Credit Valuation Adjustment

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MSc Thesis
Trinity 2011

Acknowledgements

I would like to thank my supervisor Professor Ben Hambly for being so generous with his time and superb mind and for emailing me when he did. I would also like to apologise to A. Hull and J. White.

Abstract

Credit risk has become a topical issue since the 2007 Credit Crisis, particularly for its impact on the valuation of OTC derivatives. This becomes critical when the credit risk of entities involved in a contract either as underlying or counterparty become highly correlated as is the case during macroeconomic shocks. It impacts the valuation of such contracts through an additional term, the credit valuation adjustment (CVA). This can become large with such correlation. This thesis outlines the main approaches to credit risk modelling, intensity and structural. It gives important examples of both and particular examples useful in the calculation of CVA, the intensity model of Brigo and the structural model of Hull and White. It details Brigo's market standard model independent framework for derivatives valuation with CVA. It does this for both its unilateral form where only one counterparty is credit risky and also for its bilateral form where both counterparties are credit risky. This thesis then shows how these frameworks can be applied to the valuation of a credit default swap contract (CDS). Finally, it shows how Brigo's and Hull and White's model for credit risk apply to the valuation of the CVA of CDS and draws comparisons, especially based on their ability to capture correlation effects.

Contents

Chapter 1

Introduction

This paper addresses the adjustment in derivative valuation when credit risk is taken account of. Credit risk is the risk that a party in a financial exchange will default. It has become a crucial component of derivative valuation. Whenever a derivative is traded over the counter (OTC) the default risk of all counterparties should, in principle, enter the valuation. This is because an OTC trade is privately negotiated outside of organized exchanges which means that disclosure of information between parties is unregulated. OTC derivatives are also illiquid and therefore need to be valued using a model. The OTC market is made up of banks and other sophisticated financial parties. As a result stipulations to adjust for counterparty risk in pricing are an increasingly common feature in regulation, e.g. Basel II, IAS 39 (international accounting standards system)[8].

This paper shall detail all the prerequisite financial and mathematical theory, then introduce the two main branches of credit risk modelling, intensity and structural. It will then develop a general model independent technology for the calculation of CVA. First for a contract where only one counterparty is credit risky (unilateral CVA) as was considered the case prior to the 2007 crisis when one counterparty, particularly in the case of CDS contracts, would be a monoline insurer or investment bank and thus default risk free. Since the 2007 crisis, this assumption can no longer be made and hence we develop a general model independent calculation for the case where both counterparties to a transaction are credit risky (bilateral CVA).

It will detail the definition and properties of a CDS contract and derive the unilateral and bilateral CVA in its pricing, all in a model independent framework. Then we shall apply particular intensity and structural models in the general CDS CVA framework. The market standard for the calculation of both unilateral and bilateral CVA of CDS are Brigo's intensity models [6], [5]. This is due to their easy implementation, calibration and flexibility in capturing the impact of volatility of default risk of any or all the names involved, the protection buyer, the protection seller and the reference credit and also the correlation between the default riskiness of any or all of the three names. Such effects have become important since the 2007 crisis as credit risk has become increasingly concentrated, volatile and correlated. We shall also review Hull and White's structural approach to calculating the unilateral CVA of CDS, which is easy to implement, calibrate and captures many important features of volatility and correlation as Brigo's intensity models. In comparing Brigo's formulation to that of Hull and White's we find both equivalent, just that Hull and White assume zero recovery. Thus Brigo's model is more general.

In course of our analysis we shall introduce the vital notion of "wrong-way-risk". Wrong-way-risk is the risk that the value of the underlying of a derivative is positively correlated to the value or credit worthiness of a counterparty to the derivative contract. This is a particularly pertinant and topical effect in the valuation of CDS. If the reference credit and protection seller in a CDS contract have credit risks that are highly positively correlated then the CVA must increase considerably to compensate the protection buyer for the risk that the protection seller will default at a point in time when the contract enjoys a high value due the credit riskiness of the reference credit. We shall investigate the ability of the different models introduced to capture this "wrong-way-risk" effect on CVA.

Chapter 2

Credit Risk Modelling

2.1 Credit Spreads and Zero Coupon Bond(ZCB)-Based Pricing

In this section we detail the basic mathematical and financial building blocks for any credit risk model. We shall show that default free ZCB and defaultable ZCB prices contain all the information on the distribution of the time of default of a given credit risky entity provided the defaultable ZCB has zero recovery. That is the bond-holders receives nothing from the bond-issuer upon default. Through this we can also introduce the risk premia paid to offset this credit risk, the credit spread.

This however is only theoretically useful as default free ZCBs and defaultable ZCBs with zero recovery are not traded in real markets. The former are fiction that approximated reality for some AAA debt and for the latter, recovery would only be zero in very extreme and unusual circumstances. This is the motivation for intensity and structural approaches to modelling credit risk. From these approaches the evolution of the credit spread can be implied from the market that we do observe. The concepts investigated such as hazard rates also help to appreciate the nature of and differences between intensity and structural credit risk models.

2.1.1 Survival Probability

The ideas of this section are heavily based on [[17], ch.3 and ch.5]. They are formulated on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathbb{Q})$. $\mathbb{Q}$ is not necessarily unique due to the incompleteness of credit derivatives markets but it is assumed to be the pricing measure. In other words, $\mathbb{Q}$ probabilities are state price densities, i.e. the $\mathbb{Q}$-probability of an event A at T is the price of a security that pays 1 at T. Hence, a default free ZCB that has payoff 1 in all states of the world has the following price,

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}[\exp(-\int_t^T r_u du) \cdot 1] \tag{2.1}$$

where r_u is the instantaneous short rate (default free). To be explicit the price of a default free ZCB is the expectation under $\mathbb{Q}$ of the discounted final payoff of a unit of currancy.

A defaultable ZCB, however, has the payoff,

$$Payoff = \mathbf{1}_{\tau > T} = \begin{cases} 1 & \text{if default after T, } \tau > T \\ 0 & \text{if default before T, } \tau \leq T \end{cases} \tag{2.2}$$

where τ is the random time of default. The price of a defaultable ZCB at $t < \tau$ is thus,

$$\overline{P}(t,T) = \mathbb{E}^{\mathbb{Q}}[\exp(-\int_t^T r_u du) \cdot \mathbf{1}_{\tau > T}] \tag{2.3}$$

If we assume that r is independent of τ, the default of the bond issuer and evolution of the short rate are independent, then,

$$\begin{aligned} \overline{P}(t,T) &= \mathbb{E}^{\mathbb{Q}}[\exp(-\int_t^T r_u du) \cdot \mathbf{1}_{\tau > T}] = \mathbb{E}^{\mathbb{Q}}[\exp(-\int_t^T r_u du)] = \mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{\tau > T}] \\ &= P(t,T)\mathbf{P}(t,T) \end{aligned}$$

where $\mathbf{P}(t,T) = \mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{\tau > T}]$ is the implied probability of survival of the defaultable bond issuer in the time interval $[t,T]$.

This is consistent as it implies that the price of a bond of a defaultable bond issuer is the price of bond issued by a default free bond issuer multiplied by the probability of survival of the defaultable issuer in question, $\overline{P(t,T)} = P(t,T)\mathbf{P}(t,T)$. This in turn implies that the ZCB between times $[t,T]$ issued by the defaultable issuer with survival probability 1 between times $[t,T]$ would have the same price as that of the default free issuer in $[t,T]$ by no arbitrage.

In theory then the probability of survival of a defaultable bond issuer could be implied from the market using the relationship,

$$\mathbf{P}(t,T) = \frac{\overline{P}(t,T)}{P(t,T)}$$

From this we can imply that the probability of default of the defaultable bond issuer over the time interval $[t,T]$ is,

$$\widetilde{\mathbf{P}}(t,T) = 1 - \mathbf{P}(t,T)$$

If $\mathbf{P}(t,T)$ has a right sided derivative in T then the implied probability density of the default time is,

$$\mathbf{p}(\tau \in [T, T+dt]) = -\frac{\partial}{\partial T}\mathbf{P}(t,T)dt$$

Hence, if we have default free and defaultable ZCB prices for a continuum of maturities then we have survival probabilities for all maturities and also densities for all maturities as in this case the survival probability implied by the market is a continuous function of T. From this a term structure of survival probabilities can be ascertained. As a function of the time horizon of interest T, the implied survival probabilities have the following properties,

$$\mathbf{P}(t,t) = 1$$

$\mathbf{P}(t,T)$ is a non-negative and decreasing function of T. A reasonable assumption to make is that the credit entity will default eventually, i.e. $\mathbf{P}(t,\infty) = 0$,

As a function of the present time t, $\mathbf{P}(t,T)$ is increasing in t since if the credit entity has not defaulted between $[t, t+\Delta t]$ then this information updates the projected future survival probability through the incorporation of this information into the prices of defaultable ZCBs and $\mathbf{P}(t,T)$ increases.

Other relevant default risk information will also arrive in $[t, t+\Delta t]$ and $\mathbf{P}(t+\Delta t, T)$ will be updated up or down accordingly as this information is incorporated into the market prices of defaultable ZCBs.

2.1.2 Conditional Survival Probabilities and Implied Hazard Rates

The term structure of survival probabilities $\mathbf{P}(t,T)$ is important to understanding the default risk implied by the term structure of defaultable ZCB prices. However, this has limitations when comparing default risk across different periods of time for the following reasons. Survival probabilities may be implied from overlapping tenors. If $t < T_1 < T_2$ then $\mathbf{P}(t,T_1)$ and $\mathbf{P}(t,T_2)$ both imply survival information about the time interval $[t, T_1]$. But supply and demand for $P(t,T_1)$ and $P(t,T_2)$, and $\overline{P}(t,T_1)$ and $\overline{P}(t,T_2)$ may be distinctly different obscuring the information implied by $\mathbf{P}(t,T_1)$ and $\mathbf{P}(t,T_2)$. In this context the deterministic positive change of $\mathbf{P}(t,T)$ with respect to increasing t is distortionary when comparing survival probabilities over different time intervals, i.e. the survival probability between $[T_1, T_2]$ as implied by $\mathbf{P}(T_1,T_2)$ and $\mathbf{P}(t,T_2)$. The length of the tenor $[t,T]$ can also influence the survival probabilities implied. The longer the tenor the more volatile are the survival probabilities as a small change in credit risk information will have longer to escalate and therefore be more like to be causally relevant to the likelihood of eventual default.

These issue can be resolved by conditioning which can be used to isolate default risk over intervals of time avoiding these distortions. The survival probability over $[T_1, T_2]$ conditional on all information up to time t is,

$$\mathbf{P}(t,T_1,T_2) = \frac{\mathbf{P}(t,T_2)}{\mathbf{P}(t,T_1)}$$

We justify this as follows. Define,

$\cdot A =$ survival until T_2

$\cdot B =$ survival until T_1

$\cdot A \cap B$ survival until T_1 and survival until T_2

Notice that $A = A \cap B$ as $B \subseteq A$. A credit entity that has survived until T_2 must also have survived until $T_1 \leq T_2$. This gives the conditional survival probability over $[T_1, T_2]$ as seen from t as,

$$\begin{aligned}\mathbf{P}(t, T_1, T_2) &= \frac{\mathbf{P}[A \cap B]}{\mathbf{P}[B]} = \frac{\mathbf{P}[A]}{\mathbf{P}[B]} \\ &= \frac{\mathbf{P}(t, T_2)}{\mathbf{P}(t, T_1)} \\ &= \frac{\overline{P}(t, T_2)}{P(t, T_2)} \frac{P(t, T_1)}{\widetilde{P}(t, T_1)}\end{aligned}$$

A consequence of this is that,

$$\mathbf{P}(t, T_2) = \mathbf{P}(t, T_1, T_2)\mathbf{P}(t, T_1) \tag{2.4}$$

The probability of survival from t until T_2 is equivalent to the probability of survival from t to T_1 multiplied by the probability of survival from T_1 to T_2 as seen from t.

Now that we have isolated the risk of default in a given time interval as seen from today we can clearly compare the relative risks of default over different time periods $[T_1, T_2]$ as seen from different times t. Note, in order to achieve this time intervals must be of equal length. Also notice survival probabilities do not have to be updated for the non-occurrence of defaults as time proceeds. Thus conditional survival probabilities extend the limitations of survival probabilities.

From conditional survival probabilities we construct a term structure of default risk as seen from t. The smaller the increments $[T_1, T_2]$ the greater the accuracy. However as noted, the value of default probabilities will decrease deterministically as a consequence of reducing the time interval. Therefore, it is more transparent to use default probabilities per time interval length. The conditional probability of default per time unit Δt as seen from $t < T$ is,

$$\frac{1}{\Delta t}\widetilde{\mathbf{P}}(t, T, T + \Delta t) = \frac{1}{\Delta t}(1 - \mathbf{P}(t, T, T + \Delta t)) \tag{2.5}$$

The discrete implied hazard rate of default over $[T, T + \Delta t]$ is,

$$H(t, T, T + \Delta t) := \frac{1}{\Delta t} \frac{\widetilde{\mathbf{P}}(t, T, T + \Delta t)}{\mathbf{P}(t, T, T + \Delta t)} \tag{2.6}$$

The discrete hazard rate is an odds ratio, the ratio of the probability of default over the probability of survival adjusted for the time interval. The continuous implied hazard rate of default at T is,

$$h(t,T) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} H(t,T,T+\Delta t) \tag{2.7}$$

This can be given a more convenient form if we notice that,

$$\begin{aligned}
\lim_{\Delta t \to 0} H(t,T,T+\Delta t) &= \frac{1}{\Delta t} \frac{\widetilde{\mathbf{P}}(t,T,T+\Delta t)}{\mathbf{P}(t,T,T+\Delta t)} \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{1-\mathbf{P}(t,T,T+\Delta t)}{\mathbf{P}(t,T,T+\Delta t)} \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(\frac{1}{\mathbf{P}(t,T,T+\Delta t)} - 1\right) \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{\mathbf{P}(t,T) - \mathbf{P}(t,T+\Delta t)}{\mathbf{P}(t,T+\Delta t)} \\
&= -\frac{1}{\mathbf{P}(t,T)} \frac{\partial}{\partial T} \mathbf{P}(t,T)
\end{aligned}$$

The continuous implied hazard rate of default gives the odds ratio of default against survival over an infinitesimal time interval $[T, T+dt]$. Note its relation to the probability density of default at T,

$$\mathbf{p}(\tau \in [T, T+dt]) = -\frac{\partial}{\partial T} \mathbf{P}(t,T) dt \tag{2.8}$$

both equations provide the same information but in different forms. The probability density of default at T gives the local probability that the default of the credit entity occurs at T. Here, $\mathbf{p}(\tau \in [T, T+dt]) \to 0$ as $T \to \infty$, must hold. This is not because there is no default risk but because it will be increasingly likely that default has already happened. Mathematically this must hold because the density must integrate to 1,

$$\int_0^\infty \mathbf{p}(\tau \in [T, T+dt]) d\tau = 1 \tag{2.9}$$

The continuous hazard rate of default at T is the probability density of default at T relative to the probability of survival until T where $\mathbf{P}(t,T) \to 0$ as $T \to \infty$ as well. Thus,

$$h(t,T) = -\frac{1}{\mathbf{P}(t,T)} \frac{\partial}{\partial T} \mathbf{P}(t,T) > 0 \tag{2.10}$$

almost surely at least until default actually happens when it looses meaning.

