

Height inequalities and canonical class inequalities

Minhyong Kim

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1

This lecture is concerned with a priori bounds on the size of solutions to Diophantine equations. We will present a rather random collection of results with the intention to provide a flavor of the subject, with only a glimpse of something like a unifying principle. In the next lecture, we will make a better attempt to give coherence to our exposition, by describing the relation to *canonical class inequalities* from the theory of algebraic surface. The beginning of the lecture is intended to be entirely popular, while the technical language will grow denser towards the end.

It might be best to begin with an example from ordinary number theory which is likely to be more familiar than the arithmetic of function-fields, although it is with the latter that we will be most concerned, and for which there are definite results of a general nature. Consider the equation

$$x^3 + y^3 = 1729$$

to which we seek solutions in integral coordinates. Ramanujan observed the existence of the solutions (9,10) and (1,12), and noted that these were the only ones. How might one go about proving this assertion? Here is one method: The left hand side of the equation factors into

$$(x + y)(x^2 - xy + y^2)$$

so one notes that for any solution (x, y) , the quantity $x^2 - xy + y^2$ must divide 1729. In particular, $|x^2 - xy + y^2| \leq 1729$. But $x^2 - xy + y^2 = (x - y/2)^2 + 3y^2/4$ so we get

$$3y^2/4 \leq 1729 \Rightarrow |y| \leq 2\sqrt{1729/3} \approx 48$$

and by symmetry, we have the same bound for $|x|$. Thus, we arrive at a brute force algorithm for finding all the solutions, namely, by enumerating all the

possibilities up to the given bound, and observing that the pair above are the only ones. In general, this method shows that the integer solutions to

$$x^3 + y^3 = m$$

satisfy the bounds $|x|, |y| \leq 2\sqrt{|m|/3}$. This bound can be used to verify another of Ramanujan's claim from the same anecdote, that 1729 is the smallest natural number which can be written as a sum of cubes in two different ways.

Finding solutions to Diophantine equations such as the above, or such as $x^n + y^n = z^n$ is almost always a non-trivial problem, and simple resolutions are an extreme rarity. Also, if one is looking for rational solutions in addition to integral ones, the difficulties multiply, as may be seen even in the example above: $(20760/1727, -3457/1727)$ is also a rational solution, and in fact, the equation has infinitely many rational solutions. At present one can safely say that the general problem of finding all rational solutions to a two-variable Diophantine equation has a satisfactory *systematic* resolution only for equations of genus zero and some families of genus-one equations, in addition to scattered examples (e.g., Fermat's equation, some families of modular curves, certain hyperelliptic equations). On the other hand, for equations of genus at least two, Faltings' theorem assures us of the finiteness of solutions, so it makes sense to ask the question: Given an (irreducible) equation $f(x, y) = 0$ with rational coefficients and genus at least 2, can one find an a priori bound for the size of its rational solutions? Here size of a solution (x, y) may be defined as follows: Write $(x, y) = (p/r, q/r)$ with $(p, q, r) = 1$. Then the size or *height* of (x, y) is defined to be $\sup(|p|, |q|, |r|)$. Thus, giving a bound for the height ensures that the brute force search is a finite algorithm.

This question is known as the *effective Mordell conjecture*. While it is increasingly clear that this conjecture sits inside a coherent network of arithmetic-geometric investigations (Vojta's conjectures, ABC conjectures, Lang's conjectures, etc.), it is also fair to say that the problem is very difficult, and there are no positive results at present. A goal of these two lectures will be to convince the audience, by bringing out the underlying structure of the conjecture, that it is nevertheless reasonable to believe in its validity.

For this, we are motivated to study the same question in the context of Diophantine equations over function fields, which have traditionally been a source of inspiration and techniques that often end up influencing in a decisive way the corresponding problem over number fields. Furthermore, the function field set-up has given rise to many questions of algebraic geometry interesting in their own right, independent of their provenance in number theory.

