

Pfaffian particles and strings in $SO(2N)$ gauge theories

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ABSTRACT: We introduce (generalised) Pfaffian operators into our lattice calculations of the mass spectra and confining string tensions of $SO(2N)$ gauge theories, complementing the conventional trace operators used in previous lattice calculations. In $SO(6)$ the corresponding ‘Pfaffian’ particles match the negative charge conjugation particles of $SU(4)$, thus resolving a puzzle arising from the observation that $SO(6)$ and $SU(4)$ have the same Lie algebra. The same holds true (but much more trivially) for $SO(2)$ and $U(1)$. For $SO(4)$ the Pfaffian particles are degenerate with, but orthogonal to, those obtained with the usual single trace operators. That is to say, there is a doubling of the spectrum, as one might expect given that the Lie algebra of $SO(4)$ is the same as that of $SU(2) \times SU(2)$. Additional $SO(8)$ and $SO(10)$ calculations of the Pfaffian spectrum confirm the naive expectation that these masses increase with N , so that they cease to play a role in the physics of $SO(N)$ gauge theories as $N \rightarrow \infty$. We also calculate the energies of Pfaffian ‘strings’ in these gauge theories. Although all our lattice calculations are for gauge theories in $D = 2 + 1$, similar conclusions should hold for $D = 3 + 1$.

KEYWORDS: Nonperturbative Effects, Lattice Quantum Field Theory, $1/N$ Expansion

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1 Introduction

We begin with our original motivation for the calculations of this paper. We recall that certain pairs of $SU(N)$ and $SO(N')$ gauge theories share the same Lie algebra. These pairs are $SO(3)$ and $SU(2)$, $SO(4)$ and $SU(2) \times SU(2)$, and $SO(6)$ and $SU(4)$. Whether the differing global properties of the groups in each pair affect the physics is an interesting question that has provided one of the motivations for recent calculations [1–3] of the low-lying mass spectra of $SO(N)$ gauge theories. For technical reasons (to do with the location

of the transition between strong and weak coupling physics) these have been performed in $2 + 1$ rather than $3 + 1$ dimensions. These spectra have then been compared to existing $SU(N)$ calculations in $D = 2 + 1$ [4]. Since $SO(N)$ is real, one might naively assume that there is no room for negative charge conjugation states so that, for example, there are no states in the $SO(6)$ spectrum that correspond to the $C = -$ states of $SU(4)$. And indeed one finds that if one uses the standard single (or multi) trace operators this is so [1–3]. Moreover the lightest $C = +$ spectra do appear to be consistent between the corresponding pairs of $SU(N)$ and $SO(N')$ gauge theories [1–3]. This is also the case for the planar large- N limits of $SO(N)$ and $SU(N)$ gauge theories, as predicted by the usual diagrammatic large- N counting [5, 6]. However the fact that the light $C = -$ states of $SU(4)$ do not appear to be encoded in the corresponding $SO(6)$ spectrum creates a puzzle, as emphasised in [7]. For example, a sufficiently excited $C = +$ glueball in $SU(4)$ can decay into two $C = -$ glueballs. This will contribute to its decay width (and will shift its mass). If the corresponding $C = +$ glueball in the $SO(6)$ theory is identical, then its decay products will include states composed of these two $C = -$ glueballs arbitrarily far apart. This is hard to understand if the theory does not include single particles corresponding to the $C = -$ glueballs of $SU(4)$. One can observe a similar puzzle concerning flux tubes in the fundamental representation of $SU(4)$, and hence in the spinorial of $SO(6)$, as described in detail in [7]. In this paper we shall focus on resolving the glueball puzzle, leaving the flux tube puzzle to future work.

As we shall show below, the solution to the above glueball puzzle is to be found in the fact that the $SO(6)$ gauge theory possesses an additional type of gauge invariant operator that is orthogonal to the usual trace operators. This is a generalisation of the Pfaffian operator for $SO(2N)$ gauge theories whose encoding in the AdS/CFT correspondence has been discussed in [8]. Such operators were not included in the calculations of [1, 2] and so the $SO(2N)$ spectra obtained therein are incomplete, albeit correct as far as they go. This operator plays no role in $SU(N)$ gauge theories, for reasons discussed below. In this paper we shall calculate the masses of the lightest ‘Pfaffian particles’ (following the nomenclature of [8]) in a number of $SO(2N)$ gauge theories and we will show that this resolves a number of puzzles including that of the missing ‘ $C = -$ ’ states in $SO(6)$. We will include in our calculations the pair $SO(2)$ and $U(1)$, where one can also ask how the $C = -$ states of $U(1)$ are encoded in $SO(2)$. This case has the advantage of being so trivial that one can immediately see how the ‘Pfaffian particles’ resolve the puzzle. While most of our calculations will be for $SO(2)$, $SO(4)$ and $SO(6)$, for the reasons outlined above we will also perform calculations for $SO(8)$ and $SO(10)$ so as to see what happens to the ‘Pfaffian particles’ as $N \rightarrow \infty$.

In our calculations there are a number of properties of the ‘Pfaffian’ operators that are important for us. In general we shall provide numerical rather than analytic evidence for these properties, except where we are aware of simple analytic arguments. These properties are discussed in section 3. Prior to that, in section 2, we outline how the lattice calculations are performed. Here we address the caveats concerning the accuracy of our mass calculations; it is important to bear these in mind later on in the paper when we compare the spectra of the corresponding $SO(N)$ and $SU(N')$ gauge theories.

In section 4 we perform calculations in $\text{SO}(2)$, $\text{SO}(4)$ and $\text{SO}(6)$, and compare with the results of calculations in $\text{U}(1)$, $\text{SU}(2) \times \text{SU}(2)$ and $\text{SU}(4)$ respectively. The latter calculations are carried out at bare couplings such that the mass gap is nearly the same within each $\text{SO}(N)$ and $\text{SU}(N')$ pair. This allows for the comparisons to be reasonably direct. We also perform calculations in $\text{SO}(8)$ and $\text{SO}(10)$ and use our various results to make plausible extrapolations in N . In this exploratory study we do not attempt to extrapolate our results to the continuum limit of the lattice gauge theories but instead choose bare couplings where earlier work, in both $\text{SO}(N)$ and $\text{SU}(N)$, has shown lattice corrections to be very small for the masses calculated. And we assume that the same is true for the Pfaffian particles. In addition to particle masses one can also take the Pfaffian of a non-contractible loop that winds around a spatial circle. The trace of such a loop projects onto a flux loop that winds around the circle and from its energy we can estimate the confining string tension. In section 5 we ask whether this is also true of the Pfaffian of such a loop and if so what is the flux that it carries. Since the present work is exploratory in nature (albeit with a number of interesting conclusions) there are various open questions that we have encountered but have not tried to address because of our focus on resolving the puzzles listed above. We point to a number of these questions in section 6 and discuss different ways they may be resolved. Finally in our concluding section 7 we summarise our results.

2 Lattice preliminaries

Our lattice calculations are standard and we refer to [4] for a detailed description of the $\text{SU}(N)$ calculations, and to [1, 2] for details of the $\text{SO}(N)$ calculations.

2.1 Path integral

Our Euclidean space-time is a finite cubic lattice, with a lattice spacing denoted by a and a size $l_x \times l_y \times l_t$ in lattice units. The boundary conditions for the fields are periodic. For $\text{SO}(N)$ our degrees of freedom are $N \times N$ real orthogonal matrices with unit determinant, which are assigned to the links l of the lattice. For $\text{SU}(N)$ the matrices are $N \times N$ complex unitary matrices with unit determinant. The matrix on the link l will normally be denoted by U_l , if forward going, and U_l^\dagger if backward going, although we will sometimes denote $\text{SO}(N)$ matrices by O_l . We will sometimes write U_l as $U_\mu(n)$ where μ is the direction of the link and n is an integer (or triplet of integers) labelling the site from which the link emanates in a forward going direction. The partition function is

$$Z = \int \mathcal{D}U \exp\{-\beta S[U]\} \quad (2.1)$$

where $\mathcal{D}U$ is the group Haar measure and we use the standard plaquette action for $S[U]$,

$$S[U] = \frac{1}{2} \sum_{\mu \neq \nu, n} \left\{ 1 - \frac{1}{N} \text{Tr}\{U_\mu(n)U_\nu(n + \hat{\mu})U_\mu^\dagger(n + \hat{\nu})U_\nu^\dagger(n)\} \right\} \quad (2.2)$$

where for $\text{SO}(N)$ we obviously have $O_l^\dagger = O_l^T$. When we take the (naive) continuum limit and compare to the continuum action we find that $\beta = 2N/ag^2$. We recall that in $D = 2+1$

g^2 has dimensions of mass, so that lg^2 is the dimensionless running coupling on the length scale l . On the lattice the degrees of freedom are defined on the length scale of the lattice spacing a and so ag^2 is the appropriate dimensionless running coupling.

We will use the standard plaquette action in all our calculations in this paper.

2.2 Energies from correlators

We calculate the mass spectrum from correlators of gauge invariant operators. Suppose $\psi(t)$ is a gauge invariant operator with some specific J^{PC} quantum numbers and with zero momentum, $p = 0$. Then

$$\langle \psi^\dagger(t)\psi(0) \rangle = \sum_n |\langle n|\psi|vac \rangle|^2 e^{-E_n t} \stackrel{t \rightarrow \infty}{=} |\langle 0|\psi|vac \rangle|^2 e^{-Mt} \quad (2.3)$$

where M is the mass of the lightest state with the quantum numbers of the operator ψ , and $|0\rangle$ is the corresponding state. The operator ψ will typically be a linear combination of more elementary operators. This will typically be based on a path ordered product of link matrices around some closed path \mathcal{C} that starts and ends at some site x (shorthand for $\{x, y, t\}$). Call it $\Phi_{\mathcal{C}}$. In $SO(N)$ and $SU(N)$ this will be an $N \times N$ matrix. To make it gauge invariant one can take the trace, $\text{Tr}\{\Phi_{\mathcal{C}}(x)\}$. (One can also use products of traces, which will have a larger overlap onto multiparticle states.) For $SU(N)$ traced operators provide a systematic way to calculate the full spectrum of the theory. This same technique was taken over in [1, 2] to calculate the $SO(N)$ mass spectrum but in that case it turns out to be incomplete, in an interesting way, as we shall see below.

In practice we use a large basis of such closed loop operators, which include operators that are smooth on physical length scales, and hence large in lattice units, so as to have a good overlap onto the lightest physical states. (This can be efficiently achieved by iteratively ‘blocking’ the link matrices [9, 10].) A good overlap is crucial since the numerical calculations of a correlator $\langle \psi^\dagger(t)\psi(0) \rangle$ will have statistical errors roughly independent of t while the interesting physical ‘signal’ in eq. (2.3) decreases exponentially in t . So the lightest state needs to emerge from the background of excited states at small t if it is not to be drowned in the statistical noise. This requires that the normalised overlap $|\langle 0|\psi|vac \rangle|^2 / \langle \psi^\dagger(0)\psi(0) \rangle$ should be not very small, and in practice one finds that it needs to be larger than ~ 0.5 if we are to capture useful information about the state. To obtain the linear combination of loop operators that ‘best’ approximates the wavefunction of the lowest eigenstate we apply a variational calculation based on maximising the transfer matrix, i.e. $\exp\{-aH\}$ in continuum language. This gives us an approximate ground state wavefunctional Ψ_0 and we then calculate the correlator $\langle \Psi_0^\dagger(t)\Psi_0(0) \rangle$ from which we extract our mass estimate. Here it is convenient to define an effective mass by

$$\exp\{-aM_{\text{eff}}(t)\} = \frac{\langle \Psi_0^\dagger(t+a)\Psi_0(0) \rangle}{\langle \Psi_0^\dagger(t)\Psi_0(0) \rangle}. \quad (2.4)$$

If for some $t \geq t_o$ the correlator is dominated by a single exponential, then $aM_{\text{eff}}(t)$ will become independent of t for $t \geq t_o$ and will be equal to the desired lightest mass. Thus we look for a ‘plateau’ (within errors) in the effective mass, from which we extract an estimate

of the true mass. Since the errors are roughly independent of t , the error on $aM_{\text{eff}}(t)$ will grow exponentially with t and if aM is large then there will be some guesswork involved in identifying the effective mass plateau. The typical error is to extract aM from $aM_{\text{eff}}(t)$ at too small a value of t , where it still receives significant contributions from heavier excited states. This will obviously lead to an overestimate of the mass. Moreover, since the error on $aM_{\text{eff}}(t)$ increases with t , our statistical error on this overestimated mass will be smaller than it should be. All this can also occur if the mass is not large but the overlap is small. Since the credibility of our conclusions will ultimately rest on the reliability of our masses, we will provide the explicit examples of relevant effective mass plots at appropriate points in the paper.

Once we have Ψ_0 we can repeat the variational procedure in a basis orthogonal to Ψ_0 and this will give us an approximate wavefunctional Ψ_1 for the first excited state, from whose correlator we can extract an approximation to the energy of the excited state. We can repeat the process for higher excited states.

Normally the state that is picked out by the variational procedure as the candidate ground state does indeed have the largest overlap onto the true ground state. However if the lightest state has a poor overlap onto our basis of operators, then it may appear in the larger- t tail of the correlator of an excited state. We shall see an example of this later on in the paper.

To obtain glueball masses with particular quantum numbers such as spin J and parity P we use operators with those same symmetries. Note that the limited $\pi/2$ rotational symmetry means that the ‘ $J = 0$ ’ representation of the 2D rotation group contains states that become $J = 4, 8, \dots$ in the continuum limit. Similarly for ‘ $J = 2$ ’ and ‘ $J = 1$ ’. For convenience we shall, from now on, label all the states by $J = 0$ or 2 or 1 . The parity reflections in x and y can be rotated to each other, but there is a further parity in the $x = y$ axis that cannot be — although it can be, of course, in the continuum limit. Our parity will always be in the x or y axis. Note also that for $J \neq 0$ we have parity doubling: this is exact for $J = 1^\pm$ but for $J = 2^\pm$ can be broken by finite volume or sub-leading lattice corrections. For a more detailed discussion see [11, 12].

