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Centripetal operators and Koszmider spaces

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Abstract

We study properties of Koszmider spaces and introduce a related notion of weakly Koszmider spaces. We show that if the space K is weakly Koszmider and $\mathscr{C}(K)$ is isomorphic to $\mathscr{C}(L)$ then L is also weakly Koszmider, but the analogous result does not hold for Koszmider spaces. We also show that a connected Koszmider space is strongly rigid.

Keywords: Centripetal operators; Koszmider spaces; Weakly Koszmider

1. Introduction

In this article we will consider Banach spaces of the type $\mathscr{C}(K)$ with the property that all bounded linear operators on $\mathscr{C}(K)$ are of some specific type.

We say that an operator $T : \mathscr{C}(K) \to \mathscr{C}(K)$ is *centripetal* if for any bounded disjoint sequence (f_n) in $\mathscr{C}(K)$ and any sequence (x_n) in K with $f_n(x_n) = 0$ for all n, we have $\lim_{n\to\infty}(Tf_n)(x_n) = 0$. We define a space K to be *weakly Koszmider* if every operator on $\mathscr{C}(K)$ is centripetal. This condition is weaker than the one in the definition of a *Koszmider* space introduced in [7], which requires all operators on $\mathscr{C}(K)$ to be of the form $gI + W$ with $g \in \mathscr{C}(K)$ and W weakly compact.

Spaces with these properties were studied extensively by Koszmider in [6]. He constructed several examples of weakly Koszmider spaces and, assuming (CH), also obtained a connected Koszmider space. He showed that if K is a (weakly) Koszmider space then $\mathscr{C}(K)$ is not isomorphic to any of its proper subspaces or quotients. Moreover, if K is connected Koszmider (or connected weakly Koszmider such that $K \setminus F$ is connected for any finite F) then $\mathscr{C}(K)$ is indecomposable (that is, cannot be written as a direct sum of two closed infinite-dimensional subspaces). This, in turn, gives the first example of a space $\mathscr{C}(K)$ which is not isomorphic to $\mathscr{C}(L)$ for any zero-dimensional compact space L. Later, Plebanek [7] constructed a connected Koszmider space entirely within (ZFC).

The purpose of this article is to explore further properties of weakly Koszmider and Koszmider spaces. We will describe an alternative characterisation of weakly Koszmider spaces, namely that K is weakly Koszmider if and only if the commutator $ST - TS$ of any operators $S, T : \mathscr{C}(K) \to \mathscr{C}(K)$ is weakly compact. This characteristic is invariant under Banach space isomorphisms which implies that if $\mathscr{C}(K)$ and $\mathscr{C}(L)$ are isomorphic and K is weakly Koszmider then so is L. We will show that the analogous result for Koszmider spaces fails unless we restrict ourselves to spaces

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with no isolated points, in which case the relation $\mathscr{C}(K) \sim \mathscr{C}(L)$ forces K and L to be homeomorphic. As a by-product, an example of a weakly Koszmider non-Koszmider space will be constructed. A similar example has also been obtained by Fajardo [4].

In the final section we will discuss topological properties of connected Koszmider spaces. We will show that if K is connected then $K \setminus F$ is connected for any finite F, and deduce that every connected Koszmider space K is strongly rigid, that is, every continuous function $\phi : K \to K$ is constant or the identity.

2. Notation

Our notation is fairly standard. We will be working with the spaces $\mathscr{C}(K)$ of continuous real-valued functions on a space K equipped with the supremum norm. In this case K is assumed to be compact and Hausdorff and all operators are bounded and linear. The space $\mathscr{C}(K)^*$ is identified with the space $\mathscr{M}(K)$ of Radon measures on K.

To avoid too many brackets, for any (compact Hausdorff) space we will use the following notation:

$$\mathscr{L} = \mathscr{L}^K = \mathscr{L}\big(\mathscr{C}(K)\big) = \big\{T : \mathscr{C}(K) \to \mathscr{C}(K),\ T \text{ bounded linear}\big\},$$
$$\mathscr{L}_{\mathrm{wc}} = \mathscr{L}_{\mathrm{wc}}^K = \mathscr{L}_{\mathrm{wc}}\big(\mathscr{C}(K)\big) = \big\{T : \mathscr{C}(K) \to \mathscr{C}(K),\ T \text{ weakly compact}\big\},$$
$$\mathscr{L}_M = \mathscr{L}_M^K = \mathscr{L}_M\big(\mathscr{C}(K)\big) = \big\{gI : \mathscr{C}(K) \to \mathscr{C}(K),\ g \in \mathscr{C}(K)\big\}.$$

Further, for any sets X and Y we will write

$X \sim Y$, if X and Y are isomorphic as Banach spaces;

$X \approx Y$, if X and Y are homeomorphic as topological spaces;

$X \cong Y$, if X and Y are isomorphic as rings; and

$X \leqslant Y$, if X is a subring of Y.

