Topological Reconstruction and Compactification Theory

Max F. Pitz
St Catherine's College
University of Oxford

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Abstract

This thesis investigates the topological reconstruction problem, which is inspired by the reconstruction conjecture in graph theory. We ask how much information about a topological space can be recovered from the homeomorphism types of its point-complement subspaces. If the whole space can be recovered up to homeomorphism, it is called reconstructible.

In the first part of this thesis, we investigate under which conditions compact spaces are reconstructible. It is shown that a non-reconstructible compact metrizable space must contain a dense collection of 1-point components. In particular, all metrizable continua are reconstructible. On the other hand, any first-countable compactification of countably many copies of the Cantor set is non-reconstructible, and so are all compact metrizable h-homogeneous spaces with a dense collection of 1-point components.

We then investigate which non-compact locally compact spaces are reconstructible. Our main technical result is a framework for the reconstruction of spaces with a maximal finite compactification. We show that Euclidean spaces $\mathbb{R}^n$ and all ordinals are reconstructible.

In the second part, we show that it is independent of ZFC whether the Stone-Čech remainder of the integers, ω^*, is reconstructible. Further, the property of being a normal space is consistently non-reconstructible. Under the Continuum Hypothesis, the compact Hausdorff space ω^* has a non-normal reconstruction, namely the space $\omega^* \setminus \{p\}$ for a P-point p of ω^*. More generally, the existence of an uncountable cardinal κ satisfying $\kappa = \kappa^{<\kappa}$ implies that there is a normal space with a non-normal reconstruction.

The final chapter discusses the Stone-Čech compactification and the Stone-Čech remainder of spaces $\omega^* \setminus \{x\}$. Assuming the Continuum Hypothesis, we show that for every point x of ω^*, the Stone-Čech remainder of $\omega^* \setminus \{x\}$ is an ω_2-Parovičenko space of cardinality $2^{2^{\mathfrak{c}}}$ which admits a family of $2^{\mathfrak{c}}$

disjoint open sets. This implies that under $2^{\mathfrak{c}} = \omega_2$, the Stone-Čech remainders of $\omega^* \setminus \{x\}$ are all homeomorphic, regardless of which point x gets removed.

Contents

Chapter 1

The topological reconstruction problem

1.1 The origins of the topological reconstruction problem

In this thesis we investigate the *topological reconstruction problem*: when do homeomorphism types of certain subspaces determine the original topological space up to homeomorphism?

The problem is inspired by the famous *reconstruction conjecture* in graph theory, proposed by S.M. Ulam and P.J. Kelly in 1941. Recently, this conjecture was named as one of the most influential open problems in graph theory [BN12]. Roughly speaking, it asks whether every finite graph with more than three vertices is uniquely determined by the structure of those subgraphs which are obtained by deleting a single vertex and all incident edges. These vertex-deleted subgraphs are called *cards* of the graph. Information about the reconstruction conjecture can be found in the survey by J.A. Bondy [Bon91]. The following figure illustrates the concepts used so far.

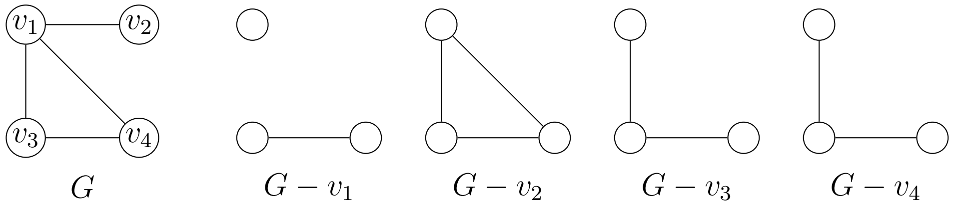

Figure 1.1: A graph G together with its four (unlabelled) cards

For the *topological reconstruction problem* we consider subspaces obtained by deleting single points, which we call cards, just like in the graph-theoretic case. With each topological space X, we associate a collection of cards—the *deck of cards*, denoted by $\mathcal{D}(X)$. For example, every card of the real numbers is homeomorphic to two disjoint copies of the reals. The topological reconstruction problem now asks how much information about a topological space can be recovered from its deck (for a more precise description of the reconstruction problem, see Section 1.2.1). Returning to our example of the reals, is it true that if all cards $X \setminus \{x\}$ of a space X are homeomorphic to two disjoint copies of the reals, must then X necessarily be homeomorphic to the reals? We set out to find criteria when spaces, or indeed, topological properties are identifiable—*reconstructible*—from their decks.

What makes this question particularly interesting is that in the topological world, contrary to the class of finite graphs, we encounter the phenomenon that objects can have proper substructures homeomorphic to the original object, and this makes the unqualified version of the reconstruction problem fail immediately.

As an illustration of this behaviour, consider the following well-known example of the countably splitting tree T_∞, which shows that the reconstruction conjecture for infinite graphs fails. Indeed, when deleting a vertex and all incident edges from T_∞, what remains are countably many copies of the original graph. So for all $n \in \mathbb{N}$, we have

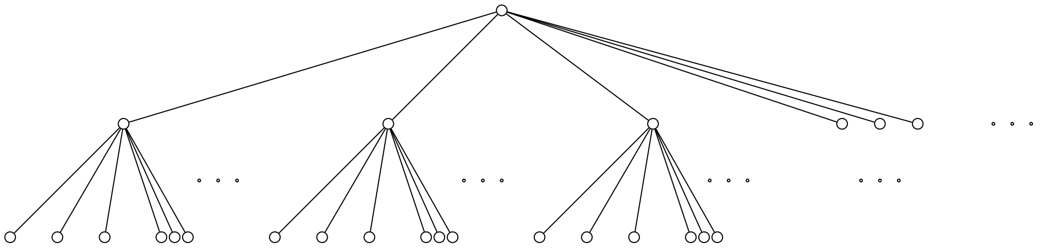

Figure 1.2: A part of T_∞

$$\mathcal{D}(T_\infty) = \mathcal{D}(n \cdot T_\infty) = \mathcal{D}(\aleph_0 \cdot T_\infty),$$

because a card of any of these graphs will always look like countably many copies of T_∞ (denoted by $\aleph_0 \cdot T_\infty$). It follows that T_∞ is not reconstructible. In fact, T_∞ has infinitely many non-isomorphic reconstructions: all graphs of the form $n \cdot T_\infty$ listed above are non-isomorphic, because they have a different number of connected components.

The topological analogue to this counterexample is given by the Cantor set C, which can be represented by $2^{\mathbb{N}}$, the space of all infinite binary sequences. We can imagine this space to correspond to the branches of the binary splitting tree. This way, we see that if for example we delete the left-most branch $(0, 0, 0, \ldots)$, the tree falls apart into countably many copies of itself: the first copy is given by all sequences starting with (1), the second copy by all sequences starting with $(0,1)$, and the nth copy by all sequences starting with $(0,0,\dots,0,1)$, and so forth. It follows that

$$\mathcal{D}(2^{\mathbb{N}}) = \left\{ \bigoplus_{n\in\omega} 2^{\mathbb{N}} \right\} = \mathcal{D}\left(\bigoplus_{n\in\omega} 2^{\mathbb{N}} \right).$$

And since the space $\bigoplus_{n\in\omega} 2^{\mathbb{N}}$ on the right-hand side is non-compact, it follows that the Cantor set is non-reconstructible.

This result is exciting, because, as we will see in this thesis, there are also many reconstructible topological spaces. In fact, I would argue that most common topological spaces are reconstructible, i.e. the behaviour of the Cantor set is the exception rather than the rule. Therefore, our main target is to find methods which allow us to decide whether certain classes of topological spaces are reconstructible or not. Ideally, we would like to find a precise topological characterisation of all reconstructible spaces—but for the moment, this aim seems to be out of reach, even for the class of compact metrizable spaces.

In the long term, we hope that our investigation of the topological reconstruction problem will help to obtain new insights regarding the original reconstruction conjecture for finite graphs and the reconstruction problem for infinite graphs. In this thesis, however, we are simply intrigued by the challenging topological problems arising in connection with topological reconstruction.

As we shall see, the topological reconstruction problem is closely connected to the study of compactifications of topological spaces. Both the theory of Stone-Čech compactifications as well as the theory of finite compactifications provide useful and elegant tools allowing us to obtain strong affirmative reconstruction results. Interestingly, this symbiosis also works in the other direction, and questions about topological reconstruction raise new problems in compactification theory, some of which we will consider in the last chapter of this thesis.

Topological reconstruction also provides surprising insights into the question which topological properties are of a truly global nature and which can rather be considered as local properties, i.e. verified from smaller fragments of a space. For example, we consider Hausdorffness to be a local property, but compactness to be a global property. However, for a property like normality, it is not at all easy to decide whether it is necessarily of a global nature. In this thesis, we present a partial answer: assuming the Continuum Hypothesis, normality is in general not reconstructible from the deck of cards—from this perspective, it is a truly global property.

1.2 A crash course in topological reconstruction

1.2.1 The set-up Embracing the terminology from graph theory, we say that a topological space Y is a *card* of another space X if Y is homeomorphic to $X \setminus \{x\}$ for some x in X. The *deck* of a space X is a transversal for the non-homeomorphic cards of X, i.e. an object recording the topologically distinct subspaces one can obtain by deleting singletons from X.

Formally, for a space X we denote by $[X]_\sim$ the homeomorphism type of X. The deck of X can then be defined as the set $\mathcal{D}(X) = \{[X \setminus \{x\}]_\sim \colon x \in X\}$, and cards of X correspond to elements of the deck of X. Note that we deliberately identify a card, which is a concrete topological space Y, with the class of spaces homeomorphic to Y. In practice this identification never causes confusion, and, as in the following examples, we will simply state which cards occur.

If Y is a card of the real line $\mathbb{R}$, then Y is homeomorphic to two copies of the real line. We write this as $\mathcal{D}(\mathbb{R}) = \{\mathbb{R} \oplus \mathbb{R}\}$. Similarly $\mathcal{D}(\mathbb{R}^n) = \{\mathbb{R}^n \setminus \{0\}\}$ for all $n \geq 1$. In the case of the unit interval I we have $\mathcal{D}(I) = \{[0,1), [0,1) \oplus [0,1)\}$. For the sphere, stereographic projection gives $\mathcal{D}(S^n) = \{\mathbb{R}^n\}$. The Cantor set has the deck $\mathcal{D}(C) = \{C \setminus \{0\}\}$. Lastly, Cantor's back-and-forth method [Eng89, 4.3.H] gives $\mathcal{D}(\mathbb{Q}) = \{\mathbb{Q}\}$ and $\mathcal{D}(P) = \{P\}$ for the rationals $\mathbb{Q}$ and the irrationals P.

We now introduce the central concept of this thesis, the notion of reconstruction. Given topological spaces X and Z, we say that Z is a *reconstruction of* X if their decks agree. A topological space X is said to be *reconstructible* if the only reconstructions of it are the spaces homeomorphic to X. In the same spirit, we say that a property of topological spaces is reconstructible if it is preserved under reconstruction.

Formally, a space X is reconstructible if $\mathcal{D}(X) = \mathcal{D}(Z)$ implies $X \cong Z$ and a property $\mathcal{P}$ of topological spaces is reconstructible if for all spaces X and Z, $\mathcal{D}(X) = \mathcal{D}(Z)$ implies "X has $\mathcal{P}$ if and only if Z has $\mathcal{P}$".

In the author's Master's thesis [Pit11], or the corresponding preprint [PS**], it is shown that all spaces introduced in the examples above are reconstructible, apart from the Cantor set: here we have $\mathcal{D}(C) = \mathcal{D}(C \setminus \{0\})$. In particular, compactness is a non-reconstructible property. Interestingly, the Cantor set is only one example of a whole family of non-reconstructible spaces. Spaces which only have λ different homeomorphism types amongst their non-empty open subspaces (for some cardinal λ) are said to be of *diversity* λ [NPR96]. The Cantor set is a compact Hausdorff space of diversity two, and it is easy to see that in fact every infinite such space X is non-reconstructible, as $\mathcal{D}(X) = \mathcal{D}(X \setminus \{x\})$. Hence, for example, the Double Arrow space D and also the product $D \times C$ are non-reconstructible [NPR96]. Note, however, that the Cantor set is the only metric example of an infinite compact Hausdorff space of diversity two [SG75].

Thus, not all spaces and properties are reconstructible. In this thesis, we investigate which precise mechanisms make some spaces reconstructible, and other spaces non-reconstructible.

1.2.2 Reconstructing separation axioms and metrizability Our first positive reconstruction result says that hereditary separation properties are reconstructible. For a precise definition of these separation properties, and other common topological concepts, we refer the reader to our standard reference, Engelking's *General Topology* [Eng89]. The topological spaces investigated in this thesis are usually at least Tychonoff.

Theorem 1.2.1 ([Pit11, 5.2.1])**.** *For topological spaces containing at least three points, hereditary separation axioms are reconstructible. Normality (resp. paracompactness) is reconstructible provided that the space has at least one normal (paracompact) card.* □

In particular, it follows that being a T_1, a Hausdorff or a Tychonoff space is reconstructible. Since the properties T_1, Hausdorff and Tychonoff are hereditary, a space X is T_1 (Hausdorff, Tychonoff) if and only if every card in the deck $\mathcal{D}(X)$ is T_1 (Hausdorff, Tychonoff). Results along these lines also hold for the other hereditary separation properties. From now on, we will assume that all spaces discussed in this thesis contain at least three points.

The reconstruction result about normality gives a condition which is sufficient but not necessary. For an uncountable cardinal κ, the Cantor cube 2^κ is an example of a reconstructible compact Hausdorff space where all cards are non-normal [Pit11, 5.2.3 & 6.2.6]. The question whether normality is reconstructible was one of the major open questions in [Pit11]. In Chapter 4 of this thesis we present a (partial) answer: normality is consistently non-reconstructible: under the Continuum Hypothesis, the Stone-Čech remainder of the integers provides a counterexample.

Finally, the property of being metrizable or even completely metrizable, is reconstructible.

Theorem 1.2.2 ([Pit11, 5.4.7 & 5.4.10])**.** *For topological spaces containing at least three points, the property of being (completely) metrizable is reconstructible.* □

1.2.3 Reconstructing local properties When investigating further topological properties, it is often useful to assume some of the lower separation axioms. Some authors even include those in the definition of the relevant property. As an example we show that *local* topological properties are reconstructible in T_1-spaces.

For a topological property $\mathcal{P}$, a topological space X and a point $x \in X$, we say that X is *locally $\mathcal{P}$ at x* (and write $(x, X) \vDash \mathcal{P}$) if the neighbourhood filter of x has a basis of sets which satisfy $\mathcal{P}$. Similarly, we say X is *locally $\mathcal{P}$* if X is locally $\mathcal{P}$ at x for all $x \in X$.

In other words, X is locally $\mathcal{P}$ if for every x and open neighbourhood U of x there is $A \subset X$ such that $x \in \operatorname{int}(A) \subset A \subset U$ and A is $\mathcal{P}$. Examples of local properties are local compactness, local connectedness and local metrizability, but also seemingly global properties like being dense-in-itself or being zero-dimensional (in the presence of regularity, local zero-dimensionality is equivalent to zero-dimensionality and every zero-dimensional T_1-space is regular).

Theorem 1.2.3. *In the realm of T_1-spaces, "being locally $\mathcal{P}$" is reconstructible for all topological properties $\mathcal{P}$.*

Proof. We prove that for a topological property $\mathcal{P}$, X is locally $\mathcal{P}$ if and only if all cards are locally $\mathcal{P}$. For the direct implication, note that a local property is hereditary with respect to open subspaces. Conversely, take an arbitrary $x \in X$ and choose $y \neq x$. A neighbourhood basis of x in the card $X \setminus \{y\}$ is a neighbourhood basis of x in X. □

Theorem 1.2.4. *In the realm of T_1-spaces, the number of points x such that the space is locally $\mathcal{P}$ at x is reconstructible for all topological properties $\mathcal{P}$.*

Proof. Let X be a T_1-space and note that for $x \in X$, we have as in the previous theorem

$$(x, X) \vDash \mathcal{P} \text{ if and only if } (x, Y) \vDash \mathcal{P}$$

for all cards $Y = X \setminus \{y\}$ with $y \neq x$. Since discrete spaces are reconstructible, we may assume without loss of generality that X is infinite. It follows

$$|\{x \in X \colon (x, X) \vDash \mathcal{P}\}| = \max\{|\{y \in Y \colon (y, Y) \vDash \mathcal{P}\}| \colon Y \in \mathcal{D}(X)\}.$$

Indeed, by our initial observation, we have $\geq$ always. If $|\{x \in X \colon (x, X) \vDash \mathcal{P}\}|$ is finite, then for any $y \in X$ with $(y, X) \nvDash \mathcal{P}$, the card $Y = X \setminus \{y\}$ witnesses equality in the above equation. And if $|\{x \in X \colon (x, X) \vDash \mathcal{P}\}|$ is infinite, then every card $Y \in \mathcal{D}(X)$ witnesses equality. □

1.2.4 Reconstructing cardinal invariants

It is obvious that the cardinality of a topological space is reconstructible; evidently, every space X contains one point more than any of its cards. Interestingly, it turns out that most other common cardinal invariants are also reconstructible.

Theorem 1.2.5 ([Pit11, §5.1])**.** *The cardinal invariants weight, density, character and cellularity are reconstructible.* □

1.2.5 Reconstruction and compactness

The example of the Cantor space shows that in general, compactness is not reconstructible. There are, however, occasions where compactness can indeed help to reconstruct topological spaces.

We begin with the observation that compact Hausdorff spaces containing an isolated point are reconstructible. For example, $\beta\mathbb{N}$ or every successor ordinal is reconstructible.

Theorem 1.2.6 ([Pit11, 6.2.1])**.** *Every compact Hausdorff space containing isolated points is reconstructible.* □

Indeed, this follows from the fact that Hausdorffness is reconstructible (Theorem 1.2.1) plus the fact that a card of a Hausdorff space is compact only if the deleted point was isolated.

Theorem 1.2.7. *If a compact Hausdorff space only has compact reconstructions then it is reconstructible.*

Proof. If a compact Hausdorff space contains an isolated point, the claim follows from Theorem 1.2.6. Otherwise, the claim follows from the fact that every reconstruction must be the 1-point compactification of some (and in fact every) card. □

This theorem explains why reconstructing compactness is so hard: in Hausdorff spaces, reconstructing compactness is in fact equivalent to reconstructing the space itself. We now come to a particular class of compact Hausdorff spaces that we can reconstruct. We call a space X a *Stone-Čech space* if it is the Stone-Čech compactification of one of its proper subspaces.

Theorem 1.2.8 ([Pit11, 6.2.5])**.** *Stone-Čech spaces are reconstructible.* □

The proof idea is as follows. By Theorem 1.2.6 we may assume that the Stone-Čech space X contains no isolated points, which is a reconstructible property by Theorem 1.2.3. Then X is the Stone-Čech compactification of at least one card $Y \in \mathcal{D}(X)$

and therefore Y does not possess a compactification with more than one additional point. If X had a non-compact reconstruction Z, then Z would be locally compact by Theorem 1.2.6, and its one-point compactification would be a 2-point compactification of Y, a contradiction. It follows that every reconstruction of X must be compact, so we can apply Theorem 1.2.7.

It is a generalisation of this approach that will be investigated in the next two chapters, yielding that many interesting classes compact and locally compact are reconstructible.

1.3 Contents of this thesis

In this thesis we explore the interplay between the theory of compactifications and the reconstruction of topological spaces. The above results already provided hints about the relationship between reconstruction and compactifications: Even though compactness itself is not reconstructible, the property whether a space has a (finite) compactification is reconstructible by Theorems 1.2.1 and 1.2.3.

We will see that the theory of compactifications provides a powerful framework allowing us to obtain many positive reconstruction results (Chapters 2 and 3). We will also demonstrate in turn that topological reconstruction provides an exciting context to generate new questions in compactification theory (Chapters 4 and 5).

This thesis consists of four chapters. The next sections contain a brief outline of the main results obtained in each chapter.

1.3.1 Chapter 2: Reconstructing compact spaces In this chapter we prove positive results for the reconstruction problem in the context of compact spaces. We begin our discussion with an introduction to the theory of finite compactifications, and then establish our Reconstruction Result for Compact Spaces, underpinning all later results in this chapter: *Compact spaces are reconstructible provided they have a card with a maximal finite compactification.*

We complement this result by showing that every connected, compact Hausdorff space whose weight is strictly smaller than its cardinality has a card with a maximal 1- or 2-point compactification. By our Reconstruction Result for Compact Spaces, every such space is reconstructible. In particular, every metrizable continuum is reconstructible. As a nice consequence we get that every graph viewed as a topological space is reconstructible in the topological sense.

With these results at hand, we set out to find a precise characterisation which compact metrizable spaces are reconstructible. Our main result is a characterisation under which conditions a compact metrizable space without isolated points has a card with a maximal finite compactification. For instance, a compact metrizable space in which the collection of 1-point components does not form a dense G_δ has a card with a maximal 1- or 2-point compactification, and hence is reconstructible. In other words, non-reconstructible compact metric spaces must be highly disconnected.

We conclude the discussion in Section 2.5 with some concrete examples of non-reconstructible compact spaces, illustrating the frontiers of our theory. Most importantly, we establish that every metrizable compactification of $C \setminus \{0\}$, and every compact metrizable h-homogeneous space with a dense set of 1-point components is non-reconstructible. From these observations we find that there are as many (non-homeomorphic) reconstructible compact metrizable spaces as there are (non-homeomorphic) non-reconstructible ones.

1.3.2 Chapter 3: Reconstructing locally compact spaces In this chapter we continue our quest for affirmative reconstruction results and inquire under which conditions non-compact, locally compact spaces are reconstructible.

In the first half of this chapter, we focus our attention on locally compact spaces with maximal finite compactifications. First, we improve Theorem 1.2.6 from above and show that every space with isolated points that has a maximal finite compactification is reconstructible. In particular, every limit ordinal of uncountable cofinality is reconstructible. We then show that if a locally compact space X with a maximal N-point compactification has a card with a maximal finite compactification, then every reconstruction of X also has a maximal N-point compactification. In the case where one card of X has an $(N+1)$-point compactification, we prove our Reconstruction Candidates Theorem: *The space X has at most N different non-homeomorphic reconstructions.* We use this "shortlist" to show that under some mild additional assumptions, every space with a maximal 1- or 2-point compactification is reconstructible.

In the second half of the chapter we show that all ordinal numbers are reconstructible, regardless of their cofinality. For this, we first show that if two ordinals are reconstructions of each other, they must be homeomorphic. This step is then complemented by the fact that the topological property of "being homeomorphic to an ordinal" is reconstructible. The crucial technical step involves a characterisation under which conditions a topological sum of spaces is homeomorphic to an ordinal.

1.3.3 Chapter 4: Reconstructing normality In this chapter we examine the Stone-Čech remainder $\omega^* = \beta\omega \setminus \omega$ of the countable discrete space ω. We show that the question whether this space is reconstructible is independent of the axioms of set theory ZFC. More precisely, we show that under the Continuum Hypothesis (CH), the spaces ω^* and $\omega^* \setminus \{p\}$ for a point p with a nested neighbourhood base are non-homeomorphic reconstructions of each other. Since $\omega^* \setminus \{p\}$ is non-normal under CH, by a result independently due to Rajagopalan and Warren, ω^* provides a consistent example that normality and paracompactness are non-reconstructible.

However, by a result by van Douwen, Kunen and van Mill, it is consistent that ω^* is a Stone-Čech space. Thus, Theorem 1.2.8 shows that ω^* is consistently reconstructible.

After proving the above reconstruction results for ω^*, we generalise our methods to the spaces S_κ, the unique κ-Parovičenko spaces of weight κ. A rigorous introduction to κ-Parovičenko spaces can be found in Section 4.2. Intuitively, however, κ-Parovičenko spaces generalise the behaviour of ω^* under CH to spaces of (potentially larger) weight κ. Replacing the role of CH, which is equivalent to $\omega_1 = \omega_1^{<\omega_1}$, the spaces S_κ exist only for cardinals κ with the property $\kappa = \kappa^{<\kappa}$.

Our main result is that assuming $\kappa = \kappa^{<\kappa}$, the spaces S_κ and $S_\kappa \setminus \{p\}$ for a point p with a nested neighbourhood base are non-homeomorphic reconstructions of each other. Since $S_\kappa \setminus \{p\}$ is non-normal, we obtain the improved result that the assumption $\kappa = \kappa^{<\kappa}$ implies that normality and paracompactness are non-reconstructible.

Our key technical result is a classification of finite compactifications of cards of ω^* under CH: Spaces $\omega^* \setminus \{x\}$ have arbitrarily large finite compactifications, and they are all homeomorphic to ω^*. Similarly, $S_\kappa \setminus \{x\}$ has arbitrarily large finite compactifications, all of which are homeomorphic to S_κ.

1.3.4 Chapter 5: The Stone-Čech compactification of $S_\kappa \setminus \{x\}$ In this chapter we investigate Stone-Čech compactifications and their remainders of cards of ω^* and S_κ. This investigation is motivated by the result mentioned above that *finite* compactifications of these cards are surprisingly well-behaved. With this in mind, it is natural to ask whether also the Stone-Čech compactifications of these spaces preserve the structural properties of ω^* and S_κ.

Our results in this chapter show that the answer is generally speaking yes. We first prove that the operation of the Stone-Čech compactification preserves the defining κ-Parovičenko properties of S_κ: The Stone-Čech compactification of $S_\kappa \setminus \{x\}$ is again a κ-Parovičenko space—but it is not homeomorphic to S_κ because its weight jumps from κ to 2^κ.

On the other hand, when focussing our attention to the remainders of these compactifications, we find that the Stone-Čech remainder of $S_\kappa \setminus \{x\}$ is a κ^+-Parovičenko space of weight 2^κ, and this holds for every point x in S_κ. As a consequence, assuming CH plus $2^{\omega_1} = \omega_2$, the Stone-Čech remainder of any $\omega^* \setminus \{x\}$ is homeomorphic to the unique ω_2-Parovičenko space of weight ω_2, independently of which point x gets removed. This is surprising, considering that subspaces $\omega^* \setminus \{x\}$ and $\omega^* \setminus \{y\}$ are typically very different. In general, we see that when assuming $\kappa^+ = 2^\kappa$ in addition to the existence of S_κ, all Stone-Čech remainders of $S_\kappa \setminus \{x\}$ are homeomorphic for all choices of x.

We complete this chapter with an investigation of cardinal invariants of the remainders of $S_\kappa \setminus \{x\}$ if we do not assume $\kappa^+ = 2^\kappa$. It turns out that cardinality, weight and cellularity are in all cases as large as possible, namely $(S_\kappa \setminus \{x\})^*$ has cardinality 2^{2^κ}, and weight and cellularity 2^κ. It remains an open problem whether the remainders of $S_\kappa \setminus \{x\}$ and $S_\kappa \setminus \{y\}$ for different points x and y can consistently be non-homeomorphic.

Chapter 2

Reconstructing compact spaces

This chapter contains results which will allow us to decide for large classes of compact spaces whether they are reconstructible. These results are motivated by our desire to understand the deeper reasons behind the result that the Cantor set is non-reconstructible. One possible abstraction of this phenomenon has already been mentioned in the introduction: The Cantor set belongs to the class of compact Hausdorff spaces of diversity two, and every such space is non-reconstructible. However, since the Cantor set is the unique compact metrizable space of diversity two, this immediately suggests the question whether the Cantor set is the only non-reconstructible compact metrizable space.

We shall see in this chapter that this question has a negative answer, and that there are many more non-reconstructible compact metrizable spaces. On the positive side, we can prove that all *connected* compact metrizable spaces are reconstructible. Indeed, we can show the even stronger result that every non-reconstructible compact metric space must contain a dense collection of 1-point components. From this, we obtain a partial uniqueness result: The Cantor set is the unique *homogeneous* non-reconstructible compact metrizable space.

However, we will prove that every metrizable compactification of $C \setminus \{0\}$ is non-reconstructible, and also that every compact metrizable h-homogeneous space with a dense collection of 1-point components is non-reconstructible. Thus, the Cantor set is by far not the only example of a non-reconstructible compact metrizable space.

This chapter is organised as follows. In Section 2.1 we recall elements of the theory of finite compactifications and show that compact spaces are reconstructible provided they have a card with a maximal finite compactification (Theorem 2.1.10).

In Section 2.2 we characterise when removing a point from a Hausdorff continuum leaves behind a card with a maximal finite compactification. With this characterisation at hand we show in Section 2.3 that every Hausdorff continuum whose cardinality is strictly bigger than its weight (in particular: every metrizable continuum) has a card with a maximal 2-point compactification and hence is reconstructible (Theorem 2.3.4). It follows that every graph viewed as a topological space is reconstructible in the topological sense.

The last two sections in this chapter investigate which compact metrizable spaces are reconstructible. In Section 2.4 we characterise all compact metrizable spaces that have a card with a maximal finite compactification (Theorem 2.4.3). As an aside we obtain the result that a compact metrizable space without isolated points has a card with a maximal finite compactification if and only if it has a card with a maximal 1- or 2-point compactification. Examining spaces where every card has arbitrarily large finite compactifications, we find sufficient conditions for a compact metrizable space to be non-reconstructible.

In Section 2.5 we present various techniques for building non-reconstructible compact spaces. This shows that there are as many non-reconstructible compact metrizable spaces as there are reconstructible ones.

2.1 Theory of maximal finite compactifications I

2.1.1 Overview In this section we investigate spaces that have a maximal finite compactification. These are spaces that have compactifications with N-point remainder, but no compactification with $N+1$-point remainder for some $N \in \omega$.

There are many examples of topological spaces with a maximal finite compactification. For example, the real line has a maximal 2-point compactification, and Euclidean spaces $\mathbb{R}^n$ for $n \geq 2$ have a maximal 1-point compactification. These examples also show that having a maximal *finite* compactification does not exclude the possibility of having infinite compactifications. It turns out that spaces with a maximal finite compactification exist in abundance: One of the main results of Section 2.3 will be that every metrizable compact connected space has a card with a maximal two-point compactification.

For different types of examples, note that the first uncountable ordinal ω_1 and, more generally, every ordinal α of uncountable cofinality are spaces with a maximal 1-point compactification, because their Stone-Čech compactification is $\alpha+1$. A similar example is given by the Tychonoff plank [Eng89, 3.12.20], which also has a one-point Stone-Čech remainder.

2.1.2 Maximal finite compactifications

A Hausdorff compactification γX of a space X is called an *N-point compactification* (for $N \in \omega$) if its remainder $\gamma X \setminus X$ has cardinality N. A *finite compactification* of X is an N-point compactification for some $N \in \omega$. We say νX is a *maximal* N-point compactification if no other finite compactification γX has a strictly larger remainder, i.e. whenever $|\gamma X \setminus X| = M$ then $M \leq N$. We agree on the convention that a compact Hausdorff space has a maximal 0-point compactification.

It is easy to see that if a space has an N-point compactification, then by identifying points in the remainder we can obtain M-point compactifications for all $1 \leq M \leq N$. Later, in Theorem 3.1.4, we will see that this observation has a converse: if νX is a maximal finite compactification of X, then every other finite compactification is a quotient of νX. Thus, in relation to all finite compactifications, νX plays the role of the Stone-Čech compactification βX—up to equivalence, it is the unique largest element. In particular, from now on we will say that νX is the maximal finite compactification.

Note that a space X has a finite compactification if and only if it is locally compact and Hausdorff. Here and in the following, we denote by αX the Alexandroff 1-point compactification of a locally compact Hausdorff space X. The following theorem, due to Magill (1965), gives an internal characterisation when a space has an N-point compactification for given $N \in \omega$.

Theorem 2.1.1 ([Mag65, 2.1]). *For $N \geq 1$, a Tychonoff space X has an N-point compactification if and only if X is locally compact and contains N non-empty open pairwise disjoint subsets G_i ($1 \leq i \leq N$) such that $K = X \setminus \bigcup_{i=1}^{N} G_i$ is compact, but the union $K \cup G_i$ is non-compact for each i.* □

Following [Mag65, 2.2], we call a family $\{K\} \cup \{G_i\}_{i=1}^{N}$ with the above properties an *N-star* . For example, $K = [-2, 2]$, $G_1 = (-\infty, -2)$ and $G_2 = (2, \infty)$ forms a 2-star of $\mathbb{R}$, but the compact $L = [-2, -1] \cup [1, 2]$ with open sets $H_1 = (-\infty, -2)$, $H_2 = (-1, 1)$ and $H_3 = (2, \infty)$ does not form a 3-star of $\mathbb{R}$, since $L \cup H_2$ is compact.

An in-depth discussion of the theory of maximal finite compactifications will follow in the next chapter, in Section 3.1. For now, we only develop the necessary theory for our first reconstruction result in Theorem 2.1.10.

In the following, we are mainly interested in the case when cards of compact Hausdorff spaces have maximal finite compactifications. For $N \geq 1$, we say a point $x \in X$ is *N-splitting* in X if $X \setminus \{x\} = X_1 \oplus \ldots \oplus X_N$ such that $x \in \overline{X_i}$ for all $i \leq N$. Further, we say that x is *locally N-splitting* (in X) if there exists a neighbourhood U

of x, i.e. a set U with $x \in \operatorname{int}(U)$, such that x is N-splitting in U. Note that whenever we have neighbourhoods U and V of x such that $V \subset U$ and x is N-splitting in U, then x is N-splitting in V as well.

Lemma 2.1.2. *A card $X \setminus \{x\}$ of a compact Hausdorff space X has an N-point compactification if and only if x is locally N-splitting in X.*

Proof. As the result is clear for isolated x, we may suppose that x is non-isolated.

For the direct implication, assume that $X \setminus \{x\}$ has an N-point compactification for some $N \geq 1$. By Theorem 2.1.1, the space $X \setminus \{x\}$ contains an N-star $\{K\} \cup \{G_i\}_{i=1}^N$. We claim that $U = X \setminus K$ is a neighbourhood as required. Indeed, since K is compact and hence closed in X, the set U is an open neighbourhood of x. Because $K \cup G_i$ is non-compact and closed in $X \setminus \{x\}$ (since it is a complement of open sets), we have $x \in \overline{G_i}^X$. It follows that $U \setminus \{x\} = G_1 \oplus \cdots \oplus G_N$ is as desired.

Conversely, if there is a neighbourhood U of x such that $U \setminus \{x\} = U_1 \oplus \cdots \oplus U_N$ with $x \in \overline{U_i}^X$ for all $i \leq N$, then $\{X \setminus U\} \cup \{U_i\}_{i=1}^N$ is an N-star of $X \setminus \{x\}$. Indeed, no set $U_i \cup X \setminus U$ can be compact because it is not closed in X. Thus, $X \setminus \{x\}$ has an N-point compactification by Theorem 2.1.1. □

2.1.3 The existence of cards with finite compactifications is a local property We begin by introducing the following new cardinal invariant for locally compact spaces. For a point x in a locally compact Hausdorff space X and a compact neighbourhood D of x, we let $\operatorname{ends}(x, D)$ be the largest number $N \in \omega$ such that $D \setminus \{x\}$ has a finite N-point compactification. If there is no largest such number, we say $\operatorname{ends}(x, D) = \infty$. Thus, $\operatorname{ends}(x, D)$ takes values in $\omega \cup \{\infty\}$.

Lemma 2.1.3. *Let X be a locally compact Hausdorff space and let $x \in X$. For any two compact neighbourhoods D_1 and D_2 of x, we have* $\operatorname{ends}(x, D_1) = \operatorname{ends}(x, D_2)$.

Proof. Since the statement is symmetric, it suffices to prove only one inequality, say "$\leq$". For this, we have to show that if $D_1 \setminus \{x\}$ has an N-point compactification, then so does $D_2 \setminus \{x\}$.

So assume that $D_1 \setminus \{x\}$ has an N-point compactification. By Lemma 2.1.2, there is a D_1-open neighbourhood $U \subset D_1$ of x witnessing that x is locally N-splitting in D_1. Note that we may assume that $U \subset \operatorname{int}_X(D_1) \cap \operatorname{int}_X(D_2)$. But then $U \subset D_2$ witnesses that x is locally N-splitting in D_2. Another application of Lemma 2.1.2 completes the proof. □

Thus, the value of $\operatorname{ends}(x, D)$ is independent of the chosen neighbourhood D. In the following, we denote by $\operatorname{ends}(x) = \operatorname{ends}(x, X)$ the value of $\operatorname{ends}(x, D)$ for an arbitrary compact $D \subset X$ with $x \in \operatorname{int}(D)$. We observe that $\operatorname{ends}(x) = 0$ if and only if x is isolated.

Corollary 2.1.4. *For a point x in a compact Hausdorff space X we have* $\operatorname{ends}(x, X) = N$ *if and only if the card $X \setminus \{x\}$ has a maximal N-point compactification.* $\square$

Corollary 2.1.5. *For a point x in a locally compact, non-compact Hausdorff space X we have* $\operatorname{ends}(x, X) = N$ *if and only if $\alpha X \setminus \{x\}$ has a maximal N-point compactification.* $\square$

Corollary 2.1.6. *A locally compact, non-compact space X has a maximal N-point compactification if and only if* $\operatorname{ends}(\infty, \alpha X) = N \geq 1$. $\square$

For a locally compact space X we write $\mathcal{R}_n(X) = \{x \in X : \operatorname{ends}(x, X) = n\}$, where $n \in \omega \cup \{\infty\}$. We make the convention that $n + \infty = \infty = \infty + n$ for all $n \in \omega \cup \{\infty\}$.

Following [Eng89, 2.4.12], for $A \subset X$ we denote by X/A the quotient space that identifies A with a single point. It is easy to verify that if X is locally compact Hausdorff and $A \subset X$ is compact, then X/A is locally compact Hausdorff as well.

Lemma 2.1.7. *Let X be a locally compact Hausdorff space X. If $x \in \mathcal{R}_n(X)$ and $y \in \mathcal{R}_m(X)$ then $\{x, y\} \in \mathcal{R}_{n+m}(X/\{x, y\})$ for all $n, m \in \omega \cup \{\infty\}$.*

Proof. Suppose first that $n, m \in \omega$. We may assume that X is compact, as otherwise, we can use Corollary 2.1.5 and work in αX. Since the quotient $X/\{x, y\}$ is compact Hausdorff, it suffices by Corollary 2.1.4 to show that $X \setminus \{x, y\}$ has a maximal $(n+m)$-point compactification.

Fix disjoint compact sets C_x and C_y such that $x \in \operatorname{int}(C_x)$ and $y \in \operatorname{int}(C_y)$. Since by assumption $C_x \setminus \{x\}$ has an n-point compactification, Lemma 2.1.2 implies that there is an X-open set U with $x \in U \subset \operatorname{int}(C_x)$ witnessing that x is locally n-splitting in C_x. Similarly, there is an open set V with $y \in V \subset \operatorname{int}(C_y)$ witnessing that y is locally m-splitting in C_y. It follows that $(U \cup V)/\{x, y\}$ is a neighbourhood witnessing that the collapsed point $\{x, y\}$ is locally $(n + m)$-splitting in $X/\{x, y\}$. By Lemma 2.1.2, the space $X \setminus \{x, y\}$ has an $(n + m)$-point compactification.

To see that this compactification is maximal, assume for a contradiction that $X \setminus \{x, y\}$ has an N-point compactification for $N = n + m + 1$. By Lemma 2.1.2, there is an open neighbourhood W such that $\{x,y\} \in W \subset X/\{x,y\}$ witnessing that $\{x,y\}$ is locally N-splitting in $X/\{x,y\}$, i.e.

$$W \setminus \{x,y\} = W_1 \oplus \ldots \oplus W_N \text{ such that } \{x,y\} \cap \overline{W_i}^X \neq \emptyset.$$

Now consider

$$I_x = \left\{1 \leq i \leq N \colon x \in \overline{W_i}^X\right\} \text{ and } I_y = \left\{1 \leq i \leq N \colon y \in \overline{W_i}^X\right\}.$$

By the pigeon hole principle, we have either $|I_x| > n$ or $|I_y| > m$. Hence, the open sets

$$U = \{x\} \cup \bigcup_{i \in I_x} W_i \text{ and } V = \{y\} \cup \bigcup_{i \in I_y} W_i$$

witness either that x is locally $(n+1)$-splitting or that y is locally $(m+1)$-splitting in X, contradicting that $x \in \mathcal{R}_n(X)$ and $y \in \mathcal{R}_m(X)$.

This completes the proof for finite n and m. In the case where either n or m is infinite, simply observe that the first part of the proof implies that $X \setminus \{x,y\}$ has arbitrarily large finite compactifications. □

Corollary 2.1.8. *Let X be a compact Hausdorff space and $x \neq y \in X$. Then $X \setminus \{x,y\}$ has a maximal N-point compactification (for $N \in \omega \cup \{\infty\}$) if and only if* $\operatorname{ends}(x,X) + \operatorname{ends}(y,X) = N$. □

2.1.4 A reconstruction result for compact spaces

Equipped with the cardinal invariant $\operatorname{ends}(x)$, we now prove our fundamental reconstruction result for compact spaces. As $\operatorname{ends}(x)$ has been defined in terms of local neighbourhoods, it is reconstructible.

Lemma 2.1.9. *In a locally compact Hausdorff space X, $|\mathcal{R}_n(X)|$ is reconstructible for all $n \in \omega \cup \{\infty\}$.*

Proof. Since $\operatorname{ends}(x,X) = n$ is a local property, the assertion follows from Theorem 1.2.4. □

Theorem 2.1.10 (Reconstruction Result for Compact Spaces)**.** *Every compact Hausdorff space that has a card with a maximal finite compactification is reconstructible.*

Proof. Suppose X is a compact Hausdorff space that has a card with a maximal finite compactification. Using Corollary 2.1.4, we see that $\mathcal{R}_n(X) \neq \emptyset$ for some $n \in \omega$. Let N be minimal such that $\mathcal{R}_N(X) \neq \emptyset$ and fix $x \in \mathcal{R}_N(X)$.

Suppose for a contradiction that Z is a non-compact reconstruction of X. By Theorem 1.2.3, the space Z is locally compact. Find $z \in Z$ such that $Z \setminus \{z\} \cong X \setminus \{x\}$. Then $Z \setminus \{z\}$ has a maximal N-point compactification, so Corollary 2.1.8 implies that $\operatorname{ends}(z, Z) + \operatorname{ends}(\infty, \alpha Z) = N$. Since Z is non-compact, we have $\operatorname{ends}(\infty, \alpha Z) \geq 1$ and hence $\operatorname{ends}(z, Z) < N$. Thus, $\mathcal{R}_n(Z) \neq \emptyset$ for some $n < N$, and hence $\mathcal{R}_n(X) \neq \emptyset$ for the same $n < N$ by Lemma 2.1.9, contradicting minimality of N.

Thus, any reconstruction of X is compact. It follows from Theorem 1.2.7 that X is reconstructible. □

The theorem gives a sufficient condition for a compact space to be reconstructible. We shall see in Section 2.3 that it applies to all metrizable continua. Even more, in the metric case, we shall characterise in Theorem 2.4.3 to which compact spaces this reconstruction result applies. In Theorem 2.4.13, however, we present an example showing that Theorem 2.1.10 does not characterise the reconstructible compact spaces.

2.2 Maximal finite compactifications and continuum theory

2.2.1 Overview In this section we give a precise characterisation when removing a point from a continuum leaves behind a card with a maximal finite compactification. Following this characterisation we examine some well-known continua and determine, for example, that cards of the pseudoarc do not have 2-point compactifications, and cards of the Stone-Čech remainder of the real line do not have 3-point compactifications. It follows from the Reconstruction Result for Compact Spaces, Theorem 2.1.10, that all these spaces are reconstructible.

2.2.2 Background from continuum theory A *metrizable continuum* is a compact connected metrizable space, and a *Hausdorff continuum* is a compact connected Hausdorff space, not necessarily metrizable. Our basic reference for continuum theory is [Nad92].