So we move on to some examples in that direction. Let $f(x, y, t)$ be polynomial in $\mathbf{C}[x, y, t]$ and suppose we are interested in finding all rational function solutions to the equation

$$f(x, y, t) = 0$$

where by solution, we mean pairs $x(t), y(t) \in \mathbf{C}(t)$ such that $f(x(t), y(t), t) = 0$

as a function of t . For example, the equation

$$y^3 = x^4 - 6tx^3 + 11t^2x^2 - 6t^3x$$

has the solution $(t, 0)$ and the equation

$$(t^4 + t)y^3 = (t^3 + 1)x^4 + tx^3 - t^4$$

has the solution (t, t) . The well-known analogy here is between the field \mathbf{Q} of rational numbers and the field $\mathbf{C}(t)$ of rational functions over the complex numbers. There have been many results that give apriori bounds for the degrees of solutions, of which we start by mentioning one, in simplified form. Let $f(x, y, t)$ be an irreducible polynomial and let X be the closure in $\mathbf{P}^2 \times \mathbf{P}^1$ of the zero set $f(x, y, t) = 0$ in $\mathbf{C}^3 = \mathbf{C}^2 \times \mathbf{C}^1$. Thus, X is equipped with a map π to \mathbf{P}^1 and can be viewed as a projective surface fibered over \mathbf{P}^1 , or a projective family of curves parametrized by \mathbf{P}^1 , where for each $t \in \mathbf{P}^1$, X_t is the closure of $f(x, y, t) = 0$ in \mathbf{P}^2 . We assume that X is a non-singular surface, and that the general fiber of $\pi : X \rightarrow \mathbf{P}^1$, that is, (the projective plane curve corresponding to) $f(x, y, t) = 0$ for a general value of t , is connected and smooth. Further assume that the (x, y) -degree d of f is at least 4 and let e be the t -degree of the polynomial f . We define a two invariants s, k by looking at each curve $f(x, y, t) = 0$ for fixed t separately, and noting that only finitely many are singular, while the rest are smooth of genus $g = (d - 1)(d - 2)/2$. Denote the number of singular fibers by s . The singular fibers may contain some components that are rational curves, and these can be found readily. Denote by k the total number of rational components obtained thus.

All the quantities just defined can be easily computed if f is given. We will measure the size of rational solutions as follows: If $(x(t), y(t))$ is a pair of rational functions, then write $x = p(t)/r(t), y = q(t)/r(t)$ where $(p, q, r) = 1$, and define $h(x, y) = \text{supdeg}(p, q, r)$. Among many results on apriori bounds, we mention one, which is in many senses the sharpest [14]:

Theorem 1 (*S.-L. Tan*) *With the assumptions above, one has the following bound for the height of solutions in $\mathbf{C}(t)$ to the equation $f(x, y, t) = 0$:*

$$h(x, y) \leq \frac{(d^2 - 3d + 1)(s - 1) + k}{d - 3}$$

This allows us in principle to find all solutions to the equation by putting N to be the right-hand side of the inequality and doing a substitution

$$(x, y) = \left(\frac{p_0 + p_1t + \cdots + p_Nt^N}{r_0 + r_1t + \cdots + r_Nt^N}, \frac{q_0 + q_1t + \cdots + q_Nt^N}{r_0 + r_1t + \cdots + r_Nt^N} \right),$$

to find all solutions via Groebner basis techniques.

Where does this kind of inequality come from and why does one get so much stronger results than the number field case? One answer might be that the

algebraic geometry of surfaces gives us powerful tools for dealing with problems of this kind. More specifically, the main technical advantage over the arithmetic version of the problem is the use of differential forms on curves and surfaces. To explain this, let us recall the set-up for translating the Diophantine problem into algebraic geometry.

Given a polynomial f as above, we can view it as defining a curve X_η of genus at least two over $\eta = \text{Spec}(\mathbf{C}(t))$ which, in turn, is the generic point of $B = \mathbf{P}^1$. Thus, we can thicken the curve into a family

$$\begin{array}{ccc} X_\eta & \hookrightarrow & X \\ \downarrow & & \downarrow f \\ \eta & \hookrightarrow & B \end{array}$$

that is, an algebraic surface fibered over B . There is actually a canonical way of constructing this fibration, which is called the regular, relatively minimal model. Solutions of the original equations can then be interpreted as diagrams

$$\begin{array}{ccc} & & X \\ & \nearrow P & \downarrow \\ B & \xrightarrow{Id} & B \end{array}$$

or *sections* of the fibration, in the terminology of algebraic geometry. We will also be more generally interested in the multi-sections of the fibration, that is, diagrams

