A Stochastic Partial Differential Equation Approach to Mortgage Backed Securities

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To Mom

I miss you

Abstract

The market for mortgage backed securities (MBS) was active and fast growing from the issuance of the first MBS in 1981. This enabled financial firms to transform risky individual mortgages into liquid and tradable market instruments. The subprime mortgage crisis of 2007 shows the need for a better understanding and development of mathematical models for these securities. The aim of this thesis is to develop a model for MBS that is flexible enough to capture both regular and subprime MBS.

The thesis considers two models, one for a single mortgage in an intensity based framework and the second for mortgage backed securities using a stochastic partial differential equation approach. In the model for a single mortgage, we capture the prepayment and default incentives of the borrower using intensity processes. Using the minimum of the two intensity processes, we develop a nonlinear equation for the mortgage rate and solve it numerically and present some case studies.

In modelling of an MBS in a structural framework using stochastic PDEs (SPDEs), we consider a large number of individuals in a mortgage pool and assume that the wealth of each individual follows a stochastic process, driven by two Brownian motions, one capturing the idiosyncratic noise of each individual and the second a common market factor. By defining the empirical measure of a large pool of these individuals we study the evolution of the limit empirical measure and derive an SPDE for the evolution of the density of the limit empirical measure.

We numerically solve the SPDE to demonstrate its flexibility in different market environments. The calibration of the model to financial data is the focus of the final part of thesis. We discuss the different parameters and demonstrate how many can be fitted to observed data. Finally, for the key model parameters, we present a strategy to estimate them given observations of the loss function and use this to determine implied model parameters of ABX.HE.

Contents

Notation

$[.,.]$	quadratic covariation
$[.]$	quadratic variation
δ_x	Dirac measure concentrated on x
$\triangle$	cemetry state
$\mathcal{P}(I)$	set of probability measure on $I = [0,1]$
$\mathcal{C}_{\mathcal{P}(I)}[0,1]$	the collection of all $\mathcal{P}(I)$-valued continuous functions on I
$C_I[0,1]$	the set of all $[0,1]$-valued continuous functions on $[0,1]$
$C^\infty([0,1])$	collection of all bounded and continuous functions on $[0,1]$ with continuous derivatives
$\bar{C}$	a class of test functions $\{\phi\} \in C^\infty([0,1])$ such that $\phi(0) = \phi(1) = 0$, $\phi'(x) \to 0$ as $x \to 0$ or $x \to 1$
$\langle .,\rangle$	duality between signed measures and function, for example $\langle \phi, \nu \rangle = \int_0^1 \phi d\nu$
$\langle .,.\rangle_0$	inner product in L^2 defined as $\langle \phi, \psi \rangle_0 = \int_0^1 \phi(x)\psi(x)\,dx$
$\lVert . \rVert_0$	L^2 norm defined as $\lVert \phi \rVert_0^2 = \int_0^1 \lvert \phi(x) \rvert^2 dx$
H_0	$= L^2([0,1])$: the Hilbert space with L^2 norm $\lVert . \rVert_0^2$ and inner product $\langle .,.\rangle_0$ defined above
$\mathcal{M}([0,1])$	set of finite Borel measures on $[0,1]$

Chapter 1

Introduction and motivation

The market for mortgage backed securities (MBS) has been one of the fastest growing and most important instruments in the US financial industry for more than 30 years. The securitization of mortgages gave rise to a secondary market for the mortgages by converting risky, non-rated individual loans into securities that are highly liquid and have low credit risks. Securitization is a process of combining a large number of individual mortgages into securities to be sold to investors. Large numbers of mortgage backed securities were issued, where the underlying mortgages were insured by government-sponsored corporations such as Fannie Mae, Ginnie Mae and Freddie Mac. Freddie Mac was privatised in 1989 and was permitted to buy privately secured mortgages in response to the growing demand of mortgages.

In the late 90's and early 2000's, the house prices increased dramatically and interest rates remained relatively low. Encouraged partly by a belief that the rise in house prices would continue in the medium to long term and partly by the US government legislation that made it easy for high-risk potential homeowners to obtain mortgages, the demand for mortgages increased. The private sector in mortgage securities expanded dramatically at this time, fueled by the issuance of subprime mortgages by Government Sponsored Enterprises (GSE) to borrowers with poor credit histories and weak documentation of income.

As the private securitization sector grew, the guidelines for mortgage qualifications began to change, and further riskier mortgages were issued where no proof of income or down payments was required. Adjustable rate mortgages (ARMs) were issued, where the borrowers were allowed to pay a variable monthly amount, chosen by the borrower. The subprime mortgage origination grew from $65 billions in 1995 to $665 billions in 2005, representing 26% increase.

It is well known that subprime mortgages performed much worse than the prime mortgages even during best times due to their risky nature. Mortgage Bankers Association (MBA)'s data shows that the delinquency rates (quarterly) for subprime mortgages were between 2.3% to 3.6% during 2003-2006 compared to 0.28-0.41% for prime mortgages. The delinquency rates on subprime mortgages increased to 5.4% by the end of 2007 and to 15% in 2010. The number of residential MBS (RMBS), especially subprime MBS, grew significantly in the run-up to the subprime mortgage crisis. Following the financial crisis in 2008, the outstanding amount of all RMBS in the US dropped from $12 trillion in the second quarter of 2008 to $11 trillion in the second quarter of 2009.

In [54], Simkovic argued that the recent crisis is at least the third failure of private mortgage securitization after the competition between non-GSE and GSE intensified in late 90's and early 2000's, and non-GSEs overtook GSE securitization in 2005, which in late 2007 and early 2008 witnessed a high number of defualts. The author argued that the dramatic changes occurred because of competition in non-GSE, making it difficult for organisations to decline loan requests for fear of losing business to other private securitization institutions.

The need for a model

The complexities involved in the securitization process also played a major role in the subprime mortgage crisis of 2008 that led to the financial crisis. After the dot com crisis of 2000, the US Federal Reserve lowered interest rates on treasury bills to \$1, and investment bankers found better investment opportunities in the housing market, which was booming. By buying thousands of mortgage loans, combining them into an MBS and selling the tranches of the MBS to other investors, banks and hedge funds, investment bankers yielded much better returns on their investment than \$1 in treasury bills would have provided. House prices had declined and a surplus of unowned houses created a difficult market, leaving investment bankers owning large numbers of largely worthless houses as homeowners defaulted. The series of securitization processes and MBS tranches from one investment bank to another made it difficult to establish ultimate liabilities, leaving investment banks with uncertain balance sheets and connected retail banks unwilling to lend.

The securitization process further complicates the modelling of MBS due to a lack of transparency in capturing the changes to servicing and underwriting standards for individual mortgages that form an MBS. An MBS contains many tranches of securities that may include different complex and difficult-to-value securities, such as interest only or principal only securities with varying characteristics. The current financial crisis made it vital to understand the securitization and characteristics of the underlying assets and to develop a mathematical model of mortgage backed securities.

The model

To understand the evolution of an MBS, we must track the underlying mortgages and the risks that affect the individual mortgages, as well as the economical factors that affect a large pool of mortgages. We will therefore model the wealth of individuals in a stochastic environment that when combined form a mortgage pool. The model should capture the default and prepayment risks that can affect MBS cash flows and comprise a dynamic model of mortgage payers that is flexible enough to capture both regular and subprime mortgage pools.

We consider the empirical measure of the system of individual mortgages with absorbing boundary conditions, and investigate the existence and uniqueness of the limit empirical measure. We then prove the absolute continuity of the limiting measure with respect to Lebesgue measure and obtain a stochastic partial differential equation for the density of the limiting measure and prove its existence and uniqueness. We aim to find the prices of mortgage backed securities by using the loss distribution which is a function of this limit empirical measure.

Modelling mortgage backed securities using empirical measures for the mortgage pool is a novel approach to model this difficult problem. The model we develop requires some background mathematical results of existence and uniqueness of the resulting stochastic partial differential equation and boundary conditions. In the stochastic analysis literature, many authors investigated the limits of empirical measures for particle systems. Kurtz and Xiong [**?**] assumed a particle system with three components with particle representation in the whole space, without boundary conditions, and provided results on existence and uniqueness for the final stochastic PDE. Lototsky [41] studied the smoothness of the solution to a parabolic equation with Dirichlet boundary conditions over a bounded interval (0,K) in C^2 provided that the initial condition is sufficiently smooth. Jin [28] and Bush et al. [8] used a similar model to Kurtz and Xiong to model large basket credit portfolios with boundary condition at the default barrier.

Outline

This work is divided into two parts. First part is a small project to model a single mortgage in an intensity framework by including both prepayment and default risks presented to the mortgage. The second and major part of the project focuses on modelling mortgage backed securities (MBS) and their modelling in a structural framework. We summarise the project as follows. In Chapter 2, we present an introduction and preliminary definitions that are associated with mortgage and mortgage backed securities. We define and explain the risks associated to these securities and present the models that market practitioners commonly use for these risks. We further discuss different types of MBS and present a literature review of these securities in Intensity based and Structural models.

Chapter 3 develops a model of a single mortgage in the intensity based framework. This model extends a preexisting model due to Gorovoy and Linetsky [23] by explicitly including mortgage default in the model using another hazard process. We also present the prerequisite definitions and results related to the hazard process in this chapter before formally introducing the model. We numerically solve the nonlinear equation for the mortgage rates and present some case studies to discuss and compare the results with that of [23]. We conclude the chapter by giving some comments on the model.

The remaining chapters in this project develop and implement a model for a large pool of the wealth of mortgage borrowers in a structural framework. Chapter 4 begins with a discussion on choosing the key factors that need to be captured in the model. We justify the choice of important factors that we need to consider to model a pool of MBS by briefly discussing the reasons of a shortfall in cash flow. We then define the model in detail for the wealth of each individual in the mortgage pool and the transformation to the distance to default. We further define the empirical measure for the pool and discuss the existence and uniqueness of the limit empirical measure.

Chapter 5 investigates the evolution of the limit empirical measure. We do this by convolving the measure with test functions and derive an evolution-equation for the limiting measure. We further provide boundary conditions on the limiting measure in this Chapter and discusses two cases for that. Assuming that the limit empirical measure has a density with respect to Lebesgue measure we drive a stochastic partial differential equation satisfied by the density. We also give an expression for the loss from the pool at any time t. We further present some results on the second moment of the limit empirical measure near both boundaries that we need in the proof of existence and uniqueness of the density with respect to Lebesgue measure in Chapter 6.

In Chapter 6, we justify our assumption that we made in Chapter 5 regarding the existence of a density with respect to Lebesgue measure. We prove that the limit empirical measure has an L^2 density at any time t with respect to Lebesgue measure provided that the initial measure has an L^2 density with respect to Lebesgue measure. We further show the uniqueness of this density in L^2. We prove this by convolving the measure valued solution with an appropriate kernel. We use the estimates on the second moment that we found in Chapter 5.

Chapter 7 is on numerical implementation and calibration of the model and is divided into two main parts. In the first section, we present the numerical algorithm to solve the stochastic PDE for the density of the pool using Monte Carlo technique. The second half of the Chapter discusses the calibration process to estimate the parameters of the model using the data from Federal Housing Finance Agency (FHFA), using which we found the parameters of the interest rate process and initial distribution. We also present a brief literature on the previous work done in pricing of the MBS. This is to recall that calibrating parameters from the market data has always been challenging in pricing MBS due to lack of data available through free sources. In the same chapter, we introduce a relatively new index of MBS namely ABX.HE index which was introduced in 2006 and only released until 2007 due to the crisis. These indices show the dramatic fall in the mortgage market. We discuss the issue of calibration to such indices, our focus is on the methods as current values of these indices are very low. We provide a detailed description of the index, its construction and how it works. We provide a strategy to use the information and data on the index to price the index and estimates for remaining set of parameters.

Chapter 8 gives a description of future work and concludes the project.

Chapter 2

Preliminaries and mathematical background

In this chapter, we present the preliminary definitions regarding mortgages and mortgage backed securities (MBS). Individual mortgages do not remain in the possession of a lender, instead the lender sells individual mortgages to an investor of a pool of mortgages. The pools and then sold to other investors as a financial security. Securitization is the process of transforming mortgages into securities. The principle advantage of securitization to investors is that they spread the risk of decreased individuals' cash flow to a larger number of parties. The securities also provide a significantly liquid market to mortgage loans with considerable diversification that attracts market participants to invest. The liquid market of the securities and diversified risks lead eventually to lower interest rates for residential mortgages. This also allows the lenders of individual mortgages to use the funds they obtain from selling the mortgages further to investors to initiate more mortgages. We discuss the basic structures of the securities and the risks involved in these securities. Further, we review the literature and present the different techniques for modelling the securities.

2.1 Mortgage

A mortgage is a loan that is secured by some specified property. It is an agreement between the lender and the borrower that the borrower will repay the loan through a series of monthly payments. If the borrower fails to make (typically three) contractual payments, the lender has the right to repossess the property to redeem the incurred loss. The borrower who takes the loan on a real estate is termed a mortgagor and is obliged to the conditions of the loan. A mortgage lender who lends money secured by a mortgage on a real estate is termed a mortgagee. Since the market of mortgage backed securities is most active in the US, we will focus on the US housing market and discuss US conventions within this framework. The more common mortgages originate with a $30-$year original term, but loans with shorter terms such as 15-year and 10-year are also sold.

The properties on which a borrower seeks a loan can be divided into residential properties and non-residential properties. Residential properties include houses that accommodate between one and four families, or apartments where more than four families reside. Non-residential properties include office buildings, retail malls, hotels, etc. Mortgages can also be classified by interest rate type. Fixed rate mortgages (FRM) have an interest rate that is set at the time of origination of the mortgage and remain fixed throughout the life of the mortgage. Adjustable rate mortgages have an interest rate that changes over the life of the mortgage. Rates are reset at particular points in the mortgage lifetime. We focus here on fixed rate mortgages in which the underlying asset is a residential property and the borrower is a homeowner.

2.2 Securitization

Securitization of mortgages is a complex process that involves combining hundreds of individual mortgages of similar characteristics that are issued by banks to individuals. The process begins with the purchase of a mortgage, which the bank then assigns to a trust. The trust groups the mortgages into pools. The trust securitizes the pool and issues mortgage backed securities, which turns the illiquid assets of individual mortgages into securities that can be bought, sold, and traded in the market.

In 1938, the US government created the government-sponsored corporation Fannie Mae to ensure a liquid secondary market for the mortgages insured by the Federal Housing Administration (FHA). In 1968, Fannie Mae split to become Fannie Mae and Ginnie Mae in order to support other insured mortgages such as those of the Veterans Administration and the Farmers Home Administration. Later in 1970 Fannie Mae was authorised to purchase private loans, and Freddie Mac was established. A passthrough security, where the principal and interest payments are paid to security investors after deduction of the servicing fee, was first generated by Ginnie Mae in 1968. This was followed in 1971 by Freddie Mac issuing its first passthrough security composed of private mortgages. Fannie Mae issued its first passthrough security, called a mortgage backed security, in 1981. Since then many other types of securities were introduced to attract investors of different types; examples include collateralised mortgage obligations (CMO), stripped MBS, adjustable rate MBS, agency and non agency MBS, and residential and commercial MBS. Investment Banks, Real Estate Mortgage Investment Conduits (REMICs) and the Real Estate Investment Trusts (REITs) are also securitize mortgages known as private-label mortgage securities.

The government sponsor trusts guarantee the cash flow of the securities in case of defaults (fully or partially). Ginnie Mae guarantees the monthly interest and principal payment of its mortgages but this only secures the mortgages underlying 4% of the total MBS issued in 2006. Fannie Mae and Freddie Mac purchased the mortgages that met certain conditions, for example certain loan to value ratio and credit history of the borrowers, and guarantee the monthly payments of the mortgages that make up 40% of the MBS industry as of 2006. The remaining 56% of mortgages are privately packed and securitized.

In 2007, 18% of Fannie Mae purchased single family mortgages were below Fair Isaac Corporation (FICO) score of 660, a score which determines the credit worthiness of the borrower. 16% were interest-only mortgages purchased by Fannie Mae. Freddie Mac also continued to lower its standard and in 2007 20% of its single family purchased mortgages were interest-only. Due to lowering credit standards and the growth in the number of subprime mortgages, more than 90% shares of both Fannie Mae and Freddie Mac fell from their one-year prior levels by August 2008. In September 2008, both Fannie Mae and Freddie Mac were placed into conservatorship of the Federal Housing Finance Agency (FHFA), transferring control of both Fannie Mae and Freddie Mac to the FHFA.

2.3 Mortgage backed securities

An MBS is a security that is backed by the interest and principal payments of a set of loans, usually called a pool. The cash flow of an MBS depends upon cash flows of the underlying mortgages in the pool. Broadly speaking, mortgage pools are created by aggregating a large number of mortgages with similar attributes such as the same maturity, interest rates, loan balance, etc. The pool can then be sold to investors in the form of different structures according to the distribution of principal and interest payments. The common structures of an MBS are: pass-through MBS, collateralized mortgage obligation (CMO) and stripped MBS.

2.3.1 Pass-through MBS

A pass-through MBS is the basic and simplest structure of an MBS and all other structures are based on this. The cash flow depends upon the individual cash flows of the mortgages in the pool. The pool can consist of between a few and thousands of mortgages. The monthly payments that are generated from the underlying mortgages are passed from the mortgagor through the issuer to the investors in the security. The issuer cuts the servicing, guarantee and other fees from these monthly payments before passing them to investors.

In this security, the issuer sells the ownership of the underlying pool of mortgages directly to the investors, and the investors are the ultimate providers of funds to homeowners.

2.3.2 Collateralized mortgage obligations

To broaden the appeal of mortgage backed products to fixed income investors, another type of MBS was developed, called a collateralized mortgage obligation. The CMO structure is a mechanism to reallocate the cash flow from the pool of mortgages into different classes- called tranches- of different maturity. The earliest CMO is a straightforward product with fixed principal, fixed interest rate and a fixed maturity. The principal payments are passed through to tranches in order of their seniority. For example, if we have classes A, B and C, the principal payment is first passed through to class A, and the principal payment to class B is paid after class A has been paid off, after which the principal payment is directed to tranche C. The interest payment is proportional to the principal remaining for each tranche. In case of prepayments, the amount repaid is directed to the tranche receiving the principal payments at the time of prepayment. Tranche A is paid off first with a shorter life providing less risk and a small return, whereas tranche C is paid off over a longer period, providing more risk and a high return.

Due to the prepayment right being given to the mortgagor, the cash flow of the pool is uncertain. To reduce the exposure to the prepayment and cash flow uncertainty other classes such as Planned Amortization Classes (PACs) and Targeted Amortization Classes are sometimes introduced. The purpose of these classes is to have a schedule of principal repayments for some range of prepayment speed to reduce the uncertainty risk of cash flow due to prepayments.

2.3.3 Stripped MBS

In a CMO, the cash flow of a pool is divided into different tranches according to seniority level. The stripped MBS creates a cash flow depending upon principal and interest payments. This divides the original pass-through security into two classes: principal only (PO) and interest only (IO). The interest only class receives no principal, and the principal only class receives no interest payments. The PO security holders receive benefits from lower interest rates because the faster the prepayments are the faster they get their principal back. On the other hand, the IO security holders prefer high interest rates resulting in lower prepayment. The IO holders make more interest in case when the principal is unpaid for a longer time.

2.4 Risks associated with MBS

In this section we discuss the risks involved in MBS. It is important to understand the general prepayment and default risks in residential MBS. Residential mortgages allow curtailments or early payments of the loan in part or in full at any time during the life of contract, mostly without penalty. Therefore the monthly cash flow of an MBS is not known in advance and presents the risk to those investing in an MBS. The valuation and risk management of residential MBS become critical and challenging due to the risks involved.

2.4.1 Prepayment and prepayment risk

The cash flow of a fixed rate MBS, including principal and interest payments, remain constant throughout the life of the security. The borrower has a right to pay more than the scheduled monthly payment at any time during the life of the security. This extra amount is used to pay down the remaining principal which shortens the length of the security as well as causing a decrease in the expected interest payments to the lender. Most of the time and especially in residential mortgage backed securities, this happens without any penalty. If the borrower pays back the whole remaining amount of the mortgage, then it is called a prepayment. If only a portion of the outstanding principal is paid off then this is called curtailment or partial prepayment.

There could be many reasons for prepayments, such as:

- the sale of the property due to death or divorce or any other reason,
- the destruction of the property due to fire or any other disaster,
- winning a lottery or personal savings over time, and
- refinancing.

The borrower can refinance his property at a lower interest rate to reduce the monthly scheduled payments. This typically happens when interest rates decline from the prevailing level to a level that after paying the transaction costs and other fees the borrower can pay off his previous loan and is able to reduce his monthly payments. In residential mortgages the borrower is a homeowner and may not be well aware of market changes and decide not to refinance even when it is optimal to refinance. This feature of MBS makes the valuation of these securities more challenging as it is not known if or when the borrower will refinance.

2.4.2 Delinquency and default

Delinquencies and defaults are also an important feature of these securities. If the borrower is unable to make the scheduled monthly payments on the stated date then the loan becomes delinquent. There are some classifications for the delinquency periods and if all the collection efforts have failed does the loan end in default. The Office of Thrift Supervision method, the most common convention to classify the delinquencies, is as follows [15]:

- $30-60$ days late: 30 days delinquent,
- $60-90$ days late: 60 days delinquent, and
- more than 90 days late: 90+ days delinquent.

If a borrower is unable to make his monthly interest or principal payments for more than 90 days then the lender may elect to serve a legal notice of default to the borrower allowing the lender to take over the property in order to repay the remaining loan.

The prepayment and default in these securities make their modelling challenging. There is a vast literature on the modelling of these securities, which we review in the next section.

2.5 Models for prepayment and default

In this section we present the models that are widely accepted by researchers in the modelling of mortgage backed securities. These models are based on empirical default and prepayment rates and are discussed in detail in [15]. The models only describe the relationship between mortgage defaults and mortgage prepayments and the age of the mortgage, but do not consider the effects of increase in the value of underlying real estate assets on default and the effects of low interest rates on prepayment.

2.5.1 Models for prepayment

The most commonly used model for prepayment by practitioners and researchers is the Public Securities Association (PSA), now known as the Bond Market Association, prepayment model. The model is based on the fact that during the early months of the contract, mortgagors are less likely to repay the loan. This model is time dependent and only considers the changes in prepayment speed with the age of the mortgage loan, but does not consider the prepayment due to refinancing at lower interest rates. On a 30-year mortgage loan the PSA standard benchmark assumes a prepayment speed of 0.2% Conditional Prepayment Rate (CPR) in the first month, with an increase of 0.2% per month per year until 6% per year is reached in the 30th month. The CPR is defined in terms of single monthly mortality rates (SMM), where SMM defines the most basic form of prepayment, as below:

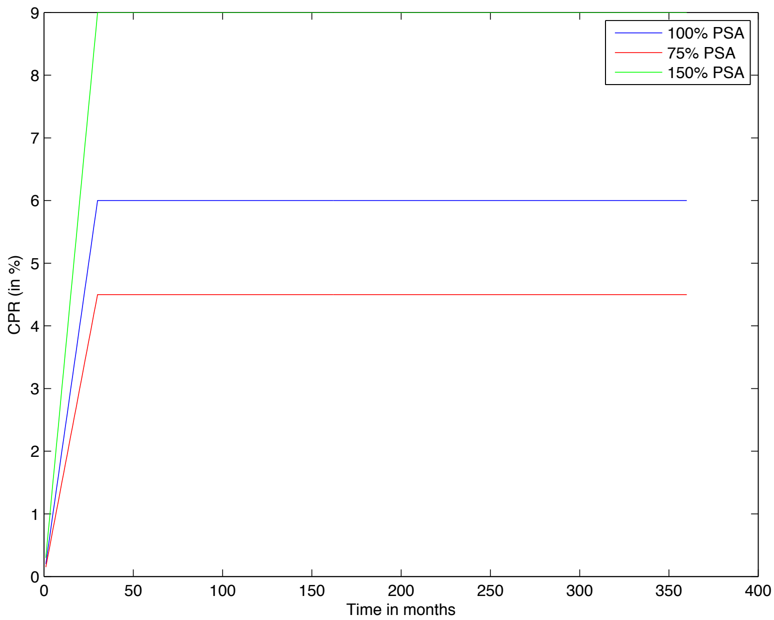


Figure 2.1: PSA prepayment curve

$$\mathrm{SMM} = \frac{\text{total payment} - [\text{scheduled interest} + \text{principal payment}]}{\text{unpaid principal balance} - \text{scheduled principal payment}}.$$

The CPR is defined as

$$\mathrm{CPR} = 1 - (1 - \mathrm{SMM})^{12}.$$

Mathematically, we write

$$b\left(at\mathbf{1}_{\{t<T^*\}} + aT^*\mathbf{1}_{\{t\geqslant T^*\}}\right). \tag{2.1}$$

$b = 1$ corresponds to a 100% PSA curve that is used as a benchmark. Other PSA parameters are $a = 0.024$ and $T^* = 30/12 = 2.5$ years. Figure 2.5.1 shows different percentages of the PSA prepayment model.

2.5.2 Model for default

The most commonly used default assumption is the PSA standard default assumption (SDA) benchmark. The model assumes annual rates of the conditional default rate (CDR) on a 30-years mortgage defined as

$$\mathrm{CDR}_t = 1 - (1 - \mathrm{Default\,rate\,for\,month\,t})^{12},$$

where the monthly default rate is the ratio of the unpaid loan balance for month t to the difference between the beginning balance for month t and scheduled monthly payment for month t. The annual 100% SDA model is defined as follows.

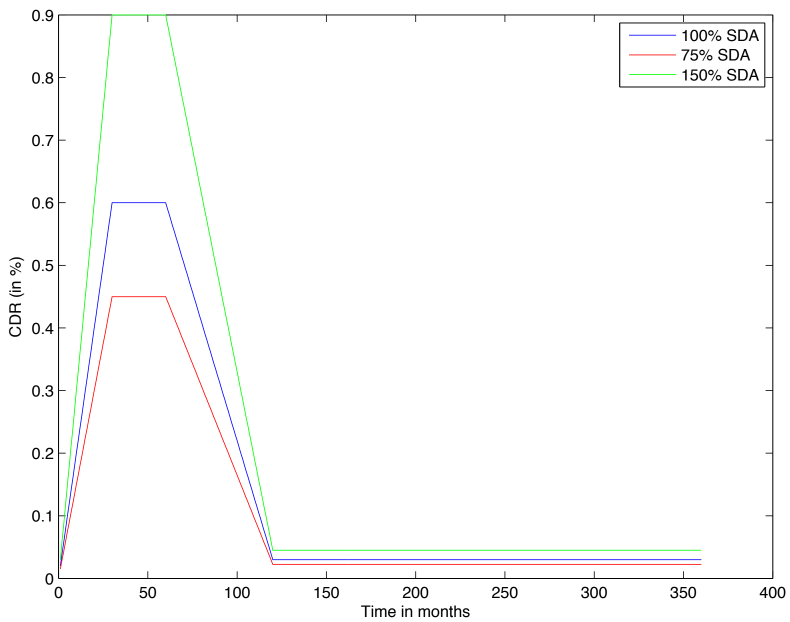


Figure 2.2: SDA Default curve

- The default rate in month 1 is 0.02% and increases by 0.02% for up to 30 months until it reaches 0.06%.
- From month 30 to 60 it remains constant, and
- then declines from month 61 to month 120 to 0.03% from 0.06%.
- From month 120 on, the default rate remains constant at 0.03%.

100% SDA, 75% SDA and 150% SDA are shown in Figure 2.5.2.

2.6 Modelling mortgage backed securities and literature review

Modelling mortgage backed securities remains a challenge to researchers in the field because of the uncertainty of cash flow due to prepayments and defaults. The dependence of the refinancing incentive on the interest rates and exogenous prepayments add complexity to the modelling such that no standard model is known to price these securities. The numerical techniques along with the theoretical models provide an approximate model for MBS. There is a vast literature on the modelling of the securities and the models can be classified into Structural or Option based models and Reduced form models.

2.6.1 Structural approach

The origin of the structural approach dates back to Merton (1973) [44] where the Black and Scholes option pricing formula was applied to value corporate liabilities. Merton assumed that a firm's asset value V_t follows a diffusion process given by

$$dV_t = rV_t dt + \sigma V_t dW_t,$$

where r is a constant interest rate, σ is a constant volatility and W_t is a Brownian motion. Assuming that a firm's asset value consists of a share S_t and a bond $Z_t(T)$ at time t with maturity T and a face value D. Then the share S_t represents a call option on the asset value of the firm V_t with maturity T and payoff $S_T = \max(V_T - D, 0)$. If at maturity T the asset values V_T of the firm is not enough to pay back the face value of debt D, then the firm defaults otherwise shareholders receive $V_T - D$. An underlying assumption is that the default only occurs at time T, allowing Merton to apply the Black and Scholes pricing formula. In later years, many refinements were made to this basic Merton model [44].

With further enhancements, the structural approach is applied to modelling of mortgage and mortgage backed securities. The process of modelling an MBS starts with the selection of structural variables of the security and describing the evolution of the variables as stochastic processes. For example, when modelling a residential MBS the key variables involved are interest rates r_t and house price H_t, the model is easily generalised to more variables. Assume that the variables follow the diffusion processes given by

$$\begin{aligned} dr_t &= \alpha\left(\theta - r_t\right)dt + \sigma\sqrt{r_t}dW_t, \\ \frac{dH_t}{H_t} &= \gamma dt + \xi dI_t, \end{aligned}$$

where α is the speed of reversion in a mean reverting CIR process, θ is a long term mean level and σ is a constant volatility. The process for house price has drift γ and a constant volatility ξ. Such stochastic processes show that H is expected to change at a constant rate γ at any time t but the actual change differs by a disturbance in economy by the instantaneous standard deviation ξ. W_t and I_t are two correlated Brownian motions such that $dW_t dI_t = \rho dt$, where ρ is a correlation factor. Simple application of stochastic calculus yield the following stochastic differential equation for valuation of any mortgage $M\left(H, r, t\right)$ whose value depends on H, r and time t.

$$\begin{aligned} \frac{1}{2}r\sigma^2\frac{\partial^2 M}{\partial r^2} + \rho H\sqrt{r}\sigma\xi\frac{\partial^2 M}{\partial r\partial H} + \frac{1}{2}\xi^2 H^2\frac{\partial^2 M}{\partial H^2} + \alpha\left(\theta - r\right)\frac{\partial M}{\partial r} \\ + \gamma H\frac{\partial M}{\partial H} + \frac{\partial M}{\partial t} - rM = 0. \end{aligned}$$

Solution to this PDE involves the early termination of the contract due to prepayment and default before maturity. The prepayment option is more like an American call option on the present value of the remaining payments with a mortgage balance as a strike price as it gives the borrower a right to prepay at any time during the life of a mortgage and receive the house. On the other hand, default is a right given to the borrower where he/she can choose not to pay the monthly payments and abandon the house. Therefore, it is modeled as a series of European put options, at the monthly payment dates, with the house price as the underlying asset and the present value of the remaining monthly payments as a strike price. Since exercising one option the borrower loses the second option, both options have to be considered jointly. For this reason, no analytical solution to the PDE is known and numerical methods must be applied to solve it.

The boundary conditions associated with the PDE are discussed by many authors and correspond to a particular numerical method used to solve the PDE. We briefly discuss the boundary conditions used by Azevedo-Pereira et al [3], Kau et al [33] and Sharp et al [53]. The default and prepayment options are related in a way that if one occurs then it diminishes the occurrence of the other. They model a joint option $J(H,r,t)$ which is

$$J(H,r,t) = P(r,H,t) + D(r,H,t),$$

where $P(r,H,t)$ is a prepayment option and $D(r,H,t)$ is a default option. The PDE is solved each month for $V(H,r,t)$ during the life of the mortgage. We let mortgage value $V(H,r,t)$ be the difference between the value of total scheduled outstanding payments $OP(r,t)$ and the joint option that is

$$V(H,r,t) = OP(r,t) - J(H,r,t).$$

At maturity, the scheduled outstanding payment is the monthly payment MP and the value of mortgage is

$$V(H,r,T) = \min(MP,H).$$

The default option D is worthless if the value of the house is greater than MP. Otherwise $D(H,r,t) = \max(MP-H,0)$ and the prepayment option $C(H,r,t) = 0$. Working backward and checking the value of all mortgage components, one needs to check the default and prepayment conditions. Default occurs if at each payment date t the value of house is less than the sum of future monthly payments and the value of mortgage after that month's payment is made. In this case $D(H, r, t) = OP(r, t) - H$ and $C(H, r, t) = 0$. The value of mortgage becomes $V^-(H, r, t) = \min(V^+(H, r, t) + MP, H)$, where V^- and V^+ are the value of mortgage before and after the monthly payment is made, respectively. More details on the boundary conditions and solutions to the PDE with these boundary conditions can be found in [3], [33], [53].

In modelling mortgage backed securities, researchers mostly have considered either of the two risks involved with these securities. They either modelled the right of prepayment by ruling out the possibility of default or considered default only and overlooked the prepayment right of the borrower. In 1981, Dunn and McConnell [13] used option pricing techniques to model these securities for the first time. They considered the mortgage as a portfolio of a non-callable mortgage loan and an American style option.

Kau et al. in [34] and [33] discuss the prepayment and default as the underlying source of uncertainty for the first time. They described the mortgage rate as a solution to a partial differential equation using option based pricing. In their models, Dunn and McConnell and Kau et al, assume that the borrower's prepayment behaviour is optimal and borrowers refinance whenever it is optimal which may not be the case in reality, especially when dealing with residential mortgages. Another assumption in developing these models was discussed by Kariya and Kobayashi [29], Nakamura [47], Kariya et al. [30], and Nakagawa and Shouda [46]. They assumed that the borrowers make a decision by looking at the economy and prepay when the mortgage and interest rates reach a certain level or threshold.

In [31], Kau and Keenan discussed a structural model for mortgages by including another stochastic process for a commercial building or multiple family house on which the mortgage is taken. They assumed that the interest rates follow the CIR process as in Stanton [56]. In addition to that they assumed that the building prices follow a lognormal process

$$\frac{dB_t}{B_t} = (\alpha_B - s)\,dt + \sigma_B dW_{B_t} + dN_t,$$

with $dW_{r_t} dW_{B_t} = \rho dt$, where dN_t is a Poisson process with frequency λ to measure the impact of a catastrophic event.

In [12] Deng discussed a modification to this model of residential mortgages. He assumed that house prices H follow a lognormal process and as

$$\frac{dH_t}{H_t} = (r - d)\,dt + \sigma_{H_t} dW_{H_t},$$

with $dW_{r_t} dW_{H_t} = \gamma\,(r, H, t)\,dt$. In all cases these processes yield a mixed PDE for the value of mortgage M_t at time t that can be solved for the mortgage rates, or for the optimal choice of the threshold levels for spot price and building price / house price. By threshold level we mean that the default occurs when the house price falls below the threshold level H^* and prepayment occurs when the interest rates fall below the threshold level r^*.

In [30], a three factor valuation model for MBS is introduced by Kariya, Ushiyama and Pliska in a discrete time setting. The authors defined three stochastic processes, one for the mortgage rate that specifies the condition on the prepayment, one for the house price that specifies the condition on the default, and a third short-term spot rate process that determines the discount factor. They introduced the prepayment condition for each borrower

$$u_t^1 := R_0 - R_t \geq d_k^1,$$

where R_0 and R_t are the initial and current mortgage rates and d_k^1 is assumed to be a constant threshold level for k^{th}-borrower. Similarly, default occurs at the first time when the difference between the (log of) current house price and the (log of) initial house price exceeds a threshold d_k^2 for k^{th}-borrower, that is

$$u_t^2 := \log P_t - \log P_0 \geq d_k^2.$$

The authors used Monte Carlo simulations to solve the model numerically, and presented results discussing the effects of each parameter on the mortgage rates.

The structural models have some disadvantages, one of which is that the borrower can prepay for reasons other than refinancing; also, many borrowers behave irrationally from the view point of option theory. Secondly, in modelling under the structural framework the time of default and prepayment become predictable but one can overcome it by introducing jumps in the model. This approach is also computationally demanding and if sequential refinancing is allowed, this will further complicate the problem. It has also been seen empirically that the observed prices of MBS are not fully explained by the existing structural models [15].

2.6.2 Intensity based approach

The second approach to model these securities is Intensity based or hazard rate models. The first model using the hazard rates is due to Schwartz and Torous [51]. This approach models prepayment and default as a random time that is governed by some hazard rate process that is estimated by the actual prepayment and default data in large mortgage pools. Then working along the simulated interest rate paths, the hazard process is used to simulate mortgage prepayments and defaults. The mortgage is valued by discounting the cash flows and averaging over the simulation paths. The valuation of these models is done by numerical schemes mostly, such as Monte Carlo simulation. A complete overview of hazard processes and other definitions is given in Chapter 3.

Schwartz and Torous [51] modelled prepayment as a function of some set of variables and estimated this function using empirical data. In 1995, Stanton [56] modelled the prepayment behavior of the borrower by capturing rational decision by borrower. By capturing rational decision of a borrower we mean that the decision of prepayment is modelled by introducing a hazard rate λ to account for prepayment due to refinancing as well as prepayment for reasons other than refinancing such as divorce. Stanton claimed that his model fits data better than Schwartz and Torous [52], and matches closely observed prepayment behaviors.

Kau et al. in [33] applied an intensity based prepayment framework to model prepayment behaviour of a borrower. They also capture the default event in their modelling. They used extensive historical data to estimate termination of mortgage in events of default and prepayment. Using CIR processes for the baseline hazard functions for default and prepayment, they modelled individual mortgage contracts in the context of interest rate modelling.

Some interesting work by Goncharov [19] models the sub-optimal prepayment behavior of borrowers, noticing the fact that a borrower can prepay if it is not optimal to prepay and may not prepay when it is optimal to prepay. Furthermore, he found a nonlinear equation for the endogenous mortgage rate, that is the mortgage rates for which the mortgage value is equal to the loan amount at the origination. Goncharov presented a numerical scheme to solve the model and test the scheme on the interest rates and specifications used by Stanton [56]. He further presented some case studies on the dependence of mortgage rate on CIR interest rates' parameters and intensity process parameters as well as discusses the dependence of numerical scheme on interest rates grid points and number of paths. No calibration to market data is done by Goncharov. He showed that the solution to the resulting equation exists and is unique, but solving the equation numerically is challenging. Later in

[21] and [20] he computed the mortgage rates using the equation for mortgage rate based prepayment assumption. He used Monte Carlo and randomized quasi Monte Carlo simulation for this purpose. Pliska in [48] discussed the discrete case of this model.

Some further contributions to intensity based modelling were recently made by Gorovoy and Linetsky [23] and Rom-Poulsen [50]. Rom-Poulsen modelled fixed rate mortgage backed securities and obtained semi-analytical solutions for valuing MBSs. He presented a multi-factor model for a default free MBS. Gorovoy and Linetsky presented a semi-analytically tractable valuation model for fixed rate residential mortgages by allowing prepayments in the model. They modelled the random prepayment time as a hazard process. Using the CIR diffusion process to model interest rates they were able to solve the model semi-analytically using Feynman-Kac semigroups to explicitly constructing eigenfunction expansions for the mortgage rates. Then using the Newton-Raphson method to solve fixed point of the mortgage equation they valued the mortgage rates.

Chapter 3

Determining the mortgage rate in a reduced form model

We work in the intensity based framework to determine the mortgage rate for a single mortgage. We follow the same strategy as Gorovoy and Linetsky in [23], where they modelled the mortgage rate for a single mortgage by introducing an intensity process for prepayment. We extend the model by allowing for mortgage default We do this by introducing another intensity process for default, and study the termination of the mortgage caused by occurrence of either default or prepayment.

In the first section, we present the preliminary definitions and results that we use later. We introduce a setup for a single mortgage without default and prepayment. Using the theory of hazard processes, we introduce the intensity processes for default and prepayments and drive a nonlinear implicit expression for the mortgage rate of a single borrower. We find numerically the mortgage rates for a given set of parameters and present some case studies to compare the results to those of Gorovoy and Linetsky [23].

3.1 Intensity-based valuation: an overview

In modelling a mortgage rate, we are looking at an intensity based model. We recall some definitions and results from Bielecki and Rutkowski [5] in modelling a credit spread under this approach. In the intensity based approach the default time is modelled as an unpredictable stopping time. Exogenous specifications of the conditional probability of default knowing that no default has yet occurred is the main tool in modelling under this framework. In most cases this is done by using hazard rates or process. In the next section we define the hazard process for one random variable and state some known results that we find useful in our work. In the second part of the section we present some definitions and results for the hazard process of the minimum of several random variables.

3.1.1 Hazard process

To define a hazard process, we start with a non-negative random variable τ on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ which satisfies $\mathbb{P}\{\tau = 0\} = 0$ and $\mathbb{P}\{\tau > t\} > 0$ for $t \in \mathbb{R}_+$. We introduce a right continuous function H by setting $H_t = \mathbf{1}_{\{\tau \leq t\}}$ that is the process $H_t = 0$ if no default occurs until t and $H_t = 1$ if default occurs before t. The associated filtration of H is $\mathcal{H}_t = \sigma(H_u : u \leq t)$ and is denoted by $\mathscr{H}$. Let $\mathscr{G} = (\mathcal{G}_t)_{t \geq 0}$ be an arbitrary filtration on $(\Omega, \mathcal{G}, \mathbb{P})$ where all filtrations are assumed to satisfy the usual conditions of right continuity and completeness. The information that is available at time t, for each $t \in \mathbb{R}_+$, is captured by the $\sigma-$ field $\mathcal{G}_t$.

We also assume that we have an auxiliary filtration $\mathscr{F}$ such that $\mathscr{G} = \mathscr{H} \vee \mathscr{F}$, that is $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$ for all $t \in \mathbb{R}_+$. We note that this auxiliary filtration is not unique as for $\mathcal{G}_t \vee \mathcal{H}_t$, we may take $\mathscr{F} = \mathscr{F}^0$ and we can also take $\mathscr{F} = \mathscr{H}$. We also assume that for every $t \in \mathbb{R}_+$ the event $\{\tau \leq t\}$ belongs to the $\sigma-$ field $\mathcal{F}_t$ and hence τ is an $\mathscr{F}-$ stopping time. In addition to that we assume that for some dates $t \in \mathbb{R}_+$, the event $\{\tau \leq t\}$ does not belong to the $\sigma-$field $\mathcal{G}_t$ which means τ is not a $\mathscr{G}$- stopping time.

We write $F_t = \mathbb{P}\{\tau \leq t \mid \mathcal{F}_t\}$ that is the probability that default occurs before time t for any $t \in \mathbb{R}_+$ and let us also denote the $\mathscr{F}$-survival process of τ by G with respect to the filtration $\mathscr{F}$ and is given as :

$$G_t = 1 - F_t = \mathbb{P}\{\tau > t \mid \mathcal{F}_t\}, \quad \forall t \in \mathbb{R}_+.$$

We this set up, we are now in a position to give the definition of a Hazard process:

Definition 1. *Assuming that $F_t < 1$ for $t \in \mathbb{R}_+$, the $\mathscr{F}$-hazard process Γ of τ under $\mathbb{P}$ is defined by the formula $1 - F_t = e^{-\Gamma_t}$.*

Equivalently, we can write it in terms of survival process as $\Gamma_t = -\ln G_t = -\ln(1 - F_t)$ for every $t \in \mathbb{R}_+$. The inequality $F_t < 1$ is assumed so that the hazard process (or to be more precise $\mathscr{F}$- hazard process) Γ is well defined.

After defining the hazard process, our next goal is to express the conditional expectations of a $\mathcal{G}$- measurable random variable Y knowing the information $\mathcal{G}_t$ at any time t. We state the following lemma from [5]:

Lemma 3.1.1. *Assume that for every $t \in \mathbb{R}_+$, we have $\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{H}_t \vee \mathcal{F}_t$. Then for any $t \in \mathbb{R}_+$ and for any $\mathcal{G}-$ measurable random variable Y we have*

$$\mathbb{E}_{\mathbb{P}}\left(\mathbf{1}_{\{\tau>t\}}Y \mid \mathcal{G}_t\right) = \mathbb{P}(\tau > t \mid \mathcal{G}_t)\,\frac{\mathbb{E}_{\mathbb{P}}\left(\mathbf{1}_{\{\tau>t\}}Y \mid \mathcal{F}_t\right)}{\mathbb{P}\{\tau > t \mid \mathcal{F}_t\}}. \tag{3.1}$$

If in addition to this $\mathcal{H}_t \subseteq \mathcal{G}_t$ then

$$\mathbb{E}_{\mathbb{P}}\left(\mathbf{1}_{\{\tau>t\}}Y \mid \mathcal{G}_t\right) = \mathbf{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}(Y \mid \mathcal{G}_t) = \mathbf{1}_{\{\tau>t\}}\frac{\mathbb{E}_{\mathbb{P}}\left(\mathbf{1}_{\{\tau>t\}}Y \mid \mathcal{F}_t\right)}{\mathbb{P}\{\tau > t \mid \mathcal{F}_t\}}. \tag{3.2}$$

A simpler and useful modification of these results involving hazard functions can be summarized in the following corollary.

Corollary 3.1.2. *We let Y be a $\mathcal{G}-$ measurable random variable and $t \leq s$,*

1. *Assume that for every $t \in \mathbb{R}_+$ we have $\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{H}_t \vee \mathcal{F}_t$, then*

$$\mathbb{E}_{\mathbb{P}}\left(\mathbf{1}_{\{\tau>s\}}Y \mid \mathcal{G}_t\right) = \mathbb{P}\left\{\tau > t \mid \mathcal{G}_t\right\} \mathbb{E}_{\mathbb{P}}\left(\mathbf{1}_{\{\tau>s\}}e^{\Gamma_t}Y \mid \mathcal{F}_t\right). \tag{3.3}$$

2. *If for any $t \in \mathbb{R}_+$, $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$ then*

$$\mathbb{E}_{\mathbb{P}}\left(\mathbf{1}_{\{\tau>s\}}Y \mid \mathcal{G}_t\right) = \mathbf{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}\left(\mathbf{1}_{\{\tau>s\}}e^{\Gamma_t}Y \mid \mathcal{F}_t\right) \tag{3.4}$$

and

$$\mathbb{E}_{\mathbb{P}}\left(\mathbf{1}_{\{t<\tau\leq s\}}Y \mid \mathcal{G}_t\right) = \mathbf{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}\left(\mathbf{1}_{\{t<\tau\leq s\}}e^{\Gamma_t}Y \mid \mathcal{F}_t\right). \tag{3.5}$$

If Y is $\mathcal{F}_s-$measurable, then

$$\mathbb{E}_{\mathbb{P}}\left(\mathbf{1}_{\{\tau>s\}}Y \mid \mathcal{G}_t\right) = \mathbf{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}\left(e^{\Gamma_t-\Gamma_s}Y \mid \mathcal{F}_t\right) \tag{3.6}$$

and

$$\mathbb{E}_{\mathbb{P}}\left(\mathbf{1}_{\{t<\tau\leq s\}}Y \mid \mathcal{G}_t\right) = \mathbf{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}\left(\left(\mathbf{1}_{\{\tau>t\}} - e^{-\Gamma_s}\right)e^{\Gamma_t}Y \mid \mathcal{F}_t\right). \tag{3.7}$$

3.1.2 Stochastic intensity of a random time

We now discuss the most widely and practical use of an absolutely continuous $\mathscr{F}$-hazard process Γ and assume that $\Gamma_t = \int_0^t \gamma_u du$ for some $\mathscr{F}$ -progressively measurable process γ. γ is referred to as the $\mathscr{F}$-intensity of a random time τ or simply the stochastic intensity of τ. As stated in [5], a continuous, $\mathscr{F}$-adapted process Γ is $\mathscr{F}$-predictable and then there exists an $\mathscr{F}$-predictable modification of the $\mathscr{F}-$intensity γ. The process $\hat{M}$, defined as

$$\hat{M}_t = H_t - \int_0^{t\wedge\tau} \gamma_u du = H_t - \int_0^t \mathbf{1}_{\{\tau\geq u\}}\gamma_u du,$$

follows a $\mathscr{G}-$martingale. This property is used as a definition of stochastic intensity of a random time. We next state a result about stochastic intensity that interpret the $\mathscr{F}$-intensity γ as the intensity of default given the information flow $\mathscr{F}$.

Corollary 3.1.3. *For any $t \leq s$, the following holds for an absolutely continuous $\mathscr{F}$-hazard process Γ of a random time τ*

$$\mathbb{P}\{\tau > s \mid \mathcal{G}_t\} = \mathbf{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}\left(e^{-\int_t^s \gamma_u du} \mid \mathcal{F}_t\right)$$

and

$$\mathbb{P}\{t < \tau \leq s \mid \mathcal{G}_t\} = \mathbf{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}\left(1 - e^{-\int_t^s \gamma_u du} \mid \mathcal{F}_t\right).$$

In next subsection, we briefly discuss the case of several random variables and give a definition of conditionally independent random times that we use in our later work.

3.1.3 Hazard process of the minimum of several random variables

We now move on to the case of several random times and suppose that we have a finite collection $\tau_1, \tau_2, \ldots, \tau_n$ of random times on a common probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and let $\mathscr{F}$ be a filtration on this probability space. We define a family of jump process $H_t^i = \mathbf{1}_{\{\tau_i \leq t\}}$ and let $\mathscr{H}^i$ denotes the filtration generated by i^{th} jump process H^i. We also introduce a larger filtration $\mathscr{G} = \mathscr{H}^1 \vee \ldots \vee \mathscr{H}^n \vee \mathscr{F}$.

Our goal is to find the hazard process for the minimum τ of default times $\tau_1, \tau_2, \ldots, \tau_n$ where we are given a finite family of random times τ_i, $i = 1, \ldots, n$ and the associated hazard processes Γ^i, $i = 1, \ldots, n$. We first state the result from Bielecki and Rutkowski [5] when the random times are mutually independent.

Lemma 3.1.4. *Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space. Let τ_i, $i = 1, \ldots, n$ be n mutually independent random times on this probability space. Assuming that τ_i admits the hazard function Γ_i, the hazard function Γ of τ is equal to the sum of the hazard functions Γ_i, $i = 1, \ldots, n.$, that is,*

$$\Gamma = \Sigma_{i=1}^n \Gamma_i.$$

So far we presented the results in a real probability space but from now onward we work under a risk neutral environment, as in [5], for presenting results for the case when the random times are not mutually independent. In [5], Bielecki and Rutkowski presented the results using the risk-neutral probability $\mathbb{Q}^*$. We re-define the set up adopted in this part of the chapter under the probability space $(\Omega, \mathcal{G}, \mathbb{Q}^*)$. Now we consider a finite collection of random times $\tau_1, \ldots, \tau_n$ defined on this probability space with $\mathbb{Q}^* \{\tau_k = 0\} = 0$ and $\mathbb{Q}^* \{\tau_k > t\} > 0$ for every $t \in \mathbb{R}_+$ and for $k = 1, \ldots, n$. We assume that no simultaneous defaults occur that is $\mathbb{Q}^* \{\tau_k = \tau_j = 0\}$ for arbitrary $k, j = 1, \ldots, n$ with $k \neq j$. We also assume that we are given some reference filtration $\mathscr{F}$ on the probability space $(\Omega, \mathcal{G}, \mathbb{Q}^*)$ and introduce an enlarged filtration $\mathscr{G}$ by setting $\mathscr{G} = \mathscr{F} \vee \mathscr{H}^1 \vee \ldots \vee \mathscr{H}^n$.

With this set up, we present the result that we use in our later work. We assume that the default times are conditionally independent to the underlying filtration. The assumption of conditional independence of default times means that fixing the common risk factors we can make the idiosyncratic risk factors independent of each other.

Definition 2. *The random times τ_i, $i = 1, \ldots, n$ are said to be conditional independent with respect to the filtration $\mathscr{F}$ under $\mathbb{Q}^*$ if and only if the*

$$\mathbb{Q}^* \{\tau_1 > t_1, \ldots, \tau_n > t_n \mid \mathcal{F}_t\} = \prod_{i=1}^{n} \mathbb{Q}^* \{\tau_i > t_i \mid \mathcal{F}_t\}$$

for any $T > 0$ and arbitrary $t_1, \ldots, t_n \in [0, T]$.

We note here that conditionally independence of default times does not imply their independence and vice versa. It is also emphasized that the property of conditionally independence may not be invariant under an equivalent change of probability measure. Now assuming that the following condition holds, we present another formulation of the above definition from [5].

We suppose that for every $T > 0$, $u \in [0, T]$, and $i = 1, \ldots, n$ we have

$$\mathbb{Q}^* \{\tau_i > u \mid \mathcal{F}_T\} = \mathbb{Q}^* \{\tau_i > u \mid \mathcal{F}_u\}.$$

Definition 3. *The random times $\tau_1, \ldots \tau_n$ are called dynamically conditionally independent with respect to $\mathscr{F}$ under $\mathbb{Q}^*$ if and only if*

$$\mathbb{Q}^* \{\tau_1 > t_1, \ldots, \tau_n > t_n \mid \mathcal{F}_T \vee \mathcal{H}_t\} = \prod_{i=1}^{n} \mathbb{Q}^* \{\tau_i > t_i \mid \mathcal{F}_T \vee \mathcal{H}_t\},$$

for any $0 \leq t < T$ and arbitrary $t_1, \ldots, t_n \in [t, T]$.

After a detailed review of the intensity based approach, we are now in a position to introduce our model in next and later sections. We start by giving a description of a model without prepayment and default and introduce these factors of uncertainty later in Section (3.3).