Finally, if $g : K \to \mathbb{R}$ is any bounded Borel function, we define the operator $gI : \mathscr{M}(K) \to \mathscr{M}(K)$ by setting

$$(gI)(\mu)(f) = \int gf \,\mathrm{d}\mu \quad \forall \mu \in \mathscr{M}(K),\ \forall f \in \mathscr{C}(K).$$

Note that when g is a continuous function, gI is the dual of the operator on $\mathscr{C}(K)$ which sends each function f in $\mathscr{C}(K)$ to gf. We shall call this operator gI again, that is,

$$gI : \mathscr{C}(K) \to \mathscr{C}(K), \quad (gI)(f) = gf.$$

3. Properties of Koszmider spaces

Definition 3.1. A space K is said to be *Koszmider* if $\mathscr{L}^K = \mathscr{L}_M^K + \mathscr{L}_{\mathrm{wc}}^K$, that is, for every bounded operator T on $\mathscr{C}(K)$, there exist a continuous function g in $\mathscr{C}(K)$ and a weakly compact operator W such that

$$T = gI + W. \tag{3.1}$$

As mentioned in the introduction, examples of Koszmider spaces can be found in [6] and [7]. For general information on $\mathscr{C}(K)$ spaces we refer the reader to [9].

We will look at Koszmider spaces from a different perspective and we will start with the following general observation. Let K be any compact Hausdorff space. Then

- $\mathscr{L}$ is a ring (with ring operations being $+$ and $\circ$),
- $\mathscr{L}_M$ is a subring of $\mathscr{L}$,
- $\mathscr{L}_{\mathrm{wc}}$ is an ideal in $\mathscr{L}$.

Note the following consequences of this fact.

Proposition 3.2. *Suppose that* $\mathscr{C}(K) \sim \mathscr{C}(L)$. *Then* $\mathscr{L}^K / \mathscr{L}_{\mathrm{wc}}^K \cong \mathscr{L}^L / \mathscr{L}_{\mathrm{wc}}^L$.

Proof. Define a map $\theta : \mathscr{L}^K \to \mathscr{L}^L/\mathscr{L}^L_{\mathrm{wc}}$ by setting $\theta(T) := JTJ^{-1} + \mathscr{L}^L_{\mathrm{wc}}$, where $J : \mathscr{C}(K) \to \mathscr{C}(L)$ is an isomorphism. Then θ is a ring homomorphism, $\mathrm{Ker}(\theta) = \{T \in \mathscr{L}^K\colon\ JTJ^{-1} \in \mathscr{L}^L_{\mathrm{wc}}\} = \mathscr{L}^K_{\mathrm{wc}}$ and $\mathrm{Im}(\theta) = \mathscr{L}^L/\mathscr{L}^L_{\mathrm{wc}}$. The result follows from the first isomorphism theorem for rings. □

Proposition 3.3. *Let K be a compact space and K' be its derived set. Then*

$$(\mathscr{L}_M + \mathscr{L}_{\mathrm{wc}})/\mathscr{L}_{\mathrm{wc}} \cong \mathscr{L}_M/(\mathscr{L}_M \cap \mathscr{L}_{\mathrm{wc}}) \cong \mathscr{C}(K'). \tag{3.2}$$

In particular, if K is Koszmider then

$$\mathscr{L}/\mathscr{L}_{\mathrm{wc}} \cong \mathscr{C}(K'). \tag{3.3}$$

For the proof we will use the following version of Claim from [6], where the corresponding result was proved for an arbitrary function $h : K \to \mathbb{R}$.

Lemma 3.4. *Let $h : K \to \mathbb{R}$ be a continuous function. Then the operator $hI : \mathscr{C}(K) \to \mathscr{C}(K)$ is weakly compact if and only if the set*

$$A^h_\varepsilon = \{x \in K\colon\ |h(x)| > \varepsilon\}$$

is finite for all $\varepsilon > 0$.

For completeness, we will include the proof of this result. Since we are only interested in continuous functions, our proof is different from the proof in [6]. It is based on the following result which will be used throughout the article and can be found in [3, p. 160, Corollary 17].

Theorem 3.5. *A bounded operator $T : \mathscr{C}(K) \to Y$ is weakly compact if and only if for every bounded disjoint sequence (f_n) in $\mathscr{C}(K)$ we have*

$$\lim_{n\to\infty} \|Tf_n\| = 0.$$

Proof of Lemma 3.4. Suppose that for some $\varepsilon > 0$ there exists a sequence (x_n) of distinct elements of K such that $|h(x_n)| > \varepsilon$ for all n. Passing to subsequences, we may assume that (x_n) is relatively discrete, so that there exist pairwise disjoint open U_n with $x_n \in U_n$ for all n. Let (f_n) be a sequence in $\mathscr{C}(K)$ separating (x_n), that is,

$$\|f_n\| = 1, \quad f_n(x_n) = 1, \quad \mathrm{supp}(f_n) \subseteq U_n$$

(here, $\mathrm{supp}(f_n) = \{x \in K\colon\ f_n(x) \neq 0\}$). Then (f_n) is a bounded disjoint sequence and $|h(x_n)f_n(x_n)| > \varepsilon$ for each n. Theorem 3.5 implies that hI is not weakly compact.