$$\begin{array}{ccc} & & X \\ & \nearrow P & \downarrow \\ T & \rightarrow & B \end{array}$$

where $T \rightarrow B$ is a finite covering, corresponding to solutions of the original equation with coefficients in some finite extension of the function field of B , i.e., an algebraic solution. Then given a projective embedding $X \hookrightarrow \mathbf{P}^n$, one can view P as being a map $P : T \rightarrow \mathbf{P}^n$ and use the geometry of X to give bounds on the degree of this map. Typically, the results are phrased in terms of bounds for

$$h(P) := \frac{1}{[T : B]} \deg(P^* \omega_{X/B}),$$

where $\omega_X := K_X \otimes f^* K_B^{-1}$ is the relative dualizing sheaf. Being intrinsic to the surface, it is easier to work with, and it is near enough to being ample to actually give height bounds.

For example, in the case of a surface in $\mathbf{P}^2 \times \mathbf{P}^1$ of bidegree (d, e) fibered over \mathbf{P}^1 , we get $\omega = \mathcal{O}(d-3, e)$. Also, when considering algebraic points, an important invariant is the relative discriminant $d(P) := (2g_T - 2)/[T : B]$, which measures the ramification in the map $T \rightarrow B$.

Then the height inequalities mentioned above follows from a geometric result,

Theorem 2 (*S.-L. Tan*) *Let X be a projective smooth surface, B a projective smooth curve. Let $f : X \rightarrow B$ such that the generic fiber is smooth and connected of genus at least two and the fibration is relatively minimal. Then for every section P , we have*

$$h(P) \leq (2g - 1)(d(P) + 3s) - \omega^2.$$

where s refers to the number of singular fibers in the fibration X/B .

This result is an immediate consequence of a logarithmic version of the Miyaoka-Yau inequality for algebraic surfaces of general type, due to Sakai and Miyaoka [11], [13]. As such, it uses the differentials Ω_X (in fact, even the log differentials) on the surface X . Once one has degree bounds of this sort, one knows that the possible rational points run through a finite algebraic family of divisor. On the other hand, if the fibration is not ‘isotrivial’, that is, is not birational to a product of curves even after a finite base extension, any rational point P in the above sense satisfies an intersection-theoretic inequality $\langle P, P \rangle < 0$. Thus, each P is rigid, so that the boundedness of the family actually implies finiteness. So inequalities of the above form are always much stronger statements than general finiteness results, that is, Manin’s theorem.

A class of results that uses the differentiation on the base B , which is often cited as being the distinguishing characteristic of function fields, involves the *Kodaira-Spencer map* of the fibration which measures the variation of the family.

There are many different kinds of K-S maps that one can associate with a family of curves, but the one that seems to have the most relation to the Diophantine properties seems to be the K-S map measuring the variation of the Jacobians (that is, the period integrals). We recall the definitions. If we have a fibration X/B as above, choose an open set $U \subset B$ such that $f : X_U \rightarrow U$ is smooth. We get an exact sequence

$$0 \rightarrow f^* \Omega_U \rightarrow \Omega_X \rightarrow \Omega_{X/U} \rightarrow 0$$

from which we get a map

$$f_*(\Omega_{X/U}) \rightarrow R^1 f_*(\mathcal{O}_X) \otimes \Omega_U$$

as the boundary map for the higher direct images. Passing to the limit over open sets U , we get

$$\text{K-S} : H^0(\Omega_{X_\eta}) \rightarrow H^1(\mathcal{O}_{X_\eta}) \otimes \Omega_\eta$$

This map is actually the cotangent map of the map to the moduli stack of principally polarized abelian varieties given by Jacobian of the curve X_η/η . It is a fact that the generic curve over a function field η will have a K-S map of full rank, that is, will be maximally varying. The Kodaira-Spencer map already occurs in the original proof that Manin [10] gave of the geometric Mordell

conjecture. This is because the K-S map is a ‘component’ of the Gauss-Manin connection. For example, the assumption that K-S map is of full rank is equivalent to the assumption that the Gauss-Manin connection is generated by the global relative differentials.

We state a few more height inequalities in this vein.

