

TRACES IN MONOIDAL DERIVATORS, AND HOMOTOPY COLIMITS

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ABSTRACT. A variant of the trace in a monoidal category is given in the setting of closed monoidal derivators, which is applicable to endomorphisms of fiberwise dualizable objects. Functoriality of this trace is established. As an application, an explicit formula is deduced for the trace of the homotopy colimit of endomorphisms over finite categories in which all endomorphisms are invertible. This result can be seen as a generalization of the additivity of traces in monoidal categories with a compatible triangulation.

1. INTRODUCTION

1.1. The additivity of traces. Let \mathcal{C} be a symmetric monoidal category which in addition is triangulated. Examples include various “stable homotopy categories” (such as the classical and equivariant in algebraic topology, the motivic in algebraic geometry) or all kinds of “derived categories” (of modules, of perfect complexes on a scheme, etc.). Let X , Y and Z be dualizable objects in \mathcal{C} ,

$$D : X \rightarrow Y \rightarrow Z \rightarrow^+$$

a distinguished triangle, and f an endomorphism of D . The *additivity of traces* is the statement that the following relation holds among the traces of the components of f :

$$\mathrm{tr}(f_Y) = \mathrm{tr}(f_X) + \mathrm{tr}(f_Z). \quad (1)$$

Well-known examples are the additivity of the Euler characteristic of finite CW-complexes ($\chi(Y) = \chi(X) + \chi(Y/X)$ for $X \subset Y$ a subcomplex) or the additivity of traces in short exact sequences of finite dimensional vector spaces. The additivity of traces should be considered as a *principle*: Although incorrect as it stands, it embodies an important idea. One should therefore try to find the right context to formulate this idea precisely and prove it.

In [19], J. Peter May made an important step in this direction. He gave a list of axioms expressing a compatibility of the monoidal and the triangulated structure, and proved that if they are satisfied, then one can always replace f by an endomorphism f' with $f'_X = f_X$ and $f'_Y = f_Y$ such that (1) holds for the components

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of f' . This result has two drawbacks though: Firstly, there is this awkwardness of f' replacing f , and secondly, the axioms are rather complicated.

As noted in [7], both these drawbacks are related to the well-known deficiencies of triangulated categories. Since the foremost example of a situation in which May's compatibility axioms hold, is when \mathcal{C} is the homotopy category of a stable monoidal model category, it should not come as a surprise that May's result can be reproved in the setting of triangulated derivators. Moreover, since triangulated derivators eliminate some of the problems encountered in triangulated categories, a more satisfying formulation of the additivity of traces should be available. We will describe it now.

Let \mathbb{D} be a closed symmetric monoidal triangulated derivator¹, and \ulcorner the free category on the following graph:

$$\begin{array}{ccc} (1, 1) & \longleftarrow & (0, 1) \\ \uparrow & & \\ (1, 0) & & \end{array} \quad (2)$$

Let A be an object of $\mathbb{D}(\ulcorner)$ with underlying diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \\ 0, & & \end{array}$$

and suppose that both X and Y are dualizable objects of $\mathbb{D}(\star)$, \star denoting the terminal category. Let f be an endomorphism of A , and denote by p_{\ulcorner} the unique functor $\ulcorner \rightarrow \star$. Then there is a distinguished triangle

$$X \rightarrow Y \rightarrow p_{\ulcorner!}A \rightarrow^+$$

in $\mathbb{D}(\star)$, $p_{\ulcorner!}A$ is also dualizable, and the following relation holds:

$$\mathrm{tr}(f_{(0,1)}) = \mathrm{tr}(f_{(1,1)}) + \mathrm{tr}(p_{\ulcorner!}f).$$

This is the main theorem of [7].

1.2. The trace of the homotopy colimit. Another advantage of the formulation in the context of derivators is that it immediately invites us to consider the additivity of traces as a mere instance of a more general principle. As a first step, we see that the condition $A_{(1,0)} = 0$ is not essential. Indeed, if A is an object of $\mathbb{D}(\ulcorner)$ whose fibers are all dualizable objects in $\mathbb{D}(\star)$ and if f is an endomorphism of A then the formula above generalizes to

$$\mathrm{tr}(p_{\ulcorner!}f) = \mathrm{tr}(f_{(0,1)}) + \mathrm{tr}(f_{(1,0)}) - \mathrm{tr}(f_{(1,1)}). \quad (3)$$

And now, in a second step, it is natural to replace the category \ulcorner by other categories I and try to see whether there still is an explicit formula for $\mathrm{tr}(p_{I!}f)$. The main result of the present article states that this is the case for finite EI-categories, i. e. finite categories in which all endomorphisms are invertible (such as groups or posets), provided that the derivator is \mathbb{Q} -linear. For each of these categories the trace of the homotopy colimit of an endomorphism of a fiberwise dualizable object can be computed as a linear combination of “local traces” (depending only on the

¹See section 2 for the definition of this notion.

fibers of the endomorphism and the action of the automorphisms of the objects in the category) with coefficients which depend only on the category and can be computed combinatorially.

As for the proof of this result, the idea is to define the trace of endomorphisms of objects not only living in $\mathbb{D}(\star)$ but in $\mathbb{D}(I)$ for general categories I . This trace should contain enough information to relate the trace of the homotopy colimit to the local traces of the endomorphism. The naive approach of considering $\mathbb{D}(I)$ as a monoidal category and taking the usual notion of the trace doesn't lead too far though since few objects in $\mathbb{D}(I)$ will be dualizable in general even if in $\mathbb{D}(\star)$ all of them are; in other words, being fiberwise dualizable does not imply being dualizable.

This is why we will replace the “internal” tensor product by an “external product”

$$\boxtimes : \mathbb{D}(I) \times \mathbb{D}(I) \rightarrow \mathbb{D}(I \times I)$$

and the internal hom by an “external hom”

$$\langle -, - \rangle : \mathbb{D}(I)^\circ \times \mathbb{D}(I) \rightarrow \mathbb{D}(I^\circ \times I),$$

which has the property that for any object A of $\mathbb{D}(I)$ and objects i, j of I

$$\langle A, A \rangle_{(i,j)} = [A_i, A_j],$$

(implying that fiberwise dualizable objects will be “dualizable with respect to the external hom”) and which also contains enough information to compute $[A, A]$ (among other desired formal properties). As soon as this bifunctor is available we can mimic the usual definition of the “internal” trace in a closed symmetric monoidal category to define an “external” trace for any endomorphism of a fiberwise dualizable object, replacing the internal by the external hom everywhere. It will turn out that this new trace encodes all local traces, and in good cases allows us to relate these to the trace of the homotopy colimit, thus yielding the sought after formula.

A “general additivity theorem” for traces, supposedly similar to the main result in this article, was announced by Kate Ponto and Michael Shulman in [7]. However, their proof should be quite different from the one presented here, relying on the technology of bicategorical traces (personal communication, December 2012).

1.3. Outline of the paper. We do not include an introduction to the theory of derivators (see for this the references given in section 2.3). However, as the definition of a derivator varies in the literature we give the axioms we use in section 2.3. Moreover, the few results on derivators we need in the article are either proved or justified by a reference to where a proof can be found. In section 2.4 we define the notion of a (closed) monoidal derivator and describe its relation to the axiomatization available in the literature. We also discuss briefly linear structures on derivators (2.5), triangulated derivators (2.6), and the interplay between triangulated and monoidal structures on derivators (2.7). Apart from this, section 2 is meant to fix the notation used in the remainder of the article.

The main body of the text starts with section 3 where the construction of the external hom mentioned above is given. The proofs of the desired formal properties of this bifunctor are lengthy and not needed in the rest of the article so they are deferred to appendix A. Next we define the external trace (section 4) and prove its functoriality (section 5). As a corollary we deduce that this trace encodes all local traces.

The main result is to be found in section 6. First we prove that in good cases the trace of the homotopy colimit is a function of the external trace (again, the uninteresting part of the story is postponed to the appendix; specifically to appendix B). In the case of finite EI-categories and a \mathbb{Q} -linear triangulated derivator, this function can be made explicit, and this leads to the formula for the trace of the homotopy colimit in terms of the local traces. Some technical hypotheses used to prove this result will be eliminated in section 7.

At several points in the article the need arises for an explicit description of an additive derivator evaluated at a finite group. Although this description is certainly well-known, we haven't been able to find it in the literature and have thus included it as appendix C.

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Joseph Ayoub not only suggested to me that there should be an explicit formula for the trace of the homotopy colimit of an endomorphism in the setting of a closed symmetric monoidal triangulated derivator but also contributed several crucial ideas and constructions. In particular, the important definition of the “external hom” mentioned above is due to him. I am also grateful to Michael Shulman for pointing out that an earlier formula was too simplistic. Finally, I was lucky to receive a very competent and detailed report by the referee which led to many improvements in the article. I would like to thank him or her warmly.

2. CONVENTIONS AND PRELIMINARIES

In this section, we recall some notions and facts (mostly related to derivators) and fix the notation used in the remainder of the article.

2.1. By a 2-category we mean a strict 2-category. The 2-category of (small) categories is denoted by **CAT** (**Cat**). Given a 2-category \mathcal{C} (encompassing the special case of a category), \mathcal{C}° denotes the 2-category with the same objects, and $\mathcal{C}^\circ(x, y) = \mathcal{C}(y, x)$ for all objects x, y . The 2-category $\mathcal{C}^{\circ, \circ}$ also has the same objects as \mathcal{C} but $\mathcal{C}^{\circ, \circ}(x, y) = \mathcal{C}(y, x)^\circ$ (see [12, p. 82]). The (possibly large) sets of objects, 1-morphisms and 2-morphisms in a 2-category \mathcal{C} are sometimes denoted by \mathcal{C}_0 , \mathcal{C}_1 , and \mathcal{C}_2 respectively.

By a 2-functor we mean a *strict* 2-functor between 2-categories. Modifications are morphisms of lax natural transformations between 2-functors (see [12, p. 82]). For fixed 2-categories \mathcal{C} and \mathcal{D} , the 2-functors from \mathcal{C} to \mathcal{D} together with lax natural transformations and modifications form a 2-category $\mathbf{Fun}_{\text{lax}}(\mathcal{C}, \mathcal{D})$.

2.2. Counits and units of adjunctions are usually denoted by adj . Given a functor $u : I \rightarrow J$, and an object $j \in J_0$, the category of *objects u-under j* is (abusively) denoted by $j \backslash I$ and the category of *objects u-over j* by I / j (see [15, 2.6]). We also need the following construction ([15, p. 223]): Given a category I , we define the *twisted arrow category* associated to I , denoted by $\text{tw}(I)$, as having objects the arrows of I and as morphisms from $i \rightarrow j$ to $i' \rightarrow j'$ pairs of morphisms making the

following square in I commutative:

$$\begin{array}{ccc} i & \longrightarrow & j \\ \uparrow & & \downarrow \\ i' & \longrightarrow & j' \end{array}.$$

There is a canonical functor $\mathrm{tw}(I) \rightarrow I^\circ \times I$. In fact, this extends canonically to a functor $\mathrm{tw}(-) : \mathbf{Cat} \rightarrow \mathbf{Cat}$ together with a natural transformation $\mathrm{tw}(-) \rightarrow (-)^\circ \times (-)$.

2.3. Let us recall the notion of a derivator. For the basic theory we refer to [17], [3], [6]. For an outline of the history of the subject see [3, p. 1385].

A full sub-2-category **Dia** of **Cat** is called a *diagram category* if:

- (Dia1) **Dia** contains the totally ordered set $\underline{2} = \{0 < 1\}$;
- (Dia2) **Dia** is closed under finite products and coproducts, and under taking the opposite category and subcategories;
- (Dia3) if $I \in \mathbf{Dia}_0$ and $i \in I_0$, then $I/i \in \mathbf{Dia}_0$;
- (Dia4) if $p : I \rightarrow J$ is a fibration (to be understood in the sense of [9, exposé VI]) whose fibers are all in **Dia**, and if $J \in \mathbf{Dia}_0$, then also $I \in \mathbf{Dia}_0$.

By (Dia2), the initial category \emptyset and the terminal category \star are both in **Dia**. We will often use that **Dia** is closed under pullbacks (as follows from (Dia2)). The smallest diagram category consists of finite posets, other typical examples include finite categories, finite-dimensional categories, all small posets or **Cat** itself.

A *prederivator* (of type **Dia**) is a 2-functor $\mathbb{D} : \mathbf{Dia}^{\circ, \circ} \rightarrow \mathbf{CAT}$ from a diagram category **Dia** to **CAT**. If \mathbb{D} is fixed in a context, **Dia** always denotes the domain of \mathbb{D} . Given a prederivator \mathbb{D} , categories $I, J \in \mathbf{Dia}_0$ and a functor $u : I \rightarrow J$, we denote by $u^* : \mathbb{D}(J) \rightarrow \mathbb{D}(I)$ the value of \mathbb{D} at u ; if u is clear from the context, we sometimes denote u^* by $|_I$. Its left and right adjoint (if they exist) are denoted by $u_!$ and u_* respectively. The unique functor $I \rightarrow \star$ is denoted by p_I . Given an object $i \in I_0$, we denote also by $i : \star \rightarrow I$ the functor pointing i . Thus for an object $A \in \mathbb{D}(I)_0$ and a morphism $f \in \mathbb{D}(I)_1$, their *fiber over i* is i^*A and i^*f , respectively, sometimes also denoted by A_i and f_i , respectively. Given a natural transformation $\eta : u \rightarrow v$ in **Dia**, we denote by η^* the value of \mathbb{D} at η . It is a natural transformation from v^* to u^* . In particular, if $h : i \rightarrow j$ is an arrow in I then we can consider it as a natural transformation from the functor $i : \star \rightarrow I$ to $j : \star \rightarrow I$, and therefore it makes sense to write h^* ; evaluated at an object $A \in \mathbb{D}(I)_0$, it yields a morphism of the fibers $A_j \rightarrow A_i$. The canonical “underlying diagram” functor $\mathbb{D}(I) \rightarrow \mathbf{CAT}(I^\circ, \mathbb{D}(\star))$ is denoted by dia_I . Finally, if \mathbb{D} is a prederivator and $J \in \mathbf{Dia}_0$, we denote by \mathbb{D}_J the prederivator $\mathbb{D}_J(-) = \mathbb{D}(- \times J)$.

A *derivator* (of type **Dia**) is a prederivator (of type **Dia**) \mathbb{D} satisfying the following list of axioms:

- (D1) \mathbb{D} takes arbitrary coproducts to products up to equivalence of categories.
- (D2) For every $I \in \mathbf{Dia}_0$, the family of functors $i^* : \mathbb{D}(I) \rightarrow \mathbb{D}(\star)$ indexed by I_0 is jointly conservative.
- (D3) For all functors $u \in \mathbf{Dia}_1$, the left and right adjoints $u_!$ and u_* to u^* exist.

- (D4) Given a functor $u : I \rightarrow J$ in **Dia** and an object $j \in J_0$, the “Beck-Chevalley” transformations associated to both comma squares

$$\begin{array}{ccc} j \backslash I & \xrightarrow{t} & I \\ p_{j \backslash I} \downarrow & \nearrow & \downarrow u \\ \star & \xrightarrow{j} & J \end{array} \quad \text{and} \quad \begin{array}{ccc} I / j & \xrightarrow{s} & I \\ p_{I / j} \downarrow & \searrow & \downarrow u \\ \star & \xrightarrow{j} & J \end{array}$$

are invertible: $p_{j \backslash I} t^* \xrightarrow{\sim} j^* u_!$ and $j^* u_* \xrightarrow{\sim} p_{I / j} s^*$.

The derivator \mathbb{D} is called *strong* if in addition

- (D5) For every $J \in \mathbf{Dia}_0$, the functor $\text{dia}_2 : \mathbb{D}_J(\underline{2}) \rightarrow \mathbf{CAT}(\underline{2}^\circ, \mathbb{D}_J(\star))$ is full and essentially surjective.

As an important example, if \mathcal{M} is a model category then the association

$$\begin{aligned} \mathbb{D}^{\mathcal{M}} : \mathbf{Cat}^{\circ, \circ} &\longrightarrow \mathbf{CAT} \\ I &\longmapsto \mathcal{M}^{I^\circ}[\mathcal{W}_I^{-1}] \end{aligned}$$

defines a strong derivator, where $\mathcal{M}^{I^\circ}[\mathcal{W}_I^{-1}]$ denotes the category obtained from \mathcal{M}^{I° by formally inverting those morphisms of presheaves which are pointwise weak equivalences. This result is due to Denis-Charles Cisinski (see [2]). If \mathbb{D} is a (strong) derivator and $J \in \mathbf{Dia}_0$ then also \mathbb{D}_J is a (strong) derivator.

One consequence of the axioms we shall often have occasion to refer to is the following result on (op)fibrations:

Lemma 1 *Given a derivator \mathbb{D} of type **Dia** and given a pullback square*

$$\begin{array}{ccc} & \xrightarrow{w} & \\ v \downarrow & & \downarrow u \\ & \xrightarrow{x} & \end{array}$$

*in **Dia** with either u a fibration or x an opfibration, the canonical “Beck-Chevalley” transformation*

$$v_! w^* \longrightarrow x^* u_! \quad (\text{or, equivalently, } u^* x_* \longrightarrow w_* v^*)$$

is invertible.

For a proof see [6, 1.30] or [10, 2.7].

2.4. By a *monoidal category* we always mean a symmetric unitary monoidal category. *Monoidal functors* between monoidal categories are functors which preserve the monoidal structure up to (coherent) natural isomorphisms; in the literature, these are sometimes called *strong monoidal functors*. *Monoidal transformations* are natural transformations preserving the monoidal structure in an obvious way. We thus arrive at the 2-category of monoidal categories **MonCAT**. The monoidal product is always denoted by \otimes and the unit by $\mathbb{1}$. If internal hom functors exist, we arrive at its closed variant **CiMonCAT**. (Notice that functors between closed categories are not required to be closed. In other words, **CiMonCAT** is a *full* sub-2-category of **MonCAT**.) The internal hom functor is always denoted by $[-, -]$.

A (closed) monoidal prederivator (of type **Dia**) is a prederivator with a factorization

$$\mathbb{D} : \mathbf{Dia}^{\circ, \circ} \rightarrow (\mathbf{Cl})\mathbf{MonCAT} \rightarrow \mathbf{CAT},$$

where $(\mathbf{Cl})\mathbf{MonCAT} \rightarrow \mathbf{CAT}$ is the forgetful functor. (Closed) monoidal prederivators were also discussed in [1, 2.1.6] and [5].

Let us now define the “external product” mentioned in the introduction. Given a monoidal prederivator \mathbb{D} and categories I, J in **Dia** we define the bifunctor

$$\begin{aligned} \boxtimes : \mathbb{D}(I) \times \mathbb{D}(J) &\rightarrow \mathbb{D}(I \times J) \\ (A, B) &\mapsto A|_{I \times J} \otimes B|_{I \times J}. \end{aligned}$$

Given two functors $u : I' \rightarrow I$ and $v : J' \rightarrow J$ in **Dia**, $A \in \mathbb{D}(I)_0$ and $B \in \mathbb{D}(J)_0$, we define a morphism

$$(u \times v)^*(A \boxtimes B) \rightarrow u^*A \boxtimes v^*B \quad (4)$$

as the composition

$$\begin{aligned} (u \times v)^*(A|_{I \times J} \otimes B|_{I \times J}) &\xrightarrow{\sim} (u \times v)^*A|_{I \times J} \otimes (u \times v)^*B|_{I \times J} \\ &= (u^*A)|_{I' \times J'} \otimes (v^*B)|_{I' \times J'}. \end{aligned}$$

Hence we see that (4) is in fact an isomorphism, and it is clear that it is also natural in A and B . Putting these and similar properties together one finds that the external product defines a pseudonatural transformation of 2-functors

$$\mathbb{D} \times \mathbb{D} \rightarrow \mathbb{D} \circ (- \times -),$$

i.e. a 1-morphism in $\mathbf{Fun}_{\text{lax}}(\mathbf{Dia}^{\circ, \circ} \times \mathbf{Dia}^{\circ, \circ}, \mathbf{CAT})$ with invertible 2-cell components.

Now fix a monoidal prederivator \mathbb{D} , a functor $u : I \rightarrow J \in \mathbf{Dia}_1$, and $A \in \mathbb{D}(I)_0$, $B \in \mathbb{D}(J)_0$. We can define the *projection morphism*

$$u_!(A \otimes u^*B) \rightarrow u_!A \otimes B \quad (5)$$

by adjunction as the composition

$$A \otimes u^*B \xrightarrow{\text{adj}} u^*u_!A \otimes u^*B \xleftarrow{\sim} u^*(u_!A \otimes B).$$

It is clearly natural in A and B . Fix a second functor $v : I' \rightarrow J'$ in **Dia** and consider the following morphism ($A \in \mathbb{D}(I)_0$, $B \in \mathbb{D}(I')_0$):

$$(u \times v)_!(A \boxtimes B) \rightarrow u_!A \boxtimes v_!B, \quad (6)$$

obtained by adjunction from

$$A \boxtimes B \xrightarrow{\text{adj}} u^*u_!A \boxtimes v^*v_!B \xleftarrow{\sim} (u \times v)^*(u_!A \boxtimes v_!B).$$

Of course, it is also natural in A and B .

Lemma 2 *Let \mathbb{D} be a monoidal prederivator which satisfies the axioms of a derivator. Then the following conditions are equivalent:*

- (1) *The projection morphism (5) is invertible for all $u = p_I$, $I \in \mathbf{Dia}_0$.*
- (2) *The projection morphism (5) is invertible for all fibrations u in **Dia**.*
- (3) *(6) is invertible for all $u, v \in \mathbf{Dia}_1$.*

If \mathbb{D} is a closed monoidal prederivator then condition (2) is also equivalent to each of the following ones:

- (4) *$u^*[B, B'] \rightarrow [u^*B, u^*B']$ is invertible for all fibrations u in **Dia**.*

(5) $[u_! A, B] \rightarrow u_*[A, u^* B]$ is invertible for all fibrations u in **Dia**.