3.2 Model setup

We consider the same framework for the mortgage without any prepayment and/or default as Goncharov [19] and Gorovoy and Linetsky in [23]. We consider a fixed rate mortgage that is originated at time $t = 0$ with maturity $T > 0$. The rate of the mortgage is fixed at the origination of the mortgage and remains the same throughout the life of the mortgage. We further assume that at the origination of the mortgage, the borrower takes a loan of P_0 dollars and continuously pays a coupon stream at a rate of $c > 0$ dollars per annum during the life of the mortgage $[0, T]$. The interest is compounded at a contractual mortgage rate $m(0, T)$ or simply m and is expressed in percent per annum. The remaining principal at any time $t \in [0, T]$ is denoted by P_t. The total payment made by the borrower over an infinitesimal interval of time dt is then cdt. This payment consists of two parts, the scheduled principal repayment $-dP_t$ and the interest payment on the remaining principal, that is $mP_t dt$. This can be written as an ordinary differential equation:

$$\frac{dP_t}{dt} = mP_t - c; \tag{3.8}$$

with the value of the loan P_0 at time $t = 0$. Solving it for P_t gives us

$$P_t = P_0 e^{mt} + \frac{c}{m}\left(1 - e^{mt}\right).$$

At the origination of the mortgage, it is expected that the mortgage would be fully amortized, that means the remaining principal at maturity would be zero, that is $P_T = 0$ at T. Using this, we obtain the coupon rate

$$c = \frac{mP_0}{1 - e^{-mT}} \tag{3.9}$$

in terms of the contractual mortgage rate. The principal payment then becomes

$$P_t = P_0\left(\frac{1 - e^{-m(T-t)}}{1 - e^{-mT}}\right), \quad t \in [0, T]. \tag{3.10}$$

The cash flow in the absence of default and prepayment is deterministic. We now define prepayment and default on the mortgage using intensity processes in the next section.

3.3 Introducing prepayment and defaults

We start by formulating the set up for introducing hazard processes for default and prepayment. We introduce a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ that carries two standard Brownian motions $\left\{W_t^I, t \geqslant 0\right\}$ and $\left\{W_t^{H^*}, t \geqslant 0\right\}$ and two exponential random variables with unit mean $e \sim$Exp (1) and $e^* \sim$Exp (1). Both random variables are independent of the two Brownian motions. The filtration generated by W_t^I is $\mathscr{F}^1 = \{\mathcal{F}_t^1,\, t \geqslant 0\}$ and the filtration generated by $W_t^{H^*}$ is denoted by $\mathscr{F}^2$ and is defined as $\mathscr{F}^2 = \{\mathcal{F}_t^2,\, t \geqslant 0\}$. We let $\mathscr{F} = \mathscr{F}^1 \vee \mathscr{F}^2$ where $\mathcal{F}_t = \mathcal{F}_t^1 \vee \mathcal{F}_t^2$.

We assume that the market is arbitrage free and consider an equivalent martingale measure $\mathbb{Q}$ as given. The market is not complete in our model and there can be more than one equivalent martingale measures. We are not dealing with finding a suitable equivalent martingale measure but that can be done by calibrating to the market data on mortgages. For the purpose of this work we assume that we are given an equivalent martingale measure $\mathbb{Q}$ and we model the instantaneous interest rates as a positive continuous process adapted to the filtration $\mathscr{F}^1$, in the same way as Gorovoy and Linetsky in [23].

We model the prepayment and default as two independent intensity processes. We introduce a positive prepayment intensity process $\{h_t, t \geqslant 0\}$ adapted to the Brownian filtration $\mathscr{F}$ for prepayment. The random time of prepayment is modelled as the first time when the hazard process $\int_0^t h_s ds$ is greater or equal to the random level e, that is,

$$\tau = \inf\left\{t \geqslant 0 : \quad \int_0^t h_u du \geqslant e\right\}. \tag{3.11}$$

At time τ, the borrower prepays all the remaining principal balance of the mortgage P_τ. We assume that there is only one prepayment during the life of the mortgage and the contract is over when the borrower prepays the remaining principal. We do not consider partial prepayments in our work.

We introduce an indicator process to keep track of the information regarding prepayment over time. This indicator process $\{N_t^1,\, t \geqslant 0\}$, $N_t^1 = \mathbf{1}_{\{t \geqslant \tau\}}$ is a one jump process that is equal to zero before prepayment and jumps to one when prepayment occurs at time τ. We denote by $\mathscr{D}^1$ the filtration generated by N_t^1 that is $\mathscr{D}^1 = \{N_t^1,\, t \geq 0\}$ and introduce an enlarged filtration $\mathscr{G}^1 = \{\mathcal{G}_t^1,\, t \geqslant 0\}$, where $\mathcal{G}_t^1 = \mathcal{F}_t^1 \vee \mathcal{D}_t^1$. Then the compensated process $N_t^1 - \int_0^{t\wedge\tau} h_u du$ is a $(\mathscr{G}^1, \mathbb{Q})$ −martingale.

We now deal with another important feature of mortgages explicitly in our model, that is default. To model default we adapt the same technique as before for prepayment and introduce a positive intensity process $\{h_t^*, t \geq 0\}$ adapted to the Brownian filtration $\mathscr{F}$. We model the random time of default τ^* as the first time when the hazard process $\int_0^t h_u^* du$ is greater than or equal to the random level $e^* \sim Exp(1)$ that is

$$\tau^* = \inf\left\{t \geq 0 : \quad \int_0^t h_u^* du \geq e^*\right\}. \tag{3.12}$$

We suppose that after a delinquency of three months, the default event occurs at time τ^* and the probability of recovering from delinquency and entering into the contract again is zero. The lender receives $\min\left(H_{\tau^*}^*, P_{\tau^*}\right)$ at time of default, where for simplification we assume that $H_{\tau^*}^*$ is the price of the house after repossession and P_{τ^*} is the outstanding principal balance at time of default.

To keep track of the information on default, we introduce an indicator process for default $\{N_t^2, t \geq 0\}$, where $N_t^2 = \mathbf{1}_{\{t \geq \tau^*\}}$,that is, the jump process is zero before default and jumps to 1 when default occurs. We also define a filtration $\mathscr{D}^2 = \{N_t^2, t \geq 0\}$ generated by N_t^2, and an enlarged filtration $\mathscr{G}^2 = \{\mathcal{G}_t^2, t \geq 0\}$ where $\mathcal{G}_t^2 = \mathcal{F}_t \vee \mathcal{D}_t^2$. Then the process

$$N_t^2 - \int_0^{t \wedge \tau^*} h_s^* ds \tag{3.13}$$

is a $(\mathscr{G}^2, \mathbb{Q})-$ martingale. We let $\mathscr{G} = \mathscr{G}^1 \vee \mathscr{G}^2$ where $\mathcal{G} = \mathcal{G}^1 \vee \mathcal{G}^2$.

Until the time of prepayment, default or maturity, whichever comes first, the lender receives the cash flow stream at the fixed rate c dollars per annum. At the time of prepayment τ, the lender receives a recovery amount equal to the principal remaining at τ, that is P_τ, and at the time of default τ^*, the lender receives minimum of $H_{\tau^*}^*$ and P_{τ^*} where $H_{\tau^*}^*$ is the house price available after repossession and P_{τ^*} is the unpaid principal balance at the time of default.

We now use the credit risk modelling framework to model the present value of an individual mortgage with continuous payments. That is at time $t < T$, we have

$$\begin{aligned} M_t &= \mathbb{E}\left[\int_t^T ce^{-\int_t^u r_s ds}\mathbf{1}_{\{\eta>u\}}du \mid \mathcal{G}_t\right] \\ &\quad +\mathbb{E}\left[\hat{P}_\eta e^{-\int_t^\eta r_s ds}\mathbf{1}_{\{t<\eta\leqslant T\}} \mid \mathcal{G}_t\right], \end{aligned} \tag{3.14}$$

where $\eta = \min(\tau, \tau^*)$ that is, η is the event which occurs first (default or prepayment) and $\hat{P}$ is the recovery amount in case of either default or prepayment and is given as,

$$\hat{P} = \begin{cases} P_\tau & \text{if } \eta = \tau, \\ \min\left(H^*_{\tau^*}, P_{\tau^*}\right) & \text{if } \eta = \tau^*. \end{cases}$$

So we have

$$M_t = \mathbf{1}_{\{\eta>t\}}\mathbb{E}\left[\int_t^T \left(c + \hat{h}_u\hat{P}_u\right) e^{-\int_t^u \left(r_s+\hat{h}_s\right)ds}du \mid \mathcal{F}_t\right], \tag{3.15}$$

where the repayment rate c and the remaining principal balance P_τ are given by equations (3.9) and (3.10) and $\hat{h}_u$ is the intensity process of the minimum of default and prepayment times. This intensity process depends on the intensity processes of default and prepayment and we will find an expression for this after introducing the interest and house price processes later in this chapter.

We now introduce the following notation for $u \geq t \geq 0$,

$$\begin{aligned} \mathcal{Q}(t,u) &:= \mathbb{E}\left[e^{-\int_t^u \left(r_s+\hat{h}_s\right)ds} \mid \mathcal{F}_t\right], \\ \mathcal{H}(t,u) &:= \mathbb{E}\left[\hat{h}_u e^{-\int_t^u \left(r_s+\hat{h}_s\right)ds} \mid \mathcal{F}_t\right]. \end{aligned} \tag{3.16}$$

The mortgage value can be re-written in new notation given no prepayment or default occur by time t as

$$M_t = \int_t^T \left[c\mathcal{Q}(t,u) + \hat{P}_u\mathcal{H}(t,u)\right] du. \tag{3.17}$$

To prevent arbitrage in the mortgage market, we assume that at the time of origination of the mortgage, i.e. at $t = 0$, the value of the mortgage is equal to the initial mortgage principal that is $M_0 = P_0$. So we obtain the following equation for the mortgage rate $m = m(0, T)$:

$$P_0 = \int_0^T \left[c\mathcal{Q}(0, u) + \hat{P}_u \mathcal{H}(0, u) \right] du,$$

where c is given in equation (3.9). An alternative form of equation (3.15) can be derived as by Goncharov in [19]. Given no prepayment or default by time $t \geq 0$, we obtain

$$\begin{aligned} M_t &= P_t - \mathbb{E}\left[\min\left(H^*_{\tau^*}, P_{\tau^*}\right) e^{-\int_t^{\tau^*}\left(r_s + \hat{h}_s\right)ds} \mathbf{1}_{\{\tau^* < T\}} \mid \mathcal{F}_t \right] \\ &\quad + \mathbb{E}\left[\int_t^T (m - r_u)\, \hat{P}_u e^{-\int_t^u \left(r_s + \hat{h}_s\right)ds} du \mid \mathcal{F}_t \right], \end{aligned} \tag{3.18}$$

where we have integrated $\int_t^T \hat{h}_u \hat{P}_u e^{-\int_t^u\left(r_s + \hat{h}_s\right)ds} du$ by using integration by parts and equation (3.8). We also used the fact that if default occurs at $\tau^* < T$ then $P_{\tau^*} = \min\left(H^*_{\tau^*} - P_{\tau^*}\right)$. If the borrower prepays before maturity then the principal remaining at time T would be zero. Using the notation as defined above, we obtain

$$\begin{aligned} M_t &= P_t - \mathbb{E}\left[\min\left(H^*_{\tau^*}, P_{\tau^*}\right) e^{-\int_t^{\tau^*}\left(r_s + \hat{h}_s\right)ds} \mathbf{1}_{\{\tau^* < T\}} \mid \mathcal{F}_t \right] \\ &\quad + \int_t^T \left[m\mathcal{Q}(t, u) - \mathcal{R}(t, u) \right] \hat{P}_u du, \end{aligned} \tag{3.19}$$

where we introduced the notation

$$\mathcal{R}(t, u) := \mathbb{E}\left[r_u e^{-\int_t^u \left(r_s + \hat{h}_s\right)ds} \mid \mathcal{F}_t \right]. \tag{3.20}$$

Note that the quantities $\mathcal{Q}(t, u)$, $\mathcal{H}(t, u)$ and $\mathcal{R}(t, u)$ are related as follows:

$$\frac{\partial \mathcal{Q}}{\partial u}(t, u) + \mathcal{H}(t, u) + \mathcal{R}(t, u) = 0. \tag{3.21}$$

Now using the expressions for P_u and c, as in (3.8) and (3.9), in equation (3.19), we obtain

$$\begin{aligned} M_t &= P_t\left(1+\int_t^T \left[m\mathcal{Q}\left(t,u\right)-\mathcal{R}\left(t,u\right)\right]\left(\frac{1-e^{-m(T-u)}}{1-e^{-m(T-t)}}\right)du\right) \\ &\quad -\mathbb{E}\left[\min\left(H^*_{\tau^*},P_{\tau^*}\right)e^{-\int_t^{\tau^*}\left(r_s+\hat{h}_s\right)ds}\mathbf{1}_{\{\tau^*<T\}}\mid\mathcal{F}_t\right]. \end{aligned} \tag{3.22}$$

On equating the right hand sides of equations (3.17) and (3.19), we obtain

$$\begin{aligned} &\int_t^T \left[\left(c-m\hat{P}_u\right)\mathcal{Q}\left(t,u\right)+\hat{P}_u\left(\mathcal{H}\left(t,u\right)+\mathcal{R}\left(t,u\right)\right)\right]du = \\ &\qquad P_t-\mathbb{E}\left[\min\left(H_{\tau^*},P_{\tau^*}\right)e^{-\int_t^{\tau^*}\left(r_s+\hat{h}_s\right)ds}\mathbf{1}_{\{\tau^*<T\}}\mid\mathcal{F}_t\right]. \end{aligned}$$

We can interpret the mortgage in the representation (3.18) as a defaultable swap with the amortizing principal P_t. At time t, the present value of the mortgage is equal to the principal remaining at time t minus the expected return if default occurs before maturity; plus the value of a swap where the borrower continuously pays the lender at a fixed mortgage rate m applied to P_u and the lender continuously pays the borrower at current interest rate r_u on P_u.

At the origination of the mortgage, the mortgage value is equal to the mortgage principal, ie, $M_0 = P_0$ at $t = 0$. So equation (3.19) becomes,

$$\begin{aligned} &\mathbb{E}\left[\min\left(H^*_{\tau^*},P_{\tau^*}\right)e^{-\int_0^{\tau^*}\left(r_s+\hat{h}_s\right)ds}\mathbf{1}_{\{\tau^*<T\}}\mid\mathcal{F}_0\right] \\ &\qquad = \int_0^T \left[m\mathcal{Q}\left(0,u\right)-\mathcal{R}\left(0,u\right)\right]\hat{P}_u du. \end{aligned} \tag{3.23}$$

We now summarize the above discussion as the following theorem.

Theorem 3.3.1. *The fixed mortgage rate m, fixed at time of initialization of the* *contract, is the solution to the following implicit equation.*

$$\begin{aligned} m &= \frac{1}{\int_0^T \mathcal{Q}(0,u)\,\hat{P}_u du} \left\{ \mathbb{E}\left[\min\left(H^*_{\tau^*}, P_{\tau^*}\right) e^{-\int_0^{\tau^*}\left(r_s+\hat{h}_s\right)ds} \mathbf{1}_{\{\tau^*<T\}} \mid r_0 \right] \right. \\ &\quad \left. + \int_0^T \mathcal{R}(0,u)\,\hat{P}_u du \right\}, \end{aligned} \tag{3.24}$$

where $\hat{P}_u = P_u$ when there is no prepayment and default.

Proof. The proof follows from (3.23) on solving for m. □

(3.24) is an equation involving m on both sides of the expression, as P_u is a function of m, and hence the right hand side also depends on m. The prepayment and default intensities may also depend on the mortgage rate. That makes the problem more complicated so we deal with a much simpler case where the prepayment and default intensities do not depend on m and numerically solve this equation using fixed point algorithm and applying Monte Carlo simulation.

Equation (3.24) is an extension of results in Gorovoy and Linetsky [23] by introducing defaults in it. If we assume that the mortgage contract has no default risk in it then our result agrees with equation (3.9) in [23] after substituting the expression for P_u that is

$$m = \frac{\int_0^T \left(1 - e^{-m(T-u)}\right) \mathcal{R}(0,u)\,du}{\int_0^T \left(1 - e^{-m(T-u)}\right) \mathcal{Q}(0,u)\,du}.$$

In our equation for the mortgage rate (3.24), the intensity process of the minimum of both default time and prepayment time appears. We will find this intensity process after defining our model specifications.

3.4 Model specifications

We now give model specifications to solve the mortgage rate equation numerically. We define the underlying processes for interest rates and house prices that incorporate in our model to specify prepayment and default intensities. The specifications lead us to an expression for the intensity process of the minimum of default and prepayment intensity. We further define the prepayment and default intensities and discuss the model for them.

1 Model for interest rate

Modelling interest rates as a stochastic mean reverting process is a common practice in the literature on mortgage modelling. As we are going to compare our case studies with Gorovoy and Linetsky [23], we are going to consider interest rates $\{r_t, t \geqslant 0\}$ as a mean reverting CIR process

$$dr_t = \kappa\left(\theta - r_t\right)dt + \sigma\sqrt{r_t}dW_I, \tag{3.25}$$

where $\kappa > 0$, $\theta > 0$, and $\sigma > 0$ are the rate of mean reversion, the long-run mean interest rate level, and interest rate volatility, respectively. The parameters of the CIR process are assumed to satisfy the Feller condition $2\kappa\theta \geqslant \sigma^2$ to ensure that the origin is an inaccessible boundary and the process stays strictly positive over time. Starting with r_0 at time $t = 0$, the interest rate r_t at any time t is likely to revert to the long-run mean interest rate level θ with a speed κ and disturbance or volatility σ.

2 Model for house price

We model the house price as a geometric Brownian motion (GBM) as in Titman and Torous [58] and Kau et al [34].

$$dH_t^* = H_t^*\left(r_t dt + \sigma_{H^*}\left(\rho dW_{H^*} + \sqrt{1-\rho^2}dW_I\right)\right), \tag{3.26}$$

here r_t is the interest rate at time t and $\sigma_{H^*} \geq 0$ is a constant volatility for the process. The interest rates effect the prices of real estate, therefore we consider that the Brownian motions driving the house price process W_{H^*} and interest rate process

W_I are correlated by the correlation factor ρ that is $dW_{H^*}dW_I = \rho dt$. We assume that the correlation between the interest rates and the house prices is positive that is $\rho \geq 0$. This means that as the interest rates increase or decrease the price of the real estate increase and decrease accordingly. A decrease in interest rates implying a decrease in the real estate prices can trigger the default on the existing mortgages, as we discuss in our default model. The decrease in the real estate prices can also make the houses affordable for the new borrowers who are looking to buy a house when the prices of house are at a lower level.

3 Model for prepayment

We assume, as in Gorovoy and Linetsky [23], that the prepayment intensity is a function of time and current interest rate r_t only, that is $h_t = h(t, r_t)$, and adopt a simple specification for this function as

$$h(r,t) = h_0(t) + \gamma (K - r_t)^+, \tag{3.27}$$

where $\gamma \geqslant 0$ and $K > 0$ are constant parameters, and $h_0(t)$ is a deterministic function of time. The reasons of prepayment that are independent of interest rates, such as prepayment in case of divorce, moving out, savings and death in family, are captured by the function $h_0(t)$, in other words this function captures the exogenous prepayment. We consider $h_0(t)$ as a PSA assumption for prepayment intensity as discussed in Chapter 2.

The term $\gamma (K - r_t)^+$ captures the endogenous prepayment due to refinancing on lower interest rate. We set the threshold $K > 0$ to the prevailing interest rate at origination in our specifications, or more precisely to $r_0 - s$, where the spread $s \geqslant 0$ accounts for the transaction costs for refinancing. $\gamma \geqslant 0$ is the constant intensity scaled parameter specifying that if there is a decline in interest rate, there is a proportional increase in prepayment intensity.

4 Model for default

In the same way as for the prepayment intensity function, we assume that the default intensity function is a function of current house price and time only, ie, $h_t^* = h^*(t, H_t^*)$. We adopt the specification for h_t^* as

$$h^*(t, H_t^*) = h_0^*(t) + \gamma^* (K^* - \log(H_t^*))^+ , \tag{3.28}$$

where $\gamma^* \geqslant 0$ and K^* are constant parameters, and h_0^* is a deterministic function of time and captures the exogenous default due to the reasons independent of interest rate. We consider the PSA assumption for the time varying part of our default as discussed in [15] and presented in Chapter 2.

The term $\gamma^* (K^* - \log(H_t^*))^+$ captures the endogenous default rate due to change in the house price over time. The idea is that the house price is a proxy for the overall economic state and when the economy is down, house prices are decreased. Similarly if the economy is good and going up the house prices are assumed to increase. This specification suggests that if the house price, at any time t, goes below the house price at time of initialization of the contract then there is a proportional increase in the default intensity and that proportion is γ^*, where $\gamma^* \geq 0$. We set the default threshold to $K^* = \log(H_0^*)$ in our work. Since our house prices follow a geometric Brownian motion. The probability that H_t^* goes below 1 is small and hence we neglect it. This makes the use of $\log(H_t^*)$ well justified and simplifies the numerical code.

The change in house price at any time can also trigger prepayment. If the difference of the current price (or the log) of the underlying house and the original price of the house on which the loan is being taken reaches a threshold level then the borrower can choose to sell the house and prepay the loan as discussed in Karia and Ushiyama in [30]. We however do not consider this prepayment factor in our modeling of mortgage rates.

Finding $\hat{h}(s, r_s)$: the hazard process of η

We now try to find the hazard process of the minimum of prepayment and default times after discussing the model specifications. We notice that if we fix the common risk factor then we can make the idiosyncratic risk factors independent of the common risk factor. In our model the interest rates define the common risk factor, and once this is fixed we can make house prices independent of it. That is knowing the information $\mathcal{F}^1$ we can find the hazard process $\hat{h}$ of η as follows.

Under probability measure $\mathbb{Q}$, $\mathbb{Q}(\eta > t \mid \mathcal{F}_t^1)$ is the probability that no default and no prepayment occurred before time t given $\mathcal{F}^1$, we have

$$\mathbb{Q}\left(\eta > t \mid \mathcal{F}_t^1\right) = \mathbb{Q}\left(\tau > t, \tau^* > t \mid \mathcal{F}_t^1\right).$$

We then use Definition 2 of conditionally independent random times to get

$$\begin{aligned}
\mathbb{Q}\left(\eta > t \mid \mathcal{F}_t^1\right) &= \mathbb{Q}\left(\tau > t \mid \mathcal{F}_t^1\right)\mathbb{Q}\left(\tau^* > t \mid \mathcal{F}_t^1\right) \\
&= \mathbb{Q}\left(\tau > t \mid \mathcal{F}_t^1\right)\mathbb{E}\left[\mathbf{1}_{\{\tau^* > t\}} \mid \mathcal{F}_t^1\right] \\
&= \mathbb{Q}\left(\tau > t \mid \mathcal{F}_t^1\right)\mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{\tau^* > t\}} \mid \mathcal{F}_t^1 \vee \mathcal{F}_t^2\right] \mid \mathcal{F}_t^1\right] \\
&= \mathbb{Q}\left(\tau > t \mid \mathcal{F}_t^1\right)\mathbb{E}\left(\mathbb{Q}\left(\tau^* > t \mid \mathcal{F}_t\right) \mid \mathcal{F}_t^1\right) \\
&= e^{-\int_0^t h_s ds}\mathbb{E}\left[e^{-\int_0^t h_s^* ds} \mid \mathcal{F}_t^1\right].
\end{aligned}$$

So we obtain

$$\mathbb{E}\left[e^{-\int_0^t \hat{h}_s ds} \mid \mathcal{F}_t^1\right] = e^{-\int_0^t h_s ds}\mathbb{E}\left[e^{-\int_0^t h_s^* ds} \mid \mathcal{F}_t^1\right]. \tag{3.29}$$

Hence we have a form that we can use to model mortgage rates. From the definition of the default intensity, we are able to solve $\mathbb{E}\left(e^{-\int_0^u h^*(s)ds} \mid \mathcal{F}_t^1\right)$. We defined the default hazard function in (3.28) with H_t^* given as in (3.26). This gives us

$$\begin{aligned} h^*(s, r_s) &= h_0^*(s) + \gamma^* \left(K^* - \ln H_0^* - \left\{ \left(r_s - \frac{1}{2}\sigma_{H^*}^2 \right) s + \sigma_{H^*} \rho W_I \right.\right. \\ &\quad \left.\left. + \sigma_{H^*} \sqrt{1-\rho^2} W_{H_s^*} \right\} \right)^+ . \end{aligned}$$

We obtain

$$\begin{aligned} \mathbb{E}\left(e^{-\int_0^u h^*(s)ds} \mid \mathcal{F}_u^1 \right) &= \mathbb{E}\left[e^{-\int_0^u \gamma^* \left(K^* - \ln H_0^* - \left(r_s - \frac{1}{2}\sigma_{H^*}^2 \right)s - \sigma_{H^*}\rho W_I + \sigma_{H^*}\sqrt{1-\rho^2} W_{H_s^*} \right)^+ ds} \right. \\ &\quad \left. e^{-\int_0^u h_0^*(s)ds} \mid \mathcal{F}_u^1 \right]. \end{aligned}$$

$$\begin{aligned} \mathbb{E}\left(e^{-\int_0^u h^*(s)ds} \mid \mathcal{F}_u^1 \right) &= \mathbb{E}\left[e^{-\int_0^u \gamma^* \left(K^* - \ln H_0^* - \left(r_s - \frac{1}{2}\sigma_{H^*}^2 \right)s - \sigma_{H^*}\rho W_I + \sigma_{H^*}\sqrt{1-\rho^2} W_{H_s^*} \right) ds} \right. \\ &\quad \left. e^{-\int_0^u h_0^*(s)ds} \mathbf{1}_{\{K^* - d^* > 0\}} \mid \mathcal{F}_u^1 \right] + \mathbb{E}\left[e^{-\int_0^u h_0^*(s)ds} \mathbf{1}_{\{K^* - d^* \le 0\}} \mid \mathcal{F}_u^1 \right]. \end{aligned}$$

where $d^* = \ln H_0^* - \left(r_s - \frac{1}{2}\sigma_{H^*}^2 \right) s - \sigma_{H^*}\rho W_I + \sigma_{H^*}\sqrt{1-\rho^2} W_{H_s^*}$.

$$\begin{aligned} \mathbb{E}\left(e^{-\int_0^u h^*(s)ds} \mid \mathcal{F}_u^1 \right) &= e^{-\int_0^u \left(h_0^*(s) + \gamma^* \left(K^* - \ln H_0^* - \left(r_s - \frac{1}{2}\sigma_{H^*} \right)s - \sigma_{H^*}\rho W_I \right) \right) ds} \\ &\quad \mathbb{E}\left[e^{-\int_0^u \sigma_{H^*}\sqrt{1-\rho^2} W_{H_s^*} ds} \mathbf{1}_{\{K^* - d^* > 0\}} \mid \mathcal{F}_u^1 \right] \\ &\quad + e^{-\int_0^u h_0^*(s)ds} \mathbb{E}\left[\mathbf{1}_{\{K^* - d^* \le 0\}} \mid \mathcal{F}_u^1 \right]. \end{aligned} \tag{3.30}$$

For the calculation and numerical purposes, we approximate the above using positive drift component in d^* that is we set our parameters such that we have a positive $-\left(r_s - \frac{1}{2}\sigma_{H^*}^2 \right)$ for long term. This is reasonable for $\theta = 0.06$, $\sigma_{H^*} = 0.36$ and $\rho = 0.45$ because in long term $K^* - \ln H_s^*$ is more likely to stay positive. Therefore we can use the following for a long term mortgage contact with above mentioned parameters.

$$\begin{aligned} \mathbb{E}\left(e^{-\int_0^u h^*(s)ds} \mid \mathcal{F}_u^1 \right) &= e^{-\int_0^u \left(h_0^*(s) + \gamma^* \left(K^* - \ln H_0^* - \left(r_s - \frac{1}{2}\sigma_{H^*} \right)s - \sigma_{H^*}\rho W_I \right) \right) ds} \\ &\quad \mathbb{E}\left[e^{-\int_0^u \sigma_{H^*}\sqrt{1-\rho^2} W_{H_s^*} ds} \mid \mathcal{F}_u^1 \right]. \end{aligned}$$

We suppose that $\alpha = \sigma_{H_s^*}\sqrt{1-\rho^2}$, we therefore have

$$\mathbb{E}\left[e^{-\int_0^u \sigma_{H^*}\sqrt{1-\rho^2}W_{H_s^*}ds} \mid \mathcal{F}_u^1\right] = \mathbb{E}\left[e^{-\alpha\int_0^u W_{H_s^*}ds} \mid \mathcal{F}_u^1\right].$$

Now using the conditional independence of $W_{H_s^*}$ on W_{I_s} and using the fact that $W_{H_s^*} \sim \mathcal{N}(0,s)$, we have $\int_0^u W_{H_s^*}ds \sim \mathcal{N}\left(0, \frac{u^3}{3}\right)$, We obtain

$$\mathbb{E}\left[e^{-\alpha\int_0^u W_{H_s^*}ds}\right] = e^{+\frac{1}{6}\alpha^2 u^3}.$$

We obtain the following result for the hazard process for the minimum of default and prepayment as follows

$$\begin{aligned}\mathbb{E}\left[e^{-\int_0^u \hat{h}_s ds} \mid \mathcal{F}_u^1\right] &= e^{-\gamma^* u\left(K^* - \ln H_0^* - \left(r_u - \frac{1}{2}\sigma_{H_s^*}^2\right)\frac{u^2}{2}\right) + \frac{1}{6}\sigma_{H^*}^2\left(1-\rho^2\right)u^3} e^{-\gamma u(K-r_u)^+} \\ &\quad e^{-\int_0^u \left(h_0(s) + h_0^*(s) - \sigma_{H_*}\rho W_{I_s}\right)ds}. \end{aligned} \tag{3.31}$$

For a set of parameters, where we do not put a restriction on $\left(r_s - \frac{1}{2}\sigma_{H^*}^2\right)$, we need to find a way to simplify the expression in 3.30 where we can use properties of a $W_{H_s^*} \sim \mathcal{N}(0,s)$. The other way is to simply the expression to use numerical methods for approximating the hazard process.

We now have an expression for the hazard process $\hat{h}(t)$ for the minimum process of default and prepayment which we can substitute in (3.24) to implement our model numerically. We can write (3.24) as

$$\begin{aligned} m &= \frac{1}{\int_0^T \mathcal{Q}(0,u)\hat{P}_u du}\left\{\mathbb{E}\left[\mathbb{E}\left[\min\left(H_{\tau^*}^*, P_{\tau^*}\right)e^{-\int_0^{\tau^*}(r_s+\hat{h}_s)ds}\mathbf{1}_{\{\tau^*<T\}}\right.\right.\right. \\ &\quad \left.\left.\left. \mid \mathcal{F}_{\tau^*}^1\right] \mid r_0\right] + \int_0^T \mathcal{R}(0,u)\hat{P}_u du\right\}. \\ &= \frac{1}{\int_0^T \mathcal{Q}(0,u)\hat{P}_u du}\left\{\mathbb{E}\left[\mathbb{E}\left[\min\left(H_{\tau^*}^*, P_{\tau^*}\right)e^{-\int_0^{\tau^*}(r_s)ds}\mathbf{1}_{\{\tau^*<T\}} \mid \mathcal{F}_{\tau^*}^1\right]\right.\right. \\ &\quad \left.\left. \mathbb{E}\left[e^{-\int_0^{\tau^*}(r_s)ds} \mid \mathcal{F}_{\tau^*}^1\right] \mid r_0\right] + \int_0^T \mathcal{R}(0,u)\hat{P}_u du\right\}. \end{aligned}$$

Then using the expression we found in (3.31) for the hazard process of the minimum of default and prepayment time, we obtain

$$
\begin{aligned}
m &= \frac{1}{\int_0^T \mathcal{Q}(0,u)\,\hat{P}_u du}\Bigg\{\mathbb{E}\Bigg[\mathbb{E}\Bigg[\min\left(H^*_{\tau^*}, P_{\tau^*}\right) e^{-\int_0^{\tau^*}\left(r_s+\left(h_0(s)+h_0^*(s)\right)\right)ds} \\
&\quad e^{-\sigma_{H_*}\rho W_{I_s} ds-\gamma^*\tau^*\left(K^*-\ln H_0^*-\left(r_{\tau^*}-\frac{1}{2}\sigma^2_{H_s^*}\right)\frac{\tau^{*2}}{2}\right)-\frac{1}{6}\sigma^2_{H^*}\left(1-\rho^2\right)\tau^{*3}} \\
&\quad e^{-\gamma\tau^*(K-r_{\tau^*})^+}\mathbf{1}_{\{\tau^*<T\}} \mid \mathcal{F}^1_{\tau^*}\Bigg] \mid r_0\Bigg] + \int_0^T \mathcal{R}(0,u)\,\hat{P}_u du\Bigg\}, \\
&= \frac{1}{\int_0^T \mathcal{Q}(0,u)\,\hat{P}_u du}\Bigg\{\mathbb{E}\Bigg[\min\left(H^*_{\tau^*}, P_{\tau^*}\right) e^{-\int_0^{\tau^*}\left(r_s+\left(h_0(s)+h_0^*(s)-\sigma_{H_*}\rho W_{I_s}\right)\right)ds} \\
&\quad e^{-\gamma^*\tau^*\left(K^*-\ln H_0^*-\left(r_{\tau^*}-\frac{1}{2}\sigma^2_{H_s^*}\right)\frac{\tau^{*2}}{2}\right)-\frac{1}{6}\sigma^2_{H^*}\left(1-\rho^2\right)\tau^{*3}-\gamma\tau^*(K-r_{\tau^*})^+} \\
&\quad \mathbf{1}_{\{\tau^*<T\}} \mid r_0\Bigg] + \int_0^T \mathcal{R}(0,u)\,\hat{P}_u du\Bigg\}. \qquad (3.32)
\end{aligned}
$$

We have an expression to find the mortgage rate of a single borrower numerically. In next section we describe the implementation of this equation numerically to find the mortgage rates. Since this is a non-linear equation we need to use a fixed point algorithm to solve it iteratively. We use a Monte Carlo simulation to simulate the interest rates and house prices. A few interesting case studies are presented in the second part of the next section. We investigate the effects of the parameters of default and prepayment on mortgage rates. We also investigate the effects of the correlation factor between the two Brownian motions as well as the volatility of the house price process.

3.5 Implementation and case studies

3.5.1 Computational implementation

In this subsection we describe briefly how we implement equation (3.32) numerically. In [23], Gorovoy and Linetsky used eigenfunction expansion of Feynman-Kac semi-group to find an expression for $\mathcal{Q}(t,u)$, $\mathcal{R}(t,u)$ and $\mathcal{H}(t,u)$ and find semi-analytical solution to the mortgage rate. We solve the non-linear equation for m numerically using Monte Carlo simulations using fixed point iteration method. We start with the initial value $m_0 = r_0$ which is an obvious choice and proceed as

$$m_i = g\left(m_{i-1}\right), \quad \text{for} \quad i = 1, 2, 3, ...$$

where $g\left(m\right)$ is the expression on right hand side of equation (3.32). Starting from $m_0 = r_0$, we simulate interest rates and house prices and find the mortgage rates using (3.32). We then check the stopping criteria for the fixed point iteration method and repeat the process with new m found at last step if the stopping criteria is not met. We stop the fixed point iterations when $\mid m_i - m_{i-1} \mid < 10^{-4}$ is reached.

Algorithm:

Keeping the parameters of the model constant, we start with $m_0 = r_0$ and

- For every path at time t, generate a random variable W_I and find

$$r_t = r_{t-1} + \kappa\left(\theta - r_{t-1}\right)\triangle t + \sigma\sqrt{r_{t-1}}\triangle W_I$$

  using Euler-Maruyama method

- Generating another random variable for W_{H^*}, find H_t^* using

$$H_t^* = H_0^* e^{\rho\triangle W_{H^*} + \sqrt{1-\rho^2}\sigma_{H^*}\triangle W_I - \frac{1}{2}\sigma_{H^*}^2\triangle t} \times \text{discounted factor}$$

- Find the intensities $h\left(t, r_t\right)$, $h^*\left(t, H_t\right)$ and $\mathbb{E}\left[e^{-\int_0^u \hat{h}_s ds} \mid \mathcal{F}_u^1\right]$

- In case there is no default, we find $\mathcal{Q}\left(0, u\right)$ and $\mathcal{R}\left(0, u\right)$ and the integrals in (3.32) and compute m

- If $\tau^* > t$ that is if default occurs, then we find the additional expectations in (3.32) and compute m

- We stop the fixed point iterations when the stopping criteria is met.

We are able to solve the expression for m in (3.32) numerically and are now in a position to study the effects of the prepayment and default parameters on the mortgage rates. We compare our results to Gorovoy and Linetsky in [23] where we consider only the prepayment incentive of the borrower.

3.5.2 Case studies

Mortgage Rates vs. Intensity Parameters

In this section, we investigate the effects of the parameters of prepayment and default on mortgage rates. The parameters of CIR and house price processes are kept fixed during this analysis. The prepayment and default intensities are considered as in equations (3.27) and (3.28), and the PSA assumptions for $h_0(t, r_t)$ and $h_0^*(t, H_t^*)$ as defined in Chapter 2. We then have two prepayment parameters γ and b, and two default parameters γ^* and b^*. The interest rate threshold is set to $k = r_0 - s$ where s is a spread of 25 basis points, subtracted to account the cost of refinancing transaction costs. The house price threshold is set at $K^* = \ln(H_0^*)$ so that if log of house prices, at any time t, goes below this threshold then the default intensity will increase with a proportion of γ^*.

We plot the effects of prepayment and default separately as well as their combined effect on mortgage rates for three different starting interest levels $r_0 = 0.03$, $r_0 = 0.06$ and $r_0 = 0.09$. On each of these graphs, we plot four term structures of mortgage rates: without prepayment and default, with exogenous prepayments or defaults (γ and γ^* equal to zero), interest rate-driven prepayment or default (b and b^* equal to zero), and both exogenous and interest rate-driven prepayment or default.

In Figure 3.1, the current interest rate r_0 is set to be 0.03 which is much lower than the assumed long-run level $\theta = 0.06$. Since we assumed a mean reverting process for interest rates, the rates are expected to increase towards the long-run mean level θ

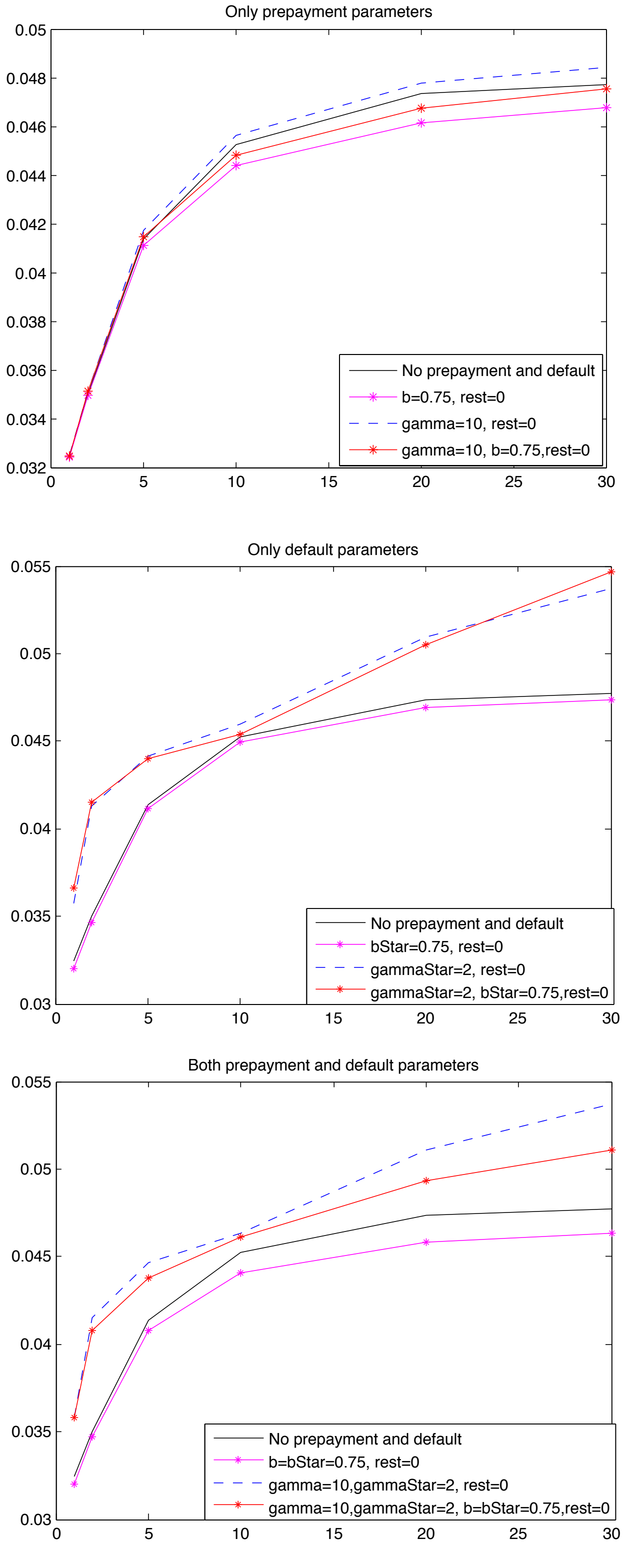


Figure 3.1: Mortgage rates $m = m(0, T)$ as function of maturity T (in years). Model Parameters: $r_0 = 0.03$, $\theta = 0.06$, $\kappa = 0.25$, $\sigma = 0.1$, $\sigma_{H^*} = 0.36$, $\rho = 0.45$, $P_0 = 200000\$$, and $H_0^* = 200000\$$. The prepayment and default parameters are given on graphs.

in future. As a result, the term structure for the mortgage rates in the case of no prepayment and default is steeply upward sloping. The first figure shows the effects of the prepayment parameters on mortgage rates and is similar to the one in Gorovoy and Linetsky [23]. The mortgage rates with exogenous prepayment ($b = 0.75, \gamma = 0$) are lower than the one with no prepayment and default, a negative spread is seen. This makes sense as the mortgagor is still paying on the low interest rates and is less likely to prepay the remaining principal. On the other hand, the mortgage rates with the interest rate-driven refinancing only ($b = 0, \gamma = 10$) are higher than the mortgage rates without prepayment and default. The prepayment spread is positive but small as the probability that the interest rates would further decline is very small due to the mean reverting property of the interest rate process. The mortgage rates with $b = 0.75$ and $\gamma = 10$ bring the prepayment spreads very close to zero as the positive prepayment spread arising from $\gamma = 10$ and $b = 0$ is cancelled by the negative prepayment spread arising from $\gamma = 0$ and $b = 0.75$.

The second figure in Figure 3.1 investigates the effects of the default intensity parameters on the mortgage rates when the current interest rate level is 3%. The driving process for the house price is correlated to the interest rate driving process with a correlation factor of $\rho = 0.45$ (in our case) as well as the drift for the house price process is assumed to be the current interest rate. This means that changes in interest rates will change our house prices at any time with volatility $\sigma_{H^*} = 0.36$. Assuming that the current interest rate is much lower than the assumed long run level, the interest rates will tend to increase and will increase the house prices in the long run. For the exogenous case we see a negative default spread. This makes sense because the mortgagor is paying the mortgage at the time of lower interest rates and is unlikely to default in time of higher interest rates when the house prices are more likely to increase. For the case when default occurs due to a change in house prices we see a positive spread of 59.68 basis points which is substantially smaller than the other two cases for interest rate levels that we are discussing.

Moving to the case when we allow both default and prepayment in our model. The third figure in Figure 3.1 exhibits the combined effect of prepayment and default on mortgage rates. We see a negative spread for the case of exogenous default and prepayments which we expected by the analysis of the prepayment and default parameters. For the case when $\gamma > 0$ and $\gamma^* > 0$ we see a positive spread of 59.87

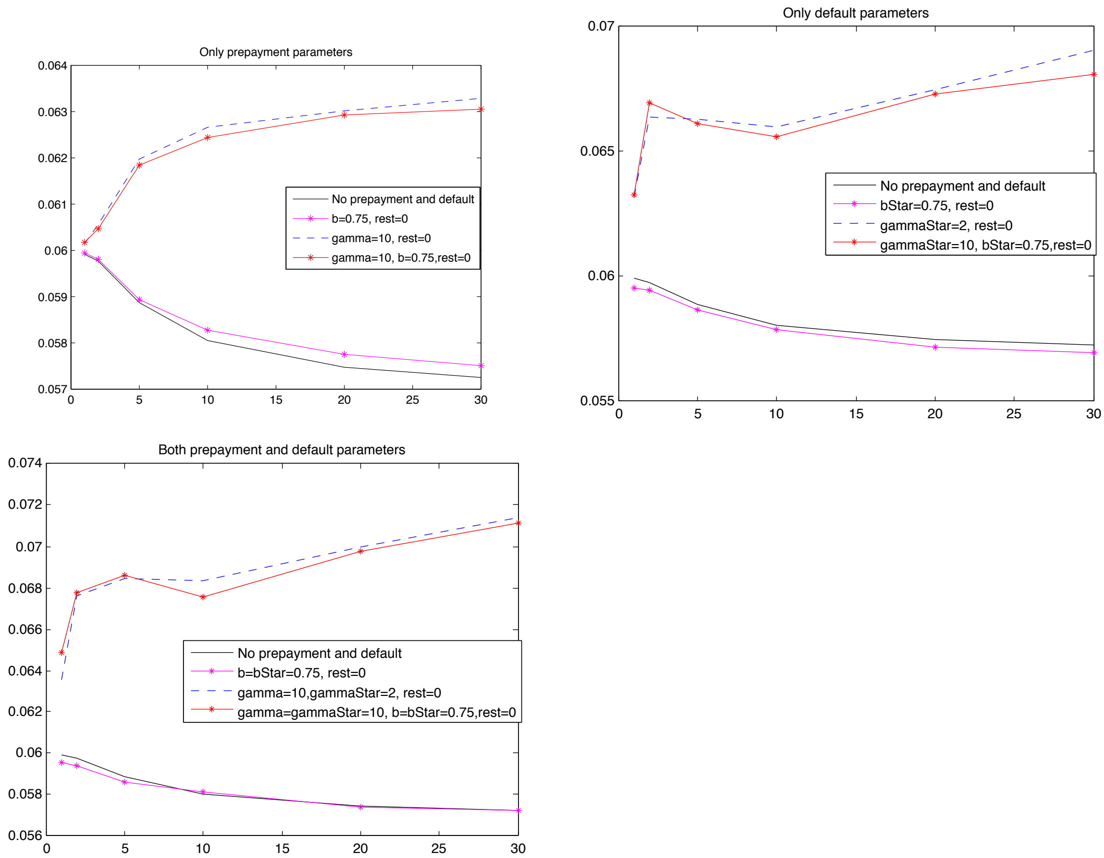


Figure 3.2: Mortgage rates $m = m(0, T)$ as function of maturity T (in years). Model Parameters: $r_0 = 0.06$, $\theta = 0.06$, $\kappa = 0.25$, $\sigma = 0.1$, $\sigma_{H^*} = 0.36$, $\rho = 0.45$, $P_0 = 200000\$$, and $H_0^* = 200000\$$. The prepayment and default parameters are given on graphs.

basis points which is closer to the default spread for $\gamma^* > 0$ which shows that in the low interest rate environment the possibility of prepaying the remaining principal before maturity is negligible when we observe prepayment and default in our model.

Next we analyze the second case when the current rate of interest is equal to the long run interest rate that is $r_0 = 0.06$. This very case is in the middle of the two extremes in prepayment and default spreads for $r_0 = 0.03$ and $r_0 = 0.09$. The term structure without prepayment and default is sloping downward due to the convexity effect. The addition of exogenous prepayment adds a very small positive spread to the mortgage rate. The increase or decrease in interest rates in the future is roughly equally likely by setting the current interest rates to the long run level. Thus the possibility of exogenous prepayment would occur at a lower or higher interest rate is equally likely so we expect a small positive spread. The prepayment due to interest rate driven refinancing does add a substantial positive prepayment spread that is larger than the spread of the lower regime with $r_0 = 0.03$ and smaller than the spread of the higher regime with $r_0 = 0.09$.

The second figure in Figure 3.2 shows the effects of default parameters in our model when the current interest rates are assumed to be equal to the long run interest rate level. The exogenous default curve still has a negative spread but this time the spread is smaller than the lower interest rate regime. This is because the probability that the interest rate will increase or decrease in the future is equally likely and hence it is roughly equally likely that exogenous default occurs at lower interest rates or at higher interest rates. Thus a small spread is seen for the exogenous defaults. On the other hand the default spread for non-exogenous case is very large and positive but this spread is larger than the spread for $r_0 = 0.03$ and is smaller than the spread for $r_0 = 0.09$. We see a hump for shorter maturity mortgage contract this is because for a shorter maturity the mortgage rates are more volatile and the probability of default increases which can be compensated by a long term behaviour in house prices. When we allow both prepayment and default in this model we get the results as in third figure in Figure 3.2. The exogenous default and prepayment spread is very small. This is because the exogenous prepayment has a small positive spread whereas the exogenous default spread has a small but negative spread.

Now moving to the case of high current interest rates, that is $r_0 = 0.09$ in Figure

3.3, the opposite behavior to the case with low current interest rates is observed. The term structure of mortgage rates without prepayment and default is downward-sloping. The interest rates are expected to decline in future towards the long run mean reverting level 0.06. The prepayment spread in the exogenous prepayment case has a moderately positive spread.

The prepayment spread in the interest driven prepayment intensity is however very large. This makes sense as the interest rates are declining in future and the borrower will intend to refinance at a lower interest rate. Hence, to compensate the reinvestment risk, the lender quotes a higher mortgage rate in the case of no prepayment and default. The exogenous rates do not substantially add up in the interest driven prepayment as due to the lower interest rates in future the mortgage is likely to be paid off. To investigate the effect of default parameters on the mortgage, in this case we set the threshold K^* to 0.09 equal to the current interest rate. A moderately positive spread can be noticed for the exogenous default. However a relatively significant spread is observed for interest driven default which declines for longer maturities. The mortgage rates are sloping downward because the interest rates tend to decline in the future and the probability of getting in to default also decreases. Since at the origination, the probability that the interest rates will decline in the short term is very small, we see a hump for the mortgages with short maturity. The interest rates will decline towards the long run interest rate level over time and we see the spread of default with that.

Finally, we investigate the effects of prepayment and default on mortgage rates for the above framework. For exogenous prepayment and default, we alter a positive spread which is expected as we see positive prepayment and default spreads in separate frameworks. It is interesting, however, to look at the interest driven prepayment and default case. A relative constant effect can be seen with a positive slope for shorter maturities and a very small downward slope for longer maturities.

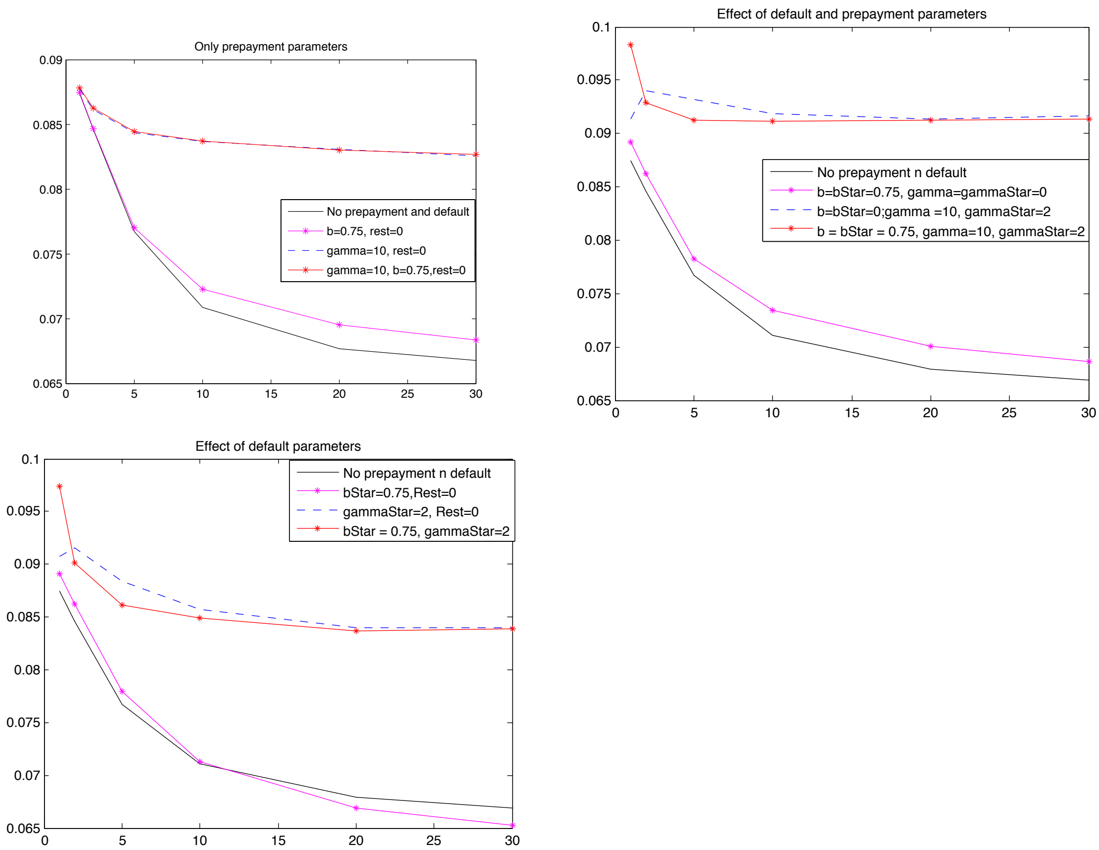


Figure 3.3: Mortgage rates $m = m(0,T)$ as function of maturity T (in years). Model Parameters: $r_0 = 0.09$, $\theta = 0.06$, $\kappa = 0.25$, $\sigma = 0.1$, $\sigma_{H^*} = 0.36$, $\rho = 0.45$, $P_0 = 200000\$$, and $H_0^* = 200000\$$. The prepayment and default parameters are given on graphs.

The effects of prepayment and default can clearly be observed in this case since the interest rates decline in future, the borrower will intend to prepay and the default risk is relatively high for small maturities, an increase in mortgage rates can be seen to compensate these two risks. This graph becomes almost constant for higher maturities as the default spread decreases over time whereas the prepayment spread increases for longer maturities.

Mortgage Rates vs. House Pricing Parameters

In this section, we investigate the relationship between the house price parameters and mortgage rates. We assumed that the house prices follow a geometric Brownian motion and that the house prices are related to the interest rates by a correlation factor ρ. Now we analyze the effects of these two parameters on mortgage rates.