If the theory is confining then there is a finite volume state consisting of a flux tube wrapped around a spatial circle. If the length of the circle is l and the ground state energy is $E(l)$ then we extract the string tension σ using

$$E(l) = \sigma l \left(1 - \frac{\pi}{3\sigma l^2}\right)^{\frac{1}{2}}. \quad (2.5)$$

This expression is a very good approximation down to very small l [13–17]. We extract $E(l)$ using operators that consist of products of link matrices around a curve that winds once around the spatial circle. This is a non-contractible loop in contrast to the contractible loops appropriate for glueball operators. For traced operators one can show that such a non-contractible loop operator has no overlap on glueballs for those theories with a non-trivial centre symmetry, such as $SU(N)$ and $SO(2k)$, if we are in the confining phase.

3 Some properties of ‘Pfaffian’ operators in $SO(2N)$

We begin with the standard definition of the Pfaffian and how it usually appears in gauge theories. We then move onto generalisations for both glueballs and winding flux tubes. To be useful in calculations of energies these operators must be gauge invariant as we shall see they are. In addition, to provide a useful addition to the standard trace operators used in previous lattice calculations [1–3], they should project onto states with which the trace operators have no, or very little, overlap. We shall show, numerically, that this is indeed the case for the $SO(N)$ groups that we consider in this paper.

3.1 The Pfaffian operator

Let A be a $2k \times 2k$ skew-symmetric matrix, i.e. $A^T = -A$. Then its Pfaffian, $\text{Pf}(A)$, is defined to be

$$\text{Pf}(A) = \frac{1}{2^k k!} \epsilon_{i_1 i_2 \dots i_{2k}} A_{i_1 i_2} A_{i_3 i_4} \dots A_{i_{2k-1} i_{2k}} \quad (3.1)$$

where $\epsilon_{i_1 i_2 \dots i_{2k}}$ is the totally antisymmetric (Levi-Civita) tensor with $2k$ indices. (We do not distinguish upper and lower indices.) Moreover for such a matrix

$$\text{Pf}(A)^2 = \det(A). \quad (3.2)$$

Recall that the generators of $SO(N)$ are skew-symmetric. So if we choose A to be an antisymmetric second rank tensor transforming in the adjoint of $SO(2k)$, e.g. the field strength, then $\text{Pf}(A)$ will be invariant under a gauge transformation $V(x)$ since it will satisfy eq. (3.2) and $\det(A)$ is gauge invariant: $\det(A(x)) \xrightarrow{V(x)} \det(V(x)A(x)V^T(x)) = \det(A(x))$ since $\det(V) = 1$. To be more precise, in order to evade the \pm ambiguity in taking the square root of $\det(A)$ in eq. (3.2) we should restrict ourselves to gauge transformations $V(x)$ that are continuously connected to the identity. Along such a path the Pfaffian will not change sign by continuity. These are the usual ‘small’ gauge transformations in contrast to the ‘large’ gauge transformations that are typically associated with topological fluctuations in one higher Euclidean dimension. (In fact a particular ‘large’ transformation under which the Pfaffian flips sign will play a role in our arguments later on in this paper.)

Following [8] we will refer to the states that this Pfaffian (and its generalisations below) couple to as ‘Pfaffian particles’. Naively these particles will be composed of k gauge bosons in $SO(2k)$ and so one might expect that they will become heavy as k grows and will become irrelevant to the light particle spectrum of $SO(2k)$ as $k \rightarrow \infty$. Although this is an expectation that we will test later on in this paper, our primary interest is in small k where these states should be important.

3.2 Generalised Pfaffians

On the lattice we do not work with the gluon fields directly but rather with $SO(N)$ matrices $\Phi(\mathcal{C}; x)$ that are obtained by multiplying the link matrices around some path \mathcal{C} that begins and ends at the site x . (In terms of continuum fields, this becomes just the path ordered exponential of the gauge fields around the contour \mathcal{C} .) We define the Pfaffian $\text{Pf}(\Phi)$ of such

a matrix Φ by replacing A with Φ in eq. (3.1):

$$\text{Pf}(\Phi(x)) = \frac{1}{2^k k!} \epsilon_{i_1 i_2 \dots i_{2k}} \Phi(x)_{i_1 i_2} \Phi(x)_{i_3 i_4} \dots \Phi(x)_{i_{2k-1} i_{2k}}. \quad (3.3)$$

This is a generalisation of the usual definition of a Pfaffian in eq. (3.1) since the $\text{SO}(N)$ matrix $\Phi(\mathcal{C}; x)$ is not skew symmetric, but rather satisfies $\Phi^T = \Phi^{-1}$. Such an operator will only be useful if it is gauge invariant which we need to show because the relation in eq. (3.2) that we used to show gauge invariance will no longer hold in general. So consider an $\text{SO}(N)$ gauge transformation $V(x)$ under which $\Phi(x) \rightarrow V(x)\Phi(x)V^T(x)$. Then the Pfaffian of Φ changes as

$$\begin{aligned} 2^k k! \text{Pf}(\Phi(x)) &= \epsilon_{i_1 i_2 \dots i_{2k}} \Phi_{i_1 i_2} \dots \Phi_{i_{2k-1} i_{2k}} \\ &\xrightarrow{V} 2^k k! \text{Pf}(V\Phi(x)V^T) \\ &= \epsilon_{i_1 i_2 \dots i_{2k}} V_{i_1, j_1} \Phi_{j_1 j_2} V_{j_2, i_2}^T \dots V_{i_{2k-1}, j_{2k-1}} \Phi_{j_{2k-1} j_{2k}} V_{j_{2k}, i_{2k}}^T \\ &= \epsilon_{i_1 i_2 \dots i_{2k}} V_{i_1, j_1} \Phi_{j_1 j_2} V_{i_2, j_2} \dots V_{i_{2k-1}, j_{2k-1}} \Phi_{j_{2k-1} j_{2k}} V_{i_{2k}, j_{2k}} \\ &= \epsilon_{i_1 i_2 \dots i_{2k}} V_{i_1, j_1} V_{i_2, j_2} \dots V_{i_{2k-1}, j_{2k-1}} V_{i_{2k}, j_{2k}} \Phi_{j_1 j_2} \dots \Phi_{j_{2k-1} j_{2k}} \\ &= \epsilon_{j_1 j_2 \dots j_{2k}} \Phi_{j_1 j_2} \dots \Phi_{j_{2k-1} j_{2k}} \\ &= 2^k k! \text{Pf}(\Phi(x)) \end{aligned} \quad (3.4)$$

where in the fourth line we have used the fact that $V_{ab}^T = V_{ba}$, and in the sixth line the identity

$$\epsilon_{i_1 i_2 \dots i_N} V_{i_1, j_1} V_{i_2, j_2} \dots V_{i_{N-1}, j_{N-1}} V_{i_N, j_N} = \epsilon_{j_1 j_2 \dots j_N} \quad (3.5)$$

for $\text{SO}(N)$ matrices V which is valid for odd N as well as for even N . Thus our generalised Pfaffian is also gauge invariant and can be a useful operator in lattice mass calculations.

From the above derivation it is clear that if we have matrices $\Phi(\mathcal{C}_i; x)$ that differ because they involve products of link matrices around different paths \mathcal{C}_i , but with all the paths still beginning and ending at the same point x , then the even more general Pfaffian, defined by

$$\text{Pf}(\{\Phi(\mathcal{C}_i; x)\}) = \frac{1}{2^k k!} \epsilon_{i_1 i_2 \dots i_{2k}} \Phi(\mathcal{C}_1; x)_{i_1 i_2} \Phi(\mathcal{C}_2; x)_{i_3 i_4} \dots \Phi(\mathcal{C}_k; x)_{i_{2k-1} i_{2k}}, \quad (3.6)$$

is also gauge invariant. So this provides an even more extensive set of operators that one can use in mass calculations.

Another more illuminating way to see the gauge invariance of an adjoint operator $\Phi(x)$ is as follows. Let us decompose Φ into a sum of symmetric and skew-symmetric pieces: $\Phi = \Phi^e + \Phi^o$ where $\Phi_{ij}^o = -\Phi_{ji}^o$ and $\Phi_{ij}^e = \Phi_{ji}^e$. Clearly $\Phi^o = 1/2(\Phi - \Phi^T)$ and $\Phi^e = 1/2(\Phi + \Phi^T)$. (Aside: since Φ is an $\text{SO}(N)$ group element, we can write $\Phi = \exp\{A\}$ where A is a skew-symmetric Lie algebra element, and expanding the exponential we see that Φ^e and Φ^o are just the sums of even and odd powers of A respectively.) Inserting this sum for each occurrence of Φ into $\text{Pf}(\Phi)$ we see that any term containing at least one Φ^e must vanish. That is to say,

$$\text{Pf}(\Phi(x)) = \text{Pf}(\Phi^e(x) + \Phi^o(x)) = \text{Pf}(\Phi^o(x)) = \{\det(\Phi^o(x))\}^{\frac{1}{2}} \quad (3.7)$$

where we can now use eq. (3.2) since Φ^o is skew-symmetric. Now under a gauge transformation $V(x)$, we have

$$\begin{aligned}\text{Pf}(\Phi(x)) &\rightarrow \text{Pf}(V(x)\Phi(x)V^T(x)) = \text{Pf}(V(x)\Phi^o(x)V^T(x)) \\ &= \det(V(x)\Phi^o(x)V^T(x))^{\frac{1}{2}} \\ &= \det(\Phi^o(x))^{\frac{1}{2}}\end{aligned}\tag{3.8}$$

since $\det(V(x)) = \det(V^T(x)) = 1$ and one can easily show that the skew symmetric piece of $V(x)\Phi(x)V^T(x)$ is just $V(x)\Phi^o(x)V^T(x)$. So we see that $\text{Pf}(\Phi(x))$ is indeed gauge invariant. (Again, we are only considering ‘small’ gauge transformations in order to avoid a possible sign ambiguity in the square root.)

3.3 Pfaffians and $\text{SU}(N)$

As an aside, we remark that eq. (3.5) also holds if V is an $\text{SU}(N)$ matrix. However in the $\text{SU}(N)$ gauge theory a gauge transformation leads to $\Phi(x)_{ab} \rightarrow V(x)_{ac}\Phi(x)_{cd}V^\dagger(x)_{db} = V(x)_{ac}\Phi(x)_{cd}V^{*\dagger}(x)_{db} = V(x)_{ac}V^*(x)_{bd}\Phi(x)_{cd}$ and so the Pfaffian of Φ is not gauge invariant since eq. (3.5) does not hold if we replace every second V in the product by its complex conjugate. On the other hand, if we introduce fundamental fields $\psi(x)$ into the $\text{SU}(N)$ gauge theory, then eq. (3.5) ensures that the operator $\epsilon_{i_1 i_2 \dots i_N} \psi_{i_1} \psi_{i_2} \dots \psi_{i_N}$ is gauge invariant. If ψ is a fermion, then this is of course just the gauge-invariant operator for a baryon in N -colour QCD. The same holds if we introduce fundamental fields into an $\text{SO}(N)$ gauge theory, for any value of N .

3.4 Pfaffians for flux tubes

The above demonstration of the gauge invariance of $\text{Pf}(\{\Phi(\mathcal{C}; x)\})$ is clearly valid irrespective of whether the curve \mathcal{C} is contractible or not. Now, as described in section 2, the traces of contractible loops form a basis for glueball operators while the traces of non-contractible loops that wind once around a periodic spatial direction typically form a basis for operators that project onto confining flux tubes that wrap once around that spatial direction. This distinction can be readily motivated using the Z_2 centre symmetry of the $\text{SO}(2k)$ gauge theory. If we multiply by $-1 \in Z_2$ the matrices $U_{\mu=x}(n_x, n_y, n_t); \forall n_y, n_t$ then a contractible loop, label it Φ_g , will transform as $\Phi_g \rightarrow \Phi_g$, since the number of factors of -1 will be even, but a non-contractible loop that winds once around the x -torus, label it Φ_l , will clearly transform as $\Phi_l \rightarrow -\Phi_l$ since the number of factors of -1 will be odd. Under this field transformation the Haar measure is invariant and so is the action since the plaquette (a contractible loop) is unchanged. So the fields have the same weight and if the Z_2 symmetry is not spontaneously broken we will necessarily have

$$\langle \text{Tr}(\Phi_g(n)) \text{Tr}(\Phi_l(n')) \rangle = -\langle \text{Tr}(\Phi_g(n)) \text{Tr}(\Phi_l(n')) \rangle = 0; \quad \forall n, n', g, l.\tag{3.9}$$

That is to say, the states produced by contractible and non-contractible loops are orthogonal to each other, as one would expect for glueballs and winding flux tubes in a confining theory.

We can now apply the same argument to the Pfaffian of Φ_g and Φ_l in the $\text{SO}(2k)$ gauge theory. From eq. (3.3) we see that under this Z_2 field transformation we have $\text{Pf}\{\Phi_g\} \rightarrow \text{Pf}\{\Phi_g\}$ while $\text{Pf}\{\Phi_l\} \rightarrow (-1)^k \text{Pf}\{\Phi_l\}$. Thus $\text{Pf}\{\Phi_x\}|vac\rangle$ is orthogonal to $\text{Pf}\{\Phi_g\}|vac\rangle$ and also to $\text{Tr}(\Phi_g(x))|vac\rangle$ if k is odd. That is to say, we expect $\text{Pf}\{\Phi_l\}$ to project onto confining flux tubes that wind around the x -torus in $\text{SO}(4k+2)$ theories but not necessarily in $\text{SO}(4k)$ theories. The former includes $\text{SO}(2)$, $\text{SO}(6)$, $\text{SO}(10)$ while the latter includes $\text{SO}(4)$, $\text{SO}(8)$.

For $\text{SO}(4k)$ theories we can instead use the more general Pfaffian in eq. (3.6). For instance for $\text{SO}(4)$ the operator $\epsilon_{ijkl}\{\Phi_g(x)\}_{ij}\{\Phi_l(x)\}_{kl}$ is gauge invariant and changes sign under the centre transformation discussed above, and so produces states orthogonal to glueball states. In general this will be the case whenever the generalised Pfaffian contains an odd number of winding loops with the remaining loops being contractible.