Conditional survival probabilities and hazard rates evolve stochastically w.r.t. t, but there evolution is only effected by default risk information in $[T_1, T_2]$ not by the fact that no default has occurred in $[t, t+\Delta t]$. If default had occurred prior to T_1 such probabilities and rates would be meaningless.

2.2 Intensity Models

Intensity based credit risk models model the risk of default as an event that arrives exogeneously. In other words the explanation for the arrival of default events is not something that the model attempts to capture. The pattern with which defaults arrive is what is modelled. In intensity based models the local implied probability of default over a small time interval is modelled as proportional to the length of the time interval. This gives intensity based models a great deal of analytic tractability. The proportionality factor is the hazard rate of default over the infinitessimal time increment. It is also as will be shown the short-term credit spread or risk premia paid above the default risk free short rate for the entities likelihood of default in the next infinitessimal time increment.

The mathematical framework for modelling intensity based credit spreads or default times with this property are Poisson Processes. The behaviour of such processes, incremental jumps, model default events as unpredictable. This means that their behaviour in the next infinitessimally small time increment comes as a complete surprise even with complete information for time converging on the time of default.

The complexity of such processes can be developed until they are capable of reproducing default/survival dynamics realistic enough to serve the purpose of CVA where the volatility and correlation between the credit risk of names is important.

The risk of default arriving in an intensity based model is modelled via a counting process $N(t)$ which increases incrementally by jumps of one unit with the arrival of events thereby counting the events. $N(t) = \sum_0^t \Delta N(s)$ where,

$$\Delta N(s) = N(t + \Delta t) - N(t) \begin{cases} 1 & \text{if event arrives in } [t, t + \Delta t) \\ 0 & \text{if no event arrives in } [t, t + \Delta t) \end{cases} \tag{2.11}$$

The frequency with which these events arrive in the time interval $[t, t + \Delta t)$ is described by a number called the intensity. The intensity is denoted $\lambda(t)$ and can be deterministic or stochastic. Since the probability of two events arriving simultaneously is zero by assumption in this framework, events arrive sequentially and the probability of more than one event arriving in a infinitessimal increment $[t, t + \Delta t)$ is zero. Intensity models assume that the probability of the arrival of such events is proportional to the intensity which is quite natural. Default for us will be an event at a single time. Thus the first jump of $N(t)$ is the event that the entity of interest has defaulted. We define τ the time of default or first jump time of $N(t)$,

$$\tau = \inf\{t \in \mathbb{R}^+ | N(t) > 0\} \tag{2.12}$$

The probability of survival between $[0, T]$, $\mathbf{P}(0, T)$, based on this counting process is then,

$$\mathbf{P}(0,T) = \mathbf{P}(N(T) = 0) \tag{2.13}$$

2.2.1 Poisson Processes

A Poisson process is a counting process in which the probability that N jumps in the next small time interval $[t, t+\Delta t]$ is proportional to Δt,

$$\mathbf{P}[N(t+\Delta t) - N(t) = 1] = \lambda \Delta t \tag{2.14}$$

where λ is a constant "intensity" process.

Also,

$$\mathbf{P}[N(t+\Delta t) - N(t) = n, n > 1] = 0 \tag{2.15}$$

and the events $\{N(k+\Delta t) - N(k) = 1\}$ and $\{N(s+\Delta t) - N(s) = 1\}$ for $[s+\Delta t, s)$ disjoint from $[k+\Delta t, k)$ are independent. In other words, in infinitessimal increments, jumps by more than one do not occur and jumps in disjoint time intervals are independent.

We can imply from the foregoing that the probability of no jump in $[t, t+\Delta t]$ is,

$$\mathbf{P}[N(t+\Delta t) - N(t) = 0] = 1 - \lambda \Delta t \tag{2.16}$$

Suppose we are interested in the probability of survival over $[t, T]$. Subdivide the interval into n equal subintervals of length $\Delta t = (T-t)/n$. By the independence of jump events across disjoint time intervals and the fact that we are only interested in one particular jump, the first jump, each subinterval of time can be viewed as an independent binomial experiment in which the process jumps with probability $\lambda \Delta t$. By independence of these arrival events over disjoint intervals, the probability of survival (no default) over the period $[t, T)$ is,

$$\begin{aligned}
\mathbf{P}[(N(T) - N(t) = 0] &= \lim_{n\to\infty} \mathbf{P}[N(t, t+\Delta t) - N(t) = 0] \cdot \mathbf{P}[N(\Delta t, t+2\Delta t) - N(\Delta t) = 0] \\
&\ldots \cdot \mathbf{P}[N((n-1)\Delta t, T) - N((n-1)\Delta t) = 0] \\
&= \lim_{n\to\infty} (1 - \lambda \Delta t)^n \\
&= \lim_{n\to\infty} (1 - \frac{1}{n}(T-t)\lambda)^n
\end{aligned}$$

Then from the fact that $(1+\frac{x}{n})^n \to \exp(x)$ as $n \to \infty$ we can infer that,

$$\mathbf{P}[N(T) = N(t)] = (1 - \Delta t \lambda)^n \to e^{-\lambda(T-t)} \tag{2.17}$$

as $n \to \infty$.

Poisson processes are designed to model rare events (e.g. natural disasters) and/or discretely countable events (e.g. radioactive decay). Default events are well suited to such models being both rare and discrete. The time of default is the time of the first jump of the Poisson process.

The probability of this first default jump occuring in $[t, T]$ can be deduced. Across our n subexperiments there are n different ways to realise one jump in $[t, T]$ each with probability $\lambda\Delta t(1-\lambda\Delta t)^{n-1}$. Thus,

$$\begin{aligned}
\mathbf{P}[N(T)-N(t)=1] &= \lim_{n\to\infty} n\cdot\lambda\Delta t(1-\lambda\Delta t)^{n-1} \\
&= \lim_{n\to\infty} n\cdot\frac{\frac{(T-t)}{n}\lambda(1-\frac{1}{n}(T-t)\lambda)^n}{(1-\frac{1}{n}(T-t)\lambda)} \\
&= \lim_{n\to\infty}\frac{(T-t)}{1-\frac{1}{n}(T-t)\lambda}\lambda(1-\frac{1}{n}(T-t)\lambda)^n \\
&\to \lambda(T-t)e^{-\lambda(T-t)}
\end{aligned}$$

as $n\to\infty$

Credit Spreads from Poisson processes

In a Poisson process model the survival probability of a credit risky name is,

$$\mathbf{P}(0,T)=\mathbf{P}[N(T)=0]=\exp(-\lambda T) \tag{2.18}$$

the hazard rates of default are,

$$\begin{aligned}
H(t,T,T+\Delta t) &= \frac{1}{\Delta t}(\exp(\lambda\Delta t)-1) \\
h(t,T) &= \lambda
\end{aligned}$$

Now we see that in intensity based models the formal interpretation of the intensity is that of the hazard rate.

Indeed, $\tau\sim\exp(\lambda)$, i.e. τ the time of default is exponentially distributed with parameter λ. Recall equation (7),

$$\mathbf{p}[\tau\in[t,t+dt]]=-\frac{\partial}{\partial T}\mathbf{P}(0,T)dt$$

as

$$\begin{aligned}
\mathbf{P}(0,T) &= e^{-\lambda T} \\
\mathbf{p}[\tau\in[t,t+dt]] &= \lambda e^{-\lambda t}dt
\end{aligned}$$

This implies that $\lambda\tau\sim\exp(1)$ i.e. $\lambda\tau$ is a standard exponential random variable (parameter 1) [[2], p.759].

$$\mathbf{P}(\tau>t)=\mathbf{P}(\lambda\tau>\lambda t)=\exp(-\lambda t) \tag{2.19}$$

Note that H and h are independent of both t and T. This implies that for constant intensity λ, the term structure of credit spreads is a flat line, neither changing in t nor T. Thus in this model the local probability of default is the same regardless of a credit instrument's maturity or current time. This is elegantly simple but not very realistic. Especially from our point of view of our requirements. In this model the volatility of an entity's credit spread, i.e. credit risk, is zero and the correlation of its credit riskiness as compared to the credit risk of other names is also zero. This model is therefore useless for modelling the effects of volatility and correlation of credit spreads on price adjustments. It is however the basis of more realistic models.

2.2.2 Inhomogeneous Poisson processes

In order to achieve a more realistic credit spread curve we allow the default intensity and therefore hazard rate to change in time. A Poisson process with time dependent intensity, $\lambda(t)$ is logically known as a time inhomogeneous Poisson process. Now the local probability of a jump is,

$$\mathbf{P}[N(t+\Delta t) - N(t) = 1] = \lambda(t)\Delta t \tag{2.20}$$

Using the fact that $\ln(1-x) = -x + O(x^2)$ for small x we can deduce the probability of no jump in $[t, T]$,

$$\begin{aligned} \mathbf{P}[N(T) - N(t) = 0] &= \lim_{n\to\infty} \Pi_{i=1}^{n}(1 - \lambda(t+i\Delta t))\Delta t) \\ \Rightarrow \ln \mathbf{P}[N(T) - N(t) = 0] &= \lim_{n\to\infty} \sum_{i=1}^{n} \ln(1 - \lambda(t+i\Delta t))\Delta t) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} -\lambda(t+i\Delta t)\Delta t \\ &\to -\int_t^T \lambda(s)ds \end{aligned}$$

as $\Delta t \to 0$

Thus,

$$\mathbf{P}[N(T) - N(t) = 0] = e^{-\int_t^T \lambda(s)ds} \tag{2.21}$$

And then akin to the time homogeneous case, for the inhomogeneous Poisson process, the density of the first jump time given no jump prior to t is,

$$\mathbf{p}[\tau \in [t, t+dt]] = \lambda(t)e^{-\int_0^T \lambda(s)ds}dt \tag{2.22}$$

and the discrete and continuous hazard rates of default are,

$$H(t, T, T+\Delta t) = \frac{1}{\Delta t}(e^{\int_T^{T+\Delta t} \lambda(s)ds} - 1) \tag{2.23}$$

$$h(t, T) = \lambda(T) \tag{2.24}$$

Define the cumulative intensity, $\Gamma(t) := \int_0^t \lambda(u)du$ also known as the Hazard function. If M_t is a standard Poisson process (intensity 1) then a time inhomogeneous Poisson process N_t with intensity $\lambda(t)$ can be defined as,

$$N_t = M_{\Gamma(t)}$$

[[2], p.761] Thus a time inhomogeneous Poisson process is a time changed standard Poisson process. Thus if N_t first jumps at τ, M_t first jumps at $\Gamma(\tau)$. But the first jump time of M_t is a standard exponential random variable. Thus, $\Gamma(\tau) =: \xi \sim \exp(1)$. This implies that, $\tau = \Gamma^{-1}(\xi)$. Therefore,

$$\mathbf{P}(\tau > t) = \mathbf{P}(\Gamma(\tau) > \Gamma(t))$$
$$= \mathbf{P}(\xi > \Gamma(t)) = \exp(-\Gamma(t))$$

This suggests that we can simulate the default time for an inhomogeneous Poisson process. Generate a unit uniform random U then,

$$\tau = \inf\{t > 0 : \exp(-\Gamma(t)) < U\}$$

This follows from the fact that $e^{-\Gamma(\tau)}$ is the cumulative distribution function of an exponential random variable until $\Gamma(\tau)$ and that cumulative distribution functions are themselves uniformly distributed. Thus we have the following equivalence amongst events,

$$\{\tau < t\} = \{\Gamma(\tau) < \Gamma(t)\} = \{-\Gamma(t) < -\Gamma(\tau)\}$$
$$= \{e^{-\Gamma(t)} < e^{-\Gamma(\tau)}\} = \{e^{-\Gamma(t)} < U\}$$

This is an informal proof of existence as by simulating the default time we have effectively constructed an inhomogeneous Poisson process [[2], p761-5].

Assuming the time of default is the first jump of an inhomogeneous Poisson process, the price of a defaultable ZCB, can be deduced.

$$\begin{aligned}\overline{P}(0,T) &= \mathbb{E}^{\mathbb{Q}}[\exp(-\int_0^T r(s)ds)\mathbf{1}_{\{N(T)=0\}}] \\ &= \mathbb{E}^{\mathbb{Q}}[\exp(-\int_0^T r(s)ds)\mathbf{1}_{\{\tau>T\}}]\end{aligned}$$

Assuming that the default free spot rate implied by the default free ZCB price is independent of the time of default, the price of a defaultable ZCB is,

$$\begin{aligned}\overline{P}(0,T) &= \mathbb{E}^{\mathbb{Q}}[\exp(-\int_0^T r(s)ds)]\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{\{\tau>T\}}] \\ &= P(0,T)\mathbb{Q}(\tau > T) \\ &= \mathbb{E}^{\mathbb{Q}}[\exp(-\int_0^T r(s) + \lambda(s)ds)]\end{aligned}$$

For inhomogeneous Poisson Process models, the hazard rates now do depend upon the time horizon T, $\lambda(T)$, so here the term structure of credit spreads is not flat but a function of the horizon time or maturity of the contract. This in theory can be used to directly mimic any observed term structure of credit spreads and hazard rates. However, the credit spread is deterministic and thus by implication we assume that no matter our vantage point in time t we expect the credit spread to evolve in an identical fashion for all such t. Again useless for our purposes with volatility of and correlation between credit spreads intransigently $= 0$.