The left figure in Figure 3.4 below investigates the effect of ρ on mortgage rates. We keep the parameters of prepayment and default as non-zero constants. The interest rates follow the CIR process with the parameters as in the last section and the current value of interest rate is set to 3%. The threshold of prepayment and default are also same as in the last section. A moderate increase in mortgage rates can be seen as an increase in ρ but the prepayment and default spread is small. The mortgage rates are increasing because of an increase in interest rates but at the same time the default and prepayments are unlikely to happen so we see some small spreads.

Now we discuss the effects of volatility in the house price model, assuming that the house price model follows a geometric Brownian motion. The right figure in Figure 3.4 alters the effects of σ_{H^*} on mortgage rates. The figure is plotted for current interest rate at 0.03 while all other parameters of prepayment and default are kept non-zero constants. The interest rates will tend to increase towards the long run interest rate 0.06. The volatility can be seen in mortgage rates on changing the values for σ_{H^*}.

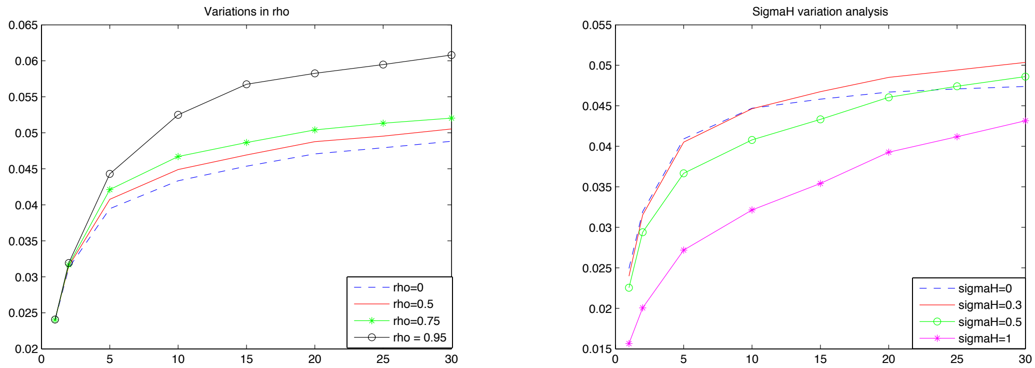


Figure 3.4: Effects of ρ and σ_{H^*} on mortgage rates. Model Parameters: $r_0 = 0.03$, $\theta = 0.06$, $\kappa = 0.25$, $\sigma = 0.1$, $b = b^* = 0.75$, $\gamma = 5$, $\gamma^* = 1$, $P_0 = 200000\$$, and $H_0^* = 200000\$$. The parameters of house price are given on the graphs

The correlation factor ρ is fixed at 0.45. We see that as we increase σ_{H^*}, the mortgage rates decline and the spread from the case where $\sigma_{H^*} = 0$ is negative. This shows that the volatility has a negative effect on the mortgage rates. This is because one cannot predict the future economy when the economy is more volatile, and when the economy is unstable the mortgage market is also down. This means that the mortgages are lent on a lower rate than in the case of a stable market that is for small value of σ_{H^*}. The volatility effect is large for the contracts with smaller maturities and the mortgage rates are low for these maturities. The volatility spread is reduced for contracts with longer maturities and for small values of σ_{H^*} this spread becomes positive, as we do not expect that the economy will remain volatile for long.

3.6 Comments

In this chapter we developed a model for a single mortgage in intensity based framework by extending a pre-existing model of Gorovoy and Linetsky [23] by including defaults in it. We presented some case studies and compared our results with that in [23]. The model presented was to find the mortgage rates of a single mortgage contract. A mortgage backed security is a large collection of these individual mortgages. The current model can be extended easily to a homogenous pool of mortgages where the value of the whole pool is equal to the value of the individual mortgage contract times the number of individual in the pool. The other assumption is to consider the prepayment intensity as a function of short rate only, as with the default intensity. Goncharov in [19] and Pliska in [48] study more a general form of prepayment intensity, assuming the prepayment intensity is as a function of current mortgage rates. Similarly, default intensity can also depend on current mortgage rates and interest rates explicitly. This represents a possible extension to the model developed in this chapter. Recent work on reduced forms models by Gu et al. in [25] and [26] can also be adapted to MBS settings. They discussed pricing of basket credit swaps in both homogeneous and two-group heterogeneous cases under the interacting intensity default contagion model. They also considered stochastic intensity of defaults and multiple defaults which can be adapted to the MBS settings with stochastic intensity and partial prepayments.

Chapter 4

A structural model for a pool of mortgages

In this chapter we work in a structural framework in a large portfolio setting as in [8] and [28]. In [8], Bush et al. applied the large portfolio limit of a multidimensional structural model to credit modelling and obtained a stochastic partial differential equation (SPDE) for the evolution of distance to default of an infinite portfolio of the underlying assets. They considered the approach of modelling the empirical measure of the asset prices assuming that the underlyings are linked through a common factor. We adopt the same strategy to model a large portfolio of individuals in a mortgage pool and find an SPDE for the evolution of the large portfolio limit of the mortgage pool.

We introduce the model and the key factors that one needs to take into account to model mortgage backed securities. We introduce two boundary conditions incorporating defaults and prepayments and one killing term for prepayments due to refinancing in our model. We further define the empirical measure associated to the model and show the results for the uniqueness and existence of the limit empirical measure of the system.

4.1 Choosing the key factors of the model

In this section we justify the choice of the variables for the model before formally introducing it. We have discussed two principal risks involved with the securities in Chapter 2 that affect the cash flow of a mortgage pool. In this section we present some studies and discuss the factors behind the defaults and prepayments, ie, the determinants of defaults and prepayments.

We start by considering an individual mortgage at the time of initialization of the contract before looking at the pool as a whole. The key factors at the initialization of single contract that play an important role in the cash flow of the pool are

- an individual's wealth or credit score,
- the size of the loan or Loan to Value ratio (LTV ratio),
- the income stream, and
- current interest rates.

Once a contract is initialized, the events that can cause early termination are defaults and prepayments. During the life of a mortgage contract, default occurs when there is a shortfall in the borrower's income and he is unable to make the scheduled monthly payments for more than three months. Prepayment, on the other hand, occurs either because of excess money to pay all the remaining mortgage principal or because of refinancing.

The individual's wealth, the LTV ratio and the income stream at the initialization ensure that the individual has the capability to repay his loan in monthly installments provided his financial situation remains the same or improves over time. If the individual's credit score falls below the set prerequisite credit score (Fair Isaac Corporation score (FICO) = 660), then the individual is more likely to default. Similarly, if the individual has taken more than 95% mortgage on his house then he is more likely to have difficulties in repaying the loans over the life of the contract.

During the life of the contract, if there is a shortfall in the individual's income stream then a default may occur. The shortfall could be due to unemployment, bankruptcy, divorce or moving home, for example. Other events that typically terminate the contract on default are reduced working hours, reduced pay, illness, separation or the death of a partner. Empirical studies show that all these events play a role in default on mortgage contracts, though the exact percentage of defaults due to each of these events is not available. Oxford Economic Forecasting [10], for example, estimated that holding everything else constant, a rise in the unemployment rate by 10% will raise defaults by 30%. In a Survey of English Housing (SEH) [17], a significant proportion (21%) that experienced mortgage default listed the death of a partner or relationship breakdown as a contributory factor; about 11% cited reduced hours and/or reduced pay as a contributory factor.

Empirical and theoretical studies suggest that the interest rates, LTV, borrower credit worthiness, loan size and other variables have an impact on the prepayment behaviour. Lai, Bogdon and Li [38] discussed the empirical results and showed that interest rate changes have a significant impact on prepayments. Prepayments due to refinancing are a good example of how interest rates affect the cash flow of a mortgage pool. Prepayments can also occur when the individual has excess money and is able to prepay all the remaining loan. The excess in money could be a result inherited wealth due to a death in the family, winning a lottery or personal savings over time. Green and Shoven [24], Campbell and Dietrich [9] and Cunningham and Capone [11] demonstrated through theoretical studies that prepayment is indeed affected by changes in interest rates, LTV ratio, mortgage age and the borrower's creditworthiness.

From this we learn that the significant factors that contribute to the risk of default and prepayment can be captured by an individual's wealth and by adding appropriate conditions for default and prepayment. We therefore consider a wealth process for each individual, taking values within certain limits, in a pool to model the mortgage backed securities. We learn that an individual can refinance on lower interest rates. Thus we introduce a killing term that removes the individual from the pool when the interest rates go below some threshold level.

So far we have analyzed the factors that affect individual mortgages, but when combining the mortgages into a pool, there are macro economic factors that might cause overall levels of defaults or prepayments to vary over time. These factors can include an overall decrease in the prices of houses or baseline interest rates. We also take into account these systematic factors that will impact all individuals at the same level in our model by introducing a common market factor. We are now in a position to describe the model formally in the next section.

4.2 The model

We see from the above analysis that the wealth of an individual is the starting point of modelling a mortgage backed security. We consider the wealth of each individual in the pool as a point, taking values in $[B_L, B_U]$ at any time t where B_L and B_U are chosen as lower and upper limits on the wealth that ensures the individual is able to pay the scheduled monthly payments but not so much that they do not need a mortgage. The individual's wealth process reaching the lower limit B_L indicates that he does not have enough money to pay the scheduled monthly payments and is in default. The upper limit B_U is set to take account of the prepayments in case the individual gets enough money to prepay all the remaining loan and decides to end the contract. The individual is absorbed at either boundary if he is in default or prepaid due to any of the reasons discussed before.

Interest rates also affect prepayments throughout the life of a pool. Refinancing occurs when interest rates decrease at any time and it is profitable to prepay the loan and refinance on lower interest rates. We include the prepayment due to refinancing in the model by introducing a variable τ_r^i for each individual. We assume that τ_r^i is an unpredictable variable and has an exponential distribution with rate of refinancing $\lambda(r_t)$ that depends on the interest rates r_t at time t.

We denote the probability space for the mortgage pool consisting of n individuals by $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$. We assume that all the individuals in our mortgage pool have same drift μ and volatility σ and are correlated to each other by a common market factor.

Before the time of default or prepayment, the dynamics of the wealth of i^{th} individual at time t under the risk neutral measure $\mathbb{P}^n$ follows a diffusion process

$$dY_t^i = Y_t^i \left\{ \mu dt + \sigma\rho dM_t + \sigma\sqrt{1-\rho^2} dW_t^i \right\} \quad i = 1, \cdots, n, \qquad t < \tau^i$$

W_t^i and M_t are standard Brownian motions on the probability space $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$ satisfying

$$d[W_t^i, M_t] = 0 \quad \forall i = 1, 2, \cdots, n$$
$$d[W_t^i, W_t^j] = \delta_{ij} dt, \quad \forall i, j = 1, 2, \cdots, n,$$

where $[.,.]$ is the quadratic covariation, and we will use $[.]$ for the quadratic variation. μ, $\sigma \geq 0$, and ρ are constants. $0 \leq \rho \leq 1$, $\forall i, i = 1, \cdots, n$. We define the stopping time τ^i as the first time that either default or prepayment occurs, that is

$$\tau^i \quad := \quad \min\left\{\tau_{B_L}^i, \tau_{B_U}^i, \tau_r^i\right\},$$

where

$$\tau_{B_L}^i := \inf\left\{t : Y_t^i \leq B_L\right\},$$

$$\tau_{B_U}^i := \inf\left\{t : Y_t^i \geq B_U\right\},$$

$$\tau_r^i := \begin{cases} \inf\left\{t : \int_0^t \lambda\left(r_s\right) ds > e^i\right\} & \text{if } t < \tau_{B_L}^i \wedge \tau_{B_U}^i \\ \infty & \text{if either } \tau_{B_L}^i \text{ or } \tau_{B_U}^i \text{ has occurred} \end{cases}$$

where $e^i \sim \exp(1)$ are independent. The interest rates r_t follow a diffusion process.

The system for the wealth of individuals in a mortgage pool then becomes

$$\begin{cases} dY_t^i = Y_t^i \left\{\mu dt + \sigma\rho dM_t + \sigma\sqrt{1-\rho^2} dW_t^i\right\}, \quad t < \tau^i \\ Y_t^i = B_L, \qquad t \geq \tau^i = \tau_{B_L}^i \\ Y_t^i = B_U, \qquad t \geq \tau^i = \tau_{B_U}^i \\ Y_t^i = \triangle, \qquad t \geq \tau^i = \tau_r^i \\ Y_0^i = b^i \qquad B_L < b^i < B_U, \end{cases} \tag{4.1}$$

where each Y_t^i is driven by the idiosyncratic noise W_t^i and the common factor M_t and b^i is the initial wealth of each individual. The individuals who choose to refinance their loan are sent to the cemetery state $\triangle$ and are removed from the pool. Y_0^i is the wealth of individual i at the beginning of the contract.

We denote the default and prepayment proportions in the pool up to time t by

$$\begin{aligned} b_{t,n}^{B_L} &= \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\left\{\tau^i=\tau_{B_L}^i \leq t\right\}} = \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\left\{Y_t^i \leq B_L\right\}}, \\ b_{t,n}^{B_U} &= \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\left\{\tau^i=\tau_{B_U}^i \leq t\right\}} = \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\left\{Y_t^i \geq B_U\right\}}. \\ b_{t,n}^{\triangle} &= \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\{\tau^i=\tau_r^i \leq t\}} = \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\left\{Y_t^i = \triangle\right\}}, \end{aligned}$$

The more popular mortgage contracts are issued for 30 years, hence we assume that the mortgages that are neither defaulted nor prepaid are fully amortized at time $T = 30$ years. We further assume that all the borrowers enter into the portfolio at the same time with the same maturity.

We make a simple transformation of the wealth of the individual and consider another system with a default boundary at 0, a prepayment boundary at 1, and a cemetery state $\triangle$. Consider the transformation

$$X_t^i = \frac{\log Y_t^i - \log B_L}{\log B_U - \log B_L}.$$

The location of the points X_t^i in the new system is the individuals' distance from the default boundary scaled by $\frac{1}{\log B_U - \log B_L}$. We obtain the dynamics of X_t^i by applying Ito's formula

$$\begin{aligned} dX_t^i &= \frac{1}{\log B_U - \log B_L}\left\{\frac{1}{Y_t^i}dY_t^i - \frac{1}{2\left(Y_t^i\right)^2}d[Y_t^i, Y_t^i]\right\} \\ &= \frac{1}{\log B_U - \log B_L}\left\{\mu dt + \sigma\sqrt{1-\rho^2}dW_t^i + \sigma\rho dM_t - \frac{1}{2}\sigma^2 dt\right\}, \\ &= \hat{\mu}dt + \hat{\sigma}\sqrt{1-\rho^2}dW_t^i + \hat{\sigma}\rho dM_t, \end{aligned}$$

where

$$\hat{\mu} = \left(\mu - \frac{1}{2}\sigma^2\right)\left(\frac{1}{\log B_U - \log B_L}\right) \tag{4.2}$$

and

$$\hat{\sigma} = \frac{\sigma}{\log B_U - \log B_L}. \tag{4.3}$$

The dynamics of X_t^i after including the default and prepayment conditions become

$$\begin{cases} dX_t^i = \hat{\mu} dt + \hat{\sigma}\sqrt{1-\rho^2} dW_t^i + \hat{\sigma}\rho dM_t, & t < \tau^i \\ X_t^i = 0, \quad t \geq \tau^i = \tau_0^i, \\ X_t^i = 1, \quad t \geq \tau^i = \tau_1^i, \\ X_t^i = \triangle, \quad t \geq \tau^i = \tau_r^i, \\ X_0^i = x^i, \quad 0 < x^i < 1, \end{cases} \tag{4.4}$$

where

$$\tau^i = \min\left(\tau_0^i, \tau_1^i, \tau_r^i\right),$$

and

$$\begin{aligned} \tau_0^i &= \inf\left\{t : X_t^i \leq 0\right\} = \inf\left\{t : Y_t^i \leq B_L\right\}, \\ \tau_1^i &= \inf\left\{t : X_t^i \geq 1\right\} = \inf\left\{t : Y_t^i \geq B_U\right\}, \end{aligned}$$

$$\tau_r^i := \begin{cases} \inf\left\{t : \int_0^t \lambda(r_s)\, ds > e^i\right\} & \text{if } t < \tau_0^i \wedge \tau_1^i \\ \infty & \text{if either } \tau_0^i \text{ or } \tau_1^i \text{ has occurred} \end{cases}$$

where as before $e^i \sim \exp(1)$ are independent and the interest rates r_t follow some diffusion process. $\triangle$ is a cemetery state, $\hat{\mu}$ and $\hat{\sigma}$ are defined by (4.2) and (4.3) and

$$x^i = \frac{\log b^i - \log B_L}{\log B_U - \log B_L} \quad \text{and} \quad 0 < x^i < 1, \forall i.$$

The default and prepayment proportions of the mortgage pool up to time $t > 0$ are given as

$$\begin{aligned} b_{t,n}^0 &= \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\left\{\tau^i = \tau_0^i \leq t\right\}} = \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\left\{X_t^i \leq 0\right\}}, \\ b_{t,n}^1 &= \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\left\{\tau^i = \tau_1^i \leq t\right\}} = \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\left\{X_t^i \geq 1\right\}}, \\ b_{t,n}^{\triangle} &= \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\{\tau^i = \tau_r^i \leq t\}} = \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\left\{X_t^i = \triangle\right\}}. \end{aligned}$$

With constant coefficients, the system in (4.4) has a unique strong solution with boundary conditions at 0 and 1.

We now define the empirical measure for the pool as

$$\nu_{n,t} = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_t^i},$$

which has a support in $[0,1]\cup\Delta$ and where δ_x is a Dirac measure at point x. Typically the mortgage pools consist of a large number of individual mortgages and our setup allows us to extend the system to an infinite system by letting $n \to \infty$. Therefore, we investigate the limit empirical measure whose density satisfies an SPDE. We are interested in the limit empirical measure as $n \to \infty$, that is for a large number of individuals in a mortgage pool which also has a support in $[0,1] \cup \Delta$. We prove the result for the existence of the limit empirical measure in the next section.

4.3 Existence of the limit empirical measure

In this section we show the existence of the limit empirical measure and the limit default proportions. We give the definition of exchangeability as well as Hewitt and Savage's and de Finetti's theorems that we need in our proof of the existence of the limit empirical measure.

We define the notion of exchangeability for a complete, separable metric space S as

Definition 4. *An infinite sequence $\{\xi_1, \xi_2, \cdots\}$ of S-valued random variables is exchangeable if for any finite number n and any two finite sequences $i_1, i_2, \cdots, i_n$ and $j_1, j_2, \cdots, j_n$, the subsequences $\xi_{i_1}, \cdots, \xi_{i_n}$ and $\xi_{j_1}, \cdots, \xi_{j_n}$ have the same distribution. A sequence $\xi_1, \xi_2, \cdots$ is exchangeable if every finite subfamily $\{\xi_1, \cdots, \xi_m\}$ is exchangeable.*

We give the following version of Hewitt and Savage's theorem as given by Lauritzen in [39].

Theorem 4.3.1 (Hewitt and Savage [27]). *Let $X_1, \cdots, X_n, \cdots$ be an exchangeable sequence of random variables with values in a complete, separable metric space S. Then there exists a probability measure ζ on the set of probability measures $\mathcal{P}(S)$, such that*

$$\mathbb{P}(X_1 \in A_1, \cdots, X_n \in A_n) = \int Q(A_1) \cdots Q(A_n) \zeta(dQ),$$

where Q is a distribution so that $X_1, \cdots, X_n, \cdots$ are independent and identically distributed with distribution Q. It further holds that ζ is the distribution function of the empirical measure

$$R(A) = \lim_{n \to \infty} R_n(A) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \chi_A(X_i), \; a.s., \quad R \sim \zeta$$

and $Q^{\otimes \infty}$ is the distribution obtained by conditioning on R:

$$\mathbb{P}(X_1 \in A_1, \cdots, X_n \in A_n \mid R) = Q(A_1) \cdots Q(A_n).$$

We now state de Finetti's theorem from [[36], Theorem 4.1] for the empirical measure of an infinite exchangeable sequence of random variables.

Theorem 4.3.2 (de Finetti). *Let $\xi_1, \xi_2, \cdots$ be an exchangeable sequence of S-valued random variables, where S is a complete and separable metric space. Then there is a $\mathcal{P}(S)$-valued random variable Ξ such that*

$$\Xi = \lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} \delta_{\xi_m}, \; a.s,$$

where $\mathcal{P}(S)$ denotes the collection of all probability measures on S. Moreover, conditioned on Ξ, $\xi_1, \xi_2, \cdots$ are i.i.d. with distribution Ξ, that is, for each $f \in \mathcal{B}(S^m)$, $m = 1, 2, \cdots$,

$$\mathbb{E}[f(\xi_1, \cdots, \xi_n) \mid \Xi] = \langle f, \Xi^m \rangle.$$

Before proceeding to the results on the existence of empirical measure, we first define the weak convergence of measure.

Definition 5 (Weak convergence)**.** *Let U be a metric space with its Borel sigma algebra Σ. A sequence P_n, $n = 1, 2, \cdots$, of probability measures on (U, Σ) converges weakly to P if*

$$\lim_{n\to\infty} \int_U f(x)\, dP_n(x) = \int_U f(x)\, dP(x)$$

for every bounded and continuous function $f : U \to \mathbb{R}$.

We now give the result for existence of empirical measure $\nu_t = \lim_{n\to\infty} \nu_{n,t}$. Let $I = [0, 1]$. We write $\mathcal{P}(I)$ for the set of probability measure on I and $C_{\mathcal{P}(I)}[0, 1]$ for the collection of all $\mathcal{P}(I)$-valued continuous functions on $[0, 1]$. The topology we use in $\mathcal{P}(I)$ is the weak convergence topology. We give a theorem for the existence of the limit empirical measure in $C_{\mathcal{P}(I)}[0, 1]$ and the existence of its limit default proportions in $C_I[0, 1]$, the set of all $[0, 1]$-valued continuous functions on $[0, 1]$, under the assumption of exchangeability.

Assumption A

By construction of the model we see that changing the labels of individuals does not matter, so we assume that $\{X_0^1, \cdots, X_0^n\}$ is a family of exchangeable, $[0, 1]$-valued random variables which are independent of W^i and M.

Theorem 4.3.3. *There exists a $C_{\mathcal{P}(I)}[0, 1]$-valued random variable ν_t such that*

$$\nu_t := \lim_{n\to\infty} \nu_{n,t} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{X_t^i}, \qquad a.s.,$$

under the assumption (A). Furthermore, the default proportions

$$\begin{aligned} b_t^0 &:= \lim_{n\to\infty} b_{t,n}^0 = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\{X_t^i\le 0\}} \\ b_t^1 &:= \lim_{n\to\infty} b_{t,n}^1 = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\{X_t^i=1\}} \\ b_t^{\triangle} &:= \lim_{n\to\infty} b_{t,n}^{\triangle} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\{X_t^i=\triangle\}} \end{aligned}$$

exist almost surely in $C_I\,[0,1]$.

Proof. We denote the system in (4.4) without absorption by $\{\tilde{X}_t^i\}$ that is $\tilde{X}_t^i = X_0^i + \hat{\mu}t + \hat{\sigma}\sqrt{1-\rho^2}W_t^i + \hat{\sigma}\rho M_t$, for all t. Then

$$X_t^i = \tilde{X}_t^i \mathbf{1}_{\left\{0<\min_{0\le s\le t}\tilde{X}_s^i,\max_{0\le s\le t}\tilde{X}_s^i<1\right\}}\mathbf{1}_{\left\{\int_0^t \lambda(r_s)ds<e^i\right\}} := F\left(\tilde{X}_s^i, e^i, 0\le s\le t\right).$$

F is independent of $i's$, therefore exchangeability of $\{X^i\}$ in $C_I\,[0,1]$ follows from the exchangeability of $\{\tilde{X}_t^i\}$ in $C_I\,[0,1]$.

By assumption $\{X_0^i\}$ is a family of exchangeable random variables and as $\tilde{X}_t^i = X_0^i + \hat{\mu}t + \hat{\sigma}\sqrt{1-\rho^2}W_t^i + \hat{\sigma}\rho M_t$, for all $t<\tau_r^i$, we have $\{\tilde{X}_t^1,\cdots,\tilde{X}_t^n\}$ is exchangeable for any $t<\tau_r^i$. It is then easy to show that for any integer N, $\{\tilde{X}_\cdot^1,\cdots,\tilde{X}_\cdot^N\}$ is exchangeable in $C_I\,[0,1]$ using induction as shown in Jin's DPhil thesis [28], and as a consequence $\{X_\cdot^1,\cdots,X_\cdot^N\}$ is exchangeable in $C_I\,[0,1]$.

The process in system (4.4) can also be absorbed at the cemetry state due to exponential killing. Assuming that the market factor M_t and interest rates r_t appearing in the rate of killing are independent, we see that $\{\tilde{X}_t^1,\cdots,\tilde{X}_t^n\}$ is again exchangeable.

Since the system in (4.4) can easily be extended to an infinite system by de Finetti's theorem, we have

$$\nu = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{X^i_\cdot} \tag{4.5}$$

exists almost surely in $\mathcal{P}\left(C_I\left[0,1\right]\right)$.

The proof of the theorem follows from Jin's DPhil thesis [[28], theorem 4.3] and from Bush et al. [[8], theorem 3.1] $\square$

We have so far defined the model and the empirical measure associated with it. We have also shown that the limit of the empirical measure exists. In the next chapter we investigate the evolution of the empirical measure and examine the behaviour of the limiting measure near the boundaries.

Chapter 5

The evolution of the limit empirical measure and SPDEs

In the previous chapter we provided a detailed description of our model for a large pool of individual mortgages and defined an empirical measure for our system of mortgages. We further showed that the limit empirical measure exists for the system. In this chapter we show that the limit empirical measure satisfies an evolution equation on $[0, 1]$. Assuming that the limit empirical measure has a density with respect to Lebesgue measure, we derive a stochastic partial differential equation satisfied by the density; we prove the existence of the density in Chapter 6. We show that the boundary conditions are satisfied by the limit empirical measure under certain assumptions for the initial density. We give estimates on the second moments of the limit empirical measure near both boundaries. The estimates are then used in the proof of the existence and uniqueness of the solution of the evolution equation.

In this chapter in order to derive the evolution equation we amend our model to take into account the jump at the time of refinancing and record the position of each individual at that time. We start by considering the distance to default process $\tilde{X}_t^i$ as a two component process $\tilde{X}_t^i = \left(X_t^i, X_{t-}^i\right)$ whose first component is our previously defined distance to default process and the second component is the left continuous version X_{t-}^i this is the same as X_t^i for all times up to the time of refinancing, and at the time of refinancing X_t^i jumps to the cemetry state $\triangle$ and $X_{t-}^i = X_{\tau_r^i}^i$ records the position where the borrower chose to refinance, that is $\tilde{X}_{\tau_r^i}^i = \left(\triangle, X_{\tau_r^i}^i\right)$, at the time of refinancing.

We define our class of test functions accordingly. Let $\bar{C}$ be a class of test functions $\left\{\tilde{\phi}\right\}$ satisfying the following conditions:

$\tilde{\phi} \in C^{\infty}\left(\triangle \cup [0,1]\right) \times [0,1] \to [\mathbb{R} \times \mathbb{R}]$ s.t.

$$\tilde{\phi}(x,y) = \begin{cases} \phi(x)\psi(y) & x \in (0,1)\,,\ y \in (0,1) \\ \phi(y)\psi(y) & x \in \triangle,\ y \in (0,1) \\ 0 & x = y = 0 \text{ or } x = y = 1 \text{ or } x = y = \triangle \end{cases} \tag{5.1}$$

where $\{f\} = \{\phi\}$ or $\{\psi\}$ satisfying the following conditions:

1. $f \in C^{\infty}\left([0,1]\right) \cup \{\triangle\}$ that is the collection of all bounded (and continuous) functions on $[0,1]$ with continuous derivatives
2. $f(0) = 0,\ f(1) = 0,\ f(\triangle) = 0,\ f'(x) \to 0$ as $x \to 0$ or $x \to 1$.

Moreover, for any function $\tilde{\psi}$ and the limit empirical measure $\tilde{\nu}_t$ with support in $\tilde{\phi} \in C^{\infty}\left(\triangle \cup [0,1]\right) \times [0,1]$, we write

$$\langle \psi, \nu_t \rangle = \int_0^1 \int_0^1 \tilde{\psi}(x,y)\, \tilde{\nu}_t(dx, dy)\,. \tag{5.2}$$

We now study the empirical measure defined below and derive the equation describing the evolution of the empirical measure.

5.1 Derivation of the evolution equation

The empirical measure of the system at time t is defined as

$$\tilde{\nu}_{n,t} \quad = \quad \frac{1}{n}\sum_{i=1}^{n}\delta_{\left(X_t^i, X_{t-}^i\right)}.$$

We decompose the empirical measure for time t before and after the stopping time τ^i in the events of default or prepayment whichever comes first. We then have

$$\begin{aligned}
\tilde{\nu}_{n,t} \quad &= \quad \frac{1}{n}\sum_{i=1}^{n}\delta_{\left(X_t^i, X_{t-}^i\right)}\mathbf{1}_{\{t<\tau^i\}} + \frac{1}{n}\sum_{i=1}^{n}\delta_{\left(X_t^i, X_{t-}^i\right)}\mathbf{1}_{\{t\geq\tau^i\}} \\
&= \quad \frac{1}{n}\sum_{i=1}^{n}\delta_{\left(X_t^i, X_{t-}^i\right)}\mathbf{1}_{\{t<\tau^i\}} + \frac{1}{n}\sum_{i=1}^{n}\delta_{\left(X_t^i, X_{t-}^i\right)}\mathbf{1}_{\{t\geq\tau_r^i\}} + \frac{1}{n}\sum_{i=1}^{n}\delta_{\left(X_t^i, X_{t-}^i\right)}\mathbf{1}_{\{t\geq\tau_0^i\}} \\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\delta_{\left(X_t^i, X_{t-}^i\right)}\mathbf{1}_{\{t\geq\tau_1^i\}} \\
&= \quad \frac{1}{n}\sum_{i=1}^{n}\delta_{\left(X_t^i, X_{t-}^i\right)}\mathbf{1}_{\{t<\tau^i\}} + \frac{1}{n}\sum_{i=1}^{n}\delta_{\left(\triangle, X_{\tau_r^i}^i\right)}\mathbf{1}_{\{t=\tau_r^i\}} + \sum_{i=1}^{n}\delta_{(\triangle,\triangle)}\mathbf{1}_{\{t>\tau_r^i\}} \\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\delta_{(0,0)}\mathbf{1}_{\{t\geq\tau_0^i\}} + \frac{1}{n}\sum_{i=1}^{n}\delta_{(1,1)}\mathbf{1}_{\{t\geq\tau_1^i\}} \\
:= \quad & \tilde{\nu}_{n,t}^{+} + \tilde{\nu}_{n,t}^{\triangle} + \tilde{\nu}_{n,t}^{\triangle'} + \tilde{\nu}_{n,t}^{0} + \tilde{\nu}_{n,t}^{1},
\end{aligned} \tag{5.3}$$

where

$$\begin{aligned}
\tilde{\nu}_{n,t}^{+} \quad &= \quad \frac{1}{n}\sum_{i=1}^{n}\delta_{\left(X_t^i, X_{t-}^i\right)}\mathbf{1}_{\{t<\tau^i\}}, \\
\tilde{\nu}_{n,t}^{\triangle} \quad &= \quad \frac{1}{n}\sum_{i=1}^{n}\delta_{\left(\triangle, X_{\tau_r^i}^i\right)}\mathbf{1}_{\{t=\tau_r^i\}}, \\
\tilde{\nu}_{n,t}^{\triangle'} \quad &= \quad \frac{1}{n}\sum_{i=1}^{n}\delta_{(\triangle,\triangle)}\mathbf{1}_{\{t>\tau_r^i\}}, \\
\tilde{\nu}_{n,t}^{0} \quad &= \quad \frac{1}{n}\sum_{i=1}^{n}\delta_{(0,0)}\mathbf{1}_{\{t\geq\tau_0^i\}},
\end{aligned}$$

and,

$$\tilde{\nu}_{n,t}^{1} \quad = \quad \frac{1}{n}\sum_{i=1}^{n}\delta_{(1,1)}\mathbf{1}_{\{t\geq\tau_1^i\}}.$$

$\tilde{\nu}^{+}_{n,t}$ is the empirical measure of individuals that are neither defaulted nor prepaid. $\tilde{\nu}^{0}_{n,t}$ measures the mass at the absorbing boundary 0 by time t, that is the delta measure at 0 weighted by the proportion of the individuals that are defaulted by time t. $\tilde{\nu}^{1}_{n,t}$ measures the mass at the absorbing boundary 1 by time t that is essentially the delta measure at 1 weighted by the proportion of individuals that are prepaid by time t. $\tilde{\nu}^{\triangle}_{n,t}$ measures the mass of the individuals that are refinanced at time t.

For any test function $\tilde{\phi} \in \bar{C}$, we use the notation (5.2) and the decomposition of the empirical measure in (5.3) to get,

$$\begin{aligned}
\left\langle \tilde{\phi}, \tilde{\nu}_{n,t} \right\rangle &= \left\langle \tilde{\phi}, \tilde{\nu}^{+}_{n,t} \right\rangle + \left\langle \tilde{\phi}, \tilde{\nu}^{0}_{n,t} \right\rangle + \left\langle \tilde{\phi}, \tilde{\nu}^{1}_{n,t} \right\rangle + \left\langle \tilde{\phi}, \tilde{\nu}^{\triangle}_{n,t} \right\rangle + \left\langle \tilde{\phi}, \tilde{\nu}^{\triangle'}_{n,t} \right\rangle, \\
&= \frac{1}{n}\sum_{i=1}^{n} \tilde{\phi}\left(X^i_t, X^i_{t-}\right) \mathbf{1}_{\{t<\tau^i\}} + \frac{1}{n}\sum_{i=1}^{n} \tilde{\phi}\left(0,0\right) \mathbf{1}_{\{t\geq\tau^i_0\}} + \frac{1}{n}\sum_{i=1}^{n} \tilde{\phi}\left(1,1\right) \mathbf{1}_{\{t\geq\tau^i_1\}} \\
&\quad + \frac{1}{n}\sum_{i=1}^{n} \tilde{\phi}\left(\triangle, X^i_{\tau^i_r}\right) \mathbf{1}_{\{t=\tau^i_r\}} + \frac{1}{n}\sum_{i=1}^{n} \tilde{\phi}\left(\triangle,\triangle\right) \mathbf{1}_{\{t>\tau^i_r\}}.
\end{aligned}$$

Thus

$$\begin{aligned}
\left\langle \tilde{\phi}, \tilde{\nu}_{n,t} \right\rangle &= \frac{1}{n}\sum_{i=1}^{n} \tilde{\phi}\left(X^i_t, X^i_{t-}\right) \mathbf{1}_{\{t<\tau^i\}} + \frac{1}{n}\sum_{i=1}^{n} \tilde{\phi}\left(\triangle, X^i_{\tau^i_r}\right) \mathbf{1}_{\{t\geq\tau^i_r\}} \\
&= \langle \tilde{\phi}, \tilde{\nu}^{+}_{n,t} \rangle + \langle \tilde{\phi}, \tilde{\nu}^{\triangle'}_{n,t} \rangle,
\end{aligned} \tag{5.4}$$

as $\tilde{\phi}\left(0,0\right) = \tilde{\phi}\left(1,1\right) = \tilde{\phi}\left(\triangle,\triangle\right) = 0$.

Applying Ito's formula for semi-martingales to the process $\frac{1}{n}\sum_{i=1}^{n} \tilde{\phi}\left(X^i_t, X^i_{t-}\right) \mathbf{1}_{\{t<\tau^i\}}$ in (5.4), we have

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} \tilde{\phi}\left(X^i_t, X^i_{t-}\right) \mathbf{1}_{\{t<\tau^i\}} &= \frac{1}{n}\sum_{i=1}^{n} \tilde{\phi}\left(X^i_0, X^i_0\right) \mathbf{1}_{\{0<\tau^i\}} + \frac{1}{n}\sum_{i=1}^{n} \int_0^t \tilde{\phi}\left(X^i_{s-}, X^i_{s-}\right) d\mathbf{1}_{\{s<\tau^i\}} \\
&\quad + \frac{1}{n}\sum_{i=1}^{n} \int_0^t \mathbf{1}_{\{s<\tau^i\}} d\tilde{\phi}\left(X^i_{s-}, X^i_{s-}\right) \\
&\quad + \frac{1}{n}\sum_{i=1}^{n} \sum_{0\leq s\leq t} \Delta\tilde{\phi}\left(X^i_s, X^i_{s-}\right) \Delta\mathbf{1}_{\{s<\tau^i\}},
\end{aligned}$$

$$:= \frac{1}{n}\sum_{i=1}^{n}\tilde{\phi}\left(X_0^i, X_0^i\right)\mathbf{1}_{\{0<\tau^i\}} + I_1 + I_2 + I_3, \tag{5.5}$$

where

$$\begin{aligned}
I_1 &= \frac{1}{n}\sum_{i=1}^{n}\int_0^t \tilde{\phi}\left(X_{s-}^i, X_{s-}^i\right) d\mathbf{1}_{\{s<\tau^i\}},\\
I_2 &= \frac{1}{n}\sum_{i=1}^{n}\int_0^t \mathbf{1}_{\{s<\tau^i\}} d\tilde{\phi}\left(X_{s-}^i, X_{s-}^i\right),\\
I_3 &= \frac{1}{n}\sum_{i=1}^{n}\sum_{0\le s\le t}\Delta\tilde{\phi}\left(X_s^i, X_{s-}^i\right)\Delta\mathbf{1}_{\{s<\tau^i\}}.
\end{aligned}$$

Our aim is to derive an evolution equation for the limit empirical measure. The next step is to simplify the expressions for I_1, I_2, and I_3.

We start with

$$I_1 = \frac{1}{n}\sum_{i=1}^{n}\int_0^t \tilde{\phi}\left(X_{s-}^i, X_{s-}^i\right) d\mathbf{1}_{\{s<\tau^i\}},$$

where τ^i is the first time when either default or prepayment occurs. On using the definition of τ^i and using $\tilde{X}_{s-}^i = \left(X_{s-}^i, X_{s-}^i\right)$, we have

$$\begin{aligned}
I_1 &= \frac{1}{n}\sum_{i=1}^{n}\int_0^t \tilde{\phi}\left(\tilde{X}_{s-}^i\right) d\left(\mathbf{1}_{\left\{s<\min\left(\tau_0^i,\tau_1^i,\tau_r^i\right)\right\}}\right),\\
&= \frac{1}{n}\sum_{i=1}^{n}\int_0^t \tilde{\phi}\left(\tilde{X}_{s-}^i\right)\mathbf{1}_{\left\{s<\min\left(\tau_0^i,\tau_1^i\right)\right\}} d\mathbf{1}_{\{s<\tau_r^i\}}\\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\int_0^t \tilde{\phi}\left(\tilde{X}_{s-}^i\right)\mathbf{1}_{\{s<\tau_r^i\}} d\mathbf{1}_{\left\{s<\min\left(\tau_0^i,\tau_1^i\right)\right\}},\\
&:= I_1^1 + I_1^2,
\end{aligned}$$

where

$$\begin{aligned} I_1^1 &= \frac{1}{n}\sum_{i=1}^{n}\int_0^t \tilde{\phi}\left(\tilde{X}_{s-}^i\right)\mathbf{1}_{\left\{s<\min\left(\tau_0^i,\tau_1^i\right)\right\}}d\mathbf{1}_{\{s<\tau_r^i\}}, \\ I_1^2 &= \frac{1}{n}\sum_{i=1}^{n}\int_0^t \tilde{\phi}\left(\tilde{X}_{s-}^i\right)\mathbf{1}_{\{s<\tau_r^i\}}d\mathbf{1}_{\left\{s<\min\left(\tau_0^i,\tau_1^i\right)\right\}}. \end{aligned}$$

We are interested in simplifying the above two expressions. Let $s<\min\left(\tau_0^i,\tau_1^i\right)$, such that

$$I_1^1 = \frac{1}{n}\sum_{i=1}^{n}\int_0^t \tilde{\phi}\left(\tilde{X}_{s-}^i\right)d\mathbf{1}_{\{s<\tau_r^i\}}.$$

By definition, $\tau_r^i \sim \exp\left(\lambda\left(r_u\right)\right)$ that gives us

$$d\mathbf{1}_{\{\tau_r^i>s\}} = e^{-\int_0^s \lambda(r_u)du}dL_s - \lambda\left(r_s\right)\mathbf{1}_{\{\tau_r^i>s\}}ds,$$

where $L_s = e^{\int_0^s \lambda(r_u)du}\mathbf{1}_{\{\tau_r^i>s\}}$ is a martingale.

Therefore,

$$I_1^1 = \frac{1}{n}\sum_{i=1}^{n}\int_0^t \phi\left(X_{s-}^i\right)\psi\left(y\right)\mathbf{1}_{\left\{s<\min\left(\tau_0^i,\tau_1^i\right)\right\}}\left\{e^{-\int_0^s \lambda(r_u)du}dL_s - \lambda\left(r_s\right)\mathbf{1}_{\{\tau_r^i>s\}}ds\right\}. \tag{5.6}$$

We now simplify I_1^2 and use $\mathbf{1}_{\left\{s<\min\left(\tau_0^i,\tau_1^i\right)\right\}} = \mathbf{1}_{\left\{s<\tau_0^i\right\}}\mathbf{1}_{\left\{s<\tau_1^i\right\}}$

$$\begin{aligned} I_1^2 &= \frac{1}{n}\sum_{i=1}^{n}\int_0^t \tilde{\phi}\left(\tilde{X}_{s-}^i\right)\mathbf{1}_{\{s<\tau_r^i\}}d\mathbf{1}_{\left\{s<\min\left(\tau_0^i,\tau_1^i\right)\right\}} \\ &= \frac{1}{n}\sum_{i=1}^{n}\int_0^t \tilde{\phi}\left(\tilde{X}_{s-}^i\right)\mathbf{1}_{\left\{s<\min\left(\tau_r^i,\tau_1^i\right)\right\}}d\mathbf{1}_{\left\{s<\tau_0^i\right\}} \\ &\quad +\frac{1}{n}\sum_{i=1}^{n}\int_0^t \tilde{\phi}\left(\tilde{X}_{s-}^i\right)\mathbf{1}_{\left\{s<\min\left(\tau_r^i,\tau_0^i\right)\right\}}d\mathbf{1}_{\left\{s<\tau_1^i\right\}}, \end{aligned}$$

where τ_0^i (or τ_1^i) is the first time when the process of distance to default for the i^{th} individual touches the default (or prepayment) boundary. Therefore

$$\begin{aligned}
I_1^2 &= \frac{1}{n}\sum_{i=1}^{n}\int_0^t \tilde{\phi}\left(\tilde{X}^i_{\tau_0^i}\right)\mathbf{1}_{\left\{s<\min\left(\tau_r^i,\tau_1^i\right)\right\}}d\mathbf{1}_{\left\{s\geq\tau_0^i\right\}} \\
&\quad+\frac{1}{n}\sum_{i=1}^{n}\int_0^t \tilde{\phi}\left(\tilde{X}^i_{\tau_1^i}\right)\mathbf{1}_{\left\{s<\min\left(\tau_r^i,\tau_0^i\right)\right\}}d\mathbf{1}_{\left\{s\geq\tau_1^i\right\}} \\
&= \frac{1}{n}\sum_{i=1}^{n}\int_0^t \phi\left(0\right)\psi\left(0\right)\mathbf{1}_{\left\{s<\min\left(\tau_r^i,\tau_1^i\right)\right\}}d\mathbf{1}_{\left\{s\geq\tau_0^i\right\}} \\
&\quad+\frac{1}{n}\sum_{i=1}^{n}\int_0^t \phi\left(1\right)\psi\left(1\right)\mathbf{1}_{\left\{s<\min\left(\tau_r^i,\tau_0^i\right)\right\}}d\mathbf{1}_{\left\{s\geq\tau_1^i\right\}} \\
&= 0,
\end{aligned}$$

where we used (5.1) along with $\phi\left(0\right)=\phi\left(1\right)=0$.

Substituting for I_1^1 and I_1^2 in (5.6), we obtain

$$I_1=\frac{1}{n}\sum_{i=1}^{n}\int_0^t \phi\left(X^i_{s-}\right)\psi\left(X^i_{s-}\right)\mathbf{1}_{\left\{s<\min\left(\tau_0^i,\tau_1^i\right)\right\}}\left\{e^{-\int_0^s\lambda(r_u)du}dL_s-\lambda\left(r_s\right)\mathbf{1}_{\{\tau_r^i>s\}}ds\right\}. \tag{5.7}$$

After simplifying I_1, we simplify I_2 in (5.5) where

$$I_2=\frac{1}{n}\sum_{i=1}^{n}\int_0^t \mathbf{1}_{\{s<\tau^i\}}d\tilde{\phi}\left(\tilde{X}^i_{s-}\right).$$

By Ito's formula for continuous process and using (5.1) for $x\in(0,1)$ as the individuals are neither defaulted nor prepaid and we have $\tilde{\phi}\left(\tilde{X}^i_t\right)=\phi\left(X^i_t\right)\psi\left(X^i_t\right)$. We have

$$\begin{aligned}
I_2 &= \frac{1}{n}\sum_{i=1}^{n}\int_0^t \mathbf{1}_{\{s<\tau^i\}}\left\{\psi\left(X^i_{s-}\right)d\phi\left(X^i_{s-}\right)+\phi\left(X^i_{s-}\right)d\psi\left(X^i_{s-}\right)\right. \\
&\quad\left.+\frac{1}{2}d\phi\left(X^i_{s-}\right)d\psi\left(X^i_{s-}\right)\right\} \\
&= \frac{1}{n}\sum_{i=1}^{n}\int_0^t \mathbf{1}_{\{s<\tau^i\}}\psi\left(X^i_{s-}\right)\left[\phi'\left(X^i_s\right)dX^i_s+\frac{1}{2}\phi''\left(X^i_s\right)d[X^i_s]\right]+
\end{aligned}$$

$$\frac{1}{n}\sum_{i=1}^{n}\int_0^t \mathbf{1}_{\{s<\tau^i\}}\phi\left(X_{s-}^i\right)\left[\psi^{'}\left(X_s^i\right)dX_s^i+\frac{1}{2}\psi^{''}\left(X_s^i\right)d[X_s^i]\right]+$$
$$\frac{1}{n}\sum_{i=1}^{n}\int_0^t \mathbf{1}_{\{s<\tau^i\}}d\phi\left(X_{s-}^i\right)d\psi\left(X_{s-}^i\right)$$

which on using (4.4) for $s<\tau^i$ becomes,

$$\begin{aligned}
= \quad & \frac{1}{n}\sum_{i=1}^{n}\int_0^t \mathbf{1}_{\{s<\tau^i\}}\psi\left(X_s^i\right)\left[\phi^{'}\left(X_s^i\right)\left(\hat{\mu}ds+\hat{\sigma}\sqrt{1-\rho^2}dW_s^i+\hat{\sigma}\rho dM_s\right)\right. \\
& \quad \left.+\frac{\hat{\sigma}^2}{2}\phi^{''}\left(X_s^i\right)ds\right]+\frac{1}{n}\sum_{i=1}^{n}\int_0^t \mathbf{1}_{\{s<\tau^i\}}\phi\left(X_s^i\right) \\
& \quad \left[\psi^{'}\left(X_s^i\right)\left(\hat{\mu}ds+\hat{\sigma}\sqrt{1-\rho^2}dW_s^i+\hat{\sigma}\rho dM_s\right)+\frac{\hat{\sigma}^2}{2}\psi^{''}\left(X_s^i\right)ds\right]+ \\
& \quad \frac{1}{n}\sum_{i=1}^{n}\int_0^t \mathbf{1}_{\{s<\tau^i\}}\frac{1}{2}\hat{\sigma}^2\phi^{'}\left(X_s^i\right)\psi^{'}\left(X_s^i\right)ds,
\end{aligned}$$

$$\begin{aligned}
= \quad & \frac{1}{n}\sum_{i=1}^{n}\int_0^t \left(\hat{\mu}\phi^{'}\left(X_s^i\right)+\frac{\hat{\sigma}^2}{2}\phi^{''}\left(X_s^i\right)\right)\psi\left(X_s^i\right)\mathbf{1}_{\{s<\tau^i\}}ds+\frac{1}{n}\sum_{i=1}^{n}\int_0^t \hat{\sigma}\sqrt{1-\rho^2} \\
& \phi^{'}\left(X_s^i\right)\mathbf{1}_{\{s<\tau^i\}}\psi\left(X_s^i\right)dW_s^i+\frac{1}{n}\sum_{i=1}^{n}\int_0^t \hat{\sigma}\rho\phi^{'}\left(X_s^i\right)\mathbf{1}_{\{s<\tau^i\}}\psi\left(X_s^i\right)dM_s+ \\
& \frac{1}{n}\sum_{i=1}^{n}\int_0^t \left(\hat{\mu}\psi^{'}\left(X_s^i\right)+\frac{\hat{\sigma}^2}{2}\psi^{''}\left(X_s^i\right)\right)\phi\left(X_s^i\right)\mathbf{1}_{\{s<\tau^i\}}ds+\frac{1}{n}\sum_{i=1}^{n}\int_0^t \hat{\sigma}\sqrt{1-\rho^2} \\
& \psi^{'}\left(X_s^i\right)\mathbf{1}_{\{s<\tau^i\}}\phi\left(X_s^i\right)dW_s^i+\frac{1}{n}\sum_{i=1}^{n}\int_0^t \hat{\sigma}\rho\psi^{'}\left(X_s^i\right)\mathbf{1}_{\{s<\tau^i\}}\phi\left(X_s^i\right)dM_s+ \\
& \frac{1}{n}\sum_{i=1}^{n}\int_0^t \mathbf{1}_{\{s<\tau^i\}}\frac{1}{2}\hat{\sigma}^2\phi^{'}\left(X_s^i\right)\psi^{'}\left(X_s^i\right)ds,
\end{aligned}$$

$$\begin{aligned}
= \quad & \int_0^t \langle\hat{\mu}\phi^{'}(x)\psi(x)+\frac{\hat{\sigma}^2}{2}\phi^{''}(x)\psi(x)+\hat{\mu}\psi^{'}(x)\phi(x)+\frac{\hat{\sigma}^2}{2}\psi^{''}(x)\phi(x)+ \\
& \frac{1}{2}\hat{\sigma}^2\phi^{'}(x)\psi^{'}(x),\tilde{\nu}_{n,s}^{+}\rangle ds+\int_0^t \frac{1}{n}\sum_{i=1}^{n}\hat{\sigma}\sqrt{1-\rho^2}\left(\phi^{'}\left(X_s^i\right)\psi\left(X_s^i\right)+\psi^{'}\left(X_s^i\right)\phi\left(X_s^i\right)\right) \\
& \mathbf{1}_{\{s<\tau^i\}}dW_s^i+\int_0^s \langle\hat{\sigma}\rho\phi^{'}(x)\psi(x)+\hat{\sigma}\rho\psi^{'}(x)\phi(x),\tilde{\nu}_{n,s}^{+}\rangle dM_s. \qquad (5.8)
\end{aligned}$$

Finally we simplify I_3 in (5.5),

$$
\begin{aligned}
I_3 &= \frac{1}{n}\sum_{i=1}^{n}\sum_{0\le s\le t}\Delta\tilde{\phi}\left(X_s^i, X_{s-}^i\right)\Delta\mathbf{1}_{\{s<\tau^i\}}, \\
&= \frac{1}{n}\sum_{i=1}^{n}\sum_{0\le s\le t}\Delta\tilde{\phi}\left(X_s^i, X_{s-}^i\right)\mathbf{1}_{\left\{s<\min\left(\tau_0^i,\tau_1^i\right)\right\}}\Delta\mathbf{1}_{\{s<\tau_r^i\}} + \frac{1}{n}\sum_{i=1}^{n}\sum_{0\le s\le t}\Delta\tilde{\phi}\left(X_s^i, X_{s-}^i\right) \\
&\quad \mathbf{1}_{\left\{s<\min\left(\tau_r^i,\tau_1^i\right)\right\}}\Delta\mathbf{1}_{\left\{s<\tau_0^i\right\}} + \frac{1}{n}\sum_{i=1}^{n}\sum_{0\le s\le t}\Delta\tilde{\phi}\left(X_s^i, X_{s-}^i\right)\mathbf{1}_{\left\{s<\min\left(\tau_r^i,\tau_0^i\right)\right\}}\mathbf{1}_{\left\{s<\tau_1^i\right\}} \\
&= \frac{1}{n}\sum_{i=1}^{n}\sum_{0\le s\le t}\Delta\tilde{\phi}\left(X_{\tau_r^i}^i, X_{\tau_{r-}^i}^i\right)\mathbf{1}_{\left\{s<\min\left(\tau_0^i,\tau_1^i\right)\right\}}\Delta\mathbf{1}_{\{s\ge\tau_r^i\}} + \frac{1}{n}\sum_{i=1}^{n}\sum_{0\le s\le t}\Delta\tilde{\phi}\left(\tilde{X}_{\tau_0^i}^i\right) \\
&\quad \mathbf{1}_{\left\{s<\min\left(\tau_r^i,\tau_1^i\right)\right\}}\mathbf{1}_{\left\{s\ge\tau_0^i\right\}} + \frac{1}{n}\sum_{i=1}^{n}\sum_{0\le s\le t}\Delta\tilde{\phi}\left(\tilde{X}_{\tau_1^i}^i\right)\mathbf{1}_{\left\{s<\min\left(\tau_r^i,\tau_0^i\right)\right\}}\mathbf{1}_{\left\{s\ge\tau_1^i\right\}} \\
&= \frac{1}{n}\sum_{i=1}^{n}\sum_{0\le s\le t}\left(\tilde{\phi}\left(X_{\tau_r^i}^i, X_{\tau_{r-}^i}^i\right) - \tilde{\phi}\left(X_{\tau_{r-}^i}^i, X_{\tau_{r--}^i}^i\right)\right)\mathbf{1}_{\left\{s<\min\left(\tau_0^i,\tau_1^i\right)\right\}}\Delta\mathbf{1}_{\{s\ge\tau_r^i\}} \\
&= \frac{1}{n}\sum_{i=1}^{n}\sum_{0\le s\le t}\left(\phi\left(X_{\tau_{r-}^i}^i\right)\psi\left(X_{\tau_{r-}^i}^i\right) - \phi\left(X_{\tau_{r-}^i}^t\right)\psi\left(X_{\tau_{r--}^i}^i\right)\right)\mathbf{1}_{\left\{s<\min\left(\tau_0^i,\tau_1^i\right)\right\}}\Delta\mathbf{1}_{\{s=\tau_r^i\}} \\
&= \frac{1}{n}\sum_{i=1}^{n}\sum_{0\le s\le t}\phi\left(X_{\tau_{r-}^i}^i\right)\left(\psi\left(X_{\tau_{r-}^i}^i\right) - \psi\left(X_{\tau_{r--}^i}^i\right)\right)\mathbf{1}_{\left\{s<\min\left(\tau_0^i,\tau_1^i\right)\right\}}\Delta\mathbf{1}_{\{s=\tau_r^i\}} \\
&= 0. \qquad (5.9)
\end{aligned}
$$

where we used (5.1) and the fact that ψ is a left continuous process with $\psi\left(X_{\tau_-^i}^i\right) = \psi\left(X_{\tau_{--}^i}^i\right)$.