3.5 Pfaffians: orthogonality

Clearly the above Pfaffian operators will only be really useful if they have large projections onto states that our standard traced operators do not. In that case our usual lattice calculation with traced operators will completely miss these states, and the Pfaffian operators will be essential for their identification.

To see if this is the case or not we have performed numerical calculations of overlaps of the form

$$O_{IJ} = \frac{\langle \text{Tr}(\Phi_I(0)) \text{Pf}(\Phi_J(0)) \rangle}{\langle \text{Tr}(\Phi_I(0)) \text{Tr}(\Phi_I(0)) \rangle^{\frac{1}{2}} \langle \text{Pf}(\Phi_J(0)) \text{Pf}(\Phi_J(0)) \rangle^{\frac{1}{2}}} \quad (3.10)$$

for various loops labelled I and J and we have done so in all the $\text{SO}(2k)$ theories analysed in this paper. In this exploratory analysis we have not used all the loops in our glueball/flux tube calculations but only a limited subset which nonetheless includes operators with a good overlap onto the ground states (both Pfaffian and traced respectively). What we find is that all the overlaps so tested are consistent with zero within very small statistical errors. The implication (albeit based on a limited numerical calculation) is that the Pfaffian operators do project onto states that will be numerically invisible to the traced operators, and that their use is therefore essential for the identification of these states.

There is a simple argument to strengthen this conclusion. Consider an $\text{SO}(2k)$ matrix $\Phi(\mathcal{C}, n)$ obtained from the product of link matrices around the contour \mathcal{C} that is open at the site n . As we have seen above we have $\text{Pf}(\Phi) = \text{Pf}(\Phi^o)$ where $\Phi^o = 1/2(\Phi - \Phi^T)$. Now we expect that the matrices Φ and Φ^T should be equally likely in the integration over all fields. That is to say for each field that Φ and Φ^T take some values, there is another field, with equal weight, for which they take values Φ' and Φ'^T such that $\Phi' = \Phi^T$ and $\Phi'^T = \Phi$. Now clearly $\text{Pf}(\Phi') = (-1)^k \text{Pf}(\Phi)$, since the Pfaffian contains a product of k elements of Φ^o . Since on the other hand $\text{Tr}(\Phi') = \text{Tr}(\Phi^T) = \text{Tr}(\Phi)$ it immediately follows that

$$\langle \text{Tr}(\Phi(x)) \text{Pf}(\Phi(x)) \rangle = 0, \quad k = \text{odd}. \quad (3.11)$$

That is to say the operator $\text{Pf}(\Phi)$ is exactly orthogonal to the operator $\text{Tr}(\Phi)$ in $\text{SO}(2k)$ gauge theories when k is odd. This adds support to the idea that the spectrum divides

into two sectors, one accessed by using the traces of closed loops and the other by using the Pfaffians of these loops.

The above argument is, however, not enough to show that the spectrum divides into two sectors. What we wish to do is to show that $\text{Tr}(\tilde{\Phi})$ and $\text{Pf}(\Phi)$ are orthogonal for any two operators $\tilde{\Phi}$ and Φ . To show this the relevant observation is that, as discussed in section 4.1 of [8], our $\text{SO}(N)$ lattice gauge theory is in fact symmetric under $O(N)$ and not just $\text{SO}(N)$ and it is the Z_2 of $O(N)/\text{SO}(N)$ that provides the quantum number dividing the spectrum into two sectors.¹ It is worth being more specific here. $O(N)$ differs from $\text{SO}(N)$ by the inclusion of elements V with $\det(V) = -1$. To generate these elements consider the matrix O_p which is the unit matrix except that the $(1,1)$ element is -1 . This matrix clearly has $\det(O_p) = -1$ and, when multiplied by all the elements of $\text{SO}(N)$, generates the $\det(V) = -1$ sector of $O(N)$. So we will use this specific matrix below, with no loss of generality. Now under the global gauge transformation $V(x) = O_p$ the $\text{SO}(N)$ matrix on the link l transforms as $U_l \rightarrow \tilde{U}_l = O_p U_l O_p$, using the fact that $O_p^T = O_p$. Since a product of $O(N)$ matrices is in $O(N)$ and since $\det \tilde{U}_l = \{\det O_p\}^2 \det U_l = 1$, the transformed matrix \tilde{U}_l is in $\text{SO}(N)$. Moreover it is clear that the trace of the ordered product of links around a plaquette is unchanged under this transformation as is the Haar integration measure since $d(\tilde{U}_l) = d(\{U_l \tilde{U}_l^{-1}\} \tilde{U}_l) = d(U_l)$. Thus this transformation is a symmetry of the partition function. If we now perform some simple algebra, we see that the transformed matrix \tilde{U}_l is identical to the original matrix U_l except that the first row is multiplied by -1 and so is the first column. Note that this means that the $(1,1)$ element is multiplied by $(-1)^2 = +1$. This of course applies to any $\text{SO}(N)$ matrix. In particular it applies to a matrix $U_{\mathcal{C}}$ obtained by multiplying the link matrices around a closed path \mathcal{C} . It is now trivial to see that

$$\text{Pf}\{\tilde{U}_{\mathcal{C}}\} = -\text{Pf}\{U_{\mathcal{C}}\}. \quad (3.12)$$

We note that this follows from the property of the Pfaffian $\text{Pf}\{A\}$ in eq. (3.1) that if a particular matrix element A_{ij} appears in a non-zero term of $\text{Pf}\{A\}$, then any other element A_{kl} in that term must have $k \neq i, j$ and $l \neq i, j$, i.e. we have one (non-diagonal) element from the first row or one from the first column, but not from both. Note that the argument will also apply to our generalised Pfaffians in eq. (3.6). We can now complete the argument by noting that the trace is unchanged by this transformation, so the symmetry implies that

$$\langle \text{Tr}\{U_{\mathcal{C}'}\} \text{Pf}\{U_{\mathcal{C}}\} \rangle = \langle \text{Tr}\{\tilde{U}_{\mathcal{C}'}\} \text{Pf}\{\tilde{U}_{\mathcal{C}}\} \rangle = -\langle \text{Tr}\{U_{\mathcal{C}'}\} \text{Pf}\{U_{\mathcal{C}}\} \rangle = 0 \quad (3.13)$$

for any paths \mathcal{C}' and \mathcal{C} , confirming our more limited numerical demonstration above. Thus the spectrum of the theory will contain two sectors, one obtained from correlators of traces of closed loops and the other from correlators of Pfaffians of closed loops, with the latter containing our ‘Pfaffian’ particles.

The above argument assumes that in our ‘zero’ temperature $\text{SO}(N)$ gauge theory this Z_2 symmetry is explicit and not spontaneously broken. Everything we calculate is consistent with that being the case. A good way to do better would be to identify where the symmetry is broken (possibly for small spatial volumes or at high temperature) so as

¹I am grateful to the referee for pointing this out to me.

to identify useful lattice order parameters for the symmetry and then to identify the phase transition where the symmetry is restored. However such a dedicated investigation lies outside the scope of the present paper.

3.6 Some other properties

We recall that the cyclic property of the trace, i.e. $\text{Tr}\{\Phi_1\Phi_2\} = \text{Tr}\{\Phi_2\Phi_1\}$, means that the value of $\text{Tr}\{\Phi(\mathcal{C}, n)\}$ will be independent of the site n along \mathcal{C} at which we choose to take the trace. We have verified numerically, for simple loops, that for the $\text{SO}(2k)$ gauge theories considered in this paper the same is true for $\text{Pf}\{\Phi(\mathcal{C}, n)\}$: its value is independent of choice of site n along \mathcal{C} .

4 Pfaffian particles in $\text{SO}(2N)$ gauge theories

In this section we will compare the light particle masses as calculated using the standard trace operators with those obtained using the Pfaffian operators introduced above. We compare these spectra with those of the corresponding unitary gauge theories in those cases where the unitary and $\text{SO}(2k)$ theories share the same Lie algebra, i.e. $\text{SO}(2)$ and $\text{U}(1)$, $\text{SO}(4)$ and $\text{SU}(2) \times \text{SU}(2)$, $\text{SO}(6)$ and $\text{SU}(4)$. We also calculate the two types of particle spectra in $\text{SO}(8)$ and $\text{SO}(10)$ gauge theories so as to be able to say something about their dependence on k as $k \rightarrow \infty$.

The single trace operators whose correlators we calculate are linear combinations of the traces of our basic loop operators. The linear combinations fall into separate subsets which are chosen so that they have specific J^P quantum numbers (albeit with the spin ambiguity described in section 2.2). The single Pfaffian operators whose correlators we calculate are the same linear combinations of the Pfaffians rather than the traces of these basic loop operators. These should have the same spin J as the trace operator but not necessarily the same parity, as we shall see below. In tabulating our results for $\text{SO}(8)$ and $\text{SO}(10)$ we choose to label the ‘trace’ and ‘Pfaffian’ particles obtained with a given subset of operators by the J^P quantum numbers possessed by the ‘trace’ particles. For smaller k where we also compare to a unitary theory we display the Pfaffian particles with different quantum numbers as appropriate.

4.1 $\text{SO}(2)$ and $\text{U}(1)$

The $\text{SO}(2)$ gauge theory should have the same physics as the $\text{U}(1)$ gauge theory even if this physics is of limited physical interest. We recall that in units of the energy scale provided by g^2 the $\text{U}(1)$ lattice gauge theory is a free field theory in the continuum limit $\beta = 2/ag^2 \rightarrow \infty$, but that at finite a the vacuum contains a screened dilute gas of monopole-like instantons which lead to a non-trivial mass spectrum and to a non-zero confining string tension [18–22]. These masses vanish exponentially in $\beta = 2/ag^2$ as $a \rightarrow 0$, reflecting the similar behaviour of the instanton density, since the instantons are singular Dirac monopoles.

We can write a general $\text{SO}(2)$ matrix assigned to the forward going link l as

$$O_l = \begin{pmatrix} \cos \theta_l & \sin \theta_l \\ -\sin \theta_l & \cos \theta_l \end{pmatrix} \quad (4.1)$$

where $\theta_l \in (-\pi, +\pi]$. As usual we assign O_l^\dagger when the link is backward going in a path-ordered product.

If we multiply the gauge variables around the closed path \mathcal{C} , we obtain some $\text{SO}(2)$ matrix, $O_{\mathcal{C}}$, and hence its gauge-invariant trace and Pfaffian,

$$O_{\mathcal{C}} = \begin{pmatrix} \cos \theta_{\mathcal{C}} & \sin \theta_{\mathcal{C}} \\ -\sin \theta_{\mathcal{C}} & \cos \theta_{\mathcal{C}} \end{pmatrix}; \quad \text{Tr}\{O_{\mathcal{C}}\} = 2 \cos \theta_{\mathcal{C}}; \quad \text{Pf}\{O_{\mathcal{C}}\} = 2 \sin \theta_{\mathcal{C}}. \quad (4.2)$$

The corresponding $\text{U}(1)$ theory has variables on the links that are complex phases $U_{\mathcal{C}}$:

$$U_{\mathcal{C}} = \exp i\theta_{\mathcal{C}}; \quad \text{Real}\{U_{\mathcal{C}}\} = \cos \theta_{\mathcal{C}}; \quad \text{Im}\{U_{\mathcal{C}}\} = \sin \theta_{\mathcal{C}} \quad (4.3)$$

where $\text{Real}\{U_{\mathcal{C}}\}$ projects onto the $C = +$ states, and $\text{Im}\{U_{\mathcal{C}}\}$ projects onto the $C = -$ states. (We drop the trace since we have 1×1 matrices in $\text{U}(1)$.) In $\text{SO}(2)$ we use the standard plaquette action $\beta S[O] = \beta \sum_p \{1 - \frac{1}{2} \text{Tr}\{O_p\}\} = \beta \sum_p \{1 - \cos \theta_p\}$ where O_p is the product of link matrices around the plaquette p , and in $\text{U}(1)$ we use the analogous action $\beta S[U] = \beta \sum_p \{1 - \text{Real}\{U_p\}\} = \beta \sum_p \{1 - \cos \theta_p\}$ where U_p is the product of link matrices around the plaquette p . Since the Haar measure is the same for the two theories, the theories are identical as indicated by our use of common angular variables. Now this $\text{SO}(2)$ action is invariant under $\theta_l \rightarrow -\theta_l \forall l$, so we immediately see that

$$\langle \text{Tr} O_{\mathcal{C}_1}^T \text{Pf} O_{\mathcal{C}_2} \rangle = \langle \text{Tr} O_{\mathcal{C}_1} \text{Pf} O_{\mathcal{C}_2} \rangle = 0 \quad \forall \mathcal{C}_1, \mathcal{C}_2 \quad (4.4)$$

and similarly if we replace the single trace by a multiple trace operator. Thus we see that in $\text{SO}(2)$ the Trace and Pfaffian project onto two separate sectors of states, and that these correspond to the $C = +$ and $C = -$ sectors of the $\text{U}(1)$ theory respectively. (And the fact that the product of 2 Pfaffians has a non-zero overlap onto a traced operator accords with the $C = \pm$ correspondence.) That is to say, if as in [1, 2] we use only single or multiple trace operators we will not be aware of the existence of a sector of states that is identical to the $C = -$ sector of $\text{U}(1)$.