In a topological space X, the (connected) *component* of a point x, denoted by $C(x) = C_X(x)$, is the union over all connected subspaces of X containing x. Since the closure of a connected set is connected, components are closed. Note that if a space has finitely many components, then components are in fact clopen. The *quasi-component* of a point x, denoted by $Q(x)$, is the intersection over all clopen sets of X containing x. Quasi-components are closed and $C(x) \subset Q(x)$ always holds. We will need two classical results from continuum theory.

Lemma 2.2.1 (Šura-Bura Lemma [Eng89, 6.1.23])**.** *In a compact Hausdorff space, components and quasi-components agree, i.e.* $C(x) = Q(x)$ *for all* $x \in X$. □

Lemma 2.2.2 (Boundary Bumping Lemma [Nad92, 5.4])**.** *The closure of every component of a non-empty proper open subset* U *of a Hausdorff continuum intersects the boundary of* U*, i.e.* $\overline{C_U(x)} \setminus U \neq \emptyset$ *for all* $x \in U$. □

2.2.3 Characterising maximal finite compactifications Let X be a topological space. A point $x \in X$ is called *separating* in X if $X \setminus \{x\}$ has a disconnection $A_1 \oplus A_2$ such that both A_1 and A_2 intersect $C_X(x)$. A point x is called *locally separating* in X if there is a neighbourhood U of x such that x is separating in U. These definitions are due to Whyburn [Why42, III.8-9].

We extend these definitions by saying that x is N*-separating* in X if $X \setminus \{x\}$ has a disconnection into N (clopen) sets $A_1 \oplus \cdots \oplus A_N$ such that all A_i intersect $C_X(x)$. Similarly, we say x is *locally* N*-separating* in X if there is a neighbourhood U of x such that x is N-separating in U. Note that if U witnesses that x is locally N-separating, then so does any neighbourhood V with $x \in V \subset U$.

A point x of a connected space X is called *cut point* if $X \setminus \{x\}$ is disconnected, and it is called N*-cut point* if $X \setminus \{x\}$ splits into at least N non-empty disjoint clopen sets. Note that in a connected space X, the notion of N-cut point and N-separating point coincide.

The following lemma establishes the connection between finite compactifications and locally separating points. Note that "locally N-separating" is a stronger condition than "locally N-splitting". So the next lemma is a natural sharpening of our previous characterisation of finite compactification of cards of a compact space in Lemma 2.1.2.

Lemma 2.2.3. *A card* $X \setminus \{x\}$ *of a Hausdorff continuum* X *has an* N*-point compactification if and only if* x *is locally* N*-separating in* X.

Proof. Let X be a Hausdorff continuum and suppose for a point x in X that the card $X \setminus \{x\}$ has an N-point compactification $Z = (X \setminus \{x\}) \cup \{\infty_1, \ldots, \infty_N\}$.

The space Z has at most N connected components. To see this, observe that by the Boundary Bumping Lemma 2.2.2, every component D of $X \setminus \{x\}$ must have $x \in \overline{D}^X$. In particular, every such component is non-compact. But since components are closed, it follows that in Z, every connected set D contains at least one of the points at infinity in its closure. And if two different components D and D' of $X \setminus \{x\}$ limit onto the same ∞_1, say, then $D \cup D' \cup \{\infty_1\}$ is a connected subset of Z. Since there are only N different points at infinity, there are at most N components in Z. In particular, every component of Z is clopen.

Now find disjoint open neighbourhoods U_i of ∞_i in Z. We may assume $U_i \subset C_Z(\infty_i)$. Since each $C_Z(\infty_i)$ is a continuum, it follows from the Boundary Bumping Lemma 2.2.2 that the component $D_i = C_{U_i}(\infty_i)$ is non-trivial. In particular, $D_i \setminus \{\infty_i\}$ is non-empty.

Now consider the quotient $Z/\{\infty_1, \ldots, \infty_N\}$ and denote the collapsed point by ∞. By uniqueness of the one-point compactification, this quotient is homeomorphic to X via a homeomorphism sending ∞ to x and being the identity elsewhere. Put

$$U = \left(\bigcup_{i=1}^{N} U_i \right) / \{\infty_1, \ldots, \infty_N\} \quad \text{and} \quad D = \left(\bigcup_{i=1}^{N} D_i \right) / \{\infty_1, \ldots, \infty_N\}.$$

By construction, we have $C_U(\infty) = D$. And since $U \setminus \{\infty\}$ disconnects into sets $U_1 \setminus \{\infty_1\} \oplus \ldots \oplus U_N \setminus \{\infty_N\}$, each of which intersects the component D in the non-empty set $D_i \setminus \{\infty_i\}$, it follows that ∞ is locally N-separating in $Z/\{\infty_1, \ldots, \infty_N\}$. Hence, so is x in X.

For the converse, note that if x is locally N-separating in X, then x is locally N-splitting, so the card $X \setminus \{x\}$ has an N-point compactification by Lemma 2.1.2. □

2.2.4 Examples: Using the order of points For a better illustration we discuss in detail which cards of well-known metrizable and Hausdorff continua have maximal finite compactifications. Using Lemma 2.2.3, we will describe two techniques allowing us to decide when a point of a continuum is at most locally N-separating.

Our first technique of finding cards of Hausdorff continua that have maximal finite compactifications uses the notion of the order of a point [Nad92, 9.3]. The *order of a point* x *in a space* X, denoted by $\operatorname{ord}(x, X)$, is the least cardinal β such that x has a neighbourhood base of open sets, each with a boundary of size at most β. If $\operatorname{ord}(x, X) = 1$ we say x is an *end point* of X.

Lemma 2.2.4. *Let X be a Hausdorff continuum. If* $\operatorname{ord}(x, X) = N$ *then x is at most locally N-separating in X.*

Proof. Suppose the point x is locally N-separating in X. There is a compact neighbourhood U of x such that $U \setminus \{x\}$ disconnects into $U_1 \oplus \cdots \oplus U_N$ such that $C_i = C_U(x) \cap U_i \neq \emptyset$. But if V is a neighbourhood of x missing at least one point from every C_i, the Boundary Bumping Lemma 2.2.2 forces the boundary of V to intersect every C_i. Thus, the boundary of V has cardinality at least N, and therefore $\operatorname{ord}(x, X)$ is at least N as well. □

There are numerous continua containing points of finite order. An *arc* is a continuum which is homeomorphic to the unit interval. A *simple closed curve* is a continuum which is homeomorphic to the unit circle in $\mathbb{R}^2$. An arc has two endpoints and all other points are of order two. Every point on a simple closed curve has order two.

A *topological graph* is a continuum which can be written as the union of finitely many arcs, any two of which are either disjoint or intersect only in one or both their end points. In particular, arcs and simple closed curves are topological graphs. The following structure theorem shows that topological graphs consist exclusively of points of finite order.

Theorem 2.2.5 ([Nad92, 9.10])**.** *A continuum is a topological graph if and only if all points of it have finite order, and all but finitely many points are of order two.* □

The theorem also shows that if we start with a usual connected graph, i.e. a finite set of vertices together with a symmetric binary relation representing edges, and identify edges with unit intervals, we get a topological graph. The converse direction is also true, but note that there is some ambiguity, as vertices of degree two and their two incident edges get absorbed in a single arc.

Corollary 2.2.6. *All cards of a topological graph have maximal finite compactifications. Hence, every topological graph is reconstructible.*

Proof. By Theorem 2.2.5 every point of a topological graph is of finite order. Lemmas 2.2.3 and 2.2.4 imply that every card has a maximal finite compactification. The reconstruction result now follows from Theorem 2.1.10. □

It is not hard to prove that for a topological graph X we have $\operatorname{ord}(x, X) = \operatorname{ends}(x, X)$, i.e. the order of a point is not only an upper bound but a precise measure for the largest finite compactification.

To see that this need not always be the case, consider the *topologists' sine curve*

$$T = \{0\} \times [-1, 1] \cup \left\{(x, \sin(x^{-1}))\colon x \in (0, 1]\right\} \subset \mathbb{R}^2.$$

The space T is a metrizable continuum, which is neither locally connected nor path-connected. It is clear that points of the form $(x, \sin(x^{-1}))$ are of order one or two, and that removing such a point from T leaves behind cards with maximal 1- and 2-point compactifications. However, the remaining points of the form $(0, x)$ have order $\aleph_0$ in T, but cards $T \setminus \{(0, x)\}$ always have a maximal 1-point compactification. Indeed, it is not hard to verify that no point of the form $(0, x)$ is locally 2-separating in T.

Corollary 2.2.7. *Every card of the topologists' sine curve has a maximal* 1- *or a maximal* 2-*point compactification. In particular, the topologists' sine curve is reconstructible.* □

As a final illustration of our first technique we consider *linearly ordered topological spaces* (LOTS) (see [Eng89, 1.7.4]). Such spaces are hereditarily normal (T_5), but generally not metrizable.

Corollary 2.2.8. *Every card of a compact connected LOTS has a maximal* 1-*point or* 2-*point compactification. In particular, compact connected LOTS are reconstructible.*

Proof. Every compact connected LOTS has precisely two endpoints, and all other points are of order two. □

2.2.5 Examples: Using sub cut points

Our second technique of finding cards of Hausdorff continua that have maximal finite compactifications uses the notion of sub cut points. A point of a Hausdorff continuum is called *sub cut point* if it is a cut point of some subcontinuum [DH95]. More generally, we call a point a *sub* N-*cut point* if it is an N-cut point of some subcontinuum. The following observation is immediate from the definitions.

Lemma 2.2.9. *Let* X *be a Hausdorff continuum. If* x *is locally* N-*separating in* X *then* x *is a sub* N-*cut point of* X. □

We describe two applications. First, consider the pseudoarc P. The pseudoarc is a famous example of a hereditarily indecomposable continuum. It is topologically characterised as the unique metrizable hereditarily indecomposable arc-like continuum [Lew99]. Recall that a continuum is said to be *indecomposable* if it cannot be written as the union of two proper subcontinua and that it is *hereditarily indecomposable* if every subcontinuum is indecomposable.

Corollary 2.2.10. *Every card of a hereditarily indecomposable Hausdorff continuum has a maximal* 1-*point compactification.*

Proof. An indecomposable continuum cannot have cut points. Thus, a hereditarily indecomposable continuum cannot contain sub cut points, and hence does not contain locally separating points by Lemma 2.2.9. The result now follows from Lemma 2.2.3. □

Corollary 2.2.11. *The pseudoarc, and more generally every hereditarily indecomposable Hausdorff continuum is reconstructible.* □

As a second application, we consider the Stone-Čech remainder $\mathbb{H}^*$ of the half-line $\mathbb{H} = [0, \infty)$. The space $\mathbb{H}^*$ is a non-metrizable Hausdorff continuum. Recall that a continuum X is said to be *irreducible between two points* x and y if there is no proper subcontinuum of X that contains both x and y. Note that such a continuum does not contain 3-cut points: if $X \setminus \{z\}$ splits into 3 clopen subsets C_1, C_2 and C_3 then without loss of generality $\{x, y\}$ is contained in the subcontinuum $C_1 \cup \{z\} \cup C_2$.

We recall the following theorem by van Douwen about the structure of subcontinua of $\mathbb{H}^*$.

Theorem 2.2.12 ([Hart92, 5.8]). *A subcontinuum of* $\mathbb{H}^*$ *is either indecomposable or it is irreducible between two points.* ☐

Corollary 2.2.13. *Every card of* $\mathbb{H}^*$ *has a maximal* 1- *or a maximal* 2-*point compactification. Thus,* $\mathbb{H}^*$ *is reconstructible.*

Proof. A continuum which is irreducible between two points cannot contain 3-cut points. Thus, by Theorem 2.2.12, the space $\mathbb{H}^*$ does not contain sub 3-cut points, and hence contains no locally 3-separating points either, by Lemma 2.2.9. Hence it follows from Lemma 2.2.3 that every card of $\mathbb{H}^*$ has a maximal 1- or a maximal 2-point compactification. ☐

This result stands in stark contrast to the behaviour of ω^* and indicates how crucial connectedness is for the reconstruction of topological spaces. Indeed, the space $\mathbb{H}^*$ is reconstructible, whereas it is independent of ZFC whether the Stone-Čech remainder of the integers, ω^*, is reconstructible (Theorems 4.5.2 and 4.5.9).

The present proof of the reconstruction result of $\mathbb{H}^*$ depends on the highly nontrivial Theorem 2.2.12. In the next section, in Theorem 2.3.4, we will present an elementary argument why $\mathbb{H}^*$ is reconstructible.

Finally, we remark that van Douwen showed that there are points in $\mathbb{H}^*$ which are not sub cut points [vD77]. Thus, there will be cards with maximal 1-point compactifications. It is well-known that $\mathbb{H}^*$ does contain sub 2-cut points [DH95]—but it is not clear whether these points can be locally separating.

Question 2.2.14. *Do all cards of* $\mathbb{H}^*$ *have maximal* 1-*point compactifications?*

2.3 Metrizable continua are reconstructible

We now show that metrizable continua are reconstructible. More generally, we show that every Hausdorff continuum is reconstructible as long as its cardinality is bigger than its weight. We aim to prove that every such continuum has a card with a maximal 2-point compactification. Using Lemma 2.2.3, we will analyse which points of a continuum are locally separating.

For metrizable continua, Whyburn proved in [Why42, III(9.2)] that in fact all but countably many of the locally separating points are of order two. Thus, using Lemma 2.2.4, we see that metrizable continua are reconstructible. Whyburn's proof, however, seems to be irrevocably tied to metric spaces.

In the following, we will present an alternative approach which has the benefit of generalising Whyburn's result to large classes of non-metrizable continua. It builds on two lemmas, which generalise results originally obtained by Whyburn in the separable metric case.

2.3.1 Counting locally 3-separating points We prove that in a T_1 space X, the number of locally-3-separating points is at most the weight of X.

Lemma 2.3.1 (For metrizable continua: Whyburn [Why28])**.** *The number of* 3*-cut points in a connected* T_1*-space* X *is not larger than the density of* X*.*

Proof. Assume for a contradiction that X is a connected T_1-space of density κ that contains more than κ-many 3-cut points. Let D be a dense subset of X of size κ. For each 3-cut point x in X, choose a partition $C_{x,i}$ $(i = 1, 2, 3)$ of $Y_x = X \setminus \{x\}$ into disjoint non-empty Y_x-clopen sets and find points $d_{x,i} \in D \cap C_{x,i}$. Since X is T_1, the sets $C_{x,i}$ are X-open and $\overline{C_{x,i}} = C_{x,i} \cup \{x\}$ is connected.

By the pigeon-hole principle, there are distinct 3-cut points x and y of X such that $\{d_{x,1}, d_{x,2}, d_{x,3}\} = \{d_{y,1}, d_{y,2}, d_{y,3}\}$ and without loss of generality we may assume $d_{x,i} = d_{y,i}$ for all $i = 1, 2, 3$. In particular $C_{x,i} \cap C_{y,i} \neq \emptyset$. Then $F_x = C_{x,1} \cup C_{x,2} \cup \{x\}$ is connected and must contain y as otherwise F_x is a connected subset of Y_y and hence must be contained in one of $C_{y,i}$ contradicting either $C_{x,1} \cap C_{y,1} \neq \emptyset$ or $C_{x,2} \cap C_{y,2} \neq \emptyset$. But then $C_{x,1} \cup C_{x,2}$ is X-open and contains y (as $x \neq y$) so $y \notin \overline{C_{x,3}}$. Similarly $x \notin \overline{C_{y,3}}$ and hence $\overline{C_{x,3} \cap C_{y,3}} \subseteq \overline{C_{x,3}} \cap \overline{C_{y,3}} \setminus \{x, y\} = C_{x,3} \cap C_{y,3}$. But then $C_{x,3} \cap C_{y,3}$ is a proper, non-empty X-clopen subset of X, a contradiction. □

Lemma 2.3.2 (For separable metric spaces: Whyburn [Why42, III(8.1)])**.** *In a* T_1 *space* X*, the number of components containing separating points of* X *does not exceed the weight of* X*.*

Proof. Suppose for a contradiction that in a T_1 space X of weight κ there is a collection of distinct components $\{C_\alpha \colon \alpha < \kappa^+\}$, each containing a separating point x_α of X. For all α, fix a disconnection $X \setminus \{x_\alpha\} = U_\alpha \oplus V_\alpha$ and points $a_\alpha \in U_\alpha \cap C_\alpha$ and $b_\alpha \in V_\alpha \cap C_\alpha$, witnessing that x_α is separating in X.

Now consider the space $Y = \{(a_\alpha, b_\alpha) \colon \alpha < \kappa^+\} \subset X^2$. Since X^2 has weight κ, the space Y cannot be discrete. Thus, there exists (a_γ, b_γ) which is contained in the closure of $Y \setminus \{(a_\gamma, b_\gamma)\}$. Considering the card $X \setminus \{x_\gamma\} = U_\gamma \oplus V_\gamma$, we note that for all $\alpha \neq \gamma$ the connected component C_α is completely contained in one of U_γ or V_γ. Thus, the sets

$$Y_U = \{(a_\alpha, b_\alpha) \colon C_\alpha \subset U_\gamma\} \quad \text{and} \quad Y_V = \{(a_\alpha, b_\alpha) \colon C_\alpha \subset V_\gamma\}$$

split $Y \setminus \{(a_\gamma, b_\gamma)\}$ into two disjoint sets, and hence (a_γ, b_γ) will be in the closure of one of them, say, Y_U. However, all b_α with $(a_\alpha, b_\alpha) \in Y_U$ lie in U_γ, yielding $b_\gamma \in \overline{\{b_\alpha \colon (a_\alpha, b_\alpha) \in Y_U\}} \subset \overline{U}_\gamma = U_\gamma \cup \{x_\gamma\}$, a contradiction. □

Theorem 2.3.3 (For locally compact separable metric spaces: Whyburn [Why42, III(9.2)])**.** *The number of locally* 3*-separating points in a* T_1 *space* X *does not exceed the weight of* X*.*

Proof. Fix a base $\mathcal{B}$ of size $w(X)$ and suppose towards a contradiction that there is a collection $S \subset X$ with $|S| > w(X)$ such that every point $x \in S$ is locally 3-separating in X. This means for every $x \in S$ there is a basic open neighbourhood $U_x \in \mathcal{B}$ which it 3-separates. Since $|S| > w(X)$, there exists a basic open $U \in \mathcal{B}$ and a subset $S' \subset S$ with $|S'| > w(X)$ such that $U = U_x$ for all $x \in S'$.

Applying Lemma 2.3.2 to U, we see that there is a component C of U such that $|S' \cap C| > w(X)$. But since $d(C) \leq w(X)$, the connected T_1 space C now contains more than $d(C)$ many 3-separating points. Since C is connected, all these 3-separating points are in fact 3-cut points, contradicting Lemma 2.3.1. □

2.3.2 A reconstruction result for continua

We combine the previous observations to obtain the following reconstruction result.

Theorem 2.3.4. *Every Hausdorff continuum* X *with* $w(X) < |X|$ *is reconstructible. In particular, every metrizable continuum is reconstructible.*

Proof. By Theorem 2.3.3 and Lemma 2.2.3, every Hausdorff continuum X with $w(X) < |X|$ has cards with maximal 1- or 2-point compactifications. Thus, they are reconstructible by Theorem 2.1.10. As every non-trivial metrizable continuum is second-countable and has cardinality $\mathfrak{c}$, the result about metrizable continua follows. $\square$

Note that $\mathbb{H}^*$, the Stone-Čech remainder of the positive half line, satisfies $\mathfrak{c} = w(\mathbb{H}^*) < |\mathbb{H}^*| = 2^{\mathfrak{c}}$. Thus, we see again that $\mathbb{H}^*$ is reconstructible.

2.3.3 Reconstructing non-metrizable continua

Recall that always $w(X) \leq |X|$ for compact Hausdorff spaces X [Eng89, 3.1.21]. Thus, it remains to investigate whether Hausdorff continua are reconstructible when weight equals cardinality.

Question 2.3.5. *Does every Hausdorff continuum with $w(X) = |X|$ have a card with a maximal finite compactification?*

Examples of Hausdorff continua with $w(X) = |X|$ are given by the *long line* on $\mathfrak{c} + 1$, by the lexicographical ordered square or (consistently) by compact connected Souslin lines. However, these particular examples are linearly ordered spaces and hence are reconstructible by Corollary 2.2.8. Alternatively, using Corollary 2.1.4, we could also argue that every compact Hausdorff space containing a compact connected metrizable neighbourhood has a card with a maximal finite compactification and hence is reconstructible.

Gary Gruenhage pointed out to me a further class of continua with $w(X) = |X|$. Starting with a compact Hausdorff space X, the *cone over* X is the quotient $cone(X) = (X \times I)/(X \times \{1\})$. If X has cardinality and weight $\kappa \geq \mathfrak{c}$, then $cone(X)$ provides an example of a Hausdorff continuum where weight equals cardinality. However, using Lemma 2.2.9 it is easy to see that any point of the form $(x, \frac{1}{2})$ is at most locally 2-separating, and hence all cones over compact Hausdorff spaces are reconstructible.

Another idea is to try building continua with cardinality equalling weight by using products. However, in our final observation we show that non-trivial products of Hausdorff continua are always reconstructible. For this, we need the following lemma about the non-existence of cut points in products.

Lemma 2.3.6. *Let $|I| \geq 2$ and for all $i \in I$ suppose that X_i is a non-trivial connected space. Then $X = \prod_{i \in I} X_i$ has no cut points.*

Proof. The case of $|I| = 2$ is straightforward, and by induction we get the result for all finite products. So suppose that I is infinite and pick a point x of X. We claim that $X \setminus \{x\}$ is connected. Pick y in X such that $x_i \neq y_i$ for all coordinates i. Recall from the proof that connectedness is productive [Eng89, 6.1.15] that the σ-product at y

$$\sigma(y, X) = \{z \in X : |\{i : z_i \neq y_i\}| \text{ is finite}\}$$

is connected and dense in X. Since I was infinite, x is not contained in the σ-product at y and hence it follows that $X \setminus \{x\}$ has a dense connected subset, so is itself connected. $\square$

Theorem 2.3.7. *Let $|I| \geq 2$ and for all $i \in I$ suppose that X_i is a non-trivial Hausdorff continuum. Then $\prod_{i \in I} X_i$ contains no locally separating points.*

Proof. Let $x \in X$ and suppose for a contradiction there is a basic open neighbourhood $U = \bigcap_{i \in F} \pi_i^{-1}(U_i)$ of x (for $F \subset I$ finite) such that x is a cut-point of $C_{\overline{U}}(x)$. Recall that $\overline{U} = \bigcap_{i \in F} \pi_i^{-1}(\overline{U_i})$ by [Eng89, 2.3.3].

Claim. *We have $C_{\overline{U}}(x) = \bigcap_{i \in F} \pi_i^{-1}(C_{\overline{U_i}}(x_i))$.*

First, since connectedness is productive [Eng89, 6.1.15], the set $\bigcap_{i \in F} \pi_i^{-1}(C_{\overline{U_i}}(x_i))$ is a connected subset of $\overline{U}$ containing the point x. Conversely, for every point $y \in \overline{U} \setminus \bigcap_{i \in F} \pi_i^{-1}(C_{\overline{U_i}}(x_i))$ we have $y_i \notin C_{\overline{U_i}}(x_i)$ for some index i, and therefore by the Šura-Bura Lemma 2.2.1 there is a clopen subset V of $\overline{U_i}$ separating y_i from $C_{\overline{U_i}}(x_i)$. It follows that $\pi_i^{-1}(V) \cap \overline{U}$ is a clopen subset of U separating y from x. Thus, $y \notin C_{\overline{U}}(x)$, completing the proof of the claim.

So now, since x is a cut-point of its component in $\overline{U}$, the claim implies that x is a cut-point of the space $\bigcap_{i \in F} \pi_i^{-1}(C_{\overline{U_i}}(x_i))$. However, since every $C_{\overline{U_i}}(x_i)$ is non-trivial by the Boundary Bumping Lemma 2.2.2, the component of x in $\overline{U}$ is a non-trivial product of connected spaces, and therefore does not have cut-points by Lemma 2.3.6, a contradiction. $\square$

Corollary 2.3.8. *Every non-trivial product of non-trivial Hausdorff continua is reconstructible.* $\square$

2.4 Reconstruction results for compact metrizable spaces

2.4.1 Overview In this section we investigate the dividing line between reconstructible and non-reconstructible spaces within the class of compact metrizable spaces. Since compact spaces with isolated points are reconstructible, we focus on compact metrizable spaces without isolated points.

Our investigation is motivated by the following dichotomy. On one side we have the result that if a compact metrizable space without isolated points is *totally disconnected* then it is a Cantor set and hence non-reconstructible. At the other end of the scale, we have proven in Theorem 2.3.4 that if such a space is *connected* then it will be reconstructible.

In the following we explore spaces between those two extremes. We prove that a compact metrizable space has a card with a maximal finite compactification if and only if the space does not contain a dense G_δ of 1-point components. It follows that non-reconstructible compact metrizable spaces are highly disconnected.

2.4.2 Cards with a maximal finite compactification We show that the characterisation in Lemma 2.2.3 of finite compactifications of cards $X \setminus \{x\}$ in terms of the property of x being locally separating can be lifted from continua to compact metrizable spaces where all components are large.

It is not hard to see that for a compact space, 'having no arbitrarily small components' describes a topological property, i.e. a property of metrizable spaces, and not just of metric spaces: A sequence $\{B_n : n \in \omega\}$ of disjoint subsets of a space X is said to *converge to a point* x if for every neighbourhood U of x there exists $N \in \omega$ such that $B_n \subseteq U$ for every $n \geq N$. One checks that in a compact metric space in which there are arbitrarily small components, compactness implies that there is a point which is the limit of a sequence of components. This cannot happen if components are uniformly big.

Lemma 2.4.1 (Second Šura-Bura Lemma)**.** *Let X be a compact Hausdorff space, and $C \subset X$ a component. Then C has a clopen neighbourhood base in X, i.e. for every open set $U \supset C$ there is a clopen set V such that $C \subset V \subset U$.*

Proof. The proof is a straightforward application of the Šura-Bura Lemma 2.2.1 and compactness. For details, see e.g. [vM01, A.10.1]. □

Lemma 2.4.2. *Let X be a compact metric space and suppose there exists $\delta > 0$ such that the diameter of every component is at least δ. Then $X \setminus \{x\}$ has an N-point compactification if and only if x is locally N-separating in X.*

The assumption about the non-existence of small components is necessary, as the example $\{0\} \times [-1, 1] \cup \{\{(1/n)\} \times [-1/n, 1/n] : n \in \omega\} \subset \mathbb{R}^2$ shows. It is not hard to see that the point $(0, 0)$ is not 3-separating, but removing it gives a card with arbitrarily large finite compactifications.

Proof. The backwards direction is immediate from Lemma 2.1.2, so we focus on the direct implication. Let us assume that $X \setminus \{x\}$ has an N-point compactification. By Lemma 2.1.2 and regularity, there is a compact neighbourhood $U = \overline{B_\epsilon(x)}$ of x such that $U \setminus \{x\} = U_1 \oplus \cdots \oplus U_N$ and $x \in \overline{U_i}^X$ for all i. Assume without loss of generality that $\epsilon < \delta/2$. We have to show that $C_U(x)$, the component of x in U, intersects every U_i. Suppose for a contradiction this is not the case, i.e. that say $C(x) \cap U_1 = \emptyset$. Since x lies in the closure of U_1, we can find a sequence $\{x_n : n \in \omega\} \subset U_1$ converging to x.

Let $C_{U_1}(x_n)$ denote the component of x_n in U_1. Note that if $x \in \overline{C_{U_1}(x_n)}$ then $C_{U_1}(x_n) \subset C_U(x)$, a contradiction. Next, since $\operatorname{diam}(C(x_n)) > 2\epsilon$, we have $C(x_n) \not\subseteq U$. Hence, applying the Boundary Bumping Lemma 2.2.2 to $C(x_n) \cap U_1$ yields that $C_{U_1}(x_n)$ limits onto the boundary of U. Hence, whenever $d(x, x_n) < \epsilon/2$ then $\operatorname{diam}(C_{U_1}(x_n)) \geq \epsilon/2$.

To conclude the proof, note that $C_U(x) \cap U_1 = \emptyset$ implies that x is a one-point component of $\overline{U_1}$. By Lemma 2.4.1 there is a clopen neighbourhood V of x in $\overline{U_1}$ such that the diameter of V is at most $\epsilon/4$. However, we have $x_n \in V$ for n large enough, implying that $\operatorname{diam}(C_{U_1}(x_n)) \leq \operatorname{diam}(V) \leq \epsilon/4$, a contradiction. □

2.4.3 When a compact metrizable space has cards with maximal finite compactifications Our next theorem characterises under which conditions a compact metrizable space has a card with a maximal finite compactification. For this result I profited from numerous conversations with Paul Gartside when visiting the University of Pittsburgh in summer 2014.

Theorem 2.4.3. *For a compact metrizable space X without isolated points the following are equivalent:*

(1) X has a card with a maximal 1- or 2-point compactification,

(2) X has a card with a maximal finite compactification,

(3) the union of all 1-point components of X does not form a dense subspace, and

(4) the union of all 1-point components of X does not form a dense G_δ of size $\mathfrak{c}$.

Proof. The implications (1) ⇒ (2) and (3) ⇒ (4) are trivial.

For (2) ⇒ (3), assume that the subspace formed by the 1-point components is dense in X. Let x be an arbitrary point of X. We need to show that the card $X \setminus \{x\}$ has arbitrarily large finite compactifications.

Fix a nested neighbourhood base $\{V_n : n \in \omega\}$ of x such that $\overline{V_{n+1}} \subsetneq V_n$. Since the 1-point components are dense, it follows from Lemma 2.4.1 that for all n there is a non-empty clopen set $F_n \subsetneq V_n \setminus \overline{V_{n+1}}$. It follows that whenever $A_1, \ldots, A_{N-1}$ is a partition of ω into infinite subsets, the sets

$$G_i = \bigcup_{n \in A_i} F_n \quad \text{and} \quad G_0 = (X \setminus \{x\}) \setminus \bigcup_{n \in \omega} F_n$$

form a partition of $X \setminus \{x\}$ into N non-empty non-compact clopen subsets. By Theorem 2.1.1, the card $X \setminus \{x\}$ has an N-point compactification. Since N was arbitrary, the card $X \setminus \{x\}$ does not have a maximal finite compactification.

For (4) ⇒ (1), we will prove the contrapositive. Assume that all cards of X have a three-point compactification. Let $C(x)$ denote the connected component of x in X and put, for $n \in \mathbb{N}$,

$$X_n = \{x \in X \colon \operatorname{diam}(C(x)) \geq 1/n\}.$$

Every X_n is a closed subset of X. To see this, suppose for a contradiction that x lies in the closure of X_n and $\operatorname{diam}(C(x)) < 1/n$. Find $\epsilon > 0$ such that $\operatorname{diam}(C(x)) + 2\epsilon < 1/n$. By the Second Šura-Bura Lemma 2.4.1, there is a clopen set F of X such that

$$C(x) \subseteq F \subseteq B_\epsilon(C(x)).$$

Since F is a neighbourhood of x, there is a point $y \in F \cap X_n$ witnessing that x lies in the closure of X_n. Note that $y \in C(y) \subset Q(y) \subset F$, and hence that

$$\operatorname{diam}(C(y)) \leq \operatorname{diam}(F) \leq \operatorname{diam}(B_\epsilon(C(x))) \leq \operatorname{diam}(C(x)) + 2\epsilon < 1/n,$$

contradicting that $y \in X_n$. This shows that X_n is closed.

We now argue that every X_n has empty interior in X. Otherwise, since X is locally compact without isolated points, the subset X_n has uncountable interior. Since X_n is closed, it is compact. As $X \setminus \{x\}$ has a 3-point compactification for all x, Corollary 2.1.4 implies that $X_n \setminus \{x\}$ has a 3-point compactification for all $x \in \operatorname{int}(X_n)$. However, since all components of X_n have diameter at least $1/n$, Lemma 2.4.2 implies that all $x \in \operatorname{int}(X_n)$ are locally 3-separating in X_n. But by Theorem 2.3.3, no compact metrizable space contains uncountably many locally 3-separating points, a contradiction. This shows that every X_n has empty interior in X.

Finally, the set of 1-point components of X equals

$$A = \bigcap_{n \in \mathbb{N}} X \setminus X_n.$$

As we have shown that every $X \setminus X_n$ is open dense, the Baire Category Theorem [Eng89, 3.9.4] implies that A is a dense G_δ in X. And being a G_δ of a completely metrizable space is equivalent to being completely metrizable [Eng89, 4.3.23/24]. Hence, the subset of 1-point components is completely metrizable without isolated points, and in particular has cardinality $\mathfrak{c}$ [Kech95, 6.A]. ☐

Corollary 2.4.4. *Every compact metrizable space without isolated points in which the union of all* 1*-point components of* X *does not form a dense* G_δ *of cardinality* $\mathfrak{c}$ *is reconstructible.*

Proof. By Theorems 2.4.3 and 2.1.10. ☐

Corollary 2.4.5. *The Cantor set is characterised topologically as the unique compact metrizable homogeneous non-reconstructible space.* ☐

2.4.4 Universal sequences of clopen sets

The previous section has shown that a compact metrizable space is reconstructible unless its 1-point components form a dense G_δ. In this section, therefore, we turn our attention to precisely these spaces. Our analysis leads to Theorem 2.4.13, where we decide for spaces where the collection of 1-point components has non-empty interior whether they are reconstructible.

Recall that a sequence $\{B_n : n \in \omega\}$ of disjoint subsets of a space X converges to a point x if for every neighbourhood U of x there exists $N \in \omega$ such that $B_n \subseteq U$ for all $n \geq N$. Suppose T and $\{T_n : n \in w\}$ are topological spaces. We say that a sequence $\{B_n : n \in \omega\}$ of disjoint subsets of a space X is of *type* (T_n) if $B_n \cong T_n$ for all $n \in \omega$. Similarly, we say the sequence $\{B_n : n \in \omega\}$ is of *type* (T) if $B_n \cong T$ for all $n \in \omega$.

Finally, we say that a topological space X has a *universal sequence* (of type (T_n)) if every point $x \in X$ is the limit of a sequence of disjoint clopen sets of type (T_n). Similarly, we say X has a *constant universal sequence* if every point $x \in X$ is the limit of a sequence of disjoint clopen sets of the same constant type (T).

Lemma 2.4.6 (Universal Sequence Lemma)**.** *Every non-reconstructible compact metrizable space has a universal sequence.*

Proof. Suppose that Z is a non-homeomorphic reconstruction of X. Note that Z is non-compact (Theorem 1.2.7), locally compact (Theorem 1.2.3), separable (Theorem 1.2.5) and metrizable (Theorem 1.2.2). It follows from separability that the 1-point compactification αZ is metrizable [Wil04, 23C]. Moreover, since X contains 1-point components by Corollary 2.4.4, it follows that every component of Z is compact. In particular, the point $\infty \in \alpha Z$ is a 1-point component. By Lemma 2.4.1, the point ∞ has a (countable) neighbourhood base of clopen sets in αZ, and hence Z can be written as $\bigoplus_{n\in\omega} T_n$ for disjoint clopen compact subsets $T_n \subset Y$. We will show that a tail of $\{T_n \colon n \in \omega\}$ is the type of the desired universal sequence.

Claim. *For every $x \in X$ there is $N \in \omega$ such that x is the limit of a sequence of disjoint clopen sets of type $(T_n)_{n>N}$.*

To see the claim, note that $X \setminus \{x\} \cong Z \setminus \{y\}$ for some suitable $y \in Z$. But if $y \in T_N$ then $\{T_n \colon n > N\}$ is an infinite sequence of disjoint compact clopen sets in $X \setminus \{x\}$ such that their union is a closed set. Compactness of X now implies that $\{T_n \colon n > N\}$ converges to $x \in X$.

Claim. *There is $N \in \omega$ such that every $x \in X$ is the limit of a sequence of disjoint clopen sets of type $(T_n)_{n>N}$.*

To see the claim, fix $z \neq z' \in X$, and consider open neighbourhoods U and V of these points with disjoint closures. By the previous claim there exist N_z and $N_{z'}$ such that there is a sequence $\{B_n \colon n > N_z\}$ completely contained inside U converging to z, and a sequence $\{B'_n \colon n > N_{z'}\}$ contained inside V converging to z' such that $B_n \cong B'_n \cong T_n$ for all $n > N = \max\{N_z, N_{z'}\}$.

To see that this N is as required, pick any $x \in X$. Without loss of generality, $x \notin \overline{U}$. By the first claim, there is a sequence of disjoint clopen sets of type $(T_n)_{n>N_x}$ converging to x, all contained in $X \setminus \overline{U}$. If $N_x \leq N$, simply delete the first few elements. If $N_x > N$ then fill up the front of the sequence with clopen sets $\{B_n \colon N < n \leq N_x\}$. This proves claim and lemma. □

Theorem 2.4.7. *Every pseudocompact Hausdorff space with a constant universal sequence is non-reconstructible.*

Proof. Let X be a pseudocompact Hausdorff space and assume that every point of X has a sequence of disjoint clopen sets of type (T) converging to it. We show that $Z = X \oplus (\omega \times T)$ is a reconstruction of X. Since Z clearly is not pseudocompact, it follows that X is non-reconstructible.

Claim. *For any open $U \subset X$ and $x \in U$ we have $U \setminus \{x\} \cong U \setminus \{x\} \oplus (\omega \times T)$.*

To prove the claim, note that there is a sequence of disjoint clopen sets $B_n \subset U$ all homeomorphic to T converging to x. It follows that $\{x\} \cup \bigcup_{n\in\omega} B_n$ and $U \setminus \bigcup_{n\in\omega} B_n$ are closed subsets of U intersecting only in the point x. Since

$$\{x\} \cup \bigcup_{n\in\omega} B_n \cong \{x\} \cup \bigcup_{n\in\omega} B_{2n},$$

because both sides are homeomorphic to the 1-point compactification of $\omega \times T$, we conclude with the help of the Pasting Lemma [Wil04, 7.6] that

$$U \setminus \{x\} \cong U \setminus \{x\} \oplus \bigoplus_{n\in\omega} B_{2n+1} \cong U \setminus \{x\} \oplus (\omega \times T),$$

establishing the first claim. Since X contains clopen copies of T, the claim applies to both X and T.

Claim. *We have $X \cong X \oplus T$.*

To see this claim, pick a sequence of disjoint clopen sets $B_n \subset X$ converging to some $x \in X$. Since

$$\{x\} \cup \bigcup_{n\geq 0} B_n \cong \{x\} \cup \bigcup_{n\geq 1} B_n,$$

we can conclude once again with the help of the Pasting Lemma that indeed $X \cong X \oplus B_0 \cong X \oplus T$, and the claim is established.

We now complete the proof of the theorem. To prove $\mathcal{D}(X) \subseteq \mathcal{D}(Z)$, consider a card $X \setminus \{x\}$. The first claim implies $X \setminus \{x\} \cong X \setminus \{x\} \oplus (\omega \times T) \in \mathcal{D}(Z)$, as required.

For the other inclusion, it suffices to show that $X \oplus (w \times T) \setminus (0,t)$ is a card of X. But since $T \setminus \{t\} \cong T \setminus \{t\} \oplus (\omega \times T)$ by the first claim and $X \cong X \oplus T$ by the second claim, it follows that $X \oplus (w \times T) \setminus (0,t) \cong X \oplus T \setminus \{t\} \cong X \setminus \{x\} \in \mathcal{D}(X)$, as required. ☐

2.4.5 h-homogeneous and π-homogeneous spaces A space X is called *h-homogeneous*, or *strongly homogeneous*, if every non-empty clopen subset of X is homeomorphic to X. For example, the Cantor set, the rationals and the irrationals are h-homogeneous, and so is every compact space of diversity two. Note that if a compact h-homogeneous first-countable space different from the Cantor set contains a dense subset of 1-point components, then the collection of 1-point components automatically forms a dense, co-dense G_δ.

In the following we will show as an application of our results on universal sequences that every h-homogeneous first-countable compact space with a dense subspace of 1-point components is non-reconstructible. In fact, we prove the stronger result that first-countable Hausdorff compact π-homogeneous spaces without isolated points are non-reconstructible.

Recall that a collection $\mathcal{B}$ of open sets is called a π-base for X if for every open set $U \subset X$ there is $B \in \mathcal{B}$ such that $B \subset U$. A π-base $\mathcal{B}$ is called a *covering* π-base if $\bigcup \mathcal{B} = X$. We call a space *π-homogeneous* if it has a clopen π-base of pairwise homeomorphic elements.

Lemma 2.4.8 (Matveev [Mat98])**.** *Assume that X has a π-base consisting of clopen sets that are homeomorphic to X. If there exists a sequence $\{U_n : n \in \omega\}$ of non-empty open subsets of X converging to a point of X then X is h-homogeneous.* □

For a nice proof see also [Med11, Appendix A, Prop. 17]. We remark that any compactification with non-trivial connected remainder of the Cantor set minus a point is an example of a π-homogeneous space without isolated points which is not h-homogeneous. We now relate π-homogeneity and universal sequences in the realm of first-countable spaces.

Lemma 2.4.9. *Let X be a first-countable compact Hausdorff space without isolated points. Then X is π-homogeneous if and only if X has a constant universal sequence.*

Proof. For the direct implication, assume X is π-homogeneous with all π-basic elements homeomorphic to T. Let $x \in X$ and fix a neighbourhood base $\{U_n : n \in \omega\}$ of x consisting of open sets U_n with $\overline{U_{n+1}} \subsetneq U_n$. For every $n \in \omega$, use π-homogeneity to find a clopen set $B_n \subset U_n \setminus \overline{U_{n+1}}$ with $B_n \cong T$. The sequence $\{B_n : n \in \omega\}$ is as required.

Conversely, we show that clopen sets homeomorphic to T form a π-basis for X. Consider some non-empty set U of X. Pick $x \in U$ and find a sequence of disjoint clopen sets $\{B_n : n \in \omega\}$ in X all homeomorphic to T converging to x. There exists $N \in w$ such that $B_n \subset U$ for all $n \in N$. Thus, for instance we find $B_{N+1} \subset U$ and the proof is complete. □

Corollary 2.4.10. *Every π-homogeneous, first-countable, compact space without isolated points is non-reconstructible.*

Proof. Immediate from the previous lemma and Theorem 2.4.7. □

Corollary 2.4.11. *Every h-homogeneous, first-countable, compact space without isolated points in which the 1-point components are dense is non-reconstructible.*

Proof. Every non-trivial h-homogenous space has no isolated points. Also, the existence of a dense set of 1-point components together with the Second Šura-Bura Lemma 2.4.1 implies that X is π-homogeneous. Now apply the previous corollary. □

2.4.6 Reconstructing spaces where the one-point components have non-empty interior

Let X be a compact metrizable space without isolated points with a dense G_δ of 1-point components, which we will denote by A throughout this section. The next lemma describes how the different A can look like.

Lemma 2.4.12. *Let X be a compact metrizable space without isolated points where A denotes the dense G_δ of 1-point components.*

(1) If $\operatorname{int}(A)$ *is non-empty then* $\operatorname{int}(A)$ *is homeomorphic to C or to $C \setminus \{0\}$, and*

(2) if $\operatorname{int}(A)$ *is empty, then A is homeomorphic to the Baire space* $\mathbb{N}^{\mathbb{N}}$.

Proof. Observe that in both cases, A is zero dimensional by the Second Šura-Bura Lemma 2.4.1. For the first assertion, note that $\operatorname{int}(A)$ is an open subset of X and hence locally compact, metrizable and without isolated points. Thus, depending on whether $\operatorname{int}(A)$ is compact (closed) or not, it is either homeomorphic to C or to $C \setminus \{0\}$ by [Eng89, 6.2.A(c)].