Definition 3 A (closed) monoidal derivator is a (closed) monoidal prederivator which satisfies the axioms of a derivator as well as the equivalent conditions of Lemma 2.

Proof of Lemma 2. Assume condition (1). Let $u : I \rightarrow J$ be a fibration in **Dia** and consider, for any $j \in J_0$, the following pullback square:

$$\begin{array}{ccc} I_j & \xrightarrow{w} & I \\ p_{I_j} \downarrow & & \downarrow u \\ \star & \xrightarrow{j} & J. \end{array}$$

Since u is a fibration the base change morphism $p_{I_j!} w^* \rightarrow j^* u_!$ is an isomorphism, by Lemma 1. Hence for any $A \in \mathbb{D}(I)_0$, $B \in \mathbb{D}(J)_0$, all vertical morphisms in the commutative diagram below are invertible:

$$\begin{array}{ccc} j^* u_!(A \otimes u^* B) & \xrightarrow{\quad} & j^*(u_! A \otimes B) \\ \uparrow & & \downarrow \\ p_{I_j!} w^*(A \otimes u^* B) & & j^* u_! A \otimes j^* B \\ \downarrow & & \uparrow \\ p_{I_j!}(w^* A \otimes w^* u^* B) & & p_{I_j!} w^* A \otimes j^* B \\ \parallel & & \parallel \\ p_{I_j!}(w^* A \otimes p_{I_j}^* j^* B) & \xrightarrow{\quad} & p_{I_j!} w^* A \otimes j^* B. \end{array}$$

By assumption, the bottom horizontal arrow is an isomorphism hence so is the top one. Condition (2) now follows from (D2).

For condition (3), write $u \times v = (u \times 1) \circ (1 \times v)$ hence by symmetry of the monoidal product we reduce to the case where $u = 1_I$, $v : J' \rightarrow J$. We use (D2), thus let $i \in I_0$, $j \in J_0$. The fiber of (6) over (i, j) is easily seen to be the following composition (w denotes the fibration $i \setminus I \times j \setminus J' \rightarrow i \setminus I$):

$$\begin{aligned} (i, j)^*(1_I \times v)_!(A|_{I \times J'} \otimes B|_{I \times J'}) &\xleftarrow{\sim} p_{i \setminus I!} w_!(A|_{i \setminus I \times j \setminus J'} \otimes B|_{i \setminus I \times j \setminus J'}) \\ &\xrightarrow{\sim} p_{i \setminus I!}(A|_{i \setminus I} \otimes w_! B|_{i \setminus I \times j \setminus J'}) \\ &\xrightarrow{\sim} p_{i \setminus I!}(A|_{i \setminus I} \otimes p_{i \setminus I}^* p_{j \setminus J'}^! B|_{j \setminus J'}) \\ &\xrightarrow{\sim} p_{i \setminus I!} A|_{i \setminus I} \otimes p_{j \setminus J'}^! B|_{j \setminus J'} \\ &\xrightarrow{\sim} i^* A \otimes j^* v_! B \\ &\xleftarrow{\sim} (i, j)^*(A|_{I \times J} \otimes (v_! B)|_{I \times J}) \end{aligned}$$

The first, the third and the fifth arrow come from (D4), while the second and the fourth are invertible by condition (2), the last one is clearly invertible.

Putting $u = p_I$, $v = 1_\star$ in condition (3), one obtains precisely condition (1). This finishes the proof of the first statement in the lemma.

From now on we assume that \mathbb{D} is a closed monoidal prederivator. For condition (4), notice that $u^* \circ [B, -] \rightarrow [u^* B, -] \circ u^*$ corresponds via the adjunctions

$$u_! \circ (- \otimes u^* B) \dashv [u^* B, -] \circ u^* \quad \text{and} \quad (- \otimes B) \circ u_! \dashv u^* \circ [B, -]$$

to the projection morphism

$$u_!(- \otimes u^* B) \rightarrow u_! - \otimes B.$$

And similarly, the morphism $[u_! A, -] \rightarrow u_* \circ [A, -] \circ u^*$ corresponds via the adjunctions

$$u_! A \otimes - \dashv [u_! A, -] \quad \text{and} \quad u_! \circ (A \otimes -) \circ u^* \dashv u_* \circ [A, -] \circ u^*$$

to the projection morphism

$$u_!(A \otimes u^* -) \rightarrow u_! A \otimes -.$$

Hence conditions (4) and (5) are both equivalent to condition (2). (For more details, see [1, 2.1.144, 2.1.146].) \square

In contrast to this, in a closed monoidal prederivator, the canonical morphism

$$[A, u_* B] \rightarrow u_* [u^* A, B] \tag{7}$$

is *always* invertible, even if u is not a fibration.

If \mathbb{D} is a (strong) derivator of type **Cat** then precomposition with the 2-functor $(-)^{\circ} : \mathbf{Cat}^{\circ} \rightarrow \mathbf{Cat}^{\circ, \circ}$ defines a (strong) derivator $\overline{\mathbb{D}}$ in the sense of [7], and conversely starting with a (strong) derivator in their sense, precomposition with $(-)^{\circ}$ yields a (strong) derivator of type **Cat**. By Lemma 2, \mathbb{D} being monoidal corresponds to $\overline{\mathbb{D}}$ being symmetric monoidal. By [7, 8.8] then, \mathbb{D} being closed monoidal corresponds to $\overline{\mathbb{D}}$ being closed symmetric monoidal. In particular, [7, 9.11] establishes that if \mathcal{M} is a symmetric monoidal cofibrantly generated model category then the induced derivator $\mathbb{D}^{\mathcal{M}}$ is a closed monoidal, strong derivator (of type **Cat**).

Again, if \mathbb{D} is a (closed) monoidal derivator, then so is \mathbb{D}_J for any $J \in \mathbf{Dia}_0$.

2.5. A few words on linear structures on derivators (see [5, section 3] for the details). An *additive derivator* is a derivator \mathbb{D} such that $\mathbb{D}(\star)$ is an additive category. It follows that $\mathbb{D}(I), u^*, u_*, u_!$ are additive for all $I \in \mathbf{Dia}_0$, $u \in \mathbf{Dia}_1$. We define $R_{\mathbb{D}}$ to be the unital ring $\mathbb{D}(\star)(\mathbb{1}, \mathbb{1})$.

If \mathbb{D} is additive and monoidal, then $R_{\mathbb{D}}$ is a commutative ring and $\mathbb{D}(I)$ is canonically endowed with an $R_{\mathbb{D}}$ -linear structure for any $I \in \mathbf{Dia}_0$, making $u^*, u_*, u_!$ all $R_{\mathbb{D}}$ -linear functors, $u \in \mathbf{Dia}_1$. Given $f \in \mathbb{D}(I)(A, B)$, $\lambda \in R_{\mathbb{D}}$, λf is defined by

$$A \xleftarrow{\sim} p_I^* \mathbb{1} \otimes A \xrightarrow{p_I^* \lambda \otimes f} p_I^* \mathbb{1} \otimes B \xrightarrow{\sim} B.$$

2.6. We now recall the notion of a triangulated derivator. Let \square be the partially ordered set considered as a category:

$$\begin{array}{ccc} (1, 1) & \longleftarrow & (0, 1) \\ \uparrow & & \uparrow \\ (1, 0) & \longleftarrow & (0, 0), \end{array}$$

and \lrcorner the full subcategory defined by the complement of $(1, 1)$. Thus there are two canonical embeddings $i_{\lrcorner} : \lrcorner \rightarrow \square$ and $i_{\ulcorner} : \ulcorner \rightarrow \square$. We say that an object $A \in \mathbb{D}(\square)_0$ is cartesian (resp. cocartesian) if the unit

$$A \rightarrow i_{\lrcorner*} i_{\lrcorner}^* A \quad (\text{resp. the counit } i_{\ulcorner!} i_{\ulcorner}^* A \rightarrow A)$$

is an isomorphism.

A *triangulated derivator* is a strong derivator \mathbb{D} such that $\mathbb{D}(\star)$ is pointed and objects in $\mathbb{D}(\square)$ are cartesian if and only if they are cocartesian. If \mathcal{M} is a stable model category, then the derivator $\mathbb{D}^{\mathcal{M}}$ associated to \mathcal{M} is triangulated. Also, if \mathbb{D} is a triangulated derivator then so is \mathbb{D}_J for any $J \in \mathbf{Dia}_0$. The name comes from the fact that any triangulated derivator factors canonically through the forgetful functor $\mathbf{TrCAT} \rightarrow \mathbf{CAT}$ from triangulated categories to \mathbf{CAT} . In particular, every triangulated derivator is additive. This result is due to Georges Maltsiniotis ([18, Théorème 1]; see also [6, 4.15, 4.19]), and the triangulated structure is given explicitly. We will need the description of it on $\mathbb{D}(\star)$. (The description on $\mathbb{D}(I)$ can then be deduced by replacing \mathbb{D} by \mathbb{D}_I .) Thus given an object $A \in \mathbb{D}(\star)_0$ one defines canonically an object in $\mathbb{D}(\square)$ with underlying diagram

$$\begin{array}{ccc} A & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma A, \end{array}$$

some $\Sigma A \in \mathbb{D}(\star)_0$, as $i_{\ulcorner!}(1, 1)_* A$. Then we can define the suspension functor $\Sigma : \mathbb{D}(\star) \rightarrow \mathbb{D}(\star)$ as $(0, 0)^* i_{\ulcorner!}(1, 1)_*$. Moreover, if we denote by \sqcap the partial order considered as a category

$$\begin{array}{ccccc} (2, 1) & \longleftarrow & (1, 1) & \longleftarrow & (0, 1) \\ \uparrow & & \uparrow & & \uparrow \\ (2, 0) & \longleftarrow & (1, 0) & \longleftarrow & (0, 0), \end{array}$$

there are three canonical embeddings $i : \square \rightarrow \sqcap$ and we say that an object $A \in \mathbb{D}(\sqcap)_0$ is a triangle if $A_{(2,0)} \cong A_{(0,1)} \cong 0$ and $i^* A$ is (co)cartesian for all three embeddings. It then follows that one has a canonical isomorphism $A_{(0,0)} \cong \Sigma A_{(2,1)}$ (see the proof of Lemma 5 below) and therefore a triangle in $\mathbb{D}(\star)$:

$$A_{(2,1)} \rightarrow A_{(1,1)} \rightarrow A_{(1,0)} \rightarrow \Sigma A_{(2,1)}. \quad (8)$$

The distinguished triangles are those isomorphic to one of the form of (8).

2.7. We are also interested in some aspects of the interplay between monoidal and triangulated structures on derivators.

Definition 4 A (closed) *monoidal triangulated derivator* is a (closed) monoidal and triangulated derivator.

Under the correspondence $\mathbb{D} \rightsquigarrow \overline{\mathbb{D}}$ above, a closed monoidal triangulated derivator of type \mathbf{Cat} corresponds to a “closed symmetric monoidal, strong, stable derivator” in [7]. Translating the results in [7] back to our setting we see that every such derivator factors canonically through $\mathbf{ClMonTrCAT}$, the 2-category of closed

monoidal categories with a “compatible” triangulation (in the sense of [19]), such that the following diagram commutes:

$$\begin{array}{ccccc}
 & & \mathbf{TrCAT} & & \\
 & \nearrow & & \nwarrow & \\
 \mathbf{Dia}^{\circ, \circ} & \longrightarrow & \mathbf{ClMonTrCAT} & & \mathbf{CAT} \\
 & & \searrow & \nearrow & \\
 & & \mathbf{ClMonCAT} & &
 \end{array}$$

Here, it is understood that following the path on the upper part of the diagram yields the canonical factorization of the triangulated derivator, while the path through the lower part yields the factorization of the monoidal prederivator. All we will need from this statement is the following lemma.

Lemma 5 ([7, 4.1, 4.8]) *Let \mathbb{D} be a monoidal triangulated derivator and $I \in \mathbf{Dia}_0$. Then the monoidal product $\otimes : \mathbb{D}(I) \times \mathbb{D}(I) \rightarrow \mathbb{D}(I)$ is canonically triangulated in both variables.*

Proof. First of all, replacing \mathbb{D} by \mathbb{D}_I we reduce to the case $I = \star$. Moreover, by symmetry of the monoidal product we may fix $B \in \mathbb{D}(\star)_0$ and only prove $- \otimes B$ to be triangulated. Then the condition that the projection morphism $p_{J!}(A \otimes p_J^* B) \rightarrow p_{J!} A \otimes B$ be invertible for all $A \in \mathbb{D}(J)_0$ in the case of a finite discrete category J says precisely that $- \otimes B$ is additive.

The following claim is also a consequence of our definition of a monoidal derivator:

(*) Let $A \in \mathbb{D}(\square)_0$ be a cocartesian object. Then also $A \boxtimes B$ is cocartesian.

Indeed, this follows from the following factorization of the counit morphism:

$$\begin{aligned}
 i_{\Gamma!} i_{\Gamma}^* (A \boxtimes B) &\xrightarrow{\sim} i_{\Gamma!} (i_{\Gamma}^* A \boxtimes B) \\
 &\xrightarrow[\sim]{(6)} i_{\Gamma!} i_{\Gamma}^* A \boxtimes B \\
 &\xrightarrow[\sim]{\text{adj}} A \boxtimes B.
 \end{aligned}$$

Now let $A \in \mathbb{D}(\star)_0$ be an arbitrary object and consider $C = i_{\Gamma!}(1, 1)_* A \in \mathbb{D}(\square)_0$. Since $i_{\Gamma!}$ is fully faithful (this is an easy computation; see [3, 7.1]), C is cocartesian, and by (*), this is also true of $C \boxtimes B$. Moreover $(C \boxtimes B)_{(1,0)} \cong C_{(1,0)} \otimes B \cong 0$ and, similarly, $(C \boxtimes B)_{(0,1)} \cong 0$. It follows from the following claim (**) that $\Sigma(A \otimes B) \cong \Sigma(C_{(1,1)} \otimes B)$ is isomorphic to $C_{(0,0)} \otimes B \cong \Sigma A \otimes B$, naturally in A .

(**) Let $A \in \mathbb{D}(\square)_0$ be a cocartesian object with $A_{(1,0)} \cong A_{(0,1)} \cong 0$. Then there is a canonical isomorphism $\Sigma(A_{(1,1)}) \cong A_{(0,0)}$, natural in A .

The condition that those fibers vanish implies that the counit of adjunction $i_{\Gamma}^* A \rightarrow (1, 1)_*(1, 1)^* i_{\Gamma}^* A$ is invertible (again, an easy computation, cf. [3, 8.11]). But the left hand side becomes isomorphic to A after applying $i_{\Gamma!}$ by assumption, so we get the required isomorphism after applying $(0, 0)^* i_{\Gamma!}$.

Now let D be a distinguished triangle in $\mathbb{D}(\star)$, i.e. D is associated to a triangle $A \in \mathbb{D}(\square)_0$. Essentially by (*), $A \boxtimes B$ is again a triangle, and essentially by (**), the distinguished triangle associated to $A \boxtimes B$ is isomorphic to $D \otimes B$. \square

2.8. For I an object of \mathbf{Cat} , throughout the article we fix the notation as in the following diagram where both squares are pullback squares:

$$\begin{array}{ccc}
 2\mathrm{tw}(I) & \xrightarrow{r_2} & \mathrm{tw}(I)^\circ \\
 r_1 \downarrow & & \downarrow q_2 \\
 \mathrm{tw}(I) & \xrightarrow{q_1} & I^\circ \times I \xrightarrow{p_2} I \\
 & p_1 \downarrow & \downarrow p_I \\
 & I^\circ & \xrightarrow{p_{I^\circ}} \star.
 \end{array} \tag{\Delta_I}$$

Explicitly, the objects of $2\mathrm{tw}(I)$ are pairs of arrows in I of the form

$$i \rightleftarrows i',$$

and morphisms from this object to $j \rightleftarrows j'$ are pairs of morphisms $(i \leftarrow j, i' \rightarrow j')$ rendering the following two squares commutative:

$$\begin{array}{ccc}
 i & \longrightarrow & i' \\
 \uparrow & & \downarrow \\
 j & \longrightarrow & j'
 \end{array}, \quad
 \begin{array}{ccc}
 i & \longleftarrow & i' \\
 \uparrow & & \downarrow \\
 j & \longleftarrow & j'
 \end{array}.$$

Note that if I lies in some diagram category then so does the whole diagram (Δ_I) .

3. EXTERNAL HOM

Fix a closed monoidal derivator \mathbb{D} of type **Dia**. As explained in the introduction we would like to define an “external hom” functor which will play an essential role in the definition of the trace. It should behave with respect to the external product as does the internal hom with respect to the internal product (i.e. the monoidal structure). As a first indication of its nature, the external hom of $A \in \mathbb{D}(I)_0$ and $B \in \mathbb{D}(J)_0$ should be an object of $\mathbb{D}(I^\circ \times J)$, denoted by $\langle A, B \rangle$. Additionally, we would like the fibers of $\langle A, B \rangle$ to compute the internal hom of the fibers of A and B , because fiberwise dualizability should imply dualizability with respect to $\langle -, - \rangle$; moreover, $[A, B]$ should be expressible in terms of $\langle A, B \rangle$ in the case $I = J$. These and other desired properties of the external product are satisfied by the following construction which is due to Joseph Ayoub.

Given small categories I and J in **Dia**, we fix the following notation, all functors being the obvious ones:

$$\begin{array}{ccc}
 \mathrm{tw}(I) \times J & \xrightarrow{r} & J \\
 p \downarrow & \searrow q & \\
 I^\circ \times J & & I.
 \end{array} \tag{\Pi_{I,J}}$$

For any A in $\mathbb{D}(I)_0$ and B in $\mathbb{D}(J)_0$ set

$$\langle A, B \rangle := p_*[q^* A, r^* B].$$

This defines a bifunctor

$$\langle -, - \rangle : \mathbb{D}(I)^\circ \times \mathbb{D}(J) \rightarrow \mathbb{D}(I^\circ \times J),$$

whose properties we are going to list now. For the proofs the reader is referred to appendix A.

Naturality For functors $u : I' \rightarrow I$ and $v : J' \rightarrow J$ in **Dia** there is an invertible morphism

$$\Psi : (u^\circ \times v)^* \langle A, B \rangle \xrightarrow{\sim} \langle u^* A, v^* B \rangle,$$

natural in $A \in \mathbb{D}(I)_0$, $B \in \mathbb{D}(J)_0$. Moreover, Ψ behaves well with respect to functors and natural transformations in **Dia**. In other words, $\langle -, - \rangle$ defines a 1-morphism in $\mathbf{Fun}_{\text{lax}}(\mathbf{Dia}^\circ \times \mathbf{Dia}^{\circ, \circ}, \mathbf{CAT})$ from $\mathbb{D}(-)^\circ \times \mathbb{D}(-)$ to $\mathbb{D}(-^\circ \times -)$ with invertible 2-cell components (i.e. a pseudonatural transformation).

Internal hom In the case $I = J$ there is an invertible morphism

$$\Theta : [A, B] \xrightarrow{\sim} p_{2*} q_{2*} q_2^* \langle A, B \rangle \quad (\text{with the notation of } (\Delta_I)),$$

natural in A and $B \in \mathbb{D}(I)_0$. Moreover, for any functor $u : I' \rightarrow I$ in **Dia**, the canonical arrow $u^*[A, B] \rightarrow [u^*A, u^*B]$ is compatible with Ψ via Θ . In other words, Θ defines an invertible 2-morphism in $\mathbf{Fun}_{\text{lax}}((\mathbf{Dia}_{\leq 1})^\circ, \mathbf{CAT})$ between 1-morphisms from $\mathbb{D}(-)^\circ \times \mathbb{D}(-)$ to $\mathbb{D}(-)$. Here, $\mathbf{Dia}_{\leq 1}$ is the 2-subcategory of **Dia** obtained by removing all non-identity 2-cells.

External product Given categories $I_{(k)}$, $k = 1, \dots, 4$, in **Dia**, $A_k \in \mathbb{D}(I_{(k)})_0$, there is a morphism

$$\Xi : \langle A_1, A_2 \rangle \boxtimes \langle A_3, A_4 \rangle \rightarrow \tau^* \langle A_1 \boxtimes A_3, A_2 \boxtimes A_4 \rangle,$$

natural in all four arguments, where

$$\tau : I_{(1)}^\circ \times I_{(2)} \times I_{(3)}^\circ \times I_{(4)} \rightarrow I_{(1)}^\circ \times I_{(3)}^\circ \times I_{(2)} \times I_{(4)}$$

interchanges the two categories in the middle. Moreover, Ξ is compatible with Ψ and (4). In other words, it defines a 2-morphism in $\mathbf{Fun}_{\text{lax}}(\mathbf{Dia}^\circ \times \mathbf{Dia}^{\circ, \circ} \times \mathbf{Dia}^\circ \times \mathbf{Dia}^{\circ, \circ}, \mathbf{CAT})$ between 1-morphisms from $\mathbb{D}(-)^\circ \times \mathbb{D}(-) \times \mathbb{D}(-)^\circ \times \mathbb{D}(-)$ to $\mathbb{D}(-^\circ \times - \times -^\circ \times -)$.

Adjunction Given three categories in **Dia**, there is an invertible morphism

$$\Omega : \langle A, \langle B, C \rangle \rangle \xrightarrow{\sim} \langle A \boxtimes B, C \rangle,$$

natural in all three arguments. Moreover, Ω is compatible with Ψ and (4). In other words, it defines an invertible 2-morphism in $\mathbf{Fun}_{\text{lax}}(\mathbf{Dia}^\circ \times \mathbf{Dia}^\circ \times \mathbf{Dia}^{\circ, \circ}, \mathbf{CAT})$ between 1-morphisms from $\mathbb{D}(-)^\circ \times \mathbb{D}(-)^\circ \times \mathbb{D}(-)$ to $\mathbb{D}(-^\circ \times -^\circ \times -)$.