It is interesting to see how well one can confirm all the above with an explicit numerical calculation. We have therefore calculated the energies of the lightest glueballs and the lightest flux tube that winds once around one of the periodic spatial directions. To make the comparison direct we do so on identical $28^2 36$ lattice sizes at an identical coupling, $\beta = 2.2$, for both $\text{U}(1)$ and $\text{SO}(2)$. We first check numerically that, within errors, the states created by the Pfaffian and Trace operators in $\text{SO}(2)$ are indeed orthogonal (as shown analytically above). Our results for the masses of the various states are listed in table 1. We have placed the Pfaffian results in the row that corresponds to the J^{PC} that one expects from the fact that the Pfaffian picks out the skew-symmetric piece of the matrix operator. We see a very nice and convincing match between the masses of the $\text{SO}(2)$ Pfaffian particles and the $C = -$ particles of $\text{U}(1)$, at least for the lightest glueballs where the errors are small. (And of course the results using traced operators in $\text{SO}(2)$ agree with the $C = +$ particles of $\text{U}(1)$.) We also observe that the flux tube energies match very well. Although we only show one value for $\text{U}(1)$ there are in fact two degenerate ground states of a flux tube. That is to say if we take l_x to be an operator that winds once around (say) the

| | U(1), $\beta = 2.2$ | SO(2), $\beta = 2.2$ | |
|-----------|---------------------|----------------------|------------|
| J^{PC} | Trace | Trace | Pfaffian |
| 0^{--} | 0.2639(44) | | 0.2630(26) |
| 0^{--*} | 0.726(22) | | 0.740(13) |
| 0^{++} | 0.505(14) | 0.519(5) | |
| 0^{++*} | 0.663(10) | 0.668(7) | |
| 0^{-+} | 1.11(5) | 1.15(2) | |
| 0^{+-} | 1.36(5) | | 1.36(1) |
| 2^{++} | 0.711(6) | 0.695(10) | |
| 2^{-+} | 0.831(10) | 0.821(7) | |
| 2^{--} | 0.937(13) | | 0.930(12) |
| 2^{+-} | 1.049(17) | | 1.031(17) |
| 1^{++} | 1.28(3) | 1.29(3) | |
| 1^{-+} | 1.24(4) | 1.321(7) | |
| 1^{--} | 1.093(16) | | 1.099(13) |
| 1^{+-} | 1.045(16) | | 1.104(17) |
| l_f | 0.7492(36) | 0.7513(31) | 0.7452(49) |

Table 1. Lightest glueball masses in U(1) and SO(2) on a $28^2 36$ lattice at a coupling $\beta = 2.2$ for various spins J and parity P . For U(1) also labelled by charge conjugation C , and for SO(2) in the two sectors indicated as explained in text. Also shown is the energy of the lightest fundamental flux tube, l_f , winding around a periodic spatial direction.

x -direction, then l_x^\dagger projects onto a flux in the opposite direction. Because of the standard centre symmetry argument (which for U(1) is the whole group) we know that $\langle (l_x^\dagger)^\dagger l_x \rangle = 0$ so that these states are orthogonal and degenerate. So we can choose to use the $l_x \pm l_x^\dagger$ basis, which corresponds to $C = \pm$ respectively and which are clearly orthogonal and degenerate. It is these that correspond to the Trace and Pfaffian flux tubes of SO(2) listed in table 1. Of course this numerical demonstration that the Pfaffian particles of SO(2) correspond to the $C = -$ states of U(1) is trivial given our earlier discussion. However it provides a check on the reliability of our numerical calculations, which is useful for the larger groups considered below.

4.2 SO(4) and SU(2) \times SU(2)

The Lie algebra of SO(4) is the same as that of SU(2) \times SU(2) and so if the different global structures of the groups are unimportant, we would expect the single particle masses of SO(4) to be the same as those of SU(2). That this is so has been confirmed, at least for the lightest masses which are under reasonable control, in the calculations of [1–3] which used correlators of single trace operators. The present case differs from that of SO(2) and U(1) discussed above in that SU(2) is (pseudo)real and therefore there are no $C = -$ states in

| | SU(2), $\beta = 13.87$ | SO(4), $\beta = 15.1$ | |
|--------------|------------------------|-----------------------|------------|
| J^P | Trace | Trace | Pfaffian |
| 0^+ | 0.4807(11) | 0.4793(27) | 0.4782(21) |
| $0^{+\star}$ | 0.6851(50) | 0.710(5) | 0.698(4) |
| 0^- | 1.002(7) | 1.037(11) | 0.972(28) |
| 2^+ | 0.781(3) | 0.800(6) | 0.795(5) |
| 2^- | 0.781(5) | 0.807(5) | 0.791(6) |
| 1^+ | 1.051(9) | 1.138(53) | 1.063(36) |
| 1^- | 1.066(10) | 1.170(15) | 1.111(16) |
| l_f | 0.5046(8) | 1.028(7) | 1.214(14) |

Table 2. Lightest glueball masses SU(2) and SO(4) on a $50^2 56$ lattice at the couplings shown for various spins J and parity P . Also shown is the energy of the lightest fundamental flux tube, l_f , winding around a periodic spatial direction.

the correspondence. However while the spectrum of $SU(2) \times SU(2)$ should contain exactly the same particle masses as SU(2), these should be doubled, with one from each of the two SU(2) groups, and this is something that was not observed in [1–3] although this fact was not remarked upon in those papers. On the other hand the SO(4) theory should contain only one vacuum state just like $SU(2) \times SU(2)$: that state should not be doubled. And the lightest flux tube, carrying the fundamental flux of SO(4), should also not be doubled. As we shall now see, including Pfaffian operators will in fact allow us to meet all these expectations.

In this calculation we shall not attempt to perform continuum extrapolations of the masses as in [1–3] but rather we choose a single coupling at which the spectrum is very close to its continuum limit. For our SO(4) calculation we choose to use $\beta = 15.1$ on a $50^2 56$ lattice, which we expect to have negligible finite volume corrections for the masses calculated. We then use the SU(2) calculations of [4] to choose a value $\beta = 13.87$, where we expect the mass gap to equal that of the SO(4) calculation within statistical errors. The results of these calculations for the lightest states are listed in table 2. We see that the very lightest Pfaffian particles listed are consistent with being degenerate with the corresponding ‘trace’ particles and with the corresponding SU(2) masses, within statistical errors. Heavier mass estimates will be afflicted by increasing systematic errors and so are less reliable. Thus we see that the Pfaffian particles indeed appear to provide us with the expected doubling in the SO(4) spectrum.

It will be useful for the reader to see examples of the effective mass plots that are behind our mass estimates. In figure 1 we show plots of $aM_{\text{eff}}(t)$ against t for the lightest $J^P = 0^+$, $J^P = 2^\pm$, and $J^P = 1^\pm$ states obtained with trace and Pfaffian operators. We simultaneously show the asymptotic mass estimates for the same states in the SU(2) theory. We see that the quality of the 0^+ comparison is excellent, that for the 2^\pm it is quite convincing, while for the 1^\pm it is indicative but the lack of a clear effective mass plateau

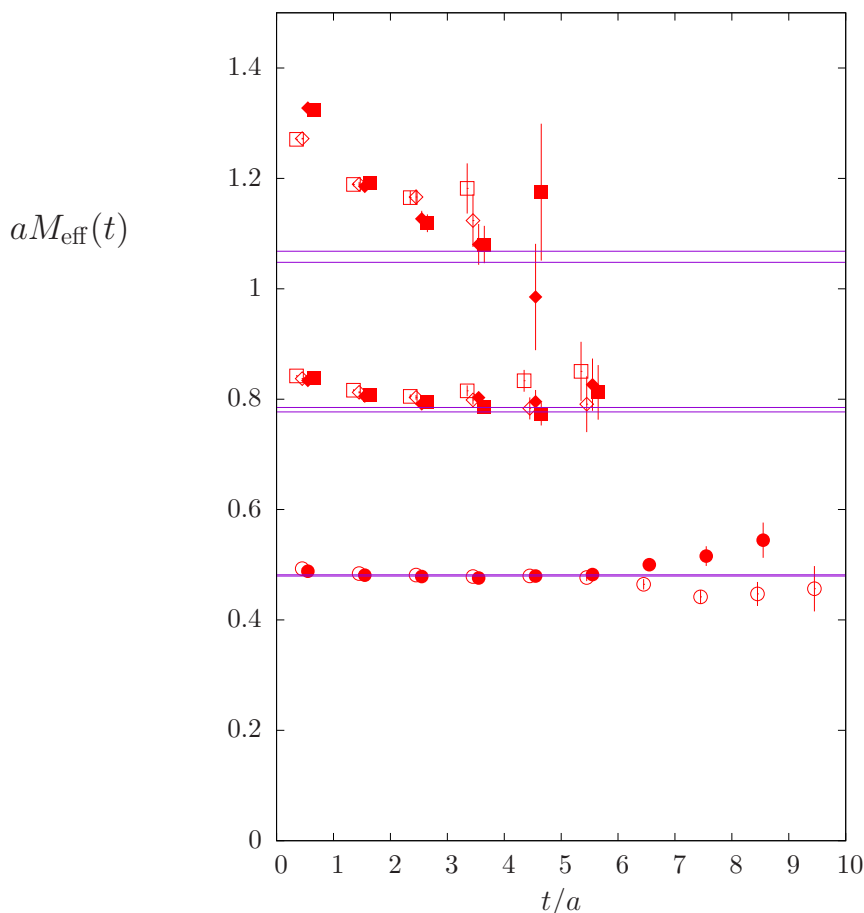


Figure 1. Effective energies of the lightest glueballs in SO(4) at $\beta = 15.1$ on a $50^2 56$ lattice: trace $J^P = 0^+$, \circ , Pfaffian ‘ 0^+ ’, \bullet , trace 2^\pm , lower \diamond , \square , Pfaffian 2^\pm , lower \blacklozenge , \blacksquare trace 1^\pm , higher \diamond , \square , Pfaffian 1^\pm , higher \blacklozenge , \blacksquare . Lines are 0^+ , 2^\pm , 1^\pm masses (in ascending order) obtained in SU(2) at $\beta = 13.87$ on the same size lattice. Points shifted for clarity.

means that it is not much more than that. (Although we note that the evidence from the plot that the $J = 1$ trace mass is degenerate with the $J = 1$ Pfaffian mass is much more convincing.) This illustrates the main issue with extracting heavier masses. We also note that some effective masses appear to increase at larger t . By the positivity of the correlator we know that this is not possible, and so this must be a statistical fluctuation even though the errors shown might suggest otherwise. This is confirmed by the fact that the 2^+ and 2^- should be degenerate. In any case, despite the caveats, this plot does illustrate quite clearly the degeneracy of the trace and Pfaffian masses.

The above comparison is on a lattice that is large enough for finite volume corrections to be invisible for the states we consider. One can also ask if the SO(4) and SU(2) particle masses remain the same when calculated on smaller volumes where some of the states are affected by finite volume corrections. So in table 3 we show the results of a calculation on a smaller $34^2 56$ lattice at the same couplings as above. The significant breaking of the 2^\pm degeneracy in SU(2) is a finite volume effect [4]. We see that the SO(4) Pfaffian spectrum

| | SU(2), $\beta = 13.87$ | SO(4), $\beta = 15.1$ | |
|-------|------------------------|-----------------------|------------|
| J^P | Trace | Trace | Pfaffian |
| 0^+ | 0.4777(12) | 0.4751(30) | 0.4787(20) |
| 0^- | 1.002(10) | 1.047(11) | 1.037(10) |
| 2^+ | 0.7441(10) | 0.764(9) | 0.735(9) |
| 2^- | 0.7909(22) | 0.798(5) | 0.784(12) |
| 1^+ | 1.069(6) | 1.084(39) | 1.075(33) |
| 1^- | 1.075(3) | 1.176(16) | 1.056(46) |
| l_f | 0.5046(8) | 0.6791(29) | 0.549(22) |

Table 3. Lightest glueball masses SU(2) and SO(4) on a $34^2 56$ lattice at the couplings shown for various spins J and parity P . Also shown is the energy of the lightest flux tube, l_f , winding around a periodic spatial direction.

exhibits a similar finite volume breaking of the 2^\pm degeneracy and that it provides an acceptable match to both the SO(4) trace spectrum and to the SU(2) spectrum.

Numerically we find that the Pfaffian and trace states appear to be orthogonal. This manifests itself in a striking way in the fact that the Pfaffian 0^+ operators have zero overlap onto the vacuum while the corresponding trace operators have a very large vacuum overlap, which is subtracted in the calculation so as to expose the 0^+ ground state glueball. That is to say, the union of trace and Pfaffian operators does indeed produce pairs of degenerate glueballs but only a single vacuum state.

4.3 SO(6) and SU(4)

Earlier calculations of the light glueball spectrum in the SO(6) gauge theory [1–3] have provided strong evidence that the part of the spectrum that can be obtained from correlators of single trace operators coincides with the $C = +$ spectrum of the SU(4) gauge theory. As remarked in the Introduction, it is hard to understand how the heavier $C = +$ glueballs, which in the SU(4) theory can decay into a pair of lighter $C = -$ glueballs, can be the same in both SO(6) and SU(4) — including their decay width — unless the SO(6) gauge theory possesses particles degenerate with the $C = -$ glueballs of SU(4).

As we have just seen above, in SO(2) the Pfaffian particles are precisely the $C = -$ particles of U(1). Both SO(2) and SO(6) belong to the $SO(4k+2)$ series within which, we argued earlier, the Pfaffian operators are orthogonal to the trace operators, something that we have confirmed numerically to be the case (within small errors). Thus it is natural to conjecture that the Pfaffian particles of SO(6) correspond to the $C = -$ particles of SU(4).