Conversely, if hI is not weakly compact, by Theorem 3.5 we can find $\varepsilon > 0$, a bounded disjoint sequence (f_n) in $\mathscr{C}(K)$ and a sequence (x_n) in K such that

$$|h(x_n)f_n(x_n)| > \varepsilon$$

for each n. Hence, if M is an upper bound for $\{\|f_n\|\}$, each x_n lies in $A^h_{\varepsilon/M}$. Finally, note that since f_n are disjoint and hI is bounded, we may assume that all x_n are distinct and so $A^h_{\varepsilon/M}$ is infinite as required. □

Proof of Proposition 3.3. The first part of (3.2) is simply the second isomorphism theorem for rings applied to $\mathscr{L}$, $\mathscr{L}_M$ and $\mathscr{L}_{\mathrm{wc}}$.

To prove the second part, define a ring homomorphism

$$\theta : \mathscr{L}_M \to \mathscr{C}(K'), \quad \theta(gI) = g|_{K'}.$$

With the help of Lemma 3.4 we will show that $\mathrm{Ker}(\theta) = \mathscr{L}_M \cap \mathscr{L}_{\mathrm{wc}}$.

Suppose that $hI \in \mathrm{Ker}(\theta)$. Then each point x with $h(x) \neq 0$ is isolated. Consequently, for each $\varepsilon > 0$, the set A^h_ε, defined in Lemma 3.4, consists of isolated points. Since K is compact, A^h_ε is finite and so $hI \in \mathscr{L}_M \cap \mathscr{L}_{\mathrm{wc}}$. Conversely, if $hI \in \mathscr{L}_M \cap \mathscr{L}_{\mathrm{wc}}$ and $x \in K$ is such that $h(x) \neq 0$, then for some $\varepsilon > 0$ we have $x \in A^h_\varepsilon$. However, each A^h_ε is open and finite. Consequently, each such x is isolated and thus $hI \in \mathrm{Ker}(\theta)$ as required.

Furthermore, by Tietze's extension theorem, $\mathrm{Im}(\theta) = \mathscr{C}(K')$ and so the result follows from the first isomorphism theorem. □

This gives us a surprising result.

Theorem 3.6. *Let K and L be Koszmider spaces such that $\mathscr{C}(K) \sim \mathscr{C}(L)$. Then $K' \approx L'$. In particular, if K and L have no isolated points, then K and L are homeomorphic.*

Proof. By Propositions 3.2 and 3.3,

$$\mathscr{C}(K') \cong \mathscr{L}^K / \mathscr{L}^K_{\mathrm{wc}} \cong \mathscr{L}^L / \mathscr{L}^L_{\mathrm{wc}} \cong \mathscr{C}(L'),$$

thus, by the Banach–Stone Theorem [9, Theorem 7.8.4], $K' \approx L'$. □

Suppose now that K is Koszmider and L is an arbitrary compact space such that $\mathscr{C}(K) \sim \mathscr{C}(L)$. Can we say anything about the structure of L or $\mathscr{C}(L)$? Does this condition force L to be Koszmider or do we have $K' \approx L'$? The answer turns out to be negative in general, but positive if we restrict ourselves to spaces with no isolated points. This will be analysed at the end of next section but first we need to introduce some machinery which does not only provide the necessary background but also gives very interesting independent results.

4. Centripetal operators and weakly Koszmider spaces

4.1. Motivation

Let us start with the definition.

Definition 4.1. A bounded operator $T : \mathscr{C}(K) \to \mathscr{C}(K)$ is said to be *centripetal* if, for every bounded disjoint sequence (f_n) in $\mathscr{C}(K)$ and for every sequence (x_n) in K with $f_n(x_n) = 0$ for all n, we have

$$\lim_{n\to\infty} (Tf_n)(x_n) = 0. \tag{4.1}$$

This definition was introduced in [6] where such operators were called "weak multipliers". The term "centripetal", however, was used in early drafts of [6] and later appeared in [7]. Both terms relate to exactly the same notion.

An immediate example of centripetal operators is the identity operator or, more generally, any operator of the form gI with $g \in \mathscr{C}(K)$. Theorem 3.5 provides a less trivial class of examples, namely weakly compact operators.

Centripetal operators lie at the heart of the constructions of Koszmider spaces in [6] and [7]. More precisely, the following result was used in both articles:

Theorem 4.2. *(See [6, Theorem 2.7 and Lemma 2.8].) Let K be a compact space. The following statements are equivalent*:

(i) *K is Koszmider.*
(ii) *All operators on $\mathscr{C}(K)$ are centripetal and the space $K \setminus \{x\}$ is C^*-embedded into K for every $x \in K$.*

In particular, if all operators on $\mathscr{C}(K)$ are centripetal and K contains no open butterflies, then (ii) *holds and K is Koszmider.*

(Recall that a subspace Y of X is *C^*-embedded* into X if every bounded function in $\mathscr{C}(Y)$ can be extended to a function in $\mathscr{C}(X)$; and a point $x \in X$ is an *open butterfly* if $\{x\} = \overline{U} \cap \overline{V}$ for some open subsets U, V of X.)