Biduality For fixed $B \in \mathbb{D}(\star)_0$, there is a morphism

$$\Upsilon : A \rightarrow \langle \langle A, B \rangle, B \rangle,$$

natural in $A \in \mathbb{D}(I)_0$. Moreover, Υ defines a 2-morphism in $\mathbf{Fun}_{\text{lax}}(\mathbf{Dia}^{\circ, \circ}, \mathbf{CAT})$ between 1-endomorphisms of \mathbb{D} .

Normalization Given $J \in \mathbf{Dia}_0$, there is an invertible morphism

$$\Lambda : [p_J^* A, B] \xrightarrow{\sim} \langle A, B \rangle,$$

natural in $A \in \mathbb{D}(\star)_0$ and $B \in \mathbb{D}(J)_0$. Again, Λ is compatible with v^* for any $v : J' \rightarrow J$ in \mathbf{Dia} , therefore it defines an invertible 2-morphism in $\mathbf{Fun}_{\text{lax}}(\mathbf{Dia}^{\circ, \circ}, \mathbf{CAT})$ between 1-morphisms from $\mathbb{D}(\star)^\circ \times \mathbb{D}(-)$ to $\mathbb{D}(-)$. Moreover under this identification, all the morphisms in the statements of the previous properties reduce to the canonical morphisms in closed monoidal categories. (These morphisms are made explicit in appendix A; see p. 37.)

4. DEFINITION OF THE TRACE

Recall that in a closed monoidal category \mathcal{C} , an object A is called *dualizable* (sometimes also *strongly dualizable*) if the canonical morphism

$$[A, \mathbb{1}] \otimes B \rightarrow [A, \mathbb{1} \otimes B] \quad (9)$$

is invertible for all $B \in \mathcal{C}_0$, and in this case one defines a *coevaluation*

$$\text{coev} : \mathbb{1} \xrightarrow{\text{adj}} [A, \mathbb{1} \otimes A] \xleftarrow{\sim} [A, \mathbb{1}] \otimes A. \quad (10)$$

It has the characterizing property that the following diagram commutes (see [14, 1.4]):

$$\begin{array}{ccc} [A, \mathbb{1}] \otimes A & \xrightarrow{\text{ev}} & \mathbb{1} \\ \sim \downarrow & & \downarrow \sim \\ [[A, \mathbb{1}] \otimes A, \mathbb{1}] & \xrightarrow{[\text{coev}, \mathbb{1}]} & [\mathbb{1}, \mathbb{1}]. \end{array} \quad (11)$$

Here the vertical morphism on the left is defined as the composition

$$[A, \mathbb{1}] \otimes A \rightarrow [A, \mathbb{1}] \otimes [[A, \mathbb{1}], \mathbb{1}] \rightarrow [A \otimes [A, \mathbb{1}], \mathbb{1} \otimes \mathbb{1}] \xrightarrow{\sim} [[A, \mathbb{1}] \otimes A, \mathbb{1}], \quad (12)$$

while the one on the right is

$$\mathbb{1} \xrightarrow{\text{adj}} [\mathbb{1}, \mathbb{1} \otimes \mathbb{1}] \xrightarrow{\sim} [\mathbb{1}, \mathbb{1}].$$

$[A, \mathbb{1}]$ is called the *dual* of A , and is often denoted by A^* . Dualizability of A implies that the canonical morphism

$$A \rightarrow (A^*)^* \quad (13)$$

is invertible.

For dualizable A , the *trace map*

$$\text{tr} : \mathcal{C}(A, A) \rightarrow \mathcal{C}(\mathbb{1}, \mathbb{1})$$

sends an endomorphism f to the composition

$$\mathbb{1} \xrightarrow{\text{coev}} A^* \otimes A \xrightarrow{1 \otimes f} A^* \otimes A \xrightarrow{\text{ev}} \mathbb{1}.$$

More generally, [16] and [11] independently introduced a (*twisted*) *trace map* for any S and T in \mathcal{C} (A still assumed dualizable),

$$\text{tr} : \mathcal{C}(A \otimes S, A \otimes T) \rightarrow \mathcal{C}(S, T),$$

which sends a “twisted endomorphism” f to the composition

$$S \xleftarrow{\sim} \mathbb{1} \otimes S \xrightarrow{\text{coev} \otimes 1} A^* \otimes A \otimes S \xrightarrow{1 \otimes f} A^* \otimes A \otimes T \xrightarrow{\text{ev} \otimes 1} \mathbb{1} \otimes T \xrightarrow{\sim} T.$$

We will mimic this definition in our derivator setting. So fix a closed monoidal derivator \mathbb{D} of type **Dia**. First of all, here is our translation of dualizability:

Definition 6 Let $I \in \mathbf{Dia}_0$, $A \in \mathbb{D}(I)_0$. We say that A is *fiberwise dualizable* if A_i is dualizable for all $i \in I_0$. The *dual* of A is defined to be $\langle A, \mathbb{1}_{\mathbb{D}(\star)} \rangle \in \mathbb{D}(I^\circ)_0$, also denoted by A^\vee .

Let I and A as in the definition, A fiberwise dualizable. Then, as was the case for dualizable objects in closed monoidal categories, the morphisms corresponding to (9) and (13) are invertible (for any $B \in \mathbb{D}(I)_0$):

$$A^\vee \boxtimes B \cong \langle A, \mathbb{1} \rangle \boxtimes [p_I^* \mathbb{1}, B] \xrightarrow[\sim]{\Lambda} \langle A, \mathbb{1} \rangle \boxtimes \langle \mathbb{1}, B \rangle \xrightarrow[\sim]{\Xi} \langle A \boxtimes \mathbb{1}, \mathbb{1} \boxtimes B \rangle \cong \langle A, B \rangle, \quad (14)$$

$$\Upsilon : A \xrightarrow{\sim} (A^\vee)^\vee. \quad (15)$$

This follows from the naturality and the normalization properties of the external hom. We now go about defining a coevaluation and an evaluation morphism. This will rely on the results of the previous section.

Using the relation between internal and external hom, we can consider the composition

$$\mathbb{1}_{\mathbb{D}(I)} \xrightarrow{\text{adj}} [A, \mathbb{1} \otimes A] \xrightarrow{\sim} [A, A] \xrightarrow[\sim]{\Theta} p_{2*} q_{2*} q_2^* \langle A, A \rangle$$

and, by adjunction, we obtain

$$\text{coev} : q_{2!} \mathbb{1} \rightarrow \langle A, A \rangle \xleftarrow[\sim]{(14)} A^\vee \boxtimes A. \quad (16)$$

Next, inspired by (11), we define the evaluation morphism to be simply the dual of the coevaluation morphism. For this, notice that A being fiberwise dualizable implies that also A^\vee is. Hence there is an analogue of (12):

$$A \boxtimes A^\vee \xrightarrow[\sim]{\Upsilon} (A^\vee)^\vee \boxtimes A^\vee \xrightarrow[\sim]{(14)} \langle A^\vee, A^\vee \rangle \xrightarrow[\sim]{\Omega} \langle A^\vee \boxtimes A, \mathbb{1}_{\mathbb{D}(\star)} \rangle. \quad (17)$$

Denote by $\mu : I \times I^\circ \rightarrow I^\circ \times I$ the canonical isomorphism. Then we define

$$\begin{aligned} \text{ev} : A^\vee \boxtimes A &\xrightarrow{\sim} \mu_*(A \boxtimes A^\vee) \\ &\xrightarrow[\sim]{(17)} \mu_* \langle A^\vee \boxtimes A, \mathbb{1} \rangle \\ &\xrightarrow[\sim]{\langle \text{coev}, \mathbb{1} \rangle} \mu_* \langle q_{2!} \mathbb{1}, \mathbb{1} \rangle \\ &\xrightarrow[\sim]{\overline{\Psi}} \mu_*(q_2)_*^\circ \langle \mathbb{1}, \mathbb{1} \rangle \\ &\xrightarrow[\sim]{} q_{1*} \mathbb{1}. \end{aligned}$$

Here, $\overline{\Psi}$ is obtained by adjunction from Ψ :

$$\begin{aligned} \overline{\Psi} : \langle q_{2!} \mathbb{1}, \mathbb{1} \rangle &\xrightarrow{\text{adj}} q_{2*}^\circ q_2^{\circ*} \langle q_{2!} \mathbb{1}, \mathbb{1} \rangle \\ &\xrightarrow[\sim]{\Psi} q_{2*}^\circ \langle q_2^* q_{2!} \mathbb{1}, \mathbb{1} \rangle \\ &\xrightarrow{\text{adj}} q_{2*}^\circ \langle \mathbb{1}, \mathbb{1} \rangle. \end{aligned}$$

It follows immediately that the following diagram commutes for any $u : I' \rightarrow I$ in **Dia**:

$$\begin{array}{ccc} (u \times u^\circ)^* \langle q_{2!} \mathbb{1}, \mathbb{1} \rangle & \longrightarrow & \langle q'_{2!} \mathrm{tw}(u)^{\circ*} \mathbb{1}, \mathbb{1} \rangle \\ \bar{\Psi} \downarrow & & \downarrow \bar{\Psi} \\ (u \times u^\circ)^* q_{2*}^\circ \langle \mathbb{1}, \mathbb{1} \rangle & \longrightarrow & q_{2*}^{\circ'} \langle \mathrm{tw}(u)^{\circ*} \mathbb{1}, \mathbb{1} \rangle. \end{array} \quad (18)$$

In the sequel we will sometimes denote by the same symbol $\bar{\Psi}$ other morphisms obtained by adjunction from Ψ in a similar way. It is hoped that this will not cause any confusion.

Finally we can put all the pieces together and define the trace:

Definition 7 Let $I \in \mathbf{Dia}_0$, $A \in \mathbb{D}(I)_0$ fiberwise dualizable, and $S, T \in \mathbb{D}(I)_0$ arbitrary. Then we define the (*twisted*) *trace map*

$$\mathrm{Tr} : \mathbb{D}(I) (A \otimes S, A \otimes T) \rightarrow \mathbb{D}(I^\circ \times I) (q_{2!} \mathbb{1} \otimes p_2^* S, q_{1*} \mathbb{1} \otimes p_2^* T)$$

as the association which sends a twisted endomorphism f to the composition

$$\begin{array}{ccccc} q_{2!} \mathbb{1} \otimes p_2^* S & \xrightarrow{\mathrm{coev} \otimes 1} & (A^\vee \boxtimes A) \otimes p_2^* S & \xrightarrow{\sim} & A^\vee \boxtimes (A \otimes S) \\ & & & & \downarrow 1 \boxtimes f \\ q_{1*} \mathbb{1} \otimes p_2^* T & \xleftarrow{\mathrm{ev} \otimes 1} & (A^\vee \boxtimes A) \otimes p_2^* T & \xleftarrow{\sim} & A^\vee \boxtimes (A \otimes T) \end{array}$$

called the (*twisted*) *trace of f* .

Remark 8 Although defined in this generality, we will be interested mainly in traces of endomorphisms twisted by “constant” objects, i. e. coming from objects in $\mathbb{D}(\star)$. In this case ($S, T \in \mathbb{D}(\star)_0$), the trace map is an association

$$\mathbb{D}(I) (A \boxtimes S, A \boxtimes T) \rightarrow \mathbb{D}(I^\circ \times I) (q_{2!} \mathbb{1} \boxtimes S, q_{1*} \mathbb{1} \boxtimes T).$$

Now, let g be an element of the target of this map. It induces the composite

$$q_{2!} S|_{\mathrm{tw}(I)^\circ} \xleftarrow{\sim} q_{2!} \mathbb{1} \boxtimes S \xrightarrow{g} q_{1*} \mathbb{1} \boxtimes T \rightarrow q_{1*} T|_{\mathrm{tw}(I)}$$

and by adjunction

$$q_2^* S|_{I^\circ \times I} \rightarrow q_2^* q_{1*} q_1^* T|_{I^\circ \times I} \xrightarrow{\sim} r_{2*} r_1^* q_1^* T|_{I^\circ \times I}$$

or, by another adjunction, a morphism

$$S|_{2\mathrm{tw}(I)} \rightarrow T|_{2\mathrm{tw}(I)}. \quad (19)$$

Applying the functor $\mathrm{dia}_{2\mathrm{tw}(I)}$ we obtain an element of

$$\mathbb{D}(\star)^{(2\mathrm{tw}(I))^\circ} \left(\mathrm{dia}_{2\mathrm{tw}(I)}(S|_{2\mathrm{tw}(I)}), \mathrm{dia}_{2\mathrm{tw}(I)}(T|_{2\mathrm{tw}(I)}) \right) \cong \prod_{\pi_0(2\mathrm{tw}(I))} \mathbb{D}(\star)(S, T).$$

The component corresponding to $\gamma \in \pi_0(2\mathrm{tw}(I))$ is called the γ -*component of g* .

Lemma 9 Suppose that the following hypotheses are satisfied:

- (H1) $q_{1*} \mathbb{1} \boxtimes A \rightarrow q_{1*}(\mathbb{1} \boxtimes A)$ is invertible for all $A \in \mathbb{D}(\star)_0$;
- (H2) for each connected component γ of $2\mathrm{tw}(I)$, the functor p_γ^* is fully faithful.

Then the map

$$\mathbb{D}(I^\circ \times I)(q_{2!}\mathbb{1} \boxtimes S, q_{1*}\mathbb{1} \boxtimes T) \rightarrow \prod_{\pi_0(2\mathrm{tw}(I))} \mathbb{D}(\star)(S, T)$$

defined above is a bijection. In particular, any morphism $q_{2!}\mathbb{1} \boxtimes S \rightarrow q_{1*}\mathbb{1} \boxtimes T$ is uniquely determined by its γ -components, $\gamma \in \pi_0(2\mathrm{tw}(I))$.

Proof. (H1) implies that the morphism $g : q_{2!}\mathbb{1} \boxtimes S \rightarrow q_{1*}\mathbb{1} \boxtimes T$ in the remark above can equivalently be described by (19). Moreover, the following square commutes:

$$\begin{array}{ccc} \mathbb{D}(2\mathrm{tw}(I))(S|_{2\mathrm{tw}(I)}, T|_{2\mathrm{tw}(I)}) & \xrightarrow{\mathrm{dia}_{2\mathrm{tw}(I)}} & \mathbb{D}(\star)^{2\mathrm{tw}(I)^\circ}(S_{\mathrm{cst}}, T_{\mathrm{cst}}) \\ \sim \downarrow & & \downarrow \sim \\ \prod_{\gamma \in \pi_0(2\mathrm{tw}(I))} \mathbb{D}(\gamma)(S|_\gamma, T|_\gamma) & \xleftarrow{(p_\gamma^*)_\gamma} & \prod_{\gamma \in \pi_0(2\mathrm{tw}(I))} \mathbb{D}(\star)(S, T). \end{array}$$

Here, the left vertical arrow is invertible by (D1). (H2) now implies that the horizontal arrow on the top is a bijection. \square

In particular we see that in favorable cases (and these are the only ones we will have much to say about) the seemingly complicated twisted trace is encoded simply by a family of morphisms over the terminal category. The goal of the following section is to determine these morphisms.

5. FUNCTORIALITY OF THE TRACE

Our immediate goal is to describe the components $S \rightarrow T \in \mathbb{D}(\star)_1$ associated to the trace of a (twisted) endomorphism of a fiberwise dualizable object as explained in the previous section. However, we take the opportunity to establish a more general functoriality property of the trace (Proposition 11). Our immediate goal will be achieved as a corollary to this result.

Throughout this section we fix a category $I \in \mathbf{Dia}_0$. An object of $2\mathrm{tw}(I)$ is a pair of arrows

$$i \begin{array}{c} \xrightarrow{h_1} \\ \xleftarrow{h_2} \end{array} j \quad (20)$$

in I (cf. 2.8). There is always a morphism in $2\mathrm{tw}(I)$ from an object of the form

$$(i, h) : \quad i \begin{array}{c} \xrightarrow{1_i} \\ \xleftarrow{h} \end{array} i$$

to (20), given by the pair of arrows $(1_i, h_1)$ if $h = h_2 h_1$. Hence we can take some of the (i, h) as representatives for $\pi_0(2\mathrm{tw}(I))$ and it is sufficient to describe the component $S \rightarrow T$ corresponding to (i, h) . This motivates the following more general functoriality statement.

Let $u : I' \rightarrow I$ be a functor, $\eta : u \rightarrow u$ a natural transformation in \mathbf{Dia} ; consider the basic diagram (Δ_I) . Notice that this diagram is functorial in I hence there is a canonical morphism of diagrams $(\Delta_{I'}) \rightarrow (\Delta_I)$ and we will use the convention that the arrows in $(\Delta_{I'})$ will be distinguished from their I -counterparts by being decorated with a prime.

Definition 10 Let $S, T \in \mathbb{D}(I)_0$. Define a pullback map

$$(u, \eta)^* : \mathbb{D}(I^\circ \times I) (q_{2!} \mathbb{1} \otimes p_2^* S, q_{1*} \mathbb{1} \otimes p_2^* T) \longrightarrow \mathbb{D}(I'^\circ \times I') (q'_{2!} \mathbb{1} \otimes p_2'^* u^* S, q'_{1*} \mathbb{1} \otimes p_2'^* u^* T)$$

by sending a morphism g to the composition

$$\begin{aligned} q'_{2!} \text{tw}(u)^{\circ*} \mathbb{1} \otimes p_2'^* u^* S &\rightarrow (u^\circ \times u)^* (q_{2!} \mathbb{1} \otimes p_2^* S) \xrightarrow{g} (u^\circ \times u)^* (q_{1*} \mathbb{1} \otimes p_2^* T) \\ &\xrightarrow{(1 \times \eta)^*} (u^\circ \times u)^* (q_{1*} \mathbb{1} \otimes p_2^* T) \rightarrow q'_{1*} \text{tw}(u)^* \mathbb{1} \otimes p_2'^* u^* T. \end{aligned}$$

Proposition 11 Let u, η, S, T as above, assume $A \in \mathbb{D}(I)_0$ is fiberwise dualizable. For any $f : A \otimes S \rightarrow A \otimes T$, we have

$$(u, \eta)^* \text{Tr}(f) = \text{Tr}(\eta^* \circ u^* f), \quad (21)$$

where $\eta^* \circ u^* f$ is any of the two top paths from the top left to the bottom right in the following commutative square:

$$\begin{array}{ccc} u^* A \otimes u^* S & \xrightarrow{u^* f} & u^* A \otimes u^* T \\ \eta_{A \otimes S}^* \downarrow & & \downarrow \eta_{A \otimes T}^* \\ u^* A \otimes u^* S & \xrightarrow{u^* f} & u^* A \otimes u^* T. \end{array}$$

Proof. The two outer paths in the following diagram are exactly the two sides of (21):

$$\begin{array}{ccc} q'_{2!} \text{tw}(u)^{\circ*} \mathbb{1} \otimes p_2'^* u^* S & \xrightarrow{\quad} & (u^\circ \times u)^* (q_{2!} \mathbb{1} \otimes p_2^* S) \\ \text{coev} \downarrow & & \downarrow \text{coev} \\ (u^* A)^\vee \boxtimes u^* A \otimes p_2'^* u^* S & \xleftarrow{\Psi} & (u^\circ \times u)^* (A^\vee \boxtimes A \otimes p_2^* S) \\ 1 \boxtimes u^* f \downarrow & & \downarrow 1 \boxtimes f \\ (u^* A)^\vee \boxtimes u^* A \otimes p_2'^* u^* T & \xleftarrow{\Psi} & (u^\circ \times u)^* (A^\vee \boxtimes A \otimes p_2^* T) \\ 1 \boxtimes \eta^* \downarrow & & \downarrow 1 \times \eta^* \\ (u^* A)^\vee \boxtimes u^* A \otimes p_2'^* u^* T & \xleftarrow{\Psi} & (u^\circ \times u)^* (A^\vee \boxtimes A \otimes p_2^* T) \\ \sim \downarrow & & \downarrow \sim \\ \mu'_* \langle (u^* A)^\vee \boxtimes u^* A, \mathbb{1} \rangle \otimes p_2'^* u^* T & \xleftarrow{\Psi} & (u^\circ \times u)^* (\mu'_* \langle A^\vee \boxtimes A, \mathbb{1} \rangle \otimes p_2^* T) \\ \langle \text{coev}, \mathbb{1} \rangle \downarrow & & \downarrow \langle \text{coev}, \mathbb{1} \rangle \\ \mu'_* \langle q'_{2!} \mathbb{1}, \mathbb{1} \rangle \otimes p_2'^* u^* T & \xleftarrow{\Psi} & (u^\circ \times u)^* (\mu'_* \langle q_{2!} \mathbb{1}, \mathbb{1} \rangle \otimes p_2^* T) \\ \downarrow & & \downarrow \\ q'_{1*} \text{tw}(u)^* \mathbb{1} \otimes p_2'^* u^* T & \xleftarrow{\quad} & (u^\circ \times u)^* (q_{1*} \mathbb{1} \otimes p_2^* T). \end{array}$$

Hence it suffices to prove the commutativity of this diagram. The second and third square clearly commute, the fourth and sixth square do so by the functoriality statements in section 3 (use also (18)). The fifth square commutes if the first does so we are left to show commutativity of the first one.

By definition, coev is the composition

$$q_2! \mathbb{1} \rightarrow \langle A, A \rangle \xleftarrow{\sim} A^\vee \boxtimes A$$

and we already know that the second arrow behaves well with respect to functors in **Dia**. Thus it suffices to prove that the following diagram commutes:

$$\begin{array}{ccccc} q_2'! q_2'^* p_2'^* \mathbb{1} & \longrightarrow & q_2'! q_2'^* p_2'^* p_2'^* q_2'^* \langle u^* A, u^* A \rangle & \xrightarrow{\text{adj}} & \langle u^* A, u^* A \rangle \\ \sim \downarrow & & \uparrow \Psi & & \uparrow \Psi \\ q_2'! q_2'^* p_2'^* u^* \mathbb{1} & \longrightarrow & q_2'! q_2'^* p_2'^* u^* p_2'^* q_2'^* \langle A, A \rangle & & \\ \downarrow & & \downarrow & & \\ (u^\circ \times u)^* q_2! q_2^* p_2^* \mathbb{1} & \longrightarrow & (u^\circ \times u)^* q_2! q_2^* p_2^* p_2^* q_2^* \langle A, A \rangle & \xrightarrow{\text{adj}} & (u^\circ \times u)^* \langle A, A \rangle. \end{array}$$

The top left square commutes by the internal hom property in section 3, the bottom left square clearly commutes, and the right rectangle is also easily seen to commute. \square

Of course, in the Proposition we can take $u = i$ to be an object of I , and η to be the identity transformation. Denote the pullback morphism $(i, 1)^*$ by i^* .