Substituting (5.7), (5.8) and (5.9) into (5.5), we have

$$
\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}\tilde{\phi}\left(\tilde{X}_t^i\right)\mathbf{1}_{\{t<\tau^i\}} &= \frac{1}{n}\sum_{i=1}^{n}\tilde{\phi}\left(\tilde{X}_0^i\right)\mathbf{1}_{\{0<\tau^i\}} + \int_0^t \langle\hat{\mu}\phi'(x)\psi(x) + \frac{\hat{\sigma}^2}{2}\phi''(x)\psi(x) + \\
&\quad \hat{\mu}\psi'(x)\phi(x) + \frac{\hat{\sigma}^2}{2}\psi''(x)\phi(x) + \frac{1}{2}\hat{\sigma}^2\phi'\left(X_s^i\right)\psi'\left(X_s^i\right), \tilde{\nu}_{n,s}^{+}\rangle ds + \\
&\quad \int_0^t \frac{1}{n}\sum_{i=1}^{n}\hat{\sigma}\sqrt{1-\rho^2}\left(\phi'(x)\psi(x) + \psi'\left(X_s^i\right)\phi\left(X_s^i\right)\right) \\
&\quad \mathbf{1}_{\{s<\tau^i\}}dW_s^i + \int_0^s \langle\hat{\sigma}\rho\phi'(x)\psi(x) + \hat{\sigma}\rho\psi'(x)\phi(x), \tilde{\nu}_{n,s}^{+}\rangle dM_s \\
&\quad +\frac{1}{n}\sum_{i=1}^{n}\int_0^t \phi\left(X_{s-}^i\right)\psi\left(X_{s-}^i\right)\mathbf{1}_{\left\{s<\min\left(\tau_0^i,\tau_1^i\right)\right\}}\left\{e^{-\int_0^s \lambda(r_u)du}dL_s\right.
\end{aligned}
$$

$$-\lambda\left(r_s\right)\mathbf{1}_{\{\tau_r^i>s\}}ds\Big\}$$

and by (5.4) this becomes

$$\begin{aligned}
\langle\tilde{\phi},\tilde{\nu}_{n,t}\rangle &= \langle\tilde{\phi},\tilde{\nu}_{n,0}\rangle+\int_0^t\langle\hat{\mu}\phi^{'}\left(x\right)\psi\left(x\right)+\frac{\hat{\sigma}^2}{2}\phi^{''}\left(x\right)\psi\left(x\right)+\hat{\mu}\psi^{'}\left(x\right)\phi\left(x\right)+\\
&\quad \frac{\hat{\sigma}^2}{2}\psi^{''}\left(x\right)\phi\left(x\right)+\frac{1}{2}\hat{\sigma}^2\phi^{'}\left(X_s^i\right)\psi^{'}\left(X_s^i\right)-\lambda\left(r_u\right)\phi\left(x\right)\psi\left(x\right),\tilde{\nu}_{n,s}^{+}\rangle ds+\\
&\quad \int_0^t\frac{1}{n}\sum_{i=1}^n\hat{\sigma}\sqrt{1-\rho^2}\left(\phi^{'}\left(x\right)\psi\left(x\right)+\psi^{'}\left(X_s^i\right)\phi\left(X_s^i\right)\right)\mathbf{1}_{\{s<\tau^i\}}dW_s^i+\\
&\quad \int_0^s\langle\hat{\sigma}\rho\phi^{'}\left(x\right)\psi\left(x\right)+\hat{\sigma}\rho\psi^{'}\left(x\right)\phi\left(x\right),\tilde{\nu}_{n,s}^{+}\rangle dM_s+\\
&\quad \frac{1}{n}\sum_{i=1}^n\int_0^t\phi\left(X_{s-}^i\right)\psi\left(X_{s-}^i\right)\mathbf{1}_{\left\{s<\min\left(\tau_0^i,\tau_1^i\right)\right\}}e^{-\int_0^s\lambda(r_u)du}dL_s+\langle\tilde{\phi},\tilde{\nu}_{n,t}^{\triangle^{'}}\rangle.
\end{aligned}\tag{5.10}$$

To drive our evolution equation we now substitute $\psi\left(x\right)=1$. We also have $\nu_t\left(dx\right)=\int_0^1\tilde{\nu}_t\left(dx,dy\right)$ and

$$\langle\phi,\nu_{n,t}\rangle=\int_0^1\phi\left(x\right)\nu_{n,t}\left(dx\right)=\int_0^1\int_0^1\tilde{\phi}\left(x,y\right)\tilde{\nu}_t\left(dx,dy\right).$$

Further to that

$$\begin{aligned}
\langle\tilde{\phi},\tilde{\nu}_{n,t}^{\triangle^{'}}\rangle &= \frac{1}{n}\sum_{i=1}^n\tilde{\phi}\left(\triangle,X_{\tau_{r-}^i}^i\right)\mathbf{1}_{\{\tau_r^i=t\}}\\
&= \frac{1}{n}\sum_{i=1}^n\phi\left(X_{\tau_r^i}^i\right)\psi\left(X_{\tau_r^i}^i\right)\mathbf{1}_{\{\tau_r^i=t\}}
\end{aligned}$$

This gives for $\psi\left(x\right)=1$

$$\int_0^1\langle\tilde{\phi},\tilde{\nu}_{n,t}^{\triangle^{'}}\rangle dy \quad = \quad \int_0^1\frac{1}{n}\sum_{i=1}^n\phi\left(X_{\tau_r^i}^i\right)\mathbf{1}_{\{\tau_r^i=t\}}dy=0$$

Therefore equation (5.10) becomes

$$\langle\phi,\nu_{n,t}\rangle \quad = \quad \langle\phi,\nu_{n,0}\rangle+\int_0^t\langle\hat{\mu}\phi^{'}\left(x\right)+\frac{\hat{\sigma}^2}{2}\phi^{''}\left(x\right)-\lambda\left(r_s\right)\phi\left(x\right),\nu_{n,s}\rangle ds$$

$$
+\int_0^t \langle \hat{\sigma}\rho\phi^{'}(x), \nu_{n,s}\rangle dM_s + \frac{1}{n}\sum_{i=1}^{n}\int_0^t \phi\left(X_{s-}^i\right)\mathbf{1}_{\left\{s<\min\left(\tau_0^i,\tau_1^i\right)\right\}}
$$

$$
e^{-\int_0^s \lambda(r_u)du} dL_s + \int_0^t \frac{1}{n}\sum_{i=1}^{n}\hat{\sigma}\sqrt{1-\rho^2}\phi^{'}\left(X_s^i\right)\mathbf{1}_{\{s<\tau^i\}} dW_s^i \tag{5.11}
$$

We now return to one dimensional setup and are interested in limit behaviour of the empirical measure as the number of individuals goes to infinity. We have already shown in theorem (4.3.3) that the limit of the empirical measure and its default and prepayment proportions exist, so we have

$$
\begin{aligned}
\nu_t &= \lim_{n\to\infty}\nu_{n,t} \\
&= \lim_{n\to\infty}\nu_{n,t}^{+} + \lim_{n\to\infty}\nu_{n,t}^{\Delta} + \lim_{n\to\infty}\nu_{n,t}^{0} + \lim_{n\to\infty}\nu_{n,t}^{1} \\
&:= \nu_t^{+} + \nu_t^{\Delta} + \nu_t^{0} + \nu_t^{1},
\end{aligned}
$$

where

$$
\begin{aligned}
\nu_t^{+} &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\delta_{X_t^i}\mathbf{1}_{\{t<\tau^i\}} \quad \text{with support in } (0,1)\,; \\
\nu_t^{\Delta} &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\delta_{\Delta}\mathbf{1}_{\{t>\tau_r^i\}} = \left(\lim_{n\to\infty} b_{n,t}^{\Delta}\right)\delta_{\Delta} = b_t^{\Delta}\delta_{\Delta} \\
\nu_t^{0} &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\delta_{0}\mathbf{1}_{\left\{t>\tau_0^i\right\}} = \left(\lim_{n\to\infty} b_{n,t}^{0}\right)\delta_{0} = b_t^{0}\delta_{0} \\
\nu_t^{1} &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\delta_{1}\mathbf{1}_{\left\{t\geq\tau_1^i\right\}} = \left(\lim_{n\to\infty} b_{n,t}^{1}\right)\delta_{1} = b_t^{1}\delta_{1}.
\end{aligned}
$$

Using the limit empirical measure, (5.10) becomes

$$
\begin{aligned}
\langle\phi,\nu_t\rangle &= \lim_{n\to\infty}\langle\phi,\nu_{n,t}\rangle \\
&= \langle\phi,\nu_{n,0}\rangle + \int_0^t \lim_{n\to\infty}\langle\hat{\mu}\phi^{'}(x) + \frac{\hat{\sigma}^2}{2}\phi^{''}(x) - \lambda(r_s)\phi(x), \nu_{n,s}\rangle ds \\
&\quad + \int_0^t \lim_{n\to\infty}\langle\hat{\sigma}\rho\phi^{'}(x), \nu_{n,s}\rangle dM_s + \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\int_0^t \phi\left(X_{s-}^i\right)\mathbf{1}_{\left\{s<\min\left(\tau_0^i,\tau_1^i\right)\right\}} \\
&\quad e^{-\int_0^s \lambda(r_u)du} dL_s + \int_0^t \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\hat{\sigma}\sqrt{1-\rho^2}\phi^{'}\left(X_s^i\right)\mathbf{1}_{\{s<\tau^i\}} dW_s^i.
\end{aligned} \tag{5.12}
$$

Let us consider the last two terms in above equation and denote

$$G_t^{n,1} := \frac{1}{n}\sum_{i=1}^{n}\int_0^t \hat{\sigma}\sqrt{1-\rho^2}\phi'\left(X_s^i\right)\mathbf{1}_{\{s<\tau^i\}}dW_s^i$$

$$G_t^{n,2} := \frac{1}{n}\sum_{i=1}^{n}\int_0^t \phi\left(X_{s-}^i\right)\mathbf{1}_{\left\{s<\min\left(\tau_0^i,\tau_1^i\right)\right\}}e^{-\int_0^s \lambda(r_u)du}dL_s.$$

These two terms due to idiosyncratic noise become deterministic in the infinite limit and we have the following result.

Proposition

$G_t^{n,1}$ and $G_t^{n,2}$ are martingales and

$$G_t^{n,1} \to 0, \quad G_t^{n,2} \to 0, \quad \text{as} \quad n \to \infty.$$

Proof. We first show that $G_t^{n,1} \to 0$ as $n \to \infty$. Since ϕ' is bounded, we observe that $G_t^{n,1}$ is a martingale with quadratic variation

$$\left[G_t^{n,1}\right] = \int_0^t \frac{1}{n^2}\sum_{i=1}^{n}\hat{\sigma}^2\left(1-\rho^2\right)\left(\phi'\left(X_s^i\right)\right)^2\mathbf{1}_{\{s<\tau^i\}}ds$$

Since $\phi \in \bar{C}$, there exists a constant K_ϕ such that $|\phi'\left(X_s^i\right)|^2 \le K_\phi$. We have

$$\int_0^t \frac{1}{n}\sum_{i=1}^{n}\hat{\sigma}^2\left(1-\rho^2\right)|\phi'\left(X_s^i\right)|^2\mathbf{1}_{\{s<\tau^i\}}ds \le K_\phi^2 t$$

With probability 1 we have,

$$\lim_{n\to\infty}\int_0^t \frac{1}{n^2}\sum_{i=1}^{n}\hat{\sigma}^2\left(1-\rho^2\right)|\phi'\left(X_s^i\right)|^2\mathbf{1}_{\{s<\tau^i\}}ds \le \lim_{n\to\infty}\frac{1}{n}K_\phi^2 t = 0,$$

that is $G_t^{n,1} \to 0$ as $n \to \infty$.

It is easy to see that $G_t^{n,2}$ is a well-behaved bounded martingale provided $\lambda\left(r_u\right)$

is bounded. We consider the intensity $\lambda(r_u)$ to be bounded in our project. Using Theorem (27.6) Section IV [Rogers and Williams [49]], we have

$$\left[G_t^{n,2}\right] = \int_0^t \frac{1}{n^2} \sum_{i=1}^{n} \left(\phi\left(X_s^i\right)\right)^2 e^{-\int_0^s 2\lambda(r_u)du} \mathbf{1}_{\left\{s<\min\left(\tau_0^i,\tau_1^i\right)\right\}} ds$$

Since $\phi \in \bar{C}$, there exists a constant D_ϕ such that $|\phi\left(X_s^i\right)|^2 \leq D_\phi$ giving

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \int_0^t e^{-\int_0^s 2\lambda(r_u)du} |\phi\left(X_s^i\right)|^2 \mathbf{1}_{\left\{s<\min\left(\tau_0^i,\tau_1^i\right)\right\}} ds \leq K_{\phi,t}^2, \quad \text{a.s.,} \tag{5.13}$$

where $K_{\phi,t}^2$ is a constant that depends on D_ϕ and t. With probability 1 we have

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} \int_0^t e^{-\int_0^s 2\lambda(r_u)du} |\phi\left(X_s^i\right)|^2 \mathbf{1}_{\left\{s<\min\left(\tau_0^i,\tau_1^i\right)\right\}} ds \leq \lim_{n\to\infty} \frac{1}{n} K_{\phi,t}^2 = 0,$$

that is $G_t^{n,2} \to 0$ as $n \to \infty$.

□

We have shown that the random terms due to idiosyncratic component of the individuals become deterministic in the infinite limit and must vanish. Therefore (5.12) becomes

$$\begin{aligned}
\langle \phi, \nu_t \rangle &= \lim_{n\to\infty} \langle \phi, \nu_{n,t} \rangle \\
&= \langle \phi, \nu_{n,0} \rangle + \int_0^t \lim_{n\to\infty} \langle \hat{\mu}\phi^{'}(x) + \frac{\hat{\sigma}^2}{2}\phi^{''}(x) - \lambda(r_s)\phi(x), \nu_{n,s} \rangle ds \\
&\quad + \int_0^t \lim_{n\to\infty} \langle \hat{\sigma}\rho\phi^{'}(x), \nu_{n,s} \rangle dM_s
\end{aligned}$$

Since $\hat{\mu}$, $\hat{\sigma}$, ρ, $\lambda(r_s)$, $\phi^{'}$ and $\phi^{''}$ are bounded, by dominance convergence theorem, we have

$$\begin{aligned}
\langle \phi, \nu_t \rangle &= \langle \phi, \nu_0 \rangle + \int_0^t \langle \hat{\mu}\phi^{'}\left(X_s^i\right) + \frac{\hat{\sigma}^2}{2}\phi^{''}\left(X_s^i\right) - \lambda(r_s)\phi, \nu_s^+ \rangle ds \\
&\quad + \int_0^s \langle \hat{\sigma}\rho\phi^{'}\left(X_s^i\right), \nu_s^+ \rangle dM_s.
\end{aligned}$$

The " + " part of ν_t describes the dynamics of the individuals that are neither defaulted nor prepaid by time t. From (5.4), we have $\langle \phi, \nu_t \rangle = \langle \phi, \nu_t^+ \rangle$, that gives us the evolution of the limit empirical measure as

$$\begin{aligned} \langle \phi, \nu_t^+ \rangle &= \langle \phi, \nu_0^+ \rangle + \int_0^t \langle \hat{\mu} \phi'(x) + \frac{\hat{\sigma}^2}{2} \phi''(x) - \lambda(r_s) \phi(x), \nu_s^+ \rangle ds \\ &\quad + \int_0^t \langle \hat{\sigma} \rho \phi'(x), \nu_s^+ \rangle dM_s. \end{aligned} \tag{5.14}$$

The initial condition for the system (4.4) gives us

$$\nu_0^+ = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n \delta_{X_0^i} \mathbf{1}_{\{0 < \tau^i\}} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n \delta_{X_0^i} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n \delta_{x^i}. \tag{5.15}$$

So far we have derived the evolution of the limit empirical measure as in (5.14) with the initial condition (5.15). In the next section we derive the boundary conditions satisfied by the limit empirical measure.

5.2 Boundary condition

In this section we investigate the behaviour of the limit empirical measure near the boundaries. We investigate both the pointwise and uniform behaviour of the limit empirical measure near the boundaries and present two theorems.

We first give a lemma.

Lemma 5.2.1. *If $\{X_\varepsilon : \varepsilon \in \mathbb{R}_+\}$ is a family of random variables and there exists a constant C such that $\mathbb{E}[f(X_\varepsilon)] \leq C\varepsilon^{\delta+1}$ $\forall \varepsilon$, for a measurable extended real-valued function f and for $\delta > 0$ then*

$$\limsup_{\varepsilon \to 0} \frac{f(X_\varepsilon)}{\varepsilon} = 0.$$

Proof. By Markov's inequality, for any $\lambda > 0$ we have

$$\mathbb{P}\left\{\frac{f(X_\varepsilon)}{\varepsilon} > \lambda\right\} \leq \frac{C\varepsilon^\delta}{\lambda},$$

and for the subsequence $\varepsilon = \frac{1}{n^{2/\delta}}$,

$$\mathbb{P}\left\{\frac{f(X_\varepsilon)}{\frac{1}{n^{2/\delta}}}\right\} \leq \frac{C}{\lambda n^2}$$

that gives,

$$\sum_{n=1}^{\infty} \mathbb{P}\left\{\frac{f(X_\varepsilon)}{\frac{1}{n^{2/\delta}}}\right\} \leq \sum_{n=1}^{\infty} \frac{C}{\lambda n^2} < \infty.$$

By the first Borel-Cantelli lemma, we have

$$\lim_{n\to\infty} \sup \frac{f(X_\varepsilon)}{\frac{1}{n^{2/\delta}}} = 0, \text{ a.s..}$$

For any $\varepsilon > 0$, there exist a n such that $\frac{1}{(n+1)^{2/\delta}} \leq \varepsilon \leq \frac{1}{n^{2/\delta}}$ and hence

$$\lim_{\varepsilon\downarrow 0} \sup \frac{f(X_\varepsilon)}{\varepsilon} \leq \lim_{n\to\infty} \sup \frac{f(X_\varepsilon)}{\frac{1}{(n+1)^{2/\delta}}} = \lim_{n\to\infty} \sup \frac{f(X_\varepsilon)}{\frac{1}{n^{2/\delta}}} \frac{\frac{1}{n^{2/\delta}}}{\frac{1}{(n+1)^{2/\delta}}} = 0, \text{ a.s..}$$

Since $\frac{f(X_\varepsilon)}{\varepsilon} \geq 0$, therefore

$$\lim_{\varepsilon\downarrow 0} \frac{f(X_\varepsilon)}{\varepsilon} = 0, \text{ a.s..}$$

□

We now state and prove the theorems for the boundary behaviour of the limiting empirical measure. The next theorem discusses the result for the boundary conditions at 0 and 1 for a fixed time t.

Theorem 5.2.2. *For a fixed $t > 0$ we have*

$$a) \quad \lim_{\varepsilon\downarrow 0} \frac{\nu_t^+((0,\varepsilon))}{\varepsilon} = 0, \quad a.s.$$

$$b) \quad \lim_{\varepsilon\downarrow 0} \frac{\nu_t^+ \left((1-\varepsilon, 1)\right)}{\varepsilon} = 0 \quad a.s.$$

Proof. a) This result is already proved by Jin in [28] and Bush et al in [8] for an initial distribution of particles that are bounded away from zero.

We adapt the same technique to our settings for a fixed time t. By the definition of empirical measure and the convergence in weak topology as given in Ethier and Kurtz [[14], theorem 3.1] for an open set $(0, \varepsilon)$, we have

$$\begin{aligned} \nu_t^+ \left((0,\varepsilon)\right) &\leq \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{X_t^i} \left((0,\varepsilon)\right) \\ &= \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\left\{0<X_t^i<\varepsilon\right\}}. \end{aligned}$$

We therefore have

$$\mathbb{E}\left[\nu_t^+ \left((0,\varepsilon)\right)\right] \leq \mathbb{E}\left[\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\left\{0<X_t^i<\varepsilon\right\}}\right].$$

Since $\frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\left\{0<X_t^i<\varepsilon\right\}} \leq 1$ and by the dominated convergence theorem we can commute the expectations and the limit in above expression, that is

$$\begin{aligned} \mathbb{E}\left[\nu_t^+ \left((0,\varepsilon)\right)\right] &\leq \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[\mathbf{1}_{\left\{0<X_t^i<\varepsilon\right\}}\right], \\ &= \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{P}\left\{0 < X_t^i < \varepsilon\right\}, \end{aligned} \tag{5.16}$$

We know that

$$\begin{cases} dX_t^i = \hat{\mu} dt + \hat{\sigma}\sqrt{1-\rho^2} dW_t^i + \hat{\sigma}\rho dM_t, \quad t < \tau^i \\ X_t^i = 0, \quad t \geq \tau^i = \tau_0^i, \\ X_t^i = 1, \quad t \geq \tau^i = \tau_1^i, \\ X_t^i = \triangle, \quad t \geq \tau^i = \tau_r^i, \\ X_0^i = x^i, \quad 0 < x^i < 1. \end{cases}$$

Which on integrating from 0 to t for $t < \tau^i$ gives us

$$X_t^i = x^i + \hat{\mu}t + \hat{\sigma}\sqrt{1-\rho^2} W_t^i + \hat{\sigma}\rho M.$$

We denote the part for $t < \tau^i$ by $\hat{X}_t^i$, that is

$$\hat{X}_t^i = x^i + \hat{\mu}t + \hat{\sigma}\sqrt{1-\rho^2} W_t^i + \hat{\sigma}\rho M_t \quad \text{for}\, t < \tau^i.$$

Since $\sqrt{1-\rho^2} W_t^i + \rho M_t$ is a local martingale with quadratic variation t, by Levy's characterization $\sqrt{1-\rho^2} W_t^i + \rho M_t = B_t$ is a Brownian motion.

We have

$$\begin{aligned} \mathbb{P}\left\{0 < X_t^i < \varepsilon\right\} &= \mathbb{P}\left\{\hat{X}_t^i < \varepsilon, \inf_{0\leq s\leq t} \hat{X}_s^i > 0, \sup_{0\leq s\leq t} \hat{X}_s^i < 1, t < \tau_r^i\right\}, \\ &\leq \mathbb{P}\left\{\hat{X}_t^i < \varepsilon, \inf_{0\leq s\leq t} \hat{X}_s^i > 0\right\} \\ &= \mathbb{P}\left\{\frac{\hat{X}_t^i}{\hat{\sigma}} < \frac{\varepsilon}{\hat{\sigma}}, \inf_{0\leq s\leq t} \frac{\hat{X}_s^i}{\hat{\sigma}} > 0\right\} \\ &= \mathbb{P}\left\{\frac{x^i}{\hat{\sigma}} + \frac{\hat{\mu}t}{\hat{\sigma}} + B_t < \frac{\varepsilon}{\hat{\sigma}}, \inf_{0\leq s\leq t} \frac{x^i}{\hat{\sigma}} + \frac{\hat{\mu}s}{\hat{\sigma}} + B_s > 0\right\}, \\ &\leq J_1 \cdot J_2, \end{aligned} \tag{5.17}$$

where

$$J_1 = \left\{ \mathbb{E}^{\tilde{\mathbb{P}}} \left[\mathbf{1}_{\left\{ \frac{x^i}{\hat{\sigma}} + \tilde{B}_t < \frac{\varepsilon}{\hat{\sigma}}, \inf_{0 \le s \le t} \frac{x^i}{\hat{\sigma}} + \tilde{B}_s > 0 \right\}} \right] \right\}^{1/a}$$

and,

$$J_2 = \left\{ \mathbb{E}^{\tilde{\mathbb{P}}} \left[\frac{1}{Z_T(\hat{\mu})} \right]^b \right\}^{1/b}$$

with

$$1/a + 1/b = 1, \; a > 1, \; b > 1.$$

$\tilde{\mathbb{P}}$ is the probability distribution defined as $\tilde{\mathbb{P}}(A) := \mathbb{E}[\mathbf{1}_A Z_T(\hat{\mu})]$ for $A \in \mathcal{F}_T$ under the change of measure $Z(\hat{\mu}) := \exp\left\{-\hat{\mu} B_t - \frac{1}{2}\hat{\mu}^2 t\right\}, \quad \forall t \ge 0$. The process $\left\{\tilde{B}_t := B_t + \hat{\mu} t, 0 \le t \le T\right\}$ is a Brownian motion on $\left(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}\right)$.

The probability in (5.17) has been discussed by Jin in [28] (theorem 5.2) and Bush et al in [8] (theorem 3.3). We have

$$\tilde{\mathbb{P}}^{\tilde{x}^i} \left\{ \tilde{B}_t < \frac{\varepsilon}{\hat{\sigma}}, \inf_{0 \le s \le t} \tilde{B}_s \ge 0 \right\} \le \left(1 - e^{\frac{-2x^i \varepsilon}{\hat{\sigma}^2 t}}\right) \int_0^\varepsilon \frac{1}{\sqrt{2\pi t}\hat{\sigma}} e^{-\left(y - x^i\right)^2 / 2t\hat{\sigma}^2} dy, \tag{5.18}$$

where $\tilde{\mathbb{P}}^{x^i}$ is the probability measure associated with the initial position $\tilde{x}^i = \frac{x^i}{\hat{\sigma}}$ of X. Now for fixed t, we have

$$\begin{aligned}
\tilde{\mathbb{P}}^{\tilde{x}^i} \left\{ \tilde{B}_t < \frac{\varepsilon}{\hat{\sigma}}, \inf_{0 \le s \le t} \tilde{B}_s \ge 0 \right\} &\le \frac{2\varepsilon x^i}{\hat{\sigma}^2 t} \int_0^\varepsilon \frac{1}{\sqrt{2\pi t}\hat{\sigma}} e^{-\left(y - x^i\right)^2 / 2t\hat{\sigma}^2} dy \\
&\le \frac{2\varepsilon^2 x^i}{\hat{\sigma}^2 t} \frac{1}{\sqrt{2\pi t}\hat{\sigma}} e^{-\left(\varepsilon - x^i\right)^2 / 2t\hat{\sigma}^2} \\
&\le K_t x^i \varepsilon^2
\end{aligned}$$

where we obtain the second inequality by taking the maximum of the integrand and $K_t = \frac{2}{\hat{\sigma}^2 t}\frac{1}{\sqrt{2\pi t}\hat{\sigma}} e^{-\left(\varepsilon - x^i\right)^2/2t\hat{\sigma}^2}$ is a constant depending on fixed t and $0 < x^i < 1$.

The remaining proof is the same as in [28] and [8].

b) To show the limit behaviour of the empirical density at the prepayment boundary we again adapt the same technique as in [28] and [8]. We have, on using the definition of the limit empirical measure, that

$$\mathbb{E}\left[\nu_t^+\left((1-\varepsilon,1)\right)\right] \quad \leq \quad \mathbb{E}\left[\liminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{\left\{1-\varepsilon<X_t^i<1\right\}}\right]$$

which by dominated convergence theorem is

$$= \quad \liminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{P}\left\{1-\varepsilon<X_t^i<1\right\}.$$

We now find the probability

$$\begin{aligned}
\mathbb{P}\left\{1-\varepsilon<X_t^i<1\right\} &= \mathbb{P}\left\{\hat{X}_t^i > 1-\varepsilon,\ \inf_{0\leq t<s}\hat{X}_s^i > 0,\ \sup_{0\leq t<s}\hat{X}_t^i < 1,\ \tau_r^i > t\right\}\\
&\leq \mathbb{P}\left\{\hat{X}_t^i > 1-\varepsilon,\ \sup_{0\leq t<s}\hat{X}_t^i < 1\right\}\\
&= \mathbb{P}\left\{\frac{x^i}{\hat{\sigma}}+\frac{\hat{\mu}t}{\hat{\sigma}}+B_t > \frac{1-\varepsilon}{\hat{\sigma}},\ \sup_{0\leq s<t}\frac{x^i}{\hat{\sigma}}+\frac{\hat{\mu}s}{\hat{\sigma}}+B_s < \frac{1}{\hat{\sigma}}\right\},
\end{aligned}$$

where $\hat{X}_t^i$ is the same as defined in previous part. Using the change of measure as in [28] and as we recalled in first part and the Hölder's inequality we have

$$\begin{aligned}
&\mathbb{P}\left\{\frac{x^i}{\hat{\sigma}}+\frac{\hat{\mu}t}{\hat{\sigma}}+B_t > \frac{1-\varepsilon}{\hat{\sigma}},\ \sup_{0\leq s<t}\frac{x^i}{\hat{\sigma}}+\frac{\hat{\mu}s}{\hat{\sigma}}+B_s < \frac{1}{\hat{\sigma}}\right\}\\
&\leq \left\{\mathbb{E}^{\tilde{\mathbb{P}}}\left[\mathbf{1}_{\left\{\frac{x^i}{\hat{\sigma}}+\tilde{B}_t>\frac{1-\varepsilon}{\hat{\sigma}},\sup_{0\leq s\leq t}\frac{x^i}{\hat{\sigma}}+\tilde{B}_s<\frac{1}{\hat{\sigma}}\right\}}\right]\right\}^{1/a}\cdot\left\{\mathbb{E}^{\tilde{\mathbb{P}}}\left[\frac{1}{Z_T\left(\hat{\mu}\right)}\right]^b\right\}^{1/b},\\
&\leq J_1\cdot J_2, \qquad (5.19)
\end{aligned}$$

where

$$J_1 = \left\{ \mathbb{E}^{\tilde{\mathbb{P}}} \left[\mathbf{1}_{\left\{ \frac{x^i}{\hat{\sigma}} + \tilde{B}_t > \frac{1-\varepsilon}{\hat{\sigma}}, \sup_{0 \le s \le t} \frac{x^i}{\hat{\sigma}} + \tilde{B}_s < \frac{1}{\hat{\sigma}} \right\}} \right] \right\}^{1/a}$$

and,

$$J_2 = \left\{ \mathbb{E}^{\tilde{\mathbb{P}}} \left[\frac{1}{Z_T(\hat{\mu})} \right]^b \right\}^{1/b}$$

with

$$1/a + 1/b = 1,\ a > 1,\ b > 1.$$

To find bounds on J_1, we need to find an expression for the probability

$$\begin{aligned} \tilde{\mathbb{P}}^{\tilde{x}^i} \left\{ \tilde{B}_t > \frac{1-\varepsilon}{\hat{\sigma}}, \sup_{0 \le s \le t} \tilde{B}_s < \frac{1}{\hat{\sigma}} \right\} &= \tilde{\mathbb{P}}^{\tilde{x}^i} \left\{ \tilde{B}_t > \frac{1-\varepsilon}{\hat{\sigma}} \right\} \\ &\quad - \tilde{\mathbb{P}}^{\tilde{x}^i} \left\{ \tilde{B}_t > \frac{1-\varepsilon}{\hat{\sigma}}, \sup_{0 \le s < t} \tilde{B}_s \ge \frac{1}{\hat{\sigma}} \right\} \end{aligned}$$

By the formula 1.1.8 in Borodin and Salminen (page 154) [6], for a Brownian motion W_t :

$$\mathbb{P}^x \left(\sup_{0 \le s \le t} W_s \ge y,\ W_t \in dz \right) = \frac{1}{\sqrt{2\pi t}} e^{-(|z-y|+y-x)^2/2t} dz \quad x \le y$$

we have,

$$\begin{aligned} \tilde{\mathbb{P}}^{\tilde{x}^i} \left\{ \tilde{B}_t > \frac{1-\varepsilon}{\hat{\sigma}}, \sup_{0 \le s \le t} \tilde{B}_s \ge \frac{1}{\hat{\sigma}} \right\} &= \int_{\frac{1-\varepsilon}{\hat{\sigma}}}^{+\infty} \frac{1}{\sqrt{2\pi t}} e^{-\left(z - \frac{x^i}{\hat{\sigma}}\right)^2/2t} dz \\ &\quad - \int_{\frac{1-\varepsilon}{\hat{\sigma}}}^{+\infty} \frac{1}{\sqrt{2\pi t}} e^{-\left(|z-\frac{1}{\hat{\sigma}}| - \frac{x^i}{\hat{\sigma}} + \frac{1}{\hat{\sigma}}\right)^2/2t} dz. \end{aligned}$$

We do the transformation $\hat{\sigma} z = y$ to get

$$\tilde{\mathbb{P}}^{\tilde{x}^i} \left\{ \tilde{B}_t > \frac{1-\varepsilon}{\hat{\sigma}}, \sup_{0 \le s \le t} \tilde{B}_s < \frac{1}{\hat{\sigma}} \right\} = \int_{1-\varepsilon}^{+\infty} \frac{1}{\sqrt{2\pi t}\hat{\sigma}} e^{-\left(y - x^i\right)^2/2t\hat{\sigma}^2} dy$$

$$
\begin{aligned}
& - \int_{1-\varepsilon}^{+\infty} \frac{1}{\sqrt{2\pi t}\hat{\sigma}} e^{-\left(|y-1|-x^i+1\right)^2/2t\hat{\sigma}^2} dy \\
= & \int_{1-\varepsilon}^{+\infty} \frac{1}{\sqrt{2\pi t}\hat{\sigma}} e^{-\left(y-x^i\right)^2/2t\hat{\sigma}^2} dy \\
& - \int_{1-\varepsilon}^{1} \frac{1}{\sqrt{2\pi t}\hat{\sigma}} e^{-\left(-y-x^i+2\right)^2/2t\hat{\sigma}^2} dy \\
& - \int_{1}^{+\infty} \frac{1}{\sqrt{2\pi t}\hat{\sigma}} e^{-\left(y-x^i\right)^2/2t\hat{\sigma}^2} dy \\
= & \int_{1-\varepsilon}^{1} \frac{1}{\sqrt{2\pi t}\hat{\sigma}} \left(1 - e^{\frac{-\left(1-x^i\right)(2-2y)}{\hat{\sigma}^2 t}}\right) \\
& e^{-\left(y-x^i\right)^2/2t\hat{\sigma}^2} dy \\
\leq & \left(1 - e^{-2\varepsilon\left(1-x^i\right)/\hat{\sigma}^2 t}\right) \int_{1-\varepsilon}^{1} \frac{1}{\sqrt{2\pi t}\hat{\sigma}} e^{-\left(y-x^i\right)^2/2t\hat{\sigma}^2} dy \\
\leq & \frac{2\varepsilon\left(1-x^i\right)}{\hat{\sigma}^2 t} \int_{1-\varepsilon}^{1} \frac{1}{\sqrt{2\pi t}\hat{\sigma}} e^{-\left(y-x^i\right)^2/2t\hat{\sigma}^2} dy \qquad (5.20)
\end{aligned}
$$

For fixed t,

$$
\begin{aligned}
\tilde{\mathbb{P}}^{\tilde{x}^i}\left\{\tilde{B}_t > \frac{1-\varepsilon}{\hat{\sigma}}, \sup_{0\leq s\leq t} \tilde{B}_s < \frac{1}{\hat{\sigma}}\right\} & \leq \frac{2\varepsilon^2\left(1-x^i\right)}{\hat{\sigma}^2 t} \frac{1}{\sqrt{2\pi t}\hat{\sigma}} e^{-\left(1-\varepsilon-x^i\right)^2/2t\hat{\sigma}^2} \\
& \leq K_t \varepsilon^2 \left(1-x^i\right),
\end{aligned}
$$

where $K_t = \sqrt{\frac{2}{\pi}} \frac{1}{t^{3/2}\hat{\sigma}^3} e^{-\left(1-\varepsilon-x^i\right)^2/2t\hat{\sigma}^2}$.

We have from [28],

$$
J_2 \leq \exp\left\{\frac{1}{2}(b-1)T\hat{\mu}^2\right\}\left\{\mathbb{E}^{\tilde{P}}\left[\exp\left\{\hat{\mu}b\tilde{B}_T - \frac{1}{2}b^2\hat{\mu}^2 T\right\}\right]\right\}.
$$

Since $\exp\left\{\hat{\mu}b\tilde{B}_T - \frac{1}{2}b^2\hat{\mu}^2 T\right\}$ is a true martingale, we choose b such that $1 < b < +\infty$ to get

$$
J_2 \leq \exp\left\{\frac{1}{2}(b-1)T\hat{\mu}^2\right\} \qquad (5.21)
$$

Therefore, we have

$$
\begin{aligned}
\mathbb{E}\left[\frac{\nu_t^+\left((1-\varepsilon,1)\right)}{\varepsilon}\right] &\leq \lim_{n\to\infty} J_1 \cdot J_2 \\
&\leq K_t \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\left(1-x^i\right)\varepsilon\right)^{1/a}\cdot e^{\frac{1}{2}(b-1)T\hat{\mu}^2},
\end{aligned}
$$

Letting $a \downarrow 1$ and since $0 < x^i < 1$, $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left(1-x^i\right)$ is bounded above and we obtain

$$
\mathbb{E}\left[\frac{\nu_t^+\left((1-\varepsilon,1)\right)}{\varepsilon}\right] \leq K_t C\varepsilon.
$$

Now applying Lemma 5.2.1 we obtain for a fixed t

$$
\mathbb{E}\left[\frac{\nu_t^+\left((1-\varepsilon,1)\right)}{\varepsilon}\right] = 0, \text{ a.s.}
$$

$\square$

In the above theorem, the constants K_t can blow near the default and prepayment boundaries if we allow t to be small. To find a uniform control on ν_t^+ near the default and prepayment boundaries, we give another theorem under the following condition on the initial distribution.

Assumption A We assume that $\{x^i\}$ are drawn from a two sided tailed-distribution that is $\hat{\mathbb{P}}\left(x^i < \lambda\right) \leq \lambda^{\beta^1}$ for some $\beta^1 > 0$ and for all i. Also, $\hat{\mathbb{P}}\left(x^i > 1-\lambda\right) \leq \lambda^{\beta^2}$ for some $\beta^2 > 0$ and for all i.

Assumption (A) provides an extension to the results in [28] and Bush et al. in [8] to the initial distribution being a tailed distribution.

Theorem 5.2.3. *Under assumption (A) for β^1, $\beta^2 > 3$ and for any $T > 0$*

$$a) \qquad \lim_{\varepsilon\downarrow 0}\sup_{0<t<T}\frac{\nu_t^+((0,\varepsilon))}{\varepsilon} = 0, \quad a.s.$$

$$b) \qquad \lim_{\varepsilon\downarrow 0}\sup_{0<t<T}\frac{\nu_t^+((1-\varepsilon,1))}{\varepsilon} = 0, \quad a.s.$$

Proof. a) We have, from theorem 5.2.2(a) equation (5.19) and [[28] theorem 5.2, [8] theorem 3.3], after the change of measure

$$\begin{aligned}&\mathbb{P}\left\{\frac{x^i}{\hat\sigma}+\frac{\hat\mu t}{\hat\sigma}+B_t<\frac{\varepsilon}{\hat\sigma},\inf_{0\le s\le t}\frac{x^i}{\hat\sigma}+\frac{\hat\mu s}{\hat\sigma}+B_s>0\right\}\\ &\quad\le J_1\cdot J_2,\end{aligned}$$

where J_1 and J_2 are as before in theorem (5.2.2, part (a)) We need to find the expression for the probability in J_1 that is,

$$\begin{aligned}J_1 &= \left(\tilde{\mathbb{P}}\left\{\frac{x^i}{\hat\sigma}+\tilde B_t<\frac{\varepsilon}{\hat\sigma},\inf_{0\le s\le t}\frac{x^i}{\hat\sigma}+\tilde B_s>0\right\}\right)^{1/a}\\ &= \left(\tilde{\mathbb{P}}^{\tilde x^i}\left\{\tilde B_t<\frac{\varepsilon}{\hat\sigma},\inf_{0\le s\le t}\tilde B_s>0\right\}\right)^{1/a}.\end{aligned}$$

We decompose the probability according to whether we start near the boundary point or not, that is

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^n\tilde{\mathbb{P}}^{\tilde x^i}\left\{\tilde B_t<\frac{\varepsilon}{\hat\sigma},\inf_{0\le s\le t}\tilde B_s>0\right\} &= \frac{1}{n}\sum_{i=1}^n\tilde{\mathbb{P}}^{\tilde x^i}\left\{\tilde B_t\le\frac{\varepsilon}{\hat\sigma},\inf_{0\le s\le t}\tilde B_s>0\right\}\mathbf{1}_{\{x^i\ge\varepsilon^\alpha\}}\\ &\quad+\frac{1}{n}\sum_{i=1}^n\tilde{\mathbb{P}}^{\tilde x^i}\left\{\tilde B_t<\frac{\varepsilon}{\hat\sigma},\inf_{0\le s\le t}\tilde B_s>0\right\}\mathbf{1}_{\{x^i<\varepsilon^\alpha\}}\end{aligned}$$

We consider the cases separately. When x^i is near the default boundary that is when $x^i < \varepsilon^\alpha$, we have from Assumption (A)

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^n\tilde{\mathbb{P}}^{\tilde x^i}\left\{\tilde B_t<\frac{\varepsilon}{\hat\sigma},\inf_{0\le s\le t}\tilde B_s>0\right\}\mathbf{1}_{\{x^i<\varepsilon^\alpha\}} &\le \frac{1}{n}\sum_{i=1}^n\mathbf{1}_{\{x^i<\varepsilon^\alpha\}}\\ &\le \hat{\mathbb{P}}\left(x^i<\varepsilon^\alpha\right)<\varepsilon^{\alpha\beta^1},\end{aligned}$$

where $\hat{\mathbb{P}}$ is the probability of initial point and the last inequality holds under the assumption that $\hat{\mathbb{P}}$ is a two-sided tailed distribution.

For the first part that is for the points x^i such that $x^i \geq \varepsilon^\alpha$, we have by using (5.18)

$$\frac{1}{n}\sum_{i=1}^n \tilde{\mathbb{P}}^{\tilde{x}^i}\left\{\tilde{B}_t \leq \frac{\varepsilon}{\hat{\sigma}}, \inf_{0\leq s\leq t}\tilde{B}_s > 0\right\}\mathbf{1}_{\{x^i\geq\varepsilon^\alpha\}}$$
$$\leq \frac{1}{n}\sum_{i=1}^n \frac{2\varepsilon x^i}{\hat{\sigma}^2 t}\mathbf{1}_{\{x^i\geq\varepsilon^\alpha\}}\int_0^\varepsilon \frac{1}{\sqrt{2\pi t}\hat{\sigma}}e^{-\left(x^i-y\right)^2/2t\hat{\sigma}^2}dy$$

We want to maximize the integrand on y for $x^i \geq \varepsilon^\alpha$. For $\alpha < 1$ and $\varepsilon << 1$, we have $x^i \geq \varepsilon^\alpha > \varepsilon$. Therefore, for $y \in (0,\varepsilon)$ and $x^i > \varepsilon^\alpha$, we have

$$\min_y \left(x^i - y\right)^2/2\hat{\sigma}^2 t > \left(\varepsilon^\alpha - \varepsilon\right)^2/2\hat{\sigma}^2 t > \frac{\varepsilon^{2\alpha}}{2\hat{\sigma}^2 t}\left(1-\varepsilon^{1-\alpha}\right)^2.$$

For some fixed ε, $\left(1-\varepsilon^{1-\alpha}\right)^2$ is a constant say C'. We have,

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^n \tilde{\mathbb{P}}^{\tilde{x}^i}\left\{\tilde{B}_t < \frac{\varepsilon}{\hat{\sigma}}, \inf_{0\leq s\leq t}\tilde{B}_s > 0\right\}\mathbf{1}_{\{x^i>\varepsilon^\alpha\}}\\
&\quad\leq \frac{1}{n}\sum_{i=1}^n \frac{2\varepsilon x^i}{\hat{\sigma}^2 t}\int_0^\varepsilon \frac{1}{\sqrt{2\pi t}\hat{\sigma}}e^{-\frac{\varepsilon^{2\alpha}C'}{2t\hat{\sigma}^2}}dy\mathbf{1}_{\{x^i>\varepsilon^\alpha\}}\\
&\quad\leq \frac{2\varepsilon^2}{\hat{\sigma}^2 t}\frac{1}{\sqrt{2\pi t}\hat{\sigma}}e^{-\frac{\varepsilon^{2\alpha}C'}{2t\hat{\sigma}^2}}\\
&\quad= \sqrt{\frac{2}{\pi}}\frac{\varepsilon^2}{\hat{\sigma}^3 t^{3/2}}e^{-\frac{\varepsilon^{2\alpha}C'}{2t\hat{\sigma}^2}}
\end{aligned}$$

That is we have,

$$\frac{1}{n}\sum_{i=1}^n \tilde{\mathbb{P}}^{\tilde{x}^i}\left\{\tilde{B}_t < \frac{\varepsilon}{\hat{\sigma}}, \inf_{0\leq s\leq t}\tilde{B}_s > 0\right\} \leq \varepsilon^{\alpha\beta^1} + \sqrt{\frac{2}{\pi}}\frac{\varepsilon^2}{\hat{\sigma}^3 t^{3/2}}e^{-\frac{\varepsilon^{2\alpha}C'}{2t\hat{\sigma}^2}}$$

Optimising on time t, we have for $t = C^*\varepsilon^{2\alpha}$

$$\frac{1}{n}\sum_{i=1}^n \tilde{\mathbb{P}}^{\tilde{x}^i}\left\{\tilde{B}_t < \frac{\varepsilon}{\hat{\sigma}}, \inf_{0\leq s\leq t}\tilde{B}_s > 0\right\} \leq \varepsilon^{\alpha\beta^1} + C^*\varepsilon^{2-3\alpha}$$

which is equivalent to

$$\frac{1}{n}\sum_{i=1}^{n}\tilde{\mathbb{P}}^{\tilde{x}^i}\left\{\tilde{B}_t<\frac{\varepsilon}{\hat{\sigma}},\inf_{0\le s\le t}\tilde{B}_s>0\right\}\ \le\ C\varepsilon^{\frac{2\beta^1}{\beta^1+3}}$$

for $\alpha=\frac{2}{\beta^1+3}$.

We have, on using the bounds on J_2 from (5.21)

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\mathbb{P}\left\{\frac{x^i}{\hat{\sigma}}+\frac{\hat{\mu}t}{\hat{\sigma}}+B_t<\frac{\varepsilon}{\hat{\sigma}},\inf_{0\le s\le t}\frac{x^i}{\hat{\sigma}}+\frac{\hat{\mu}s}{\hat{\sigma}}+B_s>0\right\}\\
&=\frac{1}{n}\sum_{i=1}^{n}J_1\cdot J_2\\
&\le\left(C\varepsilon^{\frac{2\beta^1}{\beta^1+3}}\right)^{1/a}\exp\left\{\frac{1}{2}\left(b-1\right)\hat{\mu}^2T\right\}\\
&\le K_T\varepsilon^{\frac{2\beta^1}{(\beta^1+3)a}},
\end{aligned}$$

where $K_T=C^{1/a}\exp\left\{\frac{1}{2}\left(b-1\right)\hat{\mu}^2T\right\}$.

By (5.16) and (5.17),

$$\mathbb{E}\left[\nu_t^+\left((0,\varepsilon)\right)\right]\le\liminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}K_T\varepsilon^{\frac{2\beta^1}{(\beta^1+3)a}}. \tag{5.22}$$

To get our boundary condition, we need $\frac{2\beta^1}{(\beta^1+3)a}>1$ that is $\beta^1>\frac{3a}{2-a}$ and $1<a<2$. Letting $a\downarrow 1$, we have $\beta^1>3$ such that

$$\mathbb{E}\left[\frac{\nu_t^+\left((0,\varepsilon)\right)}{\varepsilon}\right]\ \le\ \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}K_T\varepsilon^{\delta},$$

for a positive small δ.

Now by lemma 5.2.1, we have

$$\limsup_{\varepsilon\downarrow 0,0<t<T}\frac{\nu_t^+\left((0,\varepsilon)\right)}{\varepsilon}=0,\ \text{a.s.}.$$

b).

To show the limit behaviour of the empirical density at the prepayment boundary we proceed in the same way as above. From theorem 5.2.2 (b), after the change of measure we have

$$\mathbb{P}\left\{\frac{x^i}{\hat{\sigma}}+\frac{\hat{\mu}t}{\hat{\sigma}}+B_t>\frac{1-\varepsilon}{\hat{\sigma}}, \sup_{0\le s<t}\frac{x^i}{\hat{\sigma}}+\frac{\hat{\mu}s}{\hat{\sigma}}+B_s<\frac{1}{\hat{\sigma}}\right\}$$
$$\le J_1\cdot J_2,$$

where J_1 and J_2 are as defined in (5.2.2, part (b)). To find bounds on J_1, we need to find an expression for the probability

$$J_1=\left(\tilde{\mathbb{P}}^{\tilde{x}^i}\left\{\tilde{B}_t>\frac{1-\varepsilon}{\hat{\sigma}}, \sup_{0\le s\le t}\tilde{B}_s<\frac{1}{\hat{\sigma}}\right\}\right)^{1/a}$$

Decomposing according to whether we start near the prepayment boundary point or not,

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\tilde{\mathbb{P}}^{\tilde{x}^i}\left\{\tilde{B}_t>\frac{1-\varepsilon}{\hat{\sigma}}, \sup_{0\le s\le t}\tilde{B}_s<\frac{1}{\hat{\sigma}}\right\}\\
&\qquad=\frac{1}{n}\sum_{i=1}^{n}\tilde{\mathbb{P}}^{\tilde{x}^i}\left\{\tilde{B}_t>\frac{1-\varepsilon}{\hat{\sigma}}, \sup_{0\le s\le t}\tilde{B}_s<\frac{1}{\hat{\sigma}}\right\}\mathbf{1}_{\{1-x^i\ge\varepsilon^\alpha\}}\\
&\qquad+\frac{1}{n}\sum_{i=1}^{n}\tilde{\mathbb{P}}^{\tilde{x}^i}\left\{\tilde{B}_t>\frac{1-\varepsilon}{\hat{\sigma}}, \sup_{0\le s\le t}\tilde{B}_s<\frac{1}{\hat{\sigma}}\right\}\mathbf{1}_{\{1-x^i<\varepsilon^\alpha\}}
\end{aligned}$$

For the case when x^i is near the prepayment boundary that is $1-x^i<\varepsilon^\alpha$, we have

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}\tilde{\mathbb{P}}^{\tilde{x}^i}\left\{\tilde{B}_t>\frac{1-\varepsilon}{\hat{\sigma}}, \sup_{0\le s\le t}\tilde{B}_s<\frac{1}{\hat{\sigma}}\right\}\mathbf{1}_{\{1-x^i<\varepsilon^\alpha\}} &\le \frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{\{1-x^i<\varepsilon^\alpha\}}\\
&\le \hat{\mathbb{P}}\left(x^i>1-\varepsilon^\alpha\right)<\varepsilon^{\alpha\beta_2},
\end{aligned}$$

where the last inequality holds under the assumption that the initial distribution for X is a tailed distribution.

Now for the first expression, we have from (5.20)

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\tilde{\mathbb{P}}^{\tilde{x}^i}\left\{\tilde{B}_t > \frac{1-\varepsilon}{\hat{\sigma}}, \sup_{0\le s\le t}\tilde{B}_s < \frac{1}{\hat{\sigma}}\right\}\mathbf{1}_{\{1-x^i\ge\varepsilon^\alpha\}} \\
&\quad \le \frac{1}{n}\sum_{i=1}^{n}\frac{2\varepsilon\left(1-x^i\right)}{\hat{\sigma}^2 t}\int_{1-\varepsilon}^{1}\frac{1}{\sqrt{2\pi t}\hat{\sigma}}e^{-\left(y-x^i\right)^2/2t\hat{\sigma}^2}dy\mathbf{1}_{\{1-x^i\ge\varepsilon^\alpha\}}
\end{aligned}$$

We want to minimize $\left(x^i-y^i\right)^2/2\hat{\sigma}^2 t$ on y for $1-x^i\ge\varepsilon^\alpha$. We have $x^i\ge\varepsilon^\alpha>\varepsilon$ for $\alpha<1$ and $\varepsilon<<1$. Therefore for $y\in(1-\varepsilon,1)$, we have

$$\min_y\left(y-x^i\right)^2/2\hat{\sigma}^2 t > \left(\varepsilon^\alpha-\varepsilon\right)^2/2\hat{\sigma}^2 t > \frac{\varepsilon^{2\alpha}}{2\hat{\sigma}^2 t}\left(1-\varepsilon^{1-\alpha}\right)^2.$$

The remaining proof follows as for the first part. □

5.3 Density of the limit empirical measure

We know from the previous section that if the limiting empirical measure ν_t^+ has a density $\upsilon(t,x)$ with respect to Lebesgue measure then this density will satisfy Dirichlet boundary conditions i.e

$$\upsilon(t,0)=\upsilon(t,1)=0.$$

In the next chapter we will show that ν_t^+ has a density $\upsilon(t,x)$ with respect to Lebesgue measure in $\mathbb{R}^+$ and $\upsilon(t,x)\in L^2\left(\mathbb{R}^+\right)$ provided that the initial measure ν_0^+ has a density $\upsilon(0,x)$ with respect to Lebesgue measure in $\mathbb{R}^+$ and $\upsilon(0,x)\in L^2\left(\mathbb{R}^+\right)$.