In the second case, A is a dense, co-dense G_δ in a compact metrizable space. It follows from [Eng89, 6.2.A(a)] that A is homeomorphic to the Baire space $\mathbb{N}^{\mathbb{N}}$. □

We now characterise under which conditions spaces in situation (1) of Lemma 2.4.12 are reconstructible.

Theorem 2.4.13. *Let X be a compact metrizable space without isolated points such that the collection of* 1*-point components $A \subset X$ forms a dense G_δ with non-empty interior. Then X is reconstructible if and only if the interior of A is not dense in X.*

Proof. For the direct implication, we prove the contrapositive. Assume that the interior of A is non-empty and not dense in X. In this case, there cannot exist a universal sequence, because any sequence of disjoint clopen sets converging to a point in the interior of A is eventually of type (C), which is not the case for a point outside the closure of $\operatorname{int}(A)$. Thus, X is reconstructible by the Universal Sequence Lemma 2.4.6.

For the converse, assume that the interior of A is dense in X. It follows from Lemma 2.4.12(1) that the clopen sets of X homeomorphic to the Cantor set form a π-base for X. Thus, X is non-reconstructible by Corollary 2.4.10. □

Note that any reconstructible space from the previous theorem will be a space where every card has arbitrarily large finite compactifications (Theorem 2.4.3). Hence, any such space witnesses that the implication of the Reconstruction Result for Compact Spaces, Theorem 2.1.10, does not reverse.

2.5 Building non-reconstructible compact spaces

From the results in Corollary 2.4.4 and Theorem 2.4.13 it follows that we now can decide for all compact metrizable spaces whether they are reconstructible—unless it is a compact metrizable space without isolated points whose set of 1-point components forms a dense, co-dense G_δ. A precise answer when such spaces are reconstructible is yet to be found.

This section explores different classes of non-reconstructible compact metrizable spaces. These examples illustrate the results from the previous section, but also indicate where our results fall short of a complete classification of all reconstructible compact metrizable spaces.

2.5.1 Compactifications of $C \setminus \{0\}$ Theorem 2.4.13 shows that every metrizable compactification of $C \setminus \{0\}$ provides an example of a non-reconstructible space. We now construct a multitude of these types of compactifications.

Recall that a subset D of X is *sequentially dense* if every point of X is the limit of a converging sequence of points in D. A space X is called *sequentially separable* if there is a countable subset $D \subset X$ such that D is sequentially dense in X. In general, the *sequential density* of a space is the least cardinal κ such that there exists a subset of X of size κ which is sequentially dense in X.

In [Tka90], V. Tkachuk proved that every compact Hausdorff space X is a remainder of a compactification of a discrete space of cardinality at most $|X|$. We present the following, slightly improved version of Tkachuk's theorem.

Theorem 2.5.1 (cf. Tkachuk [Tka90]). *For every compact Hausdorff space X of sequential density κ there is a compactification $\gamma^X\kappa$ of the discrete space of cardinality κ such that*

(1) the remainder $\gamma^X\kappa \setminus \kappa$ is homeomorphic to X, and

(2) the discrete space κ is sequentially dense in $\gamma^X\kappa$.

Proof. Let $D \subset X$ be a sequentially dense set of cardinality κ. Let $\alpha\kappa$ denote the 1-point compactification of the discrete space of cardinality κ, with the point at infinity denoted by ∞. Fix a disjoint family $\{\omega_d \colon d \in D\}$ of countably infinite subsets of κ. Now the space

$$\gamma^X\kappa = (X \times \{\infty\}) \cup \bigcup_{d \in D} (\{d\} \times \omega_d) \subset X \times \alpha\kappa$$

is a compactification of a discrete space of cardinality κ with remainder homeomorphic to X. It remains to show that κ is sequentially dense in $\gamma^X\kappa$.

For any x in X there is a sequence $\{d_n \colon n \in \omega\} \subset D$ converging to x. And since the subspace

$$\{(x,\infty)\} \cup \bigcup_{n\in w} \{(d_n,\infty)\} \cup \bigcup_{n\in w} (\{d_n\} \times \omega_{d_n})$$

is a closed subspace of $X \times \alpha\kappa$, it is compact and countable, and hence homeomorphic to a countable ordinal. Thus, there is a sequence of isolated points converging to (x,∞). ☐

Corollary 2.5.2. *For every compact Hausdorff space X of sequential density κ there is a compactification of $C \times \kappa$ such that*

(1) the remainder of that compactification is homeomorphic to X, and

(2) the compactification has a constant universal sequence of type (C).

In particular, this compactification is non-reconstructible.

Proof. For the first part, simply replace every isolated point in Theorem 2.5.1 by a clopen copy of the Cantor set. Since the resulting space has a constant universal sequence of type (C), Theorem 2.4.7 implies that every such compactification is non-reconstructible. ☐

Corollary 2.5.3. *For every compact metrizable space X there is a (non-reconstructible) metric compactification of $C \setminus \{0\}$ such that the remainder of that compactification is homeomorphic to X.* ☐

Instead of replacing every isolated point in Theorem 2.5.1 by a clopen copy of the Cantor set, we could equally replace every isolated point by a clopen copy of a fixed (pseudo-)compact Hausdorff space that has a constant universal sequence to obtain a different non-reconstructible (pseudo-)compact space.

We note that there are also pseudocompact non-compact spaces with a constant universal sequence. The following construction is mentioned in Matveev's [Mat98, p.19]. Start with the usual (pseudocompact) Ψ-space on ω [Eng89, 3.6.I], and replace every isolated point by a clopen copy of the Cantor set. Every Ψ-space is first-countable, so this gives a space with a constant universal sequence of type (C) by Lemma 2.4.9. Since this space is still pseudocompact, Theorem 2.4.7 implies that it is non-reconstructible.

An alternative proof of Corollary 2.5.3, that every compact metrizable space is the remainder of a compactification of $C \setminus \{0\}$, goes as follows. Recall that $\omega+1\times C \cong C$, and that the Cantor space surjects onto every compact metrizable space X [Wil04, 30.7]. Fix a continuous surjection $f\colon \{\omega\} \times C \to X$. Then, the adjunction space $C \cup_f X$, which is the quotient $C/\mathcal{P}$ for the upper semi-continuous decomposition $\mathcal{P} = \{f^{-1}(x)\colon x \in X\} \cup \{\{x\}\colon x \in \omega \times C\}$, is a metrizable compactification of $\omega \times C \cong C \setminus \{0\}$ with remainder homeomorphic to X [vM01, A.11.4].

2.5.2 Non-reconstructible h-homogeneous spaces I Recall from Corollary 2.4.11 that every compact metrizable h-homogeneous space with a dense collection of 1-point components is non-reconstructible. In the next few sections we construct several examples of such spaces.

Many of the following examples in fact preceded Corollary 2.4.11, and were crucial stepping stones towards finding the reconstruction results presented in Section 2.4.4.

Theorem 2.5.4 (Medini [Med11, Thm. 18])**.** *If a Hausdorff space X has a dense set of isolated points then X^κ is h-homogeneous for every infinite cardinal κ.* □

Corollary 2.5.5. *If X is a first-countable compact space with a dense set of isolated points then X^ω is non-reconstructible.*

Proof. Note that if X has a dense set of isolated points then X^ω has a dense set of 1-point components. By Theorem 2.5.4, the space X^ω is h-homogeneous, and since X^ω is first-countable, the result now follows from Corollary 2.4.11. □

In other words, for every first-countable compactification $\gamma\omega$ of the countable discrete space, the product $(\gamma\omega)^\omega$ is non-reconstructible. We refer the reader back to Theorem 2.5.1, where many such compactifications $\gamma\omega$ are constructed.

2.5.3 Non-reconstructible h-homogeneous spaces II We now present a machinery to construct further compact h-homogeneous spaces. This machinery will allow us in Section 2.5.5 to also construct examples of non-reconstructible spaces which are not necessarily h-homogeneous.

The construction takes place in the unit cube I^3 and uses as input a countable list $\mathcal{E} = \left\{(E_f, D_f)\colon f \in \mathbb{N}^{<\mathbb{N}}\right\}$ (where $\mathbb{N} = \{1, 2, \ldots\}$) of non-trivial planar continua E_f, and countable dense subsets $D_f \subset E_f$ with a fixed enumeration $D_f = \{d_{f,n}\colon n \in \mathbb{N}\}$. We may assume that every planar continuum is given a representation

$$E_f \subset \left[\frac{1}{4}, \frac{3}{4}\right]^2 \times \left\{\frac{1}{2}\right\} \subset I^3,$$

and consequently that $d_{f,n}$ has coordinates $d_{f,n} = (d^1_{f,n}, d^2_{f,n}, d^3_{f,n})$ with $d^3_{f,n} = \frac{1}{2}$. From our list $\mathcal{E}$, we will build a compact metrizable space $X_\mathcal{E} \subset I^3$ such that the non-trivial components of $X_\mathcal{E}$ are given precisely by $\left\{E_f : f \in \mathbb{N}^{<\mathbb{N}}\right\}$, and the union of the one-point components is dense and co-dense in $X_\mathcal{E}$.

The construction could potentially be generalised to take place in the Hilbert cube I^ω and work for arbitrary metrizable continua, but since there are already $\mathfrak{c}$ many pairwise non-homeomorphic planar continua [War32], there is no pressing need to do so.

So let $\mathcal{E}$ be a countable list of planar continua as above. Consider the first planar continuum $E_{\langle\emptyset\rangle} \subset I^3$ and let $d'_{\emptyset,n} = (d^1_{\emptyset,n}, d^2_{\emptyset,n}, d^3_{\emptyset,n} + 2^{-2n})$. Consider the compact space

$$E_{\langle\emptyset\rangle} \cup \left\{d'_{\emptyset,n} : n \in \mathbb{N}\right\} \subset I^3.$$

For all $n \in \mathbb{N}$, we fix open cubes (in the ∞-norm)

$$F_{\langle n\rangle} = B^\infty_{2^{-2n-1}}(d'_{\emptyset,n}) \subset I^3.$$

These cubes have the properties that $\operatorname{diam}\big(F_{\langle n\rangle}\big) = 2^{-2n}$ (in the ∞-norm), that $\overline{F_{\langle n\rangle}} \subset I^3$, that $\overline{F_{\langle n\rangle}} \cap \overline{F_{\langle m\rangle}} = \emptyset$ whenever $n \neq m$, and that $E_{\langle\emptyset\rangle} \cap F_{\langle n\rangle} = \emptyset$ for all $n \in \mathbb{N}$. Using the affine homeomorphism

$$h_{\langle n\rangle} \colon I^3 \to \overline{F_{\langle n\rangle}},\ t \mapsto (d'_{\emptyset,n} - (2^{-2n-1}, 2^{-2n-1}, 2^{-2n-1}) + t \cdot 2^{-2n}),$$

we define $E'_{\langle n\rangle} = h_{\langle n\rangle}(E_{\langle n\rangle})$, which is a copy of $E_{\langle n\rangle}$ completely contained in $F_{\langle n\rangle}$. Now let

$$X_1 = E_{\langle\emptyset\rangle} \cup \bigcup_{n\in\mathbb{N}} E'_{\langle n\rangle} \subset I^3.$$

In the second step of the recursion, simply perform the same construction for $E_{\langle n\rangle} \subset \overline{F_{\langle n\rangle}} \cong I^3$ as we did in the first step for $E'_{\langle\emptyset\rangle} \subset I^3$. In other words, we have cubes $F_{\langle n,l\rangle} \subset F_{\langle n\rangle}$ for $l \in \mathbb{N}$ with disjoint closures and copies $E'_{\langle n,l\rangle} \subset F_{\langle n,l\rangle}$ such that $E'_{\langle n\rangle} \cup \bigcup_{l\in\mathbb{N}} E'_{\langle n,l\rangle} \subset F_{\langle n\rangle}$ is a compact metrizable space. After performing this procedure for all $n \in \mathbb{N}$, we obtain the resulting space

$$X_2 = X_1 \cup \bigcup_{f\in\mathbb{N}^2} E'_f.$$

For two finite sequences f and g we denote by $f^\frown g$ the concatenation of f and g. We use recursion to obtain cubes $\left\{F_f : f \in \mathbb{N}^{<\mathbb{N}}\right\}$, linear homeomorphisms $\left\{h_f \colon I^3 \to \overline{F_f} : f \in \mathbb{N}^{<\mathbb{N}}\right\}$ and components $\left\{E'_f : f \in \mathbb{N}^{<\mathbb{N}}\right\}$ such that

(a) for all $f \in \mathbb{N}^k$ we have $\operatorname{diam}(F_f) = 2^{-2k}$ (in the ∞-norm),

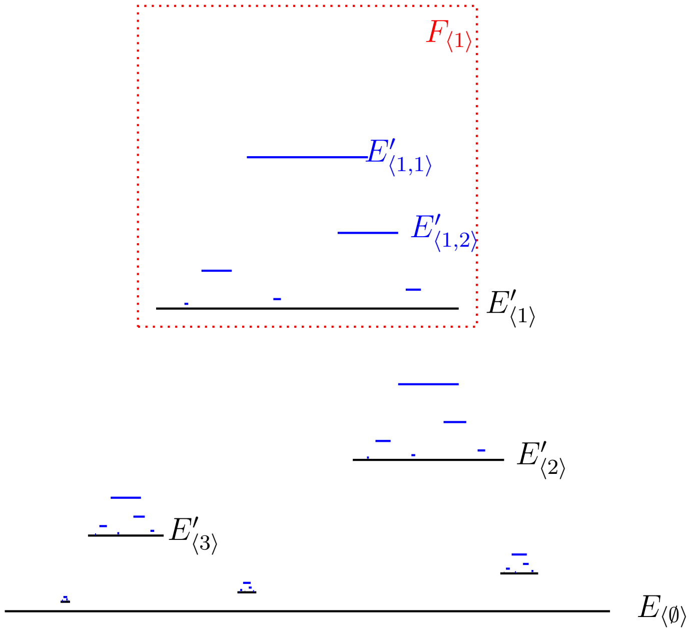

Figure 2.1: A schematic drawing of the space X_2

(b) $E'_f = h_f(E_f) \subset F_f$ for all $f \in \mathbb{N}^{<\mathbb{N}}$,

(c) $E'_f \cap \overline{F_{f^\frown n}} = \emptyset$ for all $f \in \mathbb{N}^{<\mathbb{N}}$ and $n \in \mathbb{N}$,

(d) $\overline{F_{f^\frown n}} \subset F_f$ for all $f \in \mathbb{N}^{<\mathbb{N}}$ and $n \in \mathbb{N}$, and

(e) $\overline{F_{f^\frown n}} \cap \overline{F_{f^\frown l}} = \emptyset$ for all $f \in \mathbb{N}^{<\mathbb{N}}$ and $n \neq l \in \mathbb{N}$.

For $n \in \mathbb{N}$ define recursively

$$X_n = X_{n-1} \cup \bigcup_{f \in \mathbb{N}^n} E'_f \subset I^3,$$

a compact space without one-point components. Finally, consider the compact metrizable space

$$X_{\mathcal{E}} = \overline{\bigcup_{n\in\mathbb{N}} X_n} \subset I^3.$$

The next lemma gathers some particulars about our construction.

Lemma 2.5.6. *Let $\mathcal{E} = \left\{(E_f, D_f) \colon f \in \mathbb{N}^{<\mathbb{N}}\right\}$ be a list of planar continua, and consider the compact metrizable space $X_{\mathcal{E}}$ as described above. Let A denote the union over all one-point components in $X_{\mathcal{E}}$. Then*

(1) For all $g \in \mathbb{N}^{<\mathbb{N}}$ we have $F_g \cap X_{\mathcal{E}}$ is homeomorphic to $X_{\tilde{\mathcal{E}}}$ where $\tilde{\mathcal{E}} = \left\{\tilde{E}_f \colon f \in \mathbb{N}^{<\mathbb{N}}\right\}$ with $\tilde{E}_f = E_{g^\frown f}$.

(2) The union over $\mathcal{F}_n = \{F_f \cap X_\mathcal{E} \colon f \in \mathbb{N}^n\}$ equals $X_\mathcal{E} \setminus X_{n-1}$ and hence is a dense open subspace of $X_\mathcal{E}$.

(3) The collection $\mathcal{F} = \left\{F_f \cap X_\mathcal{E} \colon f \in \mathbb{N}^{<\mathbb{N}}\right\}$ forms a clopen π-basis for $X_\mathcal{E}$.

(4) We have $A = X_\mathcal{E} \setminus \bigcup_{n\in\mathbb{N}} X_n$, and A is a dense, co-dense G_δ in $X_\mathcal{E}$, and

(5) the map $\Phi \colon \mathbb{N}^\mathbb{N} \to X_\mathcal{E},\ f \mapsto \bigcap_{n\in\mathbb{N}} F_{f|_{\{1,\dots,n\}}}$ is a homeomorphism from the Baire space onto A.

Proof. (1). The homeomorphism $h_g \colon I_3 \to \overline{F_g}$ witnesses that $h_g(X_{\mathcal{E}'}) = F_g \cap X_\mathcal{E}$.

(2). Clear by (b) and (c) above.

(3). Since $\partial F_f \cap X_\mathcal{E} = \emptyset$, every $F_f \cap X_\mathcal{E}$ is a clopen subset of $X_\mathcal{E}$. To prove that $\mathcal{F}$ is a π-base, fix an open subset $U \subset X_\mathcal{E}$. Find non-empty open V such that $\overline{V} \subset U$. By (2), we can find $F_n \in \mathcal{F}_1$ such that $F_n \cap V \neq \emptyset$. But then $F_n \cap V$ is a non-empty open subset, so we may find $F_{n,l} \in \mathcal{F}_2$ such that $F_{n,l} \cap V \neq \emptyset$. Therefore, by recursion, we may construct $f \in \mathbb{N}^\mathbb{N}$ such that

$$F_{f|_{\{1,\dots,n\}}} \cap V \neq \emptyset.$$

Now, since the diameters of the $F_{f|_{\{1,\dots,n\}}}$ converge to 0, it follows from compactness that

$$\{x_f\} = \bigcap_{n\in\mathbb{N}} F_{f|_{\{1,\dots,n\}}} \supset \bigcap_{n\in\mathbb{N}} F_{f|_{\{1,\dots,n\}}} \cap \overline{V} \neq \emptyset.$$

Hence, we have $\bigcap_{n\in\mathbb{N}} F_{f|_{\{1,\dots,n\}}} \subset \overline{V} \subset U$, so by another application of compactness there must exist $n \in \mathbb{N}$ such that $F_{f|_{\{1,\dots,n\}}} \subset U$. It follows that $\mathcal{F}$ is indeed a π-base.

(4). Since every X_n is compact, the set $X_\mathcal{E} \setminus \bigcup_{n\in\mathbb{N}} X_n$ is a G_δ. Next, note that by (3) we have that if $x \in X_\mathcal{E} \setminus X_n$ then x is contained in some clopen set $F \in \mathcal{F}_n$. In particular, $C(x) \subset F$ and therefore $C(x)$ has diameter at most 2^{-2n} by (a). It easily follows that for $x \in X_\mathcal{E} \setminus \bigcup_{n\in\mathbb{N}} X_n$, the diameter of the component of X is 0 and hence $x \in A$. Conversely, if $x \in X_n$ for some n then it is clear from the construction that the component of x is non-trivial.

Next, it follows from the Baire Category Theorem and (2) that A is dense in $X_\mathcal{E}$. And since $X_\mathcal{E}$ is defined as the closure of $\bigcup_{n\in\mathbb{N}} X_n$ in I^3 it is clear that A is co-dense.

(5). Note that as observed in Lemma 2.4.12(3) we know automatically that A must be homeomorphic to the Baire space. So we only have to show that the specific map Φ is indeed a homeomorphism onto its image. For this one checks that compactness of $X_\mathcal{E}$ implies that Φ is indeed onto, and that it is a homeomorphism because a basic open set in $\mathbb{N}^\mathbb{N}$ which specifies the first n coordinates by $g \colon n \to \mathbb{N}$ gets mapped precisely to F_g. ☐

We will now apply the above construction to special collections $\mathcal{E}$. If we choose all E_f to be identical, we get a non-reconstructible h-homogeneous space.

Theorem 2.5.7. *Suppose $\mathcal{E} = \{(E, D)\}$ for some planar continuum E and a fixed countable dense set $D \subset E$. Then the space $X_{\mathcal{E}}$ is a non-reconstructible compact metrizable h-homogeneous space.*

Proof. The sets F_f for $f \in \mathbb{N}^{<\mathbb{N}}$ form a clopen π-basis, all elements of which are homeomorphic to $X_{\mathcal{E}}$ by Lemma 2.5.6(1). Moreover, $X_{\mathcal{E}}$ has a dense set of one-point components by Lemma 2.5.6(4). Thus, $X_{\mathcal{E}}$ is strongly homogeneous by Lemma 2.4.8, and it follows from Corollary 2.4.11 that $X_{\mathcal{E}}$ is non-reconstructible. $\square$

Let X_I denote the result of the construction from above with $\mathcal{E} = \{(I, D_I)\}$ consisting of a unit interval and a fixed countable dense subset $D_I \subset I$, and X_P the result of the construction with $\mathcal{E} = \{(P, D_P)\}$ consisting of a pseudoarc and a fixed countable dense subset $D_P \subset P$. The next example shows that there are reconstructible compact metrizable spaces with a dense, co-dense G_δ of one-point components.

Theorem 2.5.8. *The compact metrizable space $Z = X_I \oplus X_P$ is an example of a reconstructible space where every card has arbitrarily large finite compactifications.*

Proof. By the Universal Sequence Lemma 2.4.6. $\square$

Another application of the machinery developed in this section will follow below in Section 2.5.5, where we construct a non-reconstructible space that does not admit a universal sequence of constant type.

2.5.4 Non-reconstructible h-homogeneous spaces III After hearing about the construction described in the previous section, Jan van Mill outlined to me an alternative way to construct compact metrizable h-homogeneous spaces with a dense set of 1-point components. In this section we work out the details of this construction. A sequence U_n of subsets of a metric space is called a *null-sequence* if $\operatorname{diam}(U_n) \to 0$ [vM01, A11].

Lemma 2.5.9. *The Cantor set C contains a null-sequence $\{C_n \colon n \in \omega\}$ of disjoint nowhere dense copies of the Cantor set such that $\bigcup C_n$ is dense in C.*

Proof. Since $(\omega + 1) \times C \cong C$, the Cantor set contains nowhere dense copies of itself. Moreover, since the Cantor set is homogeneous, every point of the Cantor set is contained in some nowhere dense copy of the Cantor set.

Now fix a countable dense subset $D = \{d_n \colon n \in \omega\}$ of C. We pick disjoint nowhere dense subsets $C_n \cong C$ making sure that $D \subset \bigcup C_n$. Suppose we have picked $\{C_n \colon n < k\}$ for some $k \in \omega$. Let d be the element with the least index in D such that d is contained in the open set $C \setminus \bigcup_{n<k} C_n$. Then choose any closed nowhere dense copy of the Cantor set C_k of diameter at most 2^{-k} such that $d \in C_k \subset C \setminus \bigcup_{n<k} C_n$. □

Note that by the Baire Category Theorem, the remaining part $C \setminus \bigcup C_n$ is a dense, co-dense G_δ of C. We now show, in Lemma 2.5.12, that all null-sequences of nowhere dense copies of the Cantor set are equivalently embedded. For this, we shall need the notion of Knaster-Reichbach covers from [vE86, 3.2.1].

Let X and Y be zero-dimensional separable metrizable spaces, $A \subset X$ and $B \subset Y$ two nowhere dense closed subsets and $f \colon A \to B$ a homeomorphism. We say that $\{(U_n, V_n) \colon n \in \omega\}$ is a *Knaster-Reichbach cover* (or short *KR-cover*) for $(X \setminus A, Y \setminus B, f)$ if $\mathcal{U} = \{U_n \colon n \in \omega\}$ and $\mathcal{V} = \{V_n \colon n \in \omega\}$ are collections of disjoint non-empty X- and Y-clopen sets covering $X \setminus A$ and $Y \setminus B$ respectively, such that, whenever $f_n \colon U_n \to V_n$ are bijections, then the bijection

$$h = f \cup \bigcup_{n \in \omega} f_n \colon X \to Y$$

is continuous at points of A, and h^{-1} is continuous at points of B.

The following lemma on the existence of KR-covers, originally due to Knaster and Reichbach in 1953 [KR53], can be found in van Engelen's book [vE86].

Lemma 2.5.10 ([vE86, 3.2.2]). *Let X and Y be zero-dimensional separable metrizable spaces, and let A and B be non-empty closed nowhere dense subspaces of X and Y respectively. If $f \colon A \to B$ is a homeomorphism, then there exists a KR-cover for $(X \setminus A, Y \setminus B, f)$.* □

The following lemma gives an alternative description of KR-covers for compact zero-dimensional spaces.

Lemma 2.5.11. *Let X and Y be compact zero-dimensional separable metrizable spaces, and let A and B be non-empty closed nowhere dense subspaces of X and Y respectively, and $f \colon A \to B$ a homeomorphism. Then a collection of pairs of clopen subsets $\{(U_n, V_n) \colon n \in \omega\}$ (with $|U_n| = |V_n|$ for all $n \in \omega$) is a KR-cover for*

$(X \setminus A, Y \setminus B, f)$ if and only if both $\{U_n \colon n \in \omega\}$ and $\{V_n \colon n \in \omega\}$ are null-sequences, and for all $x \in A$, a subsequence $\{U_{n_i} \colon i \in \omega\}$ converges to x precisely when the corresponding sequence $\{V_{n_i} \colon i \in \omega\}$ converges to $f(x)$.

Proof. First suppose that $\{(U_n, V_n) \colon n \in \omega\}$ is a KR-cover for $(X \setminus A, Y \setminus B, f)$. Let us fix bijections $f_n \colon U_n \to V_n$, and consider the resulting map $h = f \cup \bigcup_{n\in\omega} f_n$, which is continuous at points of A. Now suppose for a contradiction that $\{U_n \colon n \in \omega\}$ is not a null-sequence. Then by compactness there are points $x \neq y \in A$, a subsequence $\{U_{n_i} \colon i \in \omega\}$ and points $x_i, y_i \in U_{n_i}$ such that $x_i \to x$ and $y_i \to y$. By assumption, we have $h(x_i) \to h(x)$ and $h(y_i) \to h(y)$.

Now consider bijections $\tilde{f}_n \colon U_n \to V_n$ which equal f_n but swap $x_i \mapsto f_{n_i}(y_i)$ and $y_i \mapsto f_{n_i}(x_i)$ when $n = n_i$. It necessarily follows that $\tilde{h} = f \cup \bigcup_{n\in\omega} \tilde{f}_n$ is not continuous at points of A since $\tilde{h}(y_i) \to \tilde{h}(x)$, a contradiction.

For the second assertion, suppose for some $x \in A$ that $U_{n_i} \to x$. If $V_{n_i} \not\to h(x)$ then compactness implies that there is a subsequence $v_{n_{i_j}} \in V_{n_{i_j}}$ such that $v_{n_{i_j}} \to v \neq h(x)$. As before, we can find bijections $\tilde{f}_n$ such that the map $\tilde{h}$ is not continuous at x.

Conversely, let $f_n \colon U_n \to V_n$ be bijections and consider the corresponding map $h = f \cup \bigcup_{n\in\omega} f_n$. To show that h is continuous at points of A, let $x \in A$ and suppose that $x_i \to x$ with $x_i \notin A$. Find $\{U_{n_i} \colon i \in \omega\}$ such that $x_i \in U_{n_i}$ and suppose without loss of generality that all U_{n_i} are distinct. From $x_{n_i} \to x$ and the fact that U_n is a null-sequence, it follows that $U_{n_i} \to x$. By assumption, $V_{n_i} \to h(x)$. In particular $h(x_i) \to h(x)$, so h is continuous at x.

The case for h^{-1} is similar, and the proof is complete. □

Lemma 2.5.12. *Suppose $\{C_n \colon n \in \omega\}$ and $\{C'_n \colon n \in \omega\}$ each are null-sequences of disjoint nowhere dense copies of the Cantor set inside some Cantor set C such that both $\bigcup C_n$ and $\bigcup C'_n$ are dense in C. Then there is an autohomeomorphism h of C and a permutation π of the natural numbers such that $h(C_n) = C'_{\pi(n)}$.*

Moreover, if $g_n \colon C_n \to C$ and $g'_n \colon C \to C'_n$ are homeomorphisms witnessing that $C_n \cong C \cong C'_n$ then there are h and π as above such that $h|_{C_n} = g'_{\pi(n)} \circ g_n$.

Proof. Without loss of generality, assume that the diameters of $\{C_n \colon n \in \omega\}$ and $\{C'_n \colon n \in \omega\}$ are decreasing, and put $\epsilon_n = \max\{\operatorname{diam}(C_n), \operatorname{diam}(C'_n)\}$. By recursion we will define collections of non-empty clopen subsets $\mathcal{U} = \{U_s \colon s \in \omega^{<\omega}\}$ and $\mathcal{V} = \{V_s \colon s \in \omega^{<\omega}\}$ of X such that $U_\emptyset = C = V_\emptyset$, and

(1) $C_n \cap U_s \neq \emptyset \Rightarrow C_n \subset U_s$ and $C'_n \cap V_s \neq \emptyset \Rightarrow C'_n \subset V_s$ for all $n \in \omega$ and $s \in \omega^{<\omega}$,

(2) for all $k \in \omega$ and $s \in \omega^k$ we have $i_s = \min\{n \in \omega \colon C_n \subset U_s\} \geq k$ and $j_s = \min\{n \in \omega \colon C'_n \subset V_s\} \geq k$,

(3) the collection $\{(U_{s^\frown n}, V_{s^\frown n}) \colon n \in \omega\}$ is a KR-cover for $(U_s \setminus C_{i_s}, V_s \setminus C'_{j_s}, g'_{j_s} \circ g_{i_s})$ for all $s \in \omega^{<\omega}$,

(4) for all $k \in \omega$ and $s \in \omega^k$ we have $\operatorname{diam}(U_s) \leq 2\epsilon_k$ and $\operatorname{diam}(V_s) \leq 2\epsilon_k$.

For the recursion, given U_s and V_s for some $s \in \omega^k$ satisfying (1) – (4), we will define $\{U_{s^\frown n} \colon n \in \omega\}$ and $\{V_{s^\frown n} \colon n \in \omega\}$ such that (1) – (4) are satisfied. Set

$$I_s = \{n \in \omega \colon C_n \subset U_s\} \text{ and } J_s = \{n \in \omega \colon C'_n \subset V_s\},$$

and put

$$i_s = \min I_s \text{ and } j_s = \min J_s.$$

Inside U_s, identify each of C_n for $n \in I_s \setminus \{i_s\}$ with a single point. Since $\{C_n \colon n \in I_s\}$ is a null-sequence, the resulting quotient is a Cantor set K [vM01, A.11.6]. Let $q \colon U_s \to K$ denote the quotient map. Similarly, identifying inside V_s each of C'_n for $n \in J_s \setminus \{j_s\}$ with a single point gives a Cantor set K' with quotient map $q' \colon V_s \to K'$. By Lemma 2.5.10, there is a KR-cover $\left\{(\tilde{U}_n, \tilde{V}_n) \colon n \in \omega\right\}$ for $(K \setminus C_{i_s}, K' \setminus C'_{j_s}, g'_{j_s} \circ g_{i_s})$.

Next, since for closed maps q, every fibre $q^{-1}(x)$ has a neighbourhood base of the form $\{q^{-1}(U) \colon x \in U \text{ open}\}$ [vM01, Ex A.1.15], we can find, for every $x \in K$, a clopen neighbourhood S_x of x such that $q^{-1}(S_x) \subset B_{\epsilon_{k+1}}(q^{-1}(x))$. Since $\operatorname{diam}(C_n) \leq \epsilon_{k+1}$ for all $n \in I_s \setminus \{i_s\}$ by (2), this implies $\operatorname{diam}(q^{-1}(S_x)) \leq 2\epsilon_{k+1}$. Hence, using that $\tilde{U}_n$ is compact zero-dimensional, we can split each

$$\tilde{U}_n = S_{(n,1)} \oplus \cdots \oplus S_{(n,l_n)} \text{ and } \tilde{V}_n = T_{(n,1)} \oplus \cdots \oplus T_{(n,l_n)}$$

for some $l_n \in \omega$ such that $\operatorname{diam}\big(q^{-1}(S_{(n,l)})\big) \leq 2\epsilon_{k+1}$ and $\operatorname{diam}\big(q'^{-1}(T_{(n,l)})\big) \leq 2\epsilon_{k+1}$ for all $l \leq l_n$. It is clear that $\left\{(S_{(n,l)}, T_{(n,l)}) \colon n \in \omega, l \leq l_n\right\}$ is still a KR-cover for $(K \setminus C_{i_s}, K' \setminus C'_{j_s}, g'_{j_s} \circ g_{i_s})$ and can be enumerated as $\{(S_n, T_n) \colon n \in \omega\}$.

With the help of Lemma 2.5.11, it follows that the preimages $U_{s^\frown n} = q^{-1}(S_n)$ and $V_{s^\frown n} = q'^{-1}(T_n)$ give a KR-cover $\{(U_{s^\frown n}, V_{s^\frown n}) \colon n \in \omega\}$ for $(U_s \setminus C_{i_s}, V_s \setminus C'_{j_s}, g'_{j_s} \circ g_{i_s})$, such that (1), (3) and (4) are satisfied. Also note that since $i_{s^\frown n} \in I_s \setminus \{i_s\}$, we have for all $n \in \omega$ that $i_{s^\frown n} \geq i_s + 1 \geq k + 1$ by induction assumption on (2). This completes the recursive construction.

Observe that by (1) and (2), the sets $\{i_s \colon s \in \omega^{<\omega}\}$ and $\{j_s \colon s \in \omega^{<\omega}\}$ are enumerations of ω. This defines a permutation $\pi \colon \omega \to \omega,\ i_s \mapsto j_s$.

With the recursion completed, fix bijections $f_s \colon U_s \to V_s$ for all $s \in \omega^{<\omega}$. It follows from (3) that for all $k \in \omega$, both

$$\left\{U_s \colon s \in \omega^k\right\} \cup \left\{C_{i_s} \colon s \in \omega^{<k}\right\} \text{ and } \left\{V_s \colon s \in \omega^k\right\} \cup \left\{C'_{j_s} \colon s \in \omega^{<k}\right\}$$

are partitions of C, and hence the maps

$$h_k = \bigcup_{s \in \omega^k} f_s \cup \bigcup_{s \in \omega^{<k}} g'_{j_s} \circ g_{i_s}$$

are well-defined bijections $h_k \colon C \mapsto C$ for all $k \in \omega$. We claim that these maps satisfy

(5) $h_k(U_s) = V_s$ for all $s \in \omega^{\leq k}$, and

(6) $h_k|_{C_{i_s}} = g'_{j_s} \circ g_{i_s} \colon C_{i_s} \to C_{j_s}$ for all $s \in \omega^{<k}$.

Indeed, (6) is clear by construction, and (5) follows from (3) and a straightforward induction, as h_{k+1} is nothing else but h_k with each $f_s \colon U_s \to V_s$ for $s \in \omega^k$ replaced by the bijection $\bigcup_{n \in \omega} f_{s^\frown n} \cup (g'_{j_s} \circ g_{i_s}) \colon U_s \to V_s$.

Finally, we claim that $h = \lim_{k \to \infty} h_k \colon C \to C$ is an autohomeomorphism of C as required. For this, we first claim that

(7) $h(U_s) = V_s$ for all $s \in \omega^{<\omega}$, and

(8) $h|_{C_n} = g'_{\pi(n)} \circ g_n$.

Condition (8) follows from (6) and the definition of π. To see (7), let $x \in U_s \setminus \bigcup C_n$. By (3) and (4) there is a unique branch $t \in \omega^\omega$ such that $x \in \bigcap_{n \in \omega} U_{t|_n}$. It follows from (5) that $h_k(x) \in \bigcap_{n \leq k} V_{t|_n}$, and hence $h(x) \in \bigcap_{n \in \omega} V_{t|_n}$. Conversely, for $y \in V_s \setminus \bigcup C'_n$ there is a unique branch $t \in \omega^\omega$ such that $y \in \bigcap_{n \in \omega} V_{t|_n}$. By compactness and (4), there is a unique element $x \in \bigcap_{n \in \omega} U_{t|_n}$, and it follows $h(x) = y$ by the previous observation. This proves (7).

It follows that h is a bijection. To show that h is a homeomorphism, by compactness it suffices to show that h is continuous at every point $x \in C$. If $x \in C_n$ for some $n \in \omega$, find $s \in \omega^{<\omega}$ such that $i_s = n$. Then U_s is a clopen neighbourhood of C_n. Since $h(U_{s^\frown n}) = V_{s^\frown n}$ by (7) it follows from (3) that $h|_{U_s}$ is continuous at points of C_n, and hence so is h. And if $x \in C \setminus \bigcup C_n$ then let $t \in \omega^\omega$ be the unique branch such that $x \in \bigcap_{n \in \omega} U_{t|_n}$ and $h(x) \in \bigcap_{n \in \omega} V_{t|_n}$. By (4), the collection $\left\{V_{t|_n} \colon n \in \omega\right\}$ forms a neighbourhood base of $h(x)$. And since by (7), the preimage $h^{-1}(V_{t|_n}) = U_{t|_n}$ is open in X, it follows that h is continuous at points of $C \setminus \bigcup C_n$. This completes the proof. □

Theorem 2.5.13. *Let $\{C_n \colon n \in \omega\}$ be a null-sequence of disjoint nowhere dense copies of the Cantor set inside a Cantor set C such that $\bigcup_{n\in\omega} C_n \subset C$ is dense. Further, let E be a metrizable continuum, and let $f \colon C \to E$ be a continuous surjection. Further, fix homeomorphisms $f_n \colon C_n \to C$. Then $C/\mathcal{P}$ for*

$$\mathcal{P} = \left\{(f \circ f_n)^{-1}(e) \colon e \in E, n \in \omega\right\} \cup \left\{\{x\} \colon x \in C \setminus \bigcup C_n\right\}$$

is a compact metrizable h-homogeneous space with a dense set of 1-point components, and hence non-reconstructible.

We may think of $C/\mathcal{P}$ as a Cantor set with every C_n replaced by a copy of E.

Proof. By [vM01, Ex A.11.4], $\mathcal{P}$ is an upper semi-continuous decomposition, and hence $C/\mathcal{P}$ is compact metrizable [vM01, A.11.2]. The set $\{\{x\} \colon x \in C \setminus \bigcup C_n\}$ consists of one-point components and is dense in C by the Baire Category Theorem, and hence dense in $C/\mathcal{P}$.

Let $q \colon C \to C/\mathcal{P}$ denote the quotient map. To show that $C/\mathcal{P}$ is h-homogeneous, note that since E is connected, any clopen subset of $C/\mathcal{P}$ is of the form $q(U)$ where $U \subset C$ is clopen with the property that every C_n is either contained in U or in its complement. Hence, letting $S_U = \{n \in \omega \colon C_n \subset U\}$, we have $q(U) = U/\mathcal{P}'$ where $\mathcal{P}' = \{(f \circ f_n)^{-1}(e) \colon e \in E, n \in S_U\} \cup \{\{x\} \colon x \in U \setminus \bigcup C_n\}$. To see that $C/\mathcal{P}$ and $U/\mathcal{P}'$ are homeomorphic, use Lemma 2.5.12 to find a homeomorphism $g \colon C \to U$ such that

$$g|_{C_n} = f^{-1}_{\pi(n)} \circ f_n \colon C_n \to C_{\pi(n)}$$

for some bijection $\pi \colon \omega \to S_U$. It follows that for all $e \in E$ and $n \in \omega$, we have $(f_n \circ f)^{-1}(e) \subset C_n$ and hence

$$g\big(f \circ f_n)^{-1}(e)\big) = f^{-1}_{\pi(n)} \circ f_n\big(f \circ f_n)^{-1}(e)\big) = (f \circ f_{\pi(n)})^{-1}(e).$$

So g is a homeomorphism between C and U, which respects the partitions $\mathcal{P}$ and $\mathcal{P}'$. Therefore, $q \circ g \circ q^{-1} \colon C/\mathcal{P} \to U/\mathcal{P}'$ is a homeomorphism. This proves that $C/\mathcal{P}$ is h-homogeneous. The reconstruction result follows from Corollary 2.4.11. □

2.5.5 A non-reconstructible space without a constant universal sequence

All examples of non-reconstructible spaces we exhibited so far were non-reconstructible because they had a constant universal sequence. In this section, we show that this need not be the case. Based on the construction presented in Section 2.5.3, we now build a non-reconstructible compact metrizable space without a constant universal sequence.

Let $\{E_n \colon n \in \omega\}$ be a list of pairwise non-homeomorphic planar continua, and consider the list $\mathcal{E} = \left\{(E_f, D_f) \colon f \in \mathbb{N}^{<\mathbb{N}}\right\}$ such that for $f \colon k \to \mathbb{N}$ we have $E_f = E_{f(k-1)}$ and $D_f = D_{f(k-1)}$. Consider the space $X_{\mathcal{E}}$ as described in Section 2.5.3. Our list ensures that at the nth step in the recursion we replace isolated points always by the same continuum E_n. To make our notation more convenient, we let $Z_0 = X_{\mathcal{E}}$ and write Z_k for the compact metrizable space arising from the construction with the list of continua replaced by $\{E_n \colon n \geq k\}$.

Lemma 2.5.14. *Let $f \colon k \to \mathbb{N}$ be a finite sequence. Then $F_f \cap Z_0 \cong Z_k$.*

Proof. This is an immediate consequence of Lemma 2.5.6(1). $\square$

Lemma 2.5.15. *We have $Z_k \cong Z_k \oplus Z_m$ for all $m > k$.*

Proof. By symmetry of the constructions it is enough to prove the lemma for $k = 0$. Clearly, for fixed m, there is a clopen sequence A_n of spaces homeomorphic to Z_m converging to some $x \in E_0$. (Take an appropriate subsequence of

$$A_n = F_{(n,\underbrace{1,1,1,\dots,1,1}_{(m-1)\text{ times}})} \cap X_0$$

and use Lemma 2.5.14.) Since $\{x\} \cup \bigcup_{n\geq 0} A_n \cong \{x\} \cup \bigcup_{n\geq 1} A_n$, the result follows from the Pasting Lemma as in the proof of Theorem 2.4.7. $\square$

Lemma 2.5.16. *The space $X_{\mathcal{E}} = Z_0$ has a universal sequence of type $(Z_n)_{n>0}$.*

Proof. For every $x \in Z_0$ we will show that there is a sequence of disjoint clopen sets of type $(Z_n)_{n>0}$ converging to x.

First, assume that x forms a 1-point component of Z_0. By Lemma 2.5.6(5) there is $f \in \mathbb{N}^{\mathbb{N}}$ such that the sets $U_n = F_{f|_n} \cap X_{\mathcal{E}}$ form a (nested) neighbourhood base of x. By Lemmas 2.5.14 and 2.5.15, we have $U_n \setminus U_{n+1} \cong Z_n$, and therefore $Z_0 \setminus \{x\} \cong \bigoplus_{n\in\omega} Z_n$.

Now assume that $x \in E_0$. First note that $Z_0 \cong Z_0 \oplus Z_1$. Next, as in Lemma 2.5.15, we may find clopen sets $A_n \subset Z_0$ for $n \in \omega$ such that $A_n \cong X_1$ and $\{A_n \colon n \in \omega\}$ converges to x. Since by Lemma 2.5.15, we have $A_n \cong A_n \oplus C_n$ where $C_n \cong Z_{n+2}$ it follows that $\{Z_1\} \cup \{C_n \colon n \in \omega\}$ is a sequence of disjoint clopen sets of type $(Z_n)_{n>0}$ converging to x.