2.2.3 Stochastic Credit Spreads

As described in the previous section, default times modelled as the first jumps of inhomogeneous Poisson processes can be designed to give an exact fit to the term structure of defaultable ZCB prices, given the flexibility afforded by a time dependent intensity/hazard rate function. Recall,

$$\overline{P}(0,T) = \mathbb{E}^{\mathbb{Q}}[\exp(-\int_0^T r(s) + \lambda(s)ds)]$$

Thus the additional spread or risk premia of a defaultable ZCB over a default free ZCB is,

$$\frac{1}{T-t}\int_t^T \lambda(s)ds \tag{2.25}$$

The value of this spread for all future $\bar{t} > t$ is determined by the function $\lambda(s)$ defined when the model is designed. For practical reasons $\lambda(s)$ is chosen to be a continuous function. Thus the spread just defined will be a continuous function of both t and T. In real world markets however the credit spreads observed are not continuous but highly irregular. For this reason a more dynamic model than time inhomogeneous Poisson processes is needed to achieve this additional level of realism. In the context of investigating CVA an additional level of realism is vital as we wish to capture the dependence of CVA on this irregularity in credit spread and we want to capture the effect of credit spread volatility and the effect of correlation between such volatile spreads on price. Without capturing this in our model we would not capture the risk of such volatile movements. For such a dynamic credit spread we need a stochastic process, we thus introduce a generalisation to inhomogeneous Poisson processes, Cox processes.

2.2.4 Cox Processes

Cox processes are Poisson processes with a stochastic intensity function. Recall,

$$\mathbf{P}[N(t+dt) - N(t) = 1] = \lambda(t)dt$$

now we wish to model $\lambda(t)$ as a stochastic process. Assume $\lambda(t)$ is a diffusion process,

$$d\lambda(t) = \mu(t)dt + \sigma(t)dW(t)$$

with $\mu(t)$ and $\sigma(t)$ such that $\lambda(t) > 0$ for $\forall t$. dW is a standard Brownian Motion. Now $\lambda(t)$ has a volatility and $\lambda_1(t)$ and $\lambda_2(t)$ for two different names can be correlated and for a given realisation of $\lambda(t)$, $N(t)$ becomes an inhomogeneous Poisson process but with an intensity $\lambda(t)$ a lot less regular.

In order to introduce Cox processes formally we need to define a partition of the market filtration, $(\mathcal{M}_t)_{t\geq 0}$, the filtration containing all the economically relevant information regarding the state the world. This partition divides the market filtration into a subfiltration containing all default free market information $(\mathcal{G}_t)_{(t\geq 0)}$ and a subfiltration containing all default event information $\mathcal{F}_{(t\geq 0)}$. With the market filtration $(\mathcal{M}_t)_{(t\geq 0)} = (\mathcal{G}_t)_{(t\geq 0)} \cap (\mathcal{F}_t)_{(t\geq 0)}$. In this way we isolate the information obtained from the behaviour of $N(t)$ the Poisson process that counts default events from that of all other market observable and model dependent stochastic processes, e.g. $r(t)$, and particularly, $\lambda(t)$. In other words the causes of an entities credit riskiness dictating its behaviour and thus the information filtration $(\mathcal{G}_t)_{(t\geq 0)}$ are represented by and encapsulated in $\lambda(t)$. This is isolated from the actual default event registered by $N(t)$ and its filtration $(\mathcal{F}_t)_{(t\geq 0)}$. This leads to a stochastic yet analytically and theoretically tractable model of the effect of default events. Note that default events are exogenously determined by $(\mathcal{F}_t)_{(t\geq 0)}$ and thus causes are given no detail by the model.

A process $N(t)$ is a Cox process if $N(t)|\mathcal{G}_t$ i.e. $N(t)$ conditional on the default free economic information for all times s such that $s \in [0, t]$, $\mathcal{G}_t$, is an inhomogeneous Poisson process. Again it can be seen that we have clearly modelled the process of cause and effect as economic information $\mathcal{G}_t$ can only cause a default or jump in $N(t)$ via random realisations of $\lambda(t)$. This is a variant of the efficient markets hypothesis as it assumes that all information relevant to the default of a credit risky entity is encapsulated in $\lambda(t)$ an indirect market observable which is continuously updated. It is a convenient construction for pricing as will be detailed.

Define a probability space $(\Omega, (\mathcal{M}_t)_{(t\geq 0)}, \mathbb{Q})$ where $\mathbb{Q}$ is the pricing measure (not necessarily risk neutral) and $(\mathcal{M}_t)_{(t\geq 0)}$ is as defined. Akin to the case of homogeneous and inhomogeneous Poisson processes, and since $N(t)|\mathcal{G}_t$ is an inhomogeneous Poisson process,

$$\mathbf{P}(N(t + dt) - N(t) = 1|\mathcal{G}) = \lambda(t)dt \tag{2.26}$$

The probability of survival in a Cox process model is,

$$\begin{aligned}\mathbf{P}(0,T) &= \mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{\tau>T}] \\ &= \mathbb{E}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}}\mathbf{1}_{\tau>T}|\mathcal{G}_T]] \\ &= \mathbb{E}^{\mathbb{Q}}[\mathbf{P}(N(T)=0|\mathcal{G}_T)] \\ &= \mathbb{E}^{\mathbb{Q}}[\exp(-\int_0^T \lambda(s)ds)]\end{aligned}$$

The probability density of τ the time of the first default is,

$$\begin{aligned}\mathbf{p}[\tau\in[t,t+dt]] &= \frac{\partial}{\partial T}\mathbf{P}(0,T)dt \\ &= \frac{\partial}{\partial T}\mathbb{E}^{\mathbb{Q}}[\exp(-\int_0^T \lambda(s)ds)]dt \\ &= \mathbb{E}^{\mathbb{Q}}[\frac{\partial}{\partial T}\exp(-\int_0^T \lambda(s)ds)]dt \\ &= \mathbb{E}^{\mathbb{Q}}[\lambda(t)\exp(-\int_0^t \lambda(s)ds)]dt\end{aligned}$$

As in the case of inhomogeneous Poisson processes we define the random variable $\Lambda(t)=\int_0^t \lambda(u)du$ the cumulated intensity process or Hazard process. The first jump of a Cox process can be simulated and thus a Cox process constructed as follows. Conditional on $\mathcal{G}_\tau$,

$$\Lambda(\tau)=\xi$$

where $\xi\sim\exp(1)$. The default time is therefore defined as,

$$\tau := \Gamma^{-1}(\xi)$$

and thus can be simulated by drawing a unit uniform random variable and setting τ as,

$$\tau=\inf\{t>0:\exp(-\Gamma(t))<U\}$$

Thus the existence of Cox processes are informally confirmed [[2], p.763].

The preceding demonstrates the analytical tractability this divided filtration affords in a model of default sensitive quantities. By conditioning on the default free information filtration the problem becomes that of a model with inhomogeneous Poisson process which we know how to handle.

The price of a defaultable ZCB can now be derived as follows,

$$\begin{aligned}\overline{P}(0,T) &= \mathbb{E}_{\mathbb{Q}}[\exp(-\int_0^T r(s)ds)\cdot\mathbf{1}_{\{\tau>T\}}] \\ &= \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[\exp(-\int_0^T r(s)ds)\cdot\mathbf{1}_{\{\tau>T\}}|\mathcal{G}]]\end{aligned}$$

Since r is $\mathcal{G}$-measurable,

$$\begin{aligned}\overline{P}(O,T) &= \mathbb{E}_{\mathbb{Q}}[\exp(-\int_0^T r(s)ds)\mathbb{E}_{\mathbb{Q}}[\mathbf{1}_{\{\tau>T\}}|\mathcal{G}]] \\ &= \mathbb{E}_{\mathbb{Q}}[\exp(-\int_0^T r(s)+\lambda(s)ds)]\end{aligned}$$

and since r and λ may be correlated,

$$\overline{P}(0,T) = \mathbb{E}_{\mathbb{Q}}[\exp(-\int_0^T r(s)+\lambda(s)ds)] \neq P(0,T)\mathbb{E}_{\mathbb{Q}}[\exp(-\int_0^T \lambda(s)ds)]$$

2.2.5 Brigo Intensity Model

In this section we develop the Brigo intensity model[6][2][5]. We are interested in the effects of correlation between names and volatility of credit worthiness across names. Brigo's model is designed to capture these features. It assumes a stochastic default intensity for all credit risky names. Earlier approaches to stochastic intensity modelling generated correlation in credit spreads by modelling the noise terms of the stochastic intensity processes as correlated. Hull and White [[13], p.4] argue that this can only achieve a limited range of correlation between the defaults of contractual names. Even with perfect correlation between intensities the default correlation can be very low. To explode this limitation Brigo uses imposes a dependence structure of the default intensities in the form of a Gaussian copula. By imposing a correlation structure on the intensity processes exogenously, i.e. not via the W_i of intensity processes, Brigo is able to achieve much greater degrees of dependence.

Define a probability space $(\Omega, \mathcal{M}, \mathcal{M}_t, \mathbb{Q})$. $\mathcal{M}_t$ is the filtration of all market information i.e. observable market quantities, including credit and defaults. $\mathbb{Q}$ is the risk neutral pricing measure.

Define, name 0 the investor, name 1 the underlying and name 2 the counterparty and their default times, τ_0, τ_1 and τ_2.

The space is endowed with a subfiltration, $\mathcal{F}_t = \mathcal{M}_t/\mathcal{G}_t$, where $\mathcal{F}_t$ is the filtration generated by default event times, i.e. $\mathcal{F}_t = \sigma(\{\tau_0 \leq u\}(\vee\{\tau_1 \leq u\}) \vee \{\tau_2 \leq u\} : u \leq t)$. Definition: Stopped filtration. If τ is an $\mathcal{F}_t$ stopping time, then the stopped filtration $\mathcal{F}_\tau$ is, $\mathcal{F}_\tau = \sigma(\mathcal{F}_t \cup \{t \leq \tau\}, t \geq 0)$.

Define the default intensities of the three names, λ_i for $i = 0, 1, 2$ and the cumulative intensities, $\Lambda_i(t) = \int_0^t \lambda_i(s)ds$.

Brigo assumes for simplicity that r the default free spot rate is deterministic although all results hold under stochastic spot rates independent of default events.

Brigo assumes that the default events are modelled by Cox processes. Thus,

$$\tau_i = \Lambda_1^{-1}(\xi_i), i = 0, 1, 2$$

where ξ_i are standard (unit-mean) exponential random variables. He then defines unit uniform random variables, U_i with $U_i = 1 - \exp(-\xi_i)$.

Finally, impose a dependence structure on τ_0, τ_1 and τ_2 via a trivariate Gaussian copula $C_{\mathbf{R}}$ on U_0, U_1 and U_2,

$$C_{\mathbf{R}}(u_0, u_1, u_2) := \mathbb{Q}(U_0 < u_0, U_1 < u_1, U_2 < u_2) \tag{2.27}$$

$\mathbf{R} = [r_{i,j}]_{i,j=0,1,2}$ is a 3-dimensional correlation matrix that parametrises the trivariate Gaussian distribution.

Brigo employs CIR$^+$+ processes to model the stochastic default intensities of names "0", "1" and "2",

$$\lambda_i(t) = y_i(t) + \phi_i(t, \beta_i)$$

for $i = 0, 1, 2$, $t \geq 0$. ϕ is a deterministic function of time t, parametrised by parameter vector β_i, integrable on closed intervals, with,

$$\phi(0; \beta) = \lambda_0 - y_0$$

The main function of ϕ is to facilitate calibration of model credits spreads to market credit spreads. Brigo assumes that y_i are Cox Ingersoll Ross (CIR) processes,

$$dy_i(t) = \kappa_i(\mu_i - y_i(t))dt + \nu_i\sqrt{y_i(t)}dZ_i(t)$$

where $\kappa_i, \mu_i, \nu_i, y_i(0)$ are positive constants and $\beta_i = (\kappa_i, \mu_i, \nu_i, y_i(0))$. Z_t are $\mathbb{Q}$-BM.

In interest rate modelling the constraint $2\kappa\mu > \nu^2$ is usually assumed in order to render the origin inaccessible. However, in credit modelling it limits the CDS implied credit spread volatility that can be generated. Negative intensities are instead avoided by imposing positivity on the shift ϕ. Since λ_i are positive almost everywhere, $\Lambda_i(t) = \int_0^t \lambda_i(s)ds$ are invertible.

Correlation between the credit spreads λ_0, λ_1 and λ_2 only has a limited impact on default time correlation, so for simplicity Z_1 and Z_2 or Z_0, Z_1 and Z_2 are assumed to be independent.

2.3 Structural Models

Structural models take a very different approach to intensity models. Structural models take the economic fundamentals of a company, its debt and equity ratio and from this model the likelihood of default. In structural models the likelihood of default is therefore endogeneous, it is supposed that it is derivable from the price of a stock and balance sheet of a company. Here defaults are modelled as the financial capability of a company to repay its debt.

In Merton's original model [14] companies issue debt to finance their activities and this debt has a maturity T, i.e. must be repaid before T. At

T a default event occurs if a name is not able to pay all of its bondholder creditors in full. Thus default can only occur at T and is triggered by the name's value ending below the debt level.

Black and Cox [3] introduced first passage models that elaborate upon this. In first passage models the default time is the first instant the firm's value hits from above a deterministic or stochastic debt barrier. This barrier may be associated with legal safety covenants that force firms into early bankruptcy in the event of critical credit deterioration. This is a more realistic description of default than Merton's default at maturity.

In structural models the firm's value is modelled as a random process similar to those used to describe stocks in equity markets. In fact it is possible to observe this value or obtain it from equity market information, i.e. fundamental analysis. In this case, unlike intensity models, the default process can be completely monitored based on default free market information. Default events are no longer exogenous to the model but are explained by the model in the form of firm values hitting debt barriers. However, structural models in their basic formulations and standard barriers (final level, deterministic barrier) have few parameters in their dynamics and cannot be calibrated exactly to structured data such as CDS quotes with different maturities [[7], p.2]. This differs from intensity based models which model the credit spread directly and thus are easy to calibrate. Hull and White develop a structural model that borrows this feature of intensity based models producing a model that is equally simple to calibrate while retaining the economic intuition of a structural model.

2.3.1 Merton's Model

Merton's model is composed of a firm's value V_t, a diffusion,

$$dV_t = \mu V_t dt + \sigma V_t dW_t$$

with $V(0) = V_0$, where W is a Brownian motion.

Merton's model has a debt maturity, T, a debt face value L, and a possible company default at T if the value of the firm is $V_T < L$.

The debt value at $t < T$ is,

$$\begin{aligned} D_t &= \mathbb{E}^{\mathbb{Q}}[D(t,T)\min(V_T, L)|\mathcal{F}_t] = \mathbb{E}_t[D(t,T)[V_T - (V_T - L)^+]|\mathcal{F}_t] \\ &= \mathbb{E}^{\mathbb{Q}}[D(t,T)[L - (L - V_T)^+]|\mathcal{F}_t] = P(t,T)L - \text{Put}(t,T;V_t,L) \end{aligned}$$

where $\text{Put}(t,T;V_t,L)$ is a put option with underlying V and strike L. $D(t,T)$ is the stochastic discount factor at t with maturity T. Note, $D(t,T) = \frac{B(t)}{B(T)}$ with $B(t) = \exp(\int_0^t r_u du)$ the bank-account numeraire, r the instantaneous short rate. When interest rates are deterministic, $D(t,T) = P(t,T)$, the default free ZCB price at t for maturity T. The value of equity is therefore,

$$S_t = V_t - D_t = V_t - P(t,T)L + \text{Put}(t,T;V_t,L) = \text{Call}(t,T;V_t,L)$$

Thus in Merton's model, equity can be interpreted as a call option on the value of the firm.