Therefore we write

$$\nu_t^+=\upsilon(t,x)\,dx, \qquad x\in(0,1).$$

Substituting this in the evolution equation of the limit empirical measure (5.14), we obtain

$$\langle\phi,\upsilon(t,x)\rangle = \langle\phi,\upsilon(0,x)\rangle+\int_0^t\langle\hat{\mu}\phi'(x)+\frac{\hat{\sigma}^2}{2}\phi''(x)-\lambda(r_s)\phi(x),\upsilon(s,x)\rangle ds$$

$$+\int_0^s \langle \hat{\sigma}\rho\phi'(x), \upsilon(s,x)\rangle dM_s. \tag{5.23}$$

Using (5.2) we have,

$$\begin{aligned}\int_0^1 \phi(x)\upsilon(t,x)\,dx &= \int_0^1 \phi(x)\upsilon(0,x)\,dx + \int_0^t\int_0^1 \Big(\hat{\mu}\frac{\partial\phi(x)}{\partial x} + \frac{1}{2}\hat{\sigma}^2\frac{\partial^2\phi(x)}{\partial x^2} \\ &\quad -\lambda(r_s)\phi(x)\Big)\upsilon(s,x)\,dxds \\ &\quad +\int_0^t\int_0^1 \hat{\sigma}\rho\phi'(x)\upsilon(s,x)\,dxdM_s.\end{aligned}$$

Writing,

$$\mathcal{A} = \hat{\mu}\frac{\partial}{\partial x} + \frac{\hat{\sigma}^2}{2}\frac{\partial^2}{\partial x^2}.$$

Now using integration by parts and writing $\mathcal{A}^\dagger$ for the adjoint operator of $\mathcal{A}$ we obtain

$$\begin{aligned}\int \phi(x)\upsilon(t,x)\,dx &= \int \phi(x)\upsilon(0,x)\,dx \\ &\quad +\int_0^t\int \phi(x)\left(\mathcal{A}^\dagger - \lambda(r_s)\right)\upsilon(s,x)\,dxds \\ &\quad -\int_0^t\int \phi(x)\frac{\partial}{\partial x}\left(\hat{\sigma}\rho\upsilon(s,x)\right)dxdM_s, \\ &= \int \phi(x)\Big(\upsilon(0,x) + \int_0^t \left(\mathcal{A}^\dagger - \lambda(r_s)\right)\upsilon(s,x)\,ds \\ &\quad -\int_0^t \frac{\partial}{\partial x}\left(\hat{\sigma}\rho\upsilon(s,x)\right)dM_s\Big)\,dx. \end{aligned} \tag{5.24}$$

Where we used $\upsilon(s,0)=0$ and $\upsilon(s,1)=0$; $\phi(0)=\phi(1)=0$ and $\phi'(0)=\phi'(1)=0$.

Since (5.24) holds $\forall\phi\in\bar{C}$, therefore we must have

$$\upsilon(t,x) = \upsilon(0,x) + \int_0^t \left(\mathcal{A}^\dagger - \lambda(r_s)\right)\upsilon(s,x)\,ds - \int_0^t \frac{\partial}{\partial x}\left(\rho\hat{\sigma}\upsilon(s,x)\right)dM_s.$$

Expanding the adjoint operator

$$\mathcal{A}^{\dagger}\upsilon\left(t,x\right)=-\hat{\mu}\frac{\partial}{\partial x}\left(\upsilon\left(t,x\right)\right)+\frac{\hat{\sigma}^{2}}{2}\frac{\partial^{2}}{\partial x^{2}}\left(\upsilon\left(t,x\right)\right)$$

to get the evolution of the system (4.4) in the form of a stochastic PDE

$$\begin{aligned} d\upsilon\left(t,x\right) &= -\hat{\mu}\frac{\partial}{\partial x}\left(\upsilon\left(t,x\right)\right)dt+\frac{1}{2}\hat{\sigma}^{2}\frac{\partial^{2}}{\partial x^{2}}\left(\upsilon\left(t,x\right)\right)dt-\lambda\left(r_{t}\right)\upsilon\left(t,x\right)dt \\ &\quad -\rho\hat{\sigma}\frac{\partial}{\partial x}\upsilon\left(t,x\right)dM_{t} \qquad x\in\left(0,1\right). \end{aligned} \tag{5.25}$$

Therefore the density $\upsilon\left(t,x\right)$ is a solution to the following system:

$$\begin{cases} d\upsilon\left(t,x\right)=-\hat{\mu}\frac{\partial}{\partial x}\left(\upsilon\left(t,x\right)\right)dt+\frac{1}{2}\hat{\sigma}^{2}\frac{\partial^{2}}{\partial x^{2}}\left(\upsilon\left(t,x\right)\right)dt-\lambda\left(r_{t}\right)\upsilon\left(t,x\right)dt \\ \qquad\qquad -\rho\hat{\sigma}\frac{\partial}{\partial x}\upsilon\left(t,x\right)dM_{t} \qquad x\in\left(0,1\right), \\ \upsilon\left(0,x\right)=\upsilon_{0}\left(x\right)\in L^{2}\left(\mathbb{R}^{+}\right), \\ \upsilon\left(t,0\right)=\upsilon\left(t,1\right)=0. \end{cases} \tag{5.26}$$

This is a stochastic PDE with Dirichlet boundary conditions.

Next, we would like to find the loss distribution for the pool of individuals in an MBS. The key to pricing the pool in an MBS is to find the joint loss distribution because the price of the pool depends on the cumulative defaults and prepayments occurring in the pool. We have derived the equation that describes the evolution of the empirical measure of the limiting large portfolio. We can find the loss in the pool by calculating the total mass of the empirical measure of individuals, at any time in future, that are neither defaulted nor prepaid. For simplification, we do not consider different recovery rates in case of default and prepayment. The formula below for loss L_t can easily be extended to take into account different recovery rates for default and prepayment. We note that we can derive the expression for the loss

L_t at time t in terms of the density after integrating by parts:

$$\begin{aligned}
L_t &= 1 - \int_0^1 \upsilon(t,x)\,dx, \\
&= 1 - \int_0^1 \Big(\upsilon(0,x) - \int_0^t \hat{\mu}\frac{\partial}{\partial x}\upsilon(s,x)\,ds + \frac{1}{2}\hat{\sigma}^2 \int_0^t \frac{\partial^2}{\partial} x^2 \upsilon(s,x)\,ds \\
&\quad - \int_0^t \lambda(r_s)\,\upsilon(s,x)\,ds - \int_0^t \rho\hat{\sigma}\frac{\partial}{\partial x}\upsilon(s,x)\,dM_s \Big)\,dx, \\
&= 1 - \int_0^1 \upsilon(0,x)\,dx + \hat{\mu}\int_0^t \upsilon(s,x)\mid_{x=0}^{x=1} ds - \frac{1}{2}\hat{\sigma}^2 \int_0^t \upsilon_x(s,x)\mid_{x=0}^{x=1} ds \\
&\quad + \int_0^t \int_0^1 \lambda(r_s)\,\upsilon(s,x)\,dxds + \rho\hat{\sigma}\int_0^t \upsilon(s,x)\mid_{x=0}^{x=1} dM_s,
\end{aligned}$$

$$\begin{aligned}
L_t &= 1 - \int_0^1 \upsilon(0,x)\,dx + \frac{1}{2}\hat{\sigma}^2 \int_0^t \upsilon_x(s,0)\,ds - \frac{1}{2}\hat{\sigma}^2 \int_0^t \upsilon_x(s,1)\,ds \\
&\quad + \int_0^t \int_0^1 \lambda(r_s)\,\upsilon(s,x)\,dxds.
\end{aligned}$$

Since $x^i > 0\,\forall i$ and X_t^i is a continuous process, we can conclude that $\tau^i > 0\,\forall i$. Thus

$$L_0 = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{0\geq\tau^i\}} = 0$$

and therefore,

$$\upsilon([0,1]\cup\triangle) = 1 = \int_0^1 \upsilon(0,x)\,dx.$$

Therefore provided that the derivative of $\upsilon(t,x)$ exists with respect to x at points 0 and 1, we have

$$L_t = \frac{1}{2}\hat{\sigma}^2 \int_0^t \upsilon_x(s,0)\,ds - \frac{1}{2}\hat{\sigma}^2 \int_0^t \upsilon_x(s,1)\,ds + \int_0^t \int_0^1 \lambda(r_s)\,\upsilon(s,x)\,dxds. \quad (5.27)$$

Thus the portfolio loss can be determined by

$$L_t^n = nL_t,$$

where n is the number of individuals in the pool. We note that given the condition (5.15) we have $L_0^n = 0$. Also, due to the way the defaults and prepayments are incorporated in the model, we have

$$\begin{aligned} 0 \leq L_t \leq 1, &\qquad \text{for}\, t \geq 0, \\ P\left(L_s \geq K\right) \leq P\left(L_t \geq K\right), &\quad \text{for}\, s \leq t, \end{aligned}$$

which ensures that there is no arbitrage in the model.

5.4 Estimation of the measure behaviour near boundaries

In this section, we give estimates on the second moment of the limit empirical measure near the default and prepayment boundaries. We again have two theorems discussing these results. We need these estimates in proving the existence of the density in next chapter.

The following theorem is again due to Jin [28]. We adapt it to our settings and for a fixed time *'t'*.

Theorem 5.4.1. *We show that for a fixed time t, we have*

$$\begin{aligned} &a) \quad \mathbb{E}\left[\left(\nu_t^+\left((0,\varepsilon)\right)\right)^2\right] \leq M_t^* \varepsilon^{3+\delta}, \\ &b) \quad \mathbb{E}\left[\left(\nu_t^+\left((1-\varepsilon,1)\right)\right)^2\right] \leq N_t^* \varepsilon^{3+\delta}. \end{aligned}$$

where $\delta > 0$ and M_t^* , N_t^* are two constants depending on t.

Proof. a)

By the definition of ν_t, properties of weak convergence, we have for an open set

$(0,\varepsilon)$

$$\begin{aligned}\nu_t^+\left((0,\varepsilon)\right) &\leq \liminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\delta_{X_t^i}\left((0,\varepsilon)\right)\\ &= \liminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{\left\{0<X_t^i<\varepsilon,\inf_{0\leq s\leq t}X_s^i>0\right\}}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\mathbb{E}\left[\left(\nu_t^+\left((0,\varepsilon)\right)\right)^2\right] &\leq \mathbb{E}\left[\liminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{\left\{0<X_t^i<\varepsilon\right\}}\liminf_{m\to\infty}\frac{1}{m}\sum_{j=1}^{m}\mathbf{1}_{\left\{0<X_t^j<\varepsilon\right\}}\right]\\ &\leq \liminf_{n\to\infty}\liminf_{m\to\infty}\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{\left\{\hat{X}_t^i<\varepsilon,\inf_{0<s\leq t}\hat{X}_s^i>0\right\}}\right.\\ &\qquad\left.\frac{1}{m}\sum_{j=1}^{m}\mathbf{1}_{\left\{\hat{X}_t^j<\varepsilon,\inf_{0<s\leq t}\hat{X}_s^j>0\right\}}\right]\\ &= \liminf_{n\to\infty,m\to\infty}\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbb{E}\left[\mathbf{1}_{\left\{\hat{X}_t^i<\varepsilon,\inf_{0<s\leq t}\hat{X}_s^i>0,\hat{X}_t^j<\varepsilon,\inf_{0<s\leq t}\hat{X}_s^j>0\right\}}\right]\\ &= \liminf_{n\to\infty,m\to\infty}\frac{1}{nm}\sum_{i=1}^{n}\sum_{j\neq i,j=1}^{m}\mathbb{E}\left[\mathbf{1}_{\left\{\hat{X}_t^i<\varepsilon,\inf_{0<s\leq t}\hat{X}_s^i>0,\hat{X}_t^j<\varepsilon,\inf_{0<s\leq t}\hat{X}_s^j>0\right\}}\right].\end{aligned}$$

where $\hat{X}_t^i$ and $\hat{X}_t^j$ are the diffusion processes for $t<\tau^i$ and $t<\tau^j$ respectively and we have

$$\hat{X}_t^i = x^i+\hat{\mu}t+\hat{\sigma}\sqrt{1-\rho^2}W_t^i+\hat{\sigma}\rho M_t \stackrel{d}{=} x^i+\hat{\mu}t+\hat{\sigma}B_t^1$$

also,

$$\hat{X}_t^j = x^j+\hat{\mu}t+\hat{\sigma}\sqrt{1-\rho^2}W_t^j+\hat{\sigma}\rho M_t \stackrel{d}{=} x^j+\hat{\mu}t+\hat{\sigma}B_t^2,$$

where $B_t^1=\sqrt{1-\rho^2}W_t^i+\rho M_t$ and $B_t^2=\sqrt{1-\rho^2}W_t^j+\rho M_t$ are two correlated Brownian motions with correlation ρ^2.

This is the same setup as in Jin [28] (proposition 5.2). We now use Girsanov's theorem to change the measure to take away the drift factor from the Brownian motions $B_t = (B^1, B^2)_t^T$.

We have Radon-Nikodym density process $Z_t\left(\hat{\mu}\right)$, as found in Jin's DPhil thesis [28],

$$Z_t\left(\hat{\mu}\right) \;\; := \;\; \exp\left\{-\frac{1}{1+\rho^2}\frac{\hat{\mu}^2}{\hat{\sigma}^2}t - \frac{1}{1+\rho^2}\frac{\hat{\mu}}{\hat{\sigma}}\left(B_t^1 + B_t^2\right)\right\}.$$

which is a true martingale. Thus, we can define a probability measure $\tilde{\mathbb{P}}$ on $\mathcal{F}_T$ by

$$\tilde{\mathbb{P}}\left(A\right) := \mathbb{E}\left[\mathbf{1}_A Z_T\left(\hat{\mu}\right)\right];\;\; A \in \mathcal{F}_T.$$

Further for fixed $T \in [0, \infty)$, the process

$$\left\{\left(\tilde{B}_t^1, \tilde{B}_t^2\right) := \left(B_t^1 + \frac{\hat{\mu}}{\hat{\sigma}}t, B_t^2 + \frac{\hat{\mu}}{\hat{\sigma}}t\right),\, \mathcal{F}_t,\, 0 \le t \le T\right\}$$

is a two dimensional Brownian motion on $\left(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}\right)$.

Let us now consider the expectations under the change of probability measure,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_{\left\{\frac{x^i}{\hat{\sigma}}+\frac{\hat{\mu}}{\hat{\sigma}}t+B_t^1<\frac{\varepsilon}{\hat{\sigma}},\, \inf_{0<s\le t}\frac{x^i}{\hat{\sigma}}+\frac{\hat{\mu}}{\hat{\sigma}}s+B_s^1>0,\, \frac{x^j}{\hat{\sigma}}+\frac{\hat{\mu}}{\hat{\sigma}}t+B_t^2<\frac{\varepsilon}{\hat{\sigma}},\, \inf_{0<s\le t}\frac{x^j}{\hat{\sigma}}+\frac{\hat{\mu}}{\hat{\sigma}}s+B_s^2>0\right\}}\right] \\
&= \mathbb{E}^{\tilde{\mathbb{P}}}\left[\mathbf{1}_{\left\{\frac{x^i}{\hat{\sigma}}+\tilde{B}_t^1<\frac{\varepsilon}{\hat{\sigma}},\, \inf_{0<s\le t}\frac{x^i}{\hat{\sigma}}+\tilde{B}_s^1>0,\, \frac{x^j}{\hat{\sigma}}+\tilde{B}_t^2<\frac{\varepsilon}{\hat{\sigma}},\, \inf_{0<s\le t}\frac{x^j}{\hat{\sigma}}+\tilde{B}_s^2>0\right\}}\frac{1}{Z_T}\left(\hat{\mu}\right)\right] \\
&\le \left\{\mathbb{E}^{\tilde{\mathbb{P}}}\left[\mathbf{1}_{\left\{\frac{x^i}{\hat{\sigma}}+\tilde{B}_t^1<\frac{\varepsilon}{\hat{\sigma}},\, \inf_{0<s\le t}\frac{x^i}{\hat{\sigma}}+\tilde{B}_s^1>0,\, \frac{x^j}{\hat{\sigma}}+\tilde{B}_t^2<\frac{\varepsilon}{\hat{\sigma}},\, \inf_{0<s\le t}\frac{x^j}{\hat{\sigma}}+\tilde{B}_s^2>0\right\}}\right]\right\}^{1/a} \cdot \\
&\quad \left\{\mathbb{E}^{\tilde{\mathbb{P}}}\left[\frac{1}{Z_T}\left(\hat{\mu}\right)\right]^b\right\}^{1/b},
\end{aligned}$$

by Hölder's inequality. Therefore, we have

$$\mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_{\left\{\frac{x^i}{\hat{\sigma}}+\frac{\hat{\mu}}{\hat{\sigma}}t+B_t^1<\frac{\varepsilon}{\hat{\sigma}},\, \inf_{0<s\le t}\frac{x^i}{\hat{\sigma}}+\frac{\hat{\mu}}{\hat{\sigma}}s+B_s^1>0,\, \frac{x^j}{\hat{\sigma}}+\frac{\hat{\mu}}{\hat{\sigma}}t+B_t^2<\frac{\varepsilon}{\hat{\sigma}},\, \inf_{0<s\le t}\frac{x^j}{\hat{\sigma}}+\frac{\hat{\mu}}{\hat{\sigma}}s+B_s^2>0\right\}}\right] \;\; \le \;\; J_1 \cdot J_2,$$

where

$$J_1 \;\; = \;\; \left\{\mathbb{E}^{\tilde{\mathbb{P}}}\left[\mathbf{1}_{\left\{\frac{x^i}{\hat{\sigma}}+\tilde{B}_t^1<\frac{\varepsilon}{\hat{\sigma}},\, \inf_{0<s\le t}\frac{x^i}{\hat{\sigma}}+\tilde{B}_s^1>0,\, \frac{x^j}{\hat{\sigma}}+\tilde{B}_t^2<\frac{\varepsilon}{\hat{\sigma}},\, \inf_{0<s\le t}\frac{x^j}{\hat{\sigma}}+\tilde{B}_s^2>0\right\}}\right]\right\}^{1/a}$$

and,

$$J_2 \quad = \quad \left\{ \mathbb{E}^{\tilde{\mathbb{P}}} \left[\frac{1}{Z_T} (\hat{\mu}) \right]^b \right\}^{1/b}.$$

We now find the bounds on J_1 and J_2.

The probability in J_1 involves two correlated Brownian motions. We make a transformation from two correlated Brownian motion to a two-dimensional Brownian motion that has independent components as in Lemma 5.1 in [28]. Modifying the Lemma to our setup, we have for the probability involved in J_1,

$$\tilde{\mathbb{P}}_B \left\{ \tilde{B}_t^1 < \frac{\varepsilon}{\hat{\sigma}}, \inf_{0<s\le t} \tilde{B}_s^1 > 0, \tilde{B}_t^2 < \frac{\varepsilon}{\hat{\sigma}}, \inf_{0<s\le t} \tilde{B}_s^2 > 0 \right\}$$
$$\le 2^{1-\frac{\pi}{\alpha}} r_0{}^{\frac{\pi}{\alpha}} \int_0^{\sqrt{\frac{2(1-\rho^2)}{1-\rho^4}}\frac{\varepsilon}{\hat{\sigma}}} r^{1+\frac{\pi}{\alpha}} \frac{1}{t^{1+\frac{\pi}{\alpha}}} e^{-\frac{(r-r_0)^2 - 2rr_0}{2t}} dr, \tag{5.28}$$

where

$$\alpha = \begin{cases} \pi + \arctan\left(-\frac{\sqrt{1-\rho^4}}{\rho^2}\right) & \rho \neq 0 \\ \frac{\pi}{2} & \rho = 0 \end{cases} \tag{5.29}$$

and

$$r_0 \quad = \quad \sqrt{\frac{(x^i)^2 + (x^j)^2 - 2\rho^2 x^i x^j}{1-(\rho)^4}} \tag{5.30}$$

is the point where the new process with independent components begins at time $t = 0$. $\tilde{\mathbb{P}}_B$ is the law of initial distribution with $\tilde{B}_0^1 = \frac{x^i}{\hat{\sigma}} > 0$ and $\tilde{B}_0^2 = \frac{x^j}{\hat{\sigma}} > 0$.

We now try to find the minimum of $r^2 + r_0^2 - 3rr_0$. We have

$$r^2 + r_0^2 - 3rr_0 \ge r_0^2 - 3rr_0 > r_0^2 - 3r_0\sqrt{\frac{2(1-\rho^2)}{1-\rho^4}\frac{\varepsilon}{\hat{\sigma}}}.$$

Therefore,

$$\begin{aligned}
&\tilde{\mathbb{P}}_B\left\{\tilde{B}_t^1<\frac{\varepsilon}{\hat{\sigma}},\inf_{0<s\le t}\tilde{B}_s^1>0,\,\tilde{B}_t^2<\frac{\varepsilon}{\hat{\sigma}},\inf_{0<s\le t}\tilde{B}_s^2>0\right\}\\
&\qquad\le 2^{1-\frac{\pi}{\alpha}}{r_0}^{\frac{\pi}{\alpha}}\frac{1}{\left(2+\frac{\pi}{\alpha}\right)}\left(\sqrt{\frac{2\left(1-\rho^2\right)}{1-\rho^4}}\frac{\varepsilon}{\hat{\sigma}}\right)^{2+\frac{\pi}{\alpha}}\frac{1}{t^{1+\frac{\pi}{\alpha}}}e^{-\frac{r_0^2-3r_0\sqrt{\frac{2\left(1-\rho^2\right)}{1-\rho^4}}\frac{\varepsilon}{\hat{\sigma}}}{2t}}.
\end{aligned}$$

Now for fixed t, we have

$$\tilde{\mathbb{P}}_B\left\{\tilde{B}_t^1<\frac{\varepsilon}{\hat{\sigma}},\inf_{0<s\le t}\tilde{B}_s^1>0,\,\tilde{B}_t^2<\frac{\varepsilon}{\hat{\sigma}},\inf_{0<s\le t}\tilde{B}_s^2>0\right\}\le M_tC\varepsilon^{2+\frac{\pi}{\alpha}},$$

where $M_t=\frac{1}{t^{1+\frac{\pi}{\alpha}}}e^{-\frac{r_0^2-3r_0\sqrt{\frac{2(1-\rho^2)}{1-\rho^4}}\frac{\varepsilon}{\hat{\sigma}}}{2t}}$ and $C=2^{1-\frac{\pi}{\alpha}}\frac{1}{\left(2+\frac{\pi}{\alpha}\right)}\left(\sqrt{\frac{2(1-\rho^2)}{1-\rho^4}}\frac{1}{\hat{\sigma}}\right)^{2+\frac{\pi}{\alpha}}$.

Since $1<\frac{\pi}{\alpha}<2$, we have $2+\frac{\pi}{\alpha}>3$.

Jin found the bounds on J_2 in [28] proposition 5.2, so we have

$$J_2\ \le\ e^{(b-1)\left(\frac{1-\rho^2}{1-\rho^4}\right)\hat{\mu}^2T}:=J_T<+\infty.$$

Therefore, we have

$$\begin{aligned}
\mathbb{E}\left[\left(\nu_t^+\left((0,\varepsilon)\right)\right)^2\right]\ &\le\ J_1\cdot J_2\\
&\le\ J_TM_tC\varepsilon^{\left(2+\frac{\pi}{\alpha}\right)/a}\\
&\le\ J_TM_tC\varepsilon^{2+\frac{\pi}{\alpha}},
\end{aligned}$$

where the last inequality follows by letting $a\downarrow 1$. So, we have

$$\mathbb{E}\left[\left(\nu_t^+\left((0,\varepsilon)\right)\right)^2\right]\le M_t^*\varepsilon^{3+\delta},$$

where $M_t^*=J_TCM_t$ and $\delta>0$.

b) We now find the estimate on the second moment at the prepayment boundary.

Again on the steps of Jin [28], we have by the definition of the empirical measure, properties of weak convergence and Fatou's lemma

$$\begin{aligned}
\nu_t^+((1-\varepsilon,1)) &\leq \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \delta_{X_t^i}((1-\varepsilon,1)) \\
&= \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\left\{1-\varepsilon<X_t^i<1,\sup_{0\leq s\leq t} X_s^i<1\right\}}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\mathbb{E}\left[\left(\nu_t^+((1-\varepsilon,1))\right)^2\right] &\leq \mathbb{E}\left[\liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\left\{1-\varepsilon<X_t^i<1\right\}} \liminf_{m\to\infty} \frac{1}{m}\sum_{j=1}^{m} \mathbf{1}_{\left\{1-\varepsilon<X_t^j<1\right\}}\right] \\
&\leq \mathbb{E}\left[\liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\left\{\hat{X}_t^i>1-\varepsilon,\sup_{0<s\leq t} \hat{X}_s^i<1\right\}}\right. \\
&\qquad \left.\liminf_{m\to\infty} \frac{1}{m}\sum_{j=1}^{m} \mathbf{1}_{\left\{\hat{X}_t^j>1-\varepsilon,\sup_{0<s\leq t} \hat{X}_s^j<1\right\}}\right]
\end{aligned}$$

$$\begin{aligned}
&= \liminf_{n\to\infty,m\to\infty} \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} \mathbb{E}\left[\mathbf{1}_{\left\{\hat{X}_t^i>1-\varepsilon,\sup_{0<s\leq t} \hat{X}_s^i<1,\hat{X}_t^j>1-\varepsilon,\sup_{0<s\leq t} \hat{X}_s^j<1\right\}}\right] \\
&= \liminf_{n\to\infty,m\to\infty} \frac{1}{nm}\sum_{i=1}^{n}\sum_{j\neq i,j=1}^{m} \mathbb{E}\left[\mathbf{1}_{\left\{\hat{X}_t^i>1-\varepsilon,\sup_{0<s\leq t} \hat{X}_s^i<1,\hat{X}_t^j>1-\varepsilon,\sup_{0<s\leq t} \hat{X}_s^j<1\right\}}\right].
\end{aligned}$$

Since neither of the particles is absorbed by time t, we have

$$\begin{aligned}
X_t^i &= x^i + \hat{\mu}t + \hat{\sigma}\sqrt{1-\rho^2}W_t^i + \hat{\sigma}\rho M_t \\
X_t^j &= x^j + \hat{\mu}t + \hat{\sigma}\sqrt{1-\rho^2}W_t^j + \hat{\sigma}\rho M_t.
\end{aligned}$$

We do a transformation from upper boundary to the lower boundary by considering

$$\begin{aligned}
\hat{X}_t^i &= 1 - X_t^i = 1 - x^i - \hat{\mu}t - \hat{\sigma}\sqrt{1-\rho^2}W_t^i - \hat{\sigma}\rho M_t \stackrel{d}{=} 1 - x^i - \hat{\mu}t - \hat{\sigma}B_t^1 \\
\hat{X}_t^j &= 1 - X_t^j = 1 - x^j - \hat{\mu}t - \hat{\sigma}\sqrt{1-\rho^2}W_t^j - \hat{\sigma}\rho M_t \stackrel{d}{=} 1 - x^i - \hat{\mu}t - \hat{\sigma}B_t^2,
\end{aligned}$$

where $B_t^1 = \sqrt{1-\rho^2}W_t^i + \rho M_t$ and $B_t^2 = \sqrt{1-\rho^2}W_t^j + \rho M_t$ are two correlated Brownian motions with correlation ρ^2.

Under the above transformation, we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\nu_t\left((1-\varepsilon,1)\right)\right)^2\right] \\
&\leq \liminf_{n\to\infty,m\to\infty}\frac{1}{nm}\sum_{i=1}^{n}\sum_{j\neq i,j=1}^{m}\mathbb{E}\left[\mathbf{1}_{\left\{\hat{X}_t^i>1-\varepsilon,\sup_{0<s\leq t}\hat{X}_s^i<1,\hat{X}_t^j>1-\varepsilon,\sup_{0<s\leq t}\hat{X}_s^j<1\right\}}\right]. \\
&= \liminf_{n\to\infty,m\to\infty}\frac{1}{nm}\sum_{i=1}^{n}\sum_{j\neq i,j=1}^{m}\mathbb{E}\left[\mathbf{1}_{\left\{\hat{X}_t^i<\varepsilon,\inf_{0<s\leq t}\hat{X}_s^i>0,\hat{X}_t^j<\varepsilon,\inf_{0<s\leq t}\hat{X}_s^j>0\right\}}\right].
\end{aligned}$$

Using the same change of measure as in [28] proposition 5.2, we have the Radon-Nikodym density process $Z_t\left(\hat{\mu}\right)$

$$Z_t\left(\hat{\mu}\right) \quad = \quad \exp\left\{-\frac{1}{1+\rho^2}\frac{\hat{\mu}^2}{\hat{\sigma}^2}t+\frac{1}{1+\rho^2}\frac{\hat{\mu}}{\hat{\sigma}}\left(B_t^1+B_t^2\right)\right\}.$$

which is a true martingale. Thus, we can define a probability measure $\tilde{\mathbb{P}}$ on $\mathcal{F}_T$ by

$$\tilde{\mathbb{P}}\left(A\right):=\mathbb{E}\left[\mathbf{1}_A Z_T\left(\hat{\mu}\right)\right];\ A\in\mathcal{F}_T.$$

Further for fixed $T\in[0,\infty)$, the process

$$\left\{\left(\tilde{B}_t^1,\tilde{B}_t^2\right):=\left(-B_t^1-\frac{\hat{\mu}}{\hat{\sigma}}t,-B_t^2-\frac{\hat{\mu}}{\hat{\sigma}}t\right),\,\mathcal{F}_t,\,0\leq t\leq T\right\}$$

is a two dimensional Brownian motion on $\left(\Omega,\mathcal{F}_T,\tilde{\mathbb{P}}\right)$.

Let us now consider,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_{\left\{\frac{1-x^i}{\hat{\sigma}}-\frac{\hat{\mu}}{\hat{\sigma}}t-B_t^1<\frac{\varepsilon}{\hat{\sigma}},\inf_{0<s\leq t}\frac{1-x^i}{\hat{\sigma}}-\frac{\hat{\mu}}{\hat{\sigma}}s-B_s^1>0,\,\frac{1-xj}{\hat{\sigma}}-\frac{\hat{\mu}}{\hat{\sigma}}t-B_t^2<\frac{\varepsilon}{\hat{\sigma}},\inf_{0<s\leq t}\frac{1-xj}{\hat{\sigma}}-\frac{\hat{\mu}}{\hat{\sigma}}s-B_s^2>0\right\}}\right] \\
&=\mathbb{E}^{\tilde{\mathbb{P}}}\left[\mathbf{1}_{\left\{\frac{1-x^i}{\hat{\sigma}}+\tilde{B}_t^1<\frac{\varepsilon}{\hat{\sigma}},\inf_{0<s\leq t}\frac{1-x^i}{\hat{\sigma}}+\tilde{B}_s^1>0,\,\frac{1-xj}{\hat{\sigma}}+\tilde{B}_t^2<\frac{\varepsilon}{\hat{\sigma}},\inf_{0<s\leq t}\frac{1-xj}{\hat{\sigma}}+\tilde{B}_s^2>0\right\}}\frac{1}{Z_T}\left(\hat{\mu}\right)\right] \\
&\leq\left\{\mathbb{E}^{\tilde{\mathbb{P}}}\left[\mathbf{1}_{\left\{\frac{1-x^i}{\hat{\sigma}}+\tilde{B}_t^1<\frac{\varepsilon}{\hat{\sigma}},\inf_{0<s\leq t}\frac{1-x^i}{\hat{\sigma}}+\tilde{B}_s^1>0,\,\frac{1-xj}{\hat{\sigma}}+\tilde{B}_t^2<\frac{\varepsilon}{\hat{\sigma}},\inf_{0<s\leq t}\frac{1-xj}{\hat{\sigma}}+\tilde{B}_s^2>0\right\}}\right]\right\}^{1/a}. \\
&\quad\left\{\mathbb{E}^{\tilde{\mathbb{P}}}\left[\frac{1}{Z_T}\left(\hat{\mu}\right)\right]^b\right\}^{1/b}, \qquad (5.31)
\end{aligned}$$

by Hölder's inequality. We now transform this to a system of two uncorrelated

Brownian motions, as in part a, and let $\tilde{B}_t = \left(\tilde{B}_t^1, \tilde{B}_t^2\right)$ and consider the process $\mathbf{Z} = \sigma^{-1}\tilde{B}$, where σ is the covariance matrix:

$$\sigma = \begin{bmatrix} \sqrt{1-\rho^4} & \rho^2 \\ 0 & 1 \end{bmatrix}$$

We consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as in part (a) as $T(\mathbf{x}) = \sigma^{-1}\mathbf{x}$. In this case, $Z(t)$ begins at the point z_0 whose polar coordinates are given by

$$\begin{aligned} r_0 &= \sqrt{\frac{\left(1-x^i\right)^2 + \left(1-x^j\right)^2 - 2\rho^2\left(1-x^i\right)\left(1-x^j\right)}{1-\rho^4}} \\ \theta_0 &= \begin{cases} \pi + \arctan\left(\frac{\left(1-x^j\right)\sqrt{1-\rho^4}}{\left(1-x^i\right)-\rho^2\left(1-x^j\right)}\right) & 1-x^i < \rho^2\left(1-x^j\right) \\ \frac{\pi}{2} & 1-x^i = \rho^2\left(1-x^j\right) \\ \arctan\left(\frac{\left(1-x^j\right)\sqrt{1-\rho^4}}{\left(1-x^i\right)-\rho^2\left(1-x^j\right)}\right) & 1-x^i > \rho^2\left(1-x^j\right) \end{cases} \end{aligned}$$

One can easily verify that $0 < \theta_0 < \alpha$. Denote $\tau = \min\left(\tau^1, \tau^2\right)$ as the first exit time of Z from the wedge

$$C_\alpha = \{(r\cos\theta, r\sin\theta) \ : r > 0,\ 0 < \theta < \alpha\} \subset \mathbb{R}^2.$$

The remaining proof follows in the same way as in part (a). $\square$

We see that for a small t, $M_t = \frac{1}{t^{1+\frac{\pi}{\alpha}}} e^{-\frac{r_0^2 - 3r_0\sqrt{\frac{2\left(1-\rho^2\right)}{1-\rho^4}}\frac{\varepsilon}{\delta}}{2t}}$ can blow near the default boundary. We now look at the uniform behaviour of the second moment of the limit empirical measure near the default and prepayment boundaries under the Assumption (A). This result extends the results in Jin [28] and Bush et al. in [8] for a tailed distribution that is not bounded away from zero.

Theorem 5.4.2. *Under assumption (A), if $\beta_1, \beta_2 > \frac{3\left(1+\frac{\pi}{\alpha}\right)}{\frac{\pi}{\alpha}-1}$ then there exists $\delta > 0$*

such that for any time T

$$\begin{aligned} a) \quad & \mathbb{E}\left[\left(\nu_t^+\left((0,\varepsilon)\right)\right)^2\right] \le M\varepsilon^{3+\delta}, \\ b) \quad & \mathbb{E}\left[\left(\nu_t^+\left((1-\varepsilon,1)\right)\right)^2\right] \le N\varepsilon^{3+\delta}, \end{aligned}$$

where α is defined in (5.29).

Proof. a) We have from theorem 5.4.1 (a), after the change of measure

$$\mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_{\left\{\frac{x^i}{\hat\sigma}+\frac{\hat\mu}{\hat\sigma}t+B_t^1<\frac{\varepsilon}{\hat\sigma},\, \inf_{0<s\le t}\frac{x^i}{\hat\sigma}+\frac{\hat\mu}{\hat\sigma}s+B_s^1>0,\, \frac{x^j}{\hat\sigma}+\frac{\hat\mu}{\hat\sigma}t+B_t^2<\frac{\varepsilon}{\hat\sigma},\, \inf_{0<s\le t}\frac{x^j}{\hat\sigma}+\frac{\hat\mu}{\hat\sigma}s+B_s^2>0\right\}}\right] \quad \le \quad J_1\cdot J_2,$$

where

$$\begin{aligned} J_1 &= \left\{\mathbb{E}^{\tilde{\mathbb{P}}}\left[\mathbf{1}_{\left\{\frac{x^i}{\hat\sigma}+\tilde B_t^1<\frac{\varepsilon}{\hat\sigma},\, \inf_{0<s\le t}\frac{x^i}{\hat\sigma}+\tilde B_s^1>0,\, \frac{x^j}{\hat\sigma}+\tilde B_t^2<\frac{\varepsilon}{\hat\sigma},\, \inf_{0<s\le t}\frac{x^j}{\hat\sigma}+\tilde B_s^2>0\right\}}\right]\right\}^{1/a} \end{aligned}$$

and,

$$J_2 = \left\{\mathbb{E}^{\tilde{\mathbb{P}}}\left[\frac{1}{Z_T}(\hat\mu)\right]^b\right\}^{1/b}.$$

We now consider J_1 and find an upper bound for it.

$$\begin{aligned} J_1 &= \left\{\tilde{\mathbb{P}}\left(\frac{x^i}{\hat\sigma}+\tilde B_t^1<\frac{\varepsilon}{\hat\sigma},\, \inf_{0<s\le t}\frac{x^i}{\hat\sigma}+\tilde B_s^1>0,\, \frac{x^j}{\hat\sigma}+\tilde B_t^2<\frac{\varepsilon}{\hat\sigma},\, \inf_{0<s\le t}\frac{x^j}{\hat\sigma}+\tilde B_s^2>0\right)\right\}^{1/a} \\ &= \left\{\tilde{\mathbb{P}}_B\left(\tilde B_t^1<\frac{\varepsilon}{\hat\sigma},\, \inf_{0<s\le t}\tilde B_s^1>0,\, \tilde B_t^2<\frac{\varepsilon}{\hat\sigma},\, \inf_{0<s\le t}\tilde B_s^2>0\right)\right\}^{1/a}, \end{aligned}$$

where $\tilde{\mathbb{P}}_B$ is the law of initial distribution with $\tilde B_0^1=\frac{x^i}{\hat\sigma}>0$ and $\tilde B_0^2=\frac{x^j}{\hat\sigma}>0$.

We decompose the probability near the boundary for $r_0<\varepsilon^\gamma$ and $r_0\ge\varepsilon^\gamma$, where r_0 is the starting point of the new independent Brownian motion after the change of measure and is defined in (5.30). Therefore we have

$$\begin{aligned} & \frac{1}{nm}\sum_{i=1}^n\sum_{j=1}^m \tilde{\mathbb{P}}_B\left\{\tilde B_t^1<\frac{\varepsilon}{\hat\sigma},\, \inf_{0<s\le t}\tilde B_s^1>0,\, \tilde B_t^2<\frac{\varepsilon}{\hat\sigma},\, \inf_{0<s\le t}\tilde B_s^2>0\right\} \\ = \; & \frac{1}{nm}\sum_{i=1}^n\sum_{j=1}^m \tilde{\mathbb{P}}_B\left\{\tilde B_t^1<\frac{\varepsilon}{\hat\sigma},\, \inf_{0<s\le t}\tilde B_s^1>0,\, \tilde B_t^2<\frac{\varepsilon}{\hat\sigma},\, \inf_{0<s\le t}\tilde B_s^2>0\right\}\mathbf{1}_{\{r_0\ge\varepsilon^\gamma\}} \end{aligned}$$

$$+\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{\mathbb{P}}_B\left\{\tilde{B}_t^1<\frac{\varepsilon}{\hat{\sigma}},\inf_{0<s\leq t}\tilde{B}_s^1>0,\,\tilde{B}_t^2<\frac{\varepsilon}{\hat{\sigma}},\inf_{0<s\leq t}\tilde{B}_s^2>0\right\}\mathbf{1}_{\{r_0<\varepsilon^\gamma\}}.$$

For the second part in above equation that is for $r_0<\varepsilon^\gamma$, we have

$$\begin{aligned}
&\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{\mathbb{P}}_B\left\{\tilde{B}_t^1<\frac{\varepsilon}{\hat{\sigma}},\inf_{0<s\leq t}\tilde{B}_s^1>0,\,\tilde{B}_t^2<\frac{\varepsilon}{\hat{\sigma}},\inf_{0<s\leq t}\tilde{B}_s^2>0\right\}\mathbf{1}_{\{r_0<\varepsilon^\gamma\}}\\
&\leq\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{1}_{\{r_0<\varepsilon^\gamma\}}\\
&\leq\hat{\mathbb{P}}_B\left(r_0<\varepsilon^\gamma\right)\\
&=\hat{\mathbb{P}}_B\left\{x^{i2}+x^{j2}-2\rho^2x^ix^j<\varepsilon^{2\gamma}\left(1-\rho^4\right)\right\}\\
&\leq\hat{\mathbb{P}}_B\left\{x^i<\varepsilon^\gamma,x^j<\varepsilon^\gamma\right\}\\
&=\hat{\mathbb{P}}_B\left(x^i<\varepsilon^\gamma\right)\hat{\mathbb{P}}_B\left(x^j<\varepsilon^\gamma\right),
\end{aligned}$$

where $\hat{\mathbb{P}}_B$ is the law of initial tailed distribution starting with $B_0^1=x^i$ and $B_0^1=x^j$. The last equality holds because we assumed x^i and x^j are i.i.d.

We have, by Assumption (A),

$$\hat{\mathbb{P}}_B\left(x^i<\varepsilon^\gamma\right)\hat{\mathbb{P}}_B\left(x^j<\varepsilon^\gamma\right)\leq\varepsilon^{2\gamma\beta^1}.$$

Therefore,

$$\begin{aligned}
&\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{\mathbb{P}}_B\left\{\tilde{B}_t^1<\frac{\varepsilon}{\hat{\sigma}},\inf_{0<s\leq t}\tilde{B}_s^1>0,\,\tilde{B}_t^2<\frac{\varepsilon}{\hat{\sigma}},\inf_{0<s\leq t}\tilde{B}_s^2>0\right\}\mathbf{1}_{\{r_0<\varepsilon^\gamma\}}\\
&\leq\varepsilon^{2\gamma\beta_1}.
\end{aligned}$$

For the first part of (5.32) we have from (5.28) in theorem 5.4.1, after the transformation $T:\mathbb{R}^2\to\mathbb{R}^2$ defined by $T(\mathbf{x})=\sigma^{-1}\mathbf{x}$

$$\begin{aligned}
&\tilde{\mathbb{P}}_B\left\{\tilde{B}_t^1<\frac{\varepsilon}{\hat{\sigma}},\inf_{0<s\leq t}\tilde{B}_s^1>0,\,\tilde{B}_t^2<\frac{\varepsilon}{\hat{\sigma}},\inf_{0<s\leq t}\tilde{B}_s^2>0\right\}\\
&\leq 2^{1-\frac{\pi}{\alpha}}r_0{}^{\frac{\pi}{\alpha}}\int_0^{\sqrt{\frac{2(1-\rho^2)}{1-\rho^4}}\frac{\varepsilon}{\hat{\sigma}}}r^{1+\frac{\pi}{\alpha}}\frac{1}{t^{1+\frac{\pi}{\alpha}}}e^{-\frac{(r-r_0)^2-3rr_0}{2t}}dr.
\end{aligned}$$

The first part in (5.32) becomes

$$\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{\mathbb{P}}_B\left\{\tilde{B}_t^1<\frac{\varepsilon}{\hat{\sigma}},\ \inf_{0<s\leq t}\tilde{B}_s^1>0,\ \tilde{B}_t^2<\frac{\varepsilon}{\hat{\sigma}},\ \inf_{0<s\leq t}\tilde{B}_s^2>0\right\}\mathbf{1}_{\{r_0\geq\varepsilon^\gamma\}}$$
$$\leq\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}2^{1-\frac{\pi}{\alpha}}r_0{}^{\frac{\pi}{\alpha}}\int_0^{\sqrt{\frac{2(1-\rho^2)}{1-\rho^4}}\frac{\varepsilon}{\hat{\sigma}}}r^{1+\frac{\pi}{\alpha}}\frac{1}{t^{1+\frac{\pi}{\alpha}}}e^{-\frac{(r-r_0)^2-rr_0}{2t}}dr\mathbf{1}_{\{r_0\geq\varepsilon^\gamma\}} \tag{5.32}$$

We now try to find the minimum of $r^2+r_0^2-3rr_0$ for $r\in\left[0,\sqrt{\frac{2(1-\rho^2)}{1-\rho^4}}\frac{\varepsilon}{\hat{\sigma}}\right]$ and $r_0\geq\varepsilon^\gamma$. We obtain

$$r^2+r_0^2-3rr_0\geq r_0^2-3rr_0>\varepsilon^{2\gamma}\left(1-3\sqrt{\frac{2\left(1-\rho^2\right)}{1-\rho^4}}\frac{1}{\hat{\sigma}}\varepsilon^{1-\gamma}\right)\geq C'\varepsilon^{2\gamma},$$

for ε sufficiently small and where $C'=1-3\sqrt{\frac{2(1-\rho^2)}{1-\rho^4}}\frac{1}{\hat{\sigma}}\varepsilon^{1-\gamma}$ where we assumed $\gamma<1$. We have

$$\tilde{\mathbb{P}}_B\left\{\tilde{B}_t^1<\frac{\varepsilon}{\hat{\sigma}},\ \inf_{0<s\leq t}\tilde{B}_s^1>0,\ \tilde{B}_t^2<\frac{\varepsilon}{\hat{\sigma}},\ \inf_{0<s\leq t}\tilde{B}_s^2>0\right\}$$
$$\leq\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}2^{1-\frac{\pi}{\alpha}}r_0{}^{\frac{\pi}{\alpha}}\frac{1}{\left(2+\frac{\pi}{\alpha}\right)}\left(\sqrt{\frac{2\left(1-\rho^2\right)}{1-\rho^4}}\frac{\varepsilon}{\hat{\sigma}}\right)^{2+\frac{\pi}{\alpha}}\frac{1}{t^{1+\frac{\pi}{\alpha}}}e^{\frac{C'\varepsilon^{2\gamma}}{t}}.$$

Choosing $t=C'\varepsilon^{2\gamma}$, we obtain

$$\leq\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}2^{1-\frac{\pi}{\alpha}}r_0{}^{\frac{\pi}{\alpha}}\frac{e}{\left(2+\frac{\pi}{\alpha}\right)}\left(\sqrt{\frac{2\left(1-\rho^2\right)}{1-\rho^4}}\frac{\varepsilon}{\hat{\sigma}}\right)^{2+\frac{\pi}{\alpha}}\frac{1}{\left(C'\varepsilon^{2\gamma}\right)^{\left(1+\frac{\pi}{\alpha}\right)}}$$
$$=\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}2^{1-\frac{\pi}{\alpha}}r_0{}^{\frac{\pi}{\alpha}}\frac{e}{\left(2+\frac{\pi}{\alpha}\right)}\frac{1}{C'^{\left(1+\frac{\pi}{\alpha}\right)}}\left(\sqrt{\frac{2\left(1-\rho^2\right)}{1-\rho^4}}\frac{1}{\hat{\sigma}}\right)^{2+\frac{\pi}{\alpha}}\frac{\varepsilon^{2+\frac{\pi}{\alpha}}}{\varepsilon^{2\gamma\left(1+\frac{\pi}{\alpha}\right)}}$$
$$\leq C\varepsilon^{2+\frac{\pi}{\alpha}-2\gamma\left(1+\frac{\pi}{\alpha}\right)},$$

where $C = 2^{1-\frac{\pi}{\alpha}} \frac{e}{\left(2+\frac{\pi}{\alpha}\right)} \frac{1}{C'^{\left(1+\frac{\pi}{\alpha}\right)}} \left(\sqrt{\frac{2(1-\rho^2)}{1-\rho^4}} \frac{1}{\hat{\sigma}}\right)^{2+\frac{\pi}{\alpha}}$ and $r_0 \leq 1$.

Combining these two cases, the probability in (5.32) becomes

$$\mathbb{E}^{\tilde{P}_B}\left[\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{1}_{\left\{\tilde{B}_t^1<\frac{\varepsilon}{\hat{\sigma}},\, \inf_{0<s\leq t}\tilde{B}_s^1>0,\, \tilde{B}_t^2<\frac{\varepsilon}{\hat{\sigma}},\, \inf_{0<s\leq t}\tilde{B}_s^2>0\right\}}\right]$$
$$\leq \varepsilon^{2\gamma\beta^1} + C\varepsilon^{2+\frac{\pi}{\alpha}-2\gamma\left(1+\frac{\pi}{\alpha}\right)}$$

Now choosing $\hat{\gamma} = 2\gamma = \frac{2+\frac{\pi}{\alpha}}{\beta^1+\left(1+\frac{\pi}{\alpha}\right)}$, we obtain

$$\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{\mathbb{P}}_B\left\{\tilde{B}_t^1 < \frac{\varepsilon}{\hat{\sigma}},\ \inf_{0<s\leq t}\tilde{B}_s^1 > 0,\ \tilde{B}_t^2 < \frac{\varepsilon}{\hat{\sigma}},\ \inf_{0<s\leq t}\tilde{B}_s^2 > 0\right\}$$
$$\leq (1+C)\varepsilon^{\hat{\gamma}\beta^1},$$

or,

$$\leq (1+C)\varepsilon^{\frac{(2+\frac{\pi}{\alpha})\beta^1}{\beta^1+(1+\frac{\pi}{\alpha})}}.$$

We want to show that the power of ε is strictly greater than 3. So we let $\frac{\left(2+\frac{\pi}{\alpha}\right)\beta^1}{\beta^1+\left(1+\frac{\pi}{\alpha}\right)} = 3+\delta$ for some positive δ. Which gives $\beta^1 > \frac{3\left(1+\frac{\pi}{\alpha}\right)}{\frac{\pi}{\alpha}-1} > 6$. So we are able to obtain $\delta > 0$ for $\beta_1 > 9$ such that

$$\mathbb{E}^{\tilde{P}_B}\left[\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{1}_{\left\{\tilde{B}_t^1<\frac{\varepsilon}{\hat{\sigma}},\, \inf_{0<s\leq t}\tilde{B}_s^1>0,\, \tilde{B}_t^2<\frac{\varepsilon}{\hat{\sigma}},\, \inf_{0<s\leq t}\tilde{B}_s^2>0\right\}}\right] \leq (1+C)\varepsilon^{3+\delta}.$$

We therefore have,

$$\begin{aligned}\mathbb{E}^{\tilde{P}_B}\left[\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}J_1\right] &\leq \left\{\mathbb{E}^{x}\left[\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{1}_{\left\{\tilde{B}_t^1<\frac{\varepsilon}{\hat{\sigma}},\, \inf_{0<s\leq t}\tilde{B}_s^1>0,\, \tilde{B}_t^2<\frac{\varepsilon}{\hat{\sigma}},\, \inf_{0<s\leq t}\tilde{B}_s^2>0\right\}}\right]\right\}^{1/a} \\ &\leq A\varepsilon^{(3+\delta)/a},\end{aligned}$$

where $A = (1+C)^{1/a}$.

Using the bound on J_2 from (5.31) We have,

$$\begin{aligned}\mathbb{E}\left[\left(\nu_t^+((0,\varepsilon))\right)^2\right] &= \lim_{n\to\infty}\lim_{m\to\infty}\frac{1}{nm}\sum_{i=1}^{n}\sum_{j\neq i,j=1}^{m} J_1\cdot J_2\\ &\leq J_T A\varepsilon^{(3+\delta)/a}\\ &\leq J_T A\varepsilon^{(3+\delta)},\end{aligned}$$

where the last inequality holds by letting $a \downarrow 1$. So, we have

$$\mathbb{E}\left[\left(\nu_t^+((0,\varepsilon))\right)^2\right] \leq M\varepsilon^{(3+\delta)},$$

where $M = 1 + 2^{1-\frac{\pi}{\alpha}}\frac{e}{\left(2+\frac{\pi}{\alpha}\right)}\frac{1}{C'^{(1+\frac{\pi}{\alpha})}}\left(\sqrt{\frac{2(1-\rho^2)}{1-\rho^4}}\frac{1}{\hat{\sigma}}\right)^{2+\frac{\pi}{\alpha}}$.

b). We now prove the second part of the theorem. We proceed in the same way as in theorem 5.4.1 (b). We have by (5.31) after the change of measure

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_{\left\{\frac{1-x^i}{\hat{\sigma}}-\frac{\hat{\mu}}{\hat{\sigma}}t-B_t^1<\frac{\varepsilon}{\hat{\sigma}},\,\inf_{0<s\leq t}\frac{1-x^i}{\hat{\sigma}}-\frac{\hat{\mu}}{\hat{\sigma}}s-B_s^1>0,\,\frac{1-x^j}{\hat{\sigma}}-\frac{\hat{\mu}}{\hat{\sigma}}t-B_t^2<\frac{\varepsilon}{\hat{\sigma}},\,\inf_{0<s\leq t}\frac{1-x^j}{\hat{\sigma}}-\frac{\hat{\mu}}{\hat{\sigma}}s-B_s^2>0\right\}}\right]\\ &\leq J_1\cdot J_2,\end{aligned}$$

where

$$J_1 = \left\{\mathbb{E}^{\tilde{\mathbb{P}}}\left[\mathbf{1}_{\left\{\frac{1-x^i}{\hat{\sigma}}+\tilde{B}_t^1<\frac{\varepsilon}{\hat{\sigma}},\,\inf_{0<s\leq t}\frac{1-x^i}{\hat{\sigma}}+\tilde{B}_s^1>0,\,\frac{1-x^j}{\hat{\sigma}}+\tilde{B}_t^2<\frac{\varepsilon}{\hat{\sigma}},\,\inf_{0<s\leq t}\frac{1-x^j}{\hat{\sigma}}+\tilde{B}_s^2>0\right\}}\right]\right\}^{1/a}$$

and,

$$J_2 = \left\{\mathbb{E}^{\tilde{\mathbb{P}}}\left[\frac{1}{Z_T}(\hat{\mu})\right]^b\right\}^{1/b}.$$

We now consider J_1 and find an upper bound for it.

$$\begin{aligned}J_1 &= \left\{\mathbb{E}^{\tilde{\mathbb{P}}}\left[\mathbf{1}_{\left\{\frac{1-x^i}{\hat{\sigma}}+\tilde{B}_t^1<\frac{\varepsilon}{\hat{\sigma}},\,\inf_{0<s\leq t}\frac{1-x^i}{\hat{\sigma}}+\tilde{B}_s^1>0,\,\frac{1-x^j}{\hat{\sigma}}+\tilde{B}_t^2<\frac{\varepsilon}{\hat{\sigma}},\,\inf_{0<s\leq t}\frac{1-x^j}{\hat{\sigma}}+\tilde{B}_s^2>0\right\}}\right]\right\}^{1/a}\\ &= \left\{\tilde{\mathbb{P}}_B\left(\tilde{B}_t^1<\frac{\varepsilon}{\hat{\sigma}},\,\inf_{0<s\leq t}\tilde{B}_s^1>0,\,\tilde{B}_t^2<\frac{\varepsilon}{\hat{\sigma}},\,\inf_{0<s\leq t}\tilde{B}_s^2>0\right)\right\}^{1/a},\end{aligned}$$

where $\tilde{\mathbb{P}}_B$ is the law of initial distribution with $\tilde{B}_0^1 = \frac{1-x^i}{\hat{\sigma}} > 0$ and $\tilde{B}_0^2 = \frac{1-x^j}{\hat{\sigma}} > 0$.