Corollary 12 *Let $i \in I_0$. For any $A, S, T \in \mathbb{D}(I)_0$, A fiberwise dualizable, and for any $f : A \otimes S \rightarrow A \otimes T$, we have*

$$i^* \text{Tr}(f) = \text{tr}(f_i)$$

modulo the obvious identifications.

Proof. By the proposition, $i^* \text{Tr}(f) = \text{Tr}(i^* f)$. It remains to prove that in the case $I = \star$, the maps Tr and tr coincide. Thus assume $I = \star$ and consider the following diagram:

$$\begin{array}{ccccccc} \mathbb{1} \otimes S & \xrightarrow{\text{coev}} & A^\vee \otimes A \otimes S & \xrightarrow{f} & A^\vee \otimes A \otimes T & \xrightarrow{\text{ev}} & \mathbb{1} \otimes T \\ \sim \downarrow & & \uparrow \Lambda & & \uparrow \sim & & \uparrow \sim \\ \mathbb{1} \otimes S & \xrightarrow{\text{coev}} & A^* \otimes A \otimes S & \xrightarrow{f} & A^* \otimes A \otimes T & \xrightarrow{\text{ev}} & \mathbb{1} \otimes T. \end{array}$$

The composition of the top horizontal arrows is $\text{Tr}(f)$ while the composition of the bottom horizontal arrows is $\text{tr}(f)$. The middle square clearly commutes. The left square commutes by the normalization property of the external hom, and commutativity of the right square can be deduced from this and (11). \square

Let us come back to the situation considered at the beginning of this section. Here the proposition implies:

Corollary 13 *Let $A \in \mathbb{D}(I)_0$ fiberwise dualizable, $S, T \in \mathbb{D}(\star)_0$, $i \in I_0$, $h \in I(i, i)$, and $f : A \boxtimes S \rightarrow A \boxtimes T \in \mathbb{D}(I)_1$. Then, modulo the obvious identifications, the (i, h) -component of $\text{Tr}(f)$ is $\text{tr}(h^* \circ f_i)$.*

Proof. h defines a natural transformation $i \rightarrow i$ and we have

$$\begin{aligned} (i, h)^* \text{Tr}(f) &= \text{Tr}(h^* \circ i^* f) && \text{by the proposition above,} \\ &= \text{tr}(h^* \circ i^* f) && \text{by the previous corollary.} \end{aligned}$$

We need to prove that the left hand side computes the (i, h) -component. The pair $(1_i, h)$ defines an arrow in $\mathrm{tw}(I)^\circ$ from h to 1_i . The composition of the vertical arrows on the left of the following diagram is the (i, h) -component of $\mathrm{Tr}(f)$ while the composition of the vertical arrows on the right is $(i, h)^*\mathrm{Tr}(f)$:

$$\begin{array}{ccccc}
(i, h)^* S|_{2\mathrm{tw}(I)} & \xlongequal{\quad} & h^* S|_{\mathrm{tw}(I)^\circ} & \xlongequal{\quad} & 1_i^* S|_{\mathrm{tw}(I)^\circ} \\
\downarrow \mathrm{adj} & & \downarrow \mathrm{adj} & & \downarrow \mathrm{adj} \\
(i, h)^* r_2^* q_2^* q_{2!}(\mathbb{1} \boxtimes S) & \xlongequal{\quad} & h^* q_2^* q_{2!}(\mathbb{1} \boxtimes S) & \xleftarrow{(1_i, h)^*} & 1_i^* q_2^* q_{2!}(\mathbb{1} \boxtimes S) \\
\downarrow \sim & & \downarrow \sim & & \downarrow \sim \\
(i, h)^* r_2^* q_2^*(q_{2!}\mathbb{1} \boxtimes S) & \xlongequal{\quad} & h^* q_2^*(q_{2!}\mathbb{1} \boxtimes S) & \xleftarrow{(1_i, h)^*} & 1_i^* q_2^*(q_{2!}\mathbb{1} \boxtimes S) \\
\downarrow \mathrm{Tr}(f) & & \downarrow \mathrm{Tr}(f) & & \downarrow \mathrm{Tr}(f) \\
(i, h)^* r_2^* q_2^*(q_{1*}\mathbb{1} \boxtimes T) & \xlongequal{\quad} & h^* q_2^*(q_{1*}\mathbb{1} \boxtimes T) & \xleftarrow{(1_i, h)^*} & 1_i^* q_2^*(q_{1*}\mathbb{1} \boxtimes T) \\
\downarrow & & \downarrow & & \downarrow \\
(i, h)^* r_2^* q_2^* q_{1*}(\mathbb{1} \boxtimes T) & \xlongequal{\quad} & h^* q_2^* q_{1*}(\mathbb{1} \boxtimes T) & \xleftarrow{(1_i, h)^*} & 1_i^* q_2^* q_{1*}(\mathbb{1} \boxtimes T) \\
\downarrow \sim & & \downarrow \sim & & \downarrow (1_i \times h)^* \\
(i, h)^* r_1^* q_1^* q_{1*} T|_{\mathrm{tw}(I)} & \xlongequal{\quad} & 1_i^* q_1^* q_{1*} T|_{\mathrm{tw}(I)} & \xlongequal{\quad} & 1_i^* q_1^* q_{1*} T|_{\mathrm{tw}(I)} \\
\downarrow \mathrm{adj} & & \downarrow \mathrm{adj} & & \downarrow \mathrm{adj} \\
(i, h)^* T|_{2\mathrm{tw}(I)} & \xlongequal{\quad} & 1_i^* T|_{\mathrm{tw}(I)} & \xlongequal{\quad} & 1_i^* T|_{\mathrm{tw}(I)}.
\end{array}$$

The unlabeled arrows are the canonical ones; all squares clearly commute. \square

Knowing the components of the trace we now give a better description of the indexing set $\pi_0(2\mathrm{tw}(I))$, at least for “EI-categories”:

Definition 14 An *EI-category* I is a category whose endomorphisms are all invertible, i. e. such that for all $i \in I_0$, $G_i := I(i, i)$ is a group.

EI-categories have been of interest in studies pertaining to different fields of mathematics, especially in representation theory and algebraic topology; closest to our discussion in the sequel is their role in the study of the Euler characteristic of a category (see [4], [13]). We will see examples of EI-categories below.

Let I be an EI-category; we define the *endomorphism category* $\mathrm{end}(I)$ associated to I to be the category whose objects are endomorphisms in I and an arrow from $h \in I(i, i)$ to $k \in I(j, j)$ is a morphism $m \in I(i, j)$ such that $mh = km$. The object $h \in I(i, i)$ is sometimes denoted by (i, h) . There is also a canonical functor

$$2\mathrm{tw}(I) \rightarrow \mathrm{end}(I)$$

which takes a typical object (20) of $2\mathrm{tw}(I)$ to its composition $h_1 h_2 \in I(j, j)$. Notice that it takes (i, h) to (i, h) .

Lemma 15 *Let I be an EI-category, $h \in I(i, i)$, $k \in I(j, j)$. Then (i, h) and (j, k) lie in the same connected component of $2\mathrm{tw}(I)$ if and only if $h \cong k$ as objects of*

$\text{end}(I)$. In other words, the functor defined above induces a bijection

$$\pi_0(2\text{tw}(I)) \longleftrightarrow \text{end}(I)_0 / \cong.$$

Proof. If $m : h \rightarrow k$ is an isomorphism in $\text{end}(I)$ then (m^{-1}, m) defines a morphism in $2\text{tw}(I)$ from (i, h) to (j, k) .

For the converse we notice that $2\text{tw}(I)$ is a groupoid. Indeed, it follows from the definition of an EI-category that in a typical object (20) of $2\text{tw}(I)$, both h_1 and h_2 must be isomorphisms. From this, and using a similar argument, one deduces that the components of any morphism in $2\text{tw}(I)$ are invertible.

Given now a morphism (m_1, m_2) from (i, h) to (j, k) in $2\text{tw}(I)$, we must have $m_1 = m_2^{-1}$ and therefore m_2 defines an isomorphism from h to k in $\text{end}(I)$. \square

Example 16

- (1) Let I be a preordered set considered as a category. Clearly this is an EI-category, and $\text{end}(I) = I$. It follows that we have $\pi_0(2\text{tw}(I)) = I_0 / \cong$, the isomorphism classes of objects in I , or in other words, the (underlying set of the) poset associated to I . If the hypotheses of Lemma 9 are satisfied then the trace of an endomorphism f is just the family of the traces of the fibers $(\text{tr}(f_i))_i$, indexed by isomorphism classes of objects in I .
- (2) Let G be a group. We can consider G canonically as a category with one object, the morphisms being given by G itself, the composition being the multiplication in G . Again, this is an EI-category. Given h and k in G , an element $m \in G$ defines a morphism $m : h \rightarrow k$ if and only if it satisfies $mh = m^{-1}k$, so h and k are connected (and therefore isomorphic) in $\text{end}(I)$ if and only if they are conjugate in G . It follows that $\pi_0(2\text{tw}(I))$ can be identified with the set of conjugacy classes of G . If the hypotheses of Lemma 9 are satisfied then the trace of an endomorphism f with unique fiber e^*f is just the family of traces $(\text{tr}(h^* \circ e^*f))_{[h]}$, indexed by the conjugacy classes of G .
- (3) Generalizing the two previous examples, for an arbitrary EI-category I , $\text{end}(I)_0 / \cong$ can be identified with the disjoint union of the sets C_i of conjugacy classes of the groups $G_i = I(i, i)$ for representatives i of the isomorphism classes in I , i.e.

$$\pi_0(2\text{tw}(I)) \longleftrightarrow \coprod_{i \in I_0 / \cong} C_i.$$

If the hypotheses of Lemma 9 are satisfied then the trace of an endomorphism f is just the family of traces $(\text{tr}(h^* \circ f_i))_{i, [h]}$.

Remark 17 One can define the category $\text{end}(I)$ without the hypothesis that I be an EI-category but the previous lemma does not remain true without it. However, there is the following general alternative description of $\pi_0(2\text{tw}(I))$: Let \sim be the equivalence relation on the set $\coprod_{i \in I_0} I(i, i)$ generated by the relation $m_1 m_2 \sim m_2 m_1$, $m_1, m_2 \in I_1$ composable. Then (i, h) and (j, k) lie in the same connected component of $2\text{tw}(I)$ if and only if $h \sim k$. It follows that for arbitrary I , there is a bijection

$$\pi_0(2\text{tw}(I)) \longleftrightarrow \left(\coprod_{i \in I_0} I(i, i) \right) / \sim.$$

6. THE TRACE OF THE HOMOTOPY COLIMIT

Given a closed monoidal derivator \mathbb{D} , a category I in the domain \mathbf{Dia} of \mathbb{D} and objects A of $\mathbb{D}(I)$ fiberwise dualizable, S and T of $\mathbb{D}(\star)$, we can associate to every $f : A \boxtimes S \rightarrow A \boxtimes T$ in $\mathbb{D}(I)$ its homotopy colimit $p_{I!}f : p_{I!}A \boxtimes S \rightarrow p_{I!}A \boxtimes T$ by requiring that the following square commutes:

$$\begin{array}{ccc} p_{I!}(A \boxtimes S) & \xrightarrow{f} & p_{I!}(A \boxtimes T) \\ \textcolor{blue}{(6)} \downarrow \sim & & \sim \downarrow \textcolor{blue}{(6)} \\ p_{I!}A \boxtimes S & \xrightarrow{p_{I!}f} & p_{I!}A \boxtimes T. \end{array}$$

We will now show that, in good cases, the trace of f as defined above contains enough information to compute the trace of the homotopy colimit of f .

Definition 18 Given a morphism $g : q_{2!}\mathbb{1} \boxtimes S \rightarrow q_{1*}\mathbb{1} \boxtimes T$ as in Remark 8 (or, under the hypotheses in Lemma 9, the family of its γ -components, $\gamma \in \pi_0(2\mathrm{tw}(I))$), we associate to it a new morphism $\Phi(g) : S \rightarrow T$, provided that the morphism $p_{I!}p_{2*} \rightarrow p_{I\circ*}p_{1!}$ is invertible. (This latter morphism is obtained by adjunction from the composition

$$p_{I\circ*}p_{1!}p_{2*} \xleftarrow{\sim} p_{1!}p_{2*}p_{2*} \xrightarrow{\mathrm{adj}} p_{1!},$$

where for the first isomorphism one uses Lemma 1.) In this case $\Phi(g)$ is defined by the requirement that the following rectangle commutes:

$$\begin{array}{ccc} S & \xrightarrow{\Phi(g)} & T \\ \mathrm{adj} \downarrow & & \uparrow \mathrm{adj} \\ p_{I\circ*}p_{I\circ}^*S & & p_{I!}p_{I!}^*T \\ \mathrm{adj} \uparrow \sim & & \sim \downarrow \mathrm{adj} \\ p_{I\circ*}p_{1!}q_{2!}q_2^*p_1^*p_{I\circ}^*S & & p_{I!}p_{2*}q_{1*}q_1^*p_2^*p_{I!}^*T \\ \downarrow \sim & & \uparrow \sim \\ p_{I\circ*}p_{1!}q_{2!}(\mathbb{1} \boxtimes S) & & p_{I!}p_{2*}q_{1*}(\mathbb{1} \boxtimes T) \\ \downarrow \sim & & \uparrow \\ p_{I\circ*}p_{1!}(q_{2!}\mathbb{1} \boxtimes S) & \xrightarrow{g} p_{I\circ*}p_{1!}(q_{1*}\mathbb{1} \boxtimes T) \xleftarrow{\sim} p_{I!}p_{2*}(q_{1*}\mathbb{1} \boxtimes T). \end{array} \quad (22)$$

Here, the two (co)units of adjunctions going in the “wrong” direction are invertible by Lemma 34.

Remark 19 Suppose that the conditions (H1) and (H2) of Lemma 9 are satisfied, thus $\Phi = \Phi_{S,T}$ can be identified with a map $\prod_{\pi_0(2\mathrm{tw}(I))} \mathbb{D}(\star)(S, T) \rightarrow \mathbb{D}(\star)(S, T)$. The observation is that this map is natural in both arguments, in the following

sense: Given morphisms $S \rightarrow S'$ and $T \rightarrow T'$, the following diagram commutes:

$$\begin{array}{ccc}
 \prod_{\pi_0(2\mathrm{tw}(I))} \mathbb{D}(\star)(S', T) & \xrightarrow{\Phi_{S', T}} & \mathbb{D}(\star)(S', T) \\
 \downarrow & & \downarrow \\
 \prod_{\pi_0(2\mathrm{tw}(I))} \mathbb{D}(\star)(S, T) & \xrightarrow{\Phi_{S, T}} & \mathbb{D}(\star)(S, T) \\
 \downarrow & & \downarrow \\
 \prod_{\pi_0(2\mathrm{tw}(I))} \mathbb{D}(\star)(S, T') & \xrightarrow{\Phi_{S, T'}} & \mathbb{D}(\star)(S, T').
 \end{array}$$

This follows immediately from the definition of Φ .

Proposition 20 *Let $I \in \mathbf{Dia}_0$, and suppose that the following conditions are satisfied:*

(H3) *the morphism $p_{I!}p_{2*} \rightarrow p_{I \circ * }p_{1!}$ is invertible;*

(H4) *the morphism $p_{I \circ *} - \otimes - \rightarrow p_{I \circ *}(- \otimes p_{I^*}^* -)$ is invertible.*

If $A \in \mathbb{D}(I)_0$ is fiberwise dualizable, $S, T \in \mathbb{D}(\star)_0$, and $f : A \boxtimes S \rightarrow A \boxtimes T$, then the object $p_{I!}A$ is dualizable in $\mathbb{D}(\star)$ and the following equality holds:

$$\Phi(\mathrm{Tr}(f)) = \mathrm{tr}(p_{I!}f).$$

Proof. (H4) implies that $p_{I!}$ preserves fiberwise dualizable objects. Then the proof proceeds by decomposing (22) into smaller pieces; since it is rather long and not very enlightening we defer it to appendix B. \square

Remark 21 It is worth noting that the particular shape of diagram (22) is of no importance to us. All we will use in the sequel is that there exists a map Φ , natural in the sense of Remark 19, and which takes the trace of a (twisted) endomorphism to the trace of its homotopy colimit. The idea is the following: Suppose \mathbb{D} is additive, and let I be a category satisfying (H1)–(H4). Then Corollary 13 tells us that $\mathrm{Tr}(f)$ is completely determined by the local traces $\mathrm{tr}(h^* \circ f_i)$, $(i, h) \in \pi_0(2\mathrm{tw}(I))$. If $\pi_0(2\mathrm{tw}(I))$ is finite then, by Remark 19, we can think of Φ as a linear map which takes the input $(\mathrm{tr}(h^* \circ f_i))_{(i, h)}$ and outputs $\sum_{(i, h)} \lambda_{(i, h)} \mathrm{tr}(h^* \circ f_i) = \mathrm{tr}(p_{I!}f)$. We will obtain a formula for the trace of the homotopy colimit by determining these coefficients $\lambda_{(i, h)}$.

Let I be a finite category. The ζ -function on I is defined as the association

$$\begin{aligned}
 \zeta_I : I_0 \times I_0 &\rightarrow \mathbb{Z} \\
 (i, j) &\mapsto \#I(i, j).
 \end{aligned}$$

Following [13] we define an R -coweighting on I , R a commutative unitary ring, to be a family $(\lambda_i)_{i \in I_0}$ of elements of R such that the following equality holds for all $j \in I_0$:

$$1 = \sum_{i \in I_0} \lambda_i \zeta_I(i, j). \quad (23)$$

Not all finite categories possess an R -coweighting; and if one such exists it might not be unique. Preordered sets always possess an R -coweighting (and it is unique if and only if the preorder is a partial order), groups possess one if and only if their order is invertible in R (and in this case it is unique). One trivial reason why a coweighting may not be unique is the existence of isomorphic distinct objects in a

category. For in this case any modification of the family $(\lambda_i)_i$ which doesn't change the sum of the coefficients λ_i for isomorphic objects leaves the right hand side of (23) unchanged. On the other hand, this also means that any coweighting $(\lambda_i)_i$ on I induces a coweighting $(\rho_j)_j$ on the core of I by setting $\rho_j = \sum_{i \in I_0, i \cong j} \lambda_i$. (Here, “the” core of I is any equivalent subcategory of I which is *skeletal*, i. e. has no distinct isomorphic objects.) Conversely, any coweighting on the core induces a coweighting on I by choosing all additional coefficients to be 0. We therefore say that I admits an *essentially unique R -coweighting* if there is a unique R -coweighting on its core. In this case we sometimes speak abusively of *the* R -coweighting, especially if the context makes it clear which core is to be chosen.

For an EI-category I we continue to denote by G_i , $i \in I_0$, the group $I(i, i)$, and by C_i the set of conjugacy classes of G_i (cf. Example 16). Given $h \in G_i$, we denote by $[h] \in C_i$ the conjugacy class of h in G_i .

Definition 22 Let I be a finite EI-category. We define its *characteristic*, denoted by $\text{char}(I)$, to be the product of distinct prime factors dividing the order of the automorphism group of some object in the category, i. e.

$$\text{char}(I) = \text{rad} \left(\prod_{i \in I_0} \#G_i \right).$$

Lemma 23 (cf. [13, 1.4]) *Let I be a finite EI-category and R a commutative unitary ring. If $\text{char}(I)$ is invertible in R then there is an essentially unique R -coweighting on $\text{end}(I)$. It is given as follows:*

Choose a core $J \subset \text{end}(I)$ of objects $\{(i, h)\}$. Then

$$\lambda_{(j,k)} = \sum_{(i,h) \in J_0} \sum_{n \geq 0} (-1)^n \sum \frac{\#[h_0]}{\#G_{i_0}} \cdots \frac{\#[h_n]}{\#G_{i_n}},$$

where the last sum is over all non-degenerate paths

$$(i, h) = (i_0, h_0) \rightarrow (i_1, h_1) \rightarrow \cdots \rightarrow (i_n, h_n) = (j, k)$$

from (i, h) to (j, k) in J (i. e. the (i_l, h_l) are pairwise distinct, or, equivalently, the i_l are pairwise distinct, or, also equivalently, none of the arrows is the identity).

Proof. The data $(\zeta_J(h, k))_{h,k \in J_0}$ can be identified in an obvious way with a square matrix ζ_J with coefficients in \mathbb{Z} . For the first claim in the lemma, it suffices to prove that ζ_J is an invertible matrix in R , for then

$$\begin{aligned} \begin{bmatrix} \cdots & \lambda_{(i,h)} & \cdots \end{bmatrix} &= \begin{bmatrix} \cdots & \lambda_{(i,h)} & \cdots \end{bmatrix} (\zeta_J \zeta_J^{-1}) \\ &= \left(\begin{bmatrix} \cdots & \lambda_{(i,h)} & \cdots \end{bmatrix} \zeta_J \right) \zeta_J^{-1} \\ &= \begin{bmatrix} \cdots & 1 & \cdots \end{bmatrix} \zeta_J^{-1}. \end{aligned}$$

For any $(i, h) \in J_0$, the endomorphism monoid is $G_{(i,h)} = C_{G_i}(h)$, the centralizer of h , hence J is also a finite EI-category. This implies that we can find an object $(i, h) \in J_0$ which has no incoming arrows from other objects. Proceeding inductively we can thus choose an ordering of J_0 such that the matrix ζ_J is upper triangular. Consequently, $\det(\zeta_J) = \prod_{(i,h) \in J_0} \#C_{G_i}(h)$ is invertible in R by assumption.