To test this conjecture we have performed calculations in SO(6) and SU(4) at lattice bare couplings $\beta = 46.0$ and $\beta = 59.14$ respectively, which have been chosen so as to lead to the same mass gap (within errors) in the two theories, and also to be close to the continuum limit (for the quantities previously calculated). We use the same $46^2 48$ lattice in both cases. The resulting masses are listed in table 4. The Pfaffian particles have been placed in the

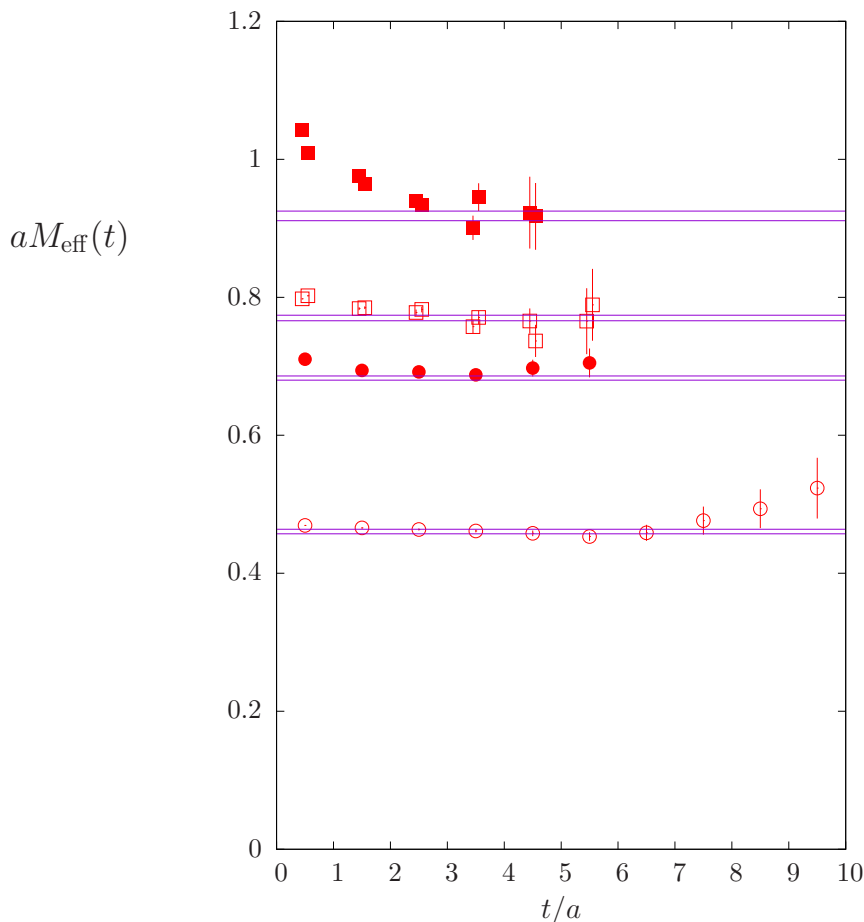


Figure 2. Effective energies of the lightest glueballs in SO(6) at $\beta = 46.0$ on a $46^2 48$ lattice: trace $J^P = 0^+$, \circ and trace 2^\pm , \square with Pfaffian ‘ 0^+ ’, \bullet , and Pfaffian ‘ 2^\pm ’, \blacksquare . Pairs of lines are $\pm 1\sigma$ bands for the $J^{PC} = 0^{++}, 0^{--}, 2^{++}, 2^{+-}$ masses (in ascending order) obtained in SU(4) at $\beta = 59.14$ on the same size lattice.

same rows as in our earlier SO(2) calculations. We observe a convincing confirmation of our conjecture that the Pfaffian particles of SO(6) correspond to the $C = -$ particles of the SU(4) gauge theory, at least for the lightest states where the error estimates are under reasonable control. In table 5 we repeat the exercise on a smaller $36^2 44$ lattice, with the same conclusion. All this provides strong numerical evidence that the spectra of the SU(4) and SO(6) gauge theories are indeed the same, with the Pfaffian particles of the latter corresponding to the $C = -$ particles of the former.

As in the case of SO(4), it is useful to display some of the effective mass plots that underpin our SO(6) mass estimates. This we do in figure 2 for the lightest trace and Pfaffian ‘ 0^+ ’ and ‘ 2^\pm ’ states. We see reasonably identifiable effective mass plateaux in all cases. (Just as in SO(4) this is not so clear for the 1^\pm states, which are significantly heavier than those shown.) In contrast to SO(4) it is clear that the lightest trace and Pfaffian particles have very different masses. We also show the asymptotic mass estimates in for the lightest $0^{++}, 0^{--}, 2^{++}, 2^{+-}$ particles in SU(4). It is quite clear from this that

| | SU(4), $\beta = 59.14$ | SO(6), $\beta = 46.0$ | |
|-----------|------------------------|-----------------------|-----------|
| J^{PC} | Trace | Trace | Pfaffian |
| 0^{++} | 0.4605(33) | 0.4612(19) | |
| 0^{++*} | 0.689(9) | 0.708(5) | |
| 0^{--} | 0.683(3) | | 0.692(4) |
| 0^{--*} | 0.857(5) | | 0.874(7) |
| 0^{-+} | 0.974(10) | 1.013(11) | |
| 0^{+-} | 1.084(14) | | 1.179(21) |
| 2^{++} | 0.770(4) | 0.759(15) | |
| 2^{-+} | 0.771(4) | 0.764(15) | |
| 2^{--} | 0.919(6) | | 0.913(21) |
| 2^{+-} | 0.917(8) | | 0.935(8) |
| 1^{++} | 1.031(39) | 1.126(14) | |
| 1^{-+} | 1.083(5) | 1.087(44) | |
| 1^{--} | 1.048(9) | | 1.094(15) |
| 1^{+-} | 1.062(5) | | 1.026(54) |
| $l_{k=1}$ | 0.5389(11) | 0.7138(89) | 1.166(55) |

Table 4. Lightest glueball masses in SU(4) and SO(6) on a $46^2 48$ lattice at the couplings shown for various spins J and parity P . For SU(4) labelled by charge conjugation C , and for SO(6) in the two sectors indicated, as explained in text. Also shown is the energy of the lightest k -string, l_k , winding around a periodic spatial direction.

while the trace particles correspond to the $C = +$ particles of SU(4), the Pfaffian particles correspond to the $C = -$ particles, as conjectured above.

4.4 SO(8) and SO(10)

For $\text{SO}(2k > 6)$ there is no $\text{SU}(N)$ group with the same Lie algebra. Since nonetheless the perturbative planar $N \rightarrow \infty$ limits of $\text{SU}(N)$ and $\text{SO}(N)$ gauge theories coincide [6] it is natural to conjecture that the ($C = +$) glueball spectra will also coincide. And there is strong numerical evidence [1–3] that this is indeed so. To get numerical evidence for the fate of Pfaffian particles in this limit we need some calculations for $2k > 6$ and so we have performed calculations in SO(8) and in SO(10).

Our calculations in SO(8) are at $\beta = 84.0$ on a $28^2 36$ lattice, and the resulting glueball mass estimates are listed in table 6. In units of the inverse mass gap the spatial size is $laM_{0^{++}} \sim 14$ which is a little smaller than the smaller of the two lattices we used for SO(6). To check for finite volume corrections we have also performed calculations on a $22^2 40$ lattice at the same coupling corresponding to the smaller spatial size $laM_{0^{++}} \sim 11$. We find that the masses are the same within errors so we assume that any finite volume effects are negligible for our purposes.

| | SU(4), $\beta = 59.14$ | SO(6), $\beta = 46.0$ | |
|------------|------------------------|-----------------------|-----------|
| J^{PC} | Trace | Trace | Pfaffian |
| 0^{++} | 0.4617(26) | 0.4639(16) | |
| 0^{++*} | 0.698(3) | 0.706(5) | |
| 0^{--} | 0.680(4) | | 0.677(4) |
| 0^{--*} | 0.849(8) | | 0.872(9) |
| 0^{-+} | 0.967(18) | 0.957(30) | |
| 0^{+-} | 1.114(11) | | 1.142(21) |
| 2^{++} | 0.772(3) | 0.778(5) | |
| 2^{-+} | 0.777(3) | 0.773(6) | |
| 2^{--} | 0.923(3) | | 0.945(10) |
| 2^{+-} | 0.923(3) | | 0.947(9) |
| 1^{++} | 1.087(17) | 1.096(14) | |
| 1^{-+} | 1.085(5) | 1.129(16) | |
| 1^{--} | 0.988(38) | | 1.083(16) |
| 1^{+-} | 1.048(15) | | 1.080(15) |
| $l_{k=1}$ | 0.4149(10) | | |
| $l_{k=2A}$ | 0.5681(14) | 0.5642(19) | |
| $l_{k=2S}$ | 0.986(14) | | 0.977(26) |

Table 5. Lightest glueball masses in SU(4) and SO(6) on a $36^2 44$ lattice at the couplings shown for various spins J and parity P . For SU(4) labelled by charge conjugation C , and for SO(6) in the two sectors indicated, as explained in text. Also shown is the energy of the lightest k -string, l_k , winding around a periodic spatial direction.

| | SO(8), $\beta = 84.0$ | | SO(10), $\beta = 120.0$ | |
|----------|-----------------------|-----------|-------------------------|-----------|
| J^P | Trace | Pfaffian | Trace | Pfaffian |
| 0^+ | 0.5101(21) | 0.990(15) | 0.6022(24) | 1.400(57) |
| 0^{+*} | 0.768(6) | 1.182(25) | 0.9210(42) | 1.73(14) |
| 0^- | 1.069(19) | 1.528(73) | 1.263(35) | 2.206(55) |
| 2^+ | 0.848(7) | 1.276(30) | 1.001(6) | 1.826(30) |
| 2^- | 0.856(5) | 1.270(24) | 1.009(5) | 1.841(27) |
| 1^+ | 1.194(21) | 1.429(36) | 1.455(15) | 2.11(6) |
| 1^- | 1.188(24) | 1.356(46) | 1.476(13) | 2.09(6) |
| l_f | 0.4829(15) | 1.126(11) | 0.5057(17) | 1.445(44) |

Table 6. Lightest glueball masses in SO(8) on a $28^2 36$ lattice and in SO(10) on a $22^2 30$ lattice at the couplings shown for various spins J and parity P . Also shown is the energy of the lightest flux loop, l_f , winding around a periodic spatial direction.

In table 6 we also list the results of our glueball mass calculation in $\text{SO}(10)$. This is on a $22^2 30$ lattice at $\beta = 120.0$. Here $laM_{0^{++}} \sim 13$ which we shall assume has no significant finite volume corrections given what we have just observed for $\text{SO}(8)$.

4.5 N dependence

Since the $\text{SO}(2k)$ Pfaffian involves a product of k fields, it is natural to expect the Pfaffian particle masses to increase $\propto k$ as k increases, just like baryons in N -colour QCD. It is interesting to test this expectation and also to see whether the sub-leading corrections to this behaviour are small, just as they have been found to be for the ‘trace’ particles [1–3].

In figure 3 we display the masses of the lightest $J = 0$, $J = 2$ and $J = 1$ Pfaffian particles in units of the lightest trace $J^P = 0^{++}$ glueball for our $\text{SO}(2k)$ groups. (Masses taken from tables 1, 2, 4, 6.) The most striking feature of this plot is the nearly linear growth with k of the mass of the $J = 0$ Pfaffian glueball: the fit in the plot is simply $M_{J=0}^{\text{Pf}}/M_{0^{++}} = 0.06 + 0.47k$. This is the lightest and hence most accurately measured of our masses. We use a subleading correction that is down by a single power of k because in $\text{SO}(N)$ gauge theories this is the case in general when one considers diagrams [6]. (Caveat: we have not shown that this is the case for Pfaffian operators but we expect it to be so.) We show similar linear fits for $J = 2$ and $J = 1$ Pfaffian particles, where we have constrained the coefficient of the linear piece to be independent of J . For $J = 2$ this appears to work well except for the very lowest value of k , while for the heavier $J = 1$ Pfaffian particles one has to go to larger k to see this linear rise. This is no surprise: as the particles become heavier the subleading correction becomes larger and presumably so do the higher order corrections in k , and this will provide an increasing curvature to the dependence of the masses on k . In summary we see from figure 3 clear evidence for the asymptotic linear growth with k for the Pfaffian particles.

The fact that the lightest Pfaffian particle has small corrections to the leading $\propto k$ behaviour encourages us to make the following simple argument for the mass of the lightest $J = 0$ Pfaffian particle in units of the lightest (trace) $J^P = 0^{++}$ mass i.e. for the ratio $m_0^{\text{Pf}}/m_{0^{++}}$. For $\text{U}(1)$ we expect the continuum theory to be a free theory of $J^{PC} = 0^{--}$ particles. The lightest 0^{++} state is composed of two non-interacting 0^{--} particles so we expect $m_0^{\text{Pf}}/m_{0^{++}} = m_{0^{--}}/m_{0^{++}} = 2$ using the identity between $\text{SO}(2)$ and $\text{U}(1)$. For $\text{SO}(4)$ we expect, as argued above, that $m_0^{\text{Pf}}/m_{0^{++}} = 1$. Assuming no corrections to the expected large- k behaviour of $m_0^{\text{Pf}}/m_{0^{++}} \propto k$ we infer that in $\text{SO}(6)$ we have $m_0^{\text{Pf}}/m_{0^{++}} = 1.5$. Since the ground state $J = 0$ Pfaffian corresponds to the 0^{--} of $\text{SU}(4)$ we have the prediction for $\text{SU}(4)$ that $m_{0^{--}}/m_{0^{++}} = 1.5$ which provides a good approximation to the calculated value [4] of $m_{0^{--}}/m_{0^{++}} = 1.465(5)$. Of course this is no ‘vanilla’ prediction: we need to use the fact that the observed corrections to the leading k dependence are small.