Finally, assume that $x \in E_f$ for some $f \in \mathbb{N}^k$. Using Lemma 2.5.14, we can apply the previous case to obtain a sequence $C_n \subset F_f$ of disjoint clopen sets of type $(Z_n)_{n>k}$ converging to x. And since $Z_0 \setminus F_f \cong Z_0 \cong Z_0 \oplus Z_1 \oplus \ldots \oplus Z_k$ by Lemma 2.5.15, the result follows. $\square$

Theorem 2.5.17. *The space $\bigoplus_{n\in\omega} Z_n$ is a non-compact reconstruction of Z_0.*

Proof. Consider a card $Z_0 \setminus \{x\}$. By Lemma 2.5.16, there is a sequence $A_n \subset Z_0$ for $n \in \omega$ of disjoint clopen sets converging to x such that $A_n \cong Z_{n+1}$. Applying Lemma 2.5.15 to A_n, we see that $A_n \cong A_n \oplus Z_{n+2}$ for all $n \in \omega$ and hence

$$Z_0 \setminus \{x\} \cong Z_0 \setminus \{x\} \oplus \bigoplus_{n\geq 2} Z_n.$$

Applying Lemma 2.5.15 to $Z_0 \setminus \{x\}$, we see that $Z_0 \setminus \{x\} \cong Z_0 \setminus \{x\} \oplus Z_1$. Thus,

$$Z_0 \setminus \{x\} \cong Z_0 \setminus \{x\} \oplus \bigoplus_{n\geq 1} Z_n$$

is a card of $\bigoplus_{n\in\omega} Z_n$.

Conversely, consider a card of $\bigoplus_{n\in\omega} Z_n$. Let us suppose that the card has been obtained by deleting a point $z_k \in Z_k$ for $k \geq 1$. Applying Lemma 2.5.15 twice gives us

$$Z_0 \oplus Z_k \setminus \{z_k\} \cong Z_0 \setminus \{z\} \cong Z_0 \setminus \{z\} \oplus Z_k.$$

It follows that

$$(\bigoplus_{n\in\omega} Z_n) \setminus \{x_k\} \cong Z_0 \setminus \{z\} \oplus \bigoplus_{n\geq 1} Z_n,$$

which is, by the above, homeomorphic to $Z_0 \setminus \{z\}$. □

Finally, Z_0 does not contain a constant universal sequence because it is not π-homogeneous (Lemma 2.4.9). Indeed, any clopen subset $B \subset Z_0$ contains a non-trivial component, say E_n. But a clopen subset of Z_0 homeomorphic to Z_k for $k > n$ does not contain a copy of E_n, and hence subsets homeomorphic to B cannot form a π-base for Z_0.

2.5.6 Counting non-reconstructible compact metrizable spaces

Let us see that from the above constructions it follows that the number of non-reconstructible compact metrizable spaces is as large as possible.

Theorem 2.5.18. *There are as many non-homeomorphic non-reconstructible compact metrizable spaces as there are reconstructible ones.*

Proof. Recall that there are $\mathfrak{c}$ non-homeomorphic metrizable continua (see e.g. [Minc10, War32]) and hence $\mathfrak{c}$ non-homeomorphic reconstructible compact metrizable spaces. However, using these non-homeomorphic continua as input for Corollary 2.5.3 or Theorem 2.5.13, we also get $\mathfrak{c}$ different non-reconstructible compact metrizable spaces.

The result now follows from the fact that since every compact metrizable space is represented by a closed subset of the Hilbert cube, there are at most $\mathfrak{c}$ non-homeomorphic compact metrizable spaces. $\square$

This theorem is especially interesting when compared to Bollobás' result for the graph theoretic reconstruction problem that almost all graphs (in the probabilistic sense) are reconstructible [Bol90]. In other words, Bollobás' result says that non-reconstructible graphs (should there exist any) are sparsely distributed. Theorem 2.5.18 says for the topological case, in terms of absolute numbers, non-reconstructible compact metrizable spaces are as frequent as reconstructible ones.

Still, it is an interesting question whether an analogue of Bollobás' result can be established for compact metrizable spaces. A common way to capture this behaviour in topology is to consider a *generic* compact metrizable space. Here, a class of spaces is called generic if all homeomorphic copies of spaces in this class form a dense G_δ in the hyperspace of the Hilbert cube [Kech95, §I.8]. However, the hyperspace of the Hilbert cube contains a dense G_δ of copies of the Cantor set [Kech95, I.8.8]. Therefore, a generic element is non-reconstructible—but only because it is homeomorphic to the Cantor set.

To answer this question properly, we need a way to talk about the distribution of *non-homeomorphic* non-reconstructible compact metrizable spaces.

2.6 Open questions

We end this chapter with a list of open problems. Some have already been mentioned along the way, and some are new.

Question 2.6.1. *What is the maximal number of non-homeomorphic reconstructions of a compact (metrizable) space? Is there an example of a compact (metrizable) space with more than one non-homeomorphic reconstruction?*

Note that this number is at most $\mathfrak{c}$. This holds, because every reconstruction is necessarily Polish (separable and completely metrizable) from the results in Section 1.2.1, and every Polish space is a G_δ in I^ω by completeness [Eng89, 4.3.24]. Since the Hilbert cube I^ω is metrizable, there are only $\mathfrak{c}$ many G_δ's, and the result follows.

Can we show that if there are N non-homeomorphic reconstructions then every point $x \in X$ is the limit of N sequences each of different type?

The next pair of questions has already been mentioned in previous parts of this chapter, and we list them here again for completeness.

Question 2.6.2. *Do all cards of* $\mathbb{H}^*$ *have maximal* 1*-point compactifications?*

Question 2.6.3. *Does every Hausdorff continuum with* $w(X) = |X|$ *have a card with a maximal finite compactification? Is there a Hausdorff continuum where every point is locally* 3*-separating?*

Finally, we would like to find an answer to the question which compact metrizable spaces are reconstructible.

Question 2.6.4. *Find a characterisation which compact metrizable spaces are reconstructible.*

Chapter 3

Reconstructing locally compact spaces

This chapter contains reconstruction results for locally compact, non-compact spaces. For compact spaces, we have seen in the previous chapter that the theory of maximal finite compactifications provides an effective tool to obtain affirmative reconstruction results. Indeed, all compact spaces that have a card with a maximal finite compactification were shown to be reconstructible. This chapter's starting point is the question whether the same result still holds for locally compact spaces: *Is it true that every locally compact space is reconstructible provided it has a card with a maximal finite compactification?* We make some progress towards answering this question. However, a complete solution is yet to be found.

From now on, a locally compact Hausdorff space with a maximal finite compactification will simply be called an *MFC-space.* Examples of MFC-spaces which are interesting in light of the results from the previous chapter are cards of well-known Stone-Čech spaces such as $\beta\omega \setminus \{x\}$ or $2^\kappa \setminus \{x\}$ for uncountable κ. But we will also have a look at spaces such as limit ordinals of uncountable cofinality, Euclidean spaces $\mathbb{R}^n$ for all $n \geq 1$, or the long line [Eng89, 3.12.19]. In the first part of this chapter, by generalising our methods from the previous chapter about maximal finite compactifications, we will see that all spaces listed above are indeed reconstructible.

Our main technical result is that if an MFC-space X with a maximal N-point compactification has an MFC-space card, then every reconstruction of X will also have a maximal N-point compactification. From this, we can show that every such space X has at most N different reconstructions, which we can name explicitly (Theorem 3.2.4). We also show that every MFC-space that contains isolated points is reconstructible.

In the second part of this chapter, we focus on the reconstruction of ordinal numbers. From the reconstruction results about MFC-spaces in the first half of this chapter, it follows that every successor ordinal and every limit ordinal of uncountable cofinality is reconstructible. We complete the investigation by showing that in fact all ordinal numbers are reconstructible. The crucial step for this proof is a characterisation in Theorem 3.3.21 under which conditions a topological sum of spaces is homeomorphic to an ordinal.

This chapter is organised as follows. Section 3.1 contains more background material on the theory of maximal finite compactifications. Section 3.2 contains affirmative reconstruction results for MFC-spaces. Finally, Section 3.3 contains proof that all ordinal numbers are reconstructible. We conclude this chapter in Section 3.4 with some open questions.

3.1 Theory of maximal finite compactifications II

We begin with a literature review about finite compactifications. All results in Section 3.1, apart from some minor observation of my own, are known.

3.1.1 Finite compactifications Recall that an *N-star* of a locally compact space X is a collection of N non-empty open pairwise disjoint subsets $\{G_i\}_{i=1}^N$ such that $K = X \setminus \bigcup_{i=1}^N G_i$ is compact, but the union $K \cup G_i$ is non-compact for each i. As seen in Theorem 2.1.1, the notion of N-stars captures the finite compactifications of X. Specifically:

Theorem 3.1.1 (see Theorem 2.1.1)**.** *For $N \geq 1$, a Tychonoff space X has an N-point compactification if and only if X is locally compact and contains an N-star.* $\square$

If γX and δX are compactifications of X and $f\colon \delta X \to \gamma X$ is the (unique) function witnessing that $\gamma X \leq \delta X$, then f is a quotient mapping [Cha76, 1.27]. In particular, every compactification of X arises as a quotient of βX by identifying points in the remainder of X. If γX is any compactification and $f\colon \beta X \to \gamma X$ the quotient mapping, then the set $F(\gamma X) = \{f^{-1}(p)\colon p \in \gamma X \setminus X\}$ is called the *β-family of γX*. Note that if γX is a finite compactification of X then the β-family $F(\gamma X)$ forms a clopen partition of $\beta X \setminus X$. The following result involving β-families is another key observation by Magill.

Theorem 3.1.2 ([Mag68, Lemma 1])**.** *For two compactifications of a space X the relation $\gamma X \leq \delta X$ holds if and only if $F(\delta X)$ refines $F(\gamma X)$.* $\square$

Corollary 3.1.3. *We have $\gamma X \approx \delta X$ if and only if $F(\gamma X) = F(\delta X)$.* ☐

Next, we show in Corollary 3.1.5 that the maximal finite compactification is, up to equivalence, unique. The result is known and appears for example in [Cha76, 6.12]. Theorem 3.1.4, however, appears to have been overlooked.

Theorem 3.1.4. *If a space X has a maximal N-point compactification νX, and γX is any other finite compactification, then $\gamma X \leq \nu X$.*

Proof. Let $F(\gamma X) = \{A_1, \ldots, A_M\}$ and $F(\nu X) = \{B_1, \ldots, B_N\}$ be the respective β-families. Assume for a contradiction that $\gamma X \not\leq \nu X$, then without loss of generality, it follows from Theorem 3.1.2 that B_1 is not contained in any of the A_i. Hence, we may assume that B_1 intersects A_1 as well as $\bigcup_{i=2}^{N} A_i$. But now,

$$\left\{ B_1 \cap A_1, B_1 \cap (\bigcup_{i=2}^{N} A_i), B_2, \ldots, B_N \right\}$$

is a clopen partition of $\beta X \setminus X$, giving rise to an $(N+1)$-point compactification of X, a contradiction. ☐

Corollary 3.1.5. *If X has a maximal N-point compactification then all N-point compactifications of X are equivalent.* ☐

Thus, every finite compactification of an MFC-space X can be obtained as a quotient of νX by identifying points in the remainder.

Finally, the following theorem indicates that there is a relationship between connectedness of a topological space and the existence of maximal finite compactifications. The proof is another simple application of β-families.

Theorem 3.1.6. *A locally compact space X has a maximal finite compactification if and only if its Stone-Čech remainder $X^* = \beta X \setminus X$ has finitely many components.*

Proof. If γX is a finite compactification of X then $\gamma X \setminus X$ is discrete, and hence $F(\gamma X)$ forms a partition of X^* consisting of clopen sets. Thus, if X^* has finitely many components, then X^* has only finitely many disjoint clopen sets, so X cannot have arbitrarily large finite compactifications.

Conversely, every partition of X^* into N disjoint clopen sets gives rise to a finite N-point compactification by identifying each clopen set with a single point. ☐

More generally, we have the following powerful result by Magill, relating finite compactifications of a locally compact space X to the number of components of $X^* = \beta X \setminus X$.

Theorem 3.1.7 ([Mag66]). *The following statements about a topological space X are equivalent:*

(1) X is locally compact and X^* has infinitely many components,

(2) X is locally compact and has a compactification with countably infinite remainder,

(3) X has an N-point compactification for all $N \in \mathbb{N}$. $\square$

As a consequence we see that a locally compact space has a maximal finite compactification if and only if it does not possess a compactification with countably infinite remainder.

Next, let X be an MFC-space and νX its maximal finite compactification. We define the *(MFC)-degree* of X to be the number $|\nu X \setminus X|$. Further, we make the convention that a compact Hausdorff space is an MFC-space of degree 0.

Theorem 3.1.8. *The class of MFC-spaces of degree at most N is invariant under perfect mappings.*

Proof. Let X be a space with a maximal N-point compactification and f a perfect mapping of X onto a topological space Y. Since Hausdorffness [Eng89, 3.7.20] and local compactness [Eng89, 3.7.21] are invariants of perfect mappings, it follows that Y is a locally compact Tychonoff space. For the remainder of the proof we follow [Mag65, 2.7]. Suppose Y contains an M-star $\{K\} \cup \{G_i\}_{i=1}^{M}$. We claim its pullback via f is an M-star in X:

(1) $f^{-1}(K)$ is compact [Eng89, 3.7.2],

(2) $f^{-1}(G_i)$ is non-empty by surjectivity and open by continuity for all i,

(3) $X = f^{-1}(K) \cup \bigcup_{i=1}^{M} f^{-1}(G_i)$ is a disjoint union and

(4) $f^{-1}(K) \cup f^{-1}(G_i)$ is non-compact for all i—otherwise, continuity would imply that also $f(f^{-1}(K) \cup f^{-1}(G_i)) = K \cup G_i$ (by surjectivity) is compact, a contradiction.

Magill's characterisation 3.1.1 now implies that $M \leq N$ and thus every finite compactification of Y has at most an N-point remainder. $\square$

To place the concept of a maximal finite compactification into further context, we relate it to the Freudenthal compactification. A subset $Y \subset X$ of a topological space X is called *zero-dimensionally embedded* if there is a base $\mathcal{B}$ of open sets for X such that $\partial B \cap Y = \emptyset$ for all $B \in \mathcal{B}$ [AN93, VI.3.4]. The *Freudenthal compactification* FX of a rim-compact (a space with a basis of open sets with compact boundaries)

Tychonoff space X is the unique maximal compactification with zero-dimensionally embedded remainder [AN93, VI.3.7]. However, it is not hard to see that for a locally compact space X, the remainder of a compactification of X is zero-dimensionally embedded if and only if the remainder is zero-dimensional. Thus, the Freudenthal compactification of a locally compact space X is the unique maximal compactification with zero-dimensional remainder. This leads us to the following observation.

Theorem 3.1.9. *Let X be a locally compact Hausdorff space. If X has a maximal finite compactification νX then $\nu X = FX$, and if a space has a finite Freudenthal compactification then it is the maximal finite compactification.*

Proof. Suppose νX is a maximal N-point compactification of X. Then clearly νX has a zero-dimensional remainder, so by maximality of the Freudenthal compactification we get $\nu X \leq FX$. Now suppose that $|FX \setminus X| > N$. Because of zero-dimensionality, there is a partition of $FX \setminus X$ into $N+1$ disjoint clopen sets. Identifying each clopen set with a single point gives an $N+1$-point compactification of X, a contradiction. It follows $\nu X = FX$.

For the second claim, observe again that every finite compactification of X has a zero-dimensional remainder. Therefore, every finite compactification is bounded by the Freudenthal compactification. Thus, if the Freudenthal compactification is finite, it is the maximal finite compactification. □

The results mentioned in Theorem 3.1.4 and Corollary 3.1.5 are now immediate from the well-known corresponding results about the Freudenthal compactification. However, it is still instructive to have direct proofs available in the case of the maximal finite compactification.

3.1.2 Almost Stone-Čech spaces We now describe an interesting subclass of MFC-spaces, namely spaces with finite Stone-Čech remainder. We call such spaces *almost Stone-Čech spaces*, or shortly *ASC-spaces*. The *ASC-degree* $|X^*|$ of an ASC-space X is the natural number corresponding to the size of the Stone-Čech remainder of X. We use the convention that a compact Hausdorff space is an MFC/ASC-space of degree zero.

A broad range of examples in general topology fit the notion of an ASC-space. The first uncountable ordinal ω_1 and, more generally, every ordinal α of uncountable cofinality is an ASC-space of degree one because its Stone-Čech compactification is $\alpha+1$. Another example is the Tychonoff plank [Eng89, 3.12.20], which also has a one-point Stone-Čech remainder. In particular, ASC-spaces are not necessarily normal.

The mechanism of the Stone-Čech compactification provides a variety of ASC-spaces. If X is a non-compact space and A a finite subset of the Stone-Čech remainder of X, then $\beta X \setminus A$ is an ASC-space of degree $|A|$. This leads to examples of ASC-spaces such as cards of $\beta\omega$, which are ASC-spaces of degree at most one. Similarly, subspaces $2^\kappa \setminus A$ and $I^\kappa \setminus A$ of Cantor- and Tychonoff-cubes for uncountable cardinals κ and finite subsets A are ASC-spaces. Theorem 3.1.16 gives a proof of this fact.

ASC-spaces of degree at most one have been investigated under the name *almost compact spaces.* The first topological characterisation of such spaces is given in [Hew47, Thm. 5]. Another reason why these spaces are interesting is that almost compact spaces have a unique uniformity [Doss49, Gál59]. Normal almost compact spaces have also been studied [FKL91].

The generalisation of almost compact spaces to ASC-spaces, however, appears not to be well-studied. The only paper about ASC-spaces is [Fir70], which generalises the internal characterisations from [Doss49, Gál59, Hew47]. We list some properties of ASC-spaces.

Lemma 3.1.10 ([GJ76, 6J])**.** *Every ASC-space is pseudocompact.*

Proof. If a space is not pseudocompact, it contains a C-embedded copy of ω [GJ76, 1.21]. This copy of ω is closed [Eng89, 1.1.11]. Thus, if X is not pseudocompact then $\omega^* \subset X^*$, and therefore $|X^*| \geq 2^{\mathfrak{c}}$. □

The next two lemmas provide a characterisation of ASC-spaces. More information on almost compact spaces, i.e. ASC-spaces of degree at most one, can be found in [GJ76, 6J] or [Eng89, 3.12.16(a)].

Lemma 3.1.11 ([Fir70, 4.1])**.** *A Tychonoff space X is an ASC-space if and only if for every infinite collection of closed sets, completely separated in pairs, at least one member is compact.* □

Lemma 3.1.12 ([Fir70, 2.2])**.** *A Tychonoff space X is an ASC-space of degree at most N if and only if for every collection of $N+1$ closed sets in X, completely separated in pairs, at least one member is compact.* □

Theorem 3.1.13. *Every Tychonoff space that is the continuous image of an ASC-space of degree N is an ASC-space of degree at most N.*

Proof. Let X be an ASC-space of degree N and $f \colon X \to Y$ a continuous map onto a Tychonoff space Y. Consider the extension $\beta f \colon \beta X \to \beta Y$. This mapping is surjective, since its compact image is closed and contains the dense subset $Y \subset \beta Y$. It follows $|Y^*| \leq |X^*|$. □

The last result also follows from the characterisation in Lemma 3.1.12. If the image Y has degree exactly N then every map in Theorem 3.1.13 is a perfect map. This holds since a map between Tychonoff spaces is perfect if and only if its Stone-Čech extension carries remainders to remainders [Eng89, 3.7.16]. Both this last observation and Theorem 3.1.13 are known for almost compact spaces [Eng89, 3.12.16(d)] and, in a slightly stronger form, for normal almost compact spaces [FKL91]. The observation of Theorem 3.1.14 appears to be new.

Theorem 3.1.14. *For every N, the class of ASC-spaces of degree at most N is invariant under perfect mappings.*

Proof. Suppose Y is a perfect image of an ASC-space X. Then Y is a Tychonoff space, as Hausdorffness [Eng89, 3.7.20] and local compactness [Eng89, 3.7.21] are invariants of perfect mappings. The result now follows from Theorem 3.1.13. □

Recall that for a locally compact space X we defined $\mathcal{R}_n(X)$ to consist of all points x with $\operatorname{ends}(x, X) = n$, i.e. all points x such that there exists a compact neighbourhood D with $x \in \operatorname{int}(D)$ such that $D \setminus \{x\}$ has a maximal n-point compactification (see Section 2.1.3). To capture this mechanism for ASC-spaces, we denote the set of points x whose removal leaves a C^*-embedded dense subset $X \setminus \{x\}$ of X by $\mathcal{R}_*(X)$. We call the set $\mathcal{R}_*(X)$ *inner remainder of X*.

If X is a Tychonoff space then $X^* \subset \mathcal{R}_*(\beta X)$ by [Eng89, 3.6.9]. The inner remainder of $\omega_1 + 1$ consists of the single point ω_1. The inner remainder of the Tychonoff plank $T = (\omega_1 + 1) \times (\omega + 1) \setminus \{(\omega_1, \omega)\}$ consists of all points in $\{\omega_1\} \times \omega$ by [Eng89, 3.12.20(c)].

A compact Hausdorff space X is a Stone-Čech space if and only if it has a non-empty inner remainder. The next lemma shows that one can think of the set $\mathcal{R}_*(X)$ as building blocks for ASC-spaces.

Lemma 3.1.15. *Let X be a Stone-Čech space. Then $X \setminus A$ is an ASC-space with $\beta(X \setminus A) = X$ if and only if A is a finite subset of $\mathcal{R}_*(X)$.*

Proof. For sufficiency, assume that $A = \{x_1, \dots, x_N\} \subset \mathcal{R}_*(X)$ and find disjoint open neighbourhoods U_i such that $x_i \in U_i$.

Now let $f \in C^*(X \setminus A)$. Since X is Tychonoff, there are continuous maps $g_i \colon X \to [0,1]$ such that $g_i(X \setminus U_i) = 1$ and $g_i(x_i) = 0$. For each i, the function $f \cdot \prod_{j \neq i} g_j$ can naturally be seen as an element of $C^*(X \setminus \{x_i\})$, and hence extended to a function f_i defined on all of X. Finally, define

$$h\colon X \to [0,1], z \mapsto \begin{cases} f_i(z) & \text{if } z \in U_i, \\ f(z) & \text{otherwise,} \end{cases}$$

which is a continuous extension of f. Necessity is clear, completing the proof. □

Let us examine an interesting class of ASC-spaces that can be built using the previous lemma.

Theorem 3.1.16. *For uncountable κ, the spaces $2^\kappa \setminus A$ and $I^\kappa \setminus A$ are ASC-spaces of degree $|A|$ for every finite subset A.*

Proof. By [Gli59, Thm. 2], both spaces 2^κ and I^κ arise as Stone-Čech compactifications. Therefore, their inner remainders are non-empty. And since both spaces are homogeneous ([vM01, 1.6.6] for I^κ), every point lies in the inner remainder. Now apply Lemma 3.1.15. □

3.1.3 Extending homeomorphisms

One of the key ingredients for the reconstruction result in Theorem 1.2.6 was that homeomorphisms between locally compact spaces extend to their respective one-point compactifications. In the case of Stone-Čech compactifications and maximal finite compactifications we have the following results.

Lemma 3.1.17. *Every homeomorphism between two Tychonoff spaces X and Y extends to a homeomorphism between βX and βY.*

Proof. Suppose $f\colon X \to Y$ is a homeomorphism between Tychonoff spaces X and Y. Then $\mathrm{id}(X) \subset \beta X$ and $f(X) \subset \beta Y$ are both Stone-Čech compactifications of X. But since all Stone-Čech compactifications are equivalent, there exists, by definition, a homeomorphism $h\colon \beta X \to \beta Y$ such that $id \circ h = f$ [Eng89, p.167]. In other words, the homeomorphism h is the desired extension of f. □

Lemma 3.1.18. *Every homeomorphism between two MFC-spaces X and Y extends to a homeomorphism between their maximal finite compactifications νX and νY.*

Proof. As in the previous proof, this follows from uniqueness of the maximal finite compactification, Corollary 3.1.5. □

In particular, the extension of a homeomorphism between MFC-spaces to their respective maximal finite compactifications maps remainders onto remainders. This simple observation will play a crucial role in the proof of Lemma 3.2.6.

We observed in Theorem 3.1.9 that for MFC-spaces, the maximal finite compactification and the Freudenthal compactification coincide. From this perspective, it is interesting that not only do homeomorphisms between rim-compact spaces extend to their respective Freudenthal compactifications (which follows immediately, as above, from the uniqueness of the Freudenthal compactification), but in fact all closed continuous functions between rim-compact spaces where fibres have compact boundaries extend to their respective Freudenthal compactifications [AN93, VI.3.17].

3.2 Reconstruction results using maximal finite compactifications

3.2.1 Reconstructing MFC-spaces with isolated points We begin with a reconstruction result for spaces containing isolated points that have a maximal finite compactification. This proof is simple, but illustrates well some of the ideas that will be needed for our main reconstruction result later.

Theorem 3.2.1. *Every MFC-space that contains isolated points is reconstructible.*

Proof. Let X be a space with a maximal N-point compactification containing an isolated point x. We proceed by induction on N. If $N = 0$ then X is a compact Hausdorff space with isolated points and hence reconstructible by Theorem 1.2.6.

For the induction step, let $N \geq 1$ and assume the theorem holds for all spaces with maximal M-point compactifications such that $M < N$. Clearly, the card $X \setminus \{x\}$ is a space with a maximal N-point compactification.

Let Z be a reconstruction of X. Recall that Z is locally compact by Theorem 1.2.3. Find $z \in Z$ such that $Z \setminus \{z\} \cong X \setminus \{x\}$. If z is isolated too, then X and Z are homeomorphic. Otherwise, any finite compactification γZ of the space Z is a finite compactification of $Z \setminus \{z\}$, and therefore $|\gamma Z \setminus Z| \leq N - 1$. Hence Z, and therefore X, are reconstructible. □

In particular, the first uncountable ordinal ω_1 and, more generally, every limit ordinal of uncountable cofinality are reconstructible. Another example to which the result applies is the Tychonoff plank [Eng89, 3.12.20]. Finally, not only is $\beta\omega$ reconstructible (Theorem 1.2.8), but we now know that all cards of $\beta\omega$ are reconstructible.

3.2.2 Reconstructing the real line

For our second reconstruction result, we make use of the uniqueness of the maximal finite compactification (Corollary 3.1.5) to give a nice proof why the space of real numbers is reconstructible. It is easily seen, for example using Theorem 3.1.1, that $Y = \mathbb{R} \oplus \mathbb{R}$ in $\mathcal{D}(\mathbb{R})$ has a maximal 4-point compactification νY.

Figure 3.1: The space νY.

Note that no reconstruction of $\mathbb{R}$ can be compact by the Reconstruction Result for Compact Spaces, Theorem 2.1.10 (or use that the one-point compactification of Y has a connected card). From the results in Section 1.2, every reconstruction Z of $\mathbb{R}$ is a non-compact, locally compact Tychonoff space without isolated points. In particular, its one-point compactification αZ is a two-point compactification of Y. By Theorem 3.1.4, αZ is a quotient of νY, where we identify points in the remainder $\nu Y \setminus Y$.

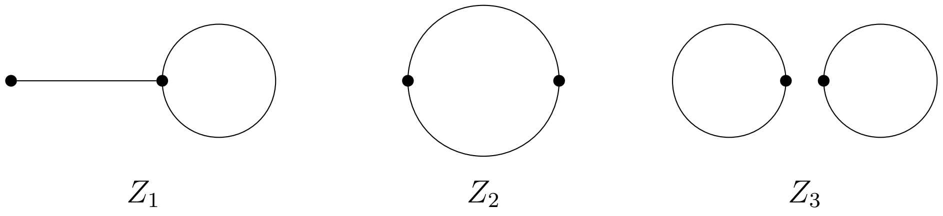

Figure 3.2: Types of two-point compactifications of $\mathbb{R} \oplus \mathbb{R}$.

The three spaces in Figure 3.2 represent all homeomorphism types of two-point compactifications of Y: when identifying three points in the remainder of νY, we end up with a space homeomorphic to Z_1. When identifying pairs of points, one gets Z_2 or Z_3, depending on whether pairs lie in different components or not.

By construction, Z must be a subspace of Z_i with one point at infinity removed. However, if one removes points at infinity from Z_1 or Z_3, one gets a space that has either a connected card, or a card with three components. Thus, the only way to obtain a reconstruction of $\mathbb{R}$ is to remove a point from Z_2. But this space is homeomorphic to $\mathbb{R}$ and hence we have shown that $\mathbb{R}$ is reconstructible.

3.2.3 Identifying potential reconstructions

We investigate the reconstruction of general MFC-spaces. By Theorem 3.2.1 it suffices to consider spaces without isolated points.

Theorem 3.2.7 is this section's main result and states that MFC-spaces X of degree 1 are reconstructible, provided that $\mathcal{R}_1(X)$ contains at least two points. Theorem 3.2.10 states under slightly stronger assumptions that also MFC-spaces of degree 2 are reconstructible.

As seen in our reconstruction result about the real line, it is useful to short-list potential reconstruction candidates of the space first and then check through them one-by-one. Theorem 3.2.4 provides such a shortlist for a variety of MFC-spaces and is the key towards all later results in this chapter. First we need some preliminary results.

Lemma 3.2.2. *Let X be a locally compact Hausdorff space. Then, for all $n \in \omega \cup \{\infty\}$,*

- *(a) for $x \neq y \in X$ we have $x \in \mathcal{R}_n(X)$ if and only if $x \in \mathcal{R}_n(X \setminus \{y\})$,*
- *(b) the cardinality $|\mathcal{R}_n(X)|$ is reconstructible,*
- *(c) if X has MFC-degree N and $x \in \mathcal{R}_n(X)$ then $X \setminus \{x\}$ has MFC-degree $N + n$,*
- *(d) if X is an MFC-space and $\mathcal{R}_n(X) \neq \emptyset$ for some $n \in \omega$ then the MFC-degree of X is reconstructible.*

Proof. Parts (a), (b) and (c) are contained in Lemmas 2.1.3, 2.1.9 and 2.1.7 respectively.

For (d), suppose X has a maximal N-point compactification. Let n be minimal such that $\mathcal{R}_n(X) \neq \emptyset$ and fix $x \in \mathcal{R}_n(X)$. By Theorem 3.2.1 we may assume that x is non-isolated, i.e. that $n \geq 1$. It follows from (c) that the card $X \setminus \{x\}$ has a maximal $(N + n)$-point compactification. Let Z be a reconstruction of X. By Theorem 1.2.3, the space Z is locally compact. Suppose γZ is a finite M-point compactification of Z. Find $z \in Z$ such that $Z \setminus \{z\} \cong Y$. It follows that γZ is a finite compactification of Y, and hence $M + \text{ends}(z, Z) \leq N + n$ by (c). And since $\mathcal{R}_m(X) = \emptyset = \mathcal{R}_m(Z)$ for all $m < n$ by (b) it follows that $M \leq N$.

Thus, we have proven that under the assumptions listed in (d), any reconstruction Z of X has a maximal finite M-point compactification with $M \leq N$. However, also Z satisfies the assumptions in (d), so the symmetric argument applied to the reconstruction X of Z gives $N \leq M$, and hence $N = M$. $\square$

We remark that Lemma 3.2.2(d) generalises the Reconstruction Result for Compact Spaces, Theorem 2.1.10. Indeed, for compact X with $\mathcal{R}_n(X) \neq \emptyset$ for some $n \in \omega$, every reconstruction will be an MFC-space of degree zero, i.e. a compact space, and hence such spaces are reconstructible by Theorem 1.2.7.

Let us see that corresponding results hold for $\mathcal{R}_*(X)$:

Lemma 3.2.3. *Let X be a locally compact Hausdorff space. Then,*

- *(a) for $x \neq y \in X$ we have $x \in \mathcal{R}_*(X)$ if and only if $x \in \mathcal{R}_*(X \setminus \{y\})$,*
- *(b) the cardinality $|\mathcal{R}_*(X)|$ is reconstructible,*
- *(c) if X has ASC-degree N and $x \in \mathcal{R}_*(X)$ then $X \setminus \{x\}$ has ASC-degree $N+1$,*
- *(d) if X is an ASC-space with $\mathcal{R}_*(X) \neq \emptyset$ then the ASC-degree of X is reconstructible.*

Proof. For (a), the proof of Lemma 3.1.15 applies, and (b) is a straightforward consequence of Theorem 1.2.4.

For (c), note that as a direct consequence of our definitions, $X \setminus \{x\}$ is dense and C^*-embedded in βX, and therefore has ASC-degree $|X^*| + 1$.

For (d), if X has isolated points it is reconstructible by Theorem 3.2.1. And if X does not have isolated points, then $|X^*| = \min\{|Y^*| \colon Y \in \mathcal{D}(X)\} - 1$. □

Theorem 3.2.4 (Reconstruction Candidates Theorem)**.** *Let X be a compact Hausdorff space without isolated points and $A = \{x_1, \ldots, x_N\} \subset \mathcal{R}_1(X)$ a finite, non-empty subset such that $\mathcal{R}_1(X) \setminus A$ is non-empty. Write $A^i = A \setminus \{x_i\}$. Then, for all $z \in \mathcal{R}_1(X) \setminus A$, every non-homeomorphic reconstruction of $X \setminus A$ is homeomorphic to one member of the candidate set*

$$\mathcal{C}_z = \left\{X \setminus \left(A^i \cup \{z\}\right) \colon 1 \leq i \leq N\right\}.$$

Proof. Let $A = \{x_1, \ldots, x_N\} \subset \mathcal{R}_1(X)$ and fix a point z in $\mathcal{R}_1(X) \setminus A$. By Lemma 3.2.2(c), the space $X \setminus A$ has a maximal N-point compactification, and its card $Y = X \setminus \{z, x_1, \ldots, x_N\}$ has X as maximal $N+1$-point compactification.

Let Z be a reconstruction of $X \setminus A$. By Lemma 3.2.2(d), the space Z has a maximal N-point compactification. In particular, Z is non-compact. Since Z is locally compact (Theorem 1.2.3) and has no isolated points, its 1-point compactification is a 2-point compactification of Y. This 2-point compactification is a quotient of the maximal finite compactification νY of Y, where the remainder $\nu Y \setminus Y$ is partitioned into two non-empty parts. Since $\nu Y = X$ by Theorem 3.1.4, there exists a non-empty subset $R \subset \{z, x_1, \ldots, x_N\}$ such that

$$Z = (Y \cup R)/R \subset X/R.$$

However, by Lemma 3.2.2(a), the space X/R is a compact Hausdorff space such that $\{z, x_1, \ldots, x_N\} \setminus R \subset \mathcal{R}_1(X/R)$. By an iterative application of Lemma 3.2.2(c) it follows that Z has MFC-degree $N + 1 - |R|$, yielding $|R| = 1$. We have shown that any non-trivial reconstruction of X is of the form $Y \cup \{x_i\}$ where $i \in \{1, \ldots, N\}$. Since this procedure can be repeated for every z in $\mathcal{R}_1(X)$, the theorem follows. □

Corollary 3.2.5. *For uncountable κ, the spaces $2^\kappa \setminus A$ and $I^\kappa \setminus A$ are reconstructible for finite A.*

Proof. It is well known that both the Cantor set 2^ω and the Hilbert cube I^ω are countably dense homogeneous, i.e. if $A, B \subset X$ are countable dense sets then there is an autohomeomorphism f of X such that $f(A) = B$. Since countable dense homogeneity implies n-point homogeneity in metric compact spaces [Ung78], both spaces are n-point homogeneous, i.e. for all finite $A, B \subset X$ with $|A| = |B| < \infty$ there is an autohomeomorphism f of X such that $f(A) = B$.

Thus, also the Cantor cube 2^κ and the Tychonoff cube I^κ are n-point homogeneous and hence the corollary follows from Theorem 3.1.16 and the Reconstruction Candidates Theorem 3.2.4. □

3.2.4 Reconstructing spaces with maximal 1- or 2-point compactifications

In this section we use the Reconstruction Candidates Theorem 3.2.4 to prove that under some mild additional assumptions, all MFC-spaces of degree 1 and 2 are reconstructible.

Lemma 3.2.6. *Let X be a compact Hausdorff space with $\{x, y\} \subset \mathcal{R}_1(X)$ and assume that $X \setminus \{x\} \ncong X \setminus \{y\}$. For every point $z \in \mathcal{R}_1(X) \setminus \{x, y\}$ we have either $X \setminus \{x, z\} \notin \mathcal{D}(X \setminus \{y\})$ or $X \setminus \{y, z\} \notin \mathcal{D}(X \setminus \{x\})$.*

Proof. Suppose the statement is false. Then there exists $z \in \mathcal{R}_1(X)$ such that $X \setminus \{x, z\}$ is a card of $X \setminus \{y\}$ and $X \setminus \{y, z\}$ is a card of $X \setminus \{x\}$. In other words, there are $z_x, z_y \in X$ such that

$$X \setminus \{x, z\} \cong X \setminus \{y, z_x\} \text{ and } X \setminus \{y, z\} \cong X \setminus \{x, z_y\}.$$

Using Lemma 3.2.2(c), these spaces are MFC-spaces of degree 2. By Lemma 3.1.18, there exist homeomorphisms f and g of X such that $f(\{x, z\}) = \{y, z_x\}$ and $g(\{y, z\}) = \{x, z_y\}$. It follows that either

$$f(x) = y \text{ or } g(y) = x \text{ or } (g^{-1} \circ f)(x) = y.$$

Thus, $X \setminus \{x\} \cong X \setminus \{y\}$ holds in all cases, a contradiction. □

Theorem 3.2.7. *Let X be a compact Hausdorff space with no isolated points such that $\{x, y, z\} \subset \mathcal{R}_1(X)$ where x, y, z are distinct. Then $X \setminus \{x\}$ is reconstructible.*

Proof. By the Reconstruction Candidates Theorem 3.2.4, every non-trivial reconstruction of $X \setminus \{x\}$ is homeomorphic to $X \setminus \{y\}$. However, if $X \setminus \{x\} \not\cong X \setminus \{y\}$ then these spaces have different decks by Lemma 3.2.6. Hence, X is reconstructible. □

Corollary 3.2.8. *Let X be a locally compact space with a maximal 1-point compactification. If $\mathcal{R}_1(X)$ contains at least two points then X is reconstructible.* □

For example, Euclidean spaces $\mathbb{R}^n$ are reconstructible for all $n \geq 2$. To generalise the result to MFC-spaces of degree two, we need the following extension of Lemma 3.2.6. Recall that a space X is *homogeneous* if for every pair of points x and y of X there exists a homeomorphism of X carrying x to y. In general, we say x and y lie in the same *orbit* of X if x can be mapped to y by a homeomorphism of X.

Lemma 3.2.9. *Let X be a compact Hausdorff space with $\{x, y, z\} \subset \mathcal{R}_1(X)$ such that $X \setminus \{x\} \not\cong X \setminus \{y\}$. If $X \setminus \{x, y\}$ and $X \setminus \{y, z\}$ are non-homeomorphic reconstructions of each other then $\{x, z\}$ is an orbit in X and all other points of $\mathcal{R}_1(X)$ are contained in the orbit of y.*

Proof. Since $X \setminus \{x, y\}$ and $X \setminus \{y, z\}$ are reconstructions of each other, for every a in $\mathcal{R}_1(X)$ there are points b and c in X such that

$$X \setminus \{x, y, a\} \cong X \setminus \{y, z, b\} \text{ and } X \setminus \{y, z, a\} \cong X \setminus \{x, y, c\}.$$

By Lemma 3.1.18, there exist homeomorphisms f and g of X such that

$$f(\{x, y, a\}) = \{y, z, b\} \text{ and } g(\{y, z, a\}) = \{x, y, c\}.$$

It is straightforward to check that the orbit of z coincides with one of the orbits of x or y. Since the problem is not symmetric in x, y and z, we have to deal with both cases separately.

Assume that z and y are contained in the same orbit. Because of $X\setminus\{x\} \not\cong X\setminus\{y\}$, neither y nor z can be mapped to x. It follows that $f(\{y,a\}) = \{y,z\}$ and $g(a) = x$, and hence $g \circ f^{-1}(\{y,z\})$ contains x, a contradiction.

Thus, we have to assume that z and x are contained in the same orbit. It follows that y cannot be mapped to either z or x, and further, a cannot be mapped to b or c because of the assumption $X \setminus \{x,y\} \not\cong X \setminus \{y,z\}$. Hence the following possibilities for the functions f and g arise.

$$\begin{aligned} &f_1\colon a \mapsto y,\ y \mapsto b,\ x \mapsto z, && g_1\colon a \mapsto x,\ z \mapsto c,\ y \mapsto y,\\ &f_2\colon a \mapsto z,\ x \mapsto b,\ y \mapsto y, && g_2\colon a \mapsto y,\ y \mapsto c,\ z \mapsto x. \end{aligned}$$

Two of the possible combinations yield a contradiction straightaway: $g_i \circ f_i^{-1}$ maps either y to x or z to y. Further, $g_1 \circ f_2^{-1}$ maps $\{y,z\}$ to $\{y,x\}$, a contradiction. Only the combination f_1 with g_2 is possible, and hence every $a \in \mathcal{R}_1(X) \setminus \{x,y,z\}$ lies in the same orbit as y. $\square$

For the next proof, note that the orbits of X form a partition of $\mathcal{R}_1(X)$.

Theorem 3.2.10. *Let X be a compact Hausdorff space with no isolated points such that $\{x,y,z,z',z''\} \subset \mathcal{R}_1(X)$ are distinct. If $\mathcal{R}_1(X)$ consists of more than one orbit, then $X \setminus \{x,y\}$ is reconstructible.*

Proof. Assume that x and y are contained in different orbits of X. By the Reconstruction Candidates Theorem 3.2.4, every reconstruction of $X \setminus \{x,y\}$ is homeomorphic to one of the three spaces

$$X \setminus \{x,y\},\ X \setminus \{x,z\} \text{ or } X \setminus \{y,z\}.$$

If $X \setminus \{y,z\}$ is a non-homeomorphic reconstruction of $X \setminus \{x,y\}$ then Lemma 3.2.9 yields that x and z form a two-point orbit and every other point in $\mathcal{R}_1(X)$ lies in the same orbit as y. In particular, the point z' lies in the same orbit as y and the orbit of y has cardinality at least three.

Applying Theorem 3.2.4 with z' in $\mathcal{R}_1(X)$ shows that either $X\setminus\{y,z'\}$ or $X\setminus\{x,z'\}$ is a non-homeomorphic reconstruction of $X \setminus \{x,y\}$. By Lemma 3.2.9, either x and z' or y and z' form a two-point orbit in X, a contradiction. Repeating the same argument for $X \setminus \{x,z\}$ completes the first part of the proof.

Now assume that x and y are contained in the same orbit. By assumption, we can find a point z in $\mathcal{R}_1(X)$ contained in a different orbit. Then both potential reconstructions $X \setminus \{x,z\}$ and $X \setminus \{y,z\}$ of $X \setminus \{x,y\}$ are reconstructible by the former argument. Hence, $X \setminus \{x,y\}$ is reconstructible. $\square$

Corollary 3.2.11. *Let X be a locally compact space with a maximal 2-point compactification. If $\mathcal{R}_1(X)$ contains at least three points which do not all lie in the same orbit of X, then X is reconstructible.* □

3.3 Ordinal numbers are reconstructible

3.3.1 Overview By Theorem 1.2.6, every compact space that contains isolated points is reconstructible. Therefore, every successor ordinal is reconstructible. By Theorem 3.2.1, every limit ordinal of uncountable cofinality is reconstructible, as these are precisely the ordinals with a one-point Stone-Čech remainder.