2.3.2 Black Cox (BC) Model

Here safety covenants are written into the debt contract. A firm is therefore obliged to repay its debt at the moment V_t hits a safety barrier H_t. Designing H_t is non-trivial. As with Merton assume a firm value with dynamics,

$$dV_t = V_t\mu_t dt + V_t\sigma_t dW(t)$$

with $V(0) = V_0$ and a debt face value of L at maturity T.

A candidate for the safety barrier is the discounted face value, $P(t,T)L$. However, a contract may be designed to allow the firm to recover between t and T and thus a lower safety barrier may be chosen. Black and Cox assume a time dependent safety barrier, $H(t)$.

In intensity based models the probability of default in an infinissimal interval $t+\Delta t$ of time from now t was proportional to Δt. In other words default in the next infinissimal time interval always retains a degree of stochasticity no matter how tiny Δt. Thus intensity based models are unpredictible. In structural models where a diffusion is used to model the firm's value the probability of default in an infinitessimal time interval from now is much smaller than order Δt. In fact, $\frac{1}{\Delta t}(1 - P(t, t+\Delta t)) \to 0$ as $\Delta t \to 0$. This is true of all diffusion based models. The reason for this is the predictability of continuous diffusion processes that cannot jump. If we start away from the default barrier then due to this continuity it is impossible for the process to reach it in the next infinitessimal time increment. Thus the survival probabilities tends to one very fast. This leads to defaults in the next infinitessimal time that are predictable. This is not satisfactory as it leads to credit spreads curves that vanish near zero. This is not observed in markets.

This shortfall of diffusion based structural models leads to jump diffusion based structural models which then gain the unpredictability of intensity based models by adding a Poisson process. With these models more realistic default probabilities over short time intervals are achieved.

2.3.3 Hull and White model

Hull and White [13] develop a structural model where the default probabilities are consistent with those implied by bond prices and CDS spreads and where the joint probability of default across a large number of names can be sampled from a multivariate normal distribution. Thus facilitating the simplicity of calibration usually associated with intensity models.

They assume a risk neutral probability of default in $[t, t+\Delta t]$ of the form $q(t)\Delta t$ as seen from time 0. Recall the hazard rate $h(t)$ is a function such that $h(t)\Delta t$ is the probability of default in $[t, t+\Delta t]$ as seen from time t. As shown (2.22) $q(t)$ and $h(t)$ are related by,

$$q(t) = h(t)e^{-\int_0^t h(u)du}$$

Hull and White assume that these have been implied from bond prices or CDS spreads.

Assume a random variable $X_i(t)$ that describes the credit quality at time t of name i, for $i = 1, 2, 3...N$. Hull and White call this the "credit index" of name i. This can be a number of measures of credit quality assume here that it is the value of a firms assets as in the usual structural models introduced thus far. Alternatively it could be some function of a continuous credit rating or something else.

Assume, $dX_i(t) = dW(t)$ with $W(t)$ a $\mathbb{Q}$-Brownian Motion and $X_i(0) = 0$. Hull and White argue that these credit indices can be assumed to be standard Brownian Motion without loss of generality as any measure of credit quality can be transformed into a standard Brownian Motion by some function [13].

A barrier is chosen such that the first passage time probability distribution is equivalent to the default probability density, $q(t)\Delta t$. To achieve this $q(t)\Delta t$ is discretised such that default can only occur at fixed times t_j, $j = 1, 2, 3, ..., n$. Define $t_0 = 0$ and,

$$\delta_i = t_i - t_{i-1}$$

for $1 \leq i \leq n$.

Also define the following quantities,q_{ji} the risk-neutral probability of default by name i at time t_j, K_{ji} the value of the default barrier for name i at time t_j and $f_{ji}(x)\Delta x$ the probability that $X_i(t_j)$ is in $[x, x+\Delta x]$ with no default prior to t_j.

This implies that the cumulative probability of name i defaulting by time t_j is

$$1 - \int_{K_{ji}}^{\infty} f_{ji}(x)dx$$

Both K_{ji} and $f_{ji}(x)$ can be found by induction on the risk-neutral default probabilities q_{ji}. In this discrete setting $X_i(t_1) \sim N(0, \delta_1)$ where $\delta_1 = t_1$. Hence,

$$f_{1i}(x) = \frac{1}{\sqrt{2\pi\delta_1}} \exp[-\frac{x^2}{2\delta_1}]$$

and

$$q_{1i} = \Phi(\frac{K_{1i}}{\sqrt{\delta_1}})$$

This implies,

$$K_{1i} = \sqrt{\delta_1}\Phi^{-1}(q_{1i})$$

Then for $j \in \{2, 3, ..., n\}$, calculate K_{ji} using the relationship,

$$q_{ji} = \int_{K_{j-1,i}}^{\infty} f_{j-1,i}(u)\Phi(\frac{K_{ji} - u)}{\sqrt{\delta_j}})du$$

This can be solved for K_{ji} using an iterative procedure. Now we can find $f_{ji}(x)$ for $x > K_{ji}$ via,

$$f_{ji}(x) = \int_{K_{j-1,i}}^{\infty} f_{j-1,i}(u)\frac{1}{\sqrt{2\pi\delta_i}}\exp[-\frac{(x-u)^2}{2\delta_i}]du$$

This is solved numerically.

Chapter 3

CVA

Now that we have developed the basic credit risk modelling approaches we turn to the notion of credit valuation adjustment (CVA). The difference between the price of a contract with default risk free counterparties and that with default risky counterparties is the CVA. Unilateral CVA is the adjustment in a valuation due to one of the two parties being considered default risky. From the point of view of the default risk free counterparty entering the financial contract, the positive risk of default before the maturity of the contract in the default risky counterparty leads to the default risk free counterparty charging a risk premium, the unilateral CVA. This risk premia is closely related to the credit spread of the risky counterparty. From the viewpoint of the default riskless counterparty it is always reduces the value. If however we consider the case where both counterparties are default risky, bilateral CVA, this risk premia may change signs depending on the relative riskiness of the two. In developing general calculations for unilateral and bilateral CVA we will find that pricing a financial contract with counterparty risk is the default free price of the contract plus a portfolio of options. We notice that even payoffs whose valuation is model independent therefore may become model dependent due to counterparty risk if the options added cannot be statically hedged. This is so because in order to price options with any degree of accuracy, a model of the volatile behaviour of the underlying is required. It also means by implication that the volatility of and correlation between the credit spreads of credit risky entities becomes crucial.

The accurate calculation of CVA has become of critical importance after the 2007 credit crisis, particularly for credit based OTC instruments like CDS. Pre crisis models of CVA neglected the correlation and volatility in the credit worthiness of contractual counterparties. Post crisis, credit spreads have become increasingly volatile due to the uncertainty from lack of transparency with regard to companies exposure to the crisis. They have also become increasingly correlated due to the concentrated interdependency and complexity inherent in the instruments involved (CDS, CDOs etc...) and the corresponding widespread default risk contagion inherent such instruments. We shall show that the volatility and correlation of credit spreads have a large impact on the level of CVA.

3.1 Wrong-Way-Risk

In order to make models analytically tractable assuming independence between various quantities can dramatically simplify taking $\mathbb{Q}$-expectations of payoffs as then the expectation operator becomes distributative across functions of independent variables. In credit risk the simplifying assumption is often made that the derivative underlying and the counterparty credit risk are independent. This can be a mistake when there is a close relationship between the behaviour of the two and they are highly correlated, or the behaviour of the two changes in a very volatile way. Both scenarios can occur when for example the economy endures large macroeconomic shocks.

"Wrong-way-risk" is the phenomena of default risk of a counterparty to a contract and the value of the underlying of the contract increasing together[16]. This is risky in the wrong way because as the value of underlying increases the counterparty to the contract becomes increasingly likely to default. Thus there is a potential for very large losses.

Examples of "Wrong-way-risk" include: an interest rate swap where the counterparty pays the floating legs and has increasing default risk, i.e. the level of floating payments increase; a CDS in which the default risk of the reference credit is positively correlated with that of the protection buyer; a put option on the stock of the selling counterparty. "Right-way-risk" in comparison is when the value of the underlying decreases with an increase in the credit riskiness of the counterparty.

Wrong way risk has large implications for CVA of CDS because the CVA of a CDS will increase considerably in the event of wrong-way-risk. This is due to fact that the amount lost given default will be at it largest due to the large credit risk of the reference credit just when the default risk of the protection seller is highest due to the positive correlation between them.

The most intractable problem in calculating wrong-way-risk is in predicting and hedging the correlation of the value of the underlying and the counterparty credit risk. According to Stein and Lee calibrating correlation to the market is problematic [18]. Derivatives do not exist that price correlation directly so the only options are historical analysis of correlation in time series data and/or by analysing the behaviour of models as compared to the market. For the latter, it must be assumed that model dependent market observables like volatility smiles are the by-product of correlation.

3.2 Unilateral CVA

In unilateral CVA one name is default free and the other default risky. Let us derive the general pricing formula for unilateral CVA from the point of view of the investor facing counterparty risk. This section is based on Brigo [6]. Denote by T the time horizon of the contract. Denote τ the default time of the credit risky counterparty. If $\tau \leq T$ the counterparty defaults and at τ the Net Present Value (NPV) of the residual payoff between $[\tau, T]$ is evaluated.

If the NPV is negative for the investor, positive for the counterparty, it is paid in full by the investor to the counterparty. If the NPV is positive for the investor, negative for the counterparty, only a recovery fraction REC of the NPV is exchanged.

Here we assume REC to be deterministic. The NPV evaluated at τ is,

$$NPV(\tau) = \mathbb{E}_\tau[\Pi(\tau, T)]$$

where $\Pi(t, T)$ are the net discounted cash flows of the claim in $[t, T]$.

Denote by $\Pi(t, T)$ the discounted payoff of a claim with a default free counterparty as seen from time t and $\Pi^D(t)$ the discounted payoff of the equivalent claim but with default risky counterparty. Then we have,

$$\begin{aligned}\Pi^D(t,T) &= \mathbf{1}_{\{\tau>T\}}\Pi(t,T) + \mathbf{1}_{\{t<\tau\leq T\}}[\Pi(t,\tau) \\ &+ D(t,\tau)\text{REC}(\text{NPV}(\tau))^+ - (-\text{NPV}(\tau))^+))]\end{aligned}$$

From this we obtain the following result,

Proposition 3.2.1 (General unilateral counterparty risk pricing formula)**.** *At time t with $t < \tau$ the price of a defaultable claim is,*

$$\mathbb{E}_t[\Pi^D(t,T)] = \mathbb{E}_t[\Pi(t,T)] - LGD\mathbb{E}_t[\mathbf{1}_{\{t<\tau\leq T\}}D(t,T)(NPV(\tau))^+] \tag{3.1}$$

where $LGD = 1 - REC$ is the Loss Given Default with REC deterministic.

Proof

We can write,

$$\Pi(t,T) = \mathbf{1}\{\tau > T\}\Pi(t,T) + \mathbf{1}\{\tau \leq T\}\Pi(t,T)$$

Thus,

$$\begin{aligned}\Pi(t,T) &+ (\text{REC}-1)\mathbf{1}_{\{t<\tau\leq T\}}D(t,T)(\text{NPV}(\tau))^+ \\ &= \mathbf{1}\{\tau > T\}\Pi(t,T) + \mathbf{1}\{\tau \leq T\}\Pi(t,T) \\ &+ \text{REC}\mathbf{1}_{\{t<\tau\leq T\}}D(t,T)(\text{NPV}(\tau))^+ - \mathbf{1}_{\{t<\tau\leq T\}}D(t,T)(\text{NPV}(\tau))^+\end{aligned}$$

Conditional on information up until τ the expectation of the first plus the third term on the right hand side,

$$\mathbf{1}\{\tau \leq T\}\Pi(t,T) - \mathbf{1}_{\{t<\tau\leq T\}}D(t,T)(NPV(\tau))^+$$

is,

$$\mathbb{E}_\tau[\mathbf{1}\{\tau \leq T\}\Pi(t,T) - \mathbf{1}_{\{t<\tau\leq T\}}D(t,T)(NPV(\tau))^+]$$
$$= \mathbb{E}_\tau[\mathbf{1}\{\tau \leq T\}[\Pi(t,T) + D(t,\tau)\pi(\tau,T) - D(t,\tau)(\mathbb{E}_\tau[\Pi(\tau,T)])^+]$$

where $\Pi(t,T) = \Pi(t,T) + D(t,\tau)\Pi(\tau,T)$

$$\begin{aligned} &= \mathbf{1}\{\tau \leq T\}[\Pi(t,\tau) + D(t,\tau)\mathbb{E}_\tau[\Pi(\tau,T) - D(t,\tau)(\mathbb{E}_\tau[\Pi(\tau,T)])^+] \\ &= \mathbf{1}\{\tau \leq T\}\Pi(t,\tau) - D(t,\tau)(\mathbb{E}_\tau[\Pi(\tau,T)])^-] \end{aligned}$$

where $(.)^- = \inf(.,0)$ is the "negative part of"-operator,

$$\begin{aligned} &= \mathbf{1}\{\tau \leq T\}\Pi(t,\tau) - D(t,\tau)(\mathbb{E}_\tau[-\Pi(\tau,T)])^+] \\ &= \mathbf{1}\{\tau \leq T\}\Pi(t,\tau) - D(t,\tau)(-NPV(\tau))^+ \end{aligned}$$

Performing a similar calculation for the other terms and taking the $\mathbb{E}_t$-expectation so that $\mathbb{E}_t[\mathbb{E}_\tau[.]] = \mathbb{E}_t[.]$ gives the result $\square$

From the general formula for unilateral CVA it is noticed that the price of a defaultable claim is the price of a default free claim minus a call option on the residual NPV with zero strike, with non zero payoff only if $\tau < T$.

Thus pricing counterparty risk adds optionality to the contract. Option valuation is model dependent unless one can find a static hedge, thus in the absence of such a hedge the valuation becomes model dependent whether the original valuation of the default free claim was or not.

3.3 Bilateral CVA

In this section we construct an arbitrage free machinery for the calculation of bilateral CVA where both counterparties to a contract are default risky. The objective is to define a general formula for the bilateral CVA that can dynamically change sign as the default risks of the respective counterparties evolve. The following is based largely on, Brigo [5].