The question of determining whether a given space K is Koszmider now splits up into two parts:

(i) Are all operators on $\mathscr{C}(K)$ centripetal?
(ii) Does a certain extra condition (e.g. absence of open butterflies) hold?

However, as it was shown in [6], many interesting properties of Koszmider spaces follow from the positive answer to only and do not depend on (ii) at all. This motivates us to introduce another, weaker, class of spaces.

Definition 4.3. A compact space K is said to be *weakly Koszmider* if every bounded linear operator on $\mathscr{C}(K)$ is centripetal.

Note that Theorem 3.5 implies that every Koszmider space is weakly Koszmider. We will see later that the converse is not true in general. Let us summarise several properties of weakly Koszmider spaces which either follow directly from the definition or have been proved in [6]. Recall that a Banach space X is said to be *Grothendieck* if every weak*-convergent sequence in X^* converges weakly.

Theorem 4.4. *Let K be a weakly Koszmider space. Then*

(i) *$\mathscr{C}(K)$ is Grothendieck. In particular, K does not contain* (*non-trivial*) *convergent sequences*;
(ii) *If $\phi : K \to K$ is continuous and Y is a subset of K such that $\phi(y) \neq y$ for all $y \in Y$, then $\phi(Y)$ is finite*;
(iii) [6] *If $T : \mathscr{C}(K) \to \mathscr{C}(K)$ is a bounded operator, then T is onto if and only if it is an isomorphism onto its range*;
(iv) [6] *$\mathscr{C}(K)$ is not isomorphic to any of its proper subspaces, nor to any of its proper quotients.*

Proof. (i) It is known (see [8]) that a space of the form $\mathscr{C}(K)$ is Grothendieck if and only if it does not contain a complemented copy of c_0. In particular, if $\mathscr{C}(K)$ is not Grothendieck, we can find a disjoint sequence $(f_n) \subseteq \mathscr{C}(K)$ of functions of norm 1 such that the space $Y = \overline{\mathrm{sp}}(f_n)$ is complemented in $\mathscr{C}(K)$. Let $P : \mathscr{C}(K) \to Y$ be the projection, $J : Y \to \mathscr{C}(K)$ be the inclusion map and $S : Y \to Y$ be the continuous linear extension of the shift operator $f_n \mapsto f_{n+1}$. Then JSP is a non-centripetal operator on $\mathscr{C}(K)$. Indeed, let (x_n) be a sequence in K such that $f_n(x_n) = 1$. Then for each n we have $f_n(x_{n+1}) = 0$ (since f_n are disjoint) and $(JSPf_n)(x_{n+1}) = f_{n+1}(x_{n+1}) = 1$.

For the second part of (i), note that if (x_n) is an infinite sequence in K converging to x, then δ_{x_n} converges to δ_x in the weak*-topology. However, (δ_{x_n}) is not weakly convergent as, by the Dieudonné–Grothendieck Theorem (see [2, VII.14]), the set $\{\delta_{x_n}\}$ is not relatively weakly compact.

(ii) Let $(x_n) \subseteq K$ be such that for each n we have $\phi(x_n) \neq x_n$ and the set $\{\phi(x_n)\}_{n\in\mathbb{N}}$ is infinite. Define a composition operator $T : \mathscr{C}(K) \to \mathscr{C}(K)$ by setting $T(f) = f \circ \phi$ for all $f \in \mathscr{C}(K)$.

Put $y_n = \phi(x_n)$. Passing to subsequences if necessary, we may assume that (y_n) are relatively isolated, that is, there exist pairwise disjoint open U_n such that $y_n \in U_n$ for all n. We may also assume that $x_n \notin U_n$. Let now (f_n) be a sequence of functions in $\mathscr{C}(K)$ separating (y_n), that is,

$$\|f_n\| = 1, \quad f_n(y_n) = 1, \quad \mathrm{supp}(f_n) \subseteq U_n.$$

Then (f_n) is a bounded disjoint sequence and $f_n(x_n) = 0$ for each n, but $(Tf_n)(x_n) = (f_n \circ \phi)(x_n) = f_n(y_n) = 1$, and so T is not centripetal.

(iii) is simply a weaker version of Theorem 2.3 from [6] which states that if K has no convergent sequences and T is a centripetal operator on $\mathscr{C}(K)$, then T onto if and only if it is an isomorphism onto its range.

(iv) is a direct consequence of (iii). Let Y be a subspace of $\mathscr{C}(K)$ and let $J : \mathscr{C}(K) \to Y \subseteq \mathscr{C}(K)$ be an isomorphism. Then by (iii), T must be onto $\mathscr{C}(K)$ and thus $Y = \mathscr{C}(K)$. The proof for quotients is similar. □

We will now proceed to describe an alternative characterisation of weakly Koszmider spaces which will provide a machinery for obtaining further properties.