In this section we complement these results and settle the case for limit ordinals with countable cofinality. We present an approach showing that all ordinals are reconstructible, regardless of their cofinality.

3.3.2 Linearly ordered topological spaces and ordinals Let $(X, \leq)$ be a linearly ordered space. The *order topology* on $(X, \leq)$ is given by the subbasis of sets of the form

$$(-\infty, x) = \{y \in X \colon y < x\} \quad \text{and} \quad (x, \infty) = \{y \in X \colon x < y\}.$$

Such spaces are called *linearly ordered topological spaces*, or LOTS. Every LOTS is a (hereditarily) normal space [Eng89, 1.7.4]. An *ordinal* is a linearly ordered space $(\alpha, \leq)$ such that α is well-ordered by $\leq$, i.e. every non-empty subset of α has a least element. An ordinal is a *successor* if it has a maximal element; otherwise it is called a *limit ordinal.* One can show that every ordinal α in its ordinal topology is locally compact, has a dense set of isolated points, and is compact if and only if α is a successor. The *cofinality* of an ordinal α is the least cardinality of a subset $X \subset \alpha$ with the property that for all $\beta \in \alpha$ there is $x \in X$ such that $\beta < x$. For further information on ordinals see for example [Kun80].

Most surprisingly, the concept of a LOTS is in fact a purely topological concept—see Wattel and van Dalen [vDW73] for a topological characterisation of LOTS. Also ordinal spaces can be characterised in purely topological terms, see [GP12].

If $(X, \leq)$ and $(Y, \leq)$ are LOTS, then the *ordered sum* $X + Y$ is given by the set $\{0\} \times X \cup \{1\} \times Y$ with the lexicographic order. Similarly, the product $X \cdot Y$ is $Y \times X$ with the lexicographical order. For later use, we note the following lemma.

Lemma 3.3.1. *Suppose $(X, \leq)$ and $(Y, \leq)$ are LOTS such that X has a right and Y a left endpoint. Then $X \oplus Y = X + Y$.* □

Recall that for ordinals, we can, besides addition, also (recursively) define multiplication and exponentiation [Kun80]. Using these notions we can state the most important structural theorem about ordinals.

Theorem 3.3.2 (Cantor Normal Form). *Each ordinal $\alpha > 0$ can be represented uniquely in the form*

$$\alpha = \omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\alpha_k} \cdot n_k,$$

with $k, n_1, \ldots, n_k$ positive natural numbers and $\alpha \geq \alpha_1 > \cdots > \alpha_k$. □

If we are only interested in the topological nature of ordinals, the Cantor Normal Form Theorem can be strengthened.

Theorem 3.3.3 ([KL06]). *Every infinite ordinal $\alpha = \omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\alpha_k} \cdot n_k$ is homeomorphic to $\omega^{\alpha_1} \cdot n_1 + \omega^{\alpha_k}$.* □

As our final observation in this introductory section we show that our earlier reconstruction results about MFC-spaces are not applicable to limit ordinals with countable cofinality.

Lemma 3.3.4. *The Stone-Čech remainder of an ordinal of countable cofinality contains a copy of ω^* and hence has cardinality at least $2^{\mathfrak{c}}$.*

Proof. Let δ be a limit ordinal of countable cofinality. Every strictly increasing cofinal sequence in δ is a closed, and hence by Tietze's Theorem a C^*-embedded copy of ω. The closure of this sequence in $\beta\delta$ will be a copy of $\beta\omega$, so the remainder of δ will contain a copy of ω^* (which has cardinality $2^{\mathfrak{c}}$). □

3.3.3 Scattered spaces A space X is called *scattered* if every non-empty subspace $A \subset X$ contains an isolated point (in the subspace topology). Every ordinal number is scattered.

Lemma 3.3.5. *The property of being scattered is reconstructible. More precisely, a space X with $|X| \geq 3$ is scattered if and only if all $Y \in \mathcal{D}(X)$ are scattered.*

Proof. For necessity let $Y \in \mathcal{D}(X)$. If $A \subset Y$, then clearly $A \subset X$, so has an isolated point. Hence, Y is scattered.

Conversely, let $A \subset X$. If $A \neq X$ then $A \subset Y$ for some card Y, so has an isolated point. This leaves us with the case $A = X$. If X is a T_1-space, any isolated point of $A \setminus \{x\}$ will also be isolated in X, and hence we are done. Being scattered, however, is reconstructible in all spaces. Suppose that no point $x \in X$ is isolated. By assumption, the card $Y = X \setminus \{y\}$ has an isolated point x. Thus, $\{x, y\}$ is an open set. Now find $z \in X \setminus \{x, y\}$. Then $\{x, y\} \subset X \setminus \{z\}$ has an isolated point, say x. Hence $\{x, z\}$ is open in X, and therefore $\{x\} = \{x, y\} \cap \{x, z\}$ is isolated in X. $\square$

The most important tool for scattered spaces is the notion of the Cantor-Bendixson rank. For a topological space X we denote by X' its *derived set*, the set of all non-isolated points of X. By recursion one defines the *α-th Cantor-Bendixson derivative* by $X^{(0)} = X$, $X^{(\alpha+1)} = (X^{(\alpha)})'$ for successors and $X^{(\lambda)} = \bigcap_{\alpha<\lambda} X^{(\alpha)}$ for limit ordinals λ, see [Kech95, 6.10].

The smallest α such that $X^{(\alpha)} = X^{(\alpha+1)}$ is called the *Cantor-Bendixson rank* of X. We denote it by $CB(X)$. If X is scattered then $X^{CB(X)} = \emptyset$. In particular, every point x of X will lie in some $X^{(\alpha)} \setminus X^{(\alpha+1)}$ for exactly one α. This α is the *scattered rank* of x in X and we denote it by $CB_X(x)$.

Lemma 3.3.6. *In a scattered Hausdorff space, the cardinality of points with a given scattered rank is reconstructible. In particular, the Cantor-Bendixson rank of a scattered space is reconstructible.*

Proof. Let X be a scattered space. Since the lemma is trivial for discrete spaces, let us assume that $CB(X)$ is at least 2. In particular, X has infinitely many isolated points. Let X be a scattered space and $Y = X \setminus \{x\}$ be a card of X. It is clear that $|Y^\alpha \setminus Y^{\alpha+1}| \leq |X^\alpha \setminus X^{\alpha+1}|$. However, if x is isolated in X then $|Y^\alpha \setminus Y^{\alpha+1}| = |X^\alpha \setminus X^{\alpha+1}|$. Thus,

$$\left|X^\alpha \setminus X^{\alpha+1}\right| = \max\left\{\left|Y^\alpha \setminus Y^{\alpha+1}\right| \colon Y \in \mathcal{D}(X)\right\}$$

is reconstructible, and so is $CB(X) = \max\{CB(Y) \colon Y \in \mathcal{D}(X)\}$. $\square$

3.3.4 Spaces with $X \in \mathcal{D}(X)$

The next two subsections contain a brief investigation of spaces occurring as their own cards. One of the results in this subsection is that if δ is an infinite ordinal then $\delta \in \mathcal{D}(\delta)$. Although this is not yet quite enough to reconstruct ordinals, it will allow us to rule out spaces as potential reconstructions that have no ordinals amongst their cards. The situation when $X \in \mathcal{D}(X)$ is interesting in its own right, so we spend some more time on it.

Lemma 3.3.7. *The following conditions guarantee that $X \in \mathcal{D}(X)$:*

(a) X contains a clopen copy of ω, or

(b) X is a Hausdorff space containing a converging sequence of isolated points, or more generally,

(c) X is a Hausdorff space containing an open copy of an infinite discrete subspace E such that $\overline{E}$ is homeomorphic to its 1-point compactification αE.

Proof. Part (a) is clear and (b) follows from (c). For (c), let D be an infinite compact subspace of X such that all but one point are isolated points of X. Let x denote this non-isolated point, and let y be an isolated point of D. Since $D \setminus \{x\}$ and $D \setminus \{x, y\}$ are homeomorphic, so will be their respective one-point compactifications D and $D \setminus \{y\}$. Denote this homeomorphism by f. The map $g \colon X \to X \setminus \{y\}$ defined by

$$g = f \cup \{\langle z, z\rangle \colon z \in X \setminus D\}$$

is built from the homeomorphisms f defined on the closed subset D and the identity defined on the closed subset $X \setminus (D \setminus \{x\})$. By the Pasting Lemma [Wil04, 7.6], the map g itself is a homeomorphism. □

Lemma 3.3.8. *If X is an infinite regular scattered space then $X \in \mathcal{D}(X)$.*

Proof. The result is trivial for discrete X. And for a point $x \in X^{(1)} \setminus X^{(2)}$ by regularity there is a clopen neighbourhood C of x which, apart from x itself only contains isolated points of X. It is straightforward to see that either C is compact or contains a clopen copy of ω. So either Lemma 3.3.7(c) or (a) applies. □

Corollary 3.3.9. *Let δ be an infinite ordinal. Then $\delta \in \mathcal{D}(\delta)$.* □

Note that the example of $C \setminus \{0\}$ for the Cantor set C shows that $\mathcal{D}(X) = \{X\}$ is possible even without isolated points. Similar examples are given by the rationals and the irrationals. Indeed, these examples are all special cases of a theorem due to van Douwen, and its natural generalisation, which A. Medini informed me of:

Theorem 3.3.10 ([vD84, Appendix 1])**.** *Assume that X is an infinite homogeneous, zero-dimensional, separable metrizable space. Then $X \setminus \{x\} \cong X$ for all $x \in X$.* □

Theorem 3.3.11 (Medini, personal communication)**.** *Assume that X is a non-pseudocompact, homogeneous, zero-dimensional, first-countable Hausdorff space. Then $X \setminus \{x\} \cong X$ for all $x \in X$.*

Proof. First observe that since X is non-pseudocompact and zero-dimensional, we can write $X = \bigoplus_{n \in \omega} X_n$ where $X_n \subset X$ are disjoint non-empty clopen sets [Med11, Lem. 3]. Now let $x \in X_0$. By homogeneity, it suffices to show that $X \setminus \{x\} \cong X$.

By zero-dimensionality and homogeneity, we can find collections of non-empty clopen sets $\{C_n \colon n \in \omega\}$ and $\{D_n \colon n \in \omega\}$ such that

- $X_0 \supset C_0 \supset C_1 \supset \cdots$,
- $\{C_n \colon n \in \omega\}$ forms a neighbourhood base at x,
- $D_n \subset X_n$ for all $n \in \omega$,
- $C_n \cong D_n$ for all $n \in \omega$.

Since $C_n \cong C_n \setminus C_{n+1} \oplus C_{n+1}$, we may write $D_n = A_n \oplus B_n$ and fix homeomorphisms $f_n \colon A_n \to C_n \setminus C_{n+1}$ and $g_n \colon B_n \to D_{n+1}$ for all $n \in \omega$.

Finally, it is easy to check that $h \colon X \to X \setminus \{x\}$,

$$h(x) = \begin{cases} f_n(x) & \text{if } x \in A_n, \\ g_n(x) & \text{if } x \in B_n, \text{ and} \\ x & \text{if } x \notin \bigcup_{n \in \omega} D_n, \end{cases}$$

is a homeomorphism as desired. □

3.3.5 Spaces with $X \notin \mathcal{D}(X)$ It is clear that a compact space without isolated points cannot occur as its own card. However, the results from the previous section raise the question whether all spaces containing infinitely many isolated points occur as cards of their own deck. It is easy to verify that Lemma 3.3.7 implies that every compact metric space that contains infinitely many isolated points occurs as a card of itself. However, in this section we will see that the answer is negative in the general case. To build such a space we start with two lemmas.

Lemma 3.3.12. *For every injective function $f \colon \omega \to \omega$ that moves infinitely many points there are disjoint copies X_f and Y_f of ω^* contained in ω^* such that βf is a bijection between X_f and Y_f.*

Proof. We can find disjoint subsets $\{x_n \colon n \in \omega\}$ and $\{y_n \colon n \in \omega\}$ of ω such that $f(x_n) = y_n$ for all natural numbers n. Then

$$X_f = \omega^* \cap \overline{\{x_n \colon n \in \omega\}}^{\beta\omega} \text{ and } Y_f = \omega^* \cap \overline{\{y_n \colon n \in \omega\}}^{\beta\omega}$$

are as required. □

Lemma 3.3.13. *Any bijection between ω and $\omega \setminus \{0\}$ moves infinitely many points.*

Proof. Let f be such a bijection. None of the points in $\{f^n(0) \colon n \in \mathbb{N}\}$ is a fixed point, as can be seen by induction: certainly, $f(0) \neq 0$, and if $f^{n+1}(0) \neq f^n(n)$ then by injectivity $f(f^{n+1}(0)) \neq f(f^n(0))$. Moreover, the set is infinite. For a contradiction, assume that $f^n(0) = f^m(0)$ for some $n > m$. As f is invertible, we would have $f^{n-m}(0) = 0$, a contradiction. □

Theorem 3.3.14. *There exists a subspace X of $\beta\omega$ of cardinality $\mathfrak{c}$ with $\omega \subset X \subset \beta\omega$ such that $X \notin \mathcal{D}(X)$.*

Proof. Let $\Psi = \{\psi_\alpha \colon \alpha < \mathfrak{c}\}$ be the set of bijections between ω and $\omega \setminus \{0\}$ and let $\Xi = \{\xi_\alpha \colon \alpha < \mathfrak{c}\}$ be the set of permutations of ω that move infinitely many points. Now enumerate $\Phi = \Psi \cup \Xi = \{\phi_\alpha \colon \alpha < \mathfrak{c}\}$ and observe that Lemma 3.3.12 applies to all functions ϕ in Φ.

For all $\alpha < \mathfrak{c}$ we define by transfinite recursion pairs $\langle x_\alpha, y_\alpha \rangle$ of different points of ω^* with $\beta\phi_\alpha(x_\alpha) = y_\alpha$ such that

$$\begin{aligned} &(\star) \qquad x_\alpha \notin \{y_\beta \colon \beta < \alpha\} \quad \text{and} \\ &(\dagger) \qquad y_\alpha \notin \{x_\beta \colon \beta < \alpha\}. \end{aligned}$$

Assume $\alpha < \mathfrak{c}$ and (x_β, y_β) have been defined as required for all $\beta < \alpha$ and let

$$F = \{y_\beta \colon \beta < \alpha\} \cup \left\{\beta\phi_\alpha^{-1}(x_\beta) \colon \beta < \alpha\right\}$$

be the set of forbidden points. By Lemma 3.3.12 there are disjoint sets X_{ϕ_α} and Y_{ϕ_α} of cardinality $2^\mathfrak{c}$ such that $\beta\phi_\alpha$ is a bijection between them. Since $|F| < \mathfrak{c}$ we may pick any $x_\alpha \in X_{\phi_\alpha} \setminus F$ and let $y_\alpha = \phi_\alpha(x_\alpha)$. We have found a pair $\langle x_\alpha, y_\alpha \rangle$ as required.

We claim that the space $X = \omega \cup \{x_\alpha \colon \alpha < \mathfrak{c}\} \subset \beta\omega$ satisfies the requirements of the theorem. Observe that by construction, for all $\alpha < \mathfrak{c}$ we have $\phi_\alpha(x_\alpha) \notin X$. This follows as $\phi_\alpha(x_\alpha) = y_\alpha$ is not added at a later stage than α by $(\star)$, and was not added in an earlier step because of $(\dagger)$.

We claim that $X \notin \mathcal{D}(X)$. Pick a card $Y \in \mathcal{D}(X)$. If Y has been obtained by deleting a point $n \in \omega$, then $Y \cong X \setminus \{0\}$. But X is not homeomorphic to $X \setminus \{0\}$, since if there existed a homeomorphism $f \colon X \to X \setminus \{0\}$, then $f|_\omega = \phi_\alpha \colon \omega \to \omega \setminus \{0\}$ for some α, and hence $f(x_\alpha) \notin X$. Similarly, X is not homeomorphic to $X \setminus \{x_\beta\}$, since any homeomorphism g between these spaces restricts to a bijection between their isolated points. Therefore $g|_\omega = \phi_\alpha$ for some α, and hence $g(x_\alpha) \notin X$. $\square$

Another interesting example of such a space is referenced in [Wat92, 3.1.54]. It seems to be originally due to Hanf and Jónsson in 1957, but has surfaced independently several times in the subsequent decades (see [Wat92, 3.1.54] for further details). This example has the additional advantage of being compact.

Theorem 3.3.15 ([Wat92, 3.1.54])**.** *Let X be the quotient space obtained from $\beta\omega \times \{0,1\}$ by identifying $(x,0) \sim (x,1)$ for all $x \in \omega^*$. Then $X \notin \mathcal{D}(X)$.* $\square$

A short proof of this theorem can also be found in [Scott80].

3.3.6 Reconstruction within the class of ordinals We first prove that non-homeomorphic ordinals cannot be reconstructions of each other.

For a locally compact scattered space X, we form its one-point compactification $\alpha(X)$, which is also scattered. In particular, the point $\infty \in \alpha(X)$ has a well-defined Cantor-Bendixson rank, which we will denote by $CB_X(\infty)$.

Lemma 3.3.16. *Limit ordinals* $\delta \cong \omega^{\delta_1} \cdot n_1 + \omega^{\delta_2}$ *have* $CB_\delta(\infty) = \delta_2$.

Proof. Without loss of generality we may assume that $\delta = \omega^\beta$. We prove simultaneously by induction on β that $CB_{\delta+1}(\delta) \geq \beta$ and that $CB_{\delta+1}(\delta) \leq \beta$.

The statement certainly holds for $\beta = 1$. If the statement holds for β, then, since $\delta = \omega^{\beta+1} = \omega^\beta \cdot \omega$, it follows that the point $\delta \in \delta + 1$ is a limit point of the set $\left\{\omega^\beta \cdot n \colon n \in \omega\right\}$, all points of which are of rank β. Thus $CB_{\delta+1}(\delta) \geq \beta + 1$. At the same time, no point in $\omega^{\beta+1} = \omega^\beta \cdot \omega$ has rank greater than β by induction assumption and therefore $CB_{\delta+1}(\delta) \leq \beta + 1$.

Now, suppose that β is a limit. From $\omega^\beta = \sup\{\omega^\gamma \colon \gamma < \beta\}$ it follows readily that $CB_{\delta+1}(\delta) \geq \beta$ and $CB_{\delta+1}(\delta) \leq \beta$. □

Lemma 3.3.17. *Let* $Y = X \setminus \{x\}$ *be a card of a non-compact, locally compact, scattered space* X. *Then* $CB_Y(\infty) = \max\{CB_X(x), CB_X(\infty)\}$.

Proof. We denote by ∞_X the point we adjoined to X, and by ∞_Y the point we adjoined to Y. The quotient $\alpha(X)/\{x, \infty_X\}$ forms a one-point compactification of Y in which ∞_Y corresponds to $\{x, \infty_X\}$. We observe that the element $\{x, \infty_X\}$ is isolated in $(\alpha(X)/\{x, \infty_X\})^{(\alpha)}$ if and only if both x and ∞_X are isolated (or no longer contained) in $\alpha(X)^{(\alpha)}$.

Translating back to $\alpha(Y)$ we see that ∞_Y has rank equal to the maximum of $CB_X(x)$ and $CB_X(\infty_X)$. □

Corollary 3.3.18. *For locally compact scattered spaces* X *the value of* $CB_X(\infty)$ *is reconstructible.*

Proof. Since X contains isolated points, it follows from Lemma 3.3.17 that $CB_X(\infty) = \min\{CB_Y(\infty) \colon Y \in \mathcal{D}(X)\}$ can be recovered from the deck of X. □

Theorem 3.3.19. *If two ordinal numbers are reconstructions of each other, they are homeomorphic.*

Proof. Let α and β be two ordinals and suppose that they are reconstructions of each other. By Lemma 3.3.3 we have

$$\alpha \cong \omega^{\alpha_1} \cdot n_1 + \omega^{\alpha_2} \quad \text{and} \quad \beta \cong \omega^{\beta_1} \cdot k_1 + \omega^{\beta_2}.$$

Now by Lemma 3.3.16, $CB_\alpha(\infty) = \alpha_2$ and $CB_\beta(\infty) = \beta_2$, and hence Corollary 3.3.18 yields $\alpha_2 = \beta_2$. Applying Lemma 3.3.6, we also conclude that $\alpha_1 = CB(\alpha) = CB(\beta) = \beta_1$ as well as $n_1 = k_1$. This shows that α and β are homeomorphic. □

Observe that it easily follows from the above result that all countable ordinals are reconstructible. Indeed, every reconstruction of a countable ordinal is again locally compact and countable by the results in Section 1.2, and hence homeomorphic to an ordinal by a classic result due to Mazurkiewicz and Sierpiński.

3.3.7 Reconstruction of ordinal numbers In this section we complete the proof that all ordinal numbers are reconstructible. Using Theorem 3.3.19, it suffices to show that every reconstruction of an ordinal number is again an ordinal.

Lemma 3.3.20. *Let α be a limit ordinal and $X \subset \alpha$ a closed subset. Then X is homeomorphic to an ordinal. If X is also unbounded (non-compact) then $cf(\alpha) = cf(X)$.*

Proof. The closed subset X of α is clearly well-ordered. We prove that the order topology on X coincides with its subspace topology. It is well known that the subspace topology is finer than the order topology of X [Eng89, 2.7.5]. So it suffices to show that for every $(a, b) \subset \alpha$ and $x \in (a, b) \cap X$ there are $c, d \in X$ such that $x \in (c, d)_X = \{y \in X : c < y < d\} \subset (a, b)$.

For this, let $d = \min\{y \in X : y > x\}$ or put $d = \infty$ if x is the maximum of X. For c, we can pick any element in $(a, x) \cap X$. However, if this set is empty, then we may put $c = \max\{y \in X : y < x\}$. This maximum exists, as X is closed and therefore contains its suprema. Moreover, $c \leq a < x$, so indeed $x \in (c, d)$.

For the second part, observe that if $f : \gamma \to X$ is cofinal in X then it is cofinal in α, i.e. $cf(\alpha) \leq cf(X)$. Conversely, if f is cofinal in α, then by defining $g(\delta) = \min\{\gamma \in X : \gamma > f(\delta)\}$ we see that g is cofinal in X and hence $cf(\alpha) \geq cf(X)$. □

The following theorem characterises when a topological sum of topological spaces is homeomorphic to an ordinal. For example, the space $\omega_1 \oplus \omega$ is not homeomorphic to an ordinal. For the theorem, recall that "being homeomorphic to an ordinal" is indeed a topological notion, see [GP12, Thm. 20].

Theorem 3.3.21. *Let $X = \bigoplus_{\alpha<\kappa} X_\alpha$ be a sum of topological spaces X_α. Then X is homeomorphic to an ordinal if and only if every X_α is homeomorphic to an ordinal and either*

- *X_1 has uncountable cofinality, all other X_α have finite cofinality (are compact) and κ is finite, or*
- *all X_α have at most countable cofinality and κ is at most countable.*

Proof. For sufficiency, in the first case we may order $X = \bigoplus_{i<n} X_i = X_2 + X_3 + \cdots + X_n + X_1$ by Lemma 3.3.1. For the second case, let $f\colon \omega \to \omega \times \kappa,\ i \mapsto (m_i, n_i)$ be a bijection. By assumption there exist for all $n < \kappa$ increasing cofinal sequences $\{a^n_m\colon m \in \omega\} \subset X_n$. For sake of notation we let $a^n_0 = -\infty$. By Lemma 3.3.20, the clopen subsets $(a^n_m, a^n_{m+1}]$ of X_n are homeomorphic to a compact ordinal with endpoints. It follows from Lemma 3.3.1 that

$$X = \bigoplus_{n<\kappa} X_n = \bigoplus_{n<\kappa} \bigoplus_{m<\omega} (a^n_m, a^n_{m+1}] \cong \sum_{i<\omega} (a^{n_i}_{m_i}, a^{n_i}_{m_i+1}]$$

is homeomorphic to an ordinal.

For necessity we argue by contradiction. Suppose X is an ordinal. It follows from Lemma 3.3.20 that every closed X_α is also an ordinal. If X_1 has uncountable cofinality, then so has X by Lemma 3.3.20. Thus, another non-compact X_α together with X_1 would be non-intersecting closed unbounded sets ("clubs") in X, a contradiction (see e.g. [Eng89, 3.1.21]). Thus, all other X_α must be compact. Next, suppose that κ is infinite. Choose isolated $x_\alpha \in X_\alpha$ and consider $\{x_n\colon n < w\}$. This set is closed in X as the clopen partition $\{X_\alpha\colon \alpha < \kappa\}$ witnesses local finiteness. But then the ordinal X contains a (non-compact) clopen copy of ω, i.e. has countable cofinality, a contradiction.

Now suppose that all ordinals X_α have at most countable cofinality. We have to show that κ is at most countable. Suppose it was uncountable. Again, choose isolated points x_α in X_α. By the same reasoning as above, $D = \{x_\alpha\colon \alpha < \kappa\}$ is closed in X. But Lemma 3.3.20 implies that D is an ordinal. However, D is an uncountable discrete space, a contradiction. □

Theorem 3.3.22. *The topological property of "being homeomorphic to an ordinal" is reconstructible.*

Proof. Let α be an ordinal and suppose for a contradiction that Z is a non-homeomorphic reconstruction of α which is not homeomorphic to an ordinal. Since compact spaces with isolated points are reconstructible, we may assume that α is a limit ordinal. By Theorem 1.2.1 and Lemma 3.3.5, Z is a regular scattered space, and therefore Lemma 3.3.8 implies that Z is an element of $\mathcal{D}(Z) = \mathcal{D}(\alpha)$. In other words, $Z \cong \alpha \setminus \{\beta\} \cong \beta \oplus \gamma$ for suitable ordinals β and γ. Such a representation is not unique, but the "right hand-side" γ inherits the cofinality from α by Lemma 3.3.20. Further, it is clear that $\beta \in \alpha$ has to be non-isolated, i.e. that β is a limit ordinal.

Case 1. *The ordinals α and γ have countable cofinality.*

If β also has countable cofinality then $Z = \beta \oplus \gamma$ is homeomorphic to an ordinal by Theorem 3.3.21, contradicting our assumption on Z. So suppose that β has uncountable cofinality. We claim that in this case, Z has no ordinals amongst its cards, contradicting Corollary 3.3.9 asserting that $\alpha \in \mathcal{D}(\alpha) = \mathcal{D}(Z)$.

Let $Y = Z \setminus \{\delta\}$ be a card of $Z = \beta \oplus \gamma$. If $\delta \in \beta$ then $\beta \setminus \{\delta\} = \delta \oplus \eta$ and η has uncountable cofinality. If $\delta \in \gamma$ then $\gamma \setminus \{\delta\} = \delta \oplus \theta$ and θ has countable cofinality. In any case, Y is not homeomorphic to an ordinal by Theorem 3.3.21.

Case 2. *The ordinals α and γ have uncountable cofinality.*

Since $\alpha \in \mathcal{D}(\alpha)$ by Corollary 3.3.9 and $\mathcal{D}(\alpha) = \mathcal{D}(Z) = \mathcal{D}(\beta \oplus \gamma)$, we have $\alpha \cong \mu \oplus \nu \oplus \xi$ or $\alpha \cong \mu \oplus \nu$ for some suitable limit ordinals μ, ν and ξ. However, the fact that one of them is of uncountable cofinality contradicts Theorem 3.3.21.

Thus, we are led to a contradiction in both cases. It follows that Z must be an ordinal. ☐

Theorem 3.3.23. *All ordinal numbers are reconstructible.*

Proof. By Theorems 3.3.19 and 3.3.22. ☐

3.4 Open questions

We list open questions which are raised by results in this section. In Theorem 3.2.7, a technical condition was used, requiring that the MFC-spaces under consideration contain at least two points in $\mathcal{R}_1(X)$. Even if this condition does not interfere with the reconstruction of the most interesting MFC-spaces, we would like to dispose of this technical condition.

Question 3.4.1. *Are all ASC-spaces X with $\mathcal{R}_*(X) \neq \emptyset$ reconstructible?*

Question 3.4.2. *Are all MFC-spaces X of MFC-degree 1 with $\mathcal{R}_1(X) \neq \emptyset$ reconstructible? If yes, are MFC-spaces X of MFC-degree 1 with $\bigcup_{n\in\omega} \mathcal{R}_n(X) \neq \emptyset$ reconstructible?*

The other technical issue we encountered was a homogeneity problem when generalising the method of Theorem 3.2.7 to MFC-spaces of larger degree. Attempts to improve Theorem 3.2.10 in that aspect have not yet been successful. Is it possible to work around this problem?

In the case of the real line, we have successfully demonstrated that techniques from the theory of maximal finite compactifications furnish reconstruction results also for non-compact spaces. Thus, an affirmative answer to Question 3.4.2 is not out of reach.

Also, a positive answer would connect back to reconstruction results of infinite graphs. There is a result about infinite graphs [NW87, NW91] showing that connected, locally finite graphs with at least two but a finite number of ends are reconstructible. Topologising such a graph by replacing edges with unit intervals gives a non-compact locally compact space with a maximal finite compactification, where every end will limit onto one point at infinity. It would be exciting if the theory of maximal finite compactifications could provide a proof that these infinite graphs are reconstructible in the topological sense.

Moving from MFC-spaces to our reconstruction results for ordinals, we raise the following questions. In the light of Lemma 3.3.4 we would like to know:

Question 3.4.3. *What is the cardinality of α^* as a function of α? A little easier: what is the cardinality of the remainder of ω_ω?*

It is not hard to see that $2^{2^{(\omega_\omega)}} \geq |\omega_\omega^*| \geq \max\{2^{\mathfrak{c}}, \omega_\omega\}$, and that consistently, it is the left hand-side (if $\mathfrak{c} = 2^{\omega_\omega}$).

Question 3.4.4. *Are regular scattered spaces reconstructible? If not, are regular scattered spaces of Cantor-Bendixson rank one reconstructible?*

An important example in the literature of a regular scattered space of Cantor-Bendixson rank one is the Mrowka- or Ψ-space [Eng89, 3.6.I]. We can show that such a space is reconstructible. Indeed, $\Psi_{\mathcal{A}}$ for a MAD family $\mathcal{A}$ is pseudocompact and contains isolated points. It follows that every reconstruction Z of $\Psi_{\mathcal{A}}$ is also pseudocompact. Moreover, it follows from Lemma 3.3.7(b) that $Z \in \mathcal{D}(\Psi_{\mathcal{A}})$. But the only pseudocompact cards of $\Psi_{\mathcal{A}}$ are the ones obtained by deleting an isolated point. It follows that Z is homeomorphic to $\Psi_{\mathcal{A}}$.

Question 3.4.5. *Can the method in Theorem 3.3.14 be improved to yield a machine for building non-reconstructible spaces?*

Chapter 4

Reconstructing normality

The purpose of this chapter is to describe two surprising results about topological reconstruction. Our first result shows that it is undecidable in ZFC whether ω^*, the Stone-Čech remainder of the integers, is reconstructible. Our second result is concerned with the question whether *normality*, one of the fundamental topological separation properties, is reconstructible. We show that ω^* provides a natural example establishing that the answer is consistently negative.

The result about reconstruction of normality is somewhat curious, especially so because all other topological separation axioms are reconstructible for spaces containing at least three points (Theorem 1.2.1). But for normality, the best we could do was to show that normality of a space is reconstructible provided the space has at least one normal card. On the positive side, this last result applies to every normal space containing a G_δ-point, since normality is hereditary with respect to F_σ subspaces [Eng89, 2.1.E]. In particular, normality is reconstructible in the realm of first-countable spaces.

However, for an uncountable cardinal κ, the Cantor cube 2^κ provides an example of a compact Hausdorff space where all cards are non-normal. Further examples of normal spaces with exclusively non-normal cards are given by

$$\Sigma(2^\kappa) = \left\{f \in 2^\kappa \colon \left|f^{-1}(1)\right| \leq \aleph_0\right\} \text{ and } \sigma(2^\kappa) = \left\{f \in 2^\kappa \colon \left|f^{-1}(1)\right| < \aleph_0\right\}$$

for uncountable κ, the subspaces of 2^κ of all sequences with countable and finite support, respectively. The space $\Sigma(2^\kappa)$ is normal, countably paracompact but not paracompact [Cor59, Thm. 1, Cor. 1 and Thm. 4]. The space $\sigma(2^\kappa)$ is σ-compact, and hence paracompact [Eng89, 5.1.2] and normal. However, since $\{\mathbf{0}\} \cup \left\{\mathbb{1}_{\{\alpha\}} \colon \alpha < \kappa\right\}$ is homeomorphic to $\alpha(\kappa)$, the 1-point compactification of the discrete space of cardinality κ, it follows that all three spaces above contain a compact, and hence closed copy of $\alpha(\aleph_1) \times \alpha(\aleph_0)$. Since $\alpha(\aleph_1) \times \alpha(\aleph_0)$ has a non-normal card, it follows from homogeneity that the deck of any of the normal spaces 2^κ, $\Sigma(2^\kappa)$ and $\sigma(2^\kappa)$ consists of a single, non-normal card.

Given that the Cantor set $C \cong 2^\omega$ is non-reconstructible, one might suspect that these large Cantor cubes 2^κ for uncountable κ are examples witnessing that normality is non-reconstructible. However, 2^κ is reconstructible if κ is uncountable [PS**, 2.1]. In fact, for uncountable κ, the space 2^κ is the Stone-Čech compactification of $\Sigma(2^\kappa)$, and hence is reconstructible by Theorem 1.2.8. It remains open whether $\Sigma(2^\kappa)$ and $\sigma(2^\kappa)$ are also reconstructible, but it seems likely.

This is where the Stone-Čech remainder of the integers ω^* enters the stage. It is a well-known open problem whether all cards of ω^* are non-normal (cf. [HvM90, Q13] and [Szy12]). Under the Continuum Hypothesis (CH), however, it is a classical theorem independently due to Rajagopalan [Raj72] and Warren [War72] that the answer is yes. Thus, under CH, the space ω^* is a further candidate to witness that normality might be non-reconstructible.

And indeed, we show below that ω^* provides a consistent example that normality is non-reconstructible. In our main result we prove that assuming CH, the space $\omega^* \setminus \{p\}$ for a P-point p of ω^* (under CH, this is a point p with a nested neighbourhood base) is a non-normal reconstruction of the normal space ω^*. Thus, CH implies that normality is non-reconstructible. We also show that CH (which is equivalent to $\omega_1 = \omega_1^{<\omega_1}$) can be weakened to the assumption that there is an uncountable cardinal κ with $\kappa = \kappa^{<\kappa}$. We will see that under this cardinal assumption, the spaces S_κ defined by Negrepontis in [Neg69] provide further examples that normality is non-reconstructible. This ties in well with the previous non-reconstruction results, as the spaces S_κ can be thought of as simultaneously generalising the behaviour of the Cantor set and of ω^* under CH.

This chapter is organised as follows. In Sections 4.1 and 4.2, we recall the relevant definitions and facts about the spaces ω^* and S_κ. In Sections 4.3 and 4.4 we turn to a closer investigation of cards of ω^* and S_κ. Our main technical result states that under CH, all finite compactifications of $\omega^* \setminus \{x\}$ are homeomorphic to ω^* and, moreover, that all but at most one point added at infinity will be P-points. Similarly, under $\kappa = \kappa^{<\kappa}$, all finite compactifications of $S_\kappa \setminus \{x\}$ are homeomorphic to S_κ and, moreover, that all but at most one point added at infinity will be P_κ-points. In particular, we have a new proof of the existence of P_κ-points in S_κ.

In Section 4.5, we prove the announced reconstruction results. We show that under CH, the space ω^* is non-reconstructible, and more generally that under $\kappa = \kappa^{<\kappa}$, the space S_κ is non-reconstructible. We also show that it follows from a theorem by van Douwen, Kunen and van Mill that it is consistent with MA+¬CH that ω^* is reconstructible. Finally, in Section 4.6 we recall why cards of ω^* fail to be normal under CH. We then prove that for uncountable κ with $\kappa = \kappa^{<\kappa}$, all cards of S_κ are non-normal. Together with the results from the earlier sections this proves that the existence of an uncountable cardinal κ with $\kappa = \kappa^{<\kappa}$ implies that normality is non-reconstructible. We conclude the chapter in Section 4.8 with some questions.

4.1 The Stone-Čech remainder of the integers ω^*

4.1.1 Overview In this chapter we explore problems related to the reconstruction of the Stone-Čech compactification $\beta\omega$ of the natural numbers and its remainder $\omega^* = \beta\omega \setminus \omega$.

Both these spaces are of major interest in analytic topology. Three books [CN74, GJ76, Wal74] have been published that deal almost exclusively with the investigation of $\beta\omega$ and its remainder ω^*. There is a detailed article by Jan van Mill in the "Handbook of Set-Theoretic Topology" [vM84] about $\beta\omega$, covering all major results obtained until the mid-1980's. Further, in [HvM90], the authors devote a whole chapter of close to 30 pages to "Open Problems on $\beta\omega$".

Before starting with the proper investigation, we provide a brief description of the spaces under consideration. Also, we recall selected facts which find their application in later parts of this work. All results presented in this chapter are known and most of them are included in [vM84]. For the few remaining results we provide separate proofs.

4.1.2 Preliminaries What are the spaces $\beta\omega$ and ω^*? Recall that a subspace $Y \subset X$ is called C^*-embedded if every continuous real-valued bounded function on Y can be extended to a continuous function on all of X. For every non-compact Tychonoff space X, its Stone-Čech compactification βX is a compact Hausdorff space in which X is dense and C^*-embedded, and $X^* = \beta X \setminus X$ is its remainder. The space βX is essentially unique and one can think of it as the largest compactification of X.

The most convenient description of $\beta\omega$ and ω^* is in terms of ultrafilters on the natural numbers. An excellent account of this construction is given by W. Rudin [Rud56]. Let $\mathcal{F}$ be the set of ultrafilters on ω. For every $U \subset \omega$ let $U^* \subset \mathcal{F}$ be the set of ultrafilters containing U as an element. The collection of all such sets U^* forms a base for a topology on $\mathcal{F}$.

Equipped with this topology, the set $\mathcal{F}$ becomes a compact Hausdorff space. Further, identifying ω with the space of principal ultrafilters gives a dense embedding of ω into $\mathcal{F}$. This copy of ω is C^*-embedded in $\mathcal{F}$, so it follows that $\mathcal{F} \cong \beta\omega$. The space ω^* can be identified with the free ultrafilters on ω. This construction is a special case of the "z-Ultrafilter construction", which describes a concrete construction of βX for any Tychonoff space X [GJ76, 6.5].

The space $\beta\omega$ has cardinality $2^{\mathfrak{c}}$ and weight $\mathfrak{c}$. Every basic open set U^* for $U \subset \omega$ is either finite or homeomorphic to $\beta\omega$ and thus clopen, i.e. the space βw is zero-dimensional. Every infinite closed subset of $\beta\omega$ contains a subset homeomorphic to $\beta\omega$ [Eng89, 3.6.14]. In particular, in $\beta\omega$ there are no non-trivial convergent sequences.

The space ω^* is a zero-dimensional compact Hausdorff space containing no isolated points. Every basic open subset of ω^* is of the form $U^* \setminus U$ for $U \subset \omega$ and thus homeomorphic to ω^*. The space ω^* contains a family of $\mathfrak{c}$ disjoint clopen sets and therefore has density $\mathfrak{c}$ [Eng89, 3.6.18].

4.1.3 The characterisation theorems The spaces $\beta\omega$ and ω^* are, in a sense, very special spaces and next to the properties mentioned above, only few additional topological properties are needed to characterise these spaces topologically. In the case of $\beta\omega$, this can be phrased in terms of the Stone-Čech property. A compact Hausdorff space X is homeomorphic to $\beta\omega$ if and only if it contains a dense open copy of ω such that every two disjoint subsets of ω have disjoint closures in X, cf. Theorem 4.1.3.

In the case of ω^*, two further properties (and a set-theoretic assumption) are needed. Recall that a subset of a Tychonoff space of the form $f^{-1}(0)$ for some real-valued continuous function f is called a *zero-set*. A *cozero-set* is the complement of a zero-set. A space X is called *F-space* if each cozero-set is C^*-embedded in X. A compact zero-dimensional F-space without isolated points in which each non-empty G_δ-set has non-empty interior is called *Parovičenko space*. The space ω^* is a Parovičenko space.

The reader should note that our notion of a Parovičenko space differs from the commonly adopted definition coined in [vDvM78], in the sense that it does not a priori include any assumptions regarding weight.

The following pair of theorems show that many Stone-Čech remainders are Parovičenko spaces; and that under CH, all Parovičenko spaces of weight $\mathfrak{c}$ are homeomorphic. This gives us the desired topological characterisation of ω^*.

Theorem 4.1.1 ([vM84, 1.2.5]). *For a zero-dimensional, locally compact, σ-compact, non-compact space X of weight at most $\mathfrak{c}$, its remainder X^* is a Parovičenko space of weight $\mathfrak{c}$.* $\square$

Theorem 4.1.2 (Parovičenko [Par63], van Douwen and van Mill [vDvM78]). *CH is equivalent to the assertion that every Parovičenko space of weight $\mathfrak{c}$ is homeomorphic to ω^*.*

4.1.4 More about F-spaces

In normal spaces, there is a practical description of the F-space property in terms of open F_σ-sets. We will need the following topological characterisation of C^*-embedded subspaces. Recall that subspaces $A, B \subset X$ are *completely separated* if there is a continuous $f\colon X \to [0,1]$ such that $A \subset f^{-1}(0)$ and $B \subset f^{-1}(1)$. Equivalently, two subspaces are completely separated if they are contained in disjoint zero-sets.

Theorem 4.1.3 (Urysohn's Extension Theorem [GJ76, 1.17]). *A subspace Y of a Tychonoff space X is C^*-embedded in X if and only if any two completely separated sets in Y are completely separated in X.* $\square$

The following results, and further equivalent characterisations of F-spaces, are contained in [GJ76, 14.25] and in the exercises [GJ76, 14N] and [Eng89, 3.6.G].

Lemma 4.1.4. *(a) A Tychonoff space is an F-space if and only if disjoint cozero-subsets are completely separated.*

(b) C^*-embedded subspaces of F-spaces are again F-spaces.

(c) A Tychonoff space X is an F-space if and only if βX is an F-space.

(d) In an F-space, disjoint open F_σ-subsets have disjoint closures, and in normal spaces both conditions are equivalent.

(e) Closed subspaces of normal F-spaces are F-spaces.

(f) Countable subsets of F-spaces are C^*-embedded.

(g) Infinite closed subspaces of compact F-spaces contain a copy of $\beta\omega$. Therefore, compact F-spaces do not contain convergent sequences.

Proof. (*a*). Suppose that X is an F-space and that U and V are disjoint cozero-sets of X. Their union is also cozero and the characteristic function $\mathbb{1}_U\colon U \cup V \to [0,1]$ is continuous. By the F-space property, the function $\mathbb{1}_U$ can be extended from $U \cup V$

to a function f on all of X. This function f witnesses that U and V are completely separated in X.

To prove the converse, let $Y \subset X$ be a cozero-set. Aiming to apply Urysohn's Extension Theorem 4.1.3, let A and B be completely separated subsets of Y. There exist disjoint cozero-sets U and V containing A and B respectively. And since Y is a cozero-set itself, U and V are in fact disjoint cozero-sets of X. By assumption, they are completely separated in X. It follows that also A and B are completely separated in X. Now apply Theorem 4.1.3.

(b). Let Y be a C^*-embedded subspace of an F-space X and let $U = g^{-1}((0,1])$ and $V = h^{-1}((0,1])$ be two disjoint cozero-sets in Y. The function $f\colon Y \to [-1,1]$ defined by $f = g - h$ assumes negative values on U and positive values on V. By the F-space property there is a function F extending f to X. By part (a), the disjoint cozero-sets $F^{-1}([-1,0))$ and $F^{-1}((0,1])$ are completely separated in X and hence so are U and V in Y. Thus, cozero-sets in Y are completely separated, so Y is an F-space by (a).