Theorem 3 (*Moriwaki [12] complemented by [7]*) *Let X and B be as above and suppose the K-S map of X/B has full rank. Then for any algebraic point P , we have*

$$h(P) \leq 4d(P) + 4c_2(X) - c_1(X)^2 - 4(g_B - 1).$$

This inequality is striking in that the dependence on the discriminant $d(P)$ is very weak. Serge Lang [9] has posed the question of finding the best possible constants for which one can prove inequalities of the form

$$h(P) \leq Ad(P) + B$$

for algebraic points P , where A and B should depend on the surface X . The results above are all motivated by this question, which is related to the algebraic geometric investigation of the geography of surfaces. In the theorem above, we see that one can take A independent of the surface X for a generic family.

Another result that should also be mentioned in this regard is the [15]

Theorem 4 (*Vojta*) *With X and B as above, we have*

$$h(P) \leq (2 + \epsilon)d(P) + O(1)$$

where $O(1)$ depends on X and $\epsilon > 0$, but not P .

This result is a function-field version of Vojta’s conjecture on algebraic points. This result can be extended to characteristic p under the assumption again that the K-S map is of full rank ([6]), which is a stronger assumption in positive characteristic than in characteristic zero (because of the possibility of inseparability). In relation to Lang’s question, one has made the A term (dependence on $d(P)$) very small at the expense of making B inexplicit.

Diophantine equations over function-fields of characteristic p are different again from the case of complex function fields. One advantage over complex function-fields, as far as the closeness to number fields goes, is the fact that one can have finite residue-fields. On the other hand, because of the presence of inseparable maps, many difficulties have a flavor entirely unique to this domain. Here is one general result:

Theorem 5 ([6]) *Suppose X/B is not isotrivial. Then*

$$h(P) \leq p^e(2g - 2)d(p) + O(\sqrt{h(p)})$$

where p^e is the degree of inseparability of the classifying map, that is the map $B \rightarrow \mathcal{M}_g$, for the family X/B .

The proof of this result also goes through the K-S map. Voloch has shown how a new proof of the Riemann hypothesis for curves over finite fields follows from this inequality. That is, the square root of the height inside the big-Oh is the same one that occurs in estimates for numbers of points on curves. So there are interesting arithmetical connections which have not at all been investigated in detail.

Because of the Frobenius maps, the assumption of non-isotriviality is important in characteristic p . That is, for example, if $X = B \times B$, and B is defined over a finite field, then some power of the Frobenius map will give a morphism $F^n : B \rightarrow B$, so morphisms $F^{kn} : B \rightarrow B$ for each $k > 0$. Thus we get infinitely many rational points of $X = B \times B \rightarrow B$. There are various more complicated ‘twisted’ situations where similar phenomena can arise. Purely inseparable maps are pervasive objects in characteristic p geometry, which must always be dealt with separately. For example, consider the following amusing problem: Given a fibration $X \rightarrow B$, how many diagrams of the following sort can one have?

$$\begin{array}{ccc} & & X \\ & \nearrow P & \downarrow \\ B & \xrightarrow{F^n} & B \end{array}$$

An elementary way of stating the question is to consider a polynomial $f(X, Y)$ with coefficients in a characteristic p function field K , and to start ‘twisting’ the equation with the Frobenius map of K , that, raising the coefficients of f to the p -th power. Denote by $f^{(n)}$ the polynomial twisted n times. Notice that if (X, Y) is a solution of $f(X, Y) = 0$, then (X^p, Y^p) is a solution of $f^{(1)} = 0$. That is, old solutions can be twisted to give solutions of the twisted equation. The question being posed, then, is whether or not we can keep getting *new* solutions as we twist f . These can be viewed as solutions to f in the purely inseparable tower K^{1/p^∞} . There is a sense in which ‘purely-inseparable points’ of this sort should not be that different from rational points since the map $F^n : B \rightarrow B$ is just a homeomorphism. This sentiment is captured by the following result, which is more natural to express directly in terms of a curve over a function field rather than a surface fibered over a curve:

Theorem 6 ([8]) *Suppose K is a function field of characteristic $p > 0$, and suppose C/K is a smooth curve of genus at least 2. Finally, suppose C is not isotrivial, that is, C does not become isomorphic to a curve defined over a finite field after some finite base-extension. Then $C(K^{1/p^\infty})$ is finite, where K^{1/p^∞} is the tower of all purely inseparable field extensions of K . In fact, if we assume that the curve has semi-stable reduction over the smooth model B of K , then the purely-inseparable points satisfy a height inequality*

$$h(P) \leq (2g_B - 2 + s)$$

where s is the number of singular fibers in the semi-stable reduction.