4.2. Alternative characterisation of weakly Koszmider spaces

It turns our that in order to check whether a given space is weakly Koszmider, it is enough to consider only the commutators of operators. Here, by a *commutator* of operators S, T we mean the operator $[S, T] := ST - TS$.

Theorem 4.5. *Let K be a compact space. The following are equivalent*:

(i) *K is weakly Koszmider.*

(ii) $\mathscr{L}/\mathscr{L}_{\mathrm{wc}}$ *is commutative, that is, for any two* $S, T : \mathscr{C}(K) \to \mathscr{C}(K)$*, their commutator* $[S, T]$ *is weakly compact.*

Let us note that the above characterisation is invariant under Banach space isomorphisms. Indeed, if $\mathscr{C}(K) \sim \mathscr{C}(L)$, then $\mathscr{L}^K/\mathscr{L}^K_{\mathrm{wc}}$ and $\mathscr{L}^L/\mathscr{L}^L_{\mathrm{wc}}$ are isomorphic as rings, and thus one of them is commutative if and only if another one is. Thus we get the following result.

Theorem 4.6. *Let* K *and* L *be compact spaces such that* $\mathscr{C}(K) \sim \mathscr{C}(L)$ *and* K *is weakly Koszmider. Then* L *is also weakly Koszmider.*

The proof of Theorem 4.5 will use an alternative characterisation of centripetal operators. For any operator T on $\mathscr{C}(K)$, we define a function $g_T : K \to \mathbb{R}$ by setting

$$g_T(x) = \left(T^*\delta_x\right)\left(\{x\}\right), \quad \forall x \in K,$$

where δ_x is the usual Dirac measure.

Theorem 4.7. *(See [6, Theorem 2.2].) An operator* $T : \mathscr{C}(K) \to \mathscr{C}(K)$ *is centripetal if and only if* $g_T I$ *is a well-defined operator on* $\mathscr{M}(K)$ *and* $T^* - g_T I$ *is weakly compact.*

Proof of Theorem 4.5. Let K be a weakly Koszmider space and S, T be operators on $\mathscr{C}(K)$. Let W_S and W_T be weakly compact operators on $\mathscr{C}(K)$ such that

$$S^* = g_S I + W_S, \quad T^* = g_T I + W_T.$$

Then

$$\begin{aligned}[S, T]^* = T^*S^* - S^*T^* &= (g_T I + W_T)(g_S I + W_S) - (g_S I + W_S)(g_T I + W_T) \\ &= (g_T I g_S I - g_S I g_T I) + V = V,\end{aligned}$$

where $V = g_T W_S + W_T g_S + W_T W_S - g_S W_T - W_S g_T - W_S W_T$ and so is weakly compact. By Gantmacher's Theorem, $[S, T]$ is also weakly compact, as required.

Conversely, suppose that K is such that $\mathscr{L}/\mathscr{L}_{\mathrm{wc}}$ is commutative. Note first that $\mathscr{C}(K)$ is Grothendieck. Indeed, if not, then, as before, we can find a disjoint sequence (f_n) of elements in $\mathscr{C}(K)$ of norm 1 such that the space $Y = \overline{\mathrm{sp}}(f_n)$ is a copy of c_0 complemented in $\mathscr{C}(K)$. Let $P : \mathscr{C}(K) \to Y$ be a projection onto Y, and let $S, T : Y \to Y$ be the continuous linear extensions of the operators $f_n \mapsto f_{n+1}$ and $f_n \mapsto f_{2n}$ respectively. Then for each n we have $\|[SP, TP](f_n)\| = \|f_{2n+1} - f_{2n+2}\| = 1$, and so, by Theorem 3.5, $[SP, TP]$ is not weakly compact.

Suppose now that K is not weakly Koszmider. Then there exist $\varepsilon > 0$, a bounded operator T on $\mathscr{C}(K)$, a bounded disjoint sequence (f_n) in $\mathscr{C}(K)$ and a sequence (x_n) in K such that for each n,

$$f_n(x_n) = 0 \quad \text{and} \quad \left|(Tf_n)(x_n)\right| > \varepsilon. \tag{4.2}$$

Without loss of generality assume that each f_n takes non-negative values only, and so, in particular, that $f_n^{1/2}$ is well-defined.

Let us now define a sequence of functionals $\phi_n : \mathscr{C}(K) \to \mathbb{R}$ by setting

$$\phi_n(g) = [gI, T]\left(f_n^{1/2}\right)(x_n) = g(x_n)\left(Tf_n^{1/2}\right)(x_n) - \left(T\left(gf_n^{1/2}\right)\right)(x_n).$$