Consider

$$
\begin{aligned}
& \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{\mathbb{P}}_B\left\{\tilde{B}_t^1<\frac{\varepsilon}{\hat{\sigma}},\ \inf_{0<s\leq t}\tilde{B}_s^1>0,\ \tilde{B}_t^2<\frac{\varepsilon}{\hat{\sigma}},\ \inf_{0<s\leq t}\tilde{B}_s^2>0\right\} \\
= \ & \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{\mathbb{P}}_B\left\{\tilde{B}_t^1<\frac{\varepsilon}{\hat{\sigma}},\ \inf_{0<s\leq t}\tilde{B}_s^1>0,\ \tilde{B}_t^2<\frac{\varepsilon}{\hat{\sigma}},\ \inf_{0<s\leq t}\tilde{B}_s^2>0\right\}\mathbf{1}_{\{r_0\geq\varepsilon^\gamma\}} \\
& \quad +\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{\mathbb{P}}_B\left\{\tilde{B}_t^1<\frac{\varepsilon}{\hat{\sigma}},\ \inf_{0<s\leq t}\tilde{B}_s^1>0,\ \tilde{B}_t^2<\frac{\varepsilon}{\hat{\sigma}},\ \inf_{0<s\leq t}\tilde{B}_s^2>0\right\}\mathbf{1}_{\{r_0<\varepsilon^\gamma\}}.
\end{aligned}
$$

For the second part above, we have

$$
\begin{aligned}
& \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{\mathbb{P}}_B\left\{\tilde{B}_t^1<\frac{\varepsilon}{\hat{\sigma}},\ \inf_{0<s\leq t}\tilde{B}_s^1>0,\ \tilde{B}_t^2<\frac{\varepsilon}{\hat{\sigma}},\ \inf_{0<s\leq t}\tilde{B}_s^2>0\right\}\mathbf{1}_{\{r_0<\varepsilon^\gamma\}} \\
& \leq \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{1}_{\{r_0<\varepsilon^\gamma\}} \\
& \leq \hat{\mathbb{P}}_B\left(r_0<\varepsilon^\gamma\right) \\
& = \hat{\mathbb{P}}_B\left\{\left(1-x^i\right)^2+\left(1-x^j\right)^2-2\rho^2\left(1-x^i\right)\left(1-x^j\right)<\varepsilon^{2\gamma}\left(1-\rho^4\right)\right\} \\
& \leq \hat{\mathbb{P}}_B\left\{\left(1-x^i\right)<\varepsilon^\gamma,\left(1-x^j\right)<\varepsilon^\gamma\right\} \\
& = \hat{\mathbb{P}}_B\left(x^i>1-\varepsilon^\gamma\right)\hat{\mathbb{P}}_B\left(x^j>1-\varepsilon^\gamma\right),
\end{aligned}
$$

where $\hat{\mathbb{P}}_B$ is the law of initial tailed distribution with $B_0^1=1-x^i$ and $B_0^1=1-x^j$. The last equality holds because we assumed x^i and x^j are i.i.d.

We have, by assumption (A),

$$
\mathbb{P}_B\left(x^i>1-\varepsilon^\gamma\right)\mathbb{P}_B\left(x^j>1-\varepsilon^\gamma\right)\leq\varepsilon^{2\gamma\beta^2}.
$$

Therefore,

$$
\begin{aligned}
& \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{\mathbb{P}}_B\left\{\tilde{B}_t^1<\frac{\varepsilon}{\hat{\sigma}},\ \inf_{0<s\leq t}\tilde{B}_s^1>0,\ \tilde{B}_t^2<\frac{\varepsilon}{\hat{\sigma}},\ \inf_{0<s\leq t}\tilde{B}_s^2>0\right\}\mathbf{1}_{\{r_0<\varepsilon^\gamma\}} \\
& \leq \varepsilon^{2\gamma\beta_2}.
\end{aligned}
$$

The remaining proof is the same as for the first part. $\square$

In this chapter we have showed that if the density of the limit empirical measure exists in L^2 with respect to Lebesgue measure then it will satisfy the Dirichlet boundary conditions. In next chapter we show that the density of the limit empirical measure exists in L^2 with respect to Lebesgue measure provided that ν_0^+ has a density in L^2 with respect to Lebesgue measure. We use the estimates on the second moment found in this chapter to prove the existence of the density in the next chapter.

Chapter 6

The existence and uniqueness of solution to SPDE

In Chapter 5, we showed that if the limit empirical measure has a density then it satisfies an SPDE with Dirichlet boundary conditions. In this chapter we prove the existence of the density of the limit empirical measure and uniqueness of the solution to the SPDE describing the behaviour of the density. From the previous Chapter we have that ν_t^+ satisfies

$$\begin{aligned} \langle \phi, \nu_t^+ \rangle &= \langle \phi, \nu_0^+ \rangle + \int_0^t \langle \hat{\mu} \phi' (x) + \frac{\hat{\sigma}^2}{2} \phi'' (x) - \lambda (r_s) \phi (x), \nu_s^+ \rangle ds \\ &\quad + \int_0^t \langle \hat{\sigma} \rho \phi' (x), \nu_s^+ \rangle dM_s. \end{aligned} \tag{6.1}$$

We want to show that for each t, ν_t^+ is absolutely continuous with respect to Lebesgue measure and has a density in L^2 so that we can write $\nu_t^+ dx = v(t,x)\, dx$ for some density v to re-characterize the evolution obtained as the stochastic PDE. We obtain the necessary estimates by first smoothing the solutions with an absorbing heat kernel. The estimates on the second moment, as found in the previous chapter, give the existence of a density $v(t,x)$ for the solution to SPDE in (6.1) under the assumption that ν_0 has a density in $L^2([0,1])$.

We define some notation and let $H_0 = L^2([0,1])$ be the usual Hilbert space with

L^2-norm $\|.\|_0$ and inner product $\langle ., . \rangle_0$ given by

$$\begin{aligned} \|\phi\|_0^2 &= \int_0^1 |\phi(x)|^2 dx \\ \langle \phi, \psi \rangle_0 &= \int_0^1 \phi(x) \psi(x) dx, \end{aligned} \tag{6.2}$$

for any ϕ and ψ in H_0. We now define what we mean by a weak solution

Definition 6 (Weak Solution)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open, $f \in L^2(\Omega)$. A function $u : \Omega \to \mathbb{R}^n$ is called a weak solution of*

$$\begin{cases} -\Delta u = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

if

$$\begin{aligned} &i)\ u \in H_0(\Omega) \\ &ii)\ \int_\Omega Du \cdot D\phi dx = \langle f, \phi \rangle_0 \qquad \forall \phi \in H_0(\Omega). \end{aligned}$$

The idea to prove the existence of an $L^2([0,1])$ density follows from Kotelenez [35] and Kurtz and Xiong [37] and lies in transforming an $\mathcal{M}([0,1])$-valued process to an H_0-valued process, by convolving the measure with the absorbing heat kernel. $\mathcal{M}([0,1])$ here denotes the set of finite Borel measure on $[0,1]$. The same idea was adapted by Jin [28] and Bush et al. [8] for their settings.

6.1 Transformation from $\mathcal{M}([0,1])$ to H_0

We formally describe the transformation from an $\mathcal{M}([0,1])$-valued process to an H_0 valued process and for any measure ρ in $\mathcal{M}([0,1])$ and $\delta > 0$, we write $\{T_\delta : \delta > 0\}$ for the Brownian semigroup

$$(T_\delta \rho)(x) = \int_0^1 G_\delta(x, y) \rho(dy), \tag{6.3}$$

where $G_\delta(x,y)$ is the absorbing heat kernel on interval $[0,1]$ and is given on page 122 of Borodin and Salminen [6] by

$$G_\delta(x,y) = \frac{1}{2\sqrt{2\pi\delta}} \sum_{n=-\infty}^{\infty} \left(e^{-\frac{((x-y)+2n)^2}{2\delta}} - e^{\frac{-(x+y+2n)^2}{2\delta}} \right), \quad \forall\, 0 < x, y < 1.$$

We use the following notation for absorbed Brownian semigroup on $C_b([0,1])$ where $C_b([0,1])$ is a collection of all bounded and continuous functions on interval $[0,1]$, we have

$$T_t\phi(x) = \int_0^1 G_t(x,y)\,\phi(y)\,dy, \qquad \forall \phi \in C_b([0,1]).$$

We first give the following results.

Lemma 6.1.1. *If $\rho \in \mathcal{M}([0,1])$ and $\delta > 0$, then $T_\delta\rho \in H_0 = L^2([0,1])$.*

Proof. To prove this, we need to calculate the L^2 norm of $T_\delta\rho$ using standard method and the property of Brownian semigroup. We have

$$\begin{aligned}
\|T_\delta\rho\|_0^2 &= \int_0^1 (|T_\delta\rho(x)|)^2\,dx \\
&= \int_0^1 \left[\int_0^1 G_\delta(x,y)\,\rho(dy) \int_0^1 G_\delta(x,z)\,\rho(dz)\right] dx \\
&= \int_0^1 \rho(dy) \int_0^1 \rho(dz) \int_0^1 G_\delta(x,y)\,G_\delta(x,z)\,dx \\
&= \int_0^1 \rho(dy) \int_0^1 \rho(dz)\,G_{2\delta}(y,z) \\
&= \int_0^1\int_0^1 G_{2\delta}(y,z)\,\rho(dy)\,\rho(dz) \\
&= \int_0^1\int_0^1 \frac{1}{4\sqrt{\pi\delta}} \sum_{n=-\infty}^{\infty} \left(e^{\frac{-((y-z)+2n)^2}{4\delta}} - e^{\frac{-(y+z+2n)^2}{4\delta}} \right) \rho(dy)\,\rho(dz) \\
&= \int_0^1\int_0^1 \frac{1}{4\sqrt{\pi\delta}} \sum_{n=-\infty}^{\infty} e^{-n^2/\delta} \left(e^{-\frac{(y-z)^2+4n(y-z)}{4\delta}} - e^{-\frac{(y+z)^2+2n(y+z)}{4\delta}} \right) \\
&\qquad \rho(dy)\,\rho(dz)
\end{aligned}$$

The integrand in above equation is a sum of exponential decays and is bounded, say by K, therefore we have

$$\|T_\delta \rho\|_0^2 \leq \frac{1}{4\sqrt{\pi\delta}} K |\rho([0,1])|^2 < \infty.$$

$\square$

Lemma 6.1.2. *T_δ is a mapping from $\bar{C}$ to $\bar{C}$ for any $\delta > 0$, that is for any $\phi \in \bar{C}$, $T_\delta \phi \in \bar{C}$.*

Proof. By definition,

$$\begin{aligned} T_\delta \phi(x) &= \int_0^1 G_\delta(x,y)\,\phi(y)\,dy \\ &= \int_0^1 \frac{1}{2\sqrt{2\pi\delta}} \sum_{n=-\infty}^{\infty} \left(e^{-\frac{(x-y+2n)^2}{2\delta}} - e^{-\frac{(x+y+2n)^2}{2\delta}} \right) \phi(y)\,dy. \end{aligned}$$

It is easy to show that $T_\delta \phi(0) = T_\delta \phi(1) = 0$. We now show that $T_\delta \phi(x) \in C^\infty([0,1])$.

We have $\phi \in \bar{C}$ which mean ϕ is bounded by K_ϕ. We use Continuity Lemma (16.1) in Heinz Bauer [4] to show the continuity of $T_\delta \phi(x)$. We see that $x \to G_\delta(x,y)\,\phi(y)$ is continuous for each $y \in [0,1]$. Furthermore $y \to G_\delta(x,y)$ is bounded, say by D, for fixed $x \in [0,1]$ and $\delta > 0$ that is

$$|G_\delta(x,y)\,\phi(y)| \leq K_\phi D, \qquad \text{for y} \in [0,1].$$

Therefore by Continuity Lemma, $T_\delta \phi(x)$ is continuous.

To show that the derivative of $T_\delta \phi(x)$ exists, we use Differentiability Lemma (16.2) in [4]. We first show that $x \to G_\delta(x,y)$ is differentiable for every fixed $y \in [0,1]$. We have for the first exponential term in $G_\delta(x,y)$

$$\frac{d}{dx} \sum_{n=-\infty}^{\infty} e^{-\frac{(x-y+2n)^2}{2\delta}} = \frac{d}{dz} \sum_{n=-\infty}^{\infty} e^{-\frac{z^2}{2\delta}} |_{z=x-y+2n}$$

$$= \sum_{n=-\infty}^{\infty} -\frac{z}{\delta} e^{\frac{-z^2}{2\delta}} |_{z=x-y+2n}$$

We can also see that for a fixed δ, the sum on the right hand side is bounded. One can check that the derivative for the second term in $G_\delta(x,y)$ exits and is bounded. Therefore we can apply Differentiability Lemma to show that

$$\frac{d}{dx} T_\delta \phi(x) = \int_0^1 \frac{1}{2\sqrt{2\pi\delta}} \sum_{n=-\infty}^{\infty} \left(\frac{d}{dz} e^{-z^2/2\delta} |_{z=x-y+2n} - \frac{d}{dz} e^{-z/2\delta} |_{z=x+y+2n} \right) \phi(y)\, dy$$

The same argument can be extended to

$$\frac{d^n}{dx^n} T_\delta \phi(x) = \int_0^1 \frac{1}{2\sqrt{2\pi\delta}} \sum_{n=-\infty}^{\infty} \left(\frac{d^n}{dz^n} e^{-z^2/2\delta} |_{z=x-y+2n} - \frac{d^n}{dz^n} e^{-z/2\delta} |_{z=x+y+2n} \right) \phi(y)\, dy$$

as

$$\frac{d^n}{dz^n} e^{-z^2/2\delta} = P_n(z)\, e^{-z^2/2\delta},$$

where $P_n(z)$ is a polynomial of degree n.

Lastly, we need to show that $T_\delta \phi(x)' \to 0$ as $x \to 0$, if $\lim_{x\to 0} \phi(x) = 0$. We have

$$\begin{aligned}
\lim_{x\to 0} \frac{d}{dx} T_\delta \phi(x) &= \lim_{x\to 0} \int_0^1 \frac{1}{2\sqrt{2\pi\delta}} \sum_{n=-\infty}^{\infty} \left(\frac{d}{dz} e^{-z^2/2\delta} |_{z=x-y+2n} - \frac{d}{dz} e^{-z/2\delta} |_{z=x+y+2n} \right) \\
&\quad \phi(y)\, dy \\
&= \lim_{x\to 0} \int_0^1 \frac{1}{2\sqrt{2\pi\delta}} \sum_{n=-\infty}^{\infty} \left(\frac{-z}{\delta} e^{-z^2/2\delta} \phi(x-z+2n)\, \mathbf{1}_{\{z\le x+2n\}} \right. \\
&\quad \left. + \frac{z}{\delta} e^{-z/2\delta} \phi(z-x-2n)\, \mathbf{1}_{\{z\le x+2n\}} \right) dy \\
&= \int_0^1 \frac{1}{2\sqrt{2\pi\delta}} \sum_{n=-\infty}^{\infty} \lim_{x\to 0} \left(\frac{-z}{\delta} e^{-z^2/2\delta} \phi(x-z+2n)\, \mathbf{1}_{\{z\le x+2n\}} \right. \\
&\quad \left. + \lim_{x\to 0} \frac{z}{\delta} e^{-z/2\delta} \phi(z-x-2n)\, \mathbf{1}_{\{z\le x+2n\}} \right) dy \\
&= 0,
\end{aligned}$$

where the third equality is due to dominance convergence theorem. Similarly we can show that $T_\delta\phi\left(x\right)^{'} \to 0$ as $x \to 1$, if $\lim_{x\to 1}\phi\left(x\right) = 0$. This shows that $T_\delta\phi\left(x\right) \in \bar{C}$. $\square$

6.2 Convolution of ν_t^+

The method adopted here is similar to Jin [28] and Bush et al in [8]. We adopt the same strategy in this section and consider the convolution of ν_t^+ with absorbing heat kernel as defined in (6.3) where ν_t^+ is a $\mathcal{M}\left([0,1]\right)$-valued solution of the evolution equation with total mass $\int_0^1 \nu_s^+(dx) < 1$ and define $Z_\delta\left(t\right) = T_\delta\nu_t^+$. We have

$$\begin{aligned}
\langle Z_\delta\left(t\right), \phi\rangle_0 &= \langle T_\delta\nu_t^+, \phi\rangle_0 \\
&= \int_0^1 T_\delta\nu_t^+\left(x\right)\phi\left(x\right)dx \\
&= \int_0^1\left[\int_0^1 G_\delta\left(x,y\right)\nu_t^+\left(dy\right)\right]\phi\left(x\right)dx \\
&= \int_0^1\left[\int_0^1 G_\delta\left(x,y\right)\phi\left(x\right)dx\right]\nu_t^+\left(dy\right) \\
&= \int_0^1 T_\delta\phi\left(y\right)\nu_t^+\left(dy\right) = \langle T_\delta\phi, \nu_t^+\rangle.
\end{aligned}$$

Now replacing $\phi \in \bar{C}$ in evolution equation (6.1) with $T_\delta\phi \in \bar{C}$, we have

$$\begin{aligned}
\langle Z_\delta\left(t\right), \phi\rangle_0 &= \langle T_\delta\phi, \nu_t^+\rangle \\
&= \langle T_\delta\phi, \nu_0^+\rangle + \int_0^t \langle\hat{\mu}\left(T_\delta\phi\right)^{'}\left(x\right) \\
&\quad + \frac{\hat{\sigma}^2}{2}\left(T_\delta\phi\right)^{''}\left(x\right) - \lambda\left(r_s\right)T_\delta\phi\left(x\right), \nu_s^+\rangle ds \\
&\quad + \int_0^t \langle\hat{\sigma}\rho\left(T_\delta\phi\right)^{'}\left(x\right), \nu_s^+\rangle dM_s. \qquad (6.4)
\end{aligned}$$

We also have $\langle T_\delta\left(\phi\right), \nu_0^+\rangle = \langle Z_\delta\left(0\right), \phi\rangle_0$.

We first show that $\langle T_\delta\phi, \nu_t^+\rangle$ is bounded and

$$\langle T_\delta\phi, \nu_t^+\rangle = \int_0^1\left[\int_0^1 \left|G_\delta\left(x,y\right)\phi\left(x\right)\right|dx\right]\nu_t^+\left(dy\right)$$

$$
\begin{aligned}
&\leq \int_0^1 \left[\int_0^1 G_\delta(x,y) \left| \phi(x) \right| dx \right] \nu_t^+(dy) \\
&\leq \int_0^1 \left[\int_0^1 G_\delta(x,y)\, dx K_\phi \right] \nu_t^+(dy) \\
&= K_\phi \int_0^1 \left[\int_0^1 G_\delta(x,y)\, dx \right] \nu_t^+(dy) \\
&\leq C_\phi \nu_t^+([0,1]) < +\infty.
\end{aligned}
$$

This implies that we can apply Fubini's theorem on the RHS of (6.4).

We define the reflected kernel $G_\delta^r(x,y)$ in $[0,1]$ by

$$
G_\delta^r(x,y) = \frac{1}{2\sqrt{2\pi\delta}} \sum_{n=-\infty}^{\infty} \left(e^{-\frac{(x-y+2n)^2}{2\delta}} + e^{\frac{-(x+y+2n)^2}{2\delta}} \right), \qquad \forall 0 < x, y < 1,
$$

and denote $T_\delta^r\left(\nu_t^+(x)\right) = \int_0^1 G_\delta^r(x,y)\, \nu_t^+(dy)$. One can easily verify that $\partial_x G_\delta(x,y) = -\partial_y G_\delta^r(x,y)$.

We simplify (6.4) for the integrands on the right hand side of the equation separately for each term to find an expression in terms of the inner product $\langle .,. \rangle_0$ defined in L^2. Starting with the second term and using the representation in (5.2) and convolution in (6.3), we have

$$
\begin{aligned}
&\langle \hat{\mu} (T_\delta\phi)'(x) + \frac{\hat{\sigma}^2}{2} (T_\delta\phi)''(x) - \lambda(r_s)\, T_\delta\phi(x), \nu_s^+ \rangle \\
&= \int_0^1 \left(\hat{\mu} (T_\delta\phi)'(x) + \frac{\hat{\sigma}^2}{2} (T_\delta\phi)''(x) - \lambda(r_s)\, T_\delta\phi(x) \right) \nu_s^+(dx) \\
&= \int_0^1 \left\{ \hat{\mu} \left(\partial_x \int_0^1 G_\delta(x,y)\, \phi(y)\, dy \right) + \frac{\hat{\sigma}^2}{2} \left(\partial_{xx} \int_0^1 G_\delta(x,y)\, \phi(y)\, dy \right) \right. \\
&\quad \left. -\lambda(r_s) \int_0^1 G_\delta(x,y)\, \phi(y)\, dy \right\} \nu_s^+(dx). \qquad (6.5)
\end{aligned}
$$

We now consider each term on the right hand side of (6.5) separately and consider,

$$
\int_0^1 \left[\hat{\mu} \left(\partial_x \int_0^1 G_\delta(x,y)\, \phi(y)\, dy \right) \right] \nu_s^+(dx)
$$

$$
\begin{aligned}
&= \hat{\mu} \int_0^1 \int_0^1 \partial_x \left(G_\delta (x, y) \right) \phi (y) \, dy \nu_s^+ (dx) \\
&= \hat{\mu} \int_0^1 \left\{ \int_0^1 -\partial_y \left(G_\delta^r (x, y) \right) \phi (y) \, dy \right\} \nu_s^+ (dx) \\
&= \hat{\mu} \int_0^1 \left\{ G_\delta^r (x, y) \phi (y) |_0^1 + \int_0^1 \phi^{'} (y) \left(G_\delta^r (x, y) \right) dy \right\} \nu_s^+ (dx) ,
\end{aligned}
$$

where the first equality holds as we showed in the proof of lemma (6.1.2) that $\partial_x G_\delta (x, y)$ exists and is integrable and we can swap the integral and derivative. We also used $\partial_x G_\delta (x, y) = -\partial_y G_\delta^r (x, y)$ and integration by parts. We have $\phi (0) = \phi (1) = 0$, therefore

$$
\begin{aligned}
&\int_0^1 \left[\hat{\mu} \left(\partial_x \int_0^1 G_\delta (x, y) \phi (y) \, dy \right) \right] \nu_s^+ (dx) \\
&\quad = \hat{\mu} \int_0^1 \int_0^1 \phi^{'} (y) \, G_\delta^r (x, y) \, dy \nu_s^+ (dx) \\
&\quad = \hat{\mu} \int_0^1 \left[\int_0^1 G_\delta^r (x, y) \, \nu_s^+ (dx) \right] \phi^{'} (y) \, dy \\
&\quad = \hat{\mu} \int_0^1 T_\delta^r \left(\nu_s^+ \right) (y) \, \phi^{'} (y) \, dy \\
&\quad = \hat{\mu} \left\{ T_\delta^r \left(\nu_s^+ \right) (y) \, \phi (y) |_0^1 - \int_0^1 \partial_y T_\delta^r \left(\nu_s^+ \right) (y) \, \phi (y) \, dy \right\} \\
&\quad = -\hat{\mu} \int_0^1 \partial_y T_\delta^r \left(\nu_s^+ \right) (y) \, \phi (y) \, dy \\
&\quad = -\hat{\mu} \langle \phi, \partial_x \left(T_\delta^r \left(\nu_s^+ \right) \right) \rangle_0 . \qquad (6.6)
\end{aligned}
$$

Consider the second term in (6.5) that is

$$
\begin{aligned}
\langle \frac{\hat{\sigma}^2}{2} \left(T_\delta \phi \right)^{''} (x) , \nu_s^+ \rangle &= \int_0^1 \frac{\hat{\sigma}^2}{2} \partial_{xx} \left(\int_0^1 G_\delta (x, y) \phi (y) \, dy \right) \nu_s^+ (dx) \\
&= \frac{\hat{\sigma}^2}{2} \int_0^1 \int_0^1 \partial_{xx} G_\delta (x, y) \phi (y) \, dy \nu_s^+ (dx) \\
&= \frac{\hat{\sigma}^2}{2} \int_0^1 \int_0^1 \partial_{yy} G_\delta (x, y) \phi (y) \, dy \nu_s^+ (dx) ,
\end{aligned}
$$

where we used $\partial_{xx}G_\delta(x,y) = \partial_{yy}G_\delta(x,y)$. Now using integration by parts and $\phi(0) = \phi(1) = 0$, repeatedly, we have

$$
\begin{aligned}
\langle \frac{\hat{\sigma}^2}{2}(T_\delta\phi)''(x), \nu_s^+\rangle &= \frac{\hat{\sigma}^2}{2}\int_0^1 \{\phi(y)\,\partial_y G_\delta(x,y)\,|_0^1 \\
&\qquad - \int_0^1 \phi'(y)\,\partial_y G_\delta(x,y)\,dy\Big\}\,\nu_s^+(dx) \\
&= -\frac{\hat{\sigma}^2}{2}\int_0^1 \left\{\int_0^1 \phi'(y)\,\partial_y G_\delta(x,y)\,dy\right\}\nu_s^+(dx) \\
&= -\frac{\hat{\sigma}^2}{2}\int_0^1 \left\{\phi'(y)\,G_\delta(x,y)\,|_0^1 - \int_0^1 \phi''(y)\,G_\delta(x,y)\,dy\right\}\nu_s^+(dx) \\
&= \frac{\hat{\sigma}^2}{2}\int_0^1 \phi''(y)\int_0^1 G_\delta(x,y)\,\nu_s^+(dx)\,dy \\
&= \frac{\hat{\sigma}^2}{2}\int_0^1 \phi''(y)\,T_\delta\left(\nu_s^+\right)dy \\
&= \frac{\hat{\sigma}^2}{2}\left\{\phi'(y)\,T_\delta\left(\nu_s^+\right)(y)\,|_0^1 - \int_0^1 \phi'(y)\,\partial_y\left(T_\delta\left(\nu_s^+\right)(y)\right)dy\right\} \\
&= -\frac{\hat{\sigma}^2}{2}\int_0^1 \phi'(y)\,\partial_y\left(T_\delta\left(\nu_s^+\right)\right)(y)\,dy \\
&= -\frac{\hat{\sigma}^2}{2}\left\{\phi(y)\,\partial_y\left(T_\delta\left(\nu_s^+\right)\right)(y)\,|_0^1 - \int_0^1 \phi(y)\,\partial_{yy}\left(T_\delta\left(\nu_s^+\right)\right)(y)\,dy\right\} \\
&= \frac{\hat{\sigma}^2}{2}\int_0^1 \phi(y)\,\partial_{yy}\left(T_\delta\left(\nu_s^+\right)\right)(y)\,dy \\
&= \frac{\hat{\sigma}^2}{2}\langle\phi, \partial_{xx}\left(T_\delta\nu_s^+\right)\rangle_0.
\end{aligned} \tag{6.7}
$$

We now consider the last term in (6.5)

$$
\begin{aligned}
-\langle\lambda(r_s)(T_\delta\phi)(x), \nu_s^+\rangle &= -\lambda(r_s)\int_0^1\int_0^1 G_\delta(x,y)\,\phi(y)\,dy\nu_s^+(dx) \\
&= -\lambda(r_s)\int_0^1 \phi(y)\int_0^1 G_\delta(x,y)\,\nu_s^+(dx)\,dy \\
&= -\lambda(r_s)\int_0^1 \phi(y)\,T_\delta\left(\nu_s^+\right)(y)\,dy \\
&= -\lambda(r_s)\langle\phi, T_\delta\left(\nu_s^+\right)\rangle_0.
\end{aligned} \tag{6.8}
$$

Substituting back into (6.5), we have

$$
\langle\hat{\mu}(T_\delta\phi)'(x) + \frac{\hat{\sigma}^2}{2}(T_\delta\phi)''(x) - \lambda(r_s)\,T_\delta\phi(x), \nu_s^+\rangle = -\hat{\mu}\langle\phi, \partial_x\left(T_\delta^r\left(\nu_s^+\right)\right)\rangle_0
$$

$$+\frac{\hat{\sigma}^2}{2}\langle\phi,\partial_{xx}\left(T_\delta\nu_s^+\right)\rangle_0-\lambda\left(r_s\right)\langle\phi,T_\delta\left(\nu_s^+\right)\rangle_0.$$

Similarly, the third term in (6.4) becomes

$$\begin{aligned}
\langle\hat{\sigma}\rho\left(T_\delta\phi\right)',\nu_s^+\rangle &= \hat{\sigma}\rho\int_0^1\partial_x\left(T_\delta\phi\right)(x)\,\nu_s^+(dx)\\
&= \hat{\sigma}\rho\int_0^1\partial_x\left(G_\delta(x,y)\,\phi(y)\,dy\right)\nu_s^+(dx)\\
&= -\hat{\sigma}\rho\int_0^1\left(\partial_y G_\delta^r(x,y)\,\phi(y)\,dy\right)\nu_s^+(dx)\\
&= -\hat{\sigma}\rho\int_0^1\left\{G_\delta^r(x,y)\,\phi(y)\,|_0^1-\int_0^1 G_\delta^r(x,y)\,\phi'(y)\,dy\right\}\nu_s^+(dx)\\
&= \hat{\sigma}\rho\int_0^1\phi'(y)\int_0^1 G_\delta^r(x,y)\,\nu_s^+(dx)\,dy\\
&= \hat{\sigma}\rho\int_0^1\phi'(y)\,T_\delta^r\nu_s^+(y)\,dy\\
&= \hat{\sigma}\rho\phi(y)\,T_\delta^r\nu_s^+(y)\,|_0^1-\hat{\sigma}\rho\int_0^1\phi(y)\,\partial_y T_\delta^r\left(\nu_s^+\right)(y)\,dy\\
&= -\hat{\sigma}\rho\int_0^1\phi(y)\,\partial_y T_\delta^r\left(\nu_s^+\right)(y)\,dy\\
&= -\hat{\sigma}\rho\langle\phi,\partial_x\left(T_\delta^r\nu_s^+\right)\rangle_0.
\end{aligned}$$

Replacing the integrands in (6.4), we obtain the following for $Z_\delta(t)$

$$\begin{aligned}
\langle Z_\delta(t),\phi\rangle_0 &= \langle Z_\delta(0),\phi\rangle_0-\hat{\mu}\int_0^t\langle\phi,\partial_x\left(T_\delta^r\nu_s^+\right)\rangle_0 ds+\frac{\hat{\sigma}^2}{2}\int_0^t\langle\phi,\partial_{xx}\left(T_\delta\nu_s^+\right)\rangle_0 ds\\
&\quad-\int_0^t\lambda\left(r_s\right)\langle\phi,T_\delta\left(\nu_s^+\right)\rangle_0 ds-\hat{\sigma}\rho\int_0^t\langle\phi,\partial_x\left(T_\delta^r\nu_s^+\right)\rangle_0 dM_s.
\end{aligned}$$

Using Ito formula for $\langle Z_\delta(t),\phi\rangle_0^2$, we have

$$d\langle Z_\delta(t),\phi\rangle_0^2=2\langle Z_\delta(t),\phi\rangle_0 d\langle Z_\delta(t),\phi\rangle_0+d\left[\langle Z_\delta(t),\phi\rangle_0,\langle Z_\delta(t),\phi\rangle_0\right].$$

This implies

$$\begin{aligned}
\langle Z_\delta(t),\phi\rangle_0^2 &= \langle Z_\delta(0),\phi\rangle_0^2+\int_0^t 2\langle Z_\delta(s),\phi\rangle_0 d\langle Z_\delta(s),\phi\rangle_0\\
&\quad+\int_0^t d\left[\langle Z_\delta(s),\phi\rangle_0,\langle Z_\delta(s),\phi\rangle_0\right]
\end{aligned}$$

$$
\begin{aligned}
&= \langle Z_\delta(0), \phi\rangle_0^2 + \int_0^t 2\langle Z_\delta(t), \phi\rangle_0 \left\{ -\hat{\mu}\langle \phi, \partial_x T_\delta^r\left(\nu_s^+\right)\rangle_0 \right. \\
&\quad \left. + \frac{\hat{\sigma}^2}{2}\langle \phi, \partial_{xx}\left(T_\delta \nu_s^+\right)\rangle_0 - \lambda(r_s)\langle \phi, T_\delta\left(\nu_s^+\right)\rangle_0 \right\} ds \\
&\quad - \hat{\sigma}\rho\langle \phi, \partial_x\left(T_\delta^r \nu_s^+\right)\rangle_0 dM_s + \hat{\sigma}^2\rho^2 \int_0^t |\langle \phi, \partial_x T_\delta^r\left(\nu_s^+\right)\rangle_0|^2 ds.
\end{aligned}
\tag{6.9}
$$

Summing over ϕ in a complete orthonormal basis of H_0, $Z_\delta(t) = \sum_{\phi\in H_0}\langle Z_\delta(t), \phi\rangle\phi$ and taking expectations, we have

$$
\begin{aligned}
\mathbb{E}||Z_\delta(t)||_0^2 &= ||Z_\delta(0)||_0^2 - 2\hat{\mu}\mathbb{E}\int_0^t \langle Z_\delta(s), \partial_x T_\delta^r\left(\nu_s^+\right)\rangle_0 ds \\
&\quad + \frac{\hat{\sigma}^2}{2}\mathbb{E}\int_0^t \langle Z_\delta(s), \partial_{xx}\left(T_\delta \nu_s^+\right)\rangle_0 ds - \mathbb{E}\int_0^t \lambda(r_s)\langle Z_\delta(s), T_\delta\left(\nu_s^+\right)\rangle_0 ds \\
&\quad + \rho^2\mathbb{E}\left[\int_0^t ||\partial_x T_\delta^r\left(\nu_s^+\right)||_0^2 ds\right] \\
&= ||Z_\delta(0)||_0^2 - 2\hat{\mu}\mathbb{E}\int_0^t \langle T_\delta\left(\nu_s^+\right), \partial_x T_\delta^r\left(\nu_s^+\right)\rangle_0 ds \\
&\quad + \frac{\hat{\sigma}^2}{2}\mathbb{E}\int_0^t \langle T_\delta\left(\nu_s^+\right), \partial_{xx}\left(T_\delta \nu_s^+\right)\rangle_0 ds - \mathbb{E}\int_0^t \lambda(r_s)||T_\delta\left(\nu_s^+\right)||_0^2 ds \\
&\quad + \rho^2\mathbb{E}\left[\int_0^t ||\partial_x T_\delta^r\left(\nu_s^+\right)||_0^2 ds\right].
\end{aligned}
\tag{6.10}
$$

We now show that we can control the integral terms on the right hand side of (6.10) in terms of the integral of $\mathbb{E}||T_\delta\left(\nu_s^+\right)||_0^2$ and a constant that goes to 0 as $\delta \to 0$.

We first present a lemma that we are going to use later which gives control on the integral terms.

Lemma 6.2.1. *We let,*

$$
Q(s) := \int_0^1 \left\{ \frac{2}{\sqrt{2\pi\delta}} \int_0^1 \sum_{n=-\infty}^{+\infty} \left(\frac{x+y+2n}{\delta}\right) e^{-\frac{(x+y+2n)^2}{2\delta}} \nu_s^+(dy) \right\}^2 dx,
$$

then

$$
Q(s) \le \int_0^{\sqrt{3/2}} \frac{2}{\pi\delta^3} e^{-\frac{1}{2\delta}z^2} \frac{z^2}{2} \nu_s^+\left((1-z, 1)\right)^2 dz
$$

$$+\int_0^{\sqrt{3/2}} \frac{1}{\pi\delta^3} z^2 e^{-\frac{z^2}{2\delta}} \nu_s^+ ((0,z))\, dz + \frac{1}{\pi\delta^3}\frac{32}{\sqrt{\pi\delta^3}} e^{-\frac{1}{2\delta}}$$
$$+\frac{2}{\pi\delta^3} C^2 e^{-\frac{4}{\delta}} + \int_0^2 \frac{2}{\pi\delta^3} C e^{-\frac{2}{\delta}} z e^{-\frac{z^2}{2\delta}} \nu_s^+ ((0,z))\, dz$$
$$+\int_0^2 \frac{2}{\pi\delta^3} C e^{-\frac{2}{\delta}} z e^{-\frac{z^2}{2\delta}} \nu_s^+ ((1-z,1))\, dz.$$

Proof. We have,

$$\begin{aligned} Q(s) &= \int_0^1 \left\{ \frac{2}{\sqrt{2\pi\delta}} \int_0^1 \sum_{n=-\infty}^{+\infty} \left(\frac{x+y+2n}{\delta} \right) e^{-\frac{(x+y+2n)^2}{2\delta}} \nu_s^+ (dy) \right\}^2 dx \\ &= \int_0^1 \left\{ \frac{2}{\sqrt{2\pi\delta}} \int_0^1 \sum_{n=-\infty}^{-2} \left(\frac{x+y+2n}{\delta} \right) e^{-\frac{(x+y+2n)^2}{2\delta}} \nu_s^+ (dy) \right. \\ &\quad + \frac{2}{\sqrt{2\pi\delta}} \int_0^1 \sum_{n=-1}^{0} \left(\frac{x+y+2n}{\delta} \right) e^{-\frac{(x+y+2n)^2}{2\delta}} \nu_s^+ (dy) \\ &\quad \left. + \frac{2}{\sqrt{2\pi\delta}} \int_0^1 \sum_{n=1}^{\infty} \left(\frac{x+y+2n}{\delta} \right) e^{-\frac{(x+y+2n)^2}{2\delta}} \nu_s^+ (dy) \right\}^2 dx \end{aligned}$$

We now look at each term on the right hand side of (6.11) and let

$$I_1 := \frac{2}{\sqrt{2\pi\delta}} \int_0^1 \sum_{n=1}^{+\infty} \left(\frac{x+y+2n}{\delta} \right) e^{-\frac{(x+y+2n)^2}{2\delta}} \nu_s^+ (dy)$$

We want to maximize the sum on $x, y \in [0,1]$, that is

$$\max_{x,y\in[0,1]} \left(\frac{x+y+2n}{\delta} \right) e^{-\frac{(x+y+2n)^2}{2\delta}} \leq \frac{2+2n}{\delta} e^{\frac{-2n^2}{\delta}}$$

This gives us

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{x+y+2n}{\delta} \right) e^{-\frac{(x+y+2n)^2}{2\delta}} &\leq \frac{e^{-\frac{2}{\delta}}}{\delta} \left\{ 4 + 6e^{-\frac{6}{\delta}} + 8e^{-\frac{16}{\delta}} + \cdots \right\} \\ &\leq C_1 \frac{e^{-\frac{2}{\delta}}}{\delta}, \end{aligned} \tag{6.11}$$

where C_1 is a constant. Similarly, for the first sum in (6.10) we let

$$I_2 := \frac{2}{\sqrt{2\pi\delta}} \int_0^1 \sum_{n=-\infty}^{-2} \left(\frac{x+y+2n}{\delta} \right) e^{-\frac{(x+y+2n)^2}{2\delta}} \nu_s^+ (dy)$$

We want to maximize the sum on $x, y \in [0,1]$, that is

$$\max_{x,y\in[0,1]} \left(\frac{x+y+2n}{\delta} \right) e^{-\frac{(x+y+2n)^2}{2\delta}} \quad = \quad \frac{2+2n}{\delta} e^{\frac{-2n^2}{\delta}}$$

This gives us

$$\begin{aligned} \sum_{n=-\infty}^{-2} \left(\frac{x+y+2n}{\delta} \right) e^{-\frac{(x+y+2n)^2}{2\delta}} &\leq \frac{e^{-\frac{8}{\delta}}}{\delta} \left\{ -2 - 4e^{-\frac{10}{\delta}} - \cdots \right\} \\ &\leq C_2 \frac{e^{-\frac{8}{\delta}}}{\delta}, \end{aligned} \tag{6.12}$$

where C_2 is a constant.

We also see that

$$\begin{aligned} Q(s) &= \int_0^1 \left(I_1^2 + I_2^2 + I_3(s)^2 \right) dx \\ &\leq \int_0^1 3I_1^2 + 3I_2^2 + 3I_3(s)^2 \, dx, \end{aligned} \tag{6.13}$$

where

$$I_3(s) = \frac{2}{\sqrt{2\pi\delta}} \int_0^1 \sum_{n=-1}^{0} \frac{2}{\sqrt{2\pi\delta}} \left(\frac{x+y+2n}{\delta} \right) e^{-\frac{(x+y+2n)^2}{2\delta}} \nu_s^+ (dy) .$$

Plugging these back into (6.11) and using (6.13), we have

$$\begin{aligned} Q(s) &\leq \int_0^1 \left\{ \int_0^1 \frac{6}{\pi\delta^3} C_2^2 e^{\frac{-16}{\delta}} \nu_s^+ (dy) + 3I_3^2 \right. \\ &\quad \left. + \int_0^1 \frac{6}{\pi\delta^3} C_1^2 e^{-\frac{4}{\delta}} \nu_s^+ (dy) \right\} dx \\ &\leq \int_0^1 \left\{ \int_0^1 \sum_{n=-1}^{0} \frac{2}{\sqrt{2\pi\delta}} \left(\frac{x+y+2n}{\delta} \right) e^{-\frac{(x+y+2n)^2}{2\delta}} \nu_s^+ (dy) \right\}^2 dx \end{aligned}$$

$$+\int_0^1 \frac{6}{\pi\delta^3} C e^{-\frac{4}{\delta}} \nu_s^+(dy)\, dx \tag{6.14}$$

We now need to find bounds on the integrals on the right hand side of the above expression. We let

$$Q_1(s) := \int_0^1 \left\{ \int_0^1 \frac{2}{\sqrt{2\pi\delta}} \sum_{n=-1}^{0} \frac{(x+y+2n)}{\delta} e^{-\frac{(x+y+2n)^2}{2\delta}} \nu_s^+(dy) \right\}^2 dx$$

We need to find a bound on Q_1. We have

$$\begin{aligned}
Q_1(s) &= \int_0^1 \int_0^1 \nu_s^+(dy_1)\, \nu_s^+(dy_2) \int_0^1 \frac{2}{\pi\delta} \sum_{n=-1}^{0} \sum_{m=-1}^{0} \frac{(x+y_1+2n)}{\delta} e^{-\frac{(x+y_1+2n)^2}{2\delta}} \\
&\quad \frac{(x+y_2+2m)}{\delta} e^{-\frac{(x+y_2+2m)^2}{2\delta}} dx \\
&= \int_0^1 \int_0^1 \nu_s^+(dy_1)\, \nu_s^+(dy_2) \int_0^1 \frac{2}{\pi\delta} \sum_{n=-1}^{0} \sum_{m=-1}^{0} \frac{1}{\delta^2} \left\{ \left(x + \frac{y_1+y_2+2m+2n}{2} \right)^2 \right. \\
&\quad \left. - \left(\frac{(y_1+2n)-(y_2+2m)}{2} \right)^2 \right\} e^{-\frac{1}{\delta}\left\{ \left(x+\frac{y_1+2n+y_2+2m}{2}\right)^2 + \left(\frac{y_2+2n-y_2-2m}{2}\right)^2 \right\}} dx
\end{aligned}$$

We now look at the case when $n=-1$, $m=-1$, we have

$$\begin{aligned}
Q_1^1(s) &:= \int_0^1 \int_0^1 \nu_s^+(dy_1)\, \nu_s^+(dy_2) \int_0^1 \frac{2}{\pi\delta^3} \left\{ \left(x + \frac{y_1+y_2-4}{2} \right)^2 - \left(\frac{y_1-y_2}{2} \right)^2 \right\} \\
&\quad e^{-\frac{1}{\delta}\left(x+\frac{y_1+y_2-4}{2}\right)^2 + \left(\frac{y_1-y_2}{2}\right)^2} dx.
\end{aligned}$$

Let

$$z^2 = \left(x + \frac{y_1+y_2-4}{2} \right)^2 + \left(\frac{y_1-y_2}{2} \right)^2.$$

We obtain,

$$Q_1^1(s) = \int_0^1 \int_0^1 \nu_s^+(dy_1)\, \nu_s^+(dy_2) \int_{\sqrt{\left(1+\frac{y_1+y_2-4}{2}\right)^2 + \left(\frac{y_1-y_2}{2}\right)^2}}^{\sqrt{\left(\frac{y_1+y_2-4}{2}\right)^2 + \left(\frac{y_1-y_2}{2}\right)^2}} \frac{2}{\pi\delta^3} \left\{ z^2 - \frac{(y_1-y_2)^2}{2} \right\}$$

$$\frac{ze^{-\frac{z^2}{\delta}}dz}{\sqrt{z^2-\left(\frac{y_1-y_2}{2}\right)^2}}$$

It is easy to see that $1 \le \frac{z}{\sqrt{z^2-\left(\frac{y_1-y_2}{2}\right)^2}} < \sqrt{2}$ when $z > \sqrt{\left(1+\frac{y_1+y_2-4}{2}\right)^2+\left(\frac{y_1-y_2}{2}\right)^2}$. Therefore, we have

$$\begin{aligned}
Q_1^1(s) &\le \int_0^1\int_0^1 \nu_s^+(dy_1)\,\nu_s^+(dy_2)\times \\
&\quad \int_{\sqrt{\left(1+\frac{y_1+y_2-4}{2}\right)^2+\left(\frac{y_1-y_2}{2}\right)^2}}^{\sqrt{\left(\frac{y_1+y_2-4}{2}\right)^2+\left(\frac{y_1-y_2}{2}\right)^2}} \frac{2\sqrt{2}}{\pi\delta^3}e^{-\frac{1}{\delta}z^2}\left(z^2-\frac{1}{2}(y_1-y_2)^2\right)dz \\
&= \int_0^1\int_0^1 \nu_s^+(dy_1)\,\nu_s^+(dy_2)\int_0^{\sqrt{\left(\frac{y_1+y_2-4}{2}\right)^2+\left(\frac{y_1-y_2}{2}\right)^2}} \\
&\quad \frac{2\sqrt{2}}{\pi\delta^3}e^{-\frac{1}{\delta}z^2}\left(z^2-\frac{1}{2}(y_1-y_2)^2\right)\mathbf{1}_{\left\{z>\sqrt{\left(1+\frac{y_1+y_2-4}{2}\right)^2+\left(\frac{y_1-y_2}{2}\right)^2}\right\}}dz \\
&\le \int_0^1\int_0^1 \nu_s^+(dy_1)\,\nu_s^+(dy_2)\int_0^{\sqrt{\left(\frac{y_1+y_2-4}{2}\right)^2+\left(\frac{y_1-y_2}{2}\right)^2}} \frac{2\sqrt{2}}{\pi\delta^3}e^{-\frac{1}{\delta}z^2}z^2 \\
&\quad \mathbf{1}_{\left\{z>\sqrt{\left(1+\frac{y_1+y_2-4}{2}\right)^2+\left(\frac{y_1-y_2}{2}\right)^2}\right\}}dz \\
&\le \int_0^1\int_0^1 \nu_s^+(dy_1)\,\nu_s^+(dy_2)\int_0^{\sqrt{3}} \frac{2\sqrt{2}}{\pi\delta^3}e^{-\frac{1}{\delta}z^2}z^2\mathbf{1}_{\left\{z^2>\left(1+\frac{y_1+y_2-4}{2}\right)^2+\left(\frac{y_1-y_2}{2}\right)^2\right\}}dz \\
&\le \int_0^{\sqrt{3}} \frac{2\sqrt{2}}{\pi\delta^3}e^{-\frac{1}{\delta}z^2}z^2dz\int_0^1\int_0^1 \mathbf{1}_{\left\{\sqrt{2}z>(1-y_1),\sqrt{2}z>(1-y_2)\right\}}\nu_s^+(dy_1)\,\nu_s^+(dy_2) \\
&\le \int_0^{\sqrt{3}} \frac{2\sqrt{2}}{\pi\delta^3}e^{-\frac{1}{\delta}z^2}z^2dz\int_0^1\int_0^1 \mathbf{1}_{\left\{1-\sqrt{2}z<y_1,1-\sqrt{2}z<y_2\right\}}\nu_s^+(dy_1)\,\nu_s^+(dy_2) \\
&\le \int_0^{\sqrt{3}} \frac{2\sqrt{2}}{\pi\delta^3}e^{-\frac{1}{\delta}z^2}z^2\left(\nu_s^+\left(\left(1-\sqrt{2}z,1\right)\right)\right)^2dz \\
&= \int_0^{\sqrt{3/2}} \frac{2}{\pi\delta^3}e^{-\frac{1}{2\delta}z^2}\frac{z^2}{2}\nu_s^+((1-z,1))^2\,dz. \qquad (6.15)
\end{aligned}$$

The case when $n=0$, $m=0$ follows from [28] and [8], we have

$$Q_1^2(s) := \int_0^1\int_0^1 \nu_s^+(dy_1)\,\nu_s^+(dy_2)\int_0^1 \frac{2}{\pi\delta^3}\left\{\left(x+\frac{y_1+y_2}{2}\right)^2-\left(\frac{y_1-y_2}{2}\right)^2\right\}$$

$$
\begin{aligned}
& e^{-\frac{1}{\delta}\left(x+\frac{y_1+y_2}{2}\right)^2+\left(\frac{y_1-y_2}{2}\right)^2}dx \\
\leq\ & \int_0^{\sqrt{3}} \frac{2}{\pi\delta^3} z^2 e^{-\frac{z^2}{2\delta}} \left(\nu_s^+\left((0,z)\right)\right)^2 dz.
\end{aligned}
$$

We now have the two cases when $n=-1$, $m=0$ and $n=0$, $m=-1$. We only do the calculations for the first one, the later follows similarly. We let,

$$
\begin{aligned}
Q_1^3(s) \ :=\ & \int_0^1 \int_0^1 \nu_s^+(dy_1)\,\nu_s^+(dy_2) \int_0^1 \frac{2}{\pi\delta^3} \left\{ \left(x+\frac{y_1+y_2-2}{2}\right)^2 \right. \\
& \left. - \left(\frac{y_1-y_2-2}{2}\right)^2 \right\} e^{-\frac{1}{\delta}\left(x+\frac{y_1+y_2-2}{2}\right)^2+\left(\frac{y_1-y_2-2}{2}\right)^2} dx.
\end{aligned}
$$

Letting

$$
\begin{aligned}
b_1 &= \frac{y_1-y_2-2}{2} \\
z &= x+\frac{y_1+y_2-2}{2}
\end{aligned}
$$

we obtain,

$$
\begin{aligned}
Q_1^3(s) \ =\ & \int_0^1 \int_0^1 \nu_s^+(dy_1)\,\nu_s^+(dy_2) \int_{\frac{y_1+y_2-2}{2}}^{\frac{y_1+y_2}{2}} \frac{2}{\pi\delta^3}\left(z^2-b_1^2\right) e^{-\frac{z^2-b_1^2}{\delta}} dz \\
\leq\ & \int_0^1 \int_0^1 \nu_s^+(dy_1)\,\nu_s^+(dy_2) \frac{2}{\pi\delta^3} \max_{y_1,y_2\in[0,1]} e^{-\frac{b_1^2}{\delta}} \int_{\frac{y_1+y_2-2}{2}}^{\frac{y_1+y_2}{2}} z^2 e^{-\frac{z^2}{\delta}} dz \\
\leq\ & \int_0^1 \int_0^1 \nu_s^+(dy_1)\,\nu_s^+(dy_2) \frac{2}{\pi\delta^3} e^{-\frac{1}{4\delta}} \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{\delta}} dz \\
=\ & \int_0^1 \int_0^1 \frac{2}{\pi\delta^3} e^{-\frac{1}{4\delta}} \sqrt{2\pi} \left(\frac{\delta}{2}\right)^{5/4} \nu_s^+(dy_1)\,\nu_s^+(dy_2) \\
\leq\ & \frac{2}{\pi\delta^3}\sqrt{2\pi}\left(\frac{\delta}{2}\right)^{5/4} e^{-\frac{1}{4\delta}} \int_0^1 \int_0^1 \nu_s^+(dy_1)\,\nu_s^+(dy_2) \\
\leq\ & \frac{2^{1/4}}{\sqrt{\pi}\delta^{5/4}} 16 e^{-\frac{1}{4\delta}},
\end{aligned}
$$

where we used the variance of Gaussian distribution.

