_1})$ is scalar and $q_{\tilde{v}_1} \not\equiv 1 \pmod{l}$.

Let X_0 denote the set of places of F^+ lying below a place of \tilde{X}_0 . We write $v_1 = \tilde{v}_1|_{F^+}$. If L/F is any Galois CM extension which is \tilde{X}_0 -split, then the analogue of assumption (7) of the theorem (where F is replaced by L and $\bar{\rho}$ by $\bar{\rho}|_{G_L}$) is satisfied.

If L^+/F^+ is a Galois, totally real, X_0 -split extension and $L = L^+ \cdot F$, then L/F is Galois CM and \tilde{X}_0 -split. We claim that we can find a soluble, totally real, X_0 -split extension such that the hypotheses of Theorem 5.1 are satisfied for $r|_{G_{L^+}}$. This will complete the proof: we deduce from Theorem 5.1 that $r|_{G_L} = \rho|_{G_L}$ is automorphic of weight λ and irreducible. The automorphy of ρ then follows by soluble descent.

Let \tilde{Y}_0 denote the set of places $\tilde{v} \neq \tilde{v}_0, \tilde{v}_0^c$ of F dividing l or at which ρ or π is ramified, and let Y_0 denote the set of places of F^+ lying below a place of \tilde{Y}_0 . Then every place of Y_0 splits in F and $Y_0 \cap (X_0 \cup \{v_0\}) = \emptyset$. For any odd integer $\delta \geq 1$, we can find a cyclic totally real extension M_0/F^+ of degree δ which is $X_0 \cup \{v_0\}$ -split and such that each place $v \in Y_0$ of F^+ is totally inert in M_0 .

Let M_1 be a totally real quadratic extension of F^+ which is $X_0 \cup \{v_0\} \cup Y_0$ -split. We will take $L = F \cdot M_0 \cdot M_1$. We claim that for a suitable choice of odd integer δ , this L will suffice for the application of Theorem 5.1. More precisely, we will apply Theorem 5.1 to the homomorphism $r|_{G_{L^+}}$ with the following data.

- S_l is the set of l -adic places of L^+ .
- $S(B)$ is the set of places of L^+ lying above v_0 .

- R is the set of prime-to- l places of L^+ lying above a place of Y_0 .
- S_a is the set of places of L^+ lying above v_1 .

Let Δ denote the Galois group of the maximal abelian pro- l extension of L unramified outside l , and let Δ_0 denote the Galois group of the maximal abelian pro- l extension of L which is unramified outside l and which is $S(B)$ -split. Then Δ_0 is naturally a quotient of Δ . Let us define

$$d_0 = \dim_{\mathbb{Q}_l} \ker(\Delta[1/l] \rightarrow \Delta_0[1/l])^{c=-1}$$

and

$$d_l = \inf_{v \in S_l} [L_v^+ : \mathbb{Q}_l].$$

In order to complete the proof, we must show that the odd integer $\delta \geq 1$ can be chosen so that the inequalities

$$d_0 > |R|n(n+1)/2 + 2$$

and

$$d_l > \sup(|R|n(n+1)/2 + 2, n(n-1)/2 + 1)$$

are simultaneously satisfied. It follows from [Mai02, Proposition 19] that $d_0 = 2\delta$ (see the proof of [Tho15, Theorem 7.1] for more details). If δ is prime to the absolute residue degrees of all the places of Y_0 , then we will have $|R| \leq 2|Y_0|$ and $d_l \geq \delta$. We will therefore be done if we can choose δ to satisfy

$$2\delta > 2|Y_0|n(n+1)/2 + 2$$

and

$$\delta > \sup(2|Y_0|n(n+1)/2 + 2, n(n-1)/2 + 1).$$

This is clearly possible and concludes the proof. □

We can use the same idea to prove the following finiteness result (compare [Tho12, Theorem 10.2]), which plays a crucial role in some of the level-raising arguments in [NT19].

THEOREM 6.2. *Let F be a CM field, let l be a prime, and let $\iota : \bar{\mathbb{Q}}_l \rightarrow \mathbb{C}$ be an isomorphism. Let S be a finite set of finite places of F^+ , containing the l -adic ones, and suppose that each place of S splits in F . Choose for each $v \in S$ a place \tilde{v} of F lying above v and let \tilde{S} denote the set of the \tilde{v} .*

Let π be a RACSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, and let K/\mathbb{Q}_l be a coefficient field such that $r_\iota(\pi)$ can be chosen to take values in $\mathrm{GL}_n(\mathcal{O})$ and extend it to a homomorphism $r : G_{F^+} \rightarrow \mathcal{G}_n(\mathcal{O})$ such that $\nu \circ r = \epsilon^{1-n} \delta_{F^+}^n$. Suppose that the following conditions are satisfied.