(c). Suppose X is an F-space, and let U be a cozero-set of βX. To show that U is C^*-embedded in βX, pick some $f \in C^*(X)$. Clearly, $U \cap X$ is a cozero-set of X. By the F-space property, $f|_{U\cap X}$ can be extended to a continuous function $F \in C^*(X)$ which again can be extended to $\beta F \in C^*(\beta X)$. Now $\beta F|_U = f$, as both functions agree on the dense subset $U \cap X$. Hence, βX is an F-space. The converse implication clearly follows from (b).

(d). Every cozero-set is an open F_σ, and in normal spaces, cozero-sets and open F_σ-sets coincide [Eng89, 1.5.13]. Since in a normal space, disjoint closed sets are completely separated by Urysohn's lemma [Eng89, 1.5.11], assertion (d) now readily follows from (a).

(e). By Tietze's Theorem [Eng89, 2.1.8], closed subspaces of normal spaces are C^*-embedded. Thus, the claim follows from (b).

(f). Suppose S is a countable subset of an F-space X. Let S_1 and S_2 be two completely separated subsets of S. Since

$$\overline{S_1}^X \cap S_2 = \emptyset = S_1 \cap \overline{S_2}^X,$$

it is not hard to see that S_1 and S_2 are contained in disjoint cozero-sets of X. By (a), the sets S_1 and S_2 are completely separated in X, and hence the assertion now follows from Urysohn's Extension Theorem 4.1.3.

(g). By part (e) it suffices to prove that infinite compact F-spaces contain a copy of $\beta\omega$. But every infinite regular space contains a countably infinite, relatively discrete subspace [Dug65, VII.2.4]. By (f), this subspace is C^*-embedded in its (compact) closure. This is our copy of $\beta\omega$. $\square$

4.1.5 Information on P-points

A *P-point* is a point p such that any countable intersection of neighbourhoods of p contains again a neighbourhood of p. In other words, p is a P-point if p is in the interior of every G_δ-set containing p. Every isolated point is a P-point. Another example of a P-point is the point ω_1 in $\omega_1 + 1$.

The existence of P-points in ω^* was first proved as a consequence of the Continuum Hypothesis in [Rud56]. The existence of P-points can also be shown under $MA+\neg CH$ (in fact, under $\mathfrak{d} = \mathfrak{c}$) [vM84, 2.5.5]. On the other hand, by a result by Shelah, it is consistent that P-points in ω^* do not exist [vM84, 2.7].

For ω^*, or, indeed, any other zero-dimensional space, we have the following folklore description of P-points.

Lemma 4.1.5. *A point $x \in \omega^*$ is a non-P-point if and only if there exists an open F_σ-set U such that $x \in \overline{U} \setminus U$.*

Proof. Let $V = \bigcap_{n<\omega} V_n$ be a G_δ-set such that $x \in V \setminus \mathrm{int}(V)$. By zero-dimensionality we may assume the V_n to be clopen, obtaining a closed G_δ-set witnessing that x is not a P-point. Its complement is an open F_σ-set as required. The converse implication is clear. $\square$

Lemma 4.1.6. [CH]. *A point p in ω^* is a P-point if and only if it has a strictly nested neighbourhood base of clopen sets.*

Proof. We begin by observing that ω^* is not first-countable at any point of ω^*, since it contains no isolated points and, by Lemma 4.1.4(g), no convergent sequences. Since ω^* has weight $\mathfrak{c}$, it follows under CH that every point of ω^* has character $\omega_1 = \mathfrak{c}$.

So let p be a P-point of ω^* and let $\{U_\alpha \colon \alpha < \omega_1\}$ be a neighbourhood base at p consisting of clopen sets. For $\alpha < \omega_1$, choose V_α inductively to be a clopen neighbourhood of p strictly contained inside of $U_\alpha \cap \bigcap_{\gamma<\alpha} V_\gamma$. This is possible, as this countable intersection of neighbourhoods of the P-point p is itself a neighbourhood. Then $\{V_\alpha \colon \alpha < \omega_1\}$ is a clopen, strictly nested neighbourhood base at p.

The converse implication is clear from the fact that points have character ω_1. $\square$

Further, note that P-points are invariant under homeomorphisms: whenever $p \in \omega^*$ is a P-point and f is an autohomeomorphism of ω^* then $f(p)$ is also a P-point. The following classic theorem by W. Rudin says that under CH, all other P-points can be reached that way.

Theorem 4.1.7 ([Rud56, 4.7]). [CH]. *For every two* P*-points of* ω^* *there is an autohomeomorphism of* ω^* *mapping one to the other.* $\square$

4.2 Negrepontis' space S_κ

4.2.1 Overview We extend our previous discussion from the Stone-Čech remainder ω^* under CH to the Stone spaces S_κ of the κ*-saturated Boolean algebras* of cardinality κ. In a nutshell, the spaces S_κ naturally extend the concept of a Parovičenko space to higher cardinalities. Recall that the Parovičenko properties entail a precise description of the behaviour of countable unions and intersections of clopen sets in ω^*: disjoint pairs of such countable unions have disjoint closures, and every such non-empty intersection has non-empty interior. The κ-Parovičenko properties essentially consist of corresponding requirements for all λ-unions and λ-intersections of clopen sets for all $\lambda < \kappa$. And as for ω^* under CH (which is equivalent to $\omega_1 = \omega_1^{<\omega_1}$), for every cardinal κ with the property $\kappa = \kappa^{<\kappa}$ there is a unique κ-Parovičenko space of weight κ, denoted by S_κ. In particular, S_{ω_1} coincides with ω^* under CH.

Our main source for the spaces S_κ and their corresponding Boolean algebras is the book "The Theory of Ultrafilters" [CN74], which contains two detailed chapters (6 & 14) about S_κ. Another source is A. Dow's [Dow85], in which he investigated κ-Parovičenko spaces without weight restrictions.

We bring to the reader's attention the following peculiarity about the spaces S_κ. They do exist only for cardinals κ satisfying $\kappa = \kappa^{<\kappa}$, the existence of which cannot be guaranteed in *ZFC* alone (apart from $\kappa = \omega$).

4.2.2 κ-Parovičenko spaces and Negrepontis' characterisation theorem In this section we rigorously define the κ-Parovičenko properties. If U is an open subset of a zero-dimensional space X then its *type* in X, denoted by $\tau(U)$ or $\tau_X(U)$, is the least cardinal number κ such that U is equal to the union of κ-many clopen sets of X. If an open set is of finite type, then its type equals one.

A zero-dimensional space X is called F_κ*-space* if every open set U of type less than κ is C^*-embedded in X. This extends the concept of an F-space. Indeed, in presence of normality, the notions of F_{ω_1} and F-spaces coincide, since cozero-sets are then precisely the open sets of countable type. We call a space a G_κ*-space* if every non-empty intersection of less than κ-many open sets has non-empty interior. The space ω^* is a G_{ω_1}-space.

Finally, a *κ-Parovičenko space* is a compact Hausdorff, zero-dimensional F_κ- and G_κ-space without isolated points. The notion of an ω_1-Parovičenko space is identical to being a Parovičenko space.

It turns out that similarly to ω^* under CH, there is a unique κ-Parovičenko space of weight κ under the cardinal assumption $\kappa = \kappa^{<\kappa}$. This generalisation of Parovičenko's Theorem 4.1.2 was proven by Negrepontis.

Theorem 4.2.1 (Negrepontis [Neg69]). *Under the assumption $\kappa = \kappa^{<\kappa}$ there is a unique κ-Parovičenko space of weight κ, which is denoted by S_κ.* □

In the countable case, S_ω always exist and is homeomorphic to the Cantor set C. Under CH, S_{ω_1} is homeomorphic to ω^*.

Note also that A. Dow has shown that κ-Parovičenko spaces of weight at least $\kappa^{<\kappa}$ always exist and that assumption $\kappa = \kappa^{<\kappa}$ is equivalent to the assertion that all κ-Parovičenko spaces of weight $\kappa^{<\kappa}$ are homeomorphic [Dow85].

4.2.3 The weight of κ-Parovičenko spaces Let us motivate the cardinal assumption $\kappa = \kappa^{<\kappa}$ in the above characterisation theorem. We will see that every κ-Parovičenko space has weight and cellularity at least $\kappa^{<\kappa}$. So in some sense, Negrepontis' charactersation theorem says that the smallest possible κ-Parovičenko spaces are topologically unique, whereas Dow's theorem says that larger κ-Parovičenko spaces are not.

Recall that $\kappa^{<\alpha} = \sup\left\{\kappa^\lambda \colon \lambda < \alpha\right\}$.

Lemma 4.2.2. *Every κ-Parovičenko space X has weight and cellularity at least $\kappa^{<\kappa}$.*

Proof. Using compactness and the G_κ-property at limit stages, and the fact that there are no isolated points at successor stages, we can embed a binary tree of clopen sets of height κ into X. Since at every level $\lambda < \kappa$ of the tree we have 2^λ disjoint clopen sets, every κ-Parovičenko space has weight at least $2^{<\kappa}$.

If κ is singular, then, using cofinal sequences, it is easy to see that X is in fact a G_{κ^+}-space, so we can intersect along branches to obtain 2^κ disjoint clopen subsets. Thus, the space has cellularity and weight at least 2^κ, which is at least as big as $\kappa^{<\kappa}$.

And if κ is regular, we claim that $2^{<\kappa} = \kappa^{<\kappa}$. If $\lambda < \kappa$ then by regularity

$$\kappa^\lambda = \bigcup_{\alpha<\kappa} \alpha^\lambda \leq \bigcup_{\alpha<\kappa} 2^{\max(\alpha,\lambda)} = 2^{<\kappa}.$$

Now observe that $\kappa \leq 2^{<\kappa}$ and hence the above string of inequalities gives

$$\kappa^{<\kappa} = \bigcup_{\lambda<\kappa} \kappa^\lambda \leq \kappa \cdot 2^{<\kappa} = \max(\kappa, 2^{<\kappa}) = 2^{<\kappa} \leq \kappa^{<\kappa}.$$ □

Let us comment briefly on the cardinal equality $\kappa = \kappa^{<\kappa}$.

Lemma 4.2.3 ([CN74, 1.27])**.** *For an infinite cardinal κ we have $\kappa = \kappa^{<\kappa}$ if and only if $\kappa = 2^{<\kappa}$ and κ is regular.* □

We remark that the existence of uncountable cardinals satisfying $\kappa = \kappa^{<\kappa}$ is independent of ZFC. Under GCH, every successor cardinal satisfies this equality. However, using Easton Forcing one can obtain a model without strong limit cardinals in which $2^\kappa = \kappa^{++}$ for every regular κ, implying that $\kappa < \kappa^{<\kappa}$ for all κ.

4.2.4 More about F_κ-spaces

As in the case of F-spaces, there is a description of the F_κ-space property in terms of disjoint open sets of type less than κ. Apart from details, the proof remains the same.

Lemma 4.2.4. *Let X be a Tychonoff space. The F_κ-property implies that every two disjoint open sets of type less than κ have disjoint closures. If X is normal, the converse also holds.*

Proof. Suppose that X is an F_κ-space and that U and V are disjoint open subsets of X of type less than κ. Their union is also an open set of type less than κ and the indicator function $\mathbb{1}_U \colon U \cup V \to [0,1]$ is continuous. By the F_κ-space property, the function $\mathbb{1}_U$ can be extended from the union of U and V to a function f on all of X. Since $U \subset f^{-1}(\{1\})$ and $V \subset f^{-1}(\{0\})$, the sets U and V have disjoint closures in X.

Now let X be normal. To prove the converse, we show that every open subset $Y \subseteq X$ of type less than κ is C^*-embedded in X. Let A and B be completely separated subsets of Y. There exist disjoint open F_σ-sets U and V of Y, containing A and B respectively. And since Y is of type less than κ in X, the sets U and V are in fact of type less than κ in X [CN74, 6.4]. By assumption, they have disjoint closure in X, which in turn are completely separated by Urysohn's lemma [Eng89, 1.5.11]. Hence, A and B are completely separated in X and applying Urysohn's Extension Theorem 4.1.3 completes the proof. □

Surprisingly, most parts of Lemma 4.1.4 do not generalise from ω^* to arbitrary Parovičenko spaces. Closed subspaces of normal F_κ-spaces do not necessarily inherit the F_κ-property: every compact F-space, and hence every compact F_κ-space, contains a C^*-embedded copy of ω [GJ76, 14N.5], and therefore a closed copy of $\beta\omega$. This closed subspace is not an F_κ-space for $\kappa > \omega_1$—this follows for example from the Butterfly Lemma 4.3.1.

However, the following lemma, due to Fine & Gillman in the case of ω^* (i.e. $\kappa = \omega_1$) and subsequently generalised by Comfort & Negrepontis to arbitrary κ, will allow us to rescue our results building upon the hereditary aspects of F-spaces.

Lemma 4.2.5 ([CN74, 14.1])**.** *Let κ be an infinite regular cardinal, and X be an F_κ-space. If U is an open subspace of X of type at most κ then U is itself an F_κ-space. Moreover, if X is compact zero-dimensional, then U is strongly zero-dimensional.* $\square$

In Theorem 5.2.1 we will partially generalise Lemma 4.1.4(b), which proved that the F-space property is inherited by C^*-embedded subspaces, and show that *dense* C^*-embedded subsets of zero-dimensional F_κ-spaces are again F_κ-spaces.

4.2.5 More on G_κ-spaces We conclude our introduction to κ-Parovičenko spaces with some equivalent descriptions of the property that a non-empty intersection of less than κ-many clopen sets has non-empty interior.

Lemma 4.2.6. *The following statements are equivalent for a zero-dimensional space X:*

(1) X is a G_κ-space,

(2) For an open subset U of X of type less than κ, no set H of X with $U \subsetneq H \subset \overline{U}$ can be open,

(3) For an open subset U of X such that $1 < \tau(U) < \kappa$, its closure $\overline{U}$ is not open,

(4) No open subset U of X with $1 < \tau(U) < \kappa$ is dense in X.

Proof. (1) $\Rightarrow$ (2): The proof of [CN74, 14.5] applies. For the details, let U be an open subset of X of type less than κ. Then there are clopen sets $\{U_\alpha \colon \alpha < \tau\}$ for $\tau < \kappa$ such that $U = \bigcup_{\alpha<\tau} U_\alpha$. Assume there is an open set H with $U \subsetneq H \subset \overline{U}$, and let $x \in H \setminus U$. Then for every $\alpha < \kappa$ there is a clopen set V_α such that $x \in V_\alpha \subset H$ and $U_\alpha \cap V_\alpha = \emptyset$. It follows $\bigcap_{\alpha<\tau} V_\alpha \subset H \setminus U$ is a non-empty intersection of less than κ-many clopen sets with empty interior, contradicting (1).

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are clear.

(4) $\Rightarrow$ (1): The proof of [CN74, 6.6] applies. For the details, suppose $V = \bigcap_{\alpha<\tau} V_\alpha$ is a non-empty intersection of $\tau < \kappa$ many clopen sets. If V is clopen, there is nothing to show. Otherwise, the set $U = \bigcup_{\alpha<\tau}(X \setminus V_\alpha)$ is an open set of infinite type. As U cannot be dense in X by (4), it follows that $V = X \setminus U$ has non-empty interior. $\square$

4.2.6 P_κ-points A *P_κ-point* is a point p such that the intersection of less than κ-many neighbourhoods of p contains again an open neighbourhood of p. A P_{ω_1}-point is therefore the same as a P-point. As in the case of P-points, every isolated point is a P_κ-point, and so is the point κ in $\kappa + 1$ for regular κ.

The existence of P_κ-points in S_κ is proven in [CN74, 6.17] by a Baire category argument. In Theorem 4.4.4 we present a new proof of the existence of P_κ-points in S_κ.

In S_κ, we have the following description of P_κ-points.

Lemma 4.2.7. *Assume $\kappa = \kappa^{<\kappa}$. A point $x \in S_\kappa$ is a non-P_κ-point if and only if there exists an open set U of type less than κ such that $x \in \overline{U} \setminus U$.*

Proof. Let $V = \bigcap_{\alpha<\beta} V_\alpha$ be an intersection of less than κ many neighbourhoods V_α of X such that $x \in V \setminus \operatorname{int}(V)$. By zero-dimensionality we may assume the V_α to be clopen and thus obtain a closed set V witnessing that x is not a P-point. Its complement is an open set of type less than κ as required. The converse implication is clear. □

Lemma 4.2.8. *Assume $\kappa = \kappa^{<\kappa}$. A P_κ-point in S_κ is characterised by having a strictly nested neighbourhood base of clopen sets of size κ.*

Proof. We begin by observing that no point in S_κ has character smaller than κ, since it is an F_κ-space without isolated points. Since S_κ has weight κ, it follows that every point of S_κ has character κ.

Let p be a P_κ-point of S_κ and let $\{U_\alpha \colon \alpha < \kappa\}$ be a neighbourhood base at p consisting of clopen sets. For $\alpha < \kappa$, choose V_α inductively to be a clopen neighbourhood of p strictly contained inside of $U_\alpha \cap \bigcap_{\gamma<\alpha} V_\gamma$. This is possible, as this intersection of less than κ many neighbourhoods of the P_κ-point p is itself a neighbourhood of p. Then $\{V_\alpha \colon \alpha < \kappa\}$ is a clopen, strictly nested neighbourhood base at p.

The converse implication follows at once from the regularity of κ (Lemma 4.2.3). □

Note that P_κ-points are invariant under homeomorphisms: whenever $p \in S_\kappa$ is a P_κ-point and f is a homeomorphism of S_κ then $f(p)$ is also a P_κ-point. The following extension of Rudin's Theorem 4.1.7, due to Comfort and Negrepontis, says that in S_κ, all other P_κ-points can be reached that way.

Theorem 4.2.9 ([CN74, 6.21])**.** *Assume $\kappa = \kappa^{<\kappa}$. For every pair of P_κ-points of S_κ there is a homeomorphism of S_κ mapping one to the other.* □

4.3 The Butterfly Lemma and its consequences

4.3.1 Butterflies in ω^* under CH We begin with the classic result that under CH, the space ω^* does not occur as the Stone-Čech compactification of any of its dense subspaces.

A point x of a Hausdorff space X is called *strong butterfly point* if its complement $X \setminus \{x\}$ can be partitioned into open sets A and B such that $\overline{A} \cap \overline{B} = \{x\}$. The sets A and B are called *wings* of the butterfly point x. Note that in $X \setminus \{x\}$, the wings A and B are clopen and non-compact.

The following lemma by Fine and Gillman [FG60, Gil67] states that under CH, every point in ω^* is a strong butterfly point. We present the proof, as later on we will use variations of this approach in Theorem 5.3.3 and Lemmas 5.3.7 and 5.4.6.

Lemma 4.3.1 (Butterfly Lemma, Fine and Gillman)**.** [CH]. *Every point in ω^* is a strong butterfly point.*

Proof. Let $x \in \omega^*$ and fix a neighbourhood base $\{U_\alpha \colon \alpha < \omega_1\}$ of x consisting of clopen sets. By transfinite recursion we define families of clopen sets $\{A_\alpha \colon \alpha < \omega_1\}$ and $\{B_\alpha \colon \alpha < \omega_1\}$ not containing x such that all (A_α, B_β)-pairs are disjoint and

$$
\begin{aligned}
&(\star) \quad X \setminus U_\alpha \subset A_\alpha \cup B_\alpha \;\text{ and} \\
&(\dagger) \quad A_\alpha \cap U_\alpha \neq \emptyset \neq B_\alpha \cap U_\alpha \;\text{ for all } \alpha < \omega_1.
\end{aligned}
$$

Once the construction is completed, we define disjoint open sets

$$A = \bigcup_{\alpha<\omega_1} A_\alpha \quad \text{and} \quad B = \bigcup_{\alpha<\omega_1} B_\alpha.$$

For any $y \neq x$, there is a $U_\alpha \not\ni y$. Hence $y \in A_\alpha \cup B_\alpha$ by $(\star)$, so $A \cup B$ covers all of $\omega^* \setminus \{x\}$. Lastly, x is a limit point of both sets A and B, since any basic neighbourhood U_α of x intersects both A and B by $(\dagger)$.

It remains to describe the recursive construction. Let $\alpha < \omega_1$ and assume that A_β and B_β have been defined for all ordinals $\beta < \alpha$. Since countable unions of clopen sets are open F_σ-sets, by the F-space property (Lemma 4.1.4(d)) there exist clopen sets C and D partitioning ω^* and containing the disjoint sets $\bigcup_{\beta<\alpha} A_\beta$ and $\bigcup_{\beta<\alpha} B_\beta$ respectively.

The set $U_\alpha \setminus \bigcup_{\beta<\alpha}(A_\beta \cup B_\beta)$ is a non-empty G_δ-set of the Parovičenko space ω^* as it contains x, and thus it has non-empty interior. Hence, inside this set we may find disjoint non-empty clopen sets C' and D' not containing x. By defining $A_\alpha = (C \setminus U_\alpha) \cup C'$ and $B_\alpha = (D \setminus U_\alpha) \cup D'$ we see that A_α and B_α are as required. $\square$

We list some consequences of the Butterfly Lemma regarding ω^* under CH. First must come the result for which the Butterfly Lemma was originally invented.

Corollary 4.3.2 (Fine and Gillman)**.** [CH]. *For every point x of ω^* the subspace $\omega^* \setminus \{x\}$ is not C^*-embedded in ω^*.*

Proof. By the Butterfly Lemma we have $\omega^* \setminus \{x\} = A \oplus B$ such that $x \in \overline{A}^{\omega^*} \cap \overline{B}^{\omega^*}$. Thus, the continuous function on $\omega^* \setminus \{x\}$ which takes value 1 on A and 0 on B cannot be extended to all of ω^*. ☐

Compact clopen sets of $\omega^* \setminus \{x\}$ are of course homeomorphic to ω^*. The next lemma describes how the non-compact clopen sets look like.

Lemma 4.3.3. [CH]. *For every x in ω^*, the one-point compactification of a clopen non-compact subset of $\omega^* \setminus \{x\}$ is homeomorphic to ω^*.*

Proof. Let A be a clopen non-compact subset of $\omega^* \setminus \{x\}$. Taking $A \cup \{x\}$, a closed subset of ω^*, as representative of its one-point compactification, we see that by Lemma 4.1.4(e) it is a zero-dimensional compact F-space of weight $\mathfrak{c}$ without isolated points.

For the G_{ω_1}-space property, suppose that $U \subset A \cup \{x\}$ is a non-empty G_δ-set. If U has empty intersection with A, then the singleton $U = \{x\}$ is a G_δ-set, and hence has countable character in the compact Hausdorff space $A \cup \{x\}$ [Eng89, 3.3.4]. It follows that there is a non-trivial sequence in ω^* converging to x, contradicting Lemma 4.1.4(g). Thus, U intersects the open set A and their intersection is a non-empty G_δ-set of ω^* with non-empty interior.

An application of Parovičenko's Theorem 4.1.2 completes the proof. ☐

In absence of CH, the above proof still shows that the one-point compactification of any clopen non-compact subset of $\omega^* \setminus \{x\}$ is a Parovičenko space of weight $\mathfrak{c}$.

It also follows that ω^* contains P-points under CH, as the next lemma shows. Even more, we have another proof of [CN68, 2.3] that under CH for every point x of ω^* there is a closed copy of ω^* contained in ω^* such that x is a P-point with respect to that copy.

Corollary 4.3.4. [CH]. *For every point x of ω^*, at least one of its wings together with x itself is a copy of ω^* such that x is a P-point with respect to that copy.*

Proof. This follows from the last lemma together with the observation that if x was a non-P-point with respect to both of its wings, it would, by Lemma 4.1.5, be in the closure of two disjoint open F_σ-sets, contradicting the F-space property of ω^*. ☐

4.3.2 Butterflies in S_κ

We now generalise the results from the previous section about $\omega^* = S_{\omega_1}$ to general S_κ, assuming $\kappa = \kappa^{<\kappa}$ throughout. We briefly demonstrate that the Butterfly Lemma and its immediate consequences carry over nicely to S_κ.

Lemma 4.3.5 (Butterfly Lemma). *Assume $\kappa = \kappa^{<\kappa}$. Every point of S_κ is a strong butterfly point.*

Proof. Let $x \in S_\kappa$ and fix a neighbourhood base $\{U_\alpha \colon \alpha < \kappa\}$ of x consisting of clopen sets. By transfinite induction we define clopen sets $\{A_\alpha \colon \alpha < \kappa\}$ and $\{B_\alpha \colon \alpha < \kappa\}$ not containing x such that all (A_α,B_β)-pairs are disjoint and

$$\begin{aligned} &(\star) \quad X \setminus U_\alpha \subset A_\alpha \cup B_\alpha \ \text{ and} \\ &(\dagger) \quad A_\alpha \cap U_\alpha \neq \emptyset \neq B_\alpha \cap U_\alpha \ \text{ for all } \alpha < \kappa. \end{aligned}$$

Once the construction is completed, we define disjoint open sets

$$A = \bigcup_{\alpha<\kappa} A_\alpha \quad \text{and} \quad B = \bigcup_{\alpha<\kappa} B_\alpha,$$

not containing x. Their union covers all of $S_\kappa \setminus \{x\}$ and both A and B contain x in their closure.

It remains to complete the recursive construction. Let $\alpha < \kappa$ and assume that A_β and B_β have been defined for all ordinals $\beta < \alpha$. It follows from the F_κ-space property (Lemma 4.2.4) that there exist clopen sets C and D partitioning S_κ and containing the sets $\bigcup_{\beta<\alpha} A_\beta$ and $\bigcup_{\beta<\alpha} B_\beta$ respectively.

The set $U_\alpha \setminus \bigcup_{\beta<\alpha}(A_\beta \cup B_\beta)$ is a non-empty intersection of less than κ-many clopen sets of the κ-Parovičenko space S_κ as it contains x. Thus, it has non-empty interior. It follows that inside this set there are disjoint non-empty clopen sets C' and D' not containing x. By defining $A_\alpha = (C \setminus U_\alpha) \cup C'$ and $B_\alpha = (D \setminus U_\alpha) \cup D'$ we see that A_α and B_α are as required. □

Corollary 4.3.6. *Assume $\kappa = \kappa^{<\kappa}$. For every point x of S_κ the subspace $S_\kappa \setminus \{x\}$ is not C^*-embedded in S_κ.* □

The remaining part of this section is devoted to the proof of the following generalisation of Lemma 4.3.3.

Lemma 4.3.7. *Assume $\kappa = \kappa^{<\kappa}$ and let x be any point in S_κ. The one-point compactification of a clopen non-compact subset of $S_\kappa \setminus \{x\}$ is homeomorphic to S_κ.*

The proof, however, is more delicate than in the case of ω^*. The challenge lies in the fact that Lemma 4.3.3 used as corner stones two facts about F-spaces which do not carry through to general F_κ-spaces: By Lemma 4.1.4, in normal spaces, the F-space property is closed-hereditary and every infinite closed subset of S_{ω_1} has the same cardinality as S_{ω_1}. Both assertions do not hold for F_κ-spaces since S_κ contains, being an infinite compact F-space, closed copies of $\beta\omega$.

The following lemma is crucial in circumventing these obstacles.

Lemma 4.3.8. *Assume $\kappa = \kappa^{<\kappa}$ and let x be any point in S_κ. A clopen, non-compact subset of $S_\kappa \setminus \{x\}$ is of S_κ-type κ.*

Proof. Suppose for a contradiction that there exists a clopen, non-compact subset A of $S_\kappa \setminus \{x\}$ of S_κ-type $\tau < \kappa$. Find a representation

$$A = \bigcup_{\alpha<\tau} A_\alpha$$

where all A_α are clopen subsets of S_κ. We first claim that there is a collection $\{V_\alpha \colon \alpha < \tau\}$ of pairwise disjoint clopen sets of S_κ such that $V_\alpha \subset A \setminus \bigcup_{\beta<\alpha} A_\beta$ for all $\alpha < \tau$.

We proceed by transfinite recursion. Choose a clopen subset V_0 in the non-empty open set $A \setminus A_0$. Now consider $\alpha < \tau$ and suppose that V_β have been defined for all $\beta < \alpha$. By minimality of τ, the set $U_\alpha = \bigcup_{\beta<\alpha} A_\beta \cup V_\beta$ is a proper subset of A, from which it follows by the G_κ-space property and Lemma 4.2.6(2) that U_α is not dense in A. Thus, there is a non-empty clopen set V_α in the interior of $A \setminus U_\alpha$. This completes the recursion and proves the claim.

Now, let f and g be disjoint cofinal subsets of τ. We define disjoint open sets

$$V_f = \bigcup_{\alpha\in f} V_\alpha \quad \text{and} \quad V_g = \bigcup_{\alpha\in g} V_\alpha$$

of type at most τ and claim that both sets limit onto x, contradicting the F_κ-space property of S_κ. Suppose the claim was false. Then $\overline{V_f}$ is a subset of $A = \bigcup_{\alpha<\tau} A_\alpha$. By compactness, there is a finite set $F \subset \tau$ such that $\overline{V_f} \subset \bigcup_{\beta\in F} A_\beta$. But there are sets V_α with arbitrarily large index contributing to V_f, a contradiction. $\square$

Proof of Lemma 4.3.7. Let A be a clopen non-compact subset of $S_\kappa \setminus \{x\}$, and denote by X the closure of A in S_κ, i.e. $X = A \cup \{x\} \subset S_\kappa$. Then X is a compact zero-dimensional space of weight κ without isolated points. Aiming to apply Negrepontis' Characterisation Theorem 4.2.1, we check for the remaining κ-Parovičenko properties.

To show X is an F_κ-space, let U and V be disjoint open sets of X of type less than κ. By normality, it suffices to show that U and V have disjoint closures in X. First suppose that x belongs to $U \cup V$. Assume $x \in U$, so that x does not belong to the closure of V. The sets $U \cap A$ and $V \cap A$ are of A-type less than κ. And since A is an F_κ-space by Lemma 4.2.5, our sets have disjoint closures in A, and therefore in X. Next, suppose that x does not belong to $U \cup V$. Then U and V are subsets of A, and consequently of S_κ-type less than κ. Thus, U and V have disjoint closures in S_κ, and hence in X. This establishes that X is an F_κ-space.

To show that X is a G_κ-space, suppose that $U = \bigcap_{\alpha<\lambda} U_\alpha$ is a non-empty set for $\lambda < \kappa$ where all U_α are clopen subsets of $X = A \cup \{x\}$. If U has empty intersection with A, then all $X \setminus U_\alpha$ are clopen subsets of S_κ. It follows that $A = \bigcup_{\alpha<\lambda} X \setminus U_\alpha$ is a clopen non-compact subspace $S_\kappa \setminus \{x\}$ of type less than κ, contradicting Lemma 4.3.8. Thus, U intersects A, and their intersection has non-empty interior in S_κ. □

Following the proof of Corollary 4.3.4, we see that for every point x of S_κ there is a closed copy of S_κ contained in S_κ such that x is a P_κ-point with respect to that copy. In particular, S_κ contains P_κ-points.

Corollary 4.3.9. *Assume $\kappa = \kappa^{<\kappa}$. For every point x of S_κ and for every butterfly around x, one of its wings together with x itself is a copy of S_κ such that x is a P_κ-point with respect to that copy.* □

4.4 Finite compactifications of cards of ω^* and S_κ

This section contains the technical groundwork for our non-reconstruction results. The main result is a characterisation of finite compactifications of cards of ω^* and S_κ.

Lemma 4.4.1. [CH]. *Every card of ω^* has a two-point compactification.*

Proof. It follows from the Butterfly Lemma 4.3.1 that for every point x in ω^*, we have $\omega^* \setminus \{x\} = A \oplus B$ where A and B are non-compact open subsets of ω^*. Considering their respective one-point compactifications αA and αB, we see that $\alpha A \oplus \alpha B$ is a two-point compactification of $\omega^* \setminus \{x\}$. □

Lemma 4.4.2. *Assume $\kappa = \kappa^{<\kappa}$. Every card of S_κ has a two-point compactification.* □

We have established that cards of ω^* under CH, and cards of S_κ under $\kappa = \kappa^{<\kappa}$ have non-trivial finite compactifications. We now give a precise characterisation how finite compactifications of cards of these spaces look like. Our aim is to prove the following pair of theorems.

Theorem 4.4.3 (Characterisation Theorem for Finite Compactifications)**.** [CH]. *For every $x \in \omega^*$ we have that*

(a) $\omega^* \setminus \{x\}$ has arbitrarily large finite compactifications,

(b) every finite compactification of $\omega^* \setminus \{x\}$ is homeomorphic to ω^*, and

(c) for every finite compactification, all but at most one point at infinity are P-points.

The case for S_κ looks exactly the same.

Theorem 4.4.4 (Characterisation Theorem for Finite Compactifications II)**.** *Assume $\kappa = \kappa^{<\kappa}$. For every $x \in S_\kappa$ we have that*

(a) $S_\kappa \setminus \{x\}$ has arbitrarily large finite compactifications,

(b) every finite compactification of $S_\kappa \setminus \{x\}$ is homeomorphic to S_κ, and

(c) for every finite compactification, all but at most one point at infinity are P_κ-points.

To prove these theorems, we begin with a sufficient condition for zero-dimensional locally compact Hausdorff spaces to have only one homeomorphism type amongst their finite compactifications.

Lemma 4.4.5. *Let X be a zero-dimensional compact Hausdorff space such that $X \oplus X$ is homeomorphic to X and*

($\star$) *for every point x of X, the one-point compactification of any clopen non-compact subset of $X \setminus \{x\}$ is homeomorphic to X.*

Then, for all x, all finite compactifications of $X \setminus \{x\}$ are homeomorphic to X.

Proof. Let Z be a finite compactification of $X \setminus \{x\}$ with remainder consisting of points $\infty_1, \dots, \infty_n$. By [Woods74, 2.3], every finite compactification of a locally compact zero-dimensional space is zero-dimensional. Hence, there is a partition of Z into n disjoint clopen sets A_i such that $\infty_i \in A_i$.

The set $A_i \setminus \{\infty_i\}$ is a clopen non-compact subspace of $X \setminus \{x\}$. Therefore, by property ($\star$) and uniqueness of the one-point compactification, it follows that A_i is homeomorphic to X. This proves, after applying $X \oplus X \cong X$ iteratively, that Z is homeomorphic to X. □

This lemma lies at the heart of our proofs for Theorems 4.4.3 and 4.4.4. Note that by Lemmas 4.3.3 and 4.3.7, both spaces ω^* under CH, and S_κ assuming $\kappa = \kappa^{<\kappa}$, do indeed have property $(\star)$. Surprisingly, despite its strong assumptions, the lemma itself applies to a variety of interesting spaces.

Recall from Section 1.2.1 that spaces which only have λ different homeomorphism types amongst their non-empty open subspaces (for some cardinal λ) are said to be of *diversity* λ. One checks that Lemma 4.4.5 applies to all compact Hausdorff spaces of diversity two, which are known to be zero-dimensional [Mio78]. In particular, Lemma 4.4.5 applies to the Cantor space C, the Alexandroff Double Arrow space D and to the product $D \times C$. Incidentally, we have seen in Section 1.2.1 these examples are also non-reconstructible.

In an infinite compact Hausdorff space X of diversity two, any subspace $X \setminus \{x\}$ is homeomorphic to $X \setminus \{x_1, \ldots, x_n\}$ and therefore has arbitrarily large finite compactifications. This is when Lemma 4.4.5 is most valuable. Our next lemma shows that not much is needed for this scenario to occur. The proof is a simple induction.

Lemma 4.4.6. *Let X be a topological space such that for all x, all finite compactifications of $X \setminus \{x\}$ are homeomorphic to X. If all spaces $X \setminus \{x\}$ have two-point compactifications, they have arbitrarily large finite compactifications.* $\square$

The following example of the Cantor cube 2^κ for uncountable κ shows that the assumptions in Lemma 4.4.6 cannot be considerably weakened. Since $\beta(2^\kappa \setminus \{x\}) = 2^\kappa$ [Gli59, Thm. 2], these spaces have a unique compactification. The cube 2^κ is a zero-dimensional compact Hausdorff space with $2^\kappa \cong 2^\kappa \oplus 2^\kappa$. For property $(\star)$, let $A \subset 2^\kappa \setminus \{x\}$ be a clopen non-compact subset. Since $2^\kappa \setminus \{x\}$ does not have a 2-point compactification, $A \cup \{x\}$ must be clopen in 2^κ. But every clopen set of 2^κ can be written as a disjoint union of finitely many product-basic open sets, which are homeomorphic to 2^κ. Hence $A \cup \{x\} \cong 2^\kappa$. We conclude that Lemma 4.4.5 applies, but restricts to the obvious assertion that the one-point compactification of $2^\kappa \setminus \{x\}$ is homeomorphic to 2^κ.

Now finally, with these lemmas established, we can complete the proofs of the main theorems in this section.

Proof of Theorem 4.4.3. Assertion (b) is an immediate consequence of Lemmas 4.4.5 and 4.3.3. For (a), note that Lemma 4.4.1 implies that every card of ω^* has a two-point compactification, which is homeomorphic to ω^* by (b). Therefore, (a) now follows from Lemma 4.4.6.

For assertion (c), suppose there is a finite compactification Z of $\omega^* \setminus \{x\}$ containing two non-P-points ∞_1 and ∞_2 at infinity. Then by Lemma 4.1.5, there are disjoint open F_σ-sets F_1 and F_2 in Z with $F_i \subset \omega^* \setminus \{x\}$ containing ∞_1 and ∞_2 in their respective boundaries. However, in ω^* the disjoint non-compact open F_σ-sets F_1 and F_2 both limit onto x. This contradicts the F-space property of ω^*. □

In ZFC, the above argument shows that every finite compactification of $\omega^* \setminus \{x\}$ is a Parovičenko space of weight $\mathfrak{c}$ such that at most one point at infinity is not a P-point. However, one cannot decide in ZFC alone whether there are finite compactifications of $\omega^* \setminus \{x\}$ other than the one-point compactification [vDKvM89].

Proof of Theorem 4.4.4. Assertions (a) and (b) follow as in the previous proof.

The proof of (c) uses the same idea as in the case of ω^*. Suppose there is a finite compactification Z of $S_\kappa \setminus \{x\}$ containing two non-P_κ-points ∞_1 and ∞_2 at infinity. Then by Lemma 4.2.7 there are disjoint open subsets F_1 and F_2 in Z of type less than κ with $F_i \subset S_\kappa \setminus \{x\}$ that contain ∞_1 and ∞_2 in their respective boundaries. However, in S_κ the disjoint non-compact open sets F_1 and F_2 of type less than κ both limit onto x, contradicting the F_κ-space property. □

4.5 Reconstruction results for ω^* and S_κ

This section contains our reconstruction results for the spaces ω^* and S_κ. We will see that it is consistent and independent of ZFC whether the space ω^* is reconstructible. For example, ω^* is reconstructible in models where ω^* is the Stone-Čech compactification of one of its cards, and is not reconstructible in models where the Continuum Hypothesis holds.

Generalising the behaviour of ω^* under CH, we show further below that assuming $\kappa = \kappa^{<\kappa}$, the spaces S_κ are always non-reconstructible.

4.5.1 The deck of ω^* We briefly describe the cards of ω^*. Recall that the cards of a space correspond to the non-homeomorphic subspaces one can obtain by deleting singletons. It turns out that cards of compact Hausdorff spaces, and in particular cards of ω^* correspond to the different orbits under the action of its autohomeomorphism group.

A space X is *homogeneous* if for every pair of points x and y of X there exists a homeomorphism of X carrying x to y. In general, we say x and y lie in the same *orbit* of X if x can be mapped to y by a homeomorphism of X. The orbits form equivalence classes and the collection of orbits is denoted by $X/_{\sim}$.

If x and y lie in the same orbit of X then deleting either x or y gives the same card. For example, Rudin's Theorem 4.1.7 yields that under CH, all cards of ω^* obtained by deleting a P-point are homeomorphic. Also, the deck of a homogeneous space consists of only one card, and $|\mathcal{D}(X)| \leq |X/_{\sim}|$ is true for any space X. We now show that for compact Hausdorff spaces, we have equality in the previous line.

Theorem 4.5.1. *Let X be a compact Hausdorff space. Then $|\mathcal{D}(X)| = |X/_{\sim}|$.*

Proof. As $|\mathcal{D}(X)| \leq |X/_{\sim}|$ by the remark preceding the theorem, it is enough to prove that $|\mathcal{D}(X)| \geq |X/_{\sim}|$. If X is homogeneous, the result is clear. If X is not homogeneous, find x and y contained in different orbits. Suppose for a contradiction that the two cards obtained by deleting x and y are homeomorphic, i.e. that there exists a homeomorphism $f\colon X \setminus \{x\} \to X \setminus \{y\}$. If x is isolated, then both $X \setminus \{x\}$ and $X \setminus \{y\}$ are compact and hence y must be isolated, too. But then, both points x and y lie in the same orbit, a contradiction.

Thus, we may assume that both x and y are non-isolated. Then X is a one-point compactification of both $X \setminus \{x\}$ and $X \setminus \{y\}$. But since all one-point compactifications of a locally compact space are homeomorphic by a map carrying remainders onto remainders [Eng89, 3.5.11], the map $f \cup \{\langle x, y\rangle\}$ is a homeomorphism of X. Thus, x and y are contained in the same orbit, a contradiction. □

Z. Frolík proved (in ZFC) that every orbit in ω^* is of size $\mathfrak{c}$, and therefore that ω^* has $2^{\mathfrak{c}}$-many orbits [Fro67]. It follows that $|\mathcal{D}(\omega^*)| = 2^{\mathfrak{c}} = |\omega^*|$, i.e. the space ω^* has the maximal possible number of different cards.

4.5.2 Reconstructing ω^*

In this section we prove the following theorem.

Theorem 4.5.2. [CH]. *The space ω^* is non-reconstructible. For a P-point p, the space $\omega^* \setminus \{p\}$ is a non-homeomorphic reconstruction of ω^*.*

For the proof we need three lemmas describing the behaviour of quotients of Parovičenko spaces X when identifying a subset A with a single point. Write X/A for the quotient space induced by the partition $\{A\} \cup \{\{x\} : x \in X \setminus A\}$.

Lemma 4.5.3. *Let X be a compact Hausdorff space and $A \subset X$ a closed, non-open subset of X. Then X/A is a one-point compactification of $X \setminus A$. Moreover, if X is zero-dimensional, then so is X/A.*

Proof. First, a quotient space of a compact space is compact. Further, X/A is Hausdorff as X is regular and A is closed. Since A is not open, the map $X \setminus A \hookrightarrow X/A$, sending $x \mapsto \{x\}$ is a dense embedding with a one-point remainder.

For zero-dimensionality, we show that $A \subset X$ has a neighbourhood base of clopen sets. So let U be an open neighbourhood of A. Assuming that X is zero-dimensional, for every $x \in A$ there is a clopen set such that $x \in C(x) \subseteq U$. The clopen cover $\{C(x) \colon x \in A\}$ of the compact set A has a finite subcover. Its union is a clopen set between A and U. $\square$

The next lemma shows in which cases quotients preserve the G_δ-property of Parovičenko spaces. A similar lemma appears without proof in [vM84, 1.4.2]. Note that if X is compact Hausdorff and $A \subset X$ is closed, the quotient map $\pi \colon X \to X/A$ is a continuous map from a compact space to a Hausdorff space, and therefore closed.