Note that $(f_n^{1/2})$ is a bounded disjoint sequence of functions, and by (4.2)

$$\left|\phi_n\left(f_n^{1/2}\right)\right| = \left|f_n^{1/2}(x_n) \cdot \left(Tf_n^{1/2}\right)(x_n) - \left(T\left(f_n^{1/2}f_n^{1/2}\right)\right)(x_n)\right| = \left|T(f_n)(x_n)\right| > \varepsilon$$

for each n. Thus, by the Dieudonné–Grothendieck Theorem (see [2, VII.14]), the set $\{\phi_n\colon n \in \mathbb{N}\}$ is not relatively weakly compact and, in particular, the sequence (ϕ_n) is not weakly convergent. But $\mathscr{C}(K)$ is Grothendieck, hence (ϕ_n) is also not weak*-convergent, so there exists g in $\mathscr{C}(K)$ such that $\phi_n(g) \not\to 0$. Passing to subsequences, if necessary, we can find $\delta > 0$ such that for each n,

$$\delta < \left|\phi_n(g)\right| = \left|[gI, T]\left(f_n^{1/2}\right)(x_n)\right|,$$

which means that

$$\left\|[gI,T]\left(f_n^{1/2}\right)\right\| > \delta,$$

and so, by Theorem 3.5, the operator $[gI,T]$ is not weakly compact which is a contradiction. □

4.3. Further properties of Koszmider and weakly Koszmider spaces

We will start with showing that the classes of Koszmider and weakly Koszmider spaces do not coincide.

Proposition 4.8. *There exists a weakly Koszmider non-Koszmider space.*

More precisely, let K be a (weakly) Koszmider space and x_1, x_2 be distinct non-isolated points in K. Form a quotient space L by identifying x_1 and x_2 (that is, we define an equivalence relation R on K by saying that xRy if and only if $\{x,y\} = \{x_1,x_2\}$ or $x = y$, and consider $L = K/R$). Then L is weakly Koszmider but not Koszmider.

The proof will use the following auxiliary result.

Lemma 4.9. *Let Y be a subspace of X of finite codimension. Then*

$$\mathscr{L}(X)/\mathscr{L}_{\mathrm{wc}}(X) \cong \mathscr{L}(Y)/\mathscr{L}_{\mathrm{wc}}(Y).$$

Proof. Let $P : X \to Y$ be a projection onto Y, and $J : Y \to X$ be the inclusion map. Let also denote the identity maps on X and Y by I_X and I_Y respectively. Note that $PJ = I_Y$, and that $I_X - JP$ has finite rank, hence, in particular, is weakly compact.

For any $T \in \mathscr{L}(X)$ we set $\tilde{T} = PTJ$. Then $\tilde{T}$ is an element of $\mathscr{L}(Y)$ and for any $S, T \in \mathscr{L}(X)$ we have

$$\widetilde{S+T} = \tilde{S} + \tilde{T} \quad \text{and} \quad \widetilde{ST} - \tilde{S}\tilde{T} = PS(I_X - JP)TJ \in \mathscr{L}_{\mathrm{wc}}(Y),$$

and so we can define the following ring homomorphism:

$$\theta : \mathscr{L}(X) \to \mathscr{L}(Y)/\mathscr{L}_{\mathrm{wc}}(Y), \quad T \mapsto \tilde{T} + \mathscr{L}_{\mathrm{wc}}(Y).$$

Note that if $S \in \mathscr{L}(Y)$ then $JSP \in \mathscr{L}(X)$ and $\widetilde{JSP} = PJSPJ = I_Y S I_Y = S$, and so $\mathrm{Im}(\theta) = \mathscr{L}(Y)/\mathscr{L}_{\mathrm{wc}}(Y)$.

Furthermore, $\mathrm{Ker}(\theta) = \mathscr{L}_{\mathrm{wc}}(X)$. Indeed, if $T \in \mathrm{Ker}(\theta)$, then $\tilde{T} \in \mathscr{L}_{\mathrm{wc}}(Y)$. Hence the operator $J\tilde{T}P = JPTJP$ is weakly compact. But on the other hand, so is $(I_X - JP)TJP$, being an operator of a finite rank, thus TJP is weakly compact. Similarly, $T(I_X - JP)$ is weakly compact, and thus $T \in \mathscr{L}_{\mathrm{wc}}(X)$. Conversely, of course, if $T \in \mathscr{L}_{\mathrm{wc}}(X)$ then $\tilde{T} \in \mathscr{L}_{\mathrm{wc}}(Y)$, that is, $T \in \mathrm{Ker}(\theta)$.

The result now follows from the first isomorphism theorem for rings. □

Proof of Proposition 4.8. First of all, note that the space $\mathscr{C}(L)$ is isomorphic to a subspace of $\mathscr{C}(K)$ of codimension 1, namely, to the subspace $V = \{f \in \mathscr{C}(K)\colon\ f(x_1) = f(x_2)\}$, and thus, by the above lemma and Theorem 4.5, L is weakly Koszmider.

On the other hand, L is not Koszmider. Indeed, let $\pi : K \to L$ be the quotient map and let $f : K \to [0,1]$ be a continuous function separating x_1 and x_2, that is, $f(x_1) = 0$ and $f(x_2) = 1$. Consider the point $\star = \pi(x_i)$ and the function $g = (f \circ \pi^{-1})|_{L\setminus\{\star\}} : L\setminus\{\star\} \to [0,1]$. Then g is a continuous bounded function. However, there does not exists a continuous extension of g onto the whole of L, because any such extension $\tilde{g}$ would have to satisfy $\tilde{g}(\star) = f(x_1) = 0$ and $\tilde{g}(\star) = f(x_2) = 1$ (since both x_1 and x_2 are non-isolated). This means that $L\setminus\{\star\}$ is not C^*-embedded into L and hence, by Theorem 4.2, L is not Koszmider. □

As mentioned in the introduction, another example of a weakly Koszmider non-Koszmider space has been independently obtained by Fajardo in his PhD thesis [4]. His example is, in fact, the same as ours, but the proof that the resulting space is weakly Koszmider but not Koszmider is different.