- (i) π is ι -ordinary. For each place $v \in S_l$, $\bar{r}|_{G_{F_{\tilde{v}}}}$ is the trivial representation.
- (ii) π is unramified outside S .
- (iii) There exists a place $v_0 \nmid l$ of S such that $\pi_{\tilde{v}_0}$ is an unramified twist of the Steinberg representation. Moreover, $q_{v_0} \equiv 1 \pmod{l}$ and $\bar{r}|_{G_{F_{\tilde{v}_0}}}$ is the trivial representation.
- (iv) Let $\bar{\rho} = \bar{r}|_{G_{F,S}}$. There is an isomorphism $\bar{\rho}^{ss} \cong \bar{\rho}_1 \oplus \cdots \oplus \bar{\rho}_d$, where each $\bar{\rho}_i$ is absolutely irreducible and satisfies $\bar{\rho}_i^c \cong \bar{\rho}_i^{\vee} \epsilon^{1-n}$ (in particular, \bar{r} is Schur and therefore $\bar{\rho}$ is semisimple).

- (v) $F(\zeta_l)$ is not contained in $\bar{F}^{\ker \text{ad}(\bar{\rho}^{\text{ss}})}$ and F is not contained in $F^+(\zeta_l)$. For each $1 \leq i, j \leq d$, $\bar{\rho}_i|_{G_{F(\zeta_l)}}$ is absolutely irreducible and $\bar{\rho}_i|_{G_{F(\zeta_l)}} \not\cong \bar{\rho}_j|_{G_{F(\zeta_l)}}$ if $i \neq j$. Moreover, $\bar{\rho}$ is primitive and $\bar{\rho}(G_F)$ has no quotient of order l .
- (vi) $l > 3$ and $l \nmid n$.

Define the global deformation problem

$$\mathcal{S} = (F/F^+, S, \tilde{S}, \Lambda, \bar{r}, \epsilon^{1-n} \delta_{F/F^+}^n, \{R_v^\Delta\}_{v \in S_l} \cup \{R_v^\square\}_{v \in S - (S_l \cup \{v_0\})} \cup \{R_{v_0}^{\text{St}}\}).$$

Then $R_{\mathcal{S}}^{\text{univ}}$ is a finite Λ -algebra.

Proof. Let $R = S - S_l \cup \{v_0\}$, $S(B) = \{v_0\}$. If L/F is a CM extension, let S_L denote the set of places of L^+ above S , \tilde{S}_L the set of places of L above \tilde{S} , and define R_L, \tilde{R}_L etc. similarly. If L satisfies the following conditions:

- $\bar{r}|_{L^+}$ is Schur;
- for each place $v \in R_L$, $q_v \equiv 1 \pmod{l}$ and $\bar{r}|_{G_{L_v}}$ is the trivial representation;
- for each place $v \in R$ and each place $w|v$ of L^+ , the induced map $R_{L_v}^\square \rightarrow R_{F_v}^\square$ factors through the quotient $R_{L_v}^\square \rightarrow R_{L_v}^1$,

then we can define the global deformation problem

$$\mathcal{S}_L = (L/L^+, S_L, \tilde{S}_L, \Lambda_L, \bar{r}|_{G_{L^+}}, \epsilon^{1-n} \delta_{L/L^+}^n, \{R_v^\Delta\}_{v \in S_{l,L}} \cup \{R_v^1\}_{v \in R_L} \cup \{R_{v_0}^{\text{St}}\}_{v \in S(B)_L})$$

and restriction from F^+ to L^+ determines a finite morphism $R_{\mathcal{S}_L}^{\text{univ}} \rightarrow R_{\mathcal{S}}^{\text{univ}}$. If moreover L satisfies the conditions of Theorem 5.1, then Corollary 5.4 will imply that $R_{\mathcal{S}_L}^{\text{univ}}$ is a finite Λ_L -algebra and hence that $R_{\mathcal{S}}^{\text{univ}}$ is a finite Λ -algebra. Such an extension L/F can be constructed in exactly the same manner as in the proof of Theorem 6.1. □

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