The proof in [13, 1.4] goes through word for word to establish the formula given in the lemma (the relation between “Möbius inversion” and coweighting is given in [13, p. 28]). \square

Example 24

- (1) Let I be a finite skeletal category with no non-identity endomorphisms (e.g. a partially ordered set). Then for any ring R there is a unique R -coweighting on $I = \text{end}(I)$ given by (cf. [13, 1.5])

$$\lambda_j = \sum_{i \in I_0} \sum_{n \geq 0} (-1)^n \# \{ \text{non-degenerate paths of length } n \text{ from } i \text{ to } j \}$$

for any $j \in I_0$.

- (2) Let $I = G$ be a finite group. By Example 16, the objects of the core of $\text{end}(G)$ can be identified with the conjugacy classes of G . For a $\mathbb{Z}[1/\#G]$ -algebra R , the R -coweighting on $\text{end}(G)$ is given by

$$\lambda_{[k]} = \frac{\#[k]}{\#G}$$

for any conjugacy class $[k]$ of G .

Example 25 Let us go back to the situation considered in the introduction: Let \mathfrak{r} be the category of (2). It follows from the first example above that for any ring R , the unique R -coweighting on $\mathfrak{r} = \text{end}(\mathfrak{r})$ is given by

$$\begin{array}{ccc} -1 & \longleftarrow & 1 \\ \uparrow & & \\ 1 & & \end{array}$$

and one notices that these are precisely the coefficients in the formula for the trace of the homotopy colimit (3). This is an instance of the following theorem.

Theorem 26 *Let \mathbb{D} be a closed monoidal triangulated derivator of type **Dia**, let I be a finite EI-category in **Dia** and suppose that $\text{char}(I)$ is invertible in $R_{\mathbb{D}}$. If $S, T \in \mathbb{D}(\star)_0$, $f : A \boxtimes S \rightarrow A \boxtimes T \in \mathbb{D}(I)_1$, with $A \in \mathbb{D}(I)_0$ fiberwise dualizable, then the object $p_{I!}A$ is dualizable in $\mathbb{D}(\star)$, and we have*

$$\text{tr}(p_{I!}f) = \sum_{\substack{i \in I_0 / \cong \\ [h] \in C_i}} \lambda_{(i,h)} \text{tr}(h^* \circ f_i)$$

where $(\lambda_{(i,h)})_{(i,h)}$ is the $R_{\mathbb{D}}$ -coweighting on $\text{end}(I)$.

We will prove the theorem under the additional assumption that all of the hypotheses (H1)–(H4) are satisfied. In the next section we will show that they in fact automatically hold (Proposition 29).

Proof. We view $\pi_0(2\text{tw}(I))$ as the set of pairs (i, h) where i runs through a full set of representatives for the isomorphism classes of objects of I , and h runs through a full set of representatives for the conjugacy classes of G_i (use Example 16).

Lemma 9 tells us that we may consider Φ as a group homomorphism

$$\prod_{\pi_0(2\text{tw}(I))} \mathbb{D}(\star)(S, T) \rightarrow \mathbb{D}(\star)(S, T).$$

We first assume $S = T$, set $R = \mathbb{D}(\star)(S, S)$. In this case, Remark 19 tells us that Φ is both left and right R -linear hence there exist $\lambda_{(i,h)} \in Z(R)$, the center of R ,

such that for every $g = (g_{(i,h)})_{(i,h)}$ in the domain,

$$\Phi(g) = \sum_{(i,h) \in \pi_0(2\text{tw}(I))} \lambda_{(i,h)} g_{(i,h)}.$$

In particular, if $g = \text{Tr}(f)$ we get

$$\begin{aligned} \text{tr}(p_{I!}f) &= \Phi(\text{Tr}(f)) && \text{by Proposition 20,} \\ &= \sum_{(i,h)} \lambda_{(i,h)} \text{tr}(h^* \circ f_i) && \text{by Corollary 13.} \end{aligned} \quad (24)$$

Now, fix $(j,k) \in \pi_0(2\text{tw}(I))$. Below we will define a specific endomorphism f satisfying

$$\text{tr}(p_{I!}f) = 1_S \quad (25)$$

and

$$\text{tr}(h^* \circ f_i) = \zeta_{\text{end}(I)}(h,k) \quad (26)$$

for any $(i,h) \in \pi_0(2\text{tw}(I))$. Letting $(j,k) \in \pi_0(2\text{tw}(I))$ vary, (24) thus says that the $\lambda_{(i,h)}$ define a $Z(R)$ -coweighting on the core of $\text{end}(I)$ and by Lemma 23 this is unique (by assumption, $\text{char}(I)$ is invertible in $R_{\mathbb{D}}$ but then it must also be invertible in $Z(R)$). It must therefore be (the image of) the unique $R_{\mathbb{D}}$ -coweighting on the core of $\text{end}(I)$ and this would complete the proof of the theorem in the case $S = T$.

Before we come to the construction of f , let us explain how the general case (i.e. when not necessarily $S = T$) can be deduced. Set $U = S \oplus T$ and denote by $\iota : S \rightarrow U$ and $\pi : U \rightarrow T$ the canonical inclusion and projection, respectively. By Remark 19, the following diagram commutes:

$$\begin{array}{ccc} \prod_{\pi_0(2\text{tw}(I))} \mathbb{D}(\star)(U, U) & \xrightarrow{\Phi_{U,U}} & \mathbb{D}(\star)(U, U) \\ \downarrow \iota^* & & \downarrow \iota^* \\ \prod_{\pi_0(2\text{tw}(I))} \mathbb{D}(\star)(S, U) & \xrightarrow{\Phi_{S,U}} & \mathbb{D}(\star)(S, U) \\ \downarrow \pi_* & & \downarrow \pi_* \\ \prod_{\pi_0(2\text{tw}(I))} \mathbb{D}(\star)(S, T) & \xrightarrow{\Phi_{S,T}} & \mathbb{D}(\star)(S, T). \end{array}$$

Given a family $(g_{(i,h)})_{(i,h)}$ in the bottom left, there is a canonical lift $(\widetilde{g_{(i,h)}})_{(i,h)}$ in the top left, similarly for the right hand side. In particular, given S, T, A, f as in

the statement of the theorem,

$$\begin{aligned}
 \mathrm{tr}(p_{I!}f) &= \Phi_{S,T}(\mathrm{Tr}(f)) && \text{by the proposition,} \\
 &= \Phi_{S,T}(\pi_* \iota^* (\widetilde{\mathrm{Tr}(f)}_{(i,h)}))_{(i,h)} \\
 &= \pi_* \iota^* \Phi_{U,U}((\widetilde{\mathrm{Tr}(f)}_{(i,h)}))_{(i,h)} \\
 &= \pi_* \iota^* \sum_{(i,h)} \lambda_{(i,h)} \mathrm{tr}(\widetilde{h^* \circ f_i}) && \text{by the previous argument,} \\
 &= \sum_{(i,h)} \pi \lambda_{(i,h)} \mathrm{tr}(\widetilde{h^* \circ f_i}) \iota \\
 &= \sum_{(i,h)} \lambda_{(i,h)} \pi \mathrm{tr}(\widetilde{h^* \circ f_i}) \iota \\
 &= \sum_{(i,h)} \lambda_{(i,h)} \mathrm{tr}(h^* \circ f_i).
 \end{aligned}$$

This completes the argument in the general case.

Now we come to the construction of the endomorphism f mentioned above. We will freely use the fact that for any finite group $G \in \mathbf{Dia}_0$ whose order is invertible in $R_{\mathbb{D}}$ (such as G_i for all $i \in I_0$ by assumption), the underlying diagram functor

$$\mathrm{dia}_G : \mathbb{D}(G) \rightarrow \mathbf{CAT}(G^\circ, \mathbb{D}(\star))$$

is fully faithful. We postpone the proof of this to appendix C.

Fix $(j, k) \in \pi_0(2\mathrm{tw}(I))$. Denote by $e_j : \star \rightarrow G_j$ the unique functor; by (Dia2), this is a functor in \mathbf{Dia} . Then $e_{j!}S$ is the right regular representation of G_j° associated to S (for more details, see appendix C); we denote the action by $r_{(-)}$. Left translation by k, l_k , defines a G_j° -endomorphism of $e_{j!}S$. By transitivity of the action,

$$p_{G_j!}l_k : S = p_{G_j!}e_{j!}S = e_{j!}S/G_j^\circ \rightarrow e_{j!}S/G_j^\circ = p_{G_j!}e_{j!}S = S$$

is just the identity.

Let $\bar{j} : G_j \rightarrow I$ be the fully faithful inclusion pointing j and set $f = \bar{j}_!l_k$. To be completely precise, we should set $A = j_!\mathbb{1}$, and f to be the endomorphism of $A \boxtimes S$ induced by $\bar{j}_!l_k$ via the canonical isomorphism

$$j_!\mathbb{1} \boxtimes S \xleftarrow[\sim]{(6)} j_!(\mathbb{1} \boxtimes S) \xrightarrow{\sim} j_!S \xrightarrow{\sim} \bar{j}_!e_{j!}S.$$

However, for the sake of clarity, we will continue to use this identification implicitly.

Then we have

$$\begin{aligned}
 \mathrm{tr}(p_{I!}\bar{j}_!l_k) &= \mathrm{tr}(p_{G_j!}l_k) \\
 &= \mathrm{tr}(1_S) && \text{as seen above,} \\
 &= 1_S
 \end{aligned}$$

i. e. (25) holds.

For (26) we must understand $h^* \circ i^* \bar{j}_!l_k$. Write $S(m)$ for the stabilizer subgroup of $m \in I(i, j)$ in G_j and consider the following comma square in \mathbf{Dia}

$$\begin{array}{ccc}
 \coprod_m S(m) & \xrightarrow{w} & G_j \\
 p \downarrow & \eta \nearrow & \downarrow \bar{j} \\
 \star & \xrightarrow{i} & I
 \end{array}$$

where the disjoint union is indexed by a full set of representatives for the G_j -orbits of $I(i, j)$, w is the canonical inclusion on each component, and η is m on the component of m . Under the identification $i^* \bar{j}_! \cong p_! w^*$ (by (D4)),

$$i^* \bar{j}_! e_{j!} S \cong p_! w^* e_{j!} S \cong \bigoplus_m (e_{j!} S / S(m)) \cong \bigoplus_m \bigoplus_{G_j / S(m)} S,$$

and $i^* \bar{j}_! l_k$ corresponds to the morphism which takes the $gS(m)$ -summand identically to the $k^{-1}gS(m)$ -summand. It follows that under the identification $i^* \bar{j}_! e_{j!} S \cong i^* j_! S \cong \bigoplus_{I(i, j)} S$ (again by (D4)), it corresponds to the morphism which takes the m -summand identically to the $k^{-1}m$ -summand.

Writing out explicitly the Beck-Chevalley transformation above we obtain the horizontal arrows in the following diagram:

$$\begin{array}{ccccccc} \bigoplus_{I(i, j)} & \xrightarrow{\text{adj}} & \bigoplus_{I(i, j)} j^* j_! & \xrightarrow{(m^*)_m} & \bigoplus_{I(i, j)} i^* j_! & \xrightarrow{\Sigma} & i^* j_! \\ m \mapsto mh \downarrow & & \downarrow m \mapsto mh & & & & \downarrow h^* \\ \bigoplus_{I(i, j)} & \xrightarrow{\text{adj}} & \bigoplus_{I(i, j)} j^* j_! & \xrightarrow{(m^*)_m} & \bigoplus_{I(i, j)} i^* j_! & \xrightarrow{\Sigma} & i^* j_!. \end{array}$$

Obviously, the diagram is commutative. In total we get that $h^* \circ i^* \bar{j}_! l_k$ corresponds to the morphism which takes the m -summand identically to the $k^{-1}mh$ -summand. It follows that the trace of this composition is equal to

$$\begin{aligned} \text{tr}(h^* \circ i^* \bar{j}_! l_k) &= \#\{m \in I(i, j) \mid k^{-1}mh = m\} \\ &= \#\text{end}(I)(h, k) \\ &= \zeta_{\text{end}(I)}(h, k). \end{aligned}$$

□

7. \mathbb{Q} -LINEARITY AND TRIANGULATION

Let \mathbb{D} be a monoidal triangulated derivator. In this section we will show that for any finite EI-category $I \in \mathbf{Dia}_0$, if $\text{char}(I)$ is invertible in $R_{\mathbb{D}}$ then all hypotheses (H1)–(H4) automatically hold. The main tool used in the proof is Lemma 27 below, in essence suggested to me by Joseph Ayoub, where it is shown how invertibility of $\text{char}(I)$ in $R_{\mathbb{D}}$ and \mathbb{D} being triangulated imply the existence of nice generators for $\mathbb{D}(I)$. In fact, this is the only place in the article where the triangulated structure plays any role.

Recall that a subcategory of a triangulated category is called *thick* if it is a triangulated subcategory and closed under direct factors. If \mathcal{T} is a triangulated category and $S \subset \mathcal{T}_0$ a family of objects we denote by

$$\langle S \rangle \quad (\text{resp. } \langle S \rangle_s)$$

the triangulated (resp. thick) subcategory generated by S .

Let I be an EI-category. If $i \in I_0$ is an object we denote by G_i its automorphism group $I(i, i)$, and by $\bar{i} : G_i \rightarrow I$ the fully faithful embedding of the “point” i into I . This is to distinguish it from the inclusion $i : \star \rightarrow I$.

Lemma 27 *Let \mathbb{D} be a triangulated derivator, and let $I \in \mathbf{Dia}_0$ be a finite EI-category. Then we have the following equality:*

$$\mathbb{D}(I) = \langle \bar{i}_! A \mid i \in I_0, A \in \mathbb{D}(G_i)_0 \rangle.$$

Suppose that for all $i \in I_0$, the canonical functor $e_i : \star \rightarrow G_i$ induces a faithful functor $e_i^* : \mathbb{D}(G_i) \rightarrow \mathbb{D}(\star)$. Then we also have the following equality:

$$\mathbb{D}(I) = \langle i_! A \mid i \in I_0, A \in \mathbb{D}(\star)_0 \rangle_s.$$

All these statements remain true if we replace $(-)_!$ by $(-)_*$ everywhere.

Remark 28 We will prove in appendix C that if n is invertible in $R_{\mathbb{D}}$ then $e : \star \rightarrow G$ induces a faithful functor $e^* : \mathbb{D}(G) \rightarrow \mathbb{D}(\star)$ for any group $G \in \mathbf{Dia}_0$ of order n . In particular, if $\text{char}(I)$ is invertible in $R_{\mathbb{D}}$ then the second equality in Lemma 27 holds.

Proof of Lemma 27. Note that since $I \in \mathbf{Dia}_0$ so is G_i , $i \in I_0$, by (Dia2). Therefore, the statement of the lemma at least makes sense.

The first equality is proved by induction on the number n of objects in I . Clearly, we may assume I to be skeletal. If $n = 1$, the claim is obviously true. If $n > 1$ we find an object $i \in I_0$ which is maximal in the sense that the implication $I(i, j) \neq \emptyset \Rightarrow i = j$ holds. For any $B \in \mathbb{D}(I)_0$, consider the morphism

$$\text{adj} : \bar{i}_! i^* B \rightarrow B$$

and let C be the cone. One checks easily that $i^* \text{adj}$ is an isomorphism hence $i^* C = 0$ which implies that C is of the form $u_! B'$, some $B' \in \mathbb{D}(U)_0$ where $u : U \hookrightarrow I$ is the open embedding of the full subcategory of objects different from i in I (see [3, 8.11]). By induction, $B' \in \langle \bar{j}_! A \mid j \in U_0, A \in \mathbb{D}(G_j)_0 \rangle$, hence it suffices to prove

$$u_! \langle \bar{j}_! A \mid j \in U_0, A \in \mathbb{D}(G_j)_0 \rangle \subset \langle \bar{j}_! A \mid j \in I_0, A \in \mathbb{D}(G_j)_0 \rangle.$$

But this follows from the fact that $u_!$ is a triangulated functor and $u_! \bar{j}_! = \bar{j}_!$.

For the second equality, it will follow from the first as soon as we prove, for each $i \in I_0$,

$$\mathbb{D}(G_i) = \langle e_{i!} A \mid A \in \mathbb{D}(\star)_0 \rangle_s. \quad (27)$$

So let $B \in \mathbb{D}(G_i)_0$ and consider the counit $e_{i!} e_i^* B \rightarrow B$. By assumption this is an epimorphism. But in a triangulated category every epimorphism is complemented, i. e.

$$e_{i!} e_i^* B \cong B \oplus B',$$

some $B' \in \mathbb{D}(G_i)_0$. This proves (27) and hence the second equality.

The last claim of the lemma can be established by dualizing the whole proof. \square

Proposition 29 Let \mathbb{D} be a monoidal triangulated derivator, let I be a finite EI -category in \mathbf{Dia} , and suppose that $\text{char}(I)$ is invertible in $R_{\mathbb{D}}$. Then all hypotheses (H1)–(H4) are satisfied.

Proof. (1) For (H1) we will prove more generally that

$$q_{1*} A \otimes T|_{I^\circ \times I} \rightarrow q_{1*} (A \otimes T|_{\text{tw}(I)}) \quad (28)$$

is invertible for all $T \in \mathbb{D}(\star)_0$, $A \in \mathbb{D}(\text{tw}(I))_0$. Also we will only need that I has finite Hom-sets.

Fix $i, j \in I_0$ and consider the following pullback square:

$$\begin{array}{ccc} I(i, j) & \xrightarrow{r_{i,j}} & \text{tw}(I) \\ p_{I(i,j)} \downarrow & & \downarrow q_1 \\ \star & \xrightarrow{(i,j)} & I^\circ \times I. \end{array}$$

Since q_1 is an opfibration, Lemma 1 tells us that the first vertical morphisms on the left and on the right in the following diagram are invertible:

$$\begin{array}{ccc}
(i, j)^*(q_{1*}A \otimes T|_{I^\circ \times I}) & \longrightarrow & (i, j)^*q_{1*}(A \otimes T|_{\text{tw}(I)}) \\
\sim \downarrow & & \downarrow \sim \\
p_{I(i, j)*}r_{i, j}^*A \otimes T & \longrightarrow & p_{I(i, j)*}r_{i, j}^*(A \otimes T|_{\text{tw}(I)}) \\
\sim \downarrow & & \downarrow \sim \\
\prod_{h \in I(i, j)} h^*r_{i, j}^*A \otimes T & \longrightarrow & \prod_{h \in I(i, j)} (h^*r_{i, j}^*A \otimes T).
\end{array}$$

Here, $h : \star \rightarrow I(i, j)$ is the functor defined by the object h of the discrete category $I(i, j)$, and the axiom (D1) is used for the second vertical morphisms on the left and on the right. Clearly both squares commute. Moreover, the bottom horizontal arrow is invertible since $I(i, j)$ is finite and the internal product in $\mathbb{D}(\star)$ is additive (Lemma 5). Therefore also the top horizontal arrow is invertible which implies (by (D2), and letting i and j vary) that (28) is.

- (2) For (H2), let γ be a connected component of $2\text{tw}(I)$. Since I is a finite EI-category, γ is equivalent to a finite group G whose order divides $\text{char}(I)$. As explained in Remark 28, this implies that e_G^* is faithful ($e_G : \star \rightarrow G$). Since p_G^* is a section of e_G^* , it follows that p_G^* is fully faithful.
- (3) (H3) states that $p_{I!}p_{2*} \rightarrow p_{I^\circ*}p_{1!}$ is invertible. Since also $I^\circ \times I$ is a finite EI-category and since $\text{char}(I^\circ \times I) = \text{char}(I)$ we may prove this on objects in the image of $(1_{I^\circ} \times i)_!$, where $i \in I_0$, by the previous lemma. Consider the following square:

$$\begin{array}{ccc}
p_{I!}p_{2*}(1_{I^\circ} \times i)_! & \longrightarrow & p_{I^\circ*}p_{1!}(1_{I^\circ} \times i)_! \\
\uparrow & & \downarrow \sim \\
p_{I!}i_!p_{I^\circ*} & \xrightarrow{\sim} & p_{I^\circ*}.
\end{array}$$

It is easy to see that it commutes hence it suffices to prove invertibility of the left vertical arrow.

For this we use (D2), so fix $j \in I_0$ an object. Then

$$\begin{aligned}
j^*i_!p_{I^\circ*} &\cong \oplus_{I(j, i)} p_{I^\circ*}, \\
j^*p_{2*}(1_{I^\circ} \times i)_! &\cong p_{I^\circ*}(1_{I^\circ} \times j)^*(1_{I^\circ} \times i)_! \\
&\cong p_{I^\circ*} \oplus_{I(j, i)}.
\end{aligned}$$

The claim follows since $p_{I^\circ*}$ is additive. (It is easy to see that this identification is compatible with the vertical arrow above.)

- (4) Since also I° is a finite EI-category and $\text{char}(I^\circ) = \text{char}(I)$, we may replace I by I° . (H4) then is the statement that

$$p_{I*}A \otimes B \rightarrow p_{I*}(A \otimes p_I^*B)$$

is invertible, and by Lemma 27 we may assume $A = i_*C$, some $C \in \mathbb{D}(\star)_0$ and $i \in I_0$ (here we use that $- \otimes B$ and $- \otimes p_I^*B$ both take distinguished triangles to distinguished triangles, by Lemma 5).