Using the construction from Proposition 4.8, we may finally answer the questions posed in the end of the last chapter.

Proposition 4.10. *Let K be a Koszmider space. There exists a non-Koszmider space L such that $\mathscr{C}(K) \sim \mathscr{C}(L)$.*

Proof. As in Proposition 4.8, choose two non-isolated points $x_1, x_2 \in K$ and form K_1 by identifying x_1 and x_2 into a point $\star$. Then K_1 is a weakly Koszmider non-Koszmider space and $\mathscr{C}(K_1)$ is isomorphic to a hyperplane of $\mathscr{C}(K)$, that is,

$$\mathscr{C}(K) \sim \mathscr{C}(K_1) \oplus \mathbb{R}. \tag{4.3}$$

Now, pick any point $w \notin K_1$ and form L by adding w to K_1 as an isolated point. That is, $L = K_1 \cup \{w\}$ and the topology on L is generated by sets open in K_1, and by $\{w\}$. Since $K_1 \backslash \{\star\}$ is not C^*-embedded into K_1, it follows that $L \backslash \{\star\}$ is not C^*-embedded into L and so L is not Koszmider. However, since w is isolated,

$$\mathscr{C}(L) \sim \mathscr{C}(K_1) \oplus \mathbb{R}, \tag{4.4}$$

which, combined with (4.3), gives us that $\mathscr{C}(L) \sim \mathscr{C}(K)$. □

Note that since isolated points do not change centripetality of operators or the property of being C^*-embedded, we cannot have $K' \approx L'$ for the spaces K and L constructed above.

Let us finish the section with a positive result. Let K be a Koszmider space and let L be a compact space with $\mathscr{C}(K) \sim \mathscr{C}(L)$. Assume also that K and L have no isolated points, that is, $K' = K$ and $L' = L$. Combining Propositions 3.3 and 3.2 and noting that $(\mathscr{L}_M + \mathscr{L}_{\mathrm{wc}})/\mathscr{L}_{\mathrm{wc}}$ is a subring of $\mathscr{L}/\mathscr{L}_{\mathrm{wc}}$, we get

$$\begin{aligned}\mathscr{C}(K) &= \mathscr{C}(K') \cong \mathscr{L}^K/\mathscr{L}^K_{\mathrm{wc}} \cong \mathscr{L}^L/\mathscr{L}^L_{\mathrm{wc}} \geqslant \left(\mathscr{L}^L_M + \mathscr{L}^L_{\mathrm{wc}}\right)/\mathscr{L}^L_{\mathrm{wc}} \\ &\cong \mathscr{L}^L_M/\left(\mathscr{L}^L_M \cap \mathscr{L}^L_{\mathrm{wc}}\right) \cong \mathscr{C}(L') \\ &= \mathscr{C}(L),\end{aligned}$$

so $\mathscr{C}(L)$ is ring-isomorphic to a subspace Y of $\mathscr{C}(K)$ and is Banach-space isomorphic to $\mathscr{C}(K)$. Part (iii) of Theorem 4.4 implies that $Y = \mathscr{C}(K)$, that is, $\mathscr{C}(K) \cong \mathscr{C}(L)$, and hence $K \approx L$. In particular, of course, this implies that L is also Koszmider.

Summarising all of the above, we get the following result

Theorem 4.11. *Let K and L be compact spaces with $\mathscr{C}(K) \sim \mathscr{C}(L)$. Then*

(i) *If K is weakly Koszmider, then so is L.*
(ii) *If K is Koszmider and K and L have no isolated points then $K \approx L$ and L is also Koszmider.*

5. Strong rigidity of connected Koszmider spaces

We already mentioned in the introduction that connected Koszmider spaces play a special role in the theory of Banach spaces as they provide the first counterexample to the following conjecture from [9]:

> Let K be a compact Hausdorff space. Does there exist a zero-dimensional space L such that $\mathscr{C}(K) \sim \mathscr{C}(L)$?

For more details we refer the reader to [6], where Koszmider proves an even stronger result, namely, if K is weakly Koszmider space such that $K \backslash F$ is connected for any finite set F, then $\mathscr{C}(K) \not\sim \mathscr{C}(L)$ to any zero-dimensional space L. Koszmider also constructs an example of such a space.

It turns out that connected Koszmider spaces are also interesting from a topological point of view, because the condition of having few operators on $\mathscr{C}(K)$ forces K to have few continuous functions on itself. Let us introduce one more definition.

Definition 5.1. A topological space K is said to be *strongly rigid* if the only continuous non-constant function from K to itself is the identity.