All the results mentioned use differential forms in one or more serious ways, posing for arithmeticians the problem of finding a number field substitute for differentiation. This problem has occupied the efforts many number-theorists for the last few decades.

2

The point of view we will emphasize here for understanding the height inequalities mentioned in the previous lecture is that they are all instances of *canonical class inequalities*. That is, if one didn't know of it beforehand, one would expect such height inequalities to exist, simply because of the pervasiveness of canonical class inequalities in algebraic geometry.

To elaborate a bit, a class of geometric objects will typically come with a set of numerical invariants which are often related by various natural formulas. As an example, one may consider Noether's formula for algebraic surfaces. In contrast to such equalities, one also has general *inequalities* between numerical invariants which are of aid in classification theory, and which are often more elusive than the equalities.

For example, for a surface of general type X over a field of characteristic zero, we have [11],

$$c_1(X)^2 \leq 3c_2(X),$$

known as the Miyaoka-Yau inequality, which arose from a long sequence of results due to Van-de-Ven, Bogomolov, Miyaoka and Yau [1].

That is, one attempts to classify surfaces of general type according to the pair of numerical invariants (c_1^2, c_2) . Those surfaces with a given pair of invariants form a moduli space in a natural way (Gieseker). However, it is still an ongoing project in classification theory to determine which of these schemes are non-empty, that is, which pairs of integers (n, m) actually occur as (c_1^2, c_2) . This problem is often referred to as one of determining the 'geography' of the surfaces of general type. If one plots the possible values in the plane, the M-Y inequality says that all the values lie above the line $c_2 = 3c_1^2$. For reference, we quote from the book of Barth-Peters-Van-de-Ven [1] some of the other facts known about the distribution of the (c_1^2, c_2) 's for surfaces of general type, i.e. some other canonical class inequalities:

- Clearly, $c_1^2 + c_2 \cong 0 \pmod{12}$;
- $c_1^2 > 0$ and $c_2 > 0$;
- $5c_1^2 - c_2 + 36 \geq 0$ if c_1^2 is even;
- $5c_1^2 - c_2 + 30 \geq 0$ if c_1^2 is odd; (Noether's inequalities).

The point we wish to emphasize is that such equalities and inequalities between numerical invariants are pervasive in geometry.

Another class of canonical class inequalities is associated to the geometry of the moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus $g \geq 2$. On $\overline{\mathcal{M}}_g$, there are several natural divisors classes. One is the class Δ of the boundary, corresponding to the

singular curves, and two more classes arise from the universal curve $f : \mathcal{C} \rightarrow \overline{\mathcal{M}}_g$. One is the first chern class of the Hodge bundle, $\lambda = c_1(\pi_*(\omega))$, where ω is the relative dualizing sheaf of the universal curve, and the other is the direct image $k = \pi_*(\omega \cdot \omega)$ of the self-intersection of ω which gives a divisor on $\overline{\mathcal{M}}_g$. It is well known that these divisors (and the universal curve) are only defined on the corresponding moduli stack, but there is also a well-known isomorphism of rational chow groups, which is where our classes lie. They are also \mathbf{Q} -Cartier divisors, and hence can be intersected with curves. From these classes one can construct numerical invariants for any family of stable curves parametrized by a projective curve. That is, if $f : X \rightarrow B$ is such a family, then we get a map $B \rightarrow \overline{\mathcal{M}}_g$, and so numbers δ_X, λ_X and k_X , obtained by pulling back the \mathbf{Q} -Cartier divisors above and taking degree. These can be interpreted directly in terms of the family itself: $k_X = \langle \omega_{X/B}^2 \rangle$ and $\lambda_X = \text{deg} f_* \omega_{X/B}$. δ_X is perhaps best explained by taking a minimal desingularization $p : X' \rightarrow X$ and taking the number of singularities in the fibers of $X' \rightarrow B$. Note that here, we are taking into account the fact that the original X may have been singular as a surface, not just have singularities in the fibers. X' is usually referred to as the regular semi-stable model of X . In any case, this way we arrive at natural invariants of a family X/B .