Clearly, the following square commutes:

$$\begin{array}{ccc} p_{I*}i_*C \otimes B & \longrightarrow & p_{I*}(i_*C \otimes p_I^*B) \\ \sim \downarrow & & \downarrow \\ C \otimes B & \xleftarrow{\sim} & p_{I*}i_*(C \otimes B), \end{array}$$

hence it suffices to prove invertible the vertical arrow on the right. Again we use (D2), so let $j \in I_0$ an object. Then:

$$\begin{aligned} j^*i_*C \otimes j^*p_I^*B &\cong \oplus_{I(i,j)} C \otimes B, \\ j^*i_*(C \otimes B) &\cong \oplus_{I(i,j)} (C \otimes B). \end{aligned}$$

Again, the claim follows from the additivity of the functor $- \otimes B$. \square

APPENDIX A. PROPERTIES OF THE EXTERNAL HOM

In this section we want to give proofs for the properties of the external hom listed in section 3. We take them up one by one. Throughout the section we fix a closed monoidal derivator \mathbb{D} of type **Dia**.

Naturality Given $u : I' \rightarrow I$ and $v : J' \rightarrow J$ in **Dia** there is an induced morphism of diagrams $(\Pi_{I',J'}) \rightarrow (\Pi_{I,J})$ and we distinguish the morphisms in the former from their counterparts in the latter by decorating them with a prime. We deduce a morphism

$$\begin{aligned} \Psi_{A,B}^{u,v} : (u^\circ \times v)^* \langle A, B \rangle &= (u^\circ \times v)^* p_*[q^*A, r^*B] \\ &\rightarrow p'_*(\text{tw}(u) \times v)^*[q^*A, r^*B] \\ &\rightarrow p'_*[(\text{tw}(u) \times v)^*q^*A, (\text{tw}(u) \times v)^*r^*B] \\ &= \langle u^*A, v^*B \rangle. \end{aligned}$$

Clearly, this morphism is natural in A and B , moreover it behaves well with respect to identities and composition of functors as well as natural transformations in $\mathbf{Dia}^\circ \times \mathbf{Dia}^{\circ,\circ}$ so that we have defined a lax natural transformation. The following proposition thus concludes the proof of the naturality property.

Proposition 30 *For u, v and A, B as above the morphism $\Psi_{A,B}^{u,v}$ is invertible.*

Proof. We proceed in several steps.

- (1) Let $i \in I'_0, j \in J'_0$. It suffices to prove that $(i, j)^*$ applied to the morphism $\Psi_{A,B}^{u,v}$ is invertible. But this means that it suffices to prove that $\Psi_{-, -}^{u^i, v^j}$ and $\Psi_{-, -}^{i, j}$ are invertible; in other words we may assume $I' = J' = \star, u = i \in I_0, v = j \in J_0$.
- (2) We factor $(i, j) : \star \xrightarrow{i} I \xrightarrow{1_I \times j} I \times J$, and first deal with $\Psi_{-, -}^{1_I, j}$. In this case, the square

$$\begin{array}{ccc} \text{tw}(I) & \xrightarrow{1_{\text{tw}(I)} \times j} & \text{tw}(I) \times J \\ p' \downarrow & & \downarrow p \\ I^\circ & \xrightarrow{1_{I^\circ} \times j} & I^\circ \times J \end{array}$$

is a pullback square, and p an opfibration, therefore the first arrow in the definition of Ψ is invertible (Lemma 1). For the second arrow in the definition, it suffices to prove invertible

$$(1_{\text{tw}(I)} \times j)^*[(1_{\text{tw}(I)} \times p_J)^* -, -] \rightarrow [-, (1_{\text{tw}(I)} \times j)^* -].$$

By passing to $\mathbb{D}_{\text{tw}(I)}$ we may thus assume $I = \star$ and prove instead invertible

$$j^*[p_J^* -, -] \rightarrow [-, j^* -].$$

By adjunction, this corresponds to the morphism

$$(1_\star \times j)_!(- \boxtimes -) \xrightarrow{(6)} - \boxtimes j_! -$$

which we know to be invertible.

- (3) Thus from now on we may assume $J = \star$. Factor $i : \star \xrightarrow{1_i} i \backslash I \xrightarrow{t} I$. Exactly the same argument as in the previous step shows that the first arrow in the definition of $\Psi^{t, 1_\star}$ is invertible. Moreover, $\text{tw}(t)$ is a fibration hence, by Lemma 2, also the second arrow in the definition of $\Psi^{t, 1_\star}$ is invertible.
- (4) From now on, we may assume that I has initial object i and we need to prove $\Psi^{i, 1_\star}$ invertible. Consider the following diagram:

$$\begin{array}{ccccccc} i^* p_*[q^* -, p_{\text{tw}(I)}^* -] & \longrightarrow & 1_i^*[q^* -, p_{\text{tw}(I)}^* -] & \longrightarrow & [1_i^* q^* -, 1_i^* p_{\text{tw}(I)}^* -] & = & [i^* -, -] \\ & \searrow \sim & \uparrow & & \uparrow & & \uparrow \sim \\ & & p_{\text{tw}(I)*}[q^* -, p_{\text{tw}(I)}^* -] & \xleftarrow{\sim} & [p_{\text{tw}(I)!} q^* -, -] & \xleftarrow{\sim} & [p_{I!} -, -]. \end{array}$$

The composition of the top horizontal arrows is nothing but $\Psi^{i, 1_\star}$. The triangle on the left arises from the Beck-Chevalley transformations associated to the squares

$$\begin{array}{ccc} \star & \xrightarrow{1_i} & \text{tw}(I) \\ \parallel & \downarrow p_{\text{tw}(I)} & \downarrow p \\ \star & \xrightarrow{i} & I^\circ \end{array}$$

It follows that the triangle commutes and the slanted morphism is invertible by (D4). The first bottom horizontal arrow is invertible by Lemma 2, the second one arises from the counit $q_! q^* \rightarrow 1$ which is invertible by Lemma 34. The middle vertical arrow is induced by the “dual” of the left vertical arrow, $1_i^* \xrightarrow{\text{adj}} 1_i^* p_{\text{tw}(I)}^* p_{\text{tw}(I)!} = p_{\text{tw}(I)!}$. The commutativity of the left square is therefore immediate, as is the commutativity of the right square (the right vertical arrow is induced by the canonical identification $i^* \cong p_{I!}$ as i is an initial object of I).

□

Internal hom We now want to show that in case $I = J \in \mathbf{Dia}_0$, internal hom can be expressed in terms of external hom. Consider the following category $3I$: Objects are two composable arrows in I and morphisms from the top to the bottom are of the form:

$$\begin{array}{ccccc} i_2 & \longrightarrow & i_1 & \longrightarrow & i_0 \\ \downarrow & & \uparrow & & \downarrow \\ j_2 & \longrightarrow & j_1 & \longrightarrow & j_0. \end{array}$$

We have canonical functors $t_k : 3I \rightarrow I$, $k = 0, 2$. Moreover, there are functors $p' : 3I \rightarrow \mathrm{tw}(I)^\circ$ and $q' : 3I \rightarrow \mathrm{tw}(I) \times I$, the first one forgetting the 0-component, the second one mapping the two components 0 and 1 to $\mathrm{tw}(I)$ and component 2 to I . It is easy to see that one gets a pullback square:

$$\begin{array}{ccc} 3I & \xrightarrow{p'} & \mathrm{tw}(I)^\circ \\ q' \downarrow & & \downarrow q_2 \\ \mathrm{tw}(I) \times I & \xrightarrow{p} & I^\circ \times I. \end{array}$$

Notice that there is a canonical natural transformation $t_2 \rightarrow t_0$ and hence one can define the following morphism:

$$\begin{aligned} \Theta_{A,B}^I : [A, B] &\xrightarrow{\mathrm{adj}} [t_2! t_2^* A, B] \\ &\rightarrow [t_2! t_0^* A, B] \\ &\rightarrow t_{2*} [t_0^* A, t_2^* B] \\ &\xleftarrow{\sim} p_{2*} q_{2*} p'_* [q'^* q^* A, q'^* r^* B] \\ &\xleftarrow{\sim} p_{2*} q_{2*} p'_* q'^* [q^* A, r^* B] \\ &\xleftarrow{\sim} p_{2*} q_{2*} q_2^* p_* [q^* A, r^* B]. \end{aligned} \tag{29}$$

Here the last isomorphism is due to Lemma 1 and q_2 being a fibration. Therefore also q' is a fibration and Lemma 2 gives us the second to last isomorphism.

Again, $\Theta_{A,B}^I$ is clearly natural in A and B and one checks easily (if tediously) that the following diagram commutes for any $u : I' \rightarrow I$ in \mathbf{Dia}_1 :

$$\begin{array}{ccccc} u^*[A, B] & \xrightarrow{\Theta^I} & u^* p_{2*} q_{2*} q_2^* \langle A, B \rangle & \xrightarrow{\quad \quad \quad} & p'_{2*} q'_{2*} \mathrm{tw}(u)^\circ q_2^* \langle A, B \rangle \\ \downarrow & & & & \parallel \\ [u^* A, u^* B] & \xrightarrow{\Theta_{I'}} & p'_{2*} q'_{2*} q_2^* \langle u^* A, u^* B \rangle & \xleftarrow{\quad \quad \quad \Psi \quad \quad \quad} & p'_{2*} q'_{2*} q_2^* (u^\circ \times u)^* \langle A, B \rangle. \end{array}$$

It follows that if we take the composition of the dotted arrows in the diagram as components of the 2-cells for the lax natural transformation $p_{2*} q_{2*} q_2^* \langle -, - \rangle$, then Θ defines a modification as claimed in section 3. It now remains to prove that it is invertible.

Proposition 31 $\Theta_{A,B}^I$ is invertible for all I , A and B as above.

Proof. It is easy to see that t_2 is a fibration. Hence it follows from Lemma 2 that the third arrow in (29) is invertible, and it now suffices to prove that

$$t_2! t_0^* \rightarrow t_2! t_2^* \xrightarrow{\mathrm{adj}} 1 \tag{30}$$

is invertible. Let $i \in I_0$ be an arbitrary object. We will show that i^* applied to (30) is invertible which is enough for the claim by (D2).

Consider the following two diagrams:

$$\begin{array}{ccc}
 3I_i & \xrightarrow{w} & 3I \\
 p_{3I_i} \downarrow & & \downarrow t_2 \\
 \star & \xrightarrow{i} & I,
 \end{array}
 \qquad
 \begin{array}{ccc}
 3I_i & \xrightarrow{w} & 3I \\
 u \downarrow & & \downarrow t_2 \\
 i \backslash I & \xrightarrow{\quad} & I.
 \end{array}$$

\Downarrow
 v

The first one is a pullback square, in the second one u is defined by $u(i \rightarrow i_1 \rightarrow i_0) = i \rightarrow i_0$, while $v(i \rightarrow i_0) = i_0$ and $ip_{i \backslash I} \rightarrow v$ is the canonical natural transformation. This second diagram is commutative in the sense that $ip_{i \backslash I}u \rightarrow vv$ is equal to $t_2w \rightarrow t_0w$. Consequently the second inner square on the left of the following diagram commutes:

$$\begin{array}{ccccc}
 i^*t_2!t_0^* & \longrightarrow & i^*t_2!t_2^* & \xrightarrow{\text{adj}} & i^* \\
 \uparrow \sim & & \uparrow \sim & & \uparrow \text{adj} \\
 p_{3I_i!}w^*t_0^* & \longrightarrow & p_{3I_i!}w^*t_2^* & = & p_{3I_i!}p_{3I_i}^*i^* \\
 \downarrow \sim & & \downarrow \sim & \swarrow \sim & \\
 p_{i \backslash I!}u!u^*v^* & \longrightarrow & p_{i \backslash I!}u!u^*p_{i \backslash I}^*i^* & & \\
 \downarrow \text{adj} & & \downarrow \text{adj} & & \downarrow \sim \\
 p_{i \backslash I!}v^* & \xrightarrow{\sim} & p_{i \backslash I!}p_{i \backslash I}^*i^* & \xrightarrow{\sim} & i^*
 \end{array}$$

The rest is clearly commutative. Moreover, the top row is the fiber of (30) over i . The isomorphism of functors $p_{i \backslash I!} \cong 1_i^*$ (1_i being the initial object of $i \backslash I$) implies that the bottom horizontal as well as the bent arrow induced by the counit of the adjunction $p_{i \backslash I!} \dashv p_{i \backslash I}^*$ are invertible, hence it suffices to prove $u!u^* \rightarrow 1$ an isomorphism. But this is true since u admits a fully faithful right adjoint

$$\begin{aligned}
 i \backslash I &\longrightarrow 3I_i \\
 (i \rightarrow j) &\longmapsto (i \xrightarrow{1_i} i \rightarrow j).
 \end{aligned}$$

□

External product Recall that for any closed monoidal category there is a canonical morphism

$$[A_1, A_2] \otimes [A_3, A_4] \rightarrow [A_1 \otimes A_3, A_2 \otimes A_4] \quad (31)$$

defined by adjunction as follows:

$$\begin{aligned}
 ([A_1, A_2] \otimes [A_3, A_4]) \otimes (A_1 \otimes A_3) &\xrightarrow{\sim} ([A_1, A_2] \otimes A_1) \otimes ([A_3, A_4] \otimes A_3) \\
 &\xrightarrow{\text{ev} \otimes \text{ev}} A_2 \otimes A_4.
 \end{aligned}$$

From this we deduce for $A_1, A_3 \in \mathbb{D}(I)_0$, $A_2, A_4 \in \mathbb{D}(J)_0$ ($I, J \in \mathbf{Dia}_0$):

$$\begin{aligned}
 \langle A_1, A_2 \rangle \otimes \langle A_3, A_4 \rangle &= p_*[q^*A_1, r^*A_2] \otimes p_*[q^*A_3, r^*A_4] \\
 &\longrightarrow p_*([q^*A_1, r^*A_2] \otimes [q^*A_3, r^*A_4]) \\
 &\xrightarrow{(31)} p_*([q^*A_1 \otimes q^*A_3, r^*A_2 \otimes r^*A_4]) \\
 &\xrightarrow{\sim} \langle A_1 \otimes A_3, A_2 \otimes A_4 \rangle.
 \end{aligned} \tag{32}$$

Now, fix categories $I_{(k)}$, $k = 1, \dots, 4$ in \mathbf{Dia} and objects $A_k \in \mathbb{D}(I_{(k)})_0$. Set $K = I_{(1)}^\circ \times I_{(2)} \times I_{(3)}^\circ \times I_{(4)}$. We can now finally define the morphism Ξ :

$$\begin{aligned}
 \Xi_{A_1, A_2, A_3, A_4}^{I_{(1)}, I_{(2)}, I_{(3)}, I_{(4)}} : \quad &\langle A_1, A_2 \rangle \boxtimes \langle A_3, A_4 \rangle = \langle A_1, A_2 \rangle|_K \otimes \langle A_3, A_4 \rangle|_K \\
 &\xrightarrow[\sim]{\Psi} \tau^* \langle A_1|_{I_{(1)} \times I_{(3)}}, A_2|_{I_{(2)} \times I_{(4)}} \rangle \otimes \tau^* \langle A_3|_{I_{(1)} \times I_{(3)}}, A_4|_{I_{(2)} \times I_{(4)}} \rangle \\
 &\xleftarrow[\sim]{\tau^*} \left(\langle A_1|_{I_{(1)} \times I_{(3)}}, A_2|_{I_{(2)} \times I_{(4)}} \rangle \otimes \langle A_3|_{I_{(1)} \times I_{(3)}}, A_4|_{I_{(2)} \times I_{(4)}} \rangle \right) \\
 &\xrightarrow{(32)} \tau^* \langle A_1 \boxtimes A_3, A_2 \boxtimes A_4 \rangle.
 \end{aligned} \tag{33}$$

Clearly, $\Xi^{I_{(1)}, I_{(2)}, I_{(3)}, I_{(4)}}$ is a natural transformation. To conclude the proof of the external product property it remains to verify the following lemma.

Lemma 32 *Let $u_k : I'_{(k)} \rightarrow I_{(k)}$, $k = 1, \dots, 4$. Then the following diagram commutes:*

$$\begin{array}{ccc}
 (u_1^\circ \times u_3^\circ \times u_2 \times u_4)^* (\langle A_1, A_2 \rangle \boxtimes \langle A_3, A_4 \rangle) & \xrightarrow{\Xi} & (u_1^\circ \times u_3^\circ \times u_2 \times u_4)^* \tau^* \langle A_1 \boxtimes A_3, A_2 \boxtimes A_4 \rangle \\
 \Psi \downarrow \sim & & \sim \downarrow \Psi \\
 \langle u_1^* A_1, u_2^* A_2 \rangle \boxtimes \langle u_3^* A_3, u_4^* A_4 \rangle & \xrightarrow{\Xi} & \tau^* \langle u_1^* A_1 \boxtimes u_3^* A_3, u_2^* A_2 \boxtimes u_4^* A_4 \rangle.
 \end{array}$$

Proof. By decomposing the horizontal arrows according to their definition in (33) one immediately reduces to showing that (32) behaves well with respect to the functors u_k ; in other words one reduces to showing that for $A_1, A_3 \in \mathbb{D}(I)_0$, $A_2, A_4 \in \mathbb{D}(J)_0$ and functors $u : I' \rightarrow I$, $v : J' \rightarrow J$, the following diagram commutes:

$$\begin{array}{ccc}
 (u^\circ \times v)^* (\langle A_1, A_2 \rangle \otimes \langle A_3, A_4 \rangle) & \xrightarrow{(32)} & (u^\circ \times v)^* \langle A_1 \otimes A_3, A_2 \otimes A_4 \rangle \\
 \Psi \downarrow \sim & & \sim \downarrow \Psi \\
 \langle u^* A_1, v^* A_2 \rangle \otimes \langle u^* A_3, v^* A_4 \rangle & \xrightarrow{(32)} & \langle u^* A_1 \otimes u^* A_3, v^* A_2 \otimes v^* A_4 \rangle.
 \end{array}$$

Since the unit and counit of the adjunction $p^* \dashv p_*$ behave well with respect to pulling back along $u^\circ \times v$ and $\text{tw}(u) \times v$ one reduces further to showing that (31) is functorial in this sense which is clear. \square

Adjunction Fix three categories I, J, K in **Dia**, and objects $A \in \mathbb{D}(I)_0, B \in \mathbb{D}(J)_0, C \in \mathbb{D}(K)_0$. Fix also the following notation:

$$\begin{array}{ccccc}
 J & \xleftarrow{\quad} & I \times J & \xrightarrow{\quad} & I \\
 \uparrow q & & \uparrow q'' & & \uparrow q' \\
 \text{tw}(J) \times K & \xleftarrow{\beta} & \text{tw}(I \times J) \times K & \xrightarrow{\alpha} & \text{tw}(I) \times J^\circ \times K \\
 \downarrow r & \swarrow r'' & \downarrow p'' & \swarrow p' & \\
 K & & I^\circ \times J^\circ \times K & & \\
 \downarrow p & \searrow r' & & & \\
 & J^\circ \times K & & &
 \end{array}$$

Then the morphism in the statement of the adjunction property is given by:

$$\begin{aligned}
 \Omega_{A,B,C}^{I,J,K} : p'_*[q'^*A, r'^*p_*[q^*B, r^*C]] &\xrightarrow{\sim} p'_*[q'^*A, \alpha_*\beta^*[q^*B, r^*C]] \\
 &\xrightarrow{\sim} p'_*\alpha_*[\alpha^*q'^*A, [\beta^*q^*B, \beta^*r^*C]] \\
 &\xrightarrow{\sim} p'_*\alpha_*[\alpha^*q'^*A \otimes \beta^*q^*B, \beta^*r^*C] \\
 &\xrightarrow{\sim} p''_*[q''^*(A|_{I \times J} \otimes B|_{I \times J}), r''^*C].
 \end{aligned}$$

It is clear that this morphism is natural in the three arguments. Moreover, as above it is straightforward to check that it behaves well with respect to functors $u : I' \rightarrow I, v : J' \rightarrow J, w : K' \rightarrow K$.

Biduality Fix $B \in \mathbb{D}(\star)_0, I \in \mathbf{Dia}_0$ and $A \in \mathbb{D}(I)_0$. We also fix the following notation:

$$\begin{array}{ccc}
 & \star & \\
 \bar{r} \nearrow & & \nwarrow r \\
 \text{tw}(I^\circ) & \xrightarrow{\mu} & \text{tw}(I) \\
 \bar{q} \downarrow & & \downarrow q \\
 I^\circ & \xleftarrow{p} & I \\
 & \searrow \bar{p} &
 \end{array}$$

Here, μ is the isomorphism of categories taking $j \rightarrow i$ in I° to $i \rightarrow j$ in I . We then define the morphism mentioned in the statement of the biduality property,

$$\Upsilon_A^I : A \rightarrow \langle \langle A, B \rangle, B \rangle, \quad (34)$$

by adjunction as follows:

$$\begin{aligned}
 \bar{p}^*A \otimes \bar{q}^*p_*[q^*A, r^*B] &= \bar{p}^*A \otimes \mu^*p_*p_*[q^*A, r^*B] \\
 &\xrightarrow{\text{adj}} \bar{p}^*A \otimes \mu^*[q^*A, r^*B] \\
 &\longrightarrow \bar{p}^*A \otimes [\bar{p}^*A, \bar{r}^*B] \\
 &\xrightarrow{\text{ev}} \bar{r}^*B.
 \end{aligned}$$

This is clearly natural in A . If $u : I' \rightarrow I$ is a functor in **Dia** we define a morphism

$$u^*\langle \langle A, B \rangle, B \rangle \xrightarrow{\Psi} \langle u^*\langle A, B \rangle, B \rangle \xleftarrow{\Psi} \langle \langle u^*A, B \rangle, B \rangle.$$

As we know by the naturality property, this morphism is invertible, natural in A , and behaves well with respect to identity and composition of functors as well as natural transformations in **Dia**. Therefore we have defined a pseudonatural transformation $\langle\langle -, B \rangle, B\rangle$. To check that (34) defines a modification of pseudonatural transformations as claimed in section 3 it suffices to prove the following lemma.