Define the two names of the financial contract, name "0" (investor) and name "2" (counterparty). If the instrument exchanged is default risky we define the underlying reference credit as name "1". We denote the default times of the three names, τ_0, (τ_1) and τ_2.

Define T the maturity of the contract and the stopping time $\tau = \min\{\tau_0, \tau_2\}$. If $\tau > T$, then neither investor nor counterparty default over the life of the contract and the contract is completed without contingency. On the contrary, if $\tau < T$ then either the investor or counterparty or both default before T. At τ the net present value of the (NPV) of the residual payoff until maturity is computed.

If $\tau = \tau_2$ and NPV < 0 (resp. NPV > 0) for the investor (resp. defaulted counterparty), it is fully paid (resp. received) by the investor (resp. defaulted counterparty). If NPV > 0 (resp. NPV < 0) for the investor (resp. defaulted counterparty), only a recovery fraction REC_2 of the NPV is exchanged, received (paid) by the investor (defaulted counterparty).

If $\tau = \tau_0$, and $NPV > 0$ (resp. $NPV < 0$) for the investor (defaulted counterparty), it is fully received (paid) by the investor (defaulted counterparty). If $NPV < 0$ (resp. $NPV > 0$) for the investor (defaulted counterparty), only a recovery fraction REC_0 of the NPV is exchanged, paid (received) by the investor (defaulted counterparty).

Here we define all payoffs from the point of view of the investor.

Define $\Pi(t,T)$ the net discounted cash flows of the claim without default between times $[t,T]$.

Define, $\mathrm{NPV}(\tau_i) = \mathbb{E}_{\tau_i}\{\Pi(\tau_i,T)\}$, $i = 0,\ 2$, the residual discounted Net Present Value given default by name "i" at $t = \tau_i$.

Denote $D(t,T)$ the price of a default free ZCB with maturity T.

Again, denote $\Pi(t,T)$ the discounted payoff for an equivalent claim with default-free names and $\Pi^D(t,T)$ the value of the same contract with default risky names.

Then the general payoff under bilateral counterparty default risk can be defined,

$$\Pi^D(t,T) = 1_{\{T\le \min\{\tau_0,\tau_2\}\}}\Pi(t,T)$$
$$+1_{\{\tau_0\le \min\{\tau_2,T\}\}}[\Pi(t,\tau_2) + D(t,\tau_2)(\mathrm{REC}_2(\mathrm{NPV}(\tau_2))^+ - (-\mathrm{NPV}(\tau_2))^+)]$$
$$+1_{\{\tau_2\le \min\{\tau_0,T\}\}}[\Pi(t,\tau_0) + D(t,\tau_0)((\mathrm{NPV}(\tau_0))^+ - \mathrm{REC}_0(-\mathrm{NPV}(\tau_0))^+)]$$

This leads to the following result,

Proposition 3.3.1 (General bilateral counterparty risk pricing formula)**.** *At time t conditional on $\{\tau > t\}$, the price of a generic payoff under bilateral counterparty risk is,*

$$\begin{aligned}\mathbb{E}_t[\Pi^D(t,T)] &= \mathbb{E}_t[\Pi(t,T)] \\ &+ \mathbb{E}_t[LGD_0 1_{\{\tau_2\le min\{\tau_0,T\}\}} D(t,\tau_0)[-NPV(\tau_0)]^+] \\ &- \mathbb{E}_t[LGD_2 1_{\{\tau_0\le min\{\tau_2,T\}\}} D(t,\tau_0)[NPV(\tau_2)]^+]\end{aligned}$$

where $LGD_i = 1 - REC_i$ is the Loss Given Default of name i.

The proof is similar to that of the general unilateral counterparty risk pricing formula and is included in an appendix 6.0.2.

Here, the recovery fraction REC_i can be stochastic and correlated with the default indicator process. The value of a defaultable claim is the value of the corresponding default free claim plus a long position in a put option (zero strike) on the residual NPV given the investor is earliest to default plus a short position in a call option (strike zero) on the residual NPV given the counterparty is earliest to default.

The adjustment is called bilateral counterparty risk credit valuation adjustment (BCVA-I) and is the value of CVA as seen from the investors viewpoint,

$$\begin{aligned} \text{BCVA-I} &= \mathbb{E}_t[\text{LGD}_0 1_{\{\tau_2 \le \min\{\tau_0,T\}\}} D(t,\tau_0)[-NPV(\tau_0)]^+\} \\ &- \mathbb{E}_t\{\text{LGD}_2 1_{\{\tau_0 \le \min\{\tau_2,T\}\}} D(t,\tau_0)[NPV(\tau_2)]^+] \end{aligned}$$

It may be positive or negative depending on whether the counterparty is more or less likely to default than the investor. Hence, its sign will depend upon the relative volatilities of and correlation between the default times, τ_i, $i = 0{,}2$.

Bilateral CVA has the advantage over unilateral CVA of being symmetric, i.e. the CVA calculation of the counterparty is minus the CVA of the investor, BCVA-C = −BCVA-I. On the other hand, if either name calculated the CVA assuming themselves to be default-free then the adjustments would for the investor name "0" be the unilateral CVA,

$$\mathbb{E}_t\{\text{LGD}_2 1_{\tau_2 < T} D(t,\tau_2)[NPV(\tau_2)]^+\}$$

and for the counterparty name "2",

$$\mathbb{E}_t\{\text{LGD}_0 1_{\tau_0 < T} D(t,\tau_0)[-NPV(\tau_0)]^+\}$$

Only in the case of bilateral CVA do the two parties agree on the CVA.

This bestows another advantage on bilateral CVA, it can change sign. CVA with a default-free name is always positive and subtracted from the point of view of that name. The bilateral CVA can change sign over time depending on the relative credit quality of the names. Thus bilateral CVA can explain market phenomena, that unilateral CVA leaves paradoxical.

For example, Citigroup in a press release on the last quarter revenues of 2009 reported a positive mark-to-market due its worsening credit quality: "Revenues also included [...] a net 2.5USD billion positive CVA on derivative positions, excluding monolines, mainly due to a widening of Citi's CDS spreads" [5]. So CVA on Citigroup positions increased due to a decline in its credit quality, "a widening of its CDS spreads". This is known as the DVA puzzle [1]. Where DVA or "default valuation adjustment" is what we have been calling the bilateral CVA as calculated by the previously risk-free counterparty.

The puzzle questions how a company can profit from its credit worthiness declining. The bilateral CVA formula of Brigo solves the puzzle. Assuming the CVA to a deal is agreed at inception, if the credit quality of name "0" subsequently declines then name "0" registers a gain and name "2" the counterparty registers a loss. If the deal was re-evaluated at that point in time then it would be calculated that name "0" had managed to enter a contract too cheaply and name 2 at a price that was too high. Hence name

"0" gains a positive mark-to-market. This use of the term mark-to-market seems only to make sense in accounting terms, it is not clear that it means that name "0" could actually realise the gain in the market. In order for this name "0" would have to enter the opposite position with the same counterparty for the adjusted value of the contract or of course default.

Apart from the DVA puzzle, Bilateral CVA poses problems for conventional risk neutral pricing theory. Gregory points out that the measure used in taking expectations of bilateral CVA is a subtle question [12]. He objects to the use of risk-neutral pricing in bilateral CVA. A unilateral CVA term can be hedged and thus priced by buying CDS protection on the protection seller. If the credit riskiness of the protection seller goes up the unilateral CVA *deducted* must go up thus making the contract lose value for the protection buyer. However the value of CDS protection bought on the credit risk of the protection seller as a hedge would also go up thus hedging this reduction. Likewise if the credit riskiness of the protection seller went down the CVA deducted would decrease and the price of the contract would rise. Hedging this the value of the CDS protection bought on the protection seller would fall because now the protection seller is less likely to default. This completes the hedge.

The problem with bilateral CVA is that the additional term adjusting for the credit risk of the protection buyer would require the protection buyer to sell CDS protection on its own debt. If its credit risk went up the CVA deducted would decrease and the value of the contract to the protection buyer would rise. In this case the value of the CDS protection sold on its own credit risk would decline in value thus hedging the upward move of the original contract. Likewise if its credit quality went down the CVA deducted would increase and the value of the contract would decrease. Now the value of the CDS protection sold on itself would rise as they would be less likely to have to payout. This completes the hedge.

However, finding a buyer for the seller of CDS protection with its own debt as underlying is not very likely. Who would buy protection from an insurance company on the event of that same insurance company defaulting. This would involve paying for perfect wrong-way-risk. Positive correlation between the protection seller and the reference credit of exactly one. In some sort of perfect market this should not matter to the buyer if properly corrected for via unilateral or bilateral CVA. But as markets are not perfect a buyer for protection against the credit risk of an entity would find insurance sold by parties other than that entity more competitive than the entity themselves. This also assumes that financial regulation would allow such a contract even over the counter.

If this is not possible it may be possible for a protection buyer to sell protection on debt whose default risk is highly correlated with its own. Gregory argues that a CVA should be added only according to the hedgeable components of the default risk involved. He argues in the extreme case when the component involving an institutions own credit risk cannot be hedged bilateral CVA should equal unilateral CVA.

This also raises the question of whether the pricing measures used by the two counterparties can be assumed to be the same. If their ability to hedge their respective positions is different.

In one sense the ability of the two counterparties to hedge the two components of the bilateral CVA faced is identical but opposite. Since in order to hedge they both have to buy CDS protection on the others credit risk and sell CDS protection on their own.

In another sense it differs as buying CDS protection might be more accurately priced on one name than the other when buying protection on ones opposite. And the two names respective ability to find market substitutes for selling protection on their own debt could differ considerably. This implies that the market used to hedge are incomplete in different ways depending on counterparty. Even if markets were identically incomplete for both they may choose different measures based on their idiosyncratic preferences.

These considerations would lead to the two counterparties using different pricing measures to calculate the same bilateral CVA. This undermines the ability of bilateral CVA to gain agreement between counterparties on the correct price and destroys its symmetry.

As mentioned Gregory suggests that in the extreme case when the component of the bilateral CVA involving an institution's own credit risk cannot be hedged, the value of this component should be set equal to zero thus obtaining the original unilateral CVA. Is this satisfactory? If both counterparties are credit risky which counterparty is going to be willing to accept a price that does not take into account the others credit risk but does its own.

Morini Prampolini ([15], footnote 1, p. 11) remark that a bank can buy back its own bonds which is equivalent they claim to selling insurance on ones own debt. This they argue is different from a company selling CDS protection on its own debt because it is "fully funded", i.e. the company's own credit risk is not a factor. This they assert means that it avoids the restrictions imposed on the selling by a company of CDS protection on itself where its own credit risk is a factor. For banks the selling of their own bonds is a standard and important activity. What is more assuming that banks have a sufficient number of bonds outstanding the implementation of such a strategy is feasible. Although this may entail difficulties that differ from bank to bank. They argue that the additional term in bilateral CVA over unilateral CVA is not a "default benefit moral hazard" but part of the fair value leading to fair value marking-to-market.

Chapter 4

Credit Default Swap (CDS)

In this section we formally introduce CDS contracts and their payoff. A single name Credit Default Swap (CDS) is an insurance contract on the default of a single reference credit between a protection buyer (investor) and a protection seller (counterparty). In contemporary CDS contracts neither the protection seller nor protection buyer are obliged to have investments in the underlying reference credit. In the event of reference credit default the protection seller pays the protection buyer a default payment. It is designed to reimburse the loss a physical investor would incure upon the default of the reference credit. If no default occurs before the expiration of the contract the protection seller pays nothing. The protection buyer pays an insurance premium for this protection, a fee at fixed intervals until default of the reference credit or maturity.

A CDS is designed to isolate the reference credit default risk from other risks like market risk. This creates a market in which the credit risk of the reference credit can be effectively and efficiently priced and hedged. A protection buyer with exposure to the reference credit retains the market risk but is hedged by the CDS against the default risk. The protection seller only has exposure to default risk. By specifying the set of reference credits the protection buyer and seller can cover the buyers default risk completely.

Despite increased standardisation and use of standard ISDA specifications, CDS have the advantage of flexiblity of default definition. The protection buyer and seller can agree to a definition that suits them.

The CDS spread is the rate paid by the protection buyer to the protection seller. The actual amount paid is the spread multiplied by the notional. Typical term structures for payments are quarterly or semi-annually. If default occurs in the period between payment dates, the interpolated accrued fee up to the time of default is paid.

CDS have become basic credit products in the same vein as interest rate swaps (IRS) and forward rate agreements (FRA) in the fixed income world. Mathematical modelling is no longer needed to price the most liquid CDS but rather to calibrate models for the purpose of pricing and hedging more complex credit derivatives. However, this does not hold true for OTC CDS as these do not constitute liquidly traded assets. It is in this context that we are concerned with counterparty risk adjustments.

In more formal mathematical terms we define a CDS contract [2], between three names, the protection buyer, name "0", the reference credit, name "1" and the protection seller, name "2". If name "1" defaults at time $\tau = \tau_1$ with $T_a < \tau < T_b$, "2" pays to "0" a certain deterministic or stochastic amount $LGD = LGD_1$ (Loss Given Default of the reference credit "1") of the notional. This is known as the protection leg.

In return, "0" pays "2" a rate S at times, T_{a+1},...,T_b or until default τ_1. This is known as the premium leg.

The spread S is the evaluated at the inception of the contract and is the rate that makes the contract fair, i.e. the rate at which the present value of the contracted cashflows from the protection and the premium leg are equal. CDS are quoted via "S".

The discounted payoff of a CDS from the view point of the protection buyer can be thus formalised,

$$\Pi(t) := \mathbf{1}_{T_a<\tau_1<T_b} D(t,\tau_1)\mathrm{LGD}_1$$

$$-D(t,\tau_1)(\tau_1 - T_{\beta(\tau_1)-1})S\mathbf{1}_{T_a<\tau_1<T_b} - \sum_{i=a+1}^{b} D(t,T_i)\alpha_i S\mathbf{1}_{\tau_1\geq T_i}$$

where $t \in [T_{\beta(t)-1}, T_{\beta(t)})$, $\beta(t)$ is the first premium payment date after t and α_i is the year fraction between T_i and T_{i-1}. Thus the first term in (37) is the interpolated accrued protection fee up until default time τ_1 after the last payment date.