One general inequality due to Cornalba-Harris [2] and Xiao X is that for any stable family, we have:

$$(1 - 1/g)\delta \leq (2 + 1/g)\omega^2$$

That is, ω^2 is bounded below in term of the number of singularities. On the other hand, the invariants for X' as a surface can be related to the invariants of the family X/B , where we assume that B also has genus at least two, so that X' will be of general type. The relation arises from the exact sequence

$$0 \rightarrow f^* \Omega_B \rightarrow \Omega'_{X'} \rightarrow I_Z \omega_{X'} \rightarrow 0$$

where I_Z is the sheaf of ideals for the singular points of the fiber in X'/B . One also has $p^* \omega_X = \omega_{X'}$ ($X' \rightarrow X$ is a crepant resolution), so that $\omega_X^2 = \omega_{X'}^2$. If we put these facts together, we find that $c_1^2(X') = (\omega_X + K_B)^2$ while $c_2(X') = \omega_X \cdot K_B + \delta$, where we will abuse notation a bit by basically confusing X and X' . So Noether's formula, for example, reads

$$12 \text{deg} f_*(\omega) = \omega^2 + \delta,$$

while Noether's inequality translates into:

$$\delta \leq 5\omega^2 + 9(2g - 2)(2g_B - 2) + 36$$

which is similar to the C-H-X inequality, but weaker. This version of Noether's formula is also responsible for the fact that our three invariants are not independent, so that any other relation involving two of them, such as the preceding

inequality, actually has obvious consequences for the remaining one. How does the M-Y inequality apply to families? Simple arithmetic yields

$$\omega^2 \leq (2g - 2)(2g_B - 2) + 3\delta,$$

an *upper* bound for ω^2 in terms of delta. To summarize, canonical class inequalities in two different settings, that of surfaces of general type, and that of families of curves, are related in a natural way. We just described the relation in one direction, that is, from surfaces to curves, but it is not hard to work out implications that the C-H-X inequality, for example, has for Chern numbers of surfaces, by choosing a canonical model and a Lefschetz fibration.

One last result in this vein we mention is an improvement on the C-H-X inequality due to Eisenbud, Harris, and Mumford, which holds for generic families of stable curves:

$$\delta \leq (1 + o(1/g))\omega^2.$$

What do these results have to do with height inequalities? And why was it stated above that height inequalities are natural things to expect? The point is that height inequalities for rational points are nothing but canonical class inequalities for *families of pointed curves*. That is, they relate the invariants arising from $\overline{\mathcal{M}}_{g,1}$, the moduli space of stable pointed curves. For a pointed stable curve

$$\begin{array}{ccc} & & X \\ & P & \\ & \nearrow & \downarrow f \\ B & \xrightarrow{Id} & B \end{array}$$

the natural analogue of the sheaf ω is the sheaf $\omega(P)$. One can give many justification for this analogy, but the most important one is perhaps that $f_*(\omega^{\otimes 2}(P))$ at the generic point is the cotangent space to $\overline{\mathcal{M}}_{g,1}$ at the point corresponding to the pointed curve: that is, $\omega(P)$ governs the variation of a pointed curve in the same way that ω governs the variation of a curve. Thus, one gets the new invariants, $\deg f_*(\omega(P))$, $(\omega(P).\omega(P))$, and others which need not concern us at the moment. Now

$$(\omega(P).\omega(P)) = \omega^2 + 2(\omega.P) + P^2 = \omega^2 + (\omega.P)$$

since $P^2 = -(\omega.P)$ by the adjunction formula. So the height of the point P is the most natural new invariant that arises out of the situation when considering pointed curves rather than curves.

On the non-fibered side of the story, the relevant theory is that of *log surfaces*, the category of pairs (X, D) , consisting of a surface X and a divisor D on X . A point of view which has become very current in higher-dimensional classification theory, and espoused by Kollar in his Santa Cruz lectures, is that this category of pairs is a more natural and fundamental framework for classification than that of varieties by themselves, especially in the study of singularities.