Lemma 33 *With the notation above the following diagram commutes:*

$$\begin{array}{ccc} u^*A & \xrightarrow{\Upsilon} & u^*\langle\langle A, B \rangle, B\rangle \\ \Upsilon \downarrow & & \sim \downarrow \Psi \\ \langle\langle u^*A, B \rangle, B\rangle & \xrightarrow[\Psi]{\sim} & \langle u^{\circ*}\langle A, B \rangle, B \rangle. \end{array}$$

Proof. Using adjunction, the square can be equivalently written as the outer rectangle of the following diagram:

$$\begin{array}{ccc} \bar{p}'^*u^*A \otimes \bar{q}'^*u^{\circ*}p_*[q^*A, r^*B] & \longrightarrow & \text{tw}(u^{\circ})^*(\bar{p}^*A \otimes \bar{q}^*p_*[q^*A, r^*B]) \longrightarrow \dots \\ \downarrow & & \downarrow \\ \bar{p}'^*u^*A \otimes [\bar{p}'^*u^*A, \bar{r}'^*B] & & \text{tw}(u^{\circ})^*(\bar{p}^*A \otimes [\bar{p}^*A, \bar{r}^*B]) \\ \text{ev} \downarrow & & \text{ev} \downarrow \\ \bar{r}'^*B & \xlongequal{\quad\quad\quad} & \text{tw}(u^{\circ})^*\bar{r}^*B \xlongequal{\quad\quad\quad} \dots \end{array}$$

$$\begin{array}{ccc} \dots \longrightarrow \text{tw}(u^{\circ})^*(\bar{p}^*\langle\langle A, B \rangle, B\rangle \otimes \bar{q}^*\langle A, B \rangle) & \xleftarrow{\sim} & \bar{p}'^*u^*\langle\langle A, B \rangle, B\rangle \otimes \bar{q}'^*u^{\circ*}\langle A, B \rangle \\ \downarrow & & \downarrow \\ \text{tw}(u^{\circ})^*([\bar{q}^*\langle A, B \rangle, \bar{r}^*B] \otimes \bar{q}^*\langle A, B \rangle) & & [\bar{q}'^*u^{\circ*}\langle A, B \rangle, \bar{r}'^*B] \otimes \bar{q}'^*u^{\circ*}\langle A, B \rangle \\ \text{ev} \downarrow & & \text{ev} \downarrow \\ \dots \xlongequal{\quad\quad\quad} \text{tw}(u^{\circ})^*\bar{r}^*B & \xlongequal{\quad\quad\quad} & \bar{r}'^*B. \end{array}$$

All three parts are easily seen to commute. \square

Normalization Given $J \in \mathbf{Dia}_0$, $A \in \mathbb{D}(\star)_0$ and $B \in \mathbb{D}(J)_0$, the morphism $\Lambda_{A,B}^J$ is the canonical identification induced by the strict functoriality of \mathbb{D} :

$$[p_J^*A, B] \xrightarrow{\sim} 1_{J*}[p_J^*A, B] = \langle A, B \rangle.$$

Clearly, this is natural in A and B , and behaves well with respect to functors $v : J' \rightarrow J$. The last claim in section 3 about Λ explicitly amounts to the following:

- for $A, B \in \mathbb{D}(\star)_0$, Θ is the canonical composition

$$[A, B] \xrightarrow{\sim} 1_*[A, B] \xrightarrow{\sim} 1_*1_*[A, B] \xrightarrow{\Lambda} 1_*1_*\langle A, B \rangle$$

where 1 is the unique endofunctor of the terminal category \star ;

- for $A, C \in \mathbb{D}(\star)_0$, $B \in \mathbb{D}(I)_0$, $D \in \mathbb{D}(J)_0$, Ξ fits into the commutative diagram:

$$\begin{array}{ccc}
\langle A, B \rangle \boxtimes \langle C, D \rangle & \xrightarrow{\Xi} & \langle A \boxtimes C, B \boxtimes D \rangle \\
\uparrow \Lambda \sim & & \sim \uparrow \Lambda \\
[p_I^* A, B]_{I \times J} \otimes [p_J^* C, D]_{I \times J} & & [p_{I \times J}^* (A \otimes C), B]_{I \times J} \otimes D_{I \times J} \\
\sim \downarrow & & \downarrow \sim \\
[p_{I \times J}^* A, B]_{I \times J} \otimes [p_{I \times J}^* C, D]_{I \times J} & \xrightarrow{(31)} & [p_{I \times J}^* A \otimes p_{I \times J}^* C, B]_{I \times J} \otimes D_{I \times J}.
\end{array}$$

- for $A, B \in \mathbb{D}(\star)_0$ and $C \in \mathbb{D}(J)_0$, Ω fits into the commutative diagram:

$$\begin{array}{ccc}
\langle A, \langle B, C \rangle \rangle & \xrightarrow[\sim]{\Omega} & \langle A \otimes B, C \rangle \\
\uparrow \Lambda \sim & & \sim \uparrow \Lambda \\
[p_J^* A, [p_J^* B, C]] & & [p_J^* (A \otimes B), C] \\
& \searrow \sim & \downarrow \sim \\
& & [p_J^* A \otimes p_J^* B, C].
\end{array}$$

- for $A, B \in \mathbb{D}(\star)_0$, Υ is identified with the morphism $A \rightarrow [[A, B], B]$ which by adjunction corresponds to $\text{ev} : A \otimes [A, B] \rightarrow B$.

All these statements follow easily from the constructions in this section.

APPENDIX B. THE EXTERNAL TRACE AND HOMOTOPY COLIMITS

In this section the proof of Proposition 20 will be given. Throughout we fix a closed monoidal derivator \mathbb{D} of type **Dia**. We start with a preliminary result, already needed to define the association Φ on page 22.

Lemma 34 *Let $I \in \mathbf{Dia}_0$. Then the following three morphisms are invertible:*

- (1) $p_{1!} q_{2!} q_2^* p_1^* \rightarrow 1$ (counit of adjunction),
- (2) $1 \rightarrow p_{2*} q_{1*} q_1^* p_2^*$ (unit of adjunction),
- (3) $\overline{\Psi} : [p_{I!} A, B] \rightarrow p_{I \circ *}(A, B)$ for $A \in \mathbb{D}(I)_0$, $B \in \mathbb{D}(\star)_0$.

Proof. For the first morphism, fix $i \in I_0$ and consider the following pullback square:

$$\begin{array}{ccc}
\text{tw}(I)_i^\circ & \longrightarrow & \text{tw}(I)^\circ \\
p_i \downarrow & & \downarrow p_1 q_2 \\
\star & \xrightarrow{i} & I^0.
\end{array}$$

Since q_2 and p_1 are both fibrations so is their composition and by Lemma 1 the Beck-Chevalley transformation corresponding to the square above is invertible. It follows that for the counit $p_{1!} q_{2!} q_2^* p_1^* \rightarrow 1$ to be invertible it is necessary and sufficient that $p_{i!} p_i^* \rightarrow 1$ is (for all $i \in I_0$, by (D2)). This is equivalent to $1 \rightarrow p_{i*} p_i^*$ being invertible, and this is true since $\text{tw}(I)_i^\circ = I/i$ and thus $p_{i*} = 1_i^*$. The second morphism in the statement of the Lemma is treated in the same way.

For the last morphism, we consider the following factorization:

$$\begin{array}{ccccc}
[p_{I!}A, B] & \xrightarrow{\text{adj}} & p_{I^\circ*}p_{I!}^*[p_{I!}A, B] & \xrightarrow{\Psi \circ \Lambda} & p_{I^\circ*}\langle p_{I!}^*p_{I!}A, B \rangle \\
\downarrow \sim & & \downarrow \sim & & \downarrow \text{adj} \\
p_{I*}[A, p_I^*B] & \xrightarrow{\text{adj}} & p_{I^\circ*}p_{I!}^*p_{I*}[A, p_I^*B] & & p_{I^\circ*}\langle A, B \rangle \\
\downarrow \Theta \sim & & \downarrow \Theta \sim & & \downarrow \text{adj} \\
p_{I*}p_{2*}q_{2*}q_2^*\langle A, p_I^*B \rangle & \xrightarrow{\text{adj}} & p_{I^\circ*}p_{I!}^*p_{I*}p_{2*}q_{2*}q_2^*\langle A, p_I^*B \rangle & & \\
\uparrow \Psi \sim & & \uparrow \Psi \sim & & \\
p_{I^\circ*}p_{1*}q_{2*}q_2^*p_1^*\langle A, B \rangle & \xleftarrow{\text{adj}} & p_{I^\circ*}p_{I!}^*p_{I^\circ*}p_{1*}q_{2*}q_2^*p_1^*\langle A, B \rangle & \xleftarrow{\text{adj}} & p_{I^\circ*}p_{1*}q_{2*}q_2^*p_1^*\langle A, B \rangle.
\end{array}$$

Notice that all the vertical arrows on the left are invertible (the first one by Lemma 2, the second and third by the results of section 3) as is the vertical arrow on the bottom right by part 1 of the lemma. And the composition of the horizontal arrows at the bottom is the identity so we only need to prove commutativity of the diagram.

This is clear for the left half of the diagram while the right half may be decomposed as follows:

$$\begin{array}{c}
p_{I^\circ}^*[p_{I!}A, B] \xrightarrow[\sim]{\Lambda} p_{I^\circ}^*\langle p_{I!}A, B \rangle \xrightarrow{\Psi} \dots \\
\downarrow \text{adj} \quad \quad \quad \downarrow \text{adj} \\
p_{I^\circ}^*p_{I*}p_{I!}^*[p_{I!}A, B] \xrightarrow[\sim]{\Lambda} p_{I^\circ}^*p_{I*}p_{I!}^*\langle p_{I!}A, B \rangle \xrightarrow{\text{adj}} \dots \\
\downarrow \quad \quad \quad \textcircled{1} \\
p_{I^\circ}^*p_{I*}[p_I^*p_{I!}A, p_I^*B] \xrightarrow[\sim]{\Theta} p_{I^\circ}^*p_{I*}p_{2*}q_{2*}q_2^*\langle p_I^*p_{I!}A, p_I^*B \rangle \xrightarrow{\sim} \dots \\
\downarrow \text{adj} \quad \quad \quad \downarrow \text{adj} \\
p_{I^\circ}^*p_{I*}[A, p_I^*B] \xrightarrow[\sim]{\Theta} p_{I^\circ}^*p_{I*}p_{2*}q_{2*}q_2^*\langle A, p_I^*B \rangle \xrightarrow{\sim} \dots
\end{array}$$

$$\begin{array}{ccc}
\dots & \longrightarrow & \langle p_I^*p_{I!}A, B \rangle \xrightarrow{\text{adj}} \langle A, B \rangle \\
& & \downarrow \text{adj} \\
\dots & \longrightarrow & p_{I^\circ}^*p_{I*}p_{2*}q_{2*}q_2^*(p_I^\circ \times p_I)^*(p_{I!}A, B) \quad p_{1*}q_{2*}q_2^*p_1^*\langle A, B \rangle \\
& & \downarrow \sim \Psi \quad \downarrow \Psi \\
\dots & \longrightarrow & p_{I^\circ}^*p_{I^\circ*}p_{1*}q_{2*}q_2^*\langle p_I^*p_{I!}A, p_I^*B \rangle \\
& & \downarrow \text{adj} \\
\dots & \longrightarrow & p_{I^\circ}^*p_{I^\circ*}p_{1*}q_{2*}q_2^*\langle A, p_I^*B \rangle \xrightarrow{\text{adj}} p_{1*}q_{2*}q_2^*\langle A, p_I^*B \rangle.
\end{array}$$

Everything except possibly ① clearly commutes; and ① does so by the internal hom property in section 3. \square

From now on we take the assumptions of Proposition 20 to be satisfied. First we prove:

Lemma 35 *$p_{I!}A$ is dualizable.*

Proof. We are given an object B in $\mathbb{D}(\star)$ and we need to show that the top arrow in the following diagram is invertible:

$$\begin{array}{ccc}
 [p_{I!}A, \mathbb{1}] \otimes B & \longrightarrow & [p_{I!}A, \mathbb{1} \otimes B] \\
 \downarrow \bar{\Psi} \sim & & \sim \downarrow \bar{\Psi} \\
 p_{I^{\circ}*} \langle A, \mathbb{1} \rangle \otimes B & & p_{I^{\circ}*} \langle A, \mathbb{1} \otimes B \rangle \\
 \downarrow \sim & & \parallel \\
 p_{I^{\circ}*} (\langle A, \mathbb{1} \rangle \boxtimes B) & \xrightarrow[\Xi]{\sim} & p_{I^{\circ}*} \langle A, \mathbb{1} \boxtimes B \rangle.
 \end{array}$$

The two arrows labeled $\bar{\Psi}$ are invertible by the previous lemma, as is the vertical arrow on the bottom left by hypothesis (H4). Given $i \in I_0$, the fiber over i of the morphism $\Xi : \langle A, \mathbb{1} \rangle \boxtimes B \rightarrow \langle A, \mathbb{1} \boxtimes B \rangle$ corresponds to the morphism $[i^*A, \mathbb{1}] \otimes B \rightarrow [i^*A, \mathbb{1} \otimes B]$ by the external product and normalization properties in section 3. The latter morphism is invertible since A is fiberwise dualizable hence also the bottom horizontal arrow in the diagram is invertible (by (D2)). It now suffices to prove its commutativity which we leave as an easy exercise. \square

To prove commutativity of the diagram (22) with $g = \text{Tr}(f)$ and the top horizontal arrow replaced by $\text{Tr}(p_{I!}f)$ we decompose $\text{Tr}(f)$ into coevaluation, the morphism induced by f and evaluation, and similarly for $\text{Tr}(p_{I!}f)$. Schematically:

$$\begin{array}{ccccc}
 S & \xrightarrow{\text{coev}} & (p_{I!}A)^* \otimes p_{I!}A \otimes S & \xrightarrow{p_{I!}f} & \dots \\
 \downarrow & & \downarrow & & \\
 p_{I^{\circ}*}p_{1!}(q_{2!}\mathbb{1} \otimes S|_{I^{\circ} \times I}) & \xrightarrow{\text{coev}} & p_{I^{\circ}*}p_{1!}(A^{\vee} \boxtimes A \otimes S|_{I^{\circ} \times I}) & \xrightarrow{f} & \dots \\
 & & \uparrow & & \\
 \dots & \longrightarrow & (p_{I!}A)^* \otimes p_{I!}A \otimes T & \xrightarrow{\text{ev}} & T \\
 & & \uparrow & & \uparrow \\
 \dots & \longrightarrow & p_{I^{\circ}*}p_{1!}(A^{\vee} \boxtimes A \otimes T|_{I^{\circ} \times I}) & \xrightarrow{\sim_{\text{oev}}} & p_{I!}p_{2*}(q_{1*}\mathbb{1} \otimes T|_{I^{\circ} \times I}).
 \end{array} \tag{35}$$

The vertical morphisms in the middle will be described below but we can already say here that they will be easily seen to make the square in the middle commute. Now the fact that we have isomorphisms

$$p_{I^{\circ}*}(- \otimes p_{I^{\circ}}^* -) \cong p_{I^{\circ}*} - \otimes -, \quad p_{1!}(- \otimes p_1^* -) \cong p_{1!} - \otimes -$$

allows us to neglect the twisting:

Lemma 36 *We may assume $S = T = \mathbb{1}$.*

Proof. Consider the following diagram:

$$\begin{array}{ccc}
 \mathbb{1} \otimes S & \xrightarrow{\text{coev}} & (p_{I!}A)^* \otimes p_{I!}A \otimes S \\
 \downarrow & & \downarrow \\
 p_{I^{\circ}*}p_{1!}q_{2!}\mathbb{1} \otimes S & \xrightarrow{\text{coev}} & p_{I^{\circ}*}p_{1!}(A^{\vee} \boxtimes A) \otimes S \\
 \downarrow \sim & & \downarrow \sim \\
 p_{I^{\circ}*}p_{1!}(q_{2!}\mathbb{1} \otimes p_2^*p_I^*S) & \xrightarrow{\text{coev}} & p_{I^{\circ}*}p_{1!}(A^{\vee} \boxtimes A \otimes p_2^*p_I^*S).
 \end{array}$$

It is easy to check that the composition of the two vertical morphisms on the left equals the left vertical morphism in (35). Moreover the bottom square clearly commutes thus we are left to prove the commutativity of the top square but this does not depend on S . A similar argument shows that we may assume $T = \mathbb{1}$. \square

Lemma 37 *The left square in (35) commutes.*

Proof. By the previous lemma we may assume $S = \mathbb{1}$. Again, we factor the co-evaluation morphisms on the top and bottom into two parts as in (10) and (16) respectively. This decomposes the left square in (35) into two parts which we consider separately.

By adjunction, the first one may be expanded as follows (the arrows labeled with a small Greek letter will be defined below):

$$\begin{array}{c}
 \begin{array}{ccccc}
 p_{I^\circ}^* \mathbb{1} & \xrightarrow{\text{adj}} & p_{I^\circ}^* [p_{I!} A, p_{I!} A] & \xrightarrow{\alpha} & \dots \\
 \uparrow \text{adj} \sim & & \uparrow \sim \text{adj} & & \\
 (p_1 q_2)! (p_1 q_2)^* \mathbb{1} & \xrightarrow{\text{adj}} & (p_1 q_2)! (p_1 q_2)^* p_{I^\circ}^* [p_{I!} A, p_{I!} A] & \xrightarrow{\alpha} & \dots \\
 \downarrow \sim & & \parallel & \textcircled{1} & \\
 (p_1 q_2)! (p_2 q_2)^* \mathbb{1} & \xrightarrow{\text{adj}} & (p_1 q_2)! (p_2 q_2)^* p_I^* [p_{I!} A, p_{I!} A] & \xrightarrow{\beta} & \dots \\
 \parallel & & & & \\
 (p_1 q_2)! (p_2 q_2)^* \mathbb{1} & \xrightarrow{\Theta \circ \text{adj}} & \dots & &
 \end{array} \\
 \\
 \begin{array}{ccccc}
 \dots & \longrightarrow & \langle p_I^* p_{I!} A, p_{I!} A \rangle & \xrightarrow{\text{adj}} & \langle A, p_{I!} A \rangle \\
 & & \uparrow \sim \text{adj} & & \uparrow \text{adj} \\
 \dots & \longrightarrow & (p_1 q_2)! (p_1 q_2)^* \langle p_I^* p_{I!} A, p_{I!} A \rangle & & \\
 & & \uparrow \gamma & & \\
 \dots & \longrightarrow & (p_1 q_2)! (p_2 q_2)^* (p_2 q_2)_* q_2^* \langle p_I^* p_{I!} A, p_I^* p_{I!} A \rangle & & p_{1!} p_1^* \langle A, p_{I!} A \rangle \\
 & & \downarrow \text{adj} & & \downarrow \sim \Psi \\
 & & (p_1 q_2)! (p_2 q_2)^* (p_2 q_2)_* q_2^* \langle A, p_I^* p_{I!} A \rangle & \xrightarrow{\text{adj}} & p_{1!} \langle A, p_I^* p_{I!} A \rangle \\
 & & \uparrow \text{adj} & & \uparrow \text{adj} \\
 \dots & \longrightarrow & (p_1 q_2)! (p_2 q_2)^* (p_2 q_2)_* q_2^* \langle A, A \rangle & \xrightarrow{\text{adj}} & p_{1!} \langle A, A \rangle,
 \end{array}
 \end{array}$$

and the second one as follows:

$$\begin{array}{ccc}
 p_{I^\circ}^* [p_{I!} A, p_{I!} A] & \xleftarrow{\sim} & p_{I^\circ}^* ([p_{I!} A, \mathbb{1}] \otimes p_{I!} A) \\
 \downarrow \text{adj} \circ \alpha & \textcircled{2} & \downarrow \delta \\
 \langle A, p_{I!} A \rangle & \xleftarrow[\textcircled{14}]{\sim} & \langle A, \mathbb{1} \rangle \otimes p_{I^\circ}^* p_{I!} A \\
 \uparrow \bar{\Psi} \sim & \textcircled{3} & \uparrow \sim \\
 p_{1!} \langle A, A \rangle & \xleftarrow[\textcircled{14}]{\sim} & p_{1!} (p_1^* \langle A, \mathbb{1} \rangle \otimes p_2^* A).
 \end{array}$$

Notice first that these two diagrams indeed “glue” together. Thus it suffices to show commutativity of the rectangles marked with a number (the other ones are easily seen to commute).

① may be expanded as follows (set $B = p_{I!}A$):

$$\begin{array}{ccccc}
 q_2^* p_1^* p_{I^*}^* [B, B] & \xrightarrow[\sim]{\Lambda} & q_2^* p_1^* p_{I^*}^* \langle B, B \rangle & \xrightarrow[\sim]{\Psi} & q_2^* p_1^* \langle p_I^* B, B \rangle \\
 \parallel & & & & \downarrow \sim \Psi \\
 q_2^* p_2^* p_I^* [B, B] & \xrightarrow[\sim]{\Lambda} & q_2^* p_2^* p_I^* \langle B, B \rangle & \xrightarrow[\sim]{\Psi} & q_2^* \langle p_I^* B, p_I^* B \rangle \\
 \parallel & & & & \uparrow \text{adj} \\
 q_2^* p_2^* p_I^* [B, B] & \xrightarrow[\sim]{} & q_2^* p_2^* [p_I^* B, p_I^* B] & \xrightarrow[\sim]{\Theta} & q_2^* p_2^* p_{2*} q_{2*} q_2^* \langle p_I^* B, p_I^* B \rangle.
 \end{array}$$

The top rectangle commutes by the naturality property, the bottom rectangle by the internal hom property of section 3.