Lemma 4.5.4. *Suppose X has the property that non-empty G_δ-sets have non-empty interior. Let $A \subset X$ be a closed, nowhere dense subset of X. Then X/A also has the property that non-empty G_δ-sets have non-empty interior.*

Proof. Let U be a non-empty G_δ of X/A. We have to show that it has non-empty interior. Since π is continuous and surjective, $\pi^{-1}(U)$ is a non-empty G_δ-set of X. By assumption, it has non-empty interior. Since A is closed and nowhere dense, the set $\pi^{-1}(U) \setminus A$ also has non-empty interior. Observing that $\pi(\operatorname{int}(\pi^{-1}(U)) \setminus A)$ is an open subset of U completes the proof. $\square$

Our last lemma tells us under which conditions collapsing a subset to a single point leaves the F-space property intact. The result is a slight generalisation of [vM84, 1.4.1].

Lemma 4.5.5. *Let X be a compact F-space and $A \subset X$ a closed subset containing at most one non-P-point of X. Then X/A is an F-space.*

Proof. By Lemma 4.5.3, the space X/A is normal. To establish the F-space property, it therefore suffices by Lemma 4.1.4(d) to verify that disjoint open F_σ-sets have disjoint closures.

So let U and V be disjoint open F_σ-sets of X/A. Since $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are disjoint open F_σ-sets of X, they have disjoint closures in X.

Suppose that $\{A\} \in U \cup V$. Without loss of generality, we have $A \subset \pi^{-1}(U)$ and hence $\pi(\overline{\pi^{-1}(U)}) \cap \pi(\overline{\pi^{-1}(V)}) = \emptyset$. Since π is a closed surjective map, we have $\overline{U} \subseteq \pi(\overline{\pi^{-1}(U)})$ and hence $\overline{U} \cap \overline{V} = \emptyset$.

Now suppose that $\{A\} \notin U \cup V$. Then A does not intersect $\pi^{-1}(U) \cup \pi^{-1}(V)$. Note that for all P-points $p \in A$ it follows from Lemma 4.1.5 that

$$p \notin \overline{\pi^{-1}(U)} \cup \overline{\pi^{-1}(V)}.$$

Finally, since $\pi^{-1}(U)$ and $\pi^{-1}(V)$ have disjoint closures in X, the single non-P-point of A cannot be contained in both of them. Thus, we may assume without loss of generality that $\{A\} \notin \pi(\overline{\pi^{-1}(U)})$. Therefore, $\pi(\overline{\pi^{-1}(U)}) \cap \pi(\overline{\pi^{-1}(V)}) = \emptyset$, implying, as before, that U and V have disjoint closures in X/A. ☐

Proof of Theorem 4.5.2. We prove that for a P-point p of ω^*, the space $\omega^* \setminus \{p\}$ is a non-homeomorphic reconstruction of ω^*. Let us first show $\mathcal{D}(\omega^* \setminus \{p\}) \subseteq \mathcal{D}(\omega^*)$.

For this inclusion, pick any card $\omega^* \setminus \{p, x\}$ in $\mathcal{D}(\omega^* \setminus \{p\})$. We claim that its one-point compactification $X = \{\infty\} \cup (\omega^* \setminus \{p, x\})$ is a Parovičenko space. Then, by Theorem 4.1.2, there is a homeomorphism $f \colon X \to \omega^*$. It follows

$$\omega^* \setminus \{p, x\} \cong X \setminus \{\infty\} \cong \omega^* \setminus \{f(\infty)\},$$

establishing that $\omega^* \setminus \{p, x\} \in \mathcal{D}(\omega^*)$.

To see that X is a Parovičenko space note that X is a compact space of weight $\mathfrak{c}$ without isolated points. By Lemma 4.5.3, we may take ω^*/A with $A = \{p, x\}$ as a representative for X, showing that X is zero-dimensional. Further, by Lemmas 4.5.4 and 4.5.5, the space ω^*/A is an F-space with the property that non-empty G_δ-sets have non-empty interior. Thus, X is Parovičenko, completing the proof of the first inclusion.

We now establish the reverse inclusion $\mathcal{D}(\omega^* \setminus \{p\}) \supseteq \mathcal{D}(\omega^*)$. For this, let $\omega^* \setminus \{x\}$ be any card in $\mathcal{D}(\omega^*)$. It follows from Theorem 4.4.3 that there exist points ∞_1 and ∞_2 of ω^*, of which ∞_1 is a P-point, such that

$$\omega^* \setminus \{x\} \cong \omega^* \setminus \{\infty_1, \infty_2\}.$$

By Rudin's Theorem 4.1.7, there exists a homeomorphism f of ω^* carrying ∞_1 to p. Then $\omega^* \setminus \{x\} \cong \omega^* \setminus \{p, f(\infty_2)\}$, and hence $\omega^* \setminus \{x\}$ is a card in $\mathcal{D}(\omega^* \setminus \{p\})$. ☐

The proof that S_κ is non-reconstructible is very similar. The following two lemmas are straightforward adaptions of Lemma 4.5.4 and 4.5.5 respectively.

Lemma 4.5.6. *Suppose X has the property that every non-empty intersection of fewer than κ many open sets has non-empty interior. Let $A \subset X$ be a closed, nowhere dense subset of X. Then X/A also has the property that every non-empty intersection of fewer than κ many open sets has non-empty interior.* ☐

Lemma 4.5.7. *Let X be a compact F_κ-space and $A \subset X$ a closed subset containing at most one non-P_κ-point of X. Then X/A is an F_κ-space.* □

Theorem 4.5.8. *Assume $\kappa = \kappa^{<\kappa}$. The space S_κ is non-reconstructible. Indeed, for a P_κ-point p, the space $S_\kappa \setminus \{p\}$ is a non-homeomorphic reconstruction of S_κ.*

Proof. To prove the inclusion $\mathcal{D}(S_\kappa \setminus \{p\}) \subseteq \mathcal{D}(S_\kappa)$, pick any card $S_\kappa \setminus \{p, x\}$ in $\mathcal{D}(S_\kappa \setminus \{p\})$. Using Lemmas 4.5.6 and 4.5.7 we see that its one-point compactification $X = \{\infty\} \cup (S_\kappa \setminus \{p, x\})$ is a κ-Parovičenko space. By Theorem 4.2.1 there is a homeomorphism $f\colon X \to S_\kappa$. It follows

$$S_\kappa \setminus \{p, x\} \cong X \setminus \{\infty\} \cong S_\kappa \setminus \{f(\infty)\},$$

establishing that $S_\kappa \setminus \{p, x\} \in \mathcal{D}(S_\kappa)$.

For the reverse inclusion, let $S_\kappa \setminus \{x\}$ be a card in $\mathcal{D}(S_\kappa)$. It follows from Theorem 4.4.4 that there are points ∞_1 and ∞_2 of S_κ, with ∞_1 being a P_κ-point, such that

$$S_\kappa \setminus \{x\} \cong S_\kappa \setminus \{\infty_1, \infty_2\}.$$

By Theorem 4.2.9, there exists a homeomorphism f of S_κ carrying ∞_1 to p. Then $S_\kappa \setminus \{x\} \cong S_\kappa \setminus \{p, f(\infty_2)\}$, and hence $S_\kappa \setminus \{x\}$ is a card in $\mathcal{D}(S_\kappa \setminus \{p\})$. □

We conclude this section with the result that ω^* is consistently reconstructible. In particular, together with the results above we see that the question whether ω^* is reconstructible is independent of the axioms of set theory ZFC. Note also that for showing that ω^* is non-reconstructible, the assumption CH cannot be weakened to Martin's axiom (MA).

Theorem 4.5.9 ([vDKvM89]). *It is consistent with $MA + \mathfrak{c} = \omega_2$ that for all $x \in \omega^*$ we have $\beta(\omega^* \setminus \{x\}) = \omega^*$.* □

Theorem 4.5.10. *It is consistent with $MA + \mathfrak{c} = \omega_2$ that ω^* is reconstructible.*

Proof. By Theorem 1.2.8, every compact Hausdorff space arising as a non-trivial Stone-Čech compactification is reconstructible. Hence, the reconstruction result follows from the previous theorem. □

For the interested reader we remark that after Theorem 4.5.9, further models with the same property have been found. Indeed, Malykhin has established that the same is true in the $\aleph_2$-Cohen model [Mal93]. See also [DH02, 3.6].

4.6 Normality is consistently non-reconstructible

4.6.1 Non-normality of cards of ω^* and S_κ This section contains a proof of the result that under CH, every card of ω^* is non-normal, and that for uncountable κ with $\kappa = \kappa^{<\kappa}$, every card of S_κ is non-normal.

In case of ω^*, this result is originally due to Rajagopalan and Warren. Proofs can be found in [vM86, Raj72, War72]. An account of [Raj72] is contained in [Wal74, 7.2-7.4]. In the following, we directly prove the more general result that cards of S_κ are non-normal. By Corollary 4.3.9 we may focus our attention on cards obtained by deleting a P_κ-point of S_κ. From there on, we adapt Warren's result [War72, I.1] that under CH, cards of ω^* obtained by deleting P-points are non-normal.

Theorem 4.6.1. *Assume $\kappa = \kappa^{<\kappa}$. If κ is uncountable then every card $S_\kappa \setminus \{x\}$ of S_κ is non-normal.*

Proof. Since normality is closed-hereditary, it suffices by Corollary 4.3.9 to prove the theorem for cards that have been obtained by deleting a P_κ-point of S_κ. So let p be a P_κ-point in S_κ. We show that $S_\kappa \setminus \{p\}$ is non-normal. Fix a strictly decreasing neighbourhood base $\{U_\alpha \colon \alpha < \kappa\}$ of p consisting of clopen sets. Pick P_κ-points p_α inside the non-empty sets $V_\alpha = U_\alpha \setminus U_{\alpha+1}$, which is possible as V_α is homeomorphic to S_κ. Again, for each $p_\alpha \in V_\alpha$ we fix a nested neighbourhood base $\{V_{\alpha,\beta} \colon \beta < \kappa\}$ of clopen sets, such that $V_{\alpha,0} = V_\alpha$.

We now describe two closed disjoint sets A and B of $S_\kappa \setminus \{p\}$ that cannot be separated by open sets, showing that this space is non-normal. Define, for each limit ordinal $\lambda < \kappa$, the sets

$$B_\lambda = \overline{\bigcup_{\alpha<\lambda} (V_\alpha \setminus V_{\alpha,\lambda})} \cap \bigcap_{\alpha<\lambda} U_\alpha.$$

The closures are of course taken in $S_\kappa \setminus \{p\}$. Then put

$$A = \overline{\{p_\alpha \colon \alpha < \kappa\}} \quad \text{and} \quad B = \bigcup_{\lambda<\kappa} B_\lambda.$$

Let us see why A and B are disjoint. Consider the disjoint sets $\bigcup_{\alpha<\lambda} V_\alpha \setminus V_{\alpha,\lambda}$ and $\bigcup_{\alpha<\lambda} V_{\alpha,\lambda}$, the second of which being a superset of $\{p_\alpha \colon \alpha < \lambda\}$. Both sets are of S_κ-type less than κ and hence have disjoint closures in S_κ by Lemma 4.2.4. It follows that A and B are disjoint, since if $q \in A \cap B$ then $q \notin U_\lambda$ for some limit ordinal λ, and we obtain a contradiction from

$$q \in B \cap A \cap (S_\kappa \setminus U_\lambda) \subset \overline{\bigcup_{\alpha<\lambda} V_\alpha \setminus V_{\alpha,\lambda}} \cap \overline{\bigcup_{\alpha<\lambda} V_{\alpha,\lambda}} = \emptyset.$$

We now show that B is closed. Suppose that q lies in $\overline{B}$. Since $V_\alpha \cap B = \emptyset$ for all $\alpha < \kappa$ it follows that $q \in \bigcap_{\alpha<\mu} U_\alpha \setminus U_\mu$ for some limit ordinal $\mu < \kappa$. Note that

$$B = \bigcup_{\lambda<\mu} B_\lambda \cup B_\mu \cup \bigcup_{\mu<\lambda<\kappa} B_\lambda.$$

Since the last term is a subset of the closed set U_μ, the point q is contained in the closure of the first two terms. However, it follows from the construction that $\overline{\bigcup_{\lambda<\mu} B_\lambda} \cap \bigcap_{\alpha<\mu} U_\alpha \setminus U_\mu \subset B_\mu$. So q lies in $\overline{B_\mu} = B_\mu$. Hence $q \in B$, and we have shown that B is closed.

To complete the proof it remains to show that A and B cannot be separated by open sets. So let U and V be open sets of $S_\kappa \setminus \{p\}$ containing A and B respectively. For every ordinal $\alpha < \kappa$ there exists $\beta_\alpha > \alpha$ such that $V_{\alpha,\beta_\alpha} \subset U$. Since κ is regular by Lemma 4.2.3, the increasing sequence defined by $\alpha_0 = 0$ and $\alpha_n = \beta_{\alpha_{n-1}}$ has a supremum $\gamma < \kappa$. Consider the set

$$W = \bigcup_{n\in\omega} (V_{\alpha_n,\beta_{\alpha_n}} \setminus V_{\alpha_n,\gamma}).$$

It follows from our construction that

$$W \subset U \quad \text{and} \quad W \subset \bigcup_{n\in\omega} (V_{\alpha_n} \setminus V_{\alpha_n,\gamma}).$$

Let us see that $\overline{W} \cap \bigcap_{\alpha<\gamma} U_\alpha$ is a non-empty subset of B_γ. For this, we only have to show that $\overline{W}$ intersects $\bigcap_{\alpha<\gamma} U_\alpha = \bigcap_{n\in\omega} U_{\alpha_n}$. But this holds, since otherwise the collection $\{S_\kappa \setminus U_{\alpha_n} : n \in \omega\}$ forms an open cover of the compact set $\overline{W}$, yielding a contradiction.

It follows that V is a neighbourhood of every point in $\overline{W} \cap \bigcap_{\alpha<\gamma} U_\alpha$, and therefore that $V \cap W \neq \emptyset$. Since $V \cap U \supseteq V \cap W$ we see that U and V cannot be disjoint, completing the proof. □

4.6.2 Normality is consistently non-reconstructible Theorem 4.5.2 established that under CH, the spaces ω^* and $\omega^* \setminus \{p\}$ for a P-point p are non-homeomorphic reconstructions of each other. Since under CH, the space $\omega^* \setminus \{p\}$ is non-normal by Theorem 4.6.1, this gives the desired result that normality is consistently non-reconstructible. Also, in presence of Hausdorffness, compactness implies paracompactness, which in turn implies normality [Eng89, 5.1.1 & 5.1.18]. Hence, it is also consistent that paracompactness is non-reconstructible.

More generally, we have the following theorem.

Theorem 4.6.2. *The existence of an uncountable cardinal κ with the property $\kappa = \kappa^{<\kappa}$ implies that normality and paracompactness are not reconstructible.*

Proof. The space S_κ is compact Hausdorff, hence paracompact and normal. By Theorem 4.6.1, the space $S_\kappa \setminus \{p\}$ for a P_κ-point p is non-normal and non-paracompact. However, by Theorem 4.5.8, both spaces are reconstructions of each other. Therefore, the properties of being normal or paracompact are not reconstructible. □

4.7 An alternative proof that ω^* is not reconstructible under CH

In this section we outline a different proof that ω^* is non-reconstructible under CH. This proof has been suggested to me in parts by an anonymous referee of a draft paper based on the results in this chapter.

Consider a compact Hausdorff space X and a non-compact Hausdorff space Z. We say a point $x \in X$ is *Z-approachable* in X if $X \setminus \{x\}$ contains a clopen copy of Z. We now show that the notion of being "Z-approachable" is a useful generalisation of our notion of "universal sequences" in the compact metrizable case, introduced in Section 2.4.4. Indeed, the reader will note some parallels in the following proof and the proof of the Universal Sequence Lemma 2.4.6.

Theorem 4.7.1. *Let X be a zero-dimensional compact Hausdorff space. If X is non-reconstructible, then there exists a non-compact Hausdorff space Z such that every point of X is Z-approachable.*

Proof. Suppose that X is non-reconstructible. By Theorem 1.2.7, X must have a non-compact reconstruction Z'. By Theorems 1.2.1 and 1.2.3, this space Z' is a zero-dimensional, locally compact Hausdorff space. We claim that for some clopen, co-compact $Z \subset Z'$, every point of X is Z-approachable.

Claim. *For every $x \in X$ and every clopen $U \ni x$ there is a clopen, co-compact $Z_x \subset Z'$ such that x is Z_x-approachable in U.*

To see the claim, note that for some suitable $z \in Z'$ there is a homeomorphism $f \colon Z' \setminus \{z\} \to X \setminus \{x\}$. By local compactness and zero-dimensionality, there is a clopen, co-compact $Z'_x \subset Z'$ with $z \notin Z'_x$. Then $Z_x = Z'_x \cap f^{-1}(U)$ is a clopen, co-compact subset of Z', and $f(Z_x) \subset U$ witnesses that x is Z_x-approachable in U.

Claim. *There is a clopen, co-compact $Z \subset Z'$ such that every point of X is Z-approachable.*

To see the claim, fix $u \neq v \in X$, and consider a clopen partition $X = U \oplus V$ with $u \in U$ and $v \in V$. By the previous claim there exist clopen, co-compact Z_u and $Z_v \subset Z'$, and clopen $A \subset U \setminus \{u\}$ and $B \subset V \setminus \{v\}$ with $A \cong Z_u$ and $B \cong Z_v$.

Now consider $Z = Z_u \cap Z_v$, a clopen, co-compact subset of Z'. To see that this Z is as required, pick any $x \in X$. Without loss of generality, $x \in U$. By the first claim, there is a clopen, co-compact $Z_x \subset Z'$ and $D \subset U \setminus \{x\}$ such that $D \cong Z_x$.

Fix homeomorphisms $f \colon Z_x \to D$ and $g \colon Z_v \to B$. Write $Z = (Z \cap Z_x) \oplus (Z \setminus Z_x)$ and note that $Z \setminus Z_x$ is a clopen, compact subset of Z'. It follows that $f(Z \cap Z_x) \cup g(Z \setminus Z_x)$ is an embedding of Z into $X \setminus \{x\}$ with clopen image as desired. □

We now prove under some additional assumptions on Z a partial converse to the above theorem.

Theorem 4.7.2. *Let X be a zero-dimensional compact Hausdorff space. Assume that there exists a non-compact Hausdorff space Z such that*

(1) every point of X is Z-approachable,

(2) $Z \cong Z \oplus Z$, and

(3) for all $z \in Z$ there exists a clopen neighbourhood $U \ni z$ such that $U \cong X$ and $Z \setminus U \cong Z$.

Then Z is a non-compact reconstruction of X.

Proof. To show $\mathcal{D}(X) \subset \mathcal{D}(Z)$, consider a card $Y = X \setminus \{x\}$ and note that

$$\begin{aligned} Y &\cong Z \oplus Y \setminus Z \quad \text{(by (1))} \\ &\cong (Z \oplus Z) \oplus Y \setminus Z = Z \oplus (Z \oplus Y \setminus Z) \quad \text{(by (2))} \\ &\cong Z \oplus Y \quad \text{(by (1))}, \end{aligned}$$

and that $Z \oplus Y \in \mathcal{D}(Z)$ by (3).

Conversely, to show $\mathcal{D}(Z) \subset \mathcal{D}(X)$, consider a card $Y' = Z \setminus \{z\}$. By (3), we have $Y' \cong Z \oplus X \setminus \{x\}$ for some suitable $x \in X$. Following the above chain of equalities backwards gives the result. □

Now denote by $\aleph_1$ the discrete space of cardinality ω_1, and identify $\aleph_1^*$ with the free ultrafilters on $\aleph_1$. The space $U(\aleph_1) \subset \aleph_1^*$ denotes the space of the *uniform* ultrafilters on $\aleph_1$, i.e. the ultrafilters whose members are subsets of $\aleph_1$ of full cardinality. Its complement $SU(\aleph_1) = \aleph_1^* \setminus U(\aleph_1)$ are the *subuniform* ultrafilters. Comfort and Negrepontis observed in [CN68, 2.4] that $\omega^* \setminus \{p\} \cong SU(\aleph_1)$ for all P-points $p \in \omega^*$ of character ω_1 (if such P-points exist, which cannot be guaranteed in ZFC).

Corollary 4.7.3. [CH]. *The space $SU(\aleph_1)$ is a non-compact reconstruction of ω^*.*

Proof. It suffices to check that $SU(\aleph_1)$ satisfies conditions $(1) - (3)$ of the previous theorem.

Under CH, requirement (1) follows from Corollary 4.3.4. To verify (2), write $\aleph_1 = A \oplus B$ such that $|A| = |B| = \omega_1$. Then

$$SU(\aleph_1) = SU(A) \oplus SU(B) \cong SU(\aleph_1) \oplus SU(\aleph_1).$$

For (3), let $x \in SU(\aleph_1)$. Since x is a subuniform ultrafilter, we can find a countably infinite $D \subset \aleph_1$ such that $x \in \overline{D}^{SU(\aleph_1)}$. Then $\overline{D}^{SU(\aleph_1)}$ is a clopen subset of $SU(\aleph_1)$ with $\overline{D}^{SU(\aleph_1)} \cong \omega^*$ and $SU(\aleph_1) \setminus \overline{D}^{SU(\aleph_1)} = SU(\aleph_1 \setminus D) \cong SU(\aleph_1)$ as required. $\square$

This proof raises the following question: Are there models in which CH fails and every point of ω^* is $SU(\aleph_1)$-approachable?

4.8 Open questions

One of the main results in this chapter has been that normality and paracompactness are consistently non-reconstructible. Are these results true in ZFC?

Question 4.8.1. *Are normality or paracompactness non-reconstructible properties?*

When looking for a ZFC example showing that normality is non-reconstructible, note that by our discussion in the introduction, such an example has to be a normal space whose cards are all non-normal. Such spaces seem to be rare. In ZFC, the only such spaces we are aware of are related to normal products $\prod_{\alpha<\kappa} X_\alpha$ where each X_α is a Hausdorff space containing at least two points, and κ is an uncountable cardinal. However, the cubes 2^κ and I^κ for uncountable κ are reconstructible [PS**, 2.1].

Question 4.8.2. *Are there further nice ZFC-examples of normal spaces such that every card is non-normal—and are they reconstructible?*

Next, we have shown that it is consistent with the negation of CH that ω^* is reconstructible. Is this always the case? Note that our present proof of the fact that ω^* is non-reconstructible under CH uses, on several occasions, the full power of Parovičenko's theorem—which itself is equivalent to CH.

Question 4.8.3. *Is CH equivalent to the assertion that ω^* is non-reconstructible?*

Note that the answer to this question is negative if we can find a model of $ZFC + \neg CH$ in which every point of ω^* is $SU(\aleph_1)$-approachable.

Finally, we have seen in Theorem 4.5.2 that under CH, the card $\omega^* \setminus \{p\}$ for a P-point p of ω^* is not reconstructible. What about the other cards in $\mathcal{D}(\omega^*)$?

Question 4.8.4. *Are cards of ω^* obtained by deleting non-P-points reconstructible?*

Note that these cards are not reconstructions of the space ω^*. Indeed, if x_1 and x_2 are non-P-points of ω^*, then $\omega^* \setminus \{x_1, x_2\} \notin \mathcal{D}(\omega^*)$. Otherwise, if $\omega^* \setminus \{x_1, x_2\}$ is homeomorphic to some $\omega^* \setminus \{x\}$, then this card $\omega^* \setminus \{x\}$ has a two-point compactification with two non-P-points x_1 and x_2 in its remainder, contradicting Theorem 4.4.3.

We remark that the answer to Question 4.8.4 is consistently yes, by Theorems 4.5.9 and 3.2.8.

Chapter 5

The Stone-Čech compactification of $S_\kappa \setminus \{x\}$

5.1 Motivation and results

In this chapter we investigate the Stone-Čech compactifications of spaces $\omega^* \setminus \{x\}$ under CH, and of $S_\kappa \setminus \{x\}$ assuming $\kappa = \kappa^{<\kappa}$. The results of this chapter have been published in [PS15].

The questions discussed in this chapter arise from our Characterisation Theorems for Finite Compactifications 4.4.3 and 4.4.4: under CH, cards of ω^* have arbitrarily large finite compactifications, and all finite compactifications are homeomorphic to ω^*. Similarly, under $\kappa = \kappa^{<\kappa}$, cards of S_κ have arbitrarily large finite compactifications, and again, all finite compactifications are homeomorphic to S_κ. That finite compactifications essentially preserve the structure of ω^* and S_κ has been the crucial ingredient for our reconstruction results in the previous chapter.

Given that the finite compactifications of cards of ω^* and S_κ are structurally well-behaved, it is natural to wonder whether the same is true about the Stone-Čech compactification. Since the Stone-Čech compactification dominates all finite compactifications, we know immediately that it must have infinite remainder. But what are the precise mechanisms that increase the size of the Stone-Čech compactification of $\omega^* \setminus \{x\}$ under CH? Can one determine its size or is this independent of ZFC+CH? Do the Stone-Čech compactifications and the remainders of $\omega^* \setminus \{x\}$ and $S_\kappa \setminus \{x\}$ reflect structural properties of ω^* and S_κ? And does this depend on which point x we remove?

What makes these problems particularly interesting is a surprising link between the Stone-Čech compactifications of $\omega^* \setminus \{x\}$ and larger κ-Parovičenko spaces. We will see that taking the Stone-Čech remainder of $S_\kappa \setminus \{x\}$ will naturally give us S_{κ^+} (under some mild cardinal assumptions). This result establishes the higher cardinal analogue of the well known result that under CH, the remainder of $S_\omega \setminus \{x\}$ is homeomorphic to S_{ω_1}, as the remainder of $C \setminus \{0\}$ is homeomorphic to ω^* under CH by Theorems 4.1.1 and 4.1.2.

5.1.1 Results of this chapter The main result of this chapter is that under CH, the Stone-Čech remainder of $\omega^* \setminus \{x\}$ is an ω_2-Parovičenko space of cardinality $2^{2^{\omega_1}}$ and contains a family of 2^{ω_1} disjoint open sets, regardless of the choice of x. In general, under $\kappa = \kappa^{<\kappa}$, the Stone-Čech remainder of $S_\kappa \setminus \{x\}$ is a κ^+-Parovičenko space of cardinality 2^{2^κ} and contains a family of 2^κ disjoint open sets. Again, this holds for every point x in S_κ.

As a consequence, assuming $2^\mathfrak{c} = \omega_2$, the Stone-Čech remainder of any $\omega^* \setminus \{x\}$ is homeomorphic to S_{ω_2}, the unique ω_2-Parovičenko space of weight ω_2, independently of which point x gets removed. This is surprising, considering that subspaces $\omega^* \setminus \{x\}$ and $\omega^* \setminus \{y\}$ are typically very different.

Our main result also improves a theorem by A. Dow, who showed in 1985 that the remainder of $S_\kappa \setminus \{p\}$ is a κ^+-Parovičenko space provided that p is a P_κ-point [Dow85]. Consequently, we now have a more comprehensive answer to a question by S. Negrepontis [Neg69, p. 522]: under $\kappa^+ = 2^\kappa$, the space S_{κ^+} can be represented as the Stone-Čech remainder of any $S_\kappa \setminus \{x\}$, regardless of whether x has a nested neighbourhood base or not.

5.1.2 Organisation This chapter is organised as follows. In Section 5.2 we show that the Stone-Čech compactification preserves the κ-Parovičenko properties but increases weight: the Stone-Čech compactification of $S_\kappa \setminus \{x\}$ is a κ-Parovičenko space of weight 2^κ.

The next two sections are concerned with the remainder of $S_\kappa \setminus \{x\}$. First we show in Section 5.3 that all remainders are κ^+-Parovičenko spaces and conclude that under additional set theoretic assumptions they are all homeomorphic. In Section 5.4 we answer some questions about cardinal invariants of these spaces and prove that the Stone-Čech remainder of $S_\kappa \setminus \{x\}$ has cardinality 2^{2^κ}.

We present two proofs of this last fact. The first exhibits a connection to the space of uniform ultrafilters on κ. The second builds on the observation that under CH, the space ω^* is a *monotone F-space*, i.e. that it is possible to assign a separating clopen partition to pairs of disjoint open F_σ-sets, in the spirit of a monotone normality operator. We also give an explicit topological construction of a family of 2^κ disjoint open sets in the Stone-Čech remainder of $S_\kappa \setminus \{x\}$. Finally, Section 5.5 concludes the paper with some open questions.

I would like to thank Alan Dow for his help with the proof of Theorem 5.3.3 at the Spring Topology and Dynamics Conference 2014 in Richmond, Virginia.

5.2 The Stone-Čech compactification of $S_\kappa \setminus \{x\}$

In this section we investigate the Stone-Čech compactification of $S_\kappa \setminus \{x\}$ and show that $\beta(S_\kappa \setminus \{x\})$ is a κ-Parovičenko space of large weight. The result is best possible in the sense that $\beta(S_\kappa \setminus \{x\})$ cannot be a κ^+-Parovičenko space, for points in $S_\kappa \setminus \{x\}$ continue to have character κ. For the precise definition of the spaces S_κ, we refer the reader to Section 4.2.

5.2.1 The F_κ-space property is hereditary with respect to C^*-embedded subspaces We first prove a theorem about F_κ-spaces which generalises the corresponding result in Lemma 4.1.4(b) for F-spaces. It is interesting to note that the F-space property is even hereditary with respect to C^*-embedded subspaces, but the same is not true for F_κ-spaces—see the remarks after Lemma 4.3.7.

Theorem 5.2.1 (Suabedissen). *A strongly zero-dimensional space is an F_κ-space if and only if its Stone-Čech compactification is an F_κ-space.*

Proof. Suppose that X is an F_κ-space, and U an open set of βX of type less than κ. To show that U is C^*-embedded in βX, fix a continuous $[0,1]$-valued function f defined on U. The set $U \cap X$ is of type less than κ in X. By assumption, $f|_{U \cap X}$ can be extended to a continuous $[0,1]$-valued function F on X which again can be extended to a continuous $[0,1]$-valued function βF on βX. Now $\beta F|_U = f$, as both functions agree on the dense subset $U \cap X$.

For the converse, assume that βX is an F_κ-space. We show by induction on λ that X is an F_λ-space for all $\lambda \leq \kappa$. For $\lambda = \omega$ there is nothing to prove. So let $\lambda \leq \kappa$ be uncountable and assume that X is an F_μ-space for all $\mu < \lambda$. Let U be an open set of X-type $\tau < \lambda$. We aim to show that U is C^*-embedded in X.

There are X-clopen sets U_α such that $U = \bigcup_{\alpha<\tau} U_\alpha$. Write $V_\beta = \bigcup_{\alpha<\beta} U_\alpha$ and $W_\beta = \bigcup_{\alpha<\beta} \overline{U}_\alpha$ where the closure is taken in βX. Note that all $\overline{U}_\alpha$ are clopen subsets of βX and that $V_\beta = W_\beta \cap X$. Let f be a continuous $[0,1]$-valued map on $V_\tau = U$. For each $\beta < \tau$, the set V_β is C^*-embedded in X by induction hypothesis, and hence, $f|_{V_\beta}$ extends to X and then to βX. Let f_β be the restriction of this extension to W_β.

Since $V_\beta = W_\beta \cap X$ is dense in W_β for all $\beta < \tau$, the function $f_\tau = \bigcup_{\beta<\tau} f_\beta$ is well-defined on W_τ. And since every $\overline{U}_a$ is a clopen subset of βX, it is not hard to check that f_τ is continuous. Thus, f_τ extends from W_τ to βX by the F_κ-space property. The restriction of this extension to X is the required extension of f. $\square$

5.2.2 Counting the number of butterflies in $S_\kappa \setminus \{x\}$

In this section we recall results from the previous chapters about butterflies in S_κ, assuming $\kappa = \kappa^{<\kappa}$ throughout. Recall that a point x of a Hausdorff space X is called *strong butterfly point* if its complement $X \setminus \{x\}$ can be partitioned into open sets A and B such that $\overline{A} \cap \overline{B} = \{x\}$. The sets A and B are called *wings* of the butterfly point x. In Lemma 4.3.5 we proved that under $\kappa = \kappa^{<\kappa}$, every point of S_κ is a strong butterfly point.

When examining the proof of the Butterfly Lemma in more detail, we see that the recursive construction can be used to build 2^κ many distinct clopen subsets of $S_\kappa \setminus \{x\}$. Indeed, in the proof of Lemma 4.3.5, for every $\alpha < \kappa$ we have the choice of adding C' or D' to A_α. The collection of all clopen subsets of a space X is denoted by $\mathcal{CO}(X)$.

Lemma 5.2.2. *Assume $\kappa = \kappa^{<\kappa}$. For every point x in the space S_κ we have $|\mathcal{CO}(S_\kappa \setminus \{x\})| = 2^\kappa$.*

Proof. Since the weight of $S_\kappa \setminus \{x\}$ equals κ, there are at most 2^κ open sets, and hence at most 2^κ clopen sets in $S_\kappa \setminus \{x\}$.

Fix a neighbourhood base $\{U_\alpha \colon \alpha < \kappa\}$ of x in S_κ consisting of clopen sets. For each sequence $f \in 2^\kappa$ we build a butterfly with wings A^f and B^f of $S_\kappa \setminus \{x\}$. Following Lemma 4.3.5 this involves defining S_κ-clopen sets A^f_α and B^f_α for $\alpha < \kappa$ such that all (A^f_α, B^f_β)-pairs are disjoint and

$$(\star) \quad X \setminus U_\alpha \subset A^f_\alpha \cup B^f_\alpha \text{ for all } \alpha < \kappa,$$

$$(\ddagger) \quad A^f_\alpha \cup B^f_\alpha = A^g_\alpha \cup B^g_\alpha \text{ for all } f, g \in 2^\kappa \text{ and } \alpha < \kappa.$$

Moreover, for all $\alpha < \kappa$ we will arrange that there are non-empty disjoint clopen sets $C'_\alpha, D'_\alpha \subset U_\alpha$ such that for all $f \in 2^\kappa$

$$(\dagger) \quad A^f_\alpha \cap U_\alpha = \begin{cases} C'_\alpha, & \text{if } f(\alpha) = 1, \\ D'_\alpha & \text{if } f(\alpha) = 0, \end{cases} \qquad B^f_\alpha \cap U_\alpha = \begin{cases} D'_\alpha, & \text{if } f(\alpha) = 1, \\ C'_\alpha, & \text{if } f(\alpha) = 0. \end{cases}$$

By defining $A^f = \bigcup_{\alpha<\kappa} A^f_\alpha$ and $B^f = \bigcup_{\alpha<\kappa} B^f_\alpha$ for every $f \in 2^\kappa$ we conclude as in the original butterfly proof that both A^f and B^f are indeed disjoint clopen subsets of $S_\kappa \setminus \{x\}$.

To prove that we have constructed 2^κ different clopen sets we verify that $A^f \neq A^g$ if $f \neq g$. Suppose for a contradiction that $A^f = A^g$ for different f and g. Let α be the least ordinal such that $f(\alpha) \neq g(\alpha)$. Without loss of generality, $f(\alpha) = 0$ and $g(\alpha) = 1$. It follows from (†) that $C'_\alpha \subset A^f$ and $C'_\alpha \subset B^g$. From $A^f = A^g$ it follows that A^g intersects its other wing B^g in C'_α, a contradiction.

It remains to describe the recursive construction. Let $\alpha < \kappa$, $f \in 2^\kappa$, and assume that A^f_β and B^f_β have been defined for all $\beta < \alpha$. As in the original Butterfly Lemma 4.3.5, inside the non-empty set $U_\alpha \setminus \bigcup_{\beta<\alpha}(A^f_\beta \cup B^f_\beta)$ there are disjoint non-empty clopen sets C'_α and D'_α, not containing x. It follows from (‡) that C'_α and D'_α can be chosen uniformly, as $\bigcup_{\beta<\alpha}(A^f_\beta \cup B^f_\beta) = \bigcup_{\beta<\alpha}(A^g_\beta \cup B^g_\beta)$ for all $f, g \in 2^\kappa$. Also, for every $f \in 2^\kappa$ there exist clopen sets C_f and D_f partitioning S_κ and containing the sets $\bigcup_{\beta<\alpha} A^f_\beta$ and $\bigcup_{\beta<\alpha} B^f_\beta$ respectively. By defining

$$A^f_\alpha = \begin{cases} (C_f \setminus U_\alpha) \cup C'_\alpha, & \text{if } f(\alpha) = 1, \\ (C_f \setminus U_\alpha) \cup D'_\alpha & \text{if } f(\alpha) = 0, \end{cases} \qquad B^f_\alpha = \begin{cases} (D_f \setminus U_\alpha) \cup D'_\alpha, & \text{if } f(\alpha) = 1, \\ (D_f \setminus U_\alpha) \cup C'_\alpha, & \text{if } f(\alpha) = 0, \end{cases}$$

we see that A^f_α and B^f_α are as required. This completes the recursion. □

Compact clopen sets of $S_\kappa \setminus \{x\}$ are of course homeomorphic to S_κ. Recall that by Lemma 4.3.7 we know exactly how the non-compact clopen sets look like. They are all homeomorphic to S_κ minus a point.

5.2.3 The Stone-Čech compactification We show that the Stone-Čech compactification of $S_\kappa \setminus \{x\}$ satisfies the κ-Parovičenko properties. However, its weight grows exponentially.

Theorem 5.2.3. *Assume $\kappa = \kappa^{<\kappa}$. For every point x of S_κ the space $\beta(S_\kappa \setminus \{x\})$ is a κ-Parovičenko space of weight 2^κ.*

Proof. Let $X = S_\kappa \setminus \{x\}$. First we verify the κ-Parovičenko properties. Clearly, βX does not contain isolated points. By [CN74, 14.1], every open subspace of S_κ is a strongly zero-dimensional F_κ-space and hence so is βX by Theorem 5.2.1.

For the G_κ-space property, let $U = \bigcap_{\alpha<\lambda} U_\alpha$ be a non-empty intersection of clopen sets in βX for some $\lambda < \kappa$. To prove that U has non-empty interior it suffices to show that it intersects X.

Assume for a contradiction that λ is minimal such that U has empty intersection with X. Consider the sets $V_\beta = \bigcap_{\alpha<\beta} U_\alpha \cap X$. Then $\bigcap_{\beta<\lambda} V_\beta$ is empty, whereas, without loss of generality, $V_\beta \setminus V_{\beta+1}$ is non-empty for all $\beta < \lambda$. This last set is a non-empty intersection of less than κ-many open sets in S_κ and therefore contains a compact clopen set W_β.

Let f and g be disjoint cofinal subsets of λ. We define disjoint open sets

$$W_f = \bigcup_{\alpha \in f} W_\alpha \quad \text{and} \quad W_g = \bigcup_{\alpha \in g} W_\alpha$$

of type at most $\lambda < \kappa$ and claim that in S_κ both sets limit onto x, contradicting the F_κ-space property. Suppose the claim was false, e.g. that W_f does not limit onto x. Then the closure of W_f in S_κ, a compact set, would be contained in $X = \bigcup_{\beta<\lambda} X \setminus V_\beta$. But by construction of W_f, this open cover of $\overline{W_f}$ does not have a finite subcover, giving the desired contradiction.

It remains to calculate the weight of $\beta(S_\kappa \setminus \{x\})$. By Theorem [CN74, 2.21] and Lemmas [CN74, 2.23(b) & 2.24], the weight of βX for a strongly zero-dimensional space X is equal to the cardinality of $\mathcal{CO}(X)$. Now apply Lemma 5.2.2.

The proof could be finished at this point, but it might be convenient to spell out this chain of the three referenced results. The proof relies on a bijective correspondence between $\mathcal{CO}(X)$ and $\mathcal{CO}(\beta X)$. Let us consider the closure operator $cl_{\beta X} \colon \mathcal{CO}(X) \to \mathcal{CO}(\beta X)$ and the restriction operator $|_X \colon \mathcal{CO}(\beta X) \to \mathcal{CO}(X)$.

We claim that both operators are mutual inverses. Let $U \in \mathcal{CO}(X)$. As U is closed in X it follows $\overline{U}^{\beta X} \cap X = U$. Conversely, let $V \in \mathcal{CO}(\beta X)$. Since V is clopen and X is dense in βX, it follows from [Eng89, 1.3.6] that $V = \overline{V \cap X}^{\beta X}$. This proves the claim and in particular that $|\mathcal{CO}(\beta X)| = |\mathcal{CO}(X)|$.

We complete the proof of the theorem. For $w(\beta X) \leq |\mathcal{CO}(\beta X)|$, observe that since βX is zero-dimensional, is has a basis $\mathcal{A}$ consisting of clopen sets. From $\mathcal{A} \subset \mathcal{CO}(\beta X)$ it follows $w(\beta X) \leq |\mathcal{CO}(\beta X)|$. For $w(\beta X) \geq |\mathcal{CO}(\beta X)|$ observe that if $\mathcal{A}$ is a basis of βX, then by compactness, every clopen set of βX is a finite union of elements of $\mathcal{A}$. It follows $|\mathcal{CO}(\beta X)| \leq \aleph_0 \cdot w(\beta X)$ and hence $|\mathcal{CO}(\beta X)| \leq w(\beta X)$. □

It is an interesting question whether it is a ZFC theorem that the Stone-Čech compactification of $\omega^* \setminus \{x\}$ is a Parovičenko space. Alan Dow showed that the assertion that all open subspaces of ω^* are strongly zero-dimensional F-spaces is equivalent to CH [Dow83]. However, we are only interested in open subspaces of the form $\omega^* \setminus \{x\}$. And these may be strongly zero-dimensional F-spaces even under the negation of CH, for example when $\omega^* \setminus \{x\}$ is C^*-embedded in ω^*.

The result about the weight also follows from the fact that the remainder of $S_\kappa \setminus \{x\}$ has weight 2^κ, see Theorem 5.4.4.

In the case of $\kappa = \omega_1$, a tempting proof of Theorem 5.2.3 can be obtained by observing that $\omega^* \setminus \{x\}$ is pseudocompact (as it cannot contain a closed copy of ω), implying that every G_δ-set of $\beta(\omega^* \setminus \{x\})$ intersects $\omega^* \setminus \{x\}$. But this approach does not seem to generalise to $S_\kappa \setminus \{x\}$ without extra effort. At the same time, both approaches are somehow intertwined: the proof of Theorem 5.2.3 shows that $S_\kappa \setminus \{x\}$ is α-pseudocompact for all $\alpha < \kappa$ [Ken62, 2.2]. For more on α-pseudocompactness see [Ken62].

5.3 Structural results about the Stone-Čech remainder of $S_\kappa \setminus \{x\}$

5.3.1 Remainders are κ^+-Parovičenko spaces The previous section has shown that $\beta(S_\kappa \setminus \{x\})$ is a κ-Parovičenko space of quite large weight. Thus, these spaces are too large for the κ-Parovičenko properties to provide meaningful topological restrictions on the variety of potential spaces of that size.

For a better understanding we therefore turn to an investigation of the remainders of these spaces. The main result of this section is that every space of the form $(S_\kappa \setminus \{x\})^*$ is a κ^+-Parovičenko space of weight 2^κ, regardless of the choice of x. It follows that under $2^\kappa = \kappa^+$, all such remainders are homeomorphic to S_{κ^+}. In particular, it is consistent with CH that all remainders of spaces of the form $\omega^* \setminus \{x\}$ are homeomorphic.

Note that by Theorem 5.2.3, $(S_\kappa \setminus \{x\})^*$ is a compact zero-dimensional space. The next lemma is the first step for establishing the remaining Parovičenko properties.