Similarly,

$$
\begin{aligned}
Q_1^4(s) &:= \int_0^1 \int_0^1 \nu_s^+(dy_1)\,\nu_s^+(dy_2) \int_0^1 \frac{1}{2\pi\delta^3} \left\{ \left(x + \frac{y_1+y_2-2}{2} \right)^2 \right. \\
&\quad \left. - \left(\frac{y_1-y_2+2}{2} \right)^2 \right\} e^{-\frac{1}{\delta}\left(x+\frac{y_1+y_2-2}{2}\right)^2 + \left(\frac{y_1-y_2+2}{2}\right)^2} dx \\
&\leq \frac{2^{1/4}}{\sqrt{\pi}\delta^{5/4}} 16 e^{-\frac{1}{4\delta}}.
\end{aligned}
$$

Combining all the cases, we have

$$
\begin{aligned}
Q_1(s) &\leq \int_0^{\sqrt{3}} \frac{2\sqrt{2}}{\pi\delta^3} e^{-\frac{1}{\delta}z^2} z^2 \left(\nu_s^+\left(\left(1-\sqrt{2}z, 1 \right) \right) \right)^2 dz \\
&\quad + \int_0^{\sqrt{3}} \frac{2}{\pi\delta^3} z^2 e^{-\frac{z^2}{2\delta}} \left(\nu_s^+((0,z)) \right)^2 dz \\
&\quad + \frac{2^{1/4}}{\sqrt{\pi}\delta^{5/4}} 32 e^{-\frac{1}{4\delta}},.
\end{aligned}
$$

Therefore on substituting all the simplified expressions back in (6.14), we obtain our result that is

$$
\begin{aligned}
Q(s) &\leq \int_0^{\sqrt{3/2}} \frac{6}{\pi\delta^3} e^{-\frac{1}{2\delta}z^2} \frac{z^2}{2} \nu_s^+((1-z,1))^2 dz \\
&\quad + \int_0^{\sqrt{3/2}} \frac{3}{\pi\delta^3} z^2 e^{-\frac{z^2}{2\delta}} \nu_s^+((0,z))^2 dz + \frac{2^{1/4}}{\sqrt{\pi}\delta^{5/4}} 96 e^{-\frac{1}{4\delta}} \\
&\quad + \frac{2}{\pi\delta^6} C^2 e^{-\frac{4}{\delta}} \qquad (6.16)
\end{aligned}
$$

□

To find a control on the expectations of the smoothed measure, we further need to find the bounds on the expectations of $Q(s)$. That is to find a bound on

$$
\begin{aligned}
\mathbb{E}[Q(s)] &\leq \int_0^{\sqrt{3/2}} \frac{6}{\pi\delta^3} e^{-\frac{1}{2\delta}z^2} z^2 \mathbb{E}\left[\nu_s^+((1-z,1))^2 \right] dz \\
&\quad + \int_0^{\sqrt{3/2}} \frac{3}{\pi\delta^3} z^2 e^{-\frac{z^2}{2\delta}} \mathbb{E}\left[\nu_s^+((0,z))^2 \right] dz
\end{aligned}
$$

$$+\frac{2^{1/4}}{\sqrt{\pi}\delta^{5/4}}96e^{-\frac{1}{4\delta}}+\frac{6}{\pi\delta^3}C^2e^{-\frac{4}{\delta}}$$

The bound on the expectations of $Q(s)$ depends on the expectations of ν_s^+. We have two cases for the bounds on the measure ν_t^+, discussed in Chapter 5, a) for fixed t and b) uniform results for any T. We have the following lemma for uniform bounds on ν_t^+ and its second moment that is the following theorem holds for any T.

Lemma 6.2.2. *Under the conditions of Theorem 5.2.3 and Theorem 5.4.2, Chapter 5, we have for any T*

$$\mathbb{E}\left[Q(s)\right] \leq D_T^1\delta^{\frac{\zeta}{2}}+\frac{D_2\varepsilon}{\delta^2}e^{-\tilde{\varepsilon}/2\delta}+\frac{2^{1/4}}{\sqrt{\pi}\delta^{5/4}}32e^{-\frac{1}{4\delta}}+\frac{2}{\pi\delta^3}C^2e^{-\frac{4}{\delta}} \tag{6.17}$$

where

$$D_T^1=\frac{2^{3+\zeta/2}}{\pi}3\left(2M+N\right)\int_0^\infty x^{5+\zeta}e^{-x^2}dx \text{ and } D_2=\frac{72\sqrt{2}}{\pi}$$

where M and N are defined in Theorem 5.4.2.

Proof. From theorems (5.2.3) and (5.4.2), there exists a $\tilde{\varepsilon}>0$ and $\zeta>0$ for $\beta_1,\beta_2>\frac{3\left(1+\frac{\pi}{\alpha}\right)}{\frac{\pi}{\alpha}-1}$ such that $\forall z<\tilde{\varepsilon}$

$$\begin{aligned}\mathbb{E}\left[\nu_t^+\left((0,z)\right)^2\right] &\leq Mz^{3+\zeta}\\ \mathbb{E}\left[\nu_t^+\left((1-z,1)\right)^2\right] &\leq Nz^{3+\zeta}\end{aligned}$$

Therefore we have on splitting the integral at $\tilde{\varepsilon}$

$$\begin{aligned}\mathbb{E}\left[Q(s)\right] \leq\; & \int_0^{\tilde{\varepsilon}}\frac{6}{\pi\delta^3}e^{-\frac{1}{2\delta}z^2}z^2\mathbb{E}\left[\nu_s^+\left((1-z,1)\right)^2\right]dz\\ &+\int_{\tilde{\varepsilon}}^{\sqrt{3/2}}\frac{6}{\pi\delta^3}e^{-\frac{1}{2\delta}z^2}z^2\mathbb{E}\left[\nu_s^+\left((1-z,1)\right)^2\right]dz\\ &+\int_0^{\tilde{\varepsilon}}\frac{3}{\pi\delta^3}z^2e^{-\frac{z^2}{2\delta}}\mathbb{E}\left[\nu_s^+\left((0,z)\right)^2\right]dz\\ &+\int_{\tilde{\varepsilon}}^{\sqrt{3/2}}\frac{3}{\pi\delta^3}z^2e^{-\frac{z^2}{2\delta}}\mathbb{E}\left[\nu_s^+\left((0,z)\right)^2\right]dz+\frac{2^{1/4}}{\sqrt{\pi}\delta^{5/4}}32e^{-\frac{1}{4\delta}}\\ &+\frac{6}{\pi\delta^3}C^2e^{-\frac{4}{\delta}}\end{aligned}$$

Now using the results for the second moments of ν_s^+ and the boundary conditions, we have

$$
\begin{aligned}
&\leq \int_0^{\tilde{\varepsilon}} \frac{6}{\pi\delta^3} e^{-\frac{1}{2\delta}z^2} z^2 M z^{3+\zeta} dz + \int_{\tilde{\varepsilon}}^{\sqrt{3/2}} \frac{6}{\pi\delta^3} e^{-\frac{1}{2\delta}z^2} z^2 (4) dz \\
&\quad + \int_0^{\tilde{\varepsilon}} \frac{3}{\pi\delta^3} z^2 e^{-\frac{z^2}{2\delta}} N z^{3+\zeta} dz + \int_{\tilde{\varepsilon}}^{\sqrt{3/2}} \frac{3}{\pi\delta^3} z^2 e^{-\frac{z^2}{2\delta}} (4) dz \\
&\quad + \frac{1}{\pi\delta^3} \frac{96}{\sqrt{\pi\delta^3}} e^{-\frac{1}{2\delta}} + \frac{6}{\pi\delta^3} C^2 e^{-\frac{4}{\delta}} \\
&= \int_0^{\tilde{\varepsilon}} \frac{3}{\pi\delta^3} e^{-\frac{1}{2\delta}z^2} z^2 (2M+N) z^{3+\zeta} dz + \int_{\tilde{\varepsilon}}^{\sqrt{3/2}} \frac{(36)}{\pi\delta^3} z^2 e^{-\frac{z^2}{2\delta}} dz \\
&\quad + frac 2^{1/4} \sqrt{\pi} \delta^{5/4} 96 e^{-\frac{1}{4\delta}} + \frac{6}{\pi\delta^3} C^2 e^{-\frac{4}{\delta}} \\
&\leq \int_0^{\infty} \frac{3}{\pi\delta^3} e^{-\frac{1}{2\delta}z^2} z^2 (2M+N) z^{3+\zeta} dz + \int_{\tilde{\varepsilon}}^{\infty} \frac{(36)}{\pi\delta^3} z^2 e^{-\frac{z^2}{2\delta}} dz \\
&\quad + \frac{2^{1/4}}{\sqrt{\pi}\delta^{5/4}} 96 e^{-\frac{1}{4\delta}} + \frac{6}{\pi\delta^3} C^2 e^{-\frac{4}{\delta}} \\
&= \frac{2^{3+\zeta/2}}{\pi\delta^{\zeta/2}} 3 (2M+N) \int_0^{\infty} x^{5+\zeta} e^{-x^2} dx + \frac{72\sqrt{2}}{\pi\delta^{3/2}} \int_{\frac{\tilde{\varepsilon}}{\sqrt{2\delta}}}^{\infty} x^2 e^{-x^2} dx \\
&\quad + \frac{2^{1/4}}{\sqrt{\pi}\delta^{5/4}} 96 e^{-\frac{1}{4\delta}} + \frac{6}{\pi\delta^3} C^2 e^{-\frac{4}{\delta}}
\end{aligned}
$$

As in [28], for $\eta > 0$

$$
\int_{\eta}^{\infty} x^2 e^{-x^2} \leq \frac{1}{2}\left(\eta + \frac{1}{\eta}\right) e^{-\eta^2}.
$$

Thus we have for $\eta = \frac{\tilde{\varepsilon}}{\sqrt{2\delta}}$,

$$
\mathbb{E}\left[Q(s)\right] \leq D_T^1 \delta^{\frac{-\zeta}{2}} + \frac{D_2 \tilde{\varepsilon}}{\delta^2} e^{-\tilde{\varepsilon}/2\delta} + \frac{2^{1/4}}{\sqrt{\pi}\delta^{5/4}} 96 e^{-\frac{1}{4\delta}} + \frac{6}{\pi\delta^3} C^2 e^{-\frac{4}{\delta}}
$$

where

$$
D_T^1 = \frac{2^{3+\zeta/2}}{\pi} 3 (2M+N) \int_0^{\infty} x^{5+\zeta} e^{-x^2} dx \text{ and } D_2 = \frac{72\sqrt{2}}{\pi}.
$$

□

Similarly for a fixed $s \in [0,T]$, we have

Lemma 6.2.3. *Under the conditions of Theorem 5.2.2 and Theorem 5.4.1, Chapter 5, we have*

$$\mathbb{E}\left[Q\left(s\right)\right] \quad \leq \quad D_s^*\delta^{-\frac{\zeta}{2}} + \frac{D_2\varepsilon}{\delta^2}e^{-\tilde{\varepsilon}/2\delta} + \frac{2^{1/4}}{\sqrt{\pi}\delta^{5/4}}96e^{-\frac{1}{4\delta}} + \frac{6}{\pi\delta^3}C^2e^{-\frac{4}{\delta}} \qquad (6.18)$$

where

$$D_s^* = \frac{2^{3+\zeta/2}}{\pi}3\left(2M_s^* + N_s^*\right)\int_0^\infty x^{5+\zeta}e^{-x^2}dx \text{ and } D_2 = \frac{72\sqrt{2}}{\pi}$$

where M_s^* and N_s^* are defined in Theorem 5.4.1.

Proof. The proof of the theorem followed in the same way as for the above theorem on using the results for a fixed time s. □

6.3 Results on existence and uniqueness

In this section, we find a control on the smoothed measure $Z_\delta\left(t\right)$ using the results given in the previous section. We have

$$\begin{aligned}
\mathbb{E}\|Z_\delta\left(t\right)\|_0^2 \quad &= \quad \|Z_\delta\left(0\right)\|_0^2 - 2\hat{\mu}\mathbb{E}\int_0^t\langle T_\delta\left(\nu_s^+\right), \partial_x T_\delta^r\left(\nu_s^+\right)\rangle_0 ds \\
&+\frac{\hat{\sigma}^2}{2}\mathbb{E}\int_0^t\langle T_\delta\left(\nu_s^+\right), \partial_{xx}\left(T_\delta\nu_s^+\right)\rangle_0 ds - \mathbb{E}\int_0^t\lambda\left(r_s\right)\|T_\delta\left(\nu_s^+\right)\|_0^2 ds \\
&+\rho^2\mathbb{E}\left[\int_0^t\|\partial_x T_\delta^r\left(\nu_s^+\right)\|_0^2 ds\right]. \qquad (6.19)
\end{aligned}$$

We find the bounds on the integrals on the right hand side of (6.19).

We start with $-2\hat{\mu}\mathbb{E}\int_0^t\langle T_\delta\left(\nu_s^+\right), \partial_x T_\delta^r\left(\nu_s^+\right)\rangle_0 ds$ and consider

$$\begin{aligned}
\langle T_\delta\left(\nu_s^+\right), \partial_x T_\delta^r\left(\nu_s^+\right)\rangle_0 \quad &= \quad \int_0^1 T_\delta\left(\nu_s^+\right)(x)\,\partial_x\left(T_\delta^r\left(x\right)\right)dx \\
&= \int_0^1 T_\delta\left(\nu_s^+\right)(x)\left(\int_0^1\partial_x G_\delta^r\left(x,y\right)\nu_s^+\left(dy\right)\right)dx \\
&= \int_0^1 T_\delta\left(\nu_s^+\right)(x)\left(\int_0^1\partial_x\right.
\end{aligned}$$

$$
\begin{aligned}
& \left\{\frac{1}{2\sqrt{2\pi\delta}}\sum_{n=-\infty}^{\infty}\left(e^{-\frac{(x-y+2n)^2}{2\delta}}+e^{\frac{-(x+y+2n)^2}{2\delta}}\right)\right\} \\
& \nu_s^+\left(dy\right)\big)\,dx \\
= & \int_0^1 T_\delta\left(\nu_s^+\right)(x)\left[\int_0^1 \frac{1}{2\sqrt{2\pi\delta}}\sum_{n=-\infty}^{\infty}\right. \\
& \left\{\frac{-\left(x-y+2n\right)}{\delta}e^{-\frac{(x-y+2n)^2}{2\delta}}-\frac{x+y+2n}{\delta}e^{-\frac{(x+y+2n)^2}{2\delta}}\right\} \\
& \left.\nu_s^+\left(dy\right)\right]dx \\
= & \int_0^1 T_\delta\left(\nu_s^+\right)(x)\left[\int_0^1\left\{\partial_x G_\delta\left(x,y\right)-\frac{1}{\sqrt{2\pi\delta}}\right.\right. \\
& \left.\left.\sum_{n=-\infty}^{+\infty}\left(\frac{x+y+2n}{\delta}\right)e^{-\frac{(x+y+2n)^2}{2\delta}}\right]\nu_s^+\left(dy\right)\right\}dx \\
= & \int_0^1 T_\delta\left(\nu_s^+\right)(x)\left\{\int_0^1 \partial_x G_\delta\left(x,y\right)\nu_s^+\left(dy\right)dx-\int_0^1\frac{1}{\sqrt{2\pi\delta}}\right. \\
& \left.\sum_{n=-\infty}^{+\infty}\left(\frac{x+y+2n}{\delta}\right)e^{-\frac{(x+y+2n)^2}{2\delta}}\nu_s^+\left(dy\right)\right\}dx \\
= & \int_0^1 T_\delta\left(\nu_s^+\right)(x)\,\partial_x T_\delta\left(\nu_s^+\right)dx-\int_0^1 T_\delta\left(\nu_s^+\right)(x)\int_0^1\frac{1}{\sqrt{2\pi\delta}} \\
& \sum_{n=-\infty}^{+\infty}\left(\frac{x+y+2n}{\delta}\right)e^{-\frac{(x+y+2n)^2}{2\delta}}\nu_s^+\left(dy\right)dx \\
= & \frac{1}{2}\left(T_\delta\left(\nu_s^+\right)(x)\right)^2-\int_0^1 T_\delta\left(\nu_s^+\right)(x)\int_0^1\frac{1}{\sqrt{2\pi\delta}} \\
& \sum_{n=-\infty}^{+\infty}\left(\frac{x+y+2n}{\delta}\right)e^{-\frac{(x+y+2n)^2}{2\delta}}\nu_s^+\left(dy\right)dx.
\end{aligned}
$$

This gives us

$$
\begin{aligned}
-2|\hat{\mu}\langle T_\delta\left(\nu_s^+\right),\partial_x T_\delta^r\left(\nu_s^+\right)\rangle_0| \leq\; & \hat{\mu}|\left(T_\delta\left(\nu_s^+\right)(x)\right)^2|+\hat{\mu}|\int_0^1\left\{\frac{2}{\sqrt{2\pi\delta}}\right. \\
& \left.\int_0^1\sum_{n=-\infty}^{+\infty}\left(\frac{x+y+2n}{\delta}\right)e^{-\frac{(x+y+2n)^2}{2\delta}}\nu_s^+\left(dy\right)\right\}^2 dx.| \\
\leq\; & \hat{\mu}|\left(T_\delta\left(\nu_s^+\right)(x)\right)^2|+\hat{\mu}|Q\left(s\right)|
\end{aligned}
$$

We now look at the remaining terms in (6.10) and consider

$$
\frac{\hat{\sigma}^2}{2}\mathbb{E}\left[\int_0^t\langle T_\delta\left(\nu_s^+\right),\partial_{xx}\left(T_\delta\nu_s^+\right)\rangle_0 ds\right]+\rho^2\mathbb{E}\left[\int_0^t\|\partial_x T_\delta^r\left(\nu_s^+\right)\|_0^2 ds\right]
$$

We have

$$\langle T_\delta\left(\nu_s^+\right), \partial_{xx} T_\delta\left(\nu_s^+\right)\rangle_0 = -\|\partial_x T_\delta \nu_s^+\|_0^2.$$

Also

$$\begin{aligned}
\|\partial_x T_\delta^r\left(\nu_s^+\right)(x)\|_0^2 &= \int_0^1 \left(\partial_x T_\delta^r\left(\nu_s^+\right)(x)\right)^2 dx \\
&= \int_0^1 \left\{\partial_x T_\delta \nu_s^+(x) - \frac{1}{\sqrt{2\pi\delta}} \int_0^1 \sum_{n=-\infty}^{+\infty} \left(\frac{x+y+2n}{\delta}\right)\right. \\
&\quad \left. e^{-\frac{(x+y+2n)^2}{2\delta}} \nu_s^+(dy)\right\}^2 dx \\
&= |\partial_x T_\delta \nu_s^+|_0^2 - 2\int_0^1 \partial_x T_\delta \nu_s^+(x) \int_0^1 \frac{1}{\sqrt{2\pi\delta}} \sum_{n=-\infty}^{+\infty} \left(\frac{x+y+2n}{\delta}\right) \\
&\quad e^{-\frac{(x+y+2n)^2}{2\delta}} \nu_s^+(dy)\, dx + \left(\frac{1}{\sqrt{2\pi\delta}} \int_0^1 \sum_{n=-\infty}^{+\infty} \left(\frac{x+y+2n}{\delta}\right)\right. \\
&\quad \left. e^{-\frac{(x+y+2n)^2}{2\delta}} \nu_s^+(dy)\right)^2 dx
\end{aligned}$$

Using these in (6.20), we have

$$\begin{aligned}
&\frac{\hat{\sigma}^2}{2}\langle T_\delta\left(\nu_s^+\right), \partial_{xx}\left(T_\delta \nu_s^+\right)\rangle_0 + \rho^2 |\partial_x T_\delta^r\left(\nu_s^+\right)|_0^2 \\
&= \left(\rho^2 - \frac{\hat{\sigma}^2}{2}\right) |\partial_x T_\delta \nu_s^+|_0^2 - 2\rho^2 \int_0^1 \partial_x T_\delta \nu_s^+(x) \int_0^1 \frac{1}{\sqrt{2\pi\delta}} \sum_{n=-\infty}^{+\infty} \\
&\quad e^{-\frac{(x+y+2n)^2}{2\delta}} \frac{x+y+2n}{\delta} \nu_s^+(dy) \\
&\quad + \rho^2 \int_0^1 \left(\int_0^1 \frac{1}{\sqrt{2\pi\delta}} \sum_{n=-\infty}^{+\infty} e^{-\frac{(x+y+2n)^2}{2\delta}} \left(\frac{x+y+2n}{\delta}\right) \nu_s^+(dy)\right)^2 dx \\
&\le \left(\rho^2 - \frac{\hat{\sigma}^2}{2}\right) |\partial_x T_\delta \nu_s^+|_0^2 + |2\rho^2 \sqrt{\frac{\frac{\hat{\sigma}^2}{2} - \rho^2}{\rho^2}} \int_0^1 \partial_x T_\delta \nu_s^+(x) \int_0^1 \frac{1}{\sqrt{2\pi\delta}} \\
&\quad \sqrt{\frac{\rho^2}{\frac{\hat{\sigma}^2}{2} - \rho^2}} \sum_{n=-\infty}^{+\infty} e^{-\frac{(x+y+2n)^2}{2\delta}} \frac{x+y+2n}{\delta} \nu_s^+(dy)\, dx| \\
&\quad + \rho^2 \int_0^1 \left(\int_0^1 \frac{1}{\sqrt{2\pi\delta}} \sum_{n=-\infty}^{+\infty} e^{-\frac{(x+y+2n)^2}{2\delta}} \left(\frac{x+y+2n}{\delta}\right) \nu_s^+(dy)\right)^2 dx \\
&\le \left(\rho^2 - \frac{\hat{\sigma}^2}{2}\right) |\partial_x T_\delta \nu_s^+|_0^2 + \rho^2 \int_0^1 \left(\sqrt{\frac{\frac{\hat{\sigma}^2}{2} - \rho^2}{\rho^2}} \partial_x T_\delta \nu_s^+(x)\right)^2 dx
\end{aligned}$$

$$+\rho^2\int_0^1\left(\int_0^1\frac{1}{\sqrt{2\pi\delta}}\sqrt{\frac{\rho^2}{\frac{\hat{\sigma}^2}{2}-\rho^2}}\sum_{n=-\infty}^{+\infty}e^{-\frac{(x+y+2n)^2}{2\delta}}\frac{x+y+2n}{\delta}\nu_s^+(dy)\right)^2dx$$
$$+\rho^2\int_0^1\left(\int_0^1\frac{1}{\sqrt{2\pi\delta}}\sum_{n=-\infty}^{+\infty}e^{-\frac{(x+y+2n)^2}{2\delta}}\left(\frac{x+y+2n}{\delta}\right)\nu_s^+(dy)\right)^2dx$$
$$=\frac{\rho^2\frac{\hat{\sigma}^2}{2}}{\frac{\hat{\sigma}^2}{2}-\rho^2}\int_0^1\left(\int_0^1\frac{1}{\sqrt{2\pi\delta}}\sum_{n=-\infty}^{+\infty}e^{-\frac{(x+y+2n)^2}{2\delta}}\left(\frac{x+y+2n}{\delta}\right)\nu_s^+(dy)\right)^2dx$$
$$=\frac{\rho^2\frac{\hat{\sigma}^2}{2}}{\frac{\hat{\sigma}^2}{2}-\rho^2}Q(s)$$

Combining the two terms and taking expectations we have

$$\mathbb{E}\left[|-2\hat{\mu}\langle T_\delta\left(\nu_s^+\right),\partial_x T_\delta^r\left(\nu_s^+\right)\rangle_0|\right]+\frac{\hat{\sigma}^2}{2}\mathbb{E}\left[\langle T_\delta\left(\nu_s^+\right),\partial_{xx}\left(T_\delta\nu_s^+\right)\rangle_0\right]$$
$$+\rho^2\mathbb{E}\left[\|\partial_x T_\delta^r\left(\nu_s^+\right)\|_0^2\right]$$
$$\leq\hat{\mu}\|\left(T_\delta\left(\nu_s^+\right)(x)\right)^2\|+\hat{\mu}|\mathbb{E}\left[Q(s)\right]|+\frac{\rho^2\frac{\hat{\sigma}^2}{2}}{\frac{\hat{\sigma}^2}{2}-\rho^2}\mathbb{E}\left[Q(s)\right]$$

We now have two results on the expectations of $Q(t)$, a) for any T and b) for fixed s. We first give the results for the uniform bounds on $Q(t)$ and show that we can find a control on $Z_\delta(t)$.

Theorem 6.3.1. *Under the conditions of Theorem 5.2.3 and Theorem 5.4.2, Chapter 5, we prove the existence and uniqueness of the density in L^2. That is assuming that ν_0^+ has an L^2 density, ν_t^+ has a density $v(t,x)\in H_0$. Furthermore, the density $v(t,x)$ is unique.*

Proof. We have from Lemma 6.2.2 for any T

$$\mathbb{E}\left[|-2\hat{\mu}\langle T_\delta\left(\nu_s^+\right),\partial_x T_\delta^r\left(\nu_s^+\right)\rangle_0|\right]+\frac{\hat{\sigma}^2}{2}\mathbb{E}\left[\langle T_\delta\left(\nu_s^+\right),\partial_{xx}\left(T_\delta\nu_s^+\right)\rangle_0\right]$$
$$+\rho^2\mathbb{E}\left[|\partial_x T_\delta^r\left(\nu_s^+\right)|_0^2\right]$$
$$\leq|\hat{\mu}|\mathbb{E}\left[\|T_\delta\left(\nu_s^+\right)\|_0^2\right]+|\hat{\mu}|\left\{D_T^1\delta^{-\frac{\zeta}{2}}+\frac{D_2\varepsilon}{\delta^2}e^{-\tilde{\varepsilon}/2\delta}+\frac{2^{1/4}}{\sqrt{\pi}\delta^{5/4}}96e^{-\frac{1}{4\delta}}\right.$$
$$\left.+\frac{6}{2\pi\delta^3}C^2e^{-\frac{4}{\delta}}\right\}$$

$$+\frac{\rho^2\frac{\hat{\sigma}^2}{2}}{\frac{\hat{\sigma}^2}{2}-\rho^2}\left[D_T^1\delta^{\frac{-\zeta}{2}}+\frac{D_2\tilde{\varepsilon}}{\delta^2}e^{-\tilde{\varepsilon}/2\delta}+\frac{2^{1/4}}{\sqrt{\pi}\delta^{5/4}}96e^{-\frac{1}{4\delta}}+\frac{6}{2\pi\delta^3}C^2e^{-\frac{4}{\delta}}\right]$$

where η, $\tilde{\varepsilon}$ and ζ are same as given before.

This gives us a control on the expectations of $Z_\delta(t)$. From (6.19) we have

$$\begin{aligned}
\mathbb{E}\|Z_\delta(t)\|_0^2 &= \|Z_\delta(0)\|_0^2-2\hat{\mu}\mathbb{E}\int_0^t\langle T_\delta(\nu_s^+),\partial_x T_\delta^r(\nu_s^+)\rangle_0 ds\\
&\quad+\frac{\hat{\sigma}^2}{2}\mathbb{E}\int_0^t\langle T_\delta(\nu_s^+),\partial_{xx}(T_\delta\nu_s^+)\rangle_0 ds-\mathbb{E}\int_0^t\lambda(r_s)\|T_\delta(\nu_s^+)\|_0^2 ds\\
&\quad+\rho^2\mathbb{E}\left[\int_0^t\|\partial_x T_\delta^r(\nu_s^+)\|_0^2 ds\right]\\
&\leq \|Z_\delta(0)\|_0^2+|\hat{\mu}|\int_0^t\mathbb{E}\left[\|T_\delta(\nu_s^+)\|_0^2\right]ds\\
&\quad+|\hat{\mu}|\left\{D_T^1\delta^{\frac{-\zeta}{2}}t+\frac{D_2\tilde{\varepsilon}}{\delta^2}e^{-\tilde{\varepsilon}/2\delta}t+\frac{2^{1/4}}{\sqrt{\pi}\delta^{5/4}}96e^{-\frac{1}{4\delta}}t+\frac{6}{2\pi\delta^3}C^2e^{-\frac{4}{\delta}}t\right\}\\
&\quad+\frac{\rho^2\frac{\hat{\sigma}^2}{2}}{\frac{\hat{\sigma}^2}{2}-\rho^2}\left[D_T^1\delta^{\frac{-\zeta}{2}}+\frac{D_2\tilde{\varepsilon}}{\delta^2}e^{-\tilde{\varepsilon}/2\delta}+\frac{2^{1/4}}{\sqrt{\pi}\delta^{5/4}}96e^{-\frac{1}{4\delta}}+\frac{6}{2\pi\delta^3}C^2e^{-\frac{4}{\delta}}\right]t\\
&:= \|Z_\delta(0)\|_0^2+\mathbb{E}\int_0^t(|\hat{\mu}|-\lambda(r_s))\|T_\delta(\nu_s^+)\|_0^2 ds+g(t,\delta) \qquad (6.20)
\end{aligned}$$

where

$$\begin{aligned}
g(t,\delta) &= |\hat{\mu}|\left\{D_T^1\delta^{\frac{-\zeta}{2}}t+\frac{D_2\varepsilon}{\delta^2}e^{-\tilde{\varepsilon}/2\delta}t+\frac{2^{1/4}}{\sqrt{\pi}\delta^{5/4}}96e^{-\frac{1}{4\delta}}t+\frac{6}{2\pi\delta^3}C^2e^{-\frac{4}{\delta}}t\right.\\
&\quad\left.+\frac{\rho^2\frac{\hat{\sigma}^2}{2}}{\frac{\hat{\sigma}^2}{2}-\rho^2}\left[D_T^1\delta^{\frac{\zeta}{2}}+\frac{D_2\varepsilon}{\delta^2}e^{-\tilde{\varepsilon}/2\delta}+\frac{2^{1/4}}{\sqrt{\pi}\delta^{5/4}}96e^{-\frac{1}{4\delta}}+\frac{6}{2\pi\delta^3}C^2e^{-\frac{4}{\delta}}\right]t,\right.
\end{aligned}$$

and $g(t,\delta)\to 0$ as $\delta\to 0$. Therefore we have,

$$\begin{aligned}
\mathbb{E}\|Z_\delta(t)\|_0^2 &\leq \|Z_\delta(0)\|_0^2+\mathbb{E}\int_0^t(|\hat{\mu}|-\lambda(r_s))\|T_\delta(\nu_s^+)\|_0^2 ds+g(t,\delta)\\
&= \|Z_\delta(0)\|_0^2+\mathbb{E}\int_0^t(|\hat{\mu}|-\lambda(r_s))\|Z_\delta(s)\|_0^2 ds+g(t,\delta).
\end{aligned}$$

Now if $|\hat{\mu}|<\lambda(r_s)$ then

$$\mathbb{E}\|Z_\delta(t)\|_0^2 \leq \|Z_\delta(0)\|_0^2+g(t,\delta),$$

and if $|\hat{\mu}| > \lambda(r_s)$ then

$$\mathbb{E}\|Z_\delta(t)\|_0^2 \quad \le \|Z_\delta(0)\|_0^2 + \mathbb{E}\int_0^t |\hat{\mu}|\|Z_\delta(s)\|_0^2 ds + g(t,\delta).$$

We can apply Gronwall's inequality to have,

$$\mathbb{E}\|Z_\delta(t)\|_0^2 \le \left(\|Z_\delta(0)\|_0^2 + g(t,\delta)\right) e^{|\hat{\mu}|t}.$$

Now let $\psi_j \in C_b(\mathbb{R}^+)$ be a complete, orthonormal system of H_0 such that we can write $\nu_t^+ = \sum_{j=1}^{\infty}\langle\psi_j, \nu_t^+\rangle\psi_j$. Then $\nu_t^+ \in L^2$ if $\sum_{j=1}^{\infty}\langle\psi_j, \nu_t^+\rangle < \infty$. We have from Fatou's Lemma,

$$\begin{aligned}
\mathbb{E}\left[\sum_j \langle\psi_j, \nu_t^+\rangle^2\right] &= \mathbb{E}\left[\sum_j \lim_{\delta\to 0}\langle\psi_j, T_\delta\nu_t^+\rangle^2\right] \\
&\le \liminf_{\delta\to 0}\mathbb{E}\|Z_\delta(t)\|_0^2 \le \|\nu_0^+\|_0^2 e^{|\hat{\mu}|t},
\end{aligned}$$

following from [28] and assuming that ν_0^+ has an L^2-density. This shows that $\nu_t^+ \in H_0$ and $\mathbb{E}\|\nu_t^+\|_0^2 < \infty$, $\forall t \ge 0$.

We now show that the measure-valued equation (5.14) has at most one solution assuming $\nu_0^+ \in H_0$. This follows directly from Jin [28] and Bush et al. [8]. The idea is to assume two solutions to the measure value equation, say ν_t^1 and ν_t^2 satisfying the boundary conditions in Theorem 5.2.3 starting with the same initial density $\nu_0 \in H_0$. From above theorem, we know that ν_t^1 and ν_t^2 exist in H_0. Let $\nu_t^+ = \nu_t^1 - \nu_t^2$ then $\nu_t^+ \in H_0$ is also a solution of (5.14). We can apply the results presented before to $|\nu_t^+| \le \nu_t^1 + \nu_t^2$, therefore

$$\mathbb{E}\|T_\delta\nu_t^+\|_0^2 \le |\hat{\mu}|\int_0^t \mathbb{E}\|T_\delta\left(|\nu_t^+|\right)\|_0^2 ds + g(t,\delta)$$

taking $\delta \to 0$, we have

$$\mathbb{E}\|T_\delta \nu_t^+\|_0^2 \leq |\hat{\mu}| \int_0^t \mathbb{E}\|\nu_s^+\|_0^2 ds = |\hat{\mu}| \int_0^t \mathbb{E}\|\nu_s^+\|_0^2 ds.$$

Now if $\upsilon(s,x)$ is the density of ν_s^+, then $|\upsilon(s,x)|$ is the density of $|\nu_s^+|$. By Gronwall's inequality we see that $\nu_t^+ \equiv 0$ which shows that there exists at most one solution to the evolution equation. $\square$

Remark

We have for a fixed t,

$$\begin{aligned}
\mathbb{E}\|Z_\delta(t)\|_0^2 \quad \leq \quad & \|Z_\delta(0)\|_0^2 + |\hat{\mu}| \int_0^t \mathbb{E}\left[\|T_\delta\left(\nu_s^+\right)\|_0^2\right] ds \\
& +|\hat{\mu}| \left\{ \int_0^t D_s^* \delta^{\frac{-\zeta}{2}} ds + \frac{D_2 \tilde{\varepsilon}}{\delta^2} e^{-\tilde{\varepsilon}/2\delta} t + \frac{1}{\pi\delta^3} \frac{96}{\sqrt{\pi\delta^3}} e^{-\frac{1}{2\delta}} t \right. \\
& \left. + \frac{6}{2\pi\delta^3} C^2 e^{-\frac{4}{\delta}} t \right\} \\
& + \frac{\rho^2 \frac{\hat{\sigma}^2}{2}}{\frac{\hat{\sigma}^2}{2} - \rho^2} \left[\int_0^t D_s^* \delta^{\frac{-\zeta}{2}} ds + \frac{D_2 \tilde{\varepsilon}}{\delta^2} e^{-\tilde{\varepsilon}/2\delta} + \frac{2^{1/4}}{\sqrt{\pi}\delta^{5/4}} 96 e^{-\frac{1}{4\delta}} \right. \\
& \left. + \frac{6}{2\pi\delta^3} C^2 e^{-\frac{4}{\delta}} \right] t - \mathbb{E} \int_0^t \lambda(r_s) \|T_\delta\left(\nu_s^+\right)\|_0^2 ds.
\end{aligned}$$

The terms on the right hand side of the above involve an integral term $\int_0^t D_s^* ds = \int_0^t 2M_s^* + N_s^* ds$. From Chapter 5, Theorem 5.4.1 we have that for small t the expressions for M_s^* and N_s^* can blow up near the boundary. Thus we cannot find a control on the right hand side of the above equation to use Gornwall inequality to show the existence of the density in H_0.

This chapter concludes the theoretical results presented in this project. We would like to comment on a recent work by Spiliopoulos et al. [55] on fluctuations of loss due to defaults in a large portfolio limit in reduced-form model of correlated default timings. They assumed that a firm defaults at a stochastic intensity that is influenced by an idiosyncratic noise, a systematic risk factor common to stochastic intensities of all firms and the portfolio loss. They further considered that the systematic risk factor follows a diffusion process and defined an empirical distribution as in equation (4.5) of the type and intensity of the firms that are alive. They further studied the limiting behaviour of the measure valued process, defined as empirical distribution of alive firms, and defined the portfolio loss in terms of the measure valued process. The work of Spiliopoulos et al. provides a natural link between the two parts of the thesis and motivates us to provide an extension by combining our work in reduced form settings and in structural framework for a large portfolio limit in future.

In the next chapter we present the numerical procedure and results to show how to model the mortgage backed securities using some limited market data. We also present a review of ABX.HE index for mortgage backed securities and present a hypothetical way to model an ABX.HE index.

Chapter 7

Numerical implementation and calibration

In previous chapters, we developed a model for a pool consisting of the wealth of individual mortgage holders, and derived a stochastic PDE for the evolution of the pool. We further showed that there exists at most one solution in L^2 that describes the evolution of the pool given the Dirichlet boundary conditions. In this chapter we present a numerical solution to the SPDE and discuss the modelling of a mortgage pool.

In the next section we numerically implement the SPDE and present an algorithm to solve it. We further show how to calibrate the model parameters from the market data. The data on mortgage backed securities is not freely available and thus we are not able to estimate a complete set of the model parameters. However, we present a step-by-step process to estimate the different sets of model parameters. We further give a review of the market index of MBS, namely ABX.HE that consists of subprime mortgages, and present a hypothetical example to show how we can price the index using our model, provided that we have the desired set of data.

7.1 Numerical implementation

In this section we present a numerical method to solve the stochastic PDE for the evolution of the limiting empirical measure of the system of individuals' distance to the default boundary as defined in (4.4). We recall the SPDE derived in Chapter 5 provided that there exists a density of the limit empirical measure, the results on existence and uniqueness of the density were formally proved in Chapter 6. We have

$$\begin{aligned} dv(t,x) &= -\hat{\mu} v_x(t,x)\,dt + \frac{1}{2}\hat{\sigma}^2 v_{xx}(t,x)\,dt - \lambda(r_t)\,v(t,x)\,dt \\ &\quad -\rho\hat{\sigma} v_x(t,x)\,dM_t \end{aligned} \tag{7.1}$$

with

$$dr_t = \alpha(\beta - r_t)\,dt + \sigma_I \sqrt{r_t}\,dI_t, \tag{7.2}$$

where $dM_t dI_t = \xi dt$. Provided that the initial density

$$v(0,x) = \nu_0(x) = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_0^i}(x)$$

is given and the boundary conditions are

$$v(t,0) = 0, \quad v(t,1) = 0.$$

Numerical methods for this type of SPDEs are discussed by Bush et al in [8] using finite element approximation by approximating a semi-finite boundary problem for the SPDE by approximating it on a finite domain. They further discussed the convergence and stability of the numerical scheme as well as present the convergence rate for the Monte Carlo estimates for the outstanding tranche notional and tranche spreads.

We use a finite difference method to find the derivatives of the empirical density $v(t,x)$. We define a grid $x_0 = 0, x_1 = \Delta x, \ldots, x_J = J\Delta x = 1$ with $\Delta x = \frac{1}{J}$ and time steps $t_0 = 0, t_1 = \Delta t, \ldots, t_M = M\Delta t$ with $\Delta t = 1/M$. We denote the value of

v at the node (t_m, x_j) by v_j^m, that is $v_j^m \approx v(t_m, x_j) \approx v(m\Delta t, j\Delta x)$. We approximate the derivatives by finite differences at grid points using the central difference method,

$$\begin{aligned}
\frac{\partial v}{\partial x}(t_m, x_j) &\approx \frac{v_{j+1}^m - v_{j-1}^m}{2\Delta x} \\
\frac{\partial^2 v}{\partial x^2}(t_m, x_j) &\approx \frac{v_{j+1}^m - 2v_j^m + v_{j-1}^m}{\Delta x^2}.
\end{aligned}$$

We use an explicit Euler scheme to obtain a discrete form of (7.1) as

$$\begin{aligned}
v_j^{m+1} - v_j^m = -\hat{\mu}\frac{v_{j+1}^m - v_{j-1}^m}{2\Delta x}\Delta t + \frac{1}{2}\hat{\sigma}^2\frac{v_{j+1}^m - 2v_j^m + v_{j-1}^m}{\Delta x^2}\Delta t \\
-\lambda(r_m)\, v_j^m \Delta t - \rho\hat{\sigma}\frac{v_{j+1}^m - v_{j-1}^m}{2\Delta x}\Delta M_m.
\end{aligned}$$

We can compute the values of v_j^{m+1} explicitly from v_{j-1}^m, v_j^m, and v_{j+1}^m as

$$v_j^{m+1} = A_j^m v_{j-1}^m + B_j^m v_j^m + C_j^m v_{j+1}^m, \tag{7.3}$$

where, for $j \leq J$,

$$\begin{aligned}
A_j^m &= \frac{1}{2\Delta x}\left(\hat{\mu} + \frac{\hat{\sigma}^2}{\Delta x}\right)\Delta t + \frac{\hat{\sigma}\rho}{2\Delta x}\Delta M_m, \\
B_j^m &= 1 - \left(\frac{\hat{\sigma}^2}{\Delta x^2} + \lambda(r_m)\right)\Delta t, \\
&\text{and} \\
C_j^m &= -\frac{1}{2\Delta x}\left(\hat{\mu} - \frac{\hat{\sigma}^2}{\Delta x}\right)\Delta t - \frac{\hat{\sigma}\rho}{2\Delta x}\Delta M_m.
\end{aligned} \tag{7.4}$$

The finite difference scheme we used to approximate the SPDE is the usual. The scheme is always numerically stable and convergent. The errors are linear over time step and quadratic over space step [45]. We assumed that the interest rates appearing in λ follow a CIR process. Many authors have discussed positive schemes for CIR simulation including Alfonsi [1], [2] and Malham and Wiese in [42]. We simulate the CIR process using the discretization below

$$r_{m+1} = r_m + \alpha \left(\beta - r_m\right) \Delta t + \sigma_I \sqrt{|r_m|} \Delta I_m. \tag{7.5}$$

We give the definition of $\lambda\left(r_t\right)$ later in the chapter.

The boundary conditions for the evolution measure are, as before,

$$v\left(t_m, 0\right) = v\left(t_m, 1\right) = 0.$$

The numerical inductive strategy to solve (7.1) is the following.

Algorithm

Working forward in time, for $1 \leq m \leq M$

- start at $m = 0$ with initial distribution $v_j^0 = v\left(t_0, x\right) = v\left(0, x\right)$
- generate I_m and M_m at each time m
- compute r_m using (7.5) and hence $\lambda\left(r_{t_m}\right)$ using definition of λ
- compute A_j^m, B_j^m, and C_j^m using (7.4) for $j = 1, \ldots, J$
- compute v_j^{m+1} using (7.3) for $j = 1, \ldots, J$
- check for boundary conditions at each m i.e $\nu_1^m = \nu_J^m = 0$
- use v_j^{m+1} found in previous step as an initial condition for v_j^{m+2} until $m = M$.

The initial distribution is of the form

$$v\left(0, x\right) = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_0^i},$$

where X_0^i is the initial distance to default of individual i at the beginning of the contract. For numerical purposes, we calibrate the initial distribution parameters that we assumed to be a beta distribution using market data on Loan to Value (LTV) ratios, as described later. We find parameters of the beta distribution and use that as a starting point for the algorithm above, instead of finding an initial distribution in the form above.

7.2 Calibration

In this section, we will calibrate the model with market data to estimate model parameters. We present the estimation of the parameters in various steps and discuss the process of estimation for each set of parameters. We also present a review of the pricing techniques adapted from the literature on MBS.

We recall the model in (4.4) and give the parameters of the model to be estimated. The model is complicated and contains many parameters due to prepayment and default incentives associated with the mortgage backed securities. The dynamics of the i^{th} individual's wealth $(i = 1, 2, \cdots n)$

$$\begin{cases} dX_t^i = \hat{\mu}dt + \hat{\sigma}\sqrt{1-\rho^2}dW_t^i + \hat{\sigma}\rho dM_i, \quad t < \tau^i \\ X_t^i = 0, \quad t \geq \tau^i = \tau_0^i, \\ X_t^i = 1, \quad t \geq \tau^i = \tau_1^i, \\ X_t^i = \triangle, \quad t \geq \tau^i = \tau_r^i, \\ X_0^i = x^i, \quad 0 < x^i < 1, \end{cases} \tag{7.6}$$

where

$$\tau^i = \min\left(\tau_0^i, \tau_1^i, \tau_r^i\right),$$

and

$$\begin{aligned} \tau_0^i &= \inf\left\{t : X_t^i \leq 0\right\}, \\ \tau_1^i &= \inf\left\{t : X_t^i \geq 1\right\}, \end{aligned}$$

$$\tau_r^i = \inf\left\{t : \int_0^t \lambda\left(r_s\right) ds > e\right\},$$

where as before $e \sim \exp(1)$ and the interest rates r_t follow the CIR diffusion process

$$dr_t = \alpha\left(\beta - r_t\right) dt + \sigma_I \sqrt{r_t} dI_t$$

and are correlated with M_t by a correlation factor ξ. $\triangle$ is a cemetery state, $\hat{\mu}$ and $\hat{\sigma}$ are defined by (4.2) and (4.3) and

$$x^i = \frac{\log b^i - \log B_L}{\log B_U - \log B_L} \quad \text{and} \quad 0 < x^i < 1, \forall i.$$

From here we can determine different sets of parameters of our model. We have three parameters of the diffusion process of distance to default of each individual, namely, $\hat{\mu}$, $\hat{\sigma}$ and ρ. We have three parameters appearing in the CIR process for interest rates α, β, and σ_I. We assumed that the interest rates are correlated with the market factor M_t which gives rise to another correlation factor ξ. The refinancing intensity $\lambda = f\left(r_t\right)$ may also have some parameters to be estimated.

We now consider the initial distribution of the wealth of individuals in a pool of mortgages. In Chapter 4 we have seen that if t is small then the bounds on the density near the default and prepayment boundaries can blow up for any initial distribution giving us a point-wise convergence. To achieve uniform convergence we assume that the initial distribution of the distance to default processes is a tailed distribution with parameters β^1, $\beta^2 > 3$. For calibration purposes we assume that the distribution of the wealth of individuals in the mortgage pool follows a beta distribution, which gives two more unknown parameters, say β^1 and β^2.

Summing up, our model consists of 9 parameters that we need to estimate from market data, as well as the refinancing rate $\lambda\left(r_t\right)$. One problem in approximating the parameters of MBS models is that the data on mortgages in not widely available for free. In the next section we review the prior literature on the pricing of mortgage backed securities and how how the problem of scarce data is dealt with in the literature.

7.2.1 Prior literature on pricing of MBS

Pricing mortgage backed securities is a difficult task due to the complexity involved in these securities and lack of available mortgage data. The models in the literature have mostly focused on some but not all aspects of these securities. No closed form solutions to mortgages or mortgage backed securities, taking default and prepayment into account, are available. Authors have typically adapted numerical techniques to solve their models using Monte Carlo simulation. In this section, we highlight the prior literature on pricing of MBS. There is no one paper in the literature that discusses the pricing of mortgage backed securities in complete rigorous detail. Most papers discuss the numerical techniques adapted to solve the model by giving a numerical example and discussing some case studies on the effects of the parameters of the model on the price of a mortgage.

The literature on MBS consists of modelling a single mortgage or a pass through security by including one or both risk factors. The literature often discusses how to solve the model numerically by example. In [29], Kariya and Kobayashi presented a numerical example to value a 30-years MBS by fixing the parameters of interest rates and prepayment incentive in their model using Monte Carlo simulation. They further investigated the effects of changes of mean reversion level of interest rate in a Vasicek model on values of MBS, and the effects of prepayment threshold on the values of MBS. They further described an estimation process to find the parameters when the prepayment history of the pool and interest rates are known, using a least square method. Kariya et al. in [30] again adapted a Monte Carlo simulation technique for a three factor model of MBS. The parameters for spot rate, mortgage rate and house price models were chosen as well as the threshold level for prepayment to investigate the effects of different parameters on MBS prices.

Kau et al. in [33] worked with a set of values for the underlying economic parameters that are reported in the previous literature by [7] and [13]. Keeping the base parameters fixed, they numerically solved the model to find a range of reasonable values of house price and interest rate volatility parameters of their model. In [32], they discussed the effects of parameters changes on the value of mortgages using explicit finite difference technique to solve the PDE in their model. McConnell and Singh in [43] also used Monte Carlo simulation to solve the PDE in their model. They used the specifications estimated by Schwartz and Torous in [51] for the two interest rate processes, and also employed the empirical prepayment function estimated by Schwartz and Torous. Using these parameters they priced different tranches of newly issued 30-year fixed rate mortgages bearing a coupon rate of 9.5%.

A well-cited paper on the empirical results is by Schwartz and Torous [51]. In their paper they work on $30-$year single family GNMA securities over the period January 1978 to November 1987 to estimate a prepayment function. Using the data on prepayment experience of GNMA securities and the method of maximum-likelihood adapted to their model of prepayment, they established the parameters of the prepayment function. Since the prepayments hugely depend on a decline of interest rates in terms of refinancing, the interest rates process also need to be estimated using market data. In [51], they assumed two interest rate processes, r the instantaneous risk-free rate of interest, and l the yield on default-free consol. Consol is a form of British government bond typically treated as a perpetual bond. Weekly observations of annualized one-month conditional default (CD) rates from 29 December 1982 to 01 April 1987 are used to approximate the parameters of interest rate process r_s. For the same time period, weekly observations of annualized running coupon yield on long term US Treasury bonds are used to estimate the parameters of l. A running coupon yield at a particular time is the coupon rate on a newly issued US Treasury bond, if the bond is just issued, otherwise it is the coupon rate of the recently issued bond. They adapted an iterative Aitken procedure to the discrete approximations of the interest rate processes to yield maximum-likelihood estimates. They use these estimates to value default-free mortgage and further argue that due to the lack of data, empirical testing of their valuation procedures was not possible.

Dunn and McConnell in [13] discussed the valuation of Government National Mortgage Association (GNMA) mortgage backed securities and presented a comparison of these securities with other types of fixed-rate bonds. Assuming a baseline set of parameters for the numerical purpose they discussed different features of GNMA securities such as amortization, prepayment and term to maturity and their effects on the prices of the GNMA securities. They emphasised a need of an empirical study to determine that the prices generated by their model are consistent with observed market prices. In [7], Buser and Hendershott also model GNMA securities and estimate the parameters of interest rates and compare their results with [13]. They also present some sensitivity analysis of the bonds to the parameters of interest rates and refinancing costs.

To summarize, the literature on the pricing of mortgage backed securities is mostly focused on the sensitivity analysis of the parameters of interest rates and other economic variables. No compact study has been developed on the pricing or calibration of MBS where the results of MBS are compared to observed MBS prices after estimating the parameters of interest rates and other stochastic processes from market data. The lack of publicly available data on mortgage backed securities and the complexities of the securities make it difficult to calibrate an MBS model and approximate the full set of parameters.

7.2.2 Market data selection

As discussed, we can divide the parameters of the model into four categories, namely parameters for the initial density β_1, β_2, parameters for the interest rate process α, β, σ_I, rate of prepayment $\lambda(r_t)$, and the model parameters $\hat{\mu}, \hat{\sigma}, \rho$ and ξ. We need a data set from which we can find all these parameters after the calibration process.

We first consider our model's basic properties. We are considering fixed rate individual mortgages taken on a residential property with a 30−year term to maturity. This means that the data set we need to look at must be of fixed rate residential mortgages issued with a 30-year maturity. This narrows down our selection criteria by excluding the commercial and adjustable rate mortgages as well as excluding short term mortgages for example 10 or 15-years mortgages.

To find the parameters of our model, we start by looking at the parameters of the initial distribution and recall that the initial density is given by

$$\nu(0, x) = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_0^i}(x)$$

where X_0^i is the initial distance to default of the individual i. To find the parameters of the initial distribution, we need to find data on the initial distance to default for each individual. The distance to default can also be calculated by using the data on the individual's wealth at the start of the contract or his/her credit worthiness. Its highly unlikely that we can find data on the wealth of each individual in the pool of mortgages. But if we assume that the Loan to Value (LTV) ratios can represent the credit worthiness of the borrower, that is a borrower would not buy an expensive property if he would not have a big income stream, then we can use LTV ratios to find the distance to default. So to find the initial distribution we need detailed data on the LTV ratios for the individuals in the pool.

The next step is to estimate the interest rate model parameters and for this we need data on the mortgage rates for 30-year fixed rate residential mortgages. To find the rates of prepayments, we need to find data on the prepayment rates for these mortgages. Further to find the remaining model parameters we need to calibrate our model to an MBS price index. To summarize, we need data on LTV ratios, mortgage rates, prepayment rates and MBS prices for fixed rate 30-year mortgage backed securities to calibrate the parameters of our model. This type of data set is not readily available from free sources and we are restricted to what is available easily. We are therefore able to find some, but not all, parameters of our model.

We use a data set from the Federal Housing Finance Agency website. The data consists of single family non-farm homes and excludes the non-amortized and refinanced mortgages. The data further excludes the loans guaranteed or insured by Federal Housing Agency (FHA) and the Veterans Administration (VA). We are using the data on conventional single-family mortgages annual national averages on all homes. This data set consists of the contract interest rates, term to maturity, mortgage loan amount, loan to value ratio and percentage distribution of estimated number of loans by loan to price ratios. One can find the data on the website of FHA, www.fhfa.com.

We need to find the initial density using LTV ratios and the data we have for LTV is not complete. This might not give us a good fit to the initial distribution but can still give a starting point. We can then capture different economic scenarios by changing the parameters of the initial density. The data on mortgage rates can give us a good fit to the find the parameters of the interest rate process. We do not have data on the prepayment rates from FHA so define a prepayment function in the coming sections and discuss how we can find the parameters for that. The final step, after finding other parameters, is to calibrate the model to some market index of MBS to find the remaining parameters of the model. We use ABX.HE index for subprime mortgages for this purpose. We also present a detailed study of these indices later in the chapter.

7.2.3 Estimating the parameters of the model

1 Initial distribution

We assumed that the initial distribution of the individual's wealth or in particular the distribution of the distance to the default of each individual is a beta distribution. The distance to default is scaled to fall in interval $(0, 1)$ where 0 is the default boundary and 1 is the prepayment boundary.

We find the parameters of initial distribution using loan to value ratios. We assume that the LTV ratio represents the credit worthiness of the borrower in a way that a borrower would not take a loan on a large property if he does not have a large income stream. We measure the likelihood of defaults and prepayments by using LTV ratios at the initialization of the contract and associate a distance to default to each borrower according to the relation:

$$X_0^i = 1 - \left\{ \frac{\text{Loan}}{\text{Value}} \right\}_0^i$$

where $\left\{ \frac{\text{Loan}}{\text{Value}} \right\}_0^i$ is the LTV ratio at the beginning of the contract for individual i. This scales the distance to default between $(0, 1)$ and positions a borrower according to the likelihood of default/prepayment. For example if a borrower takes 85% loan on his property then he is given a position at $1 - 0.85 = 0.15$ on $(0, 1)$, closer to 0, knowing that he borrowed 85% of the property's value and is more likely to default over the life of the pool. Similarly if a borrower borrows only 10% of the property's value then he is given a position 0.9 knowing that he only needs to pay 10% of the property's value and is more likely to prepay his loan.