Miyaoka and Sakai [13] have also proved a version of the M-Y inequality for log-surfaces. The generalization that is relevant for our purposes is in the situation where (X, D) is a pair such that X is smooth and D has normal crossings, and concerns the log differentials $\Omega(\log D)$. Suppose $c_1(\Omega(\log D)) = K_X + D$ is nef. Then

$$c_1(\Omega(\log D))^2 \leq 3c_2(\Omega(\log D)).$$

We outline how this leads to S.-L. Tan's inequality (The following proof was shown to me by N. Shepherd-Barron): Suppose $f : X \rightarrow B$ is as in the previous lecture, and we assume for convenience that the family is semi-stable. Let $P \subset X$ be the image of a section and let Σ be the sum of the singular fibers, and $S = f(\Sigma) \subset B$ the points lying below the singularities. Then $D = P + \Sigma$ is a normal crossing divisor, so we consider the log differentials $\Omega(\log D)$. There is an exact sequence

$$0 \rightarrow f^* \Omega_B(S) \rightarrow \Omega(\log D) \rightarrow \omega(P) \rightarrow 0$$

arising from the fact that the relative dualizing sheaf ω can be identified with the relative log differentials. Thus $c_1(\Omega(\log D)) = \omega + P + K_B + \Sigma$ so that

$$\begin{aligned} c_1^2(\Omega(\log D)) &= \omega^2 + 2 \langle \omega.P \rangle + \langle P^2 \rangle + 2 \langle \omega + P.K_B + \Sigma \rangle \\ &= \omega^2 + \langle \omega.p \rangle + 2 \langle \omega + P.K_B + \Sigma \rangle \end{aligned}$$

while

$$c_2(\Omega(\log D)) = \langle \omega + P.K_B + \Sigma \rangle .$$

So Miyaoka's inequality reads

$$\omega^2 + \langle \omega.P \rangle \leq \langle \omega + P.K_B + \Sigma \rangle = (2g - 1)(2g_B - 2 + s)$$

, since the intersection of a horizontal divisor and a divisor coming from the base is the product of the degrees. This is exactly the first height inequality we stated. One can use the lower bound for ω^2 given by C-H-X to eliminate the ω^2 in favor of the more explicit invariant δ . In any case, this derivation makes clear the point that a height inequality is nothing but a canonical class inequality. That is, and this is the main point we wish to convey, far from being a result of purely arithmetic interest, height inequalities sit squarely inside a well-established body of theory from the mainstream of algebraic geometry.

In the last fifteen years, much effort has been expended in developing an intersection theory for arithmetic schemes, that is, projective schemes that are flat over $\text{Spec}(\mathbf{Z})$. The inspiration of course has been intersection theory for varieties fibered over a curve, and this has motivated much of the reformulation of results from surface theory into a fibered setting, some of which we saw here. Once we see the results in fibered terms, they can then serve as a springboard for the study of arithmetic surface, that is, families of curves over $\text{Spec}(\mathbf{Z})$, for the formulation of conjectures and as well as for ideas and techniques that can

be emulated. The arithmetic intersection theory started by Arakelov, was developed by Faltings [4], Deligne [3], Gillet and Soulé [5], and many others and an increasing number of results from ordinary intersection theory are being carried over into this arithmetic setting. It has come to seem reasonable to believe that every result for algebraic surfaces has a compelling arithmetic analogue, whose proof will involve translations and refinements of the geometric theory. Faltings proof of Mordell's conjecture was to a large extent inspired by this viewpoint, and one spectacular application of the intersection theory was the second proof given by Vojta, which followed a line of reasoning entirely analogous to a function field theorem he had proved earlier. This of course was extended by Faltings to include finiteness theorems for certain sub-varieties of abelian varieties. In all, the benefit of having a clear understanding of the arithmetic geometry of function fields has become generally accepted, as well as the belief in a strong analogy and interplay between geometric and arithmetic theories of various sorts. Then, having demonstrated the occurrence and application of canonical class inequalities in a setting common to geometry and arithmetic, it seems reasonable to hope that canonical invariants for arithmetic surfaces will satisfy relations of a similar nature. This has already been demonstrated for a good class of equalities, including the adjunction formula, Riemann-Roch theorems, and Noether's formula [4]. Most recently, an important inequality, the positivity of ω^2 has been established by E. Ullmo. The other interesting inequalities have remained elusive, in particular, the one which would give us an effective Mordell conjecture. However, the remarkably coherent structures emphasized here, tying together the many different theories through precise analogies, can hardly allow us to believe otherwise than that they also remain merely to be discovered.

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