5 Pfaffian strings and confinement

In $\text{SU}(N)$ gauge theories there is the well-known and elegant connection between the (spontaneous breaking of the) Z_N centre symmetry of $\text{SU}(N)$ and (de)confinement. As reviewed in section 3.4 this same argument extends to the Z_2 centre of $\text{SO}(2k)$: in the confining phase

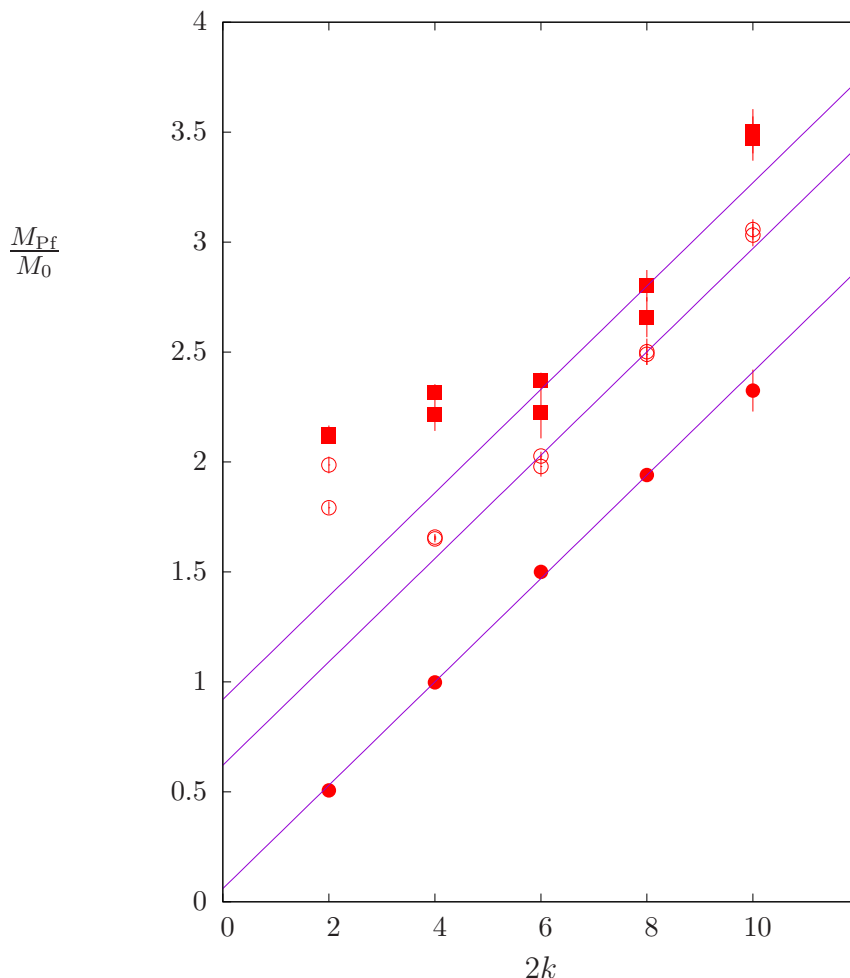


Figure 3. Lightest Pfaffian masses in units of the lightest trace $J^P = 0^+$ mass in our various $\text{SO}(2k)$ gauge theories: $J = 0$ (●), $J^P = 2^\pm$ (⊙) and $J^P = 1^\pm$ (■).

a flux loop Φ_l that winds once around the periodic x direction will satisfy $\langle \text{Tr}\{\Phi_l\} \rangle = 0$ and $\langle \text{Tr}\{\Phi_c^\dagger\} \text{Tr}\{\Phi_l\} \rangle = 0$ where Φ_c is any contractible loop. The finite volume states to which $\text{Tr}\{\Phi_l\}$ couples are flux tubes that wind around the x -torus. As shown in [1, 2] the resulting string tensions are consistent between $\text{SU}(4)$ and $\text{SO}(6)$ if one takes into account the fact that the fundamental $\text{SO}(6)$ flux corresponds to the totally antisymmetric piece of $f \otimes f$ in $\text{SU}(4)$, usually labelled as $k = 2A$. They are also consistent between $\text{SO}(4)$ and twice the string tension of $\text{SU}(2)$ (twice because of the two $\text{SU}(2)$ groups) and, indeed, between $\text{SO}(N \rightarrow \infty)$ and $\text{SU}(N \rightarrow \infty)$ when expressed, for example, in units of the mass gap in each theory [1, 2].

All this is for trace operators. What happens when we take the Pfaffian of Φ_l instead of the trace? As remarked earlier, under the Z_2 of $\text{SO}(2k)$ we have $\Phi_l \rightarrow -\Phi_l$ and hence $\text{Pf}\{\Phi_l\} \rightarrow (-1)^k \text{Pf}\{\Phi_l\}$ since $\text{Pf}\{\Phi_l\}$ contains a product of (pieces of) Φ_l k times. Thus it is natural to expect that in $\text{SO}(4k+2)$ $\text{Pf}\{\Phi_l\}$ will project onto some kind of confining flux tube wrapped around the x -torus, but not necessarily so in $\text{SO}(4k)$. That is to say,

| group | β | l | $aE_{\text{Tr}}(l)$ | $a\sqrt{\sigma_{\text{Tr}}}$ | $aE_{\text{Pf}}(l)$ | $a\sqrt{\sigma_{\text{Pf}}}$ |
|--------|---------|-----|---------------------|------------------------------|---------------------|------------------------------|
| SO(2) | 2.2 | 28 | 0.7513(31) | 0.1658(4) | 0.7452(49) | 0.1652(5) |
| SO(4) | 15.1 | 50 | 1.0284(68) | 0.1442(5) | 1.214(14) | 0.1565(9) |
| | | 46 | 0.913(18) | 0.1418(14) | 1.121(15) | 0.1569(11) |
| | | 42 | 0.8706(33) | 0.1450(3) | 0.971(12) | 0.1530(10) |
| SO(6) | 46.0 | 46 | 0.7138(89) | 0.1256(8) | 1.166(55) | 0.1600(37) |
| | | 36 | 0.5642(19) | 0.1268(3) | 0.977(26) | 0.1660(23) |
| SO(8) | 84.0 | 28 | 0.4829(15) | 0.1339(2) | 1.126(11) | 0.2022(10) |
| | | 22 | 0.3684(15) | 0.1336(3) | 0.824(17) | 0.1963(20) |
| SO(10) | 120.0 | 22 | 0.5057(17) | 0.1552(3) | 1.445(44) | 0.2584(38) |

Table 7. Energy and resulting string tension of lightest flux loop of length l in the Trace and Pfaffian sectors for the groups shown and for our largest values of l . The string tensions have been extracted using the simple ‘Nambu-Goto’ formula in eq. (2.5).

the energy of the lightest state obtained from correlators of $\text{Pf}\{\Phi_l\}$ should grow roughly linearly with the length l_x in the case of $\text{SO}(4k+2)$, but maybe not in $\text{SO}(4k)$. We shall now try to determine what actually happens in the gauge groups investigated in this paper. We begin with $\text{SO}(2)$ and $\text{SO}(6)$ which belong to the $\text{SO}(4k+2)$ series and then move on to $\text{SO}(4)$ which belongs to the $\text{SO}(4k)$ series. Finally we briefly consider larger k .

5.1 $\text{SO}(2)$ and $\text{U}(1)$

We showed earlier that the trace and Pfaffian in $\text{SO}(2)$ correspond to the real and imaginary parts of the trace in $\text{U}(1)$. In $\text{U}(1)$ the real and imaginary parts of a flux loop operator are simply $2\text{ReTr}\{\Phi_l\} = \text{Tr}\{\Phi_l\} + \text{Tr}\{\Phi_l^\dagger\}$ and $2\text{ImTr}\{\Phi_l\} = \text{Tr}\{\Phi_l\} - \text{Tr}\{\Phi_l^\dagger\}$ and these are $C = +$ and $C = -$ flux loops respectively. The usual centre symmetry argument ensures that the correlator of $\text{Tr}\{\Phi_l\}$ with $\text{Tr}\{\Phi_l^\dagger\}$ is zero, which means that the $C = +$ and $C = -$ correlators are identical and hence that the $C = +$ and $C = -$ flux loop spectra are degenerate. The same argument will hold for $\text{SU}(N > 2)$ which is why in all these cases one normally quotes a single string tension, although strictly speaking there are two equal ones corresponding to $C = +$ and $C = -$. With $\text{SO}(2)$ the $C = +$ and $C = -$ flux loops correspond to trace and Pfaffian operators respectively, which are mutually orthogonal, and produce equal string tensions as we see in table 7. As expected we also see in table 7 that this string tension equals the corresponding $\text{U}(1)$ string tension. Given the evident identity between $\text{SO}(2)$ and $\text{U}(1)$ and given the fact that the latter is well known [18–22] to possess linear confinement, we do not show here any numerical results displaying this fact.

5.2 $\text{SO}(6)$ and $\text{SU}(4)$

One expects the fundamental flux of $\text{SO}(6)$ to correspond to the $k = 2A$ flux in $\text{SU}(4)$. (As usual $k = 2A$ denotes the totally antisymmetric piece of $f \otimes f$.) In [1, 2] it was shown that, within small errors, the energy of the lightest (traced) flux loop is indeed the

| | SU(4), $\beta = 59.14$ | | | SO(6), $\beta = 46.0$ | |
|-----------------|------------------------|-------------|-------------|------------------------|------------------------|
| $l.l_\perp.l_t$ | $aE_{k=1}$ | $aE_{k=2A}$ | $aE_{k=2S}$ | $aE_{k=1}^{\text{Tr}}$ | $aE_{k=1}^{\text{Pf}}$ |
| 18.48.64 | 0.1812(9) | 0.2572(20) | 0.427(8) | 0.2546(16) | 0.426(12) |
| 22.42.52 | 0.2379(5) | 0.3316(15) | 0.561(8) | 0.3286(12) | 0.569(8) |
| 26.40.52 | 0.2911(7) | 0.3948(35) | 0.690(7) | 0.3964(19) | 0.695(9) |
| 30.40.50 | 0.3416(10) | 0.4694(10) | 0.795(19) | 0.4660(19) | 0.803(17) |
| 36.36.44 | 0.4142(17) | 0.5670(23) | 0.986(14) | 0.5642(19) | 0.977(26) |
| 46.46.48 | 0.5389(11) | 0.7331(33) | 1.23(5) | 0.7138(89) | 1.17(6) |

Table 8. Energy of lightest flux loop of length l in the Trace and Pfaffian sectors for SO(6) and in the $k = 1$, $k = 2A$ and $k = 2S$ sectors for SU(4).

same as that of the lightest $k = 2A$ flux loop in SU(4) and displays the same nearly-linear growth with length. In the minimal $k = 2$ sector of SU(4) there is in addition to the $k = 2A$ representation also the $k = 2S$ representation which is the totally symmetric piece of $f \otimes f$ [23]. (There are of course larger $k = 2$ representations obtained, for example, from $f \otimes f \otimes f \otimes \bar{f}$.) Since the trace in the $k = 2A$ representation is real while that of $k = 2S$ contains an imaginary piece, one might conjecture that the Pfaffian in SO(6) maps to the $k = 2S$ of SU(4) and that the operator $\text{Pf}\{\Phi_l\}$ projects onto winding flux tubes that correspond to those in SU(4) carrying $k = 2S$ flux. We shall now see that this conjecture is (largely) correct.

We have performed calculations in SO(6) at $\beta = 46.0$ on $l \times l_y \times l_t$ lattices and have calculated the spectrum obtained from correlators of traces and Pfaffians of operators Φ_l that wind once around the x direction of length l . As l decreases we increase the transverse size l_y and also l_t so as to minimise finite transverse volume and finite temperature effects. The lattice sizes are listed in table 8 where we also list the lightest energies we obtain from our operators of the form $\text{Pf}\{\Phi_l\}$ and $\text{Tr}\{\Phi_l\}$. These energies are plotted in figure 4 where we see that the Pfaffian energy grow linearly just like that of the usual (traced) loop. That is to say, the Pfaffian loop operator does indeed project onto states that consist of flux tubes winding around the spatial torus, just like the standard trace operator. It is however orthogonal to the latter and the string tension (slope) is clearly very different.

Just as for the glueballs it is useful to show the effective energy plots that are behind this calculation. This we do for the Pfaffian strings in figure 5 and for the trace strings in figure 6, and we compare them to our asymptotic energy estimates. It is clear that the trace energies are accurate and unambiguous. The Pfaffians have worse overlaps and the energies are larger, all of which makes the calculations much less reliable. For the Pfaffian strings the identification of effective mass plateaux is moderately convincing for $l \leq 26$ and perhaps also for $l = 30$, but for $l = 36$ and particularly for $l = 46$ one has to assume, on the basis of what one sees at smaller l , that one is close to a plateau by $t \sim 3.5a$ in order to extract any energy at all.

To make the comparison with SU(4) we perform calculations in SU(4) on exactly the same lattice sizes at the coupling $\beta = 59.14$ where the mass gap equals the mass gap in

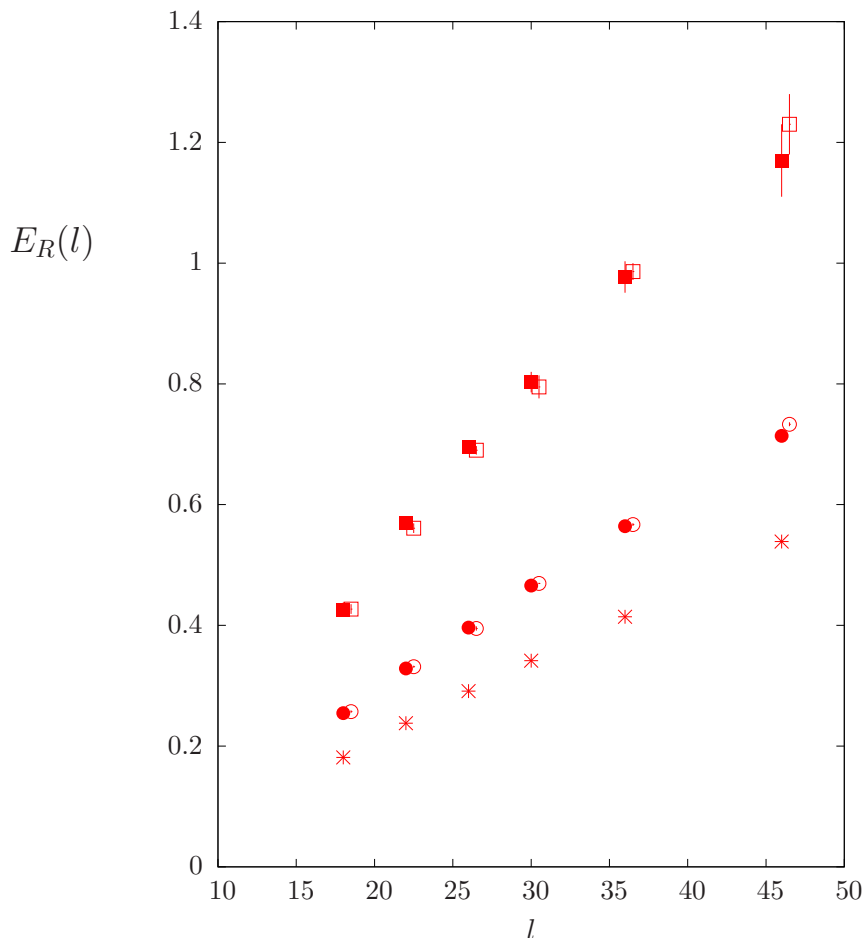


Figure 4. Energy of lightest flux tube against its length: carrying fundamental (\star), $k = 2A$ (\circ) and $k = 2S$ (\square) flux in SU(4) at $\beta = 59.14$ and trace (\bullet) and Pfaffian (\blacksquare) in SO(6) at $\beta = 46.0$. Some points shifted slightly for visibility.