For ② consider the following decomposition (by adjunction again):

$$\begin{array}{ccccc}
 [p_{I!}A, p_{I!}A] & \xleftarrow{\sim} & [p_{I!}A, \mathbb{1}] \otimes p_{I!}A & \xlongequal{\quad} & [p_{I!}A, \mathbb{1}] \otimes p_{I!}A \\
 \downarrow \Lambda \circ \text{adj} & & \downarrow & & \downarrow \Lambda \circ \text{adj} \\
 p_{I^*} p_{I^*}^* \langle p_{I!}A, p_{I!}A \rangle & \xleftarrow[\text{(14)}]{\sim} & p_{I^*} p_{I^*}^* (\langle p_{I!}A, \mathbb{1} \rangle \otimes p_{I!}A) & & p_{I^*} p_{I^*}^* \langle p_{I!}A, \mathbb{1} \rangle \otimes p_{I!}A \\
 \downarrow \Psi \sim & & \searrow \sim & & \downarrow \sim \\
 p_{I^*} \langle p_I^* p_{I!}A, p_{I!}A \rangle & \xleftarrow[\text{(14)}]{\sim} & p_{I^*} (\langle p_I^* p_{I!}A, \mathbb{1} \rangle \otimes p_I^* p_{I!}A) & & p_{I^*} (p_I^* \langle p_{I!}A, \mathbb{1} \rangle \otimes p_I^* p_{I!}A) \\
 \downarrow \text{adj} & & \downarrow \text{adj} & & \downarrow \Psi \\
 p_{I^*} \langle A, p_{I!}A \rangle & \xleftarrow[\text{(14)}]{\sim} & p_{I^*} \langle A, \mathbb{1} \rangle \otimes p_I^* p_{I!}A & & p_{I^*} \langle A, \mathbb{1} \rangle \otimes p_I^* p_{I!}A.
 \end{array}$$

The top left square commutes by the normalization property, the pentagon in the middle by the external product and normalization properties of section 3. The rest is clearly commutative. (One also needs here Lemma 34 to ensure that the morphism corresponding to δ under adjunction is invertible.)

Next, we may decompose ③ by adjunction as follows:

$$\begin{array}{ccc}
 p_I^* \langle A, p_{I!}A \rangle & \xleftarrow[\sim]{\text{(14)}} & p_I^* (\langle A, \mathbb{1} \rangle \boxtimes p_{I!}A) \\
 \downarrow \Psi \sim & & \downarrow \sim \\
 \langle A, p_I^* p_{I!}A \rangle & \xleftarrow[\sim]{\text{(14)}} & \langle A, \mathbb{1} \rangle \boxtimes p_I^* p_{I!}A \\
 \uparrow \text{adj} & & \uparrow \text{adj} \\
 \langle A, A \rangle & \xleftarrow[\sim]{\text{(14)}} & \langle A, \mathbb{1} \rangle \boxtimes A.
 \end{array}$$

Both squares commute by the external product property in section 3. \square

The following lemma completes the proof of Proposition 20.

Lemma 38 *The right square in (35) commutes.*

Proof. Again, we may assume $T = \mathbb{1}$ by the lemma above. First, (11) lets us replace the evaluation morphism on the top by the following composition (the arrows labeled with a small Greek letter will be defined below):

$$\begin{array}{ccccccc}
 (p_{I!}A)^* \otimes p_{I!}A & \xrightarrow{\sim} & (p_{I!}A)^* \otimes (p_{I!}A)^{**} & \longrightarrow & ((p_{I!}A)^* \otimes p_{I!}A)^* & \xrightarrow{\text{coev}} & \mathbb{1} \\
 \Lambda \downarrow \sim & \textcircled{4} & \bar{\Psi} \circ \Lambda \downarrow \sim & \textcircled{6} & \theta \uparrow \sim & \textcircled{7} & \\
 p_{I^{\circ}*}A^{\vee} \otimes p_{I!}A & \xrightarrow{\sim} & p_{I!}A^{\vee\vee} \otimes p_{I^{\circ}*}A^{\vee} & & p_{I!}(A^{\vee} \boxtimes p_{I!}A)^{\vee} & & \\
 \varepsilon \uparrow \sim & \textcircled{5} & \eta \uparrow \sim & & \bar{\Psi} \downarrow \sim & & \\
 p_{I^{\circ}*}p_{I!}(A^{\vee} \boxtimes A) & \xrightarrow{\sim} & p_{I!}p_{2*}\mu_*(A^{\vee\vee} \boxtimes A^{\vee}) & \xrightarrow{\Xi} & p_{I!}p_{2*}\mu_*(A^{\vee} \boxtimes A)^{\vee} & \xrightarrow{\text{coev}} & p_{I!}p_{2*}\mu_*(q_{2!}\mathbb{1}, \mathbb{1}).
 \end{array}$$

The commutativity of $\textcircled{4}$ can be checked on each tensor factor separately; only one of them is possibly non-obvious:

$$\begin{array}{ccccc}
 A & \xrightarrow{\text{adj}} & p_I^*p_{I!}A & \xrightarrow[\sim]{\Upsilon} & p_I^*\langle p_{I!}A, \mathbb{1} \rangle, \mathbb{1} \\
 \Upsilon \downarrow \sim & & \Upsilon \downarrow \sim & & \sim \downarrow \Psi \\
 \langle \langle A, \mathbb{1} \rangle, \mathbb{1} \rangle & \xrightarrow{\text{adj}} & \langle \langle p_I^*p_{I!}A, \mathbb{1} \rangle, \mathbb{1} \rangle & \xrightarrow[\sim]{\Psi} & \langle p_I^*\langle p_{I!}A, \mathbb{1} \rangle, \mathbb{1} \rangle \\
 \text{adj} \downarrow & & \bar{\Psi} \nearrow & & \bar{\Psi} \\
 \langle p_{I^{\circ}*}p_{I^{\circ}*}\langle A, \mathbb{1} \rangle, \mathbb{1} \rangle & \xleftarrow[\sim]{\Psi} & p_I^*\langle p_{I^{\circ}*}\langle A, \mathbb{1} \rangle, \mathbb{1} \rangle & &
 \end{array}$$

The two squares in the top row commute by the biduality property of section 3 while the rest is clearly commutative.

$\textcircled{5}$ may be decomposed as follows:

$$\begin{array}{ccccccc}
 p_{I^{\circ}*}A^{\vee} \otimes p_{I!}A & \xlongequal{\quad} & p_{I^{\circ}*}A^{\vee} \otimes p_{I!}A & \longrightarrow & p_{I!}A \otimes p_{I^{\circ}*}A^{\vee} & \xrightarrow{\Upsilon} & p_{I!}A^{\vee\vee} \otimes p_{I^{\circ}*}A^{\vee} \\
 \downarrow & & \uparrow & & \uparrow & & \uparrow \\
 p_{I^{\circ}*}(A^{\vee} \otimes p_{I^{\circ}*}p_{I!}A) & & p_{I!}(p_I^*p_{I^{\circ}*}A^{\vee} \otimes A) & \rightarrow & p_{I!}(A \otimes p_I^*p_{I^{\circ}*}A^{\vee}) & \xrightarrow{\Upsilon} & p_{I!}(A^{\vee\vee} \otimes p_I^*p_{I^{\circ}*}A^{\vee}) \\
 \uparrow & & \downarrow & & \downarrow & & \downarrow \\
 p_{I^{\circ}*}(A^{\vee} \otimes p_{I!}p_2^*A) & & p_{I!}(p_{2*}p_1^*A^{\vee} \otimes A) & \rightarrow & p_{I!}(A \otimes p_1^*p_2^*A^{\vee}) & \xrightarrow{\Upsilon} & p_{I!}(A^{\vee\vee} \otimes p_1^*p_2^*A^{\vee}) \\
 \uparrow & & \downarrow & & \downarrow & & \downarrow \\
 p_{I^{\circ}*}p_{I!}(A^{\vee} \boxtimes A) & \longleftarrow & p_{I!}p_{2*}(A^{\vee} \boxtimes A) & \longrightarrow & p_{I!}p_{2*}\mu_*(A \boxtimes A^{\vee}) & \xrightarrow{\Upsilon} & p_{I!}p_{2*}\mu_*(A^{\vee\vee} \boxtimes A^{\vee}).
 \end{array}$$

Here, p'_1 and p'_2 are the projections onto the factors of $I \times I^{\circ}$ and all arrows are invertible. All rectangles of this diagram are easily seen to commute (for the leftmost one may use [1, 2.1.105]).

Next we turn to ⑥. In the decomposition of it (use the normalization property of section 3 for the top horizontal arrow),

$$\begin{array}{ccc}
 (p_{I!}A)^{\vee\vee} \boxtimes (p_{I!}A)^{\vee} & \xrightarrow{\Xi} & ((p_{I!}A)^{\vee} \boxtimes p_{I!}A)^{\vee} \\
 \uparrow \overline{\Psi} & & \uparrow \overline{\Psi} \\
 (p_{I^{\circ}*}A^{\vee})^{\vee} \boxtimes (p_{I!}A)^{\vee} & \xrightarrow{\Xi} & (p_{I^{\circ}*}A^{\vee} \boxtimes p_{I!}A)^{\vee} \\
 \uparrow \overline{\Psi} & & \uparrow \overline{\Psi} \\
 p_{I!}(A^{\vee\vee}) \boxtimes (p_{I!}A)^{\vee} & & (p_{I^{\circ}*}(A^{\vee} \boxtimes p_{I!}A))^{\vee} \\
 \uparrow & & \uparrow \overline{\Psi} \\
 p_{I!}(A^{\vee\vee} \boxtimes (p_{I!}A)^{\vee}) & \xrightarrow{\Xi} & p_{I!}(A^{\vee} \boxtimes p_{I!}A)^{\vee} \\
 \downarrow & & \downarrow \\
 p_{I!}(A^{\vee\vee} \boxtimes p_{I^{\circ}*}A^{\vee}) & & p_{I!}(A^{\vee} \otimes p_{1!}p_2^*A)^{\vee} \\
 \downarrow & & \downarrow \\
 p_{I!}(A^{\vee\vee} \otimes p'_{1*}p_2'^*A^{\vee}) & & p_{I!}(p_{1!}(A^{\vee} \boxtimes A))^{\vee} \\
 \downarrow & & \downarrow \overline{\Psi} \\
 p_{I!}p_{2*}\mu_*(A^{\vee\vee} \boxtimes A^{\vee}) & \xrightarrow{\Xi} & p_{I!}p_{2*}\mu_*(A^{\vee} \boxtimes A)^{\vee},
 \end{array}$$

everything commutes by the external product property of section 3 (and adjunction). All vertical arrows are invertible.

It remains to prove the commutativity of ⑦. In the diagram

$$\begin{array}{ccccc}
 \langle (p_{I!}A)^* \otimes p_{I!}A, \mathbb{1} \rangle & \xrightarrow{\text{coev}} & \langle \mathbb{1}, \mathbb{1} \rangle & \xrightarrow{\quad} & \mathbb{1} \\
 \uparrow & & \uparrow & \swarrow & \\
 \langle p_{I^{\circ}*}p_{1!}(A^{\vee} \boxtimes A), \mathbb{1} \rangle & \xrightarrow{\text{coev}} & \langle p_{I^{\circ}*}p_{1!}q_{2!}\mathbb{1}, \mathbb{1} \rangle & & p_{I!}p_{2*}q_{1*}q_1^*p_2^*p_I^*\langle \mathbb{1}, \mathbb{1} \rangle \\
 \downarrow \overline{\Psi} & & \downarrow \overline{\Psi} & & \sim \downarrow \Psi \\
 p_{I!}p_{2*}\mu_*\langle A^{\vee} \boxtimes A, \mathbb{1} \rangle & \xrightarrow{\text{coev}} & p_{I!}p_{2*}\mu_*\langle q_{2!}\mathbb{1}, \mathbb{1} \rangle & \xrightarrow{\overline{\Psi}} & p_{I!}p_{2*}q_{1*}\langle \mathbb{1}, \mathbb{1} \rangle,
 \end{array}$$

the top left square is simply $\langle -, \mathbb{1} \rangle$ applied to the left square in (35). It follows that this square is commutative. Moreover it is easy to see that the composition of the left vertical arrows is the same as of the ones in ⑦. Thus this diagram is a decomposition of ⑦. The rest of the diagram clearly commutes. \square

APPENDIX C. $\mathbb{D}(G)$ FOR G A FINITE GROUP

The question, given a category I , whether I -diagrams and morphisms of such in the homotopy categories can be lifted (and if so whether uniquely) to the homotopy categories of I -diagrams has always been of interest (see e. g. [8, chapitre IV] or [10, p. 2]). The goal of this last section is to give a proof for the (well-known) answer in the case of I a finite group.

Proposition 39 *Let \mathbb{D} be an additive derivator of type **Dia**, let G be a finite group in **Dia** and assume that $\#G$ is invertible in $R_{\mathbb{D}}$. Then the canonical functor*

$$\text{dia}_G : \mathbb{D}(G) \rightarrow \mathbf{CAT}(G^\circ, \mathbb{D}(\star))$$

is fully faithful. If, in addition, $\mathbb{D}(G)$ is pseudo-abelian then the functor is an equivalence of categories.

Remark 40 Suppose that \mathbb{D} is triangulated and that **Dia** contains countable discrete categories. In this case $\mathbb{D}(G)$ has countable direct sums, and it follows from [20, 1.6.8] that $\mathbb{D}(G)$ is pseudo-abelian.

Proof of Proposition 39. We need to understand the two adjunctions $e_! \dashv e^*$ and $e^* \dashv e_*$ where $e : \star \rightarrow G$ is the unique functor.

Consider the following comma square where η on the component corresponding to $g \in G$ is g :

$$\begin{array}{ccc} \coprod_G \star & \xrightarrow{p} & \star \\ p \downarrow & \eta \nearrow & \downarrow e \\ \star & \xrightarrow{e} & G, \end{array}$$

By (D4), the two compositions

$$\begin{aligned} p_! p^* &\xrightarrow{\text{adj}} p_! p^* e^* e_! \xrightarrow{\eta^*} p_! p^* e^* e_! \xrightarrow{\text{adj}} e^* e_!, \\ p_* p^* &\xleftarrow{\text{adj}} p_* p^* e^* e_* \xleftarrow{\eta^*} p_* p^* e^* e_* \xleftarrow{\text{adj}} e^* e_*. \end{aligned}$$

are invertible, yielding identifications

$$e^* e_! \cong \coprod_G, \quad e^* e_* \cong \prod_G,$$

and therefore a canonical morphism $e^* e_! \rightarrow e^* e_*$ which is invertible if G is finite.

Under these identifications the (contravariant) action of G on $e^* e_!$ (obtained by applying dia_G to $e_!$) is given by right translation, and on $e^* e_*$ by left translation. Indeed, let $A \in \mathbb{D}(\star)_0$ be an arbitrary object and set $B = e^* e_! A$, fix also $g \in G$. Then the following diagram commutes where $r_g((x_h)_h) = (x_h)_{hg}$:

$$\begin{array}{ccccccc} \coprod_{h \in G} A & \xrightarrow{\text{adj}} & \coprod_{h \in G} B & \xrightarrow{\coprod_h h^*} & \coprod_{h \in G} B & \xrightarrow{\Sigma} & B \\ r_g \downarrow & & r_g \downarrow & & & & \downarrow g^* \\ \coprod_{h \in G} A & \xrightarrow{\text{adj}} & \coprod_{h \in G} B & \xrightarrow{\coprod_h h^*} & \coprod_{h \in G} B & \xrightarrow{\Sigma} & B. \end{array}$$

Thus the claim in the case of $e^* e_!$; the case of $e^* e_*$ is proved in a similar way.

Next, we would like to describe the units and counits of the adjunctions. We first deal with the unit of $e_! \dashv e^*$. Let $i : \star \rightarrow \coprod_G \star$ be the inclusion of the component

corresponding to 1_G .

$$\begin{array}{ccccc}
 p_! p^* & \xrightarrow{\text{adj}} & p_! p^* e^* e_! & \xrightarrow{\eta^*} & p_! p^* e^* e_! \\
 \uparrow \text{adj} & & \uparrow \text{adj} & \nearrow \text{adj} & \downarrow \text{adj} \\
 p_! i_! i^* p^* & \xrightarrow{\text{adj}} & p_! i_! i^* p^* e^* e_! & & \\
 \downarrow \sim & & \searrow \sim & & \\
 1 & \xrightarrow{\text{adj}} & e^* e_! & &
 \end{array}$$

The diagram clearly commutes and hence the unit $1 \rightarrow e^* e_!$ is given by the inclusion of the unit component into \coprod_G . Similarly, the counit $e^* e_* \rightarrow 1$ is the projection onto the component corresponding to 1_G .

Next, we want to describe the other two (co)units (at least after applying e^*). For this consider the composition of the unit and the counit of the adjunction,

$$e^* \rightarrow \coprod_G e^* \rightarrow e^*,$$

which we know to be the identity. By the description of the first morphism above we see that the 1_G -component of the second morphism has to be the identity. But this second morphism is also G -equivariant so the description of the G -action above implies that the morphism is the action of g on the g -component for any $g \in G$. Similarly, the counit $e^* \rightarrow \prod_G e^*$ is given by the action of g on the g -component.

We now have enough information to describe the composition

$$\xi : e_! \xrightarrow{\text{adj}} e_* e^* e_! \rightarrow e_* e^* e_* \xrightarrow{\text{adj}} e_*$$

after applying e^* . Indeed, it can then be identified with the following one:

$$\begin{array}{ccccccc}
 \coprod_G & \longrightarrow & \prod_G \coprod_G & \longrightarrow & \prod_G \prod_G & \longrightarrow & \prod_G \\
 (x_h)_h & \longmapsto & ((x_{hg^{-1}})_h)_g & \longmapsto & ((x_{hg^{-1}})_h)_g & \longmapsto & (x_{g^{-1}})_g.
 \end{array}$$

Since this morphism is invertible and e^* conservative (by (D2)), also ξ is invertible, and it thus makes sense to consider the composition

$$1 \rightarrow e_* e^* \xrightarrow{\xi^{-1}} e_! e^* \rightarrow 1. \quad (36)$$

After applying e^* it can be identified with

$$\begin{array}{ccccccc}
 e^* & \longrightarrow & \prod_G e^* & \longrightarrow & \coprod_G e^* & \longrightarrow & e^* \\
 x & \longmapsto & (g^* x)_g & \longmapsto & ((g^{-1})^* x)_g & \longmapsto & \sum_{g \in G} g^* (g^{-1})^* x = \#G \cdot x.
 \end{array}$$

If $\#G$ is invertible in $R_{\mathbb{D}}$ then this morphism and (again, by (D2)) also (36) is invertible, in particular there is, for every $B \in \mathbb{D}(G)_0$, a factorization of the identity morphism of B : $B \rightarrow e_* e^* B \rightarrow B$. For any $A \in \mathbb{D}(G)_0$, this factorization in turn induces the horizontal arrows in the following commutative diagram ($\mathcal{C} = \mathbf{CAT}(G^\circ, \mathbb{D}(\star))$, $d = \text{dia}_G$):

$$\begin{array}{ccccc}
 \mathbb{D}(G)(A, B) & \longrightarrow & \mathbb{D}(G)(A, e_* e^* B) & \longrightarrow & \mathbb{D}(G)(A, B) \\
 d \downarrow & & d \downarrow & & d \downarrow \\
 \mathcal{C}(dA, dB) & \longrightarrow & \mathcal{C}(dA, de_* e^* B) & \longrightarrow & \mathcal{C}(dA, dB).
 \end{array}$$

The first top horizontal arrow is injective hence if the middle vertical arrow is injective then so is the left vertical one. Similarly, the second bottom horizontal arrow is surjective hence if the middle vertical arrow is surjective then so is the right vertical one. Consequently, to prove fully faithfulness of dia_G it suffices to prove bijective the middle vertical arrow (for all A and B). Now, the source of this map can be identified with $\mathbb{D}(\star)(e^*A, e^*B)$ by adjunction, while the target is the set of G° -morphisms in $\mathbb{D}(\star)$ from e^*A to the left regular representation associated to e^*B — which is also $\mathbb{D}(\star)(e^*A, e^*B)$.

It remains to show essential surjectivity of dia_G . Given an object $A \in \mathbb{D}(\star)_0$ with a G° -action ρ , consider the two morphisms

$$\begin{aligned} A &\xrightarrow{\alpha} \prod_G A & \text{and} & \quad \prod_G A \xrightarrow{\beta} A \\ x &\longmapsto (\rho(g)x)_g & & \quad (x_g)_g \longmapsto \frac{1}{\#G} \sum_{g \in G} \rho(g^{-1})x_g. \end{aligned}$$

They give rise to a G° -equivariant decomposition of the identity on A :

$$1_A : A \xrightarrow{\alpha} \text{dia}_G(e_*A) \xrightarrow{\beta} A.$$

By fullness of dia_G proved above, there exists $p \in \mathbb{D}(G)(e_*A, e_*A)$ with $\text{dia}_G(p) = \alpha\beta$. By faithfulness also proved above, the equality

$$\text{dia}_G(p^2) = \text{dia}_G(p)^2 = (\alpha\beta)^2 = \alpha\beta = \text{dia}_G(p)$$

implies that p is a projector, and therefore if $\mathbb{D}(G)$ is pseudo-abelian then there is a decomposition

$$e_*A = \ker(p) \oplus \text{im}(p).$$

Let $\alpha' : \text{im}(p) \rightarrow e_*A$ be the inclusion, and $\beta' : e_*A \rightarrow \text{im}(p)$ the projection. Then

$$\begin{aligned} (\text{dia}_G(\beta')\alpha)(\beta\text{dia}_G(\alpha')) &= \text{dia}_G(\beta')\text{dia}_G(p)\text{dia}_G(\alpha') \\ &= \text{dia}_G(\beta'p\alpha') \\ &= \text{dia}_G(1_{\text{im}(p)}) \\ &= 1_{\text{dia}_G(\text{im}(p))}, \end{aligned}$$

and

$$\begin{aligned} (\beta\text{dia}_G(\alpha'))(\text{dia}_G(\beta')\alpha) &= \beta\text{dia}_G(\alpha'\beta')\alpha \\ &= \beta\text{dia}_G(p)\alpha \\ &= \beta\alpha\beta\alpha \\ &= 1_A. \end{aligned}$$

We conclude that $A \cong \text{dia}_G(\text{im}(p))$. \square

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