The price of a CDS is then computed via risk neutral valuation,

$$\begin{aligned}
\mathrm{CDS}_{a,b}(0,S,LGD_1) &= \mathbb{E}^{\mathbb{Q}}[\Pi(0))] \\
&= S[-\int_{T_a}^{T_b} D(0,t)(t - T_{\gamma(t)-1})d\mathbb{Q}_t(\tau_1 > t) \\
&+ \sum_{i=a+1}^{b} \alpha_i D(0,T_i)\mathbb{Q}_t(\tau_1 > T_i)] \\
&+ \mathrm{LGD}_1[\int_{T_a}^{T_b} D(0,t)d\mathbb{Q}_t(\tau_1 > t)]
\end{aligned}$$

4.1 the CVA of CDS

As described a classic example of wrong-way-risk occurs in a CDS contract when the default risk of the reference credit and protection seller become highly positively correlated, as may be the case during macroeconomic shocks particularly those based on credit events. Here the CVA can increase considerably. As seen from the point of view of the protection buyer, the value of the contract is at its highest when the reference credit has a high probability of default as this is when protection is paid but during such macroeconomic shocks the likelihood of the protection seller defaulting may also become concommitantly high. In such an economic climate the credit worthiness of the reference credit and the protection seller become highly positively correlated. In such a scenario, one would wish to make the CVA very large to account for such economic effects. Models that neglect this correlation between the credit riskiness of reference credit and protection seller in a CDS would miss this effect and as a result drastically underprice CVA.

Likewise uncertainty over the credit worthiness of entities as we have witnessed since the recent crisis lead to highly volatile credit spreads that also impact CVA. Volatility in credit spreads has a particularly pronounced effect when one is pricing the CVA of a CDS. In this case the calculation of CVA involves pricing an option on a CDS and volatility has a large impact on the pricing of options.

The first models of CVA were "unilateral". Unilateral CVA adjusts for the credit riskiness of one of the two counterparties to a contract, where one of the counterparties was typically considered default risk free. Pre-crisis this seemed like a reasonable assumption. Intensity based and structural models of unilateral CVA of CDS which incorporate the volatility of and correlation between the credit worthiness of the reference credit and protection seller suffice to capture all these two-way CDS effects, and we will assess them based on this ability.

Lets calculate the model independent unilateral CVA for CDS. Recall, the investor in a CDS is a protection buyer that exchanges a periodic premium S between T_a and T_b for protection against default of a reference credit name 1 between T_a and T_b. The counterparty is a protection seller that in exchange for this periodic premium pays LGD_1 in the event that name 1 defaults in T_a and T_b.

The value of this protection to the protection buyer at time 0 is,

$$\mathrm{CDS}_{a,b}(0,S,LGD_1) = S[-\int_{T_a}^{T_b} D(0,t)(t-T_{\gamma(t)-1})d\mathbb{Q}(\tau_1 > t)$$

$$+\sum_{i=a+1}^{b} \alpha_i D(0,T_i)\mathbb{Q}(\tau_1 > T_i)] + \mathrm{LGD}_1[\int_{T_a}^{T_b} D(0,t)d\mathbb{Q}(\tau_1 > t)]$$

where $\gamma(t)$ is the first payment period T_j following time t.

Denote, $\mathrm{NPV}(T_j,T_b) := \mathrm{CDS}_{a,b}(T_j,S,LGD_1)$, the residual NPV of a receiver CDS (from the point of view of the protection buyer) between T_a and T_b evaluated at time T_j, with $T_a < T_j < T_B$.

This gives,

$$\begin{aligned}
\mathrm{CDS}_{a,b}(T_j, S, LGD_1) &= 1_{\tau_1 > T_j}\{S[-\int_{\max\{T_a,T_j\}}^{T_b} D(0,t)(t - T_{\gamma(t)-1}) d\mathbb{Q}_{T_j}(\tau_1 > t) \\
&+ \sum_{i=\max\{a,j\}+1}^{b} \alpha_i D(0,T_i)\mathbb{Q}_{T_j}(\tau_1 > T_i] \\
&+ \mathrm{LGD}_1[\int_{\max\{T_a,T_j\}}^{T_b} D(0,t) d\mathbb{Q}_{T_j}(\tau_1 > t]\}
\end{aligned}$$

So the model independent unilateral CVA of a CDS is,

$$\mathbb{E}_t[CDS^D(t)] = \mathbb{E}_t[CDS(t)] - LGD_1\mathbb{E}_t[\mathbf{1}_{\{\tau_2 \le T\}} D(t,T)(CDS(\tau_2))^+]$$

The model independent bilateral CVA for CDS can be similarly derived.

$$\begin{gathered}
\mathbb{E}_t[CDS^D(t)] = \mathbb{E}_t[CDS(t)] \\
+\mathbb{E}_t[\mathrm{LGD}_0 1_{\{\tau_2 \le \min\{\tau_0, T\}\}} D(t,\tau_0)[-CDS(\tau_0)]^+] \\
-\mathbb{E}_t[\mathrm{LGD}_2 1_{\{\tau_0 \le \min\{\tau_2, T\}\}} D(t,\tau_0)[CDS(\tau_2))]^+]
\end{gathered} \tag{4.1}$$

4.2 Intensity model of CVA for CDS

In this section we develop the Brigo intensity based model for the evaluation of unilateral and bilateral CVA of a CDS. This has become the market standard.

Recall that in Brigo's intensity model default events are modelled by Cox processes. The default intensities of the three names are λ_i for $i = 0, 1, 2$, where λ_i are independent CIR^+ processes and we have cumulative intensities, $\Lambda_i(t) = \int_0^t \lambda_i(s)ds$. Thus default stopping times are defined by,

$$\tau_i = \Lambda_1^{-1}(\xi_i), i = 0, 1, 2$$

where ξ_i are standard (unit-mean) exponential random variables. Define unit uniform random variables, U_i with $U_i = 1 - \exp(-\xi_i)$. In the case of unilateral CVA, we impose a dependence structure on τ_1 and τ_2 via a bivariate Gaussian copula $C_{\mathbf{R}}$ on U_1 and U_2,

$$C_{\mathbf{R}}(u_1, u_2) := \mathbb{Q}(U_1 < u_1, U_2 < u_2)$$

with $\mathbf{R} = [r_{i,j}]_{i,j=1,2}$ a 2-dimensional correlation matrix that parametrises the bivariate Gaussian distribution. In the case of bilateral CVA, we impose a dependence structure on τ_0, τ_1 and τ_2 via a trivariate Gaussian copula $C_{\mathbf{R}}$ on U_0, U_1 and U_2,

$$C_{\mathbf{R}}(u_0, u_1, u_2) := \mathbb{Q}(U_0 < u_0, U_1 < u_1, U_2 < u_2)$$

$\mathbf{R} = [r_{i,j}]_{i,j=0,1,2}$ is a 3-dimensional correlation matrix that parametrises the trivariate Gaussian distribution.

To simplify computations later, define the integrated quantities,

$$Y_j(t) = \int_0^t y_j(s)ds, \Phi_j(t; \beta_j) = \int_0^t \phi_j(t; \beta_j)ds$$

4.2.1 Unilateral CDS CVA

Recall the general unilateral counterparty risk pricing formula,

$$\mathbb{E}_t[\Pi^D(t,T)] = \mathbb{E}_t[\Pi(t,T)] - LGD\mathbb{E}_t[\mathbf{1}_{\{t<\tau\leq T\}}D(t,T)(NPV(\tau))^+]$$

The following approximation is made to simplify calculations,

$$\begin{aligned}
\mathbb{E}_t[\Pi^D(t,T_b)] &= \mathbb{E}[\Pi(t,T_b)] - LGD\sum_{j=1}^{b}\mathbb{E}_t[\mathbf{1}_{\{T_{j-1}<\tau\leq T_j\}}D(t,\tau_2)(\mathbb{E}_{\tau_2}\Pi(\tau_2,T))^+] \\
&\approx \mathbb{E}[\Pi(t,T_b)] - LGD\sum_{j=1}^{b}\mathbb{E}_t[\mathbf{1}_{\{T_{j-1}<\tau\leq T_j\}}D(t,T_j)(\mathbb{E}_{\tau_2}\Pi(T_j,T_b))^+]
\end{aligned}$$

in the context of the present CDS pricing framework this becomes,

$$\mathbb{E}_t[CDS^D(t)] = \mathbb{E}_t[CDS(t)] - LGD_1\sum_{j=1}^{b}\mathbb{E}_t[\mathbf{1}_{\{T_{j-1}<\tau\leq T_j\}}D(t,T)[(CDS(T_j)]^+]$$

where,

$$\begin{aligned}
\mathrm{CDS}_{a,b}(T_j, S, LGD_1) &= 1_{\tau_1>T_j}\{S[-\int_{\max\{T_a,T_j\}}^{T_b} D(0,t)(t-T_{\gamma(t)-1})d\mathbb{Q}(\tau_1>t|\mathcal{F}_{T_j}) \\
&+ \sum_{i=\max\{a,j\}+1}^{b} \alpha_i D(0,T_i)\mathbb{Q}(\tau_1>T_i|\mathcal{F}_{T_j})] \\
&+ \mathrm{LGD}_1[\int_{\max\{T_a,T_j\}}^{T_b} D(0,t)d\mathbb{Q}(\tau_1>t|\mathcal{F}_{T_j})]\}
\end{aligned}$$

and define,

$$\begin{aligned}\overline{CDS}_{a,b}(T_j, S, LGD_1) &:= \{S[-\int_{\max\{T_a,T_j\}}^{T_b} D(0,t)(t - T_{\gamma(t)-1}) d\mathbb{Q}(\tau_1 > t|\mathcal{F}_{T_j}) \\ &+ \sum_{i=\max\{a,j\}+1}^{b} \alpha_i D(0,T_i)\mathbb{Q}(\tau_1 > T_i|\mathcal{F}_{T_j})] \\ &+ \text{LGD}_1[\int_{\max\{T_a,T_j\}}^{T_b} D(0,t)\end{aligned}$$

$$d\mathbb{Q}(\tau_1 > t|\mathcal{M}_{T_j})]\}$$

The T_j-CVA= $LGD_1\mathbb{E}_t[\mathbf{1}_{\{T_{j-1}<\tau\leq T_j\}}D(0,T_j)(\mathbb{E}_{\tau_2}\Pi(T_j,T_b))^+]$ can be derived by calculating,

$$\begin{aligned}&\mathbb{E}_t[\mathbf{1}_{\{T_{j-1}<\tau\leq T_j\}}[(CDS_{a,b}(T-j,S,\text{LGD}_1))^+] \\ =\ &\mathbb{E}_t[\mathbf{1}_{\{T_{j-1}<\tau\leq T_j\}}\mathbf{1}_{\tau_1>T_j}[(\overline{CDS}_{a,b}(T_j,S,\text{LGD}_1))^+] \\ =\ &\mathbb{E}[\mathbb{E}[\mathbf{1}_{\{T_{j-1}<\tau\leq T_j\}}\mathbf{1}_{\tau_1>T_j}(\overline{CDS}_{a,b}(T_j,S,\text{LGD}_1))^+|\mathcal{F}_{T_j}]] \\ =\ &\mathbb{E}[\mathbb{E}[\mathbf{1}_{\{T_{j-1}<\tau\leq T_j\}}\mathbf{1}_{\tau_1>T_j}|\mathcal{F}_{T_j}]]\mathbb{E}[(\overline{CDS}_{a,b}(T_j,S,\text{LGD}_1))^+]\end{aligned}$$

Notice that,

$$\begin{aligned}\mathbb{E}[\mathbf{1}_{\{T_{j-1}<\tau\leq T_j\}}\mathbf{1}_{\tau_1>T_j}|\mathcal{F}_{T_j}] &= \mathbb{Q}(T_{j-1}<\tau\leq T_j\cap\tau_1>T_j|\mathcal{F}_{T_j}) \\ &+ \mathbb{Q}(T_{j-1}<\tau\leq T_j|\mathcal{F}_{T_j}) - \mathbb{Q}(T_{j-1}<\tau\leq T_j\cap\tau_{1<T_j}|\mathcal{F}_{T_j}) \\ &= e^{-\Lambda_2(T_{j-1})} - e^{-\Lambda(T_j)} \\ &+ C(1-e^{-\Lambda_1(T_j)}, 1-e^{-\Lambda_2(T_{j-1})}) - C(1-e^{-\Lambda_1(T_j)}, 1-e^{-\Lambda_2(T_j)})\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{E}_t[\mathbf{1}_{\{T_{j-1}<\tau\leq T_j\}}(\mathbb{E}_{\tau_2}\Pi(T_j,T_b))^+] &= \mathbb{E}_t[\mathbf{1}_{\{T_{j-1}<\tau\leq T_j\}}(CDS_{a,b}(T_j,S,\text{LGD}_1))^+] \\ &= \mathbb{E}[(\overline{CDS}_{a,b}(T-j,S,\text{LGD}_1))^+] \\ &\cdot\ [e^{-\Lambda_2(T_{j-1})} - e^{-\Lambda(T_j)} \\ &+ [C(1-e^{-\Lambda_1(T_j)}, 1-e^{-\Lambda_2(T_{j-1})}) \\ &- C(1-e^{-\Lambda_1(T_j)}, 1-e^{-\Lambda_2(T_j)})]]\end{aligned}$$

This can be computed by simulating the process λ_1 up to T_j if we knew the formula for $\mathbb{Q}(\tau_1 \geq u|\mathcal{F}_{T_j})$ in terms of $\lambda(T_j)$, $\forall u \geq T_j$. This would be simple under $\mathcal{F}^1_{T_j}$ the default information filtration for name "1" alone. However, we must calculate this under $\mathcal{F}_{T_j}$ including the default information of name "2" as well. In the simpler case, we would find,

$$\mathbb{Q}(\tau_1 \geq u|\mathcal{F}^1_{T_j})$$

$$= \mathbf{1}_{\{\tau_1 > T_j\}} \mathbb{E}[\exp(-\int_{T_j}^{u} \lambda_1(s)ds)|\mathcal{F}^1_{T_j}] \mathbf{1}_{\{\tau_1 > T_j\}} P^{CIR^{++}}(T_j, u; y_1(T_j))$$

$$:= \mathbf{1}_{\{\tau_1 > T_j\}} \exp(-(\phi(u) - \phi(T_j))) P^{CIR}(T_j, u; y_1(T_j))$$

where $P^{CIR^{++}}$ is the defaultable ZCB price formula in the CIR++ model for λ_1 and P^{CIR} is the non-shifted defaultable ZCB price formula for y_1.