SO(6) at $\beta = 46.0$. We calculate the lightest energies of flux loops carrying fundamental ($k = 1$), $k = 2A$ and $k = 2S$ flux versus the length l of the loop. The results are displayed in figure 4. We observe that the $k = 2A$ energies of SU(4) are indeed degenerate with those of the traced loops in SO(6), as observed in earlier work [1, 2]. More interestingly we observe a similar degeneracy between the $k = 2S$ energies of SU(4) and the Pfaffian flux loops in SO(6), providing strong numerical evidence for our above conjecture.

However this conjecture can only be ‘largely’ correct. The Pfaffian and trace flux loop operators in SO(6) are orthogonal. This is not the case for $k = 2A$ and $k = 2S$ in SU(4) although in practice they are very nearly orthogonal [24]. The mixing between $k = 2A$ and $k = 2S$ flux tubes may be driven by tunnelling and it may be that this is the kind of physics in which the groups will differ even if the Lie algebras are the same. In any case while all this means that our matching between SO(6) and SU(4) flux loops is entirely adequate for most practical purposes, the theoretical underpinning is not yet complete.

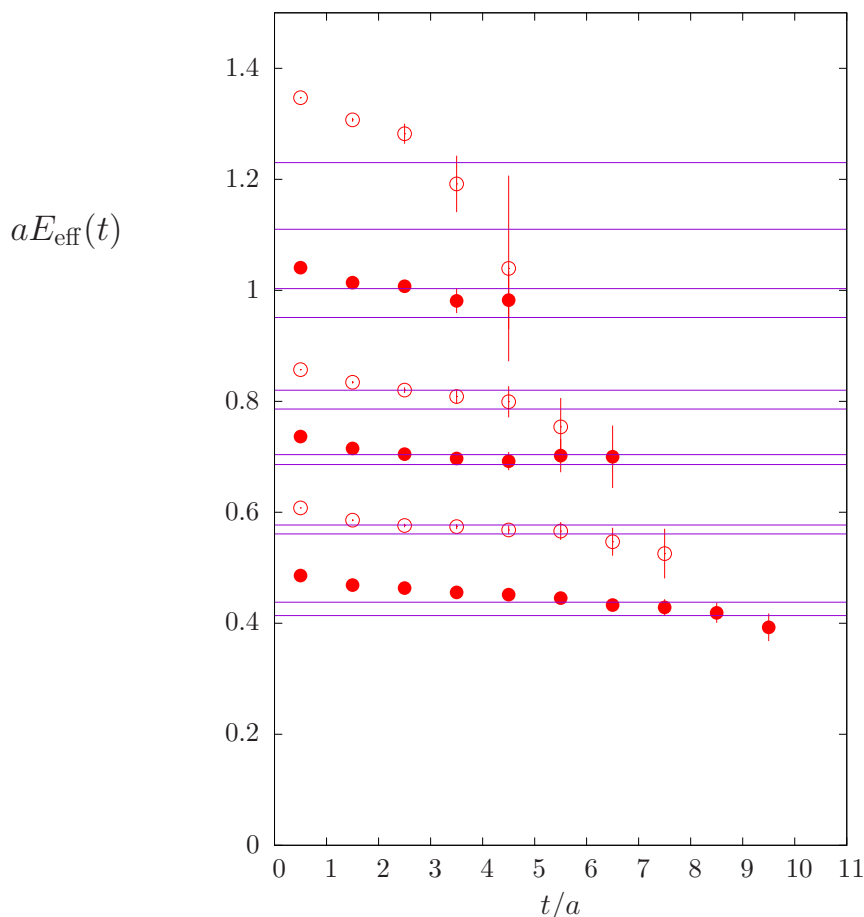


Figure 5. Effective energies of the ground state Pfaffian string in $\text{SO}(6)$ at $\beta = 46.0$ for lengths $l = 18, 22, 26, 30, 36, 46$ in order. Pairs of lines are $\pm 1\sigma$ around mean of our asymptotic energy estimates.

5.3 $\text{SO}(4)$ and $\text{SU}(2) \times \text{SU}(2)$

Unlike $\text{SO}(2)$ and $\text{SO}(6)$ the group $\text{SO}(4)$ does not belong to the $\text{SO}(4k+2)$ series so although we know that the $\text{SO}(4)$ theory is linearly confining [1, 2] we are not confident that the Pfaffian of a flux loop operator projects onto some kind of confining flux tube. To investigate this question we have performed calculations in $\text{SO}(4)$ and $\text{SU}(2)$ at $\beta = 15.1$ and $\beta = 13.87$ respectively, at which β values the $\text{SO}(4)$ and $\text{SU}(2)$ mass gaps are equal (within small errors). We perform calculations for various values of l in $\text{SO}(4)$ and for three values for $\text{SU}(2)$. (These include the smallest and largest $\text{SO}(4)$ values of l , as well as an intermediate value.)

The expectation is that the string tension in $\text{SO}(4)$ is twice that of $\text{SU}(2)$. To compare our $\text{SU}(2)$ flux loop energies to those of $\text{SO}(4)$ we proceed as follows. For each value of l we extract the $\text{SU}(2)$ string tension using the formula in eq. (2.5). We then double that string tension and calculate the flux loop with the doubled string tension, again using eq. (2.5). This energy is listed in table 9 as E_{eff} . We see from the table that these values agree reasonably well with the energies E^{Tr} of the traced flux loops in $\text{SO}(4)$, confirming the conclusions of earlier work [1, 2].

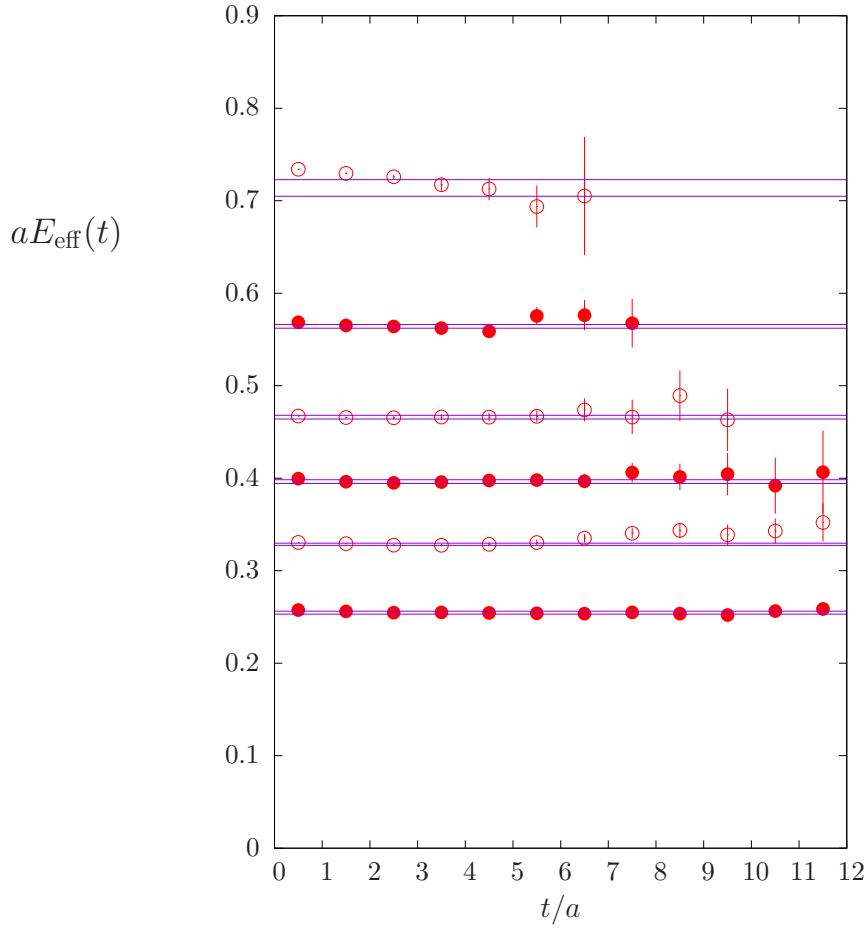


Figure 6. Effective energies of the ground state traced string in $\text{SO}(6)$ at $\beta = 46.0$ for lengths $l = 18, 22, 26, 30, 36, 46$ in order. Pairs of lines are $\pm 1\sigma$ around mean of our asymptotic energy estimates.

| | $\text{SU}(2), \beta = 59.14$ | | $\text{SO}(4), \beta = 15.1$ | | |
|-------------------|-------------------------------|-------------------|------------------------------|------------------|--------------------------|
| $l.l_{\perp}.l_t$ | $aE_{k=1}$ | aE_{eff} | aE^{Tr} | aE^{Pf} | $a\tilde{E}^{\text{Pf}}$ |
| 18.38.56 | 0.1550(7) | 0.3433(14) | 0.3094(39) | 0.3529(27) | [0.527(9)] |
| 30.30.56 | | | 0.5874(42) | 0.589(17) | 0.572(27) |
| 34.34.56 | | | 0.6791(29) | 0.679(11) | 0.549(22) |
| 38.38.56 | | | 0.7694(54) | 0.755(37) | 0.582(27) |
| 42.42.56 | | | 0.8706(33) | 0.971(12) | [0.48(14)] |
| 46.46.48 | 0.5046(8) | 1.0198(16) | 0.913(18) | 1.121(15) | [0.88(10)] |
| 50.50.56 | | | 1.0284(68) | 1.214(14) | [0.46(18)] |

Table 9. Energy of lightest flux loop of length l in the Trace and Pfaffian sectors for $\text{SO}(4)$ and in the fundamental for $\text{SU}(2)$. E_{eff} is the energy the $\text{SU}(2)$ flux loop would have if the string tension was twice as large. See text for explanation of Pfaffian energies.

Our calculation of the lightest Pfaffian flux loop turns out to be more complex. As usual our variational procedure maximises $\exp(-aE)$ and we calculate the energy of the state from the effective mass plateau of the corresponding correlator. Normally this gives us the lightest energy (as can be checked by looking at the correlators of the higher excited states). However this is not guaranteed: if the lightest state has a very poor overlap onto our basis of operators it may appear in the large t tail of a correlator that one would normally expect to correspond to a heavier excited state. This is what we find with the Pfaffian of the flux loop. In table 9 we list the energy obtained from operator that maximises $\exp(-aE)$ as E^{Pf} and, where lighter states appear in the large t tails of what should be excited states, we list the lightest of these as \tilde{E}^{Pf} . As l increases the overlap of this lightest state decreases and so it is harder to identify an effective mass plateau, so we place the $l = 46, 50$ values in brackets to indicate this uncertainty. For $l = 18$ the lightest state is indeed the one that maximises $\exp(-aE)$ so we show the next energy as \tilde{E}^{Pf} and again place it in brackets to indicate some uncertainty in this assignment. Even if we ignore the bracketed values of \tilde{E}^{Pf} it appears that the Pfaffian projects onto at least one state whose energy does not increase with l , and which is therefore particle-like rather than string-like, although its overlap is very small and appears to decrease with increasing l . In contrast, the lightest state with a substantial overlap onto our pfaffian basis has an energy E^{Pf} that increases with l as one would expect for some kind of stringy flux loop.

To better expose these spectra we plot the energies in figure 7. We observe the nearly linear rise with l of E^{Tr} and the fact that it is compatible with what one expects from $\text{SU}(2)$. The ‘normal’ Pfaffian energy, E^{Pf} , appears to be nearly degenerate with E^{Tr} at small l but appears to grow faster at the largest values of l . However this latter behaviour may be illusory: the energies are becoming large, the overlaps are mediocre and so we may be overestimating the energies by not going far enough in t to identify the effective energy plateau. The presence of a particle-like lighter state with energy \tilde{E}^{Pf} appears to be unambiguous. However a more accurate calculation is clearly needed here.

To show how reliable are the above observations, we display the effective energy plots for the traced flux loop in figure 8, for the string-like Pfaffian in figure 9 and for the particle-like Pfaffian in figure 10. The traced flux loop clearly has a very good overlap onto our basis of operators and so has convincing effective energy plateaux, except for $l = 46$ which appears to suffer a large statistical fluctuation which we try to encompass with larger errors on our final energy estimate. From figure 9 we see that the overlaps of the Pfaffian string-like states are much poorer leading to the plateaux being at larger t where the larger errors make the identification more difficult. However for the most part our energy estimates are quite plausible, even if the unquantified systematic errors, which grow with l , leave some room for doubt, particularly at larger l . Finally we see in figure 10 that the particle-like Pfaffian has a very poor overlap onto our basis for almost all l which means that identification of an effective energy ‘plateau’ around $t \sim 6a$ is very subjective. The exception is $l = 18$ where the overlap is better and we have a decent plateau — but as we remarked above, there is some uncertainty in ascribing this state to belong to the particle-like family. In any case the positivity of the correlator guarantees that $E_{\text{eff}}(t)$ always provides an upper bound for the true energy, and it is quite clear from figure 10

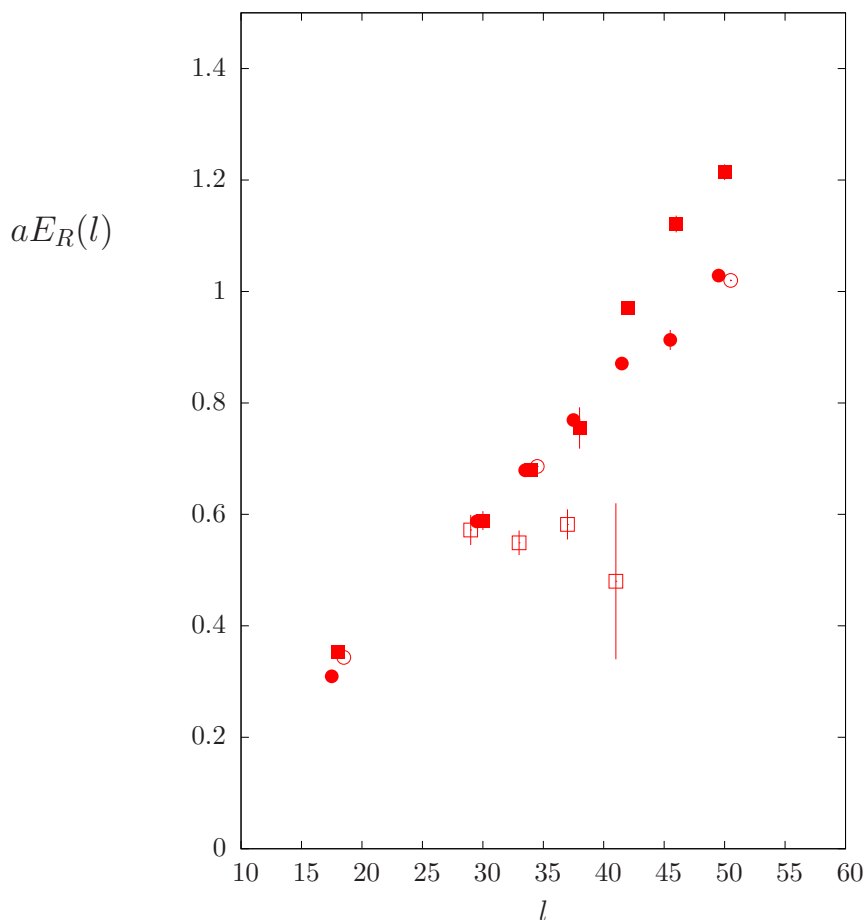


Figure 7. Energy of lightest flux tube against its length: carrying fundamental (\circ) flux in SU(2) at $\beta = 13.87$, and trace (\bullet) and Pfaffian (\square , \blacksquare) in SO(4) at $\beta = 15.1$. See text for difference between the two Pfaffian entries. Some points shifted slightly for visibility.

that this energy does not grow with l and hence this state is not some kind of confining flux loop, but must be essentially particle-like.