The data we have from the Federal Housing Agency is for the averaged Loan to

Value (LTV) ratios for the ranges 70% or less, $70-80\%$, $80-90\%$ and 90%+ and the averaged LTV of the pool for each year from 1973 to 2010. The data is weighted and by that we mean that a percentage of total mortgages, issued within a year, is given for the ranges mentioned above. We do not have data for each borrower's LTV ratio, nor do we have the total number of mortgages that are issued that year. For instance, for a given year the data we have for LTV ratios is that 26% of the mortgages have LTV ratios less than 70%, 46% of loans have an LTV between $70.1\%-80\%$, 18% of loans fall in $80.1\%-90.0\%$ and 10% have 90%+ LTV ratios. The question is how to approximate the initial distribution from this type of data. We must find two parameters of the beta distribution that results from an averaged LTV and the standard deviations from the averaged LTV ratios. Since we have the averaged LTV ratios, our aim is to find the standard deviation from the mean. We start by shifting the data set to $1-\text{LTV}$ that being our definition of the distance to default.

We use the moment matching technique for the first two moments to find the parameters of the beta distribution. We can not use maximum likelihood estimation (MLE) due to the fact that the data set we have is restrictive. We only have four numbers representing percentages of LTV ratios each year for different ranges of ratio, and it is not appropriate to apply MLE to the sample data. The data only has percentages of the LTV ratios for four intervals and there is a wide range of averaged LTV ratio for the interval (0.3,1), and we start by finding the averaged 1-LTV ratio for this interval. We label the averaged LTV ratios as x_1, x_2, x_3 and x_4 for the intervals $(0, 0.1), (0.1, 0.2), (0.2, 0.3)$ and $(0.3, 1)$, respectively, and the percentages as $\alpha_1, \alpha_2, \alpha_3$ and α_4, respectively. We then have

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = x$$

where x is the averaged (shifted) loan to value ratio. We fix the averaged LTV ratios for the first three intervals to the mean of the interval, i.e $x_1 = 0.05, x_2 = 0.15, x_3 =$

0.25 and find x_4.

We now find the mean and variance of the data for the moment matching and equate that to the mean and variance of the beta distribution. We find the variance from the data as

$$\text{Variance}_{\text{data}} = \frac{1}{4}\sqrt{\left(x - x_1\right)^2 + \left(x - x_2\right)^2 + \left(x - x_3\right)^2 + \left(x - x_4\right)^2}.$$

The density of the beta distribution with parameters α_1 and α_2 is given as

$$f\left(x, \alpha_1, \alpha_2\right) = \frac{\Gamma\left(\alpha_1 + \alpha_2\right)}{\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)} x^{\alpha_1 - 1}\left(1 - x\right)^{\alpha_2 - 1}$$

where $\Gamma\left(z\right)$ is a gamma function. We also have the mean and variance of the beta distribution as

$$\begin{aligned} \text{Mean}_{\text{Beta distribution}} &= \frac{\alpha_1}{\alpha_1 + \alpha_2} \\ \text{Variance}_{\text{Beta distribution}} &= \frac{\alpha_1 \alpha_2}{\left(\alpha_1 + \alpha_2\right)^2\left(\alpha_1 + \alpha_2 + 1\right)}. \end{aligned}$$

On equating, we obtain

$$\begin{aligned} \text{Mean}_{\text{data}} &= \frac{\alpha_1}{\alpha_1 + \alpha_2} \\ \text{Variance}_{\text{data}} &= \frac{\alpha_1 \alpha_2}{\left(\alpha_1 + \alpha_2\right)^2\left(\alpha_1 + \alpha_2 + 1\right)}. \end{aligned}$$

From the above two equations, we can find the parameters α_1, α_2 of beta distribution. The above procedure gives us an approximate distribution of distance to defaults and also gives an estimate of the initial beta distribution with $\alpha_1 \approx 14$ and $\alpha_2 \approx 44$, which also satisfies the conditions on the parameters of tailed distribution near boundaries and the second moment. This is an approximation to the initial distribution but gives a starting point. Our model is flexible enough due to the fact that by starting with a beta distribution as an initial distribution of a mortgage pool, we can capture different economic environment. For example in the case of a subprime mortgage pool, we can choose parameters of beta distribution that give a concentrated left end (with a long tail at right end) where more borrowers are at risk of default and hence are closer to the default boundary with high LTV ratios.

2 Interest rates' parameters

We start by assuming that the interest rates follow a CIR process, which is the most natural assumption for a Markovian interest rates process so as to be mean reverting and to avoid negative interest rates at any time. We have

$$dr_t = \alpha\left(\beta - r_t\right)dt + \sigma_I\sqrt{r_t}dI_t,$$

where I_t is a standard Weiner process, α, β and σ_I are constant parameters α is the speed of adjustment, β is the long term mean and σ_I is the volatility. The parameters follow the Feller condition, that is $2\alpha\beta \geq \sigma_I^2$, to avoid negative interest rates. The mean, variance and covariance of the CIR process is given as

$$\begin{aligned}
\text{Mean}_t &= \left(r_0 - \beta\right)e^{-\alpha t} + \beta \\
\text{Variance}_t &= \frac{-\sigma_I^2}{2\alpha}\left(2r_0 - \beta\right)e^{-2\alpha t} + \frac{\sigma_I^2}{\alpha}\left(r_0 - \beta\right)e^{-\alpha t} + \frac{\beta\sigma_I^2}{2\alpha} \\
\text{Covariance}_{t,s} &= \frac{\sigma_I^2}{2\alpha}\left\{2\left(r_0 - \beta\right)\left(e^{-\alpha t} - e^{-\alpha(s+t)}\right) + \beta\left(e^{-\alpha(t-s)}\right.\right. \\
&\quad \left.\left.-e^{-\alpha(t+s)}\right)\right\} \qquad \text{for } s < t
\end{aligned}$$

Using the stationarity of the CIR process and letting $t - s = k$ as constant and $t \to \infty$, we have

$$\begin{aligned}
\text{Mean}_{stationary} &= \beta \\
\text{Variance}_{stationary} &= \frac{\sigma_I^2}{2\alpha}\beta \\
\text{Cov}_{stationary}\left(r_t, r_s\right) &= \frac{\sigma_I^2}{2\alpha}\beta e^{-\alpha(t-s)}
\end{aligned} \tag{7.7}$$

The interest rate is a 30 year forward rate and we use the data on monthly mortgage rates from FHA website to find this set of parameters. We consider the data on monthly fixed rate mortgages on single family homes from 1986 to 2010. Using the moment matching technique and finding the mean from the data, we have $\beta = 0.0778$. We also have

$$\mathrm{Corr}_{stationary}\left(r_t, r_{t-k}\right) = e^{-k\alpha}$$

and obtain $\alpha = 0.33$ and $\sigma_I = 0.0502$.

3 Parameters of prepayment intensity

We now choose the function for the prepayment intensity. The factors that affect prepayments are excess money and interest rates as discussed in previous chapters. Since we are not considering partial prepayments, we say that a borrower chooses to prepay if he/she gets enough money to pay the remaining principal balance. The prepayment due to refinancing occurs when the mortgage interest rates go below a certain level giving the borrower the incentive to pay back the current principal remaining and take the loan at a lower interest rate. To accommodate these two types of prepayment, we choose

$$\lambda\left(r_t\right) = \gamma\left(K - r_t\right)^+, \tag{7.8}$$

where K and γ are two constants. If the interest rate falls below the threshold level K then we assume that the borrower refinances, that is prepays all the remaining principal. $\gamma > 0$ is the constant intensity scaled parameter specifying that if there is a decline in interest rate, there is a proportional increase in prepayment intensity.

We need to find the two parameters of the refinancing incentive appearing in the second term of equation (7.8). The threshold level K is related to the contract interest rates and since we are considering fixed rate mortgages the contract interest rates are fixed at the initialization of the contract. We fix $K = r_0 - s$, where r_0 is the fixed interest rate of the contract and s is the transaction fee and is set to 3% that is refinancing will occur at a certain rate when the interest rate falls by more than 3% from the initial rate. We chose a 3% refinancing fee in accordance with Gorovoy and Linetsky [23] who proposed the prepayment intensity we are using. They argued that it is a typical refinancing break even point according to the data on the web that discusses mortgage refinancing.The threshold level K can vary from individual to individual but we assume it is the same for each individual in our model. We still need to find the constant γ that is the proportion to which the decline in interest rates effect the refinancing. Since we do not have the data on prepayment for the same set of mortgages, we fix a certain value of γ for our numerical work.

4 Model parameters

To find the remaining set of parameters, that is the model parameters, we need to calibrate our model to the prices of MBS or some market index for MBS. As discussed before, we are unable to find the data on the prices of the MBS that is publicly available on free sources. We use ABX.HE index of subprime mortgages to calibrate to find parameters. In the next section we discuss ABX.HE index in details and present a pricing formula to price these indices.

7.3 ABX.HE indices

To find the remaining parameters of the model, we hope to calibrate the model to a subprime mortgage index ABX.HE for the residential mortgage backed securities. ABX.HE was launched in January 2006 under the administration of Markit Group and became a benchmark for market pricing of subprime mortgage related securities. Each index tracks prices of credit default swaps written on a fixed basket of 20 equally weighted US subprime RMBS pools issued within the past six months of the index launch. Each vintage of ABX.HE index consist of five subindices that correspond to tranches of different ratings from each of the twenty equally weighted

RMBS of rating categories: AAA, AA, A, BBB, and BBB.$^{-}$ A new Penultimate AAA (PENAAA) was introduced in later years to provide additional pricing information. Only four series of ABX.HE came into life (from January 2006 to July 2007) after which due to the subprime mortgage crisis, nothing further materialized. The four currently outstanding vintages of ABX.HE indices are labeled ABX.HE-2006-1, ABX.HE-2006-2, ABX.HE-2007-1 and ABX.HE-2007-2 respectively.

In the following subsections we will give a complete description of ABX.HE index, its construction, how it works and the cash flow of the index. In the next section will return to our calibration process and will calibrate the model to this index to find the remaining parameters.

7.3.1 Construction of the ABX.HE index

ABX.HE index is administer by Markit Group. The construction of each vintage of ABX.HE index begins by selecting 20 underlying subprime RMBS on the basis of a set criteria. Some of the points in the set criteria are listed below.

- A deal must be of at least $500 million at the issuance
- An average FICO score(Fair Isaac Corporation score, a number representing the credit worthiness of a person) of a deal is less than or equal to 660
- A deal must consists of at least 90% first lien loans
- The tranche average life for AAA tranche must be greater than 5 years and average life for the levels below AAA are restricted to 4-6 years at the issuance
- Each required tranche must have been issued within the six months prior to the applicable roll date (currently 19 January and 19 July)
- No more than four deals can be included with loans from the same originators
- No more than six deals can be included with the same master servicer

Index	Coupon	Price	Current Factor
ABX.HE.PENAAA.06-1	18	82.94	0.110894018
ABX.HE.AAA.06-1	18	88.65	0.714698666
ABX.HE.AA.06-1	32	48.77	0.95
ABX.HE.A.06-1	54	20.49	0.717146449
ABX.HE.BBB.06-1	154	7.03	0.270773929
ABX.HE.BBB-.06-1	267	7.14	0.148766852
ABX.HE.PENAAA.06-2	11	75.13	0.608144331
ABX.HE.AAA.06-2	11	50.63	0.964172374
ABX.HE.AA.06-2	17	17.74	0.82282109
ABX.HE.A.06-2	44	7.28	0.278545536
ABX.HE.BBB.06-2	133	7.57	0.1
ABX.HE.BBB-.06-2	242	7.04	0.082275134

Figure 7.1: Source: Markit

One can find a complete set of eligibility rules on Markit.com, in [56] and [16]. Once an index is created, the composition of the index remains static which means once the index is created there will be no substitution of reference obligations. The minimum size requirement of $500 million on each of the 20 deals gives the size of the each index series to be at least $10 billion subprime mortgage exposure at issuance.

7.3.2 How the index works

The ABX indices trade on price rather than in spread terms with a pre-determined fixed coupon that is fixed before the launch of the index. Each index tracks the price of a single credit default swap (CDS) that trades with the standardized template based on the Pay-As-You-Go (PAUG) structure of the International Swaps and Derivatives Association (ISDA). Entering into an ABX.HE index contract is analogous to buying or selling insurance on the underlying tranches.

ABX prices are determined by two payment legs as in the case of CDS contracts. The protection buyer agrees to pay fixed monthly premium to the protection seller, equal to a fixed multiple (fixed before the launch of the index) of the outstanding principal notional. In addition to the monthly premium, the protection buyer pays an upfront fee as a percentage of the outstanding notional to the protection seller if the quoted price is below par when they enter into a transaction. For example, a quoted price of \$35 means that the protection buyer is paying (Par - 35)% of the outstanding notional to the protection seller to enter into the contract. On the other hand, if the quoted prices are above par then the protection seller pays the initial payment to the protection buyer.

Since the premium rate is fixed before the launch of the index, the market prices of the index adjust to reflect changes in risk aversion or the default risk assessment of the market. A price below par means that the market cost of protection has increased since the index was first launched and an increase in quoted price means the implied spreads have tightened.

The protection seller pays the protection buyer when any interest or principal shortfalls or write-downs on the referenced obligations each period, called the floating leg. Unlike the corporate CDS, the index contract does not terminate when any of these credit events occur rather it continues with a reduced notional amount until maturity. The protection buyer reimburses the protection seller if the credit event reverses.

The prices of all the ABX series with the fixed premium rates for May 25,2011 are shown in (7.1). The first column indicates the name of the ABX.HE series, ABX.HE.AAA.06-1 is the ABX.HE series of AAA rankings issued in first quarter of 2006, namely 16th January 2006. The second column is the fixed coupon rate for each series, third column shows the price of each ABX.HE series on May 25, 2011. Current factor is the portion of outstanding principal of ABX.HE which is initially set to one.

TABX.HE.BBB Tranches	Coupon	Price	TABX.HE.BBB- Tranches	Coupon	Price
0-3	500	15.45	0-5%	500	11.91
3-7	500	17.59	5-10	500	13.68
7-12	500	19.62	10-15	500	14.18
12-20	467	21.05	15-25	500	15.00
20-35	200	21.46	25-40	267	14.65
35-100	51	25.97	40-100	72	15.27

Figure 7.2: Source: Markit

7.3.3 Tranched ABX.HE index (TABX.HE)

TABX is the tranched ABX index, which began trading in February 2007 to promote the transparency and standardization in ABX series. The index tranches consist of similar rating classes of the two most recently issued ABX series on 40 obligations (20 obligations in each series). The first set of referenced obligations to be tranched were the ABX-06-2 and ABX-07-1 indices. The tranched ABX index is only introduced for the BBB and BBB- rated ABX classes. The TABX.HE.07-1.06-2.BBB refers to the tranches consisting of the underlying series of BBB rankings for ABX.HE-06-2 and ABX.HE-07-1. The tranches of BBB and BBB-indexes are defined by the attachment and detachment points (the points of exposure to the capital structure). The prices of these tranches for the first series as of 11 September 2007 are shown in Figure 7.2.

7.3.4 Literature on ABX.HE

There exists research on the pricing techniques of ABX.HE indices. Early research focused on the mechanics of the index and hedging strategies. Fender and Scheicher [16] at the Bank of International Settlements have analysed ABX prices to study the importance of different pricing factors and used regression analysis to establish the relationship between the observed index returns and macroeconomic news. They found that the subprime mortgage securities are undervalued by as much as 60% based on corresponding write-downs on the ABX.HE index by reporting the results of a simplified CDS valuation exercise.

In 2008, Garcia and Goossens [18] presented one factor models for the ABS correlation market pricing of tranched ABX -TABX defined in section 7.3.3. They used Gaussian copula and the Levy base correlation method to price the tranches. They argued that there is no accepted standard approach to determine the prepayment for pricing purposes and used the remittance reports on prepayments.

Stanton and Wallace [57], in 2011, found that the prices for the ABX.HE indices are inconsistent with any reasonable mortgage default assumptions. On observing the changes in the credit performance of the underlying loans in the ABX.HE index they reported that the ABX.HE price changes are very weakly correlated with the changes in the credit performance of the underlying loans.

Other work includes Longstaff [40], who presented an empirical study of the pricing of subprime CDO and using data on ABX indexes of subprime CDO found a strong evidence of contagion effects. A complete description of ABX index and valuation can also be found in the Frank J Fabozzi Series [22].

Remark

Despite the newness and complexities of the index, it accurately predicted the recent financial crisis. Figure 7.3 shows prices of ABX.HE.06-1 and DOW Jones Industrial Average indices, normalised to 100 as of ABX.HE.06-1. The graph shows data from the date ABX.HE index started trading. Each significant drop in ABX.HE index coincides with a drop in the DOW Jones index. The first drop in ABX.HE indices is somewhere around March 2007 after which there is a steady fall in ABX.HE indices whereas the general market was convinced that the subprime market was capable of recovering and no dropping down of the DOW Jones index was seen just after March. The second significant drop can be seen in August 2007 when the subprime crisis intensified and the first liquidity crunch occurred in Europe. The DOW Jones index

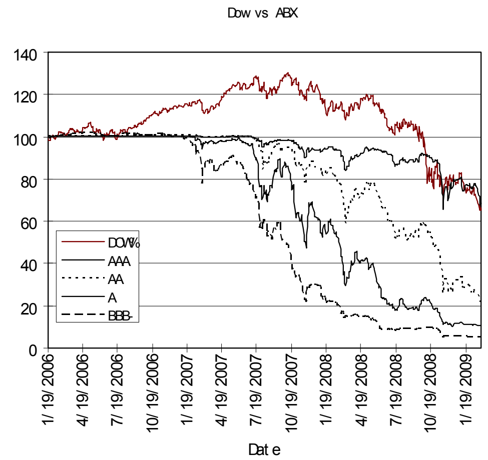


Figure 7.3: A comparison of Dow Jones and ABX.HE.06-1 indices

also saw a significant fall. Later drops in both indices includes the near-collapse of Bear Stearns in March 2008 and the Lehman Brothers default in September 2008.

7.3.5 The ABX.HE pricing formula

As described in section 7.3.2, the buyer of an ABX.HE index pays a monthly premium plus a one-time upfront fee in exchange for the floating payments from the protection seller in the event of defaults. For a CDS contract to be fairly priced at any time t, the present value of the fixed leg plus the single upfront payment paid by the protection buyer must equal to the present value of the floating leg paid by the protection seller. That is

$$\frac{Z_t\left(Par - Price_t\right)}{100} + s\sum_{i=i_t}^{n}\mathbb{E}\left(e^{-\int_t^{T_i} r_s ds}\left(Z_{T_{i-1}}\right)\right) =$$
$$\sum_{i=i_t}^{n}\mathbb{E}\left(e^{-\int_t^{T_i} r_s ds}\left[Z_{T_{i-1}}\left\{1 - \text{Prepayment}_{T_i}\right\} - Z_{T_i}\right]\right). \tag{7.9}$$

The terms on the left-hand side are a sum of the upfront fee and fixed monthly premium the protection buyer pays to the protection seller. The first term is the upfront fee that a protection buyer pays at time t as a percentage of the outstanding notional where Z_t is the outstanding notional at time t. The second term is the fixed premium leg that the buyer of the protection pays to the protection seller monthly. It comprises a fixed coupon payment each month T_i starting at time T_{i_t} on the outstanding notional $Z_{T_{i-1}}$ each month where $i_t = \inf\{j : T_j > t\}$ that is the end of the month containing date t.

The right-hand side is the value of the floating leg of ABX.HE CDS, paid by the protection seller at each date T_i to compensate for any losses due to interest and principal shortfalls to the protection buyer. It is comprised of the difference of the outstanding notional at two consecutive months T_{i-1} and T_i adjusting for any prepayments that occurred during the month from T_{i-1} to T_i. $Z_{T_{i-1}}\left\{1 - \text{Prepayment}_{T_i}\right\} - Z_{T_i}$ is the loss due to default over the time interval (T_{i-1}, T_i), where Prepayment_{T_i} is the fraction of the start-of-month principal prepaid during the month. Since we assumed that all the contracts expire at the same time after 30 years if borrowers have not prepaid or defaulted, we do not need to adjust the formula for scheduled amortizations. At the launch of the index, that is at $t = 0$, the upfront fee for the protection is zero as the market price of the ABX.HE equals par.

$$Z_t = \max(d - L_t, 0) - \max(a - L_t, 0), \tag{7.10}$$

where a and d are the attachment and detachment points of a tranche, respectively. To simulate the pricing formula, we work on one realisation of the market factor and interest rates numerically. We first simulate the loss function using (5.27) on each realisation. Once we have L_t, we find the outstanding notional Z_t using (7.10) at the monthly payment date s and averaging over the Monte Carlo simulation paths with formula (7.9) gives us the price of ABX index.

7.4 How to calibrate the model to ABX.HE indices

In this section, we discuss a strategy to calibrate the model to the ABX.HE index. Our aim is to find a way to estimate the remaining set of parameters, that is

$\hat{\mu}$, $\hat{\sigma}$, ρ and ξ. We conducted a sensitivity analysis to see the dependence of model parameters on ABX.HE prices and found out that ξ has a negligible effect on the prices. Therefore, we assume that $\xi = 0$, that is the correlation between the interest rates and the market factor is zero in the calibration process. Once we have an estimated set of parameters, we can use it to find prices of other types of MBS products.

The ABX.HE index is formulated by using equally weighted 20 underlying subprime residential mortgage backed securities (RMBS). We assume that all the underlying RMBS are driven by the same economic factor and that the pool for each MBS behaves similarly, starting with different initial distributions and the averaged weighted pool of these RMBS represents an ABX.HE pool. Starting with a known initial beta distribution at time $t = 0$, representing the initial state of an ABX.HE index pool, we work forward in time. We only need the initial distribution to start with if we enter the contract at time $t = 0$ but the problem is a bit more complicated as we can enter the contract at any time t after the initialisation of the index. This implies that if we enter the contract at any time t working forward in time to calibrate the model parameters from the index prices, we need to have a density of the pool at time t to start the calibration process. We would like to emphasize that the numerical technique presented here is ad hoc and not well defined. A more sophisticated statistical model such as maximum likelihood estimation to find the unknown parameters can be developed for a more comprehensive statistical analysis of parameters and estimation procedure.

We present a two-step process of calibration for recovering the model parameters from the observed data. Figure 7.4 shows the two steps that are involved in this process. We estimate the parameters in two steps due to the fact that one can enter the contract at any time once the index is issued and not only at the initialisation of the index. That is if the index is initialised at time $t = 0$, the investors can buy or sell protection at any time $t > 0$. The protection buyer and seller settle the difference at the time they enter the contract. We present a two-step procedure to calibrate the model parameters due to the fact that we do not have any information at time t when the investor enters the contract after the index was initialised. We start by estimating the initial distribution for the ABX index at time $t = 0$. We further assume that we can observe a loss function for $0 < s < t$ and using this information, we can have a distribution of ABX index at any time t when the investor wishes to enter the contract. At the current time t, we are therefore in a position to enter the contract and run our model forward in time to estimate the parameters of the model provided the information on the observed losses and the initial distribution of the ABX index.

Step I

We start by estimating the initial distribution from the data on LTV ratios or from the data on the ABX.HE index at time $t = 0$. We observe a loss function $L_t^{obs}(\hat{\mu}, \hat{\sigma}, \rho)$ over a interval time $0 < s < t$ where the Loss $L_0^{obs}(\hat{\mu}, \hat{\sigma}, \rho) = 0$ at time $t = 0$. Given the information about the initial distribution and observed loss, the first step is to find an approximate density of the pool at time t. That is

$$\begin{aligned}
&\text{Given } v(0,t) && x \in [0,1] \\
&\text{Observe } L_s(\hat{\mu}, \hat{\sigma}, \rho) && 0 \leq s \leq t \\
&\text{Construct } v(t,x), && \forall x
\end{aligned}$$

We can achieve it by starting with an initial distribution (estimated from the initial market data) and using a weighted least squares method to find the best fit for the loss function (numerically) to the observed loss function L_t^{obs}. Once we have the parameter estimates we can evolve the density forward in time up to t. This can give us an estimate of the density $v(t,x)$ at time t to evolve forward in time in step II. We now investigate the loss function and using the least squares method we can estimate the unknown parameters empirically from the data on the loss. We describe the process in the following.

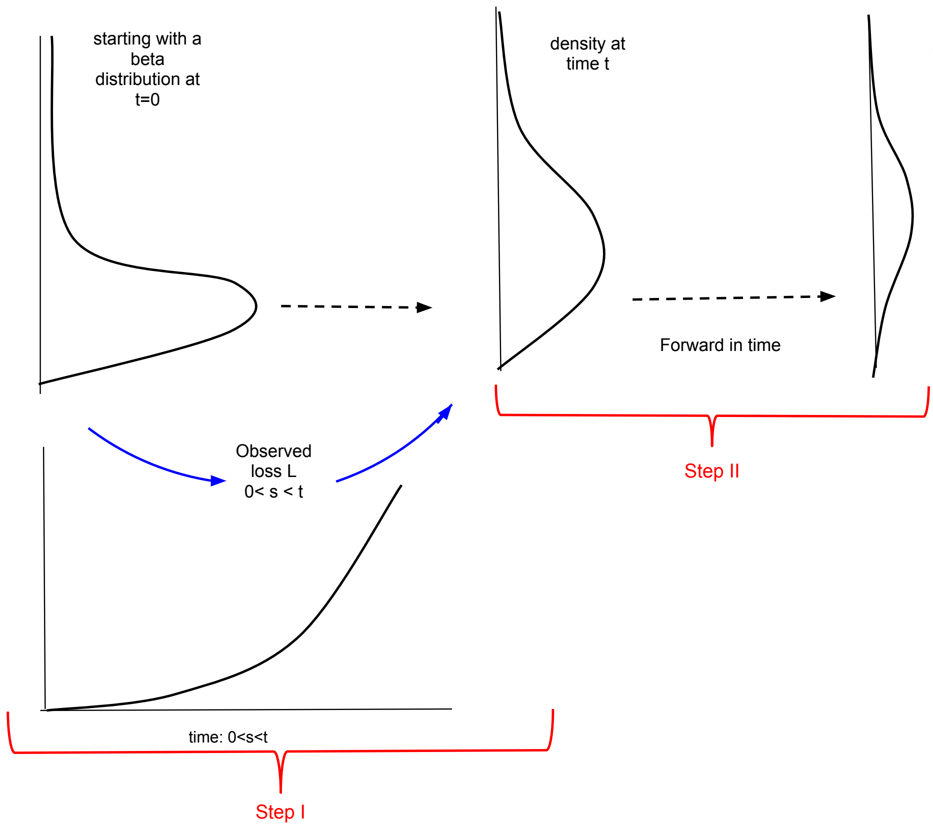


Figure 7.4: Calibration Idea

Finding parameters empirically

Assume that we have data on the loss, L^{obs}, for time $0 < s < t$ from the market and we know the parameters of the initial beta distribution. Our aim is to find a density of the pool at time t. We do this by using the least squares method on the observed loss and numerically generated loss L_s for $0 < s < t$ to find a set of empirical parameters of the model. Running the program with these parameters will give us an approximate density at time t. The idea is to minimise the sum of the squares of $\frac{\mathbb{E}[L_s]-L_s^{obs}}{\mathbb{E}[L_s]}$ for each $s \in [0,t]$ where the loss function L_s is numerically calculated using (5.27). The reason we are using expectations of loss L_s for our numerically calculated loss function is because of the randomness in the numerically calculated density. We wish to minimise the effects of the randomness due to a stochastic density in loss function when estimating model parameters.

From Figure 7.5 we can see that $\mathbb{E}\left[L_t\right]$ does not depend on ρ and we need to use

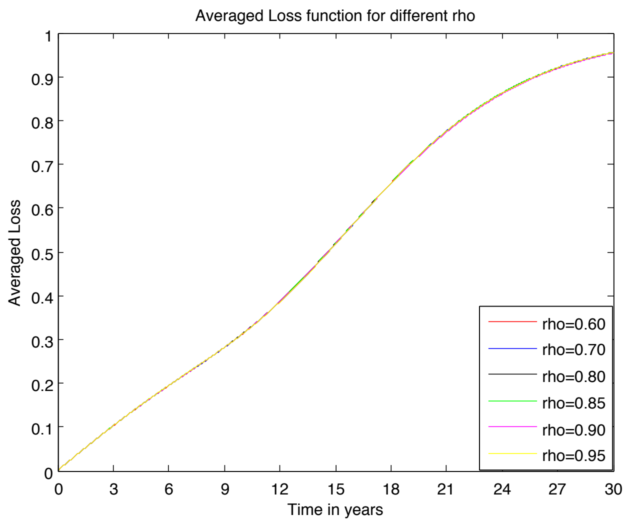


Figure 7.5: Averaged Loss function against ρ

some other function of loss that depends on ρ such as the auto-covariance of the loss. This can also be seen from the SPDE in (7.1) as by taking expectations, dM_t terms becomes zero and $\mathbb{E}\left[v\left(t,x\right)\right]$ becomes independent of ρ.

We write the algorithm for the least square method to find $\hat{\mu}$ and $\hat{\sigma}$:

- Observe Loss L_i^{obs} for $i = 1 \cdots n$
- Let $\hat{\mu}, \hat{\sigma}^2 \in \left(\underline{\hat{\mu}}, \bar{\hat{\mu}}\right) \times \left(0, \bar{\hat{\sigma}}^2\right)$ be a space of possible $\hat{\mu}$ and $\hat{\sigma}$ such that $\mid \underline{\hat{\mu}} \mid$, $\bar{\hat{\mu}}$, $\bar{\hat{\sigma}}^2 < \infty$
- For each combination $\hat{\mu}$ and $\hat{\sigma}$, find weighted sum of least squares using

$$\text{Sum of least square}\left(\hat{\mu},\hat{\sigma}\right) = \sum_{i=1}^{n}\left(\frac{\mathbb{E}\left[L_i\left(\hat{\mu},\hat{\sigma}\right)\right] - L_i^{obs}\left(\hat{\mu},\hat{\sigma}\right)}{\mathbb{E}\left[L_i\left(\hat{\mu},\hat{\sigma}\right)\right]}\right)^2$$

- $\{\hat{\mu},\hat{\sigma}\}$ at which the minimum is attained are the estimated parameters.

The division by $\mathbb{E}\left[L_i\left(\hat{\mu},\hat{\sigma}\right)\right]$ instead of the variance of $L\left(\hat{\mu},\hat{\sigma}\right)$ is used as the deviation from mean $L\left(\hat{\mu},\hat{\sigma}\right)$ is negligible and is zero for first few years of the contract as can be seen in Figure 7.6. Since we are observing losses for $t < 6$-years for calibration, we can safely use $\mathbb{E}\left[L\left(\hat{\mu},\hat{\sigma}\right)\right]$ instead of variance of $L\left(\hat{\mu},\hat{\sigma}\right)$ in the weighted least square method.

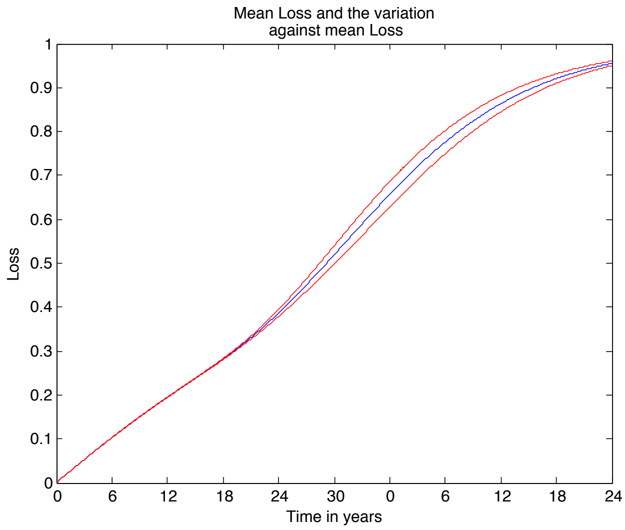


Figure 7.6: Mean loss function and the $\pm$ variance of the loss function

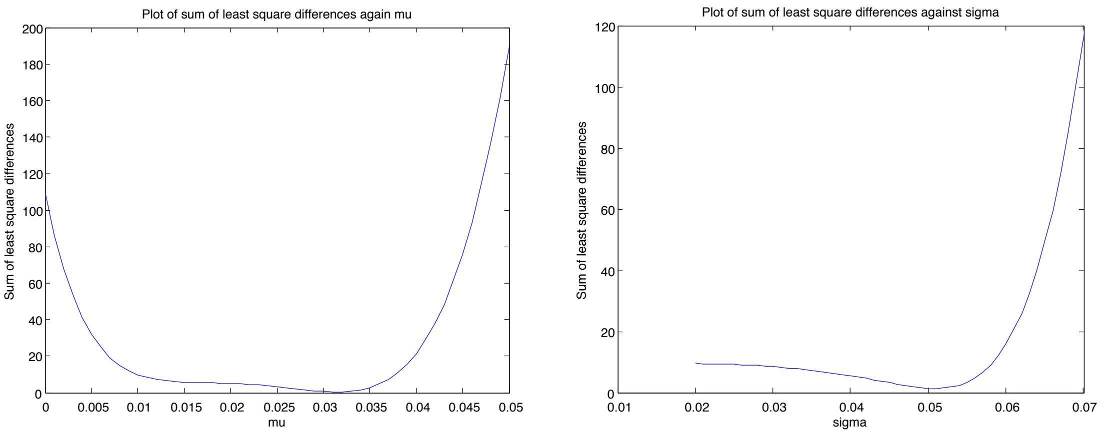


Figure 7.7: Sum of least square against $\hat{\mu}$ and $\hat{\sigma}$ for an assumed empirical loss L^{obs} using $\hat{\mu} = 0.03\,\hat{\sigma} = 0.05$ and $\rho = 0.5$.

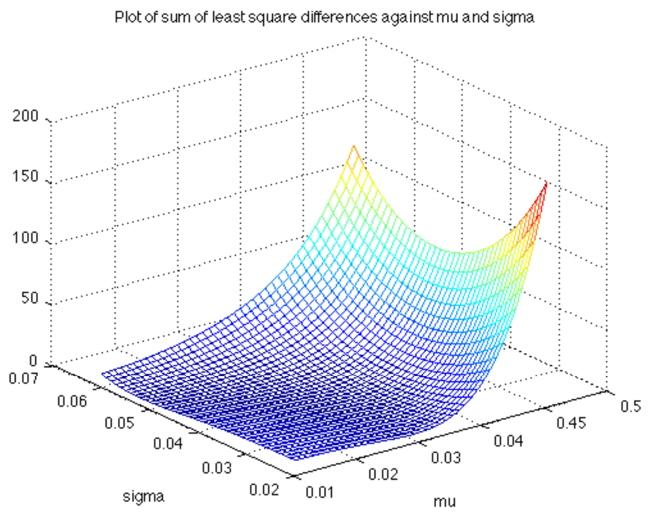
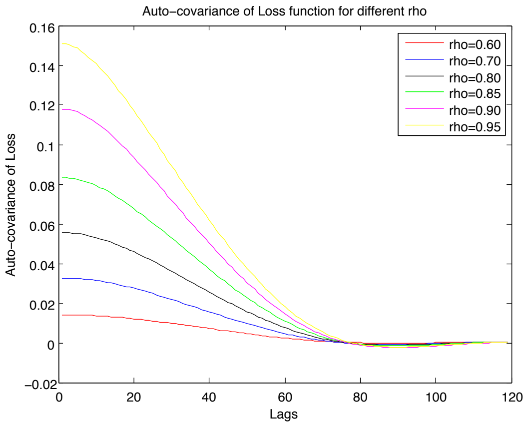


Figure 7.8: Sum of least square differences against $\hat{mu}$ and $\hat{\sigma}$

We show a sum of least squares against $\hat{\mu}$ by keeping $\hat{\sigma}$ and ρ fixed to show that we can hope to find a minimum of the sum of squares to find $\hat{\mu}$. We also show the sum of least squares for $\hat{\sigma}$ by keeping other parameters constant in Figure 7.7. The sum of least squares is checked on a example where we numerically find a loss function for a given set of parameters to use as an empirical loss function. We assume L_s^{obs}, $0 \leq s \leq t$ is for $\hat{\mu} = 0.03$, $\hat{\sigma} = 0.05$ and $\rho = 0.5$. We then try to estimate these parameters using the algorithm described above for $\hat{\mu}$, keeping $\hat{\sigma}$ constant. A similar algorithm is valid to find an estimate of $\hat{\sigma}$. Figure 7.7 shows the least square differences against $\hat{\mu}$ and $\hat{\sigma}$. We found that we can estimate the parameters from the figure. The estimated parameters $\hat{\mu} = 0.031$ and $\hat{\sigma} = 0.05$ are very close to the assumed ones. We also show a two dimensional space of $\{\hat{\mu}, \hat{\sigma}\}$ in Figure 7.8.

Finding an estimate for ρ is more difficult as $\mathbb{E}[L_t]$ is not a function of ρ. We need to find a function of loss that depends on ρ to estimate it from the data. We find out that the auto-covariance of loss is a function that depends on ρ, see Figure 7.9. Therefore we can use sum of least squares on auto-covariance of loss function to find ρ using the same algorithm for the sum of least squares as described above.



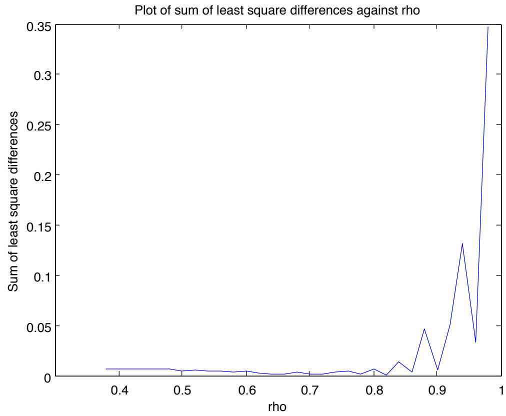


Figure 7.9: i) Auto-covariance of Loss against ρ, ii) Sum of least square of auto-covariance of loss against ρ

The weighted auto-covariance function we used is

$$\text{autocov}\left(L_s, L_{s+h}\right)) = \frac{1}{n-h} \frac{\sum_{i=1}^{n-h} \left(L_s - \mathbb{E}L_s\right)\left(L_{s+h} - \mathbb{E}L_{s+h}\right)}{\left(\mathbb{E}L_s\right)\left(\mathbb{E}L_{s+h}\right)},$$

for $n = 1, ..., N$, where N is number of monthly payments over the time $0 < t \leq T$. We can see from the figure that some more sophisticated statistical model is required to see the effects of ρ clearly and to find an estimate for ρ.

Step II

Step I gives us an empirical set of parameters for the density at time t. In step II, we evolve the pool to find an implied set of parameters. At any time T we have market prices of an ABX.HE index. We evolve the density of ABX.HE pool forward in time and match the prices generated by the model to the market prices of the index. We use the pricing formula for ABX.HE index in (7.9) to find the prices of the index using the model. Matching market prices $\text{Price}_{\text{market}}$ with the prices produced by our model will give us an estimated set of parameters $\hat{\mu}$, $\hat{\sigma}$, and ρ using least square method. The market prices of the index are nonlinear and depend on $\hat{\mu}$, $\hat{\sigma}$ and ρ. We use a least square method on a three dimensional space $S = \{\hat{\mu}, \hat{\sigma}, \rho\}$ of possible values of the unknown parameters and find the minimum of the sum of square of $\left(\text{Price}_T\left(\hat{\mu}, \hat{\sigma}, \rho\right) - \text{Price}_{\text{market}}\left(\hat{\mu}, \hat{\sigma}, \rho\right)\right)$.

Remarks

1. The dependence of loss on ρ is difficult. We may not be able to find a small $\rho < 0.50$ using the loss function, as it is eventually flat for $\rho < 0.50$. We can see from Figure 7.9 that for $\rho = 0.60$ the covariance function is almost flat. We can also see it from the SPDE as the noise is multiplied by $\hat{\sigma}$ in the dM_t term and for a small $\hat{\sigma}$ and ρ the combined effect is even smaller and makes it harder to estimate ρ.

2. The least square method we described here to find empirical and implied set of parameters from the data does not give an ideal fit. A more sophisticated statistical model such as maximum likelihood estimation to find the unknown parameters would be developed for a more comprehensive analysis of parameters and estimation procedure. The MBS pool model is quite complicated and using maximum likelihood and/or other statistical models can be numerically intensive and is a project for future research.

7.5 Pricing example

In this section, we present a pricing example using the pricing formula (7.9) to price an ABX.HE index. We use a set of parameters accordingly to produce prices that are comparable to the market prices. Since the ABX.HE index is an index of subprime mortgages, we therefore used an initial distribution that represents such a pool that is an initial distribution with a mass closer to the default boundary. The performance of the index over the last six years has shown that most of the underlying mortgages defaulted, so we use a set of parameters that induce this behaviour in the model to match the current market environment.

We check the model on ABX.HE-07-01 index to find the prices. The current prices of the index and the prices generated by the model are presented in table (7.10) with factors for two different dates. We enter the contract on 11 July 2011

ABX.HE-07-01	BBB-	BBB	A	AA	AAA	PENAAA
Fixed Coupons (in bps)	389	224	64	15	9	9
Factors (11 July 2011)	0	0.029	0.174	0.62	0.96	0.98
Factors (10 July 2012)	0	0	0.0876	0.403	0.927	0.934
Market Prices (10 July 2012)	1.67	4.04	4.50	5.44	41.03	52.30
Simulated Prices	--	3.85	4.58	14.63	40.36	101.93

Figure 7.10: Comparison of ABX.HE.07-01 Prices

with an assumed density at time t to evolve forward in time with some model parameters. We find the prices for 10 July 2012, after one year, and check the prices generated by the model with the market prices. The parameters used are $\hat{\mu} = -0.22$, $\hat{\sigma} = 0.5$ and $\rho = 0.84$. $\hat{\mu}$ determines the speed and direction of the density evolution. A negative $\hat{\mu}$ is used so that the pool evolves towards the default boundary contrary to the usual environment, where we assume that more mortgages would be paid off over time with a positive $\hat{\mu}$.

The factors on 11 July 2011 are used to find the attachment and detachment points of the six tranches, and using the parameters stated above, we evolved the density forward. We see that even with such approximations, we are able to find prices that are comparable to the market prices. Since there is nothing left in BBB- tranche, we can not find a price for this tranche. The prices for BBB, A and AAA are very close to the market prices for these tranches.

Remark

ABX.HE index is a weighted index of 20 underlying pools of mortgages with different masses and characteristics. This could be done by averaging 20 pools with different initial densities and different model parameters for each pool. This complicates the problem as it is not clear that how the resulting weighted pool would depend on the model parameters. One reason that these prices are not very close to market prices could be that we used only one pool with one set of parameters to find the prices of the index.

In this chapter, we presented an algorithm to solve the SPDE, and demonstrated the flexibility of the model for a subprime mortgage index. The calibration of the model to financial data is discussed in the second parts of the chapter. We discuss the different parameters and demonstrate how many can be fitted to observed data. Finally, for the key model parameters, we present a strategy to estimate them given observations of the loss function. We then use this to determine implied model parameters using the ABX.HE subprime mortgage index.

Chapter 8

Future work and conclusion

The thesis established a basic model for a pool of fixed rate residential mortgages. A mortgage pool is a collection of a large number of individual mortgages. We considered the wealth of each individual as a stochastic process and defined the empirical measure of a pool of these individual mortgages. We introduced many assumptions while modelling the mortgage backed securities. We therefore consider the possible extensions that could be done in future. We briefly discuss the extensions to the diffusion process of individual wealth process and to more general MBS.

8.1 Possible extensions to the model

8.1.1 Extended Brownian motion model

In our model, we considered constant coefficients of the differential equation for the wealth of individuals. We can extended the model to use more general functions in the coefficients as below.

$$\begin{cases} dX_t^i = \hat{\mu}\left(t, X_t^i\right) dt + \hat{\sigma}\left(t, X_t^i\right) \sqrt{1-\rho^2\left(t, X_t^i\right)} dW_t^i \\ \qquad\qquad +\hat{\sigma}\left(t, X_t^i\right) \rho\left(t, X_t^i\right) dM_t, \quad t < \tau^i \\ X_t^i = 0, \quad t \geq \tau^i = \tau_0^i, \\ X_t^i = 1, \quad t \geq \tau^i = \tau_1^i, \\ X_t^i = \triangle, \quad t \geq \tau^i = \tau_r^i, \\ X_0^i = x^i, \quad 0 < x^i < 1, \end{cases} \tag{8.1}$$

where as before,

$$\tau^i = \min\left(\tau_0^i, \tau_1^i, \tau_r^i\right),$$

and

$$\begin{aligned} \tau_0^i &= \inf\left\{t : X_t^i \leq 0\right\} \\ \tau_1^i &= \inf\left\{t : X_t^i \geq 1\right\} \\ \tau_r^i &= \inf\left\{t : \int_0^t \lambda\left(s, r_s\right) ds > e^i\right\}. \end{aligned}$$

This is realistic as the drift, volatility and correlation in the individual depend on time and can change over time with a change in X_t. The existence and uniqueness of the solution of above system follows from the assumption of Lipschitz and linear growth conditions on the functions $\hat{\mu}\left(t, X_t^i\right)$, $\hat{\sigma}\left(t, X_t^i\right)$ and $\rho\left(t, X_t^i\right)$. [28] discussed the extension in the context of credit modelling and the result for MBS pools can follow directly from there. Lototsky in [41] established the existence and uniqueness of a second order parabolic stochastic partial differential equation in a sufficiently smooth bounded domains. He also presented the results on existence and uniqueness of a SPDE and showed that there exists a classical solution in C^2 on a bounded domain $I = (k_1, k_2)$ with Dirichlet boundary conditions provided that the initial condition is smooth enough.

8.1.2 Partial prepayments as jumps

Allowing partial prepayments in the model is also a natural extension as mostly mortgages give the right to individuals to pay back any amount during the life of contract. This can be done by adding a small jump process that jumps up (towards the prepayment barrier) with a magnitude that is proportional to the amount that is prepaid by each borrower. Each jump towards the prepayment barrier shows that the borrower is closer to prepaying the whole borrowed amount. This can be done by adding a jump to the diffusion process of the wealth of each individual.

$$\begin{cases} dX_t^i = \hat{\mu} dt + \hat{\sigma}\sqrt{1-\rho^2} dW_t^i + \hat{\sigma}\rho dM_t + \hat{J}^i dN_t^i, \quad t < \tau^i \\ X_t^i = 0, \quad t \geq \tau^i = \tau_0^i, \\ X_t^i = 1, \quad t \geq \tau^i = \tau_1^i, \\ X_t^i = \triangle, \quad t \geq \tau^i = \tau_r^i, \\ X_0^i = x^i, \quad 0 < x^i < 1, \end{cases} \tag{8.2}$$

where M_t and W_t^i, $i = 1, \cdots, n$ are two correlated Brownian motions, τ^is are defined as before. N_t^i is a Poisson process with the intensity parameter ζ^i, independent of W^i and M_t and $\hat{J}^i > 0$ is the jump size corresponding to the amount that is prepaid by individual i. Adding a jump for each individual captures the prepayments made by each individual during the life of the contract. The jump occurs when a partial prepayment is made and the borrower gets closer to the prepayment boundary. We can again define the empirical measure as $\nu_{n,t} = \frac{1}{n}\sum_{n=1}^{\infty} \delta_{X_t^i}$. The problem again is to investigate the existence and uniqueness of the limit empirical measure and the existence and uniqueness of the solution of the SPDE.

8.1.3 Prepayment intensity

In our model, we considered that the prepayment intensity $\lambda(t, r_t)$ depends on time t and interest rates r_t, only. The prepayment due to lower interest rates, refinancing, was modeled using a preset function depending on the time and interest rates.

The prepayment can also occur due to a rise in the underlying property values as suggested in [30]. This is because a significant rise in the property can cause the sale of the property in order to withdraw equity. The borrower can sell the property and use the money to pay the outstanding principal balance. To capture this prepayment, we can introduce another variable τ_H^i for each individual that has an exponential distribution with rate $\lambda^*(t, H_t)$ where H_t follows a lognormal process

$$H_t = H_0 e^{\mu_H t + \sigma_H W_t^H},$$

where H_0 is the value of the property at time $t = 0$ that is the time when the loan is taken and W_t^H is a Brownian motion. The stopping time τ^i is defined as the first time when either prepayment or default occurs that is

$$\tau^i := \min\left(\tau_0^i, \tau_1^i, \tau_r^i, \tau_H^i\right)$$

where τ_0^i, τ_1^i and τ_r^i are defined as before and

$$\tau_H^i = \inf\left\{t : \int_0^t \lambda^*(s, H_s) > e_*^i\right\}.$$

where $e_*^i \sim \exp(1)$ are independent. $\lambda^*(t, H_t)$ can be chosen such that if the difference of the current log-price of the house H_t and the initial log-price of the house H_0 exceeds a threshold K^*, then the borrower choses to prepay the loan such as

$$\lambda^*(t, H_t) = \max\left(\log H_t - \log H_0, K^*\right).$$

8.1.4 Capturing delinquencies

In our model, we considered default as the time when the wealth of the borrower touches the default barrier. As mentioned before, the lender may choose to send a default notice to the borrower if the loan is more than 90 days delinquent. The borrower may have the right to re-enter the loan when he pays the scheduled monthly payments due at any time after the legal notice is received. Therefore a possible and realistic extension is to allow delinquencies in the model. This can be done by considering W_t^i as a Brownian motion with local time near the default boundary for each individual. In our model, we had W_t^i capturing the idiosyncratic noise for each individual. We associate a local time with the same Brownian motion to capture a delay in default due to the delinquencies by allowing the Brownian motion to stay close to the default boundary for some time. The local time for W_t^i is defined as

$$L_t^i(0) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{\{|W_s^i| < \epsilon\}} ds.$$

Let

$$\zeta^i := \inf \left\{ t \geq 0 : L_t^i(0) > e^i \right\},$$

where $e^i \sim \exp(1)$ is an independent exponential random variable. We define elastic Brownian motion $\hat{W}_s^i$ with lifetime ζ^i as

$$\hat{W}_t^i := \begin{cases} W_t^i & 0 \leq t < \zeta^i \\ \triangle^d & t \geq \zeta^i, \end{cases}$$

where $\triangle^d$ is a cemetry state.

Now the system becomes

$$\begin{cases} dX_t^i = \hat{\mu} dt + \hat{\sigma}\sqrt{1-\rho^2} dW_t^i + \hat{\sigma}\rho dM_t, \quad t < \tau^i \\ X_t^i = 0, \quad t \geq \tau^i = \tau_0^i, \\ X_t^i = 1, \quad t \geq \tau^i = \tau_1^i, \\ X_t^i = \triangle, \quad t \geq \tau^i = \tau_r^i, \\ X_0^i = x^i, \quad 0 < x^i < 1, \end{cases} \tag{8.3}$$

Define the stopping time τ^i as the first time when either of these events occur that is

$$\tau^i := \min\left\{\tau_0^i, \tau_1^i, \tau_r^i, \zeta^i\right\},$$

where τ_0^i, τ_1^i, τ_r^i and ζ^i are defined as before. We can define the empirical measure as before and study the evolution of the limit empirical measure to find the measure-evolution equation describing the evolution of the pool.

8.1.5 Covering mortgages with different amortisation time

We considered a pool of mortgages that are initialised at the same time with the same maturity. Mortgages that are initialised at different times and with different maturities can be combined into one pool. This means that there may be some loans that are not prepaid or defaulted but are taken out of the pool due to the expiry of the contract. We can capture this by introducing a time dependent killing term τ_{amt}^i (similar to the one for prepayment) that can take into account the individuals that amortise over time after their contract expires. Naturally the time τ_{amt}^i depends on the number of outstanding scheduled monthly payments or on the remaining length of the contract for each individual.

The system becomes

$$\begin{cases} dX_t^i = \hat{\mu}dt + \hat{\sigma}\sqrt{1-\rho^2}dW_t^i + \hat{\sigma}\rho dM_t, \quad t < \tau^i \\ X_t^i = 0, \quad t \geq \tau^i = \tau_0^i, \\ X_t^i = 1, \quad t \geq \tau^i = \tau_1^i, \\ X_t^i = \triangle, \quad t \geq \tau^i = \tau_r^i, \\ X_0^i = x^i, \quad 0 < x^i < 1, \end{cases} \tag{8.4}$$

where as before,

$$\tau^i = \min\left(\tau_0^i, \tau_1^i, \tau_r^i, \tau_{amt}^i\right),$$

where

$$\tau_{amt}^i = \inf\left\{t : \int_0^t \lambda_i^{amt}(s)\, ds > e_{amt}^i\right\},$$

where $e_{amt}^i \sim \exp(1)$. We can define the empirical measure of the whole pool as before and can study the evolution equation for the limit empirical measure. $\lambda_i^{amt}(t)$ here is a function of the number of outstanding scheduled monthly payments of borrower i at time t with $\lambda_i^{amt}(T) = 0$ for a loan with maturity T.

Mortgage backed securities are complex products and to capture all the properties of an MBS in one model is a difficult task. Other interesting extension could be the refinancing to other types of MBS such as a 30-year MBS can be refinanced to 10 or 15-years MBS. We are only considering fixed rate residential mortgages, a refinancing can be made to an adjustable rate mortgage. It would also be interesting to see how the model performs if the data on MBS indices is available to estimate the parameters and how it price other types of mortgages products such as TABX.HE.

8.2 Conclusion

The thesis presented a comprehensive study of residential mortgages and established two models for residential mortgages. The first model considers an individual mortgage directly affected by default and prepayments in an intensity framework. Using intensity processes for default and prepayment, we model the rate of mortgage as a nonlinear equation. Numerical implementation and some case studies are also presented to compare the results in [23].

The second and major part of this project develops a model for mortgage backed securities in a structural framework using empirical measures and considering the density of the mortgage pool. This is interesting as no significant work has been done using empirical measures in the theory of MBS and considering the limit empirical measure of the pool. Moreover we showed that a unique solution of the SPDE satisfied by the density of the MBS pool exists in L^2 with respect to Lebesgue measure given that the initial measure has a density in L^2 with respect to Lebesgue measure. We also discussed the numerical implementations and step by step calibration process in our later work. Furthermore we presented a strategy to find the last set of the parameters if the suitable data is available.

We also highlighted some possible extensions to the current project. The extensions we discussed above would further complicate the problem, not only in the mathematical sense, but also in the numerical implementation and calibration of the extended models. More sophisticated and efficient numerical models and calibration procedures could also be developed for this model.

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