De Groot [1] proved that strongly rigid Hausdorff spaces exist and Kannan and Rajagopalan [5] showed that under an extra set-theoretic assumption, it is possible to construct a Hausdorff (not necessarily compact) strongly rigid space of arbitrarily large cardinality. We will show now that connected Koszmider spaces provide another class of examples of strongly rigid spaces. For this we need to establish several intermediate results. The first one is very similar to Theorem 4.2.

Proposition 5.2. *Let K be a compact space. The following are equivalent*:

(i) *K is Koszmider.*
(ii) *All operators on $\mathscr{C}(K)$ are centripetal and the space $K\backslash F$ is C^*-embedded into K for every finite $F \subseteq K$.*

Proof. (ii) ⇒ (i) follows trivially from Theorem 4.2 and the proof of (i) ⇒ (ii) is a slight modification of the proof of Theorem 4.2 which can be found in [6], so we will only outline the main idea and leave the details to the reader.

Suppose that there exists a set $F = \{x_1, \dots, x_n\} \subseteq K$ and a continuous bounded function $h : K\backslash F \to [0, 1]$ which cannot be extended to a continuous function on the whole of K. Pick a set of functions $\{g_i\}_{1\leqslant i\leqslant n}$ in $\mathscr{C}(K)$ which satisfy the following properties:

$$\|g_i\| = 1, \quad g_i(x_j) = \delta_{ij}, \quad \sum_{i=1}^{n} g_i \equiv 1.$$

One way of constructing g_i would be the following: let $U_i \ni x_i$ be open subsets of K with mutually disjoint closures. We define continuous functions $g_i : K \to [0, 1]$ as follows:

$$g_i(x_i) = 1, \quad g_i(K\backslash U_i) \equiv 0, \quad \text{if } i \neq n,$$
$$g_n(x) = 1 - \sum_{i=1}^{n-1} g_i(x).$$

Now, for each $f \in \mathscr{C}(K)$ and $x \in K$ define

$$(Tf)(x) = \begin{cases} h(x)\sum_{i=1}^{n}[f(x) - f(x_i)]g_i(x) & \text{if } x \neq x_i, \\ 0 & \text{otherwise.} \end{cases}$$

Precisely the same argument as in [6] shows now that for no $g \in \mathscr{C}(K)$ the operator $T - gI$ is weakly compact and so K is not Koszmider. □

Corollary 5.3. *Let K be a connected Koszmider space. Then for any finite subset F of K the space $K\backslash F$ is also connected.*

Proof. Endow the space $\{0, 1\}$ with the discrete topology and consider any continuous function $g : K\backslash F \to \{0, 1\}$. By Proposition 5.2, there exists a function $\tilde{g} \in \mathscr{C}(K)$ extending g. Note that $\tilde{g}(K)$ is a non-empty connected subset of $\mathbb{R}$ which is also finite because it consists of the finite sets $g(K\backslash F)$ and $\tilde{g}(F)$. This is only possible if $\tilde{g}(K)$ is a singleton which means, in particular, that g is a constant and hence $K\backslash F$ is connected. □

We are now ready to prove the main result of the section.

Theorem 5.4. *Let K be a connected Koszmider space. Then K is strongly rigid.*

Proof. Let $\phi : K \to K$ be a continuous non-identity function. Define $S = \{x \in K\colon \phi(x) \neq x\}$. Then S is a non-empty open set and hence is infinite. It follows from part (ii) of Theorem 4.4 that $\phi(S)$ is finite, and so $K\backslash\phi(S)$ is connected.

Note that $S \subseteq K\backslash\phi(S)$. Indeed, suppose that there exists $y \in \phi(S) \cap S$. Since $\phi(S)$ is finite, we can find an open set U such that $U \cap \phi(S) = \{y\}$. Furthermore, note that $\phi(K) = (K\backslash S) \cup \phi(S)$, and hence $(U \cap S) \cap \phi(K) = \{y\}$, meaning that y is an isolated point of $\phi(K)$ which contradicts the connectedness of $\phi(K)$ (unless $\phi(K)$ is a singleton in which case ϕ is a constant map).

So, S is an open subset of $K\backslash\phi(S)$. On the other hand, since $\phi(S) \subseteq K\backslash S$, we have that $S \cup \phi(S) = \phi^{-1}(\phi(S))$, which is a closed set, and so S is closed (hence clopen) in $K\backslash\phi(S)$. As S is non-empty, the connectedness of $K\backslash\phi(S)$ implies that $S = K\backslash\phi(S)$. Thus $\phi(K) = \phi(S)$. In particular, $\phi(K)$ is non-empty, finite and connected. This is only possible if $\phi(K)$ is a singleton, that is, ϕ is a constant map, as required. □

Note that the proof also works for any weakly Koszmider space K such that $K \setminus F$ is connected whenever F is finite. An example of such a space was constructed in Section 5 of [6]. Another set of examples is provided by Proposition 4.8 and Corollary 5.3, namely, if we identify two distinct points of a connected Koszmider space, the resulting space is strongly rigid.

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