However, as discussed we must compute $\mathbb{Q}(\tau_1 \geq u|\mathcal{F}_{T_j})$. This gives,

$$\mathbb{Q}(\tau_1 \geq u|\mathcal{F}_{T_j}) = \mathbb{E}[\mathbf{1}_{\{T_{j-1} < \tau_2 \leq T_j\}} \mathbf{1}_{\{\tau_1\} > u} |\mathcal{F}_{T_j}]$$

$$= \mathbb{E}[\mathbf{1}_{\{T_{j-1} < \tau_2 \leq T_j\}} \mathbf{1}_{\{\tau_1\} > T_j} \mathbf{1}_{\{\tau_1\} > u} |\mathcal{F}_{T_j}]$$

$$= \mathbb{E}[\mathbf{1}_{\{\tau_1\} > u} |\mathcal{F}_{T_j}, T_{j-1} < \tau_2 \leq T_j, \tau_1 > T_j]$$

$$= \mathbb{E}[\mathbf{1}_{\{\tau_1\} > u} |\mathcal{F}_{T_j}, T_{j-1} < \tau_2 \leq T_j, \tau_1 > T_j] = \frac{\mathbb{Q}(\tau_1 > u, T_{j-1} < \tau_2 < T_j|\mathcal{F}_{T_j})}{\mathbb{Q}(\tau_1 > T_j, T_{j-1} < \tau_2 < T_j|\mathcal{F}_{T_j})}$$

$$= \frac{\mathbb{Q}(U_1 > 1 - e^{-\Lambda_1(u)}, 1 - e^{-\Lambda_2(T_{j-1})} < U_2 < 1 - e^{-\Lambda_2(T_j)}|\mathcal{F}_{T_j})}{\mathbb{Q}(U_1 > 1 - e^{-\Lambda_1(T_j)}, 1 - e^{-\Lambda_2(T_{j-1})} < U_2 < 1 - e^{-\Lambda_2(T_j)}|\mathcal{F}_{T_j})} =$$

$$= \frac{e^{-\Lambda_2(T_{j-1})} - e^{-\Lambda_2(T_j)} + \mathbb{E}[C(1 - e^{-\Lambda_1(u)}, e^{-\Lambda_2(T_{j-1})}) - C(1 - e^{-\Lambda_1(u)}, e^{-\Lambda_2(T_{j-1})})]}{e^{-\Lambda_2(T_{j-1})} - e^{-\Lambda_2(T_j)} + C(1 - e^{-\Lambda_1(T_j)}, e^{-\Lambda_2(T_{j-1})}) - C(1 - e^{-\Lambda_1(T_j)}, e^{-\Lambda_2(T_{j-1})})}$$

It can be checked that if the copula is the independence copula, $C(u_1, u_2) = u_1 u_2$ then this expression reduces to the simpler one prior.

Numerical results unilateral CVA of a CDS

Using the calculation of the last section Brigo performed Monte Carlo simulations for the intensity based model for unilateral CVA of a CDS to test the effects of volatility and correlation on its behaviour [6]. He first set the volatility parameter for credit risk intensity of the risk name, the protection seller to $\nu_2 = 0.01$. Thus the credit spread for the protection seller driven by λ_2 is almost deterministic.

As correlation between name "1" the underlying reference credit of the CDS and the protection seller was increased the UCVA-I for the protection buyer increased as one would expect. However, interestingly as the correlation approaches 1 it began to fall. As correlation increased between 0.9 and 0.99 the UR-CVA dropped.

Brigo gives a simple analytic explanation for this. Assume that λ_1 and λ_2 are constant. Make the natural assumption that $\lambda_1 > \lambda_2$ the underlying credit reference is risky than the protection seller insurer. With default correlation 0.99, the default triggers ξ_1 and ξ_2 are almost perfectly correlated, so also assume that $\xi_1 = \xi_2 :\approx \xi$. This implies that,

$$\Gamma_i(t) = \int_0^t \lambda_i ds = \lambda_i t$$

and since,

$$\tau_i = \Gamma^{-1}(\xi)$$

we have, $\tau_1 = \xi/\lambda_1$, $\tau_2 = \xi/\lambda_2$. As $\lambda_1 > \lambda_2$, we know that, $\xi/\lambda_1 < \xi/\lambda_2$, so that $\tau_1 < \tau_2$ almost surely. But in this scenario the UCVA of the CDS is zero since the reference credit always defaults before the protection seller.

Brigo then increases ν_1 the volatility of λ_1 from $\nu_1 = 0.1$. Now $\xi/\lambda_1 < \xi/\lambda_2$ is no longer true almost surely as the increased randomness in λ_1 generates paths on which $\lambda_1 < \lambda_2$ and hence paths on which $\tau_1 > \tau_2$. Thus, as ν_1 is increased the UCVA-I increases above 0. Increasing λ_1 further so that more and more paths are generated on which $\tau_1 > \tau_2$ with a ν_1 high, the UCVA no longer dips after the correlation reaches 0.9 and becomes monotonically increasing.

Brigo also finds that the UCVA-I vanishes for very negative correlations. This happens because here when the protection seller defaults the reference credit does not, so the premium payment for the CDS the CDS spread goes to zero and the CDS becomes worthless.

Brigo then test the unnatural scenario of a protection seller that is riskier than the reference credit. Now λ_2 tends to be greater than λ_1. Here at a correlation of 0.99 and near deterministic intensities, $\tau_1 = \xi/\lambda_1 > \xi/\lambda_2 = \tau_2$ almost surely. So the underlying reference credit always defaults before the protection seller. Now as the reference credit and the protection seller become more and more correlated the UCVA continues to increase.

Finally, Brigo increases the level of the reference credit's credit riskiness $\lambda_1(0)$. In this case, with $\lambda_1 > \lambda_2$ as is natural, as one would expect the reversion toward zero at high correlation of the UCVA happens earlier and earlier as $\lambda_1(0)$ is increased.

The analysis of Brigo clearly shows the importance of volatility of credit spreads for any CVA model hoping to take account of wrong-way-risk. In a copula model with deterministic credit spreads (a standard assumption in industry), Brigo has shown that in ignoring credit spread volatility a scenario is created where the UCVA vanishes as correlation tends to 1. Thus extreme wrong-way-risk is priced at zero, just when it should be at it highest.

4.2.2 Bilateral CVA of CDS

Let us formulate the bilateral credit valuation adjustment BCVA for a CDS. The BCVA-I at time t for a payer CDS, that is from the point of view of the protection buyer the payer of the premium S, is,

$$\text{BCVA-I}_{a,b}(t, S, LGD_{0,1,2})$$

$$= \text{LGD}_2 \mathbb{E}_t\{1_{\{\tau_2 < \tau_0\}} D(t, \tau_2)[1_{\tau_1 > \tau_2} \overline{CDS}_{a,b}(\tau_2, S, LGD_1)]^+\}$$

$$-\text{LGD}_0 \mathbb{E}_t\{1_{\{\tau_0 < \tau_2\}} D(t, \tau_0)[-1_{\tau_1 > \tau_0} \overline{CDS}_{a,b}(\tau_0, S, LGD_1)]^+\}$$

From this we see that in order to calculate the BCVA, we need to determine the value of the CDS contract on reference credit "1" at time τ_2 given that the reference credit has survived. A trivariate Gaussian copula has bivariate Gaussian marginals, thus it automatically furnishes us with the bivariate copula, $C_{1,2}$ that connects the default times τ_1 and τ_2. We also need to determine the value of the CDS at τ_0 given that the reference credit has survived. These are connected by the bivariate copula $C_{0,1}$. To calculate the expectations above it is clear from the pricing formula for CDS,

$$\mathbf{1}_{\tau_1 > T_j}\overline{CDS}_{a,b}(T_j, S_1, LGD_1)$$

$$\begin{aligned}
&= \mathbf{1}_{\tau_1 > T_j}\{S_1[-\int_{\max\{T_a,T_j\}}^{T_b} D(T_j,t)(t - T_{\gamma(t)-1} d\mathbb{Q}(\tau_1 > t|\mathcal{F}_{T_j}) \\
&+ \sum_{i=\max\{a,j\}+1}^{b} \alpha_i D(T_j,T_i)\mathbb{Q}(\tau_1 > T_i|\mathcal{F}_{T_i})] \\
&+ LGD_1[\int_{\max\{T_a,T_j\}}^{T_j} D(T_j,t) d\mathbb{Q}(\tau_1 > t|\mathcal{F}_{T_j})]\}
\end{aligned}$$

that we need to calculate,

$$1_{\{\tau_2 < \tau_0\}} 1_{\tau_1 > \tau_2} \mathbb{Q}(\tau_1 > t|\mathcal{F}_{\tau_2})$$

$$1_{\{\tau_0 < \tau_2\}} 1_{\tau_1 > \tau_0} \mathbb{Q}(\tau_1 > t|\mathcal{F}_{\tau_0})$$

Brigo calculates an explicit approximation of these quantities in a similar manner to that of unilateral CVA ([5], section 4.2 and Appendix B). As with unilateral CVA he then simulates the bilateral CVA of CDS under different volatility and correlation regimes using Monte Carlo simulation.

Numerical results bilateral CVA of a CDS

As with unilateral CVA, Brigo experiments with the effects of credit risk level $\lambda_i(0)$ and credit risk volatility ν_i and correlation in the now three names. For convenience, we use the following triple to summarise the correlation matrix of the trivariate copula, $(\rho_{01}, \rho_{02}, \rho_{12})$. We have ρ_{01} the correlation between investor and reference credit, ρ_{02} between investor and counterparty and ρ_{12} reference credit and counterparty.

Brigo tests the responses of BCVA under the following scenario. It is not clear whether it is a natural one. The credit risk of the protection buyer is low, i.e. $\lambda_0(0)$ is low and $\nu_0 = 0.1$, is near deterministic. The credit risk of the reference credit is high, the level is high and ν_1 is varied. The credit risk level of the protection seller is medium, and $\nu_2 = 0.2$. Brigo then varies ρ_{12} with $\rho_{01} = \rho_{02} = 0$, i.e. $(0, 0, varied)$. Thus in this regime Brigo varies the "wrong-way-risk" parameter, ρ_{12} the correlation between the reference credit and the protection seller. The credit risk of the protection buyer and seller are uncorrelated which would be expected in normal market conditions. That is market conditions before 2007. As the protection buyer has very low credit risk the results qualitatively replicate those derived for unilateral CVA.

BCVA-I monotonically increases with ρ_{12}. Except for the case when the credit spread volatility of the reference credit is low and its credit risk is near deterministic but higher than that of both the protection buyer and seller. Again, as $\rho_{12} \to 1$, BCVA-I$\to$ 0. This occurs for exactly the same reason as was found for unilateral CVA. The exponential triggers ξ_1 and ξ_2 are virtually identical, the intensities of both processes are virtually deterministic with $\lambda_1 > \lambda_2$ by assumption. Thus to first approximation, $\tau_1 = \frac{\xi_1}{\lambda_1} > \tau_2 = \frac{\xi_2}{\lambda_2}$ and the reference credit always defaults before the counterparty resulting in no adjustment. And just as in the unilateral case this feature disappears for high credit spread volatilities as are implied in real CDS markets.

4.3 Structural Model of CVA for CDS

In this section we apply the Hull and White structural model to the evaluation of the CVA of a CDS as they do [13]. Hull and White don't use the general CVA framework of Brigo in pricing CDS under credit risk. As a result their approach is much less general and dynamic than Brigo's and as will be shown is in fact mistaken in its valuation of the credit risk.

In their notation define, T the life of the CDS, R the expected recovery rate on name "1" the reference credit, $\theta_1(t)\Delta t$ the risk-neutral probability of the default of the reference credit before the protection buyer in $[t, t + \Delta t]$, $\theta_0(t)\Delta t$ the risk-neutral probability of the default of the protection buyer before the reference credit in $[t, t + \Delta t]$, $u(t)$ present value of CDS premium leg payments on payment dates in $[0, t]$ on a notional of 1, $e(t)$ present value of accrued premium payments in $[t*, t]$ where $t*$ is the first payment date prior to t on a notional of 1, $v(t)$ present value of 1 at t, i.e. $u(t)$, $e(t)$, $v(t)$ are relevant discount factors. w notional of the contract, s the value w that makes the initial CDS value zero, π the risk neutral probability of no default by either counterparty, $A(t)$ accrued interest on the reference credit at t. The protection buyer pays, $[u(t) + e(t)]w$ if the credit reference defaults first, the protection seller loses $[u(t)]w$ if the protection buyer defaults first, i.e. there is no accrual payment. The protection seller pays $L(t) = 1 - [1 + A(t)]R$ on reference credit first default. Based on this Hull and White price a CDS with unilateral CVA as,

$$\int_0^T v(t)L(t)\theta_1(t)dt - w \int_0^T [[u(t) + e(t)]\theta_1(t) + u(t)\theta_0(t)]dt - u(T)w\pi$$

The CDS spread s is thus,

$$s = \frac{\int_0^T v(t)L(t)\theta_1(t)dt}{\int_0^T [[u(t)+e(t)]\theta_1(t)+u(t)\theta_0(t)]dt + u(T)\pi}$$

Hull and White then use Monte Carlo methods to simulate the credit indices, $X_i(t)$ for $i = 0, 1$ for the reference credit name "1" and the protection buyer "0" and thus $\theta_i(t)$ for $i = 0, 1$. If the two names default simultaneously at a particular t_j then a fair coin is tossed to determine who defaults first.

Through their model Hull and White go on to deduce the unilateral CVA in the following way. They take the market CDS spreads directly and imply default probabilities for the reference credit and risk counterparty as is required by the Hull and White framework then use these to simulate barrier crossing default times. They then calculate S using their CDS formula with and without a risky counterparty and thus calculate the adjustment due to counterparty credit risk.

They argue that the implied spreads will not just have underlying default risk priced into them but also some counterparty default risk and thus so will the default probabilities [[13], p.6, footnote 1]. They then argue that calculating the difference in credit spreads in the CVA calculation above yields the adjustment one would have to make to take account of this.

This is slightly misleading, the default probabilities are implied from the spreads of liquidly traded, rather than OTC CDS which require no pricing model precisely because they are so liquidly traded. Although these may contain some credit risk, that of the party's whose trade set the price, this will be completely unrelated to our contract and either negligible due to the liquidity of trading or priced in via some similar CVA technology to our own. But in an OTC CDS this certainly is true.

This just highlights the fact that the market CDS spreads used may not completely isolate the credit risk of the underlying reference credit but this would not be due to the credit riskiness of our contract. CDS spreads are likely in some circumstances to suffer from the same problems as other market implied quantities, e.g. implied volatility. Such as when supply and demand effects dictate price rather than fundamental analysis. Their use in these cases is expedient rather than theoretically consistent.