Lemma 5.3.1. *Assume $\kappa = \kappa^{<\kappa}$. For every $x \in S_\kappa$ the space $(S_\kappa \setminus \{x\})^*$ has no isolated points.*

Proof. Suppose that $z \in (S_\kappa \setminus \{x\})^*$ is isolated. Then there is a clopen non-compact subset A of $S_\kappa \setminus \{x\}$ such that $A \cup \{z\}$ is compact and A is C^*-embedded in $A \cup \{z\}$. By Lemma 4.3.7, the set A is homeomorphic to $S_\kappa \setminus \{y\}$ for some $y \in S_\kappa$. However, this space does not have a one-point Stone-Čech compactification by Corollary 4.3.6, a contradiction. $\square$

Our next observation is that for a P_κ-point p, the space $S_\kappa \setminus \{p\}$ can be written as an *increasing* union of κ-many compact clopen sets, each homeomorphic to S_κ. And remainders of increasing unions of κ-Parovičenko spaces are well understood: the next theorem follows from a result by A. Dow from 1985 [Dow85, 2.2].

Theorem 5.3.2 (Dow). *Assume $\kappa = \kappa^{<\kappa}$. For a P_κ-point p of S_κ the space $(S_\kappa \setminus \{p\})^*$ is a κ^+-Parovičenko space.* ☐

Our key result is the following strengthening of Theorem 5.3.2.

Theorem 5.3.3. *Assume $\kappa = \kappa^{<\kappa}$. For every point x of S_κ the space $(S_\kappa \setminus \{x\})^*$ is a κ^+-Parovičenko space of weight 2^κ.*

We remark that our proof of Theorem 5.3.3 makes essential use of Dow's original theorem, and does not handle all cases simultaneously. Before presenting the proof, we discuss some interesting corollaries.

Corollary 5.3.4. *Assume $\kappa = \kappa^{<\kappa}$. The Stone-Čech compactifications $\beta(S_\kappa \setminus \{x\})$ and $\beta(S_\kappa \setminus \{y\})$ are homeomorphic if and only if $S_\kappa \setminus \{x\}$ and $S_\kappa \setminus \{y\}$ are homeomorphic.*

Proof. By Theorem 5.3.3, points in the ground space can be distinguished from points in the remainder of $\beta(S_\kappa \setminus \{x\})$ by their character. Thus, any homeomorphism between $\beta(S_\kappa \setminus \{x\})$ and $\beta(S_\kappa \setminus \{y\})$ restricts to a homeomorphism between $S_\kappa \setminus \{x\}$ and $S_\kappa \setminus \{y\}$. ☐

Corollary 5.3.5. *Assume $\kappa = \kappa^{<\kappa}$ and $2^\kappa = \kappa^+$. For every point x the space $(S_\kappa \setminus \{x\})^*$ is homeomorphic to S_{κ^+}.*

Proof. Write $\lambda = \kappa^+$. The condition $2^\kappa = \kappa^+$ implies $\lambda = \lambda^{<\lambda}$. Thus, for every x in S_κ, the space $(S_\kappa \setminus \{x\})^*$ is a λ-Parovičenko space of weight $\lambda = \lambda^{<\lambda}$ by Theorem 5.3.3, and hence, by Negrepontis' characterisation, homeomorphic to S_λ. ☐

Corollary 5.3.6. *Under the cardinal assumption $2^{\mathfrak{c}} = \omega_2$, the remainders of $\omega^* \setminus \{x\}$ and $\omega^* \setminus \{y\}$ are homeomorphic for all points x and y of ω^*.* ☐

This is especially interesting when compared to the fact that there are $2^{\mathfrak{c}}$ non-homeomorphic subspaces of the form $\omega^* \setminus \{x\}$, an observation that follows easily from Frolík's result [Fro67] that there are $2^{\mathfrak{c}}$ orbits in ω^* under its autohomeomorphism group, i.e. that ω^* is badly non-homogenous.

5.3.2 Proof of the main result The remaining part of this section is devoted to the proof of Theorem 5.3.3. For this, we first need a lemma about separation of disjoint open sets of small type in $S_\kappa \setminus \{x\}$. Note that a clopen non-compact set of $S_\kappa \setminus \{x\}$ is of S_κ-type κ by Lemma 4.3.8, but of $(S_\kappa \setminus \{x\})$-type 1.

Lemma 5.3.7. *Assume $\kappa = \kappa^{<\kappa}$. For any two disjoint open sets V and W of $(S_\kappa \setminus \{x\})$-type less than κ, there is a clopen set A of $S_\kappa \setminus \{x\}$ such that $V \subset A$ and $W \cap A = \emptyset$.*

Proof. The space $S_\kappa \setminus \{x\}$ is an F_κ-space by [CN74, 14.1], so V and W have disjoint closures in $S_\kappa \setminus \{x\}$. If V and W have disjoint closures in S_κ, by compactness there is a clopen set $A \subset S_\kappa$ separating V from W. Clearly, the intersection of A with $S_\kappa \setminus \{x\}$ is as required.

So assume that both V and W limit onto x. We build a butterfly such that its wings separate V and W. Let $\{U_\alpha \colon \alpha < \kappa\}$ be a clopen neighbourhood base for x in S_κ. By transfinite recursion we define S_κ-clopen sets A_α and B_α for $\alpha < \kappa$ not containing x such that all (A_β, B_γ)-, (A_β, W)- and (V, B_γ)-pairs are disjoint and $A_\beta \cup B_\beta$ covers $S_\kappa \setminus U_\beta$ for all $\beta, \gamma < \kappa$.

Once the construction is completed, we define disjoint open sets $A = \bigcup_{\alpha<\kappa} A_\alpha$ and $B = \bigcup_{\alpha<\kappa} B_\alpha$. Their union covers all of $S_\kappa \setminus \{x\}$ and $V \subset A$ and $W \subset B$ as required.

It remains to complete the recursive construction. Let $\alpha < \kappa$ and assume that A_β and B_β have been defined for all ordinals $\beta < \alpha$ satisfying the inductive requirements. Since V is of $(S_\kappa \setminus \{x\})$-type less than κ, it follows easily that $V \setminus U_\alpha$ is of S_κ-type less than κ. So the sets

$$(V \cup \bigcup_{\beta<\alpha} A_\beta) \setminus U_\alpha \quad \text{and} \quad (W \cup \bigcup_{\beta<\alpha} B_\beta) \setminus U_\alpha$$

are disjoint open sets of S_κ-type less than κ, and by the F_κ-space property there is a clopen partition (C, D) of S_κ separating them. We put $A_\alpha = C \setminus U_\alpha$ and $B_\alpha = D \setminus U_\alpha$, preserving the inductive assumptions. □

Proof of Theorem 5.3.3. Because of Theorem 5.3.2, it suffices to prove the theorem for non-P_κ-points x. So let us fix a non-P_κ-point x of S_κ and, by Lemma 4.2.7, an open subset $V \subset S_\kappa$ of type less than κ that contains x in its boundary. By the F_κ-space property, V is C^*-embedded in S_κ. In particular, if we write $X = S_\kappa \setminus \{x\}$ then the closure of V in X has a one-point Stone-Čech compactification. This means the set V limits onto precisely one point in the remainder of X. For the remaining parts of this proof, we denote this unique point in $\overline{V}^{\beta X} \setminus X$ by $\star$.

Claim 1. *For every clopen non-empty set C of X^* not containing $\star$ there is a clopen non-compact set $D \subset X$ which misses V such that $D^* = \overline{D} \setminus D = C$. Moreover, every such D is homeomorphic to $S_\kappa \setminus \{p\}$ for a P_κ-point p.*

To see that the claim holds, let C be a clopen subset of X^* not containing $\star$ and find a clopen non-compact subset E of X with $E^* = C$. There is a clopen neighbourhood U of x in S_κ such that $U \cap (E \cap V) = \emptyset$: otherwise, the closure of $E \cap V$ in βX would grow into the remainder. But

$$\overline{E \cap V}^{\beta X} \setminus X \subset \left(\overline{E}^{\beta X} \setminus X\right) \cap \left(\overline{V}^{\beta X} \setminus X\right) = C \cap \{\star\} = \emptyset,$$

a contradiction. Hence, for some suitable U, the clopen non-compact set $D = E \cap U$ of S_κ does not intersect V. And since the symmetric difference of D and E is compact, it follows from [CN74, 2.6d] that $D^* = E^* = C$, as claimed.

To see that D is homeomorphic to $S_\kappa \setminus \{p\}$ for a P_κ-point p, note that D and $X \setminus D$ form a butterfly around x in S_κ such that V witnesses that x is not a P_κ-point of $S_\kappa \setminus D$. Hence, D is homeomorphic to $S_\kappa \setminus \{p\}$ for a P_κ-point p by Corollary 4.3.9, completing the proof of Claim 1.

Claim 2. *Every compact clopen set of $X^* \setminus \{\star\}$ is a κ^+-Parovičenko space.*

By Claim 1, a compact clopen set of $X^* \setminus \{\star\}$ is of the form $(S_\kappa \setminus \{p\})^*$ for a P_κ-point p of S_κ. Claim 2 now follows from Theorem 5.3.2.

Claim 3. *The point $\star$ is a P_{κ^+}-point of X^*.*

To prove Claim 3 we show that whenever $\{C_\alpha \colon \alpha < \kappa\}$ is a collection of X^*-clopen sets not containing $\star$, there is a clopen set $B^* \subset X^*$ not containing $\star$ such that $\bigcup_{\alpha<\kappa} C_\alpha \subset B^*$.

From Claim 1, we know that for every C_α there is a clopen non-compact subset $D_\alpha \subset X$ such that $D_\alpha \cap V = \emptyset$ and $D_\alpha^* = C_\alpha$. In X, we write $F \subset^* G$ (read: F is *almost contained* in G) if there is a clopen neighbourhood U of x in S_κ, such that $F \cap U \subset G$. Write $F =^* G$ if $F \subset^* G$ and $G \subset^* F$. We will use the well-known fact that $C_\alpha \subset C_\beta$ in X^* if and only if $D_\alpha \subset^* D_\beta$ in X.

Similar to the proof of Lemma 5.3.7, our aim is to build a butterfly in X with wings A and B such that $V \subset A$ and $D_\alpha \subset^* B$ for all $\alpha < \kappa$. Clearly then, B^* is as required.

To construct such a butterfly, fix a neighbourhood base $\{U_\alpha \colon \alpha < \kappa\}$ of x in S_κ consisting of clopen sets. By recursion we will define families of S_κ-clopen sets $\{A_\alpha \colon \alpha < \kappa\}$ and $\{B_\alpha \colon \alpha < \kappa\}$ and a third family $\{E_\alpha \colon \alpha < \kappa\}$ of X-clopen sets such that for all $\alpha, \beta < \kappa$

1. $A_\alpha \cap B_\beta = \emptyset$ and $A_\alpha \cup B_\alpha = S_\kappa \setminus U_\alpha$,
2. $A_\alpha \cap E_\beta = \emptyset$ and $B_\alpha \cap V = \emptyset$,

3. $E_\alpha \subset D_\alpha$ and $E_\alpha =^* D_\alpha$.

Once the construction has been completed, it follows from (1) that $A = \bigcup_{\alpha<\kappa} A_\alpha$ and $B = \bigcup_{\alpha<\kappa} B_\alpha$ partition X into disjoint open sets. Condition (2) guarantees both $V \subset A$ and $E_\alpha \subset B$ and finally, condition (3) gives $D_\alpha \subset^* B$ as desired.

It remains to complete the recursive construction. Let $\alpha < \kappa$ and assume that A_β, B_β and E_β have been defined for all ordinals $\beta < \alpha$ satisfying the inductive assumptions. The set $S_\kappa \setminus \bigcup_{\beta<\alpha} A_\beta$ is an intersection of less than κ-many clopen sets in S_κ containing x. In particular, it is a non-empty intersection of less than κ-many clopen sets in $D_\alpha \cup \{x\}$. But by Claim 1, the point x is a P_κ-point with respect to $D_\alpha \cup \{x\}$, and hence there is a $D_\alpha \cup \{x\}$-clopen neighbourhood E'_α of x in this space such that $E'_\alpha \subset S_\kappa \setminus \bigcup_{\beta<\alpha} A_\beta$. Now put $E_\alpha = E'_\alpha \setminus \{x\}$ and note that $E_\alpha =^* D_\alpha$.

By Lemma 5.3.7 there exist X-clopen sets C and D partitioning X and containing the disjoint open sets

$$V \cup \bigcup_{\beta<\alpha} A_\beta \quad \text{and} \quad \bigcup_{\beta<\alpha} B_\beta \cup \bigcup_{\beta\leq\alpha} E_\beta$$

respectively. By defining $A_\alpha = C \setminus U_\alpha$ and $B_\alpha = D \setminus U_\alpha$ it is clear that A_α, B_α and E_α satisfy the inductive assumptions (1)-(3). The proof of Claim 3 is complete.

Claim 4. *The space X^* is a κ^+-Parovičenko space.*

For this, note that X^* is a zero-dimensional compact space without isolated points by Lemma 5.3.1. Thus, it remains to check for the F_{κ^+}- and the G_{κ^+}-space property.

Regarding the F_{κ^+}-space property, let F and G be disjoint open sets in X^* of type at most κ (which is the same as saying that F and G are of type less than κ^+). Suppose for a contradiction that there exists $z \in \overline{F} \cap \overline{G}$. By Claim 3, we must have $z \neq \star$. Pick a clopen neighbourhood C of z such that $\star \notin C$. Then $F \cap C$ and $G \cap C$ witness that C is not an F_{κ^+}-space, contradicting Claim 2.

For the G_{κ^+}-space property, suppose that $\{V_\alpha \colon \alpha < \kappa\}$ is a collection of clopen subsets of X^* such that $V = \bigcap_{\alpha<\kappa} V_\alpha$ is non-empty. If $\star \in V$ then by Claim 3, V is in fact a neighbourhood of $\star$, so has non-empty interior. Otherwise, for any $z \in V$, we can find a clopen neighbourhood C of z missing $\star$. By Claim 2, the non-empty set $C \cap V$ has non-empty interior in C, and hence in X^*. It follows that X^* is a κ^+-Parovičenko space. This proves Claim 4.

To complete the proof, recall that if a compact Hausdorff space is covered by two of its subspaces, not both subspaces can have strictly smaller weight [Eng89, 3.1.20]. Since βX has weight 2^κ by Theorem 5.2.3 and X has weight κ, the remainder X^* must have weight 2^κ as well. □

5.4 Cardinal invariants of the remainder of $S_\kappa \setminus \{x\}$

The results from the previous section have shown that under $2^\kappa = \kappa^+$, all remainders of the form $(S_\kappa \setminus \{x\})^*$ are homeomorphic to S_{κ^+}, which settles all further topological questions regarding cardinality, weight and cellularity of these spaces. However, we do not yet know what happens in absence of $2^\kappa = \kappa^+$. In this section we therefore investigate cardinal invariants of $(S_\kappa \setminus \{x\})^*$ without additional set-theoretic assumptions beyond its existence, i.e. $\kappa = \kappa^{<\kappa}$.

We first observe that for questions such as size, weight and cellularity of the remainder of $S_\kappa \setminus \{x\}$, it is enough to focus on the remainder of $S_\kappa \setminus \{p\}$ for a P_κ-point p.

Lemma 5.4.1. *Assume $\kappa = \kappa^{<\kappa}$. For every $x \in S_\kappa$ the space $(S_\kappa \setminus \{x\})^*$ contains a clopen copy of $(S_\kappa \setminus \{p\})^*$ for a P_κ-point p.*

Proof. By Corollary 4.3.9, the space $S_\kappa \setminus \{x\}$ contains a clopen copy of $S_\kappa \setminus \{p\}$ for a P_κ-point p. ☐

Every point of a κ-Parovičenko space has character at least κ and hence the whole space has, by [Eng89, 3.12.11], cardinality at least 2^κ. Thus, Theorem 5.3.2, Lemma 5.4.1 and [Eng89, 3.5.3] give us the chain of inequalities

$$2^{\kappa^+} \leq |(S_\kappa \setminus \{p\})^*| \leq |(S_\kappa \setminus \{x\})^*| \leq 2^{2^\kappa}.$$

In particular, we see once again that under $2^\kappa = \kappa^+$, the cardinality of $(S_\kappa \setminus \{x\})^*$ is clear. In the remaining part of this chapter we show without additional set-theoretic assumptions that $(S_\kappa \setminus \{p\})^*$—and hence $(S_\kappa \setminus \{x\})^*$ for all x—has maximal cardinality and cellularity.

We present two different proofs of these facts. Both approaches require interesting ideas which seem to be fundamentally different. The first proof was pointed out to me by the unknown referee of the paper corresponding to this chapter, and treats the Stone-Čech compactification as the maximal compactification, which projects onto every other compactification. Identifying the right object to project on, however, requires some creativity. Our second proof views the Stone-Čech compactification in terms of C^*-embeddedness. This proof can be considered the elementary one, using nothing beyond the basic κ-Parovičenko properties of S_κ. It also gives an intrinsic reason why the remainders in question have large cellularity: we explicitly construct 2^κ clopen sets in $S_\kappa \setminus \{p\}$ that extend to disjoint clopen sets in the remainder.

For the first proof we let κ be the discrete space of cardinality κ, and identify κ^* with the free ultrafilters on κ. The space $U(\kappa) \subset \kappa^*$ denotes the space of the *uniform* ultrafilters on κ, i.e. the ultrafilters whose members are subsets of κ of full cardinality.

Comfort and Negrepontis observed in [CN68, 2.4] that under CH we have $\omega^* \setminus \{p\} \cong \omega_1^* \setminus U(\omega_1)$ for P-points p, implying that the remainder of $\omega^* \setminus \{p\}$ has cardinality $2^{2^{\omega_1}}$ and cellularity 2^{ω_1}. Even though the equality $S_\kappa \setminus \{p\} \cong \kappa^* \setminus U(\kappa)$ does *not* generalise to larger cardinals $\kappa > \omega_1$, a similar approach can be used to compute the cardinality of $(S_\kappa \setminus \{p\})^*$.

Lemma 5.4.2. *Assume $\kappa = \kappa^{<\kappa}$. If $U \subset S_\kappa$ is open of type less than κ then every continuous function from U into any compact space extends to S_κ.*

Proof. We spell out the remark preceding [CN74, 6.22]. Let $f\colon U \to K$ be a continuous function into a compact space K. By Theorem [CN74, 6.22], the set $\overline{U}$ is a retract of S_κ. Let g be such a retraction.

By the F_κ-space property, U is C^*-embedded in S_κ, and hence $\overline{U} = \beta U$. Together, we see that the map $\beta f \circ g$ is the required extension. ☐

Lemma 5.4.3. *Assume $\kappa = \kappa^{<\kappa}$. For a P_κ-point p, there is a surjection from $S_\kappa \setminus \{p\}$ onto $\kappa^* \setminus U(\kappa)$.*

Proof. For a P_κ-point p we can write $S_\kappa \setminus \{p\} = \bigcup_{\alpha<\kappa} T_\alpha$ such that T_α is clopen in $S_\kappa \setminus \{p\}$, the union is strictly increasing and $T_\alpha \cong S_\kappa$ for all $\alpha < \kappa$. Also note that $\kappa^* \setminus U(\kappa) = \bigcup_{\alpha<\kappa} R_\alpha$ where $R_\alpha = \overline{\alpha}^{\beta\kappa} \setminus \kappa$.

We now use that S_κ is κ-co-universal, i.e. S_κ surjects onto every compact Hausdorff space of weight at most κ [CN74, 6.14]. Therefore, since $T_{\alpha+1} \setminus T_\alpha$ is homeomorphic to S_κ and $R_\alpha \cong \alpha^*$ is compact of weight $2^\alpha \leq \kappa^{<\kappa} = \kappa$, there are continuous surjections $f_{\alpha+1}\colon T_{\alpha+1} \setminus T_\alpha \to R_\alpha$ for every ordinal $\alpha < \kappa$.

And by transfinite recursion, for limits $\lambda < \kappa$ we may find by Lemma 5.4.2 a continuous $f_\lambda\colon T_\lambda \to R_\lambda$ extending $\bigcup_{\alpha<\lambda} f_\alpha\colon \bigcup_{\alpha<\lambda} T_\alpha \to R_\lambda$. It is clear that $f = \bigcup_{\alpha<\kappa} f_\alpha\colon S_\kappa \setminus \{p\} \to \kappa^* \setminus U(\kappa)$ is continuous and surjective. ☐

Theorem 5.4.4. *Assume $\kappa = \kappa^{<\kappa}$. For every point $x \in S_\kappa$ the remainder of $S_\kappa \setminus \{x\}$ has size 2^{2^κ} and contains a family of 2^κ disjoint open sets.*

Proof. Let p be a P_κ-point of S_κ. By Lemma 5.4.1 it suffices to prove the theorem for $(S_\kappa \setminus \{p\})^*$.

For the cellularity, we observe that by Lemma 4.2.2, every κ^+-Parovičenko space contains a family of 2^κ disjoint clopen sets. Now apply Theorem 5.3.2.

For the cardinality, note that by Lemma 5.4.3 there is a continuous surjection $f\colon S_\kappa \setminus \{p\} \to \kappa^* \setminus U(\kappa)$. The map f extends to a continuous surjection $\beta f\colon \beta(S_\kappa \setminus \{p\}) \to \kappa^*$ such that $U(\kappa) \subseteq \beta f((S_\kappa \setminus \{p\})^*)$. Since $U(\kappa)$ has cardinality 2^{2^κ} [CN74, 7.8], the result follows. $\square$

It is an interesting question whether the map f constructed in the previous theorem can be chosen to be perfect. If yes, [Eng89, 3.7.16] would show the existence of a continuous surjection from $(S_\kappa \setminus \{p\})^*$ onto $U(\kappa)$.

For the second proof we present an approach that is solely based on the Parovičenko properties.

The next lemma guarantees the existence of a *monotone cut operator* for all pairs of disjoint open sets of type less than κ in S_κ, in the spirit of a separation operator for monotone normality. We make this definition precise. For an ordered pair $\langle A, B\rangle$ of disjoint open sets of S_κ of type less than κ, we define a *cut* between them to be a clopen set $\mathcal{C}_{\langle A,B\rangle}$ such that

$$A \subset \mathcal{C}_{\langle A,B\rangle} \subset S_\kappa \setminus B.$$

Cuts in S_κ exist by the F_κ-space property. We write $\langle A, B\rangle \leq \langle A', B'\rangle$ if $A \subset A'$ and $B \supset B'$. A cut operator $\mathcal{C}$ is called *monotone* if $\langle A, B\rangle \leq \langle A', B'\rangle$ implies $\mathcal{C}_{\langle A,B\rangle} \subset \mathcal{C}_{\langle A',B'\rangle}$. The cut operator is called *symmetric* if $\mathcal{C}_{\langle A,B\rangle} = S_\kappa \setminus \mathcal{C}_{\langle B,A\rangle}$.

We need the following strengthening of the concept of a monotone cut operator. Let $\{U_\alpha\colon \alpha < \kappa\}$ be a decreasing neighbourhood base of clopen sets of a P_κ-point p such that $U_0 = S_\kappa$. For $\gamma < \kappa$ we call two subsets A and B of S_κ *γ-equivalent* if $A \cap U_\gamma = B \cap U_\gamma$. Also, we call A a *γ-subset* of B, and write $A \subset_\gamma B$, if $A \cap U_\gamma \subset B \cap U_\gamma$. We extend this idea to capture "local monotonicity" and write $\langle A, B\rangle \leq_\gamma \langle A', B'\rangle$ if $A \subset_\gamma A'$ and $B \supset_\gamma B'$. Note that for $\gamma \leq \delta < \kappa$, if $\langle A, B\rangle \leq_\gamma \langle A', B'\rangle$ holds then so does $\langle A, B\rangle \leq_\delta \langle A', B'\rangle$.

A cut operator $\mathcal{C}$ with the property such that for all γ, $\langle A, B\rangle \leq_\gamma \langle A', B'\rangle$ implies $\mathcal{C}_{\langle A,B\rangle} \subset_\gamma \mathcal{C}_{\langle A',B'\rangle}$ will be called a *strong monotone cut operator with respect to* $\{U_\alpha\}$. Every strong monotone cut operator is also monotone.

Lemma 5.4.5. *Assume $\kappa = \kappa^{<\kappa}$. Let $\mathcal{F}$ be the collection of all ordered pairs of disjoint open sets of S_κ of type less than κ. For every decreasing neighbourhood base of clopen sets $\{U_\alpha\colon \alpha < \kappa\}$ of a P_κ-point in S_κ with $U_0 = S_\kappa$, there exists a symmetric strong monotone cut operator $\mathcal{C}\colon \mathcal{F} \to \mathcal{CO}(S_\kappa)$ with respect to $\{U_\alpha\}$.*

Proof. By $\kappa = \kappa^{<\kappa}$ we may list $\mathcal{F} = \{\langle A_\alpha, B_\alpha\rangle\colon \alpha < \kappa\}$, such that permuted pairs are next to each other. Let $\{U_\alpha\colon \alpha < \kappa\}$ be a decreasing neighbourhood base of a

P_κ-point p consisting of clopen sets such that $U_0 = S_\kappa$. In addition we define $U_\kappa = \emptyset$, obtaining the technical advantage that for all pairs of sets, one is a γ-subset of the other for some $\gamma \leq \kappa$.

We define an operator $\mathcal{C}\colon \mathcal{F} \to \mathcal{CO}(S_\kappa)$ that satisfies for all ordinals $\beta, \delta < \kappa$:

$$\begin{aligned}
&(Cut) \quad A_\beta \subset \mathcal{C}_{\langle A_\beta, B_\beta\rangle} \subset S_\kappa \setminus B_\beta,\\
&(Sym) \quad \mathcal{C}_{\langle A_\beta, B_\beta\rangle} = S_\kappa \setminus \mathcal{C}_{\langle B_\beta, A_\beta\rangle} \text{ and}\\
&(Mon) \quad \forall \gamma < \kappa \, (\langle A_\delta, B_\delta\rangle \leq_\gamma \langle A_\beta, B_\beta\rangle \Rightarrow \mathcal{C}_{\langle A_\delta, B_\delta\rangle} \subset_\gamma \mathcal{C}_{\langle A_\beta, B_\beta\rangle}).
\end{aligned}$$

We proceed by transfinite recursion. Let $\alpha < \kappa$ and suppose we have defined cuts $\mathcal{C}_{\langle A_\beta, B_\beta\rangle}$ for all $\beta < \alpha$ satisfying the inductive assumptions for all $\beta, \delta < \alpha$.

Consider $\langle A_\alpha, B_\alpha\rangle$. If α is a successor and $\langle A_\alpha, B_\alpha\rangle = \langle B_{\alpha-1}, A_{\alpha-1}\rangle$ we define $\mathcal{C}_{\langle A_\alpha, B_\alpha\rangle} = S_\kappa \setminus \mathcal{C}_{\langle A_{\alpha-1}, B_{\alpha-1}\rangle}$. This assignment takes care of (Sym) and a straightforward calculation shows that also (Cut) and (Mon) are satisfied.

Otherwise, for all $\beta < \alpha$ we let $\gamma_\beta^\downarrow$ and $\gamma_\beta^\uparrow$ be the least ordinals such that $\langle A_\beta, B_\beta\rangle \leq_{\gamma_\beta^\downarrow} \langle A_\alpha, B_\alpha\rangle$ and $\langle A_\alpha, B_\alpha\rangle \leq_{\gamma_\beta^\uparrow} \langle A_\beta, B_\beta\rangle$. This is well-defined, as these relations are satisfied for at least κ. Let

$$\mathcal{C}_\alpha^\downarrow = \left\{ U_{\gamma_\beta^\downarrow} \cap \mathcal{C}_{\langle A_\beta, B_\beta\rangle} \colon \beta < \alpha \right\} \text{ and } \mathcal{C}_\alpha^\uparrow = \left\{ U_{\gamma_\beta^\uparrow} \setminus \mathcal{C}_{\langle A_\beta, B_\beta\rangle} \colon \beta < \alpha \right\}.$$

The idea is that these sets contain all parts of the previously defined cuts we have to be aware of in order to make our operator respect (Mon). Both sets have cardinality less than κ and consist of clopen sets.

We claim that the sets $A_\alpha \cup (\bigcup \mathcal{C}_\alpha^\downarrow)$ and $B_\alpha \cup (\bigcup \mathcal{C}_\alpha^\uparrow)$ are disjoint open sets of type less than κ. They are clearly open and of type less than κ.

We demonstrate only that $\bigcup \mathcal{C}_\alpha^\downarrow \cap \bigcup \mathcal{C}_\alpha^\uparrow = \emptyset$, as well as $A_\alpha \cap \bigcup \mathcal{C}_\alpha^\uparrow = \emptyset$, since the other cases are similar. To see that the first equality holds, we show that for any $\beta, \delta < \alpha$, the sets $U_{\gamma_\beta^\downarrow} \cap \mathcal{C}_{\langle A_\beta, B_\beta\rangle} \in \mathcal{C}_\alpha^\downarrow$ and $U_{\gamma_\delta^\uparrow} \setminus \mathcal{C}_{\langle A_\delta, B_\delta\rangle} \in \mathcal{C}_\alpha^\uparrow$ have empty intersection. By construction we have

$$\langle A_\beta, B_\beta\rangle \leq_{\gamma_\beta^\downarrow} \langle A_\alpha, B_\alpha\rangle \text{ and } \langle A_\alpha, B_\alpha\rangle \leq_{\gamma_\delta^\uparrow} \langle A_\delta, B_\delta\rangle.$$

With γ denoting the larger of $\gamma_\beta^\downarrow$ and $\gamma_\delta^\uparrow$ we may apply condition (Mon) to $\langle A_\beta, B_\beta\rangle \leq_\gamma \langle A_\delta, B_\delta\rangle$ and obtain $\mathcal{C}_{\langle A_\beta, B_\beta\rangle} \subset_\gamma \mathcal{C}_{\langle A_\delta, B_\delta\rangle}$. In particular, the sets $U_\gamma \cap \mathcal{C}_{\langle A_\beta, B_\beta\rangle}$ and $U_\gamma \setminus \mathcal{C}_{\langle A_\delta, B_\delta\rangle}$ have empty intersection, and since $U_\gamma = U_{\gamma_\beta^\downarrow} \cap U_{\gamma_\delta^\uparrow}$, the result follows.

To see the second equality, given any $U_{\gamma_\beta^\uparrow} \setminus \mathcal{C}_{\langle A_\beta, B_\beta\rangle} \in \mathcal{C}_\alpha^\uparrow$ we show that A_α and $U_{\gamma_\beta^\uparrow} \setminus \mathcal{C}_{\langle A_\beta, B_\beta\rangle}$ are disjoint. By construction we have $\langle A_\alpha, B_\alpha\rangle \leq_{\gamma_\beta^\uparrow} \langle A_\beta, B_\beta\rangle$, and therefore, by applying (Cut) to $\langle A_\beta, B_\beta\rangle$,

$$A_\alpha \cap U_{\gamma_\beta^\uparrow} \subset A_\beta \cap U_{\gamma_\beta^\uparrow} \subset \mathcal{C}_{\langle A_\beta, B_\beta\rangle} \cap U_{\gamma_\beta^\uparrow}.$$

It follows that

$$A_\alpha \cap (U_{\gamma_\beta^\uparrow} \setminus \mathcal{C}_{\langle A_\beta, B_\beta\rangle}) = \emptyset.$$

Now, since $A_\alpha \cup (\bigcup \mathcal{C}_\alpha^\downarrow)$ and $B_\alpha \cup (\bigcup \mathcal{C}_\alpha^\uparrow)$ are disjoint open sets of S_κ of type less than κ, there exist clopen sets containing the first set and not intersecting the second. Choose one and denote it by $\mathcal{C}_{\langle A_\alpha, B_\alpha\rangle}$.

This assignment clearly satisfies (Cut), so it remains to check for (Mon). Let $\beta < \alpha$ and suppose $\langle A_\beta, B_\beta\rangle \leq_\gamma \langle A_\alpha, B_\alpha\rangle$ for some $\gamma < \kappa$. Since we chose $\gamma_\beta^\downarrow$ minimal, we have $U_\gamma \subset U_{\gamma_\beta^\downarrow}$. By construction we have $U_{\gamma_\beta^\downarrow} \cap \mathcal{C}_{\langle A_\beta, B_\beta\rangle} \subset \bigcup \mathcal{C}_\alpha^\downarrow \subset \mathcal{C}_{\langle A_\alpha, B_\alpha\rangle}$ and therefore also $U_\gamma \cap \mathcal{C}_{\langle A_\beta, B_\beta\rangle} \subset \mathcal{C}_{\langle A_\alpha, B_\alpha\rangle}$. Thus, $\mathcal{C}_{\langle A_\beta, B_\beta\rangle} \subset_\gamma \mathcal{C}_{\langle A_\alpha, B_\alpha\rangle}$.

The case $\langle A_\beta, B_\beta\rangle \geq_\gamma \langle A_\alpha, B_\alpha\rangle$ is similar and the proof is complete. □

We now consider a variation of the butterfly construction which is tailored to P_κ-points. Let $\{U_\alpha \colon \alpha < \kappa\}$ be a decreasing neighbourhood base of a P_κ-point p of S_κ with $U_0 = S_\kappa$. We work through the "onion rings" $D_\alpha = U_\alpha \setminus U_{\alpha+1}$ and assign them either to the A- or the B-wing, following certain patterns. When compared to the original butterfly construction in Lemma 4.3.1, this adaptation has the advantage that one does not need to assign cuts at successor stages.

The *support* of a binary sequence $f \colon \kappa \to 2$ is the set $f^{-1}(\{1\})$.

Lemma 5.4.6. *Assume $\kappa = \kappa^{<\kappa}$ and let p be a P_κ-point of S_κ. There is a family $\{A^f \colon f \in 2^\kappa\}$ of clopen subsets of $S_\kappa \setminus \{p\}$ such that for all $f, g \in 2^\kappa$*

(1) A^f is non-empty whenever f has non-empty support,

(2) A^f is non-compact whenever f has unbounded support,

(3) $A^{1-f} \cap A^f = \emptyset$,

(4) if $f \leq g$ (pointwise) then $A^f \subset A^g$ and

(5) if $f = g$ eventually then there exists a clopen neighbourhood U of p such that $A^f \cap U = A^g \cap U$.

Proof. Let $\{U_\alpha \colon \alpha < \kappa\}$ be a decreasing neighbourhood base of p with $U_0 = S_\kappa$, and let $D_\alpha = U_\alpha \setminus U_{\alpha+1}$. Let $\mathcal{C}$ denote a fixed strong monotone cut operator from Lemma 5.4.5 with respect to $\{U_\alpha\}$. For each sequence $f \in 2^\kappa$ we build a butterfly with wings A^f and B^f around p. As always, this involves defining S_κ-clopen sets A_α^f and B_α^f for $\alpha < \kappa$ such that all (A_α^f, B_β^f)-pairs are disjoint and $A_\alpha^f \cup B_\alpha^f$ covers $S_\kappa \setminus U_\alpha$ for all α. The rules for the recursive construction are: for all ordinals $\alpha < \kappa$ set

$$A_{\alpha+1}^f = \begin{cases} A_\alpha^f \cup D_\alpha, & \text{if } f(\alpha) = 1, \\ A_\alpha^f, & \text{if } f(\alpha) = 0, \end{cases} \qquad B_{\alpha+1}^f = \begin{cases} B_\alpha^f, & \text{if } f(\alpha) = 1, \\ B_\alpha^f \cup D_\alpha, & \text{if } f(\alpha) = 0, \end{cases}$$

and if $\lambda < \kappa$ is a limit ordinal, put

$$A^f_\lambda = \mathcal{C}\left(\bigcup_{\beta<\lambda} A^f_\beta, \bigcup_{\beta<\lambda} B^f_\beta\right) \setminus U_\lambda \quad \text{and} \quad B^f_\lambda = (S_\kappa \setminus A^f_\lambda) \setminus U_\lambda.$$

The sets $A^f = \bigcup_{\alpha<\kappa} A^f_\alpha$ and $B^f = \bigcup_{\alpha<\kappa} B^f_\alpha$ are disjoint open and cover all of $S_\kappa \setminus \{p\}$. Thus, they define clopen subsets of $S_\kappa \setminus \{p\}$.

We claim the sets A^f satisfy assertions (1)-(5). It is clear that (1) is satisfied. Next, if f has unbounded support, then A^f limits onto p, i.e. is non-compact.

For (3) and (4), one shows by induction that $A^{1-f}_\alpha = B^f_\alpha$ and $A^f_\alpha \subset A^g_\alpha$ whenever $f \leq g$, using that the cut operator is symmetric and monotone, respectively.

For (5) suppose there exists an ordinal $\delta < \kappa$ such that $f(\alpha) = g(\alpha)$ for all $\alpha \geq \delta$. We show by induction that $A^f_\alpha \cap U_\delta = A^g_\alpha \cap U_\delta$. The claim is trivially true for $\alpha < \delta$. So let $\alpha \geq \delta$ and assume the claim holds for all smaller ordinals. The situation is clear for successors, so assume that α is a limit. By induction hypothesis, the pairs $\langle \bigcup_{\beta<\alpha} A^f_\beta, \bigcup_{\beta<\alpha} B^f_\beta \rangle$ and $\langle \bigcup_{\beta<\alpha} A^g_\beta, \bigcup_{\beta<\alpha} B^g_\beta \rangle$ are δ-equivalent and hence it follows from the properties of the cut operator that

$$\begin{aligned} A^f_\alpha \cap U_\delta &= \mathcal{C}(\bigcup_{\beta<\alpha} A^f_\beta, \bigcup_{\beta<\alpha} B^f_\beta) \cap (U_\delta \setminus U_\alpha) \\ &= \mathcal{C}(\bigcup_{\beta<\alpha} A^g_\beta, \bigcup_{\beta<\alpha} B^g_\beta) \cap (U_\delta \setminus U_\alpha) = A^g_\alpha \cap U_\delta. \end{aligned}$$

This completes the induction step and the proof. □

We now show how to use a family with properties (1)–(5) of Lemma 5.4.6 to push ultrafilters and almost disjoint families from κ through to the space $(S_\kappa \setminus \{p\})^*$. For a subset U of κ, let $\mathbb{1}_U \in 2^\kappa$ denote its characteristic function.

Second proof of Theorem 5.4.4. Let p be a P_κ-point of S_κ. Again, by Lemma 5.4.1, it suffices to prove the theorem for $(S_\kappa \setminus \{p\})^*$. For the cardinality, we show that there are at least 2^{2^κ}-many z-ultrafilters on $S_\kappa \setminus \{p\}$. Let $\{A^f : f \in 2^\kappa\}$ be a family of clopen sets of $S_\kappa \setminus \{p\}$ with properties (1), (3) and (4) of Lemma 5.4.6. For an ultrafilter $\mathcal{U}$ on κ, consider the family

$$\Phi(\mathcal{U}) = \{A^{\mathbb{1}_U} : U \in \mathcal{U}\}.$$

By (1) and (4), $\Phi(\mathcal{U})$ is a filter base for some clopen filter on $S_\kappa \setminus \{p\}$, and it follows from (3) that whenever $\mathcal{U}$ and $\mathcal{U}'$ are distinct ultrafilters on κ, then $\Phi(\mathcal{U})$ and $\Phi(\mathcal{U}')$ can only be extended to distinct z-ultrafilters on $S_\kappa \setminus \{p\}$. As there are 2^{2^κ} ultrafilters on κ the result follows.

For the cellularity, let $\{A^f\colon f \in 2^\kappa\}$ be a family of clopen sets of $S_\kappa \setminus \{p\}$ with properties (2)–(5) of Lemma 5.4.6. Recall that $\kappa = \kappa^{<\kappa}$ if and only if κ is regular and $\kappa = 2^{<\kappa}$ by Lemma 4.2.3. In particular, from $\kappa = 2^{<\kappa}$ we may conclude that there is an almost disjoint family $\mathcal{E}$ of size 2^κ on κ, i.e. a family whose members are subsets of κ of full cardinality such that the intersection of any two elements is of size less than κ [Kun80, II.1.3]. By property (2), the family

$$\Phi(\mathcal{E}) = \{A^{\mathbb{1}_E}\colon E \in \mathcal{E}\}$$

consists of non-compact clopen sets of $S_\kappa \setminus \{p\}$. We claim they have pairwise compact intersection. By regularity of κ, the intersection of distinct elements E and F of $\mathcal{E}$ is bounded by an ordinal $\delta < \kappa$. The function

$$f\colon \kappa \to 2,\ \alpha \mapsto \begin{cases} \mathbb{1}_F(\alpha) & \text{if } \alpha \leq \delta, \\ 1 - \mathbb{1}_E(\alpha) & \text{if } \alpha > \delta, \end{cases}$$

satisfies $\mathbb{1}_F \leq f$, and $f = 1 - \mathbb{1}_E$ eventually. By property (5), there is a clopen neighbourhood U of p such that $A^f \cap U = A^{1-\mathbb{1}_E} \cap U$. Then

$$A^{\mathbb{1}_E} \cap A^{\mathbb{1}_F} \cap U \subset A^{\mathbb{1}_E} \cap A^f \cap U = A^{\mathbb{1}_E} \cap A^{1-\mathbb{1}_E} \cap U = \emptyset,$$

where the first inclusion follows from property (4) and the last equality from (3). Thus, $A^{\mathbb{1}_E} \cap A^{\mathbb{1}_F}$ is a closed subset of the compact space $S_\kappa \setminus U$, hence compact.

It follows from [CN74, 2.6d] that whenever A and B are distinct elements in $\Phi(\mathcal{E})$ then $A^* = \overline{A} \setminus A$ and B^* are disjoint non-empty clopen subsets of $(S_\kappa \setminus \{p\})^*$. Since $\Phi(\mathcal{E})$ has cardinality 2^κ the result follows. $\square$

In fact, if X is any compact zero-dimensional F_κ-space with the property $\kappa = \kappa^{<\kappa}$, and $p \in X$ is a P_κ-point of character κ then the above methods show that $(X \setminus \{p\})^*$ has cardinality at least 2^{2^κ}, and contains a family of 2^κ many disjoint clopen sets.

The two proofs presented in this section showing that $(S_\kappa \setminus \{x\})^*$ has maximal cardinality seem to be of a very different flavour. Is there a unifying argument showing that the two approaches presented are in some sense equivalent?

5.5 Open questions

We list open questions that are motivated by results in this chapter. Corollary 5.3.6 shows that it is consistent with CH that all remainders of the form $(\omega^* \setminus \{x\})^*$ are homeomorphic. Is the negation of this result also consistent with CH?

Question 5.5.1. *Is it consistent with CH that for a P-point p and a non-P-point x of ω^* the remainders of $\omega^* \setminus \{p\}$ and $\omega^* \setminus \{x\}$ are non-homeomorphic?*

Theorem 4.5.9 shows that it is consistent with ZFC that every space $\omega^* \setminus \{x\}$ has a one-point Stone-Čech remainder. Under CH, Theorem 5.4.4 shows that the remainder of $\omega^* \setminus \{x\}$ has size $2^{2^{\omega_1}}$ for every x.

Question 5.5.2 (van Mill). *Which cardinalities, apart from 1 and $2^{2^{\omega_1}}$, can $(\omega^* \setminus \{x\})^*$ consistently have?*

In connection with Question 5.5.2 note that A. Dow has constructed a model in which $\omega^* \setminus \{x\}$ has a one-point Stone-Čech remainder if and only if x is not a P-point of ω^* [Dow97].

In Section 5.2 we mention the question whether one can prove without CH that $\omega^* \setminus \{x\}$ is a strongly zero-dimensional F-space.

Question 5.5.3. *Is it a ZFC theorem that $\beta(\omega^* \setminus \{x\})$ is a Parovičenko space?*

An affirmative answer to Question 5.5.3 would prove that $(\omega^* \setminus \{x\})^*$ is a compact F-space and hence rules out infinite cardinalities smaller than $2^{\mathfrak{c}}$ in Question 5.5.2.

Lastly, Theorem 5.3.3 gives us a large class of topologically distinct spaces whose remainders are κ^+-Parovičenko spaces. Can we find a precise description of spaces with that behaviour?

Question 5.5.4. *Is there a characterisation for which spaces X its remainder X^* is a κ^+-Parovičenko space?*

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