It can be shown that Brigo's and Hull and White's calculations of CVA are the same. Recall Brigo's formula for unilateral CVA,

$$\begin{aligned} \Pi^D(t,T) &= \mathbf{1}_{\{\tau_2 > T\}}\Pi(t,T) + \mathbf{1}_{\{t<\tau_2\leq T\}}[\Pi(t,\tau_2) \\ &+ D(t,\tau_2)\mathrm{REC}(\mathrm{NPV}(\tau_2))^+ - (-\mathrm{NPV}(\tau_2))^+))] \end{aligned}$$

Hull and White assume zero recovery thus this becomes,

$$\Pi^D(t,T) = \mathbf{1}_{\{\tau_2 > T\}}\Pi(t,T) + \mathbf{1}_{\{t<\tau_2\leq T\}}\Pi(t,\tau_2)$$

Notice that both options terms vanish. This lack of optionality is due to the assumption of zero recovery and make their formulation far less general than Brigo's. The term $(-\mathrm{NPV}(\tau_2))^+$ has vanished. This means that Hull and White assume that in the event of the risky counterparty defaulting and the remaining value of the contract having a value such that the riskless counterparty is owed money by the risk-free counterparty none of that debt will be recovered. It is questionable whether this is a realistic assumption as the creditors to the risky counterparty are likely to pursue such debts upon the risky counterparty's default. Nevertheless, Hull and White's formula for the credit risky CDS can be written as,

$$\begin{aligned}
- & \; w\int_0^T u(t)p(t=\tau_1;\tau_1<\tau_2;\tau_1<T)dt \\
- & \; wu(T)p(T<\tau_1;T<\tau_2)dt + \int_0^T v(t)L(t)p(t=\tau_1;\tau_1<\tau_2;\tau_1<T)dt \\
- & \; w\int_0^T u(t)p(t=\tau_2;\tau_2<\tau_1;\tau_2<T)dt
\end{aligned}$$

where $p(.)dt$ are the relevant probability densities. We can adapt the payoff in Brigo's formula for CVA to a CDS contract in Hull and White's notation,

$$\begin{aligned}
- & \; w\int_0^T u(t)\mathbf{1}_{t=\tau_1;\tau_1<T;}\mathbf{1}_{\tau_2>T}dt \\
- & \; wu(T)\mathbf{1}_{\tau_1>T}\mathbf{1}_{\tau_2>T} \\
+ & \; \int_0^T v(t)\mathbf{1}_{t=\tau_1;\tau_1<T}\mathbf{1}_{\tau_2>T}dt \\
- & \; w\int_0^T u(t)\mathbf{1}_{t=\tau_1;\tau_1<T;\tau_1<\tau_2}\mathbf{1}_{\tau_2<T}dt \\
- & \; wu(\tau_2)\mathbf{1}_{\tau_1>\tau_2}\mathbf{1}_{\tau_2<T} + \int_0^T v(t)\mathbf{1}_{t=\tau_1;\tau_1<\tau_2}\mathbf{1}_{\tau_2<T}
\end{aligned}$$

Since the following events are equivalent,

$$\{t=\tau_1;\tau_1<\tau_2;\tau_1<T\} = \{t=\tau_1;\tau_1<T;\tau_2>T\}\bigcup\{t=\tau_1;\tau_1<T;\tau_1<\tau_2<T\}$$

and by noticing that,

$$\begin{aligned}
& \mathbb{E}[wu(\tau_2)\mathbf{1}_{\tau_1>\tau_2}\mathbf{1}_{\tau_2<T}] \\
= & \; \mathbb{E}[w\int_0^T u(t)\mathbf{1}_{t=\tau_2;\tau_2<\tau_1;\tau_1<T}dt] \\
= & \; w\int_0^T u(t)p(t=\tau_2;\tau_2<\tau_1;\tau_1<T)dt
\end{aligned}$$

we obtain the result.

4.3.1 Numerical Results

Hull and White calculate the initial spread S for the value of a CDS contract with unilateral CVA. That is S, the insurance premium, that makes the contract between protection buyer and protection seller equal to zero. Hull and White then see how S changes with the correlation between the reference credit and protection seller. This is equivalent to the approach of Brigo. In Brigo the credit spread is assumed to have been contractually fixed at some earlier time and therefore in his formulation he is not so much valuing the CDS as re-valuing the contract after intermediary credit activity. For any $t > 0$, S as a constant can be made the subject of the CDS valuation formula by setting its value equal to zero. The value can then be compared to the original S paid and the change in value to credit risk and time evaluated. Thus is doesn't matter whether one uses the payment streams between counterparties or S to value the CDS. In practise CDS values are quoted in S.

Notice that it is the lack of optionality in Hull and White's framework that allows them through simple algebra to make S the subject of the formula. This is not possible in Brigo's model due to the fact that CDS prices are embedded in options. In Brigo's model they may have to be deduced numerically.

The valuation at $t = 0$ of S is thus just a particular case of the valuation methodology of Brigo. Hull and White restrict themselves in their numerical work to the valuation of CVA at $t = 0$ and they calculate it in S-space that is in terms of the difference between the credit spread of CDS without credit risky protection seller and that with.

They find that, as with Brigo, as the correlation is increased the difference increases, i.e. the CVA term grows. Their model thus captures the wrong-way-risk effect. The dip effect found in Brigo for high correlation and deterministic credit spreads is not replicated by Hull and White. The volatility parameters for their credit indices are set to one and are thus two high to realise the case where one always defaults before the other.

Chapter 5

Conclusion

In this investigation, an attempt has been made to give an account of valuation adjustment under credit risk, CVA. The basic models in credit risk modelling and the model independent calculation of CVA were developed and two particular models and their experimental results investigated. The market standard model of Brigo based upon his general CVA pricing framework was found to be flexible and accurate. Its experimental results proved that it took account of all the risk dynamics associated with CDS pricing under counterparty risk. The model of Hull and White was also found to be simple and easy to calibrate and to effectively capture the wrong-way-risk effect as their experimental results verify.

Chapter 6

Appendix

6.0.2 A

Proof.

Recall,

$$\begin{aligned} \Pi(t,T) &= \pi(t,T) \\ &= 1_{\{T\leq\min\{\tau_0,\tau_2\}\}}\pi(t,T) \\ &+ 1_{\{\tau_0\leq\min\{\tau_2,T\}\}}\pi(t,T) + 1_{\{\tau_2\leq\min\{\tau_0,T\}\}}\pi(t,T) \end{aligned}$$

Then by the linearity of $\mathbb{E}$-operator,

$$\begin{aligned} \mathbb{E}_t[\Pi^D(t,T)] &= \mathbb{E}_t[\Pi(t,T)] \\ &+ \mathbb{E}_t[\mathrm{LGD}_0 1_{\{\tau_2\leq\min\{\tau_0,T\}\}}D(t,\tau_0)[-NPV(\tau_0)]^+] \\ &- \mathbb{E}_t[\mathrm{LGD}_2 1_{\{\tau_0\leq\min\{\tau_2,T\}\}}D(t,\tau_0)[-NPV(\tau_2)]^+] \end{aligned}$$

becomes,

$$\begin{aligned} \mathbb{E}_t[\Pi^D(t,T)] &= \mathbb{E}_t[\Pi(t,T) \\ &+ \mathrm{LGD}_0 1_{\{\tau_2\leq\min\{\tau_0,T\}\}}D(t,\tau_0)[-NPV(\tau_0)]^+ \\ &- \mathrm{LGD}_2 1_{\{\tau_0\leq\min\{\tau_2,T\}\}}D(t,\tau_0)[-NPV(\tau_2)]^+] \end{aligned}$$

Then using (103) becomes,

$$\begin{aligned} & \mathbb{E}_t[\Pi^D(t,T)] \\ = \ & \mathbb{E}_t[1_{\{\tau_0\leq\min\{\tau_2,T\}\}}\pi(t,T) + 1_{\{\tau_2\leq\min\{\tau_0,T\}\}}\pi(t,T) + 1_{\{T\leq\min\{\tau_0,\tau_2\}\}}\pi(t,T) \\ + \ & \mathrm{LGD}_0 1_{\{\tau_2\leq\min\{\tau_0,T\}\}}D(t,\tau_0)[-NPV(\tau_0)]^+ \\ - \ & \mathrm{LGD}_2 1_{\{\tau_0\leq\min\{\tau_2,T\}\}}D(t,\tau_0)[-NPV(\tau_2)]^+] \end{aligned}$$

$$\begin{aligned}
&= \mathbb{E}[1_{\{\tau_0 \le \min\{\tau_2, T\}\}}\pi(t,T) + 1_{\{\tau_2 \le \min\{\tau_0, T\}\}}\pi(t,T) + 1_{\{T \le \min\{\tau_0, \tau_2\}\}}\pi(t,T) \\
&+ \text{LGD}_0 1_{\{\tau_2 \le \min\{\tau_0, T\}\}} D(t,\tau_0)[-NPV(\tau_0)]^+ \\
&- \text{LGD}_2 1_{\{\tau_0 \le \min\{\tau_2, T\}\}} D(t,\tau_0)[-NPV(\tau_2)]^+] \\
&= \mathbb{E}[1_{\{\tau_0 \le \min\{\tau_2, T\}\}}\pi(t,T) - \text{LGD}_2 1_{\{\tau_0 \le \min\{\tau_2, T\}\}} D(t,\tau_0)[-NPV(\tau_2)] \\
&+ 1_{\{\tau_2 \le \min\{\tau_0, T\}\}}\pi(t,T) \\
&+ \text{LGD}_0 1_{\{\tau_2 \le \min\{\tau_0, T\}\}} D(t,\tau_0)[-NPV(\tau_0)]^+ \\
&+ 1_{\{T \le \min\{\tau_0, \tau_2\}\}}\pi(t,T)]
\end{aligned}$$

using the definition of $LGD_i = 1 - REC_i$,

$$\begin{aligned}
&= \mathbb{E}[1_{\{\tau_0 \le \min\{\tau_2, T\}\}}\pi(t,T) \\
&+ (\text{REC}_2 - 1) 1_{\{\tau_0 \le \min\{\tau_2, T\}\}} D(t,\tau_0)[-NPV(\tau_2)]] \\
&+ \mathbb{E}_t[1_{\{\tau_2 \le \min\{\tau_0, T\}\}}\pi(t,T) + (1 - \text{REC}_0) 1_{\{\tau_2 \le \min\{\tau_0, T\}\}} D(t,\tau_0)[-NPV(\tau_0)]^+] \\
&+ \mathbb{E}_t[1_{\{T \le \min\{\tau_0, \tau_2\}\}}\pi(t,T)]
\end{aligned}$$

Now develop the three expectations,

$$1_{\{\tau_2 \le \min\{\tau_0, T\}\}}\pi(t,T) + (1 - \text{REC}_0) 1_{\{\tau_2 \le \min\{\tau_0, T\}\}} D(t,\tau_0)[-NPV(\tau_0)]^+$$

$$\begin{aligned}
&= 1_{\{\tau_2 \le \min\{\tau_0, T\}\}}\pi(t,T) \\
&+ 1_{\{\tau_2 \le \min\{\tau_0, T\}\}} D(t,\tau_0)[-NPV(\tau_0)]^+ \\
&- 1_{\{\tau_2 \le \min\{\tau_0, T\}\}} D(t,\tau_0)\text{REC}_0[-NPV(\tau_0)]^+
\end{aligned}$$

taking the expectation conditional on τ_0, and noticing that $\pi(t,T) = \pi(t,\tau_0) + D(t,\tau_0)\pi(\tau_0,T)$ as $\pi(s,u)$ are discounted to s. And also using $NPV(\tau_0) = \mathbb{E}_{\tau_0}[\pi(\tau_0,T)]$, gives

$$\begin{aligned}
&\mathbb{E}_{\tau_0}[1_{\{\tau_2 \le \min\{\tau_0, T\}\}}\pi(t,T) \\
&+ 1_{\{\tau_2 \le \min\{\tau_0, T\}\}} D(t,\tau_0)[-NPV(\tau_0)]^+ \\
&- 1_{\{\tau_2 \le \min\{\tau_0, T\}\}} D(t,\tau_0)\text{REC}_0[-NPV(\tau_0)]^+] \\
&= \mathbf{1}_{\{\tau_2 \le \min\{\tau_0, T\}\}} \mathbb{E}_{\tau_0}[\pi(t,\tau_0) + D(t,\tau_0)\pi(\tau_0,T) \\
&+ D(t,\tau_0)[-\mathbb{E}_{\tau_0}[\pi(\tau_0,T)]]^+ \\
&- D(t,\tau_0)\text{REC}_0[-\mathbb{E}_{\tau_0}[\pi(\tau_0,T)]]^+] \\
&= \mathbf{1}_{\{\tau_2 \le \min\{\tau_0, T\}\}}[\pi(t,\tau_0) \\
&+ D(t,\tau_0)\mathbb{E}_{\tau_0}[\pi(\tau_0,T)] + D(t,\tau_0)[-\mathbb{E}_{\tau_0}[\pi(\tau_0,T)]]^+ \\
&- D(t,\tau_0)\text{REC}_0[-\mathbb{E}_{\tau_0}[\pi(\tau_0,T)])]^+]
\end{aligned}$$

Then using $(-.)^{+} = -(.)^{-}$, with $(.)^{-} = \inf(0, .)$ the "$negative - part - of$"-operator,

$$
\begin{aligned}
&= \mathbf{1}_{\{\tau_2 \leq \min\{\tau_0, T\}\}}[\pi(t, \tau_0) \\
&+ D(t, \tau_0)\mathbb{E}_{\tau_0}[\pi(\tau_0, T)] - D(t, \tau_0)[\mathbb{E}_{\tau_0}[\pi(\tau_0, T)]]^{-} \\
&- \mathrm{REC}_0[-\mathbb{E}_{\tau_0}[\pi(\tau_0, T)]]^{+}] \\
&= 1_{\{\tau_2 \leq \min\{\tau_0, T\}\}}[\pi(t, \tau_0) \\
&+ D(t, \tau_0)[\mathrm{NPV}(\tau_0)]^{+} - \mathrm{REC}_0[-NPV(\tau_0)]^{+}]
\end{aligned}
$$

Now conditioning on t and using $\mathbb{E}_t[\mathbb{E}_{\tau_0}[.]] = \mathbb{E}_t[.]$ as $t < \tau_0$ gives,

$$
\mathbb{E}_t[1_{\{\tau_2 \leq \min\{\tau_0, T\}\}}\pi(t, \tau_0)
$$
$$
+D(t, \tau_0)[\mathrm{NPV}(\tau_0)]^{+} - \mathrm{REC}_0[-NPV(\tau_0)]^{+}]
$$

similarly we obtain,

$$
\mathbb{E}_t[1_{\{\tau_0 \leq \min\{\tau_2, T\}\}}[\pi(t, \tau_2)]
$$
$$
+D(t, \tau_2)(\mathrm{REC}_2(\mathrm{NPV}(\tau_2))^{+} - (-\mathrm{NPV}(\tau_2))^{+})]
$$

as required $\square$

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