5.4 Larger N

We have also performed some calculations of the trace and Pfaffian of flux loop operators in SO(8) and SO(10). In SO(10) we have calculated the energies for only one value of l since this belongs to the SO($4k + 2$) series where we expect both the trace and the Pfaffian of flux loop operators to project onto stringy states whose energies grow roughly linearly with l . The energy is listed in table 7 together with the string tension extracted using eq. (2.5).

For SO(8) we have two values of l . The energies and corresponding string tensions are listed in table 7. We see that the string tensions from the trace are equal within errors and that the Pfaffian string tensions are compatible with each other. That is to say, both the trace and Pfaffian of a flux loop project onto states that are stringy with an energy that grows almost linearly with l . The trace and Pfaffian string tensions are however very different. In addition although SO(8) falls into the SO($4k$) series, just like SO(4), there is

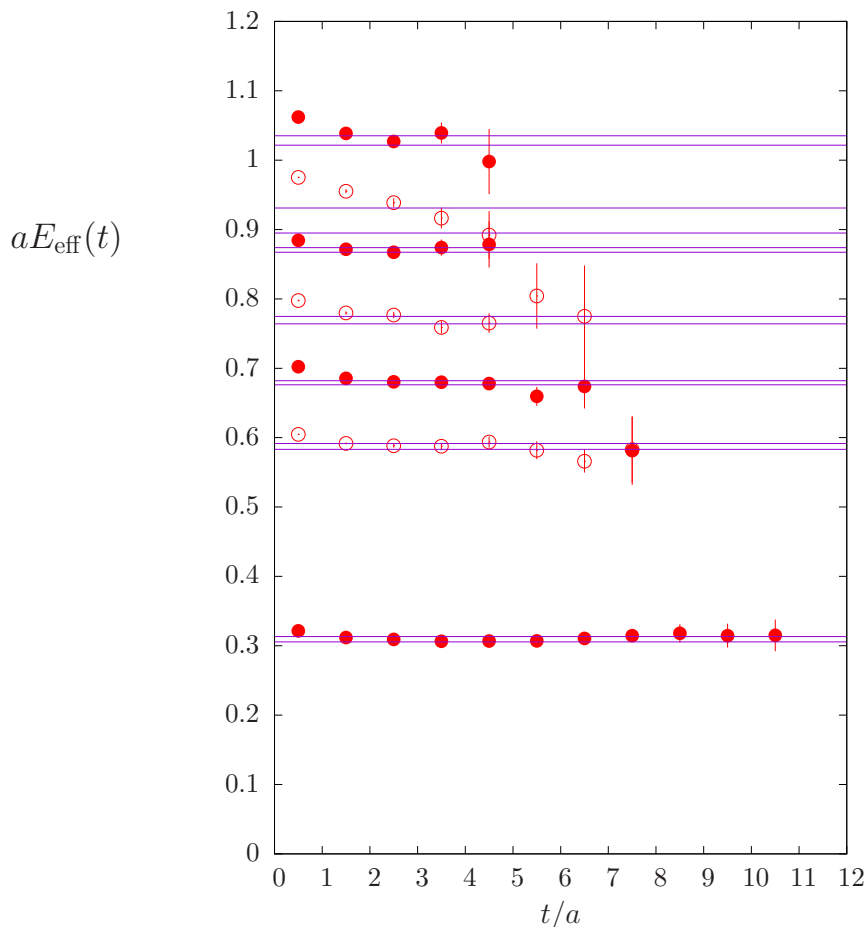


Figure 8. Effective energies of the ground state traced string in $SO(4)$ at $\beta = 15.1$ for lengths $l = 18, 30, 34, 38, 42, 46, 50$ in order. Pairs of lines are $\pm 1\sigma$ around mean of our asymptotic energy estimates.

no sign of a lighter particle-like state hidden amongst the excited state correlators. This is so despite the fact that the values of l in units of (either) string tension are in the range in which the particle-like states were readily visible in $SO(4)$. We conclude that if they are there, then the overlap must be suppressed by some power of k so that they have become invisible, within errors, for $SO(8)$. That is to say, the particle-like state appears to decouple for both large l and large k .

The dependence of the string tension on k is displayed in figure 11. We plot the two string tensions in units of the lightest trace $J^P = 0^+$ mass versus $2k$, just as we did for the particle masses in figure 3. The string tension from the trace of flux loops decreases with k , which is no surprise since one expects the $SO(\infty)$ and $SU(\infty)$ fundamental string tensions to be equal while the $SO(4)$ string tension is roughly twice that of $SU(2)$. (All in units of the mass gap.) The string tension from the Pfaffian of the same flux loops increases with k , in these units, with a nearly linear rise for larger values of k . However in units of the lightest Pfaffian particle the Pfaffian string tension also decreases with increasing k , as we see in figure 3. From the coefficients of the linearly rising pieces (units of the mass gap) we can estimate that this last ratio will asymptote to a value ~ 0.1 .

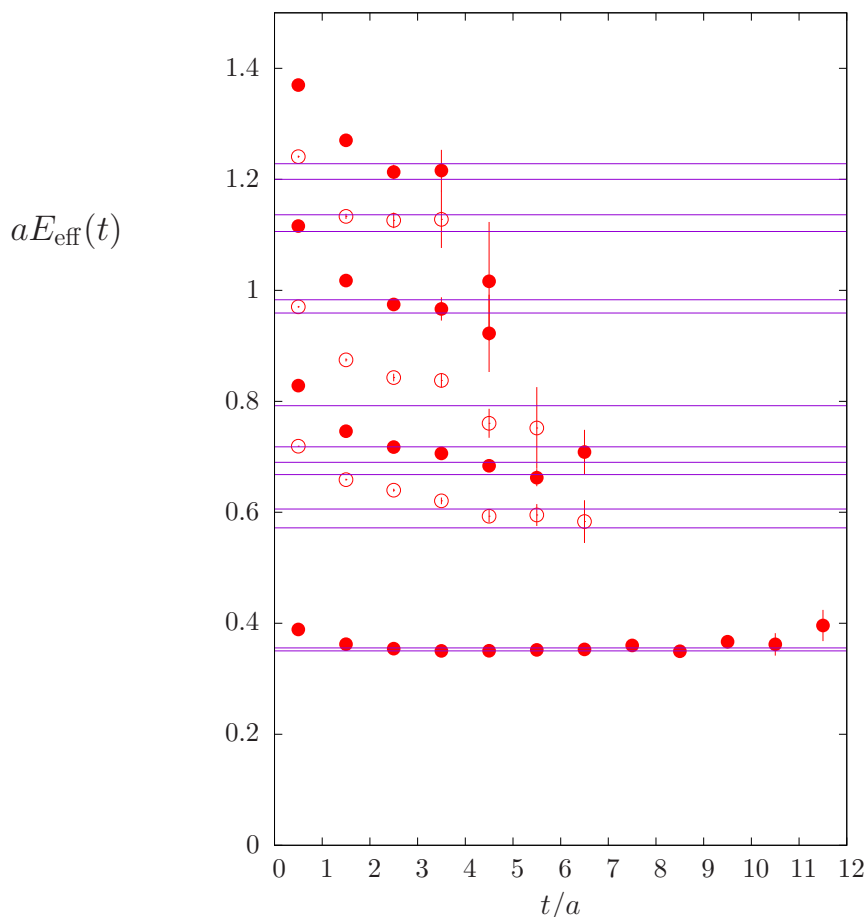


Figure 9. Effective energies of the ground state Pfaffian string in $SO(4)$ at $\beta = 15.1$ for lengths $l = 18, 30, 34, 38, 42, 46, 50$ in order. Pairs of lines are $\pm 1\sigma$ around mean of our asymptotic energy estimates.

6 Open questions

There are a number of issues that this study has brought out which we have either only partially addressed or not addressed at all. We briefly list some of them here. These are open problems which need to be addressed.

We have seen that in $SO(4)$ the Pfaffian of a flux loop operator can project onto a particle like state as well as onto a state whose energy grows roughly linearly with the length of the loop, but apparently with a string tension that is larger than the $SU(2)$ one. To understand what these states represent we need a more accurate determination of their properties.

We have pointed out that one can generalise the $SO(2k)$ Pfaffian to involve any k adjoint operators, even if they differ. The properties of such Pfaffians may be interesting. For example in $SO(4)$ an operator such as $\epsilon_{i_1 i_2 i_3 i_4} \Phi_{l, i_1 i_2}(x) \Phi_{c i_3 i_4}(x)$ where $\Phi_l(x)$ is a non-contractible flux loop operator and $\Phi_c(x)$ is a contractible loop should be exactly orthogonal to all particle-like states, something that $\text{Pf}\{\Phi_l\}$ itself is not, as we have seen.

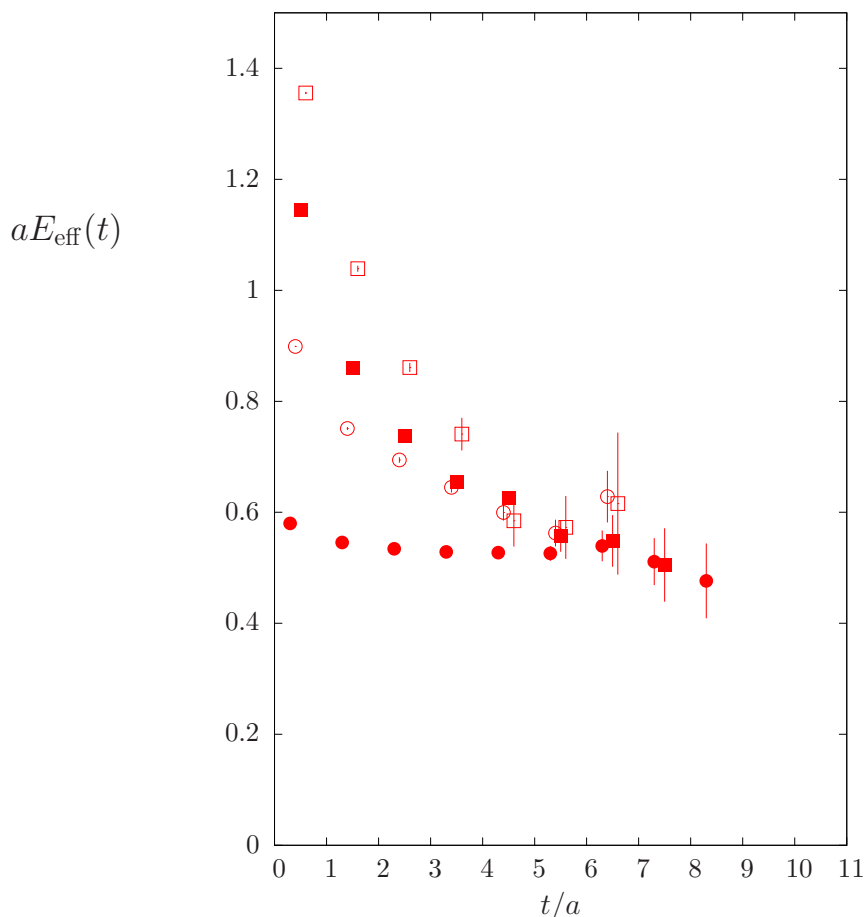


Figure 10. Effective energies of the particle-like state in the Pfaffian ‘string’ spectrum in $\text{SO}(4)$ at $\beta = 15.1$ for lengths $l = 18, \bullet$, $l = 30, \circ$, $l = 34, \blacksquare$, $l = 38, \square$. Some points shifted slightly for visibility.

We have seen numerically that in $\text{SO}(6)$ if Φ_l is a non-contractible loop operator then the operators $\text{Tr}\{\Phi_l\}$ and $\text{Pf}\{\Phi_l\}$ correspond, within small errors, to the $k = 2A$ and $k = 2S$ flux loops of $\text{SU}(4)$. This neat mapping cannot however be exact since we know that in $\text{SU}(4)$ there is a small but non-zero overlap between the $k = 2A$ and $k = 2S$ operators [24], whereas in $\text{SO}(6)$ the traced and Pfaffian operators are orthogonal. Presumably it is the two $k = 2$ orthogonal mixed states that correspond to our two types of $\text{SO}(6)$ operators — but this needs to be better understood.

A more general question concerns the $\text{SO}(2N + 1)$ gauge theories for which we have no Pfaffian operator and about which we have had nothing to say in this paper. On the other hand, earlier work has found that the physics of the $\text{SO}(2N + 1)$ and $\text{SO}(2N)$ gauge theories seems to form one continuous family, even at small N [1, 2]. Although at large N Pfaffian particles and strings become massive and so decouple from the physics, at small N they are relevant and so one can ask whether there is something in the $\text{SO}(2N + 1)$ theories that correspond to the Pfaffians in $\text{SO}(2N)$, e.g. in $\text{SO}(3)$, which has the same Lie algebra as $\text{SU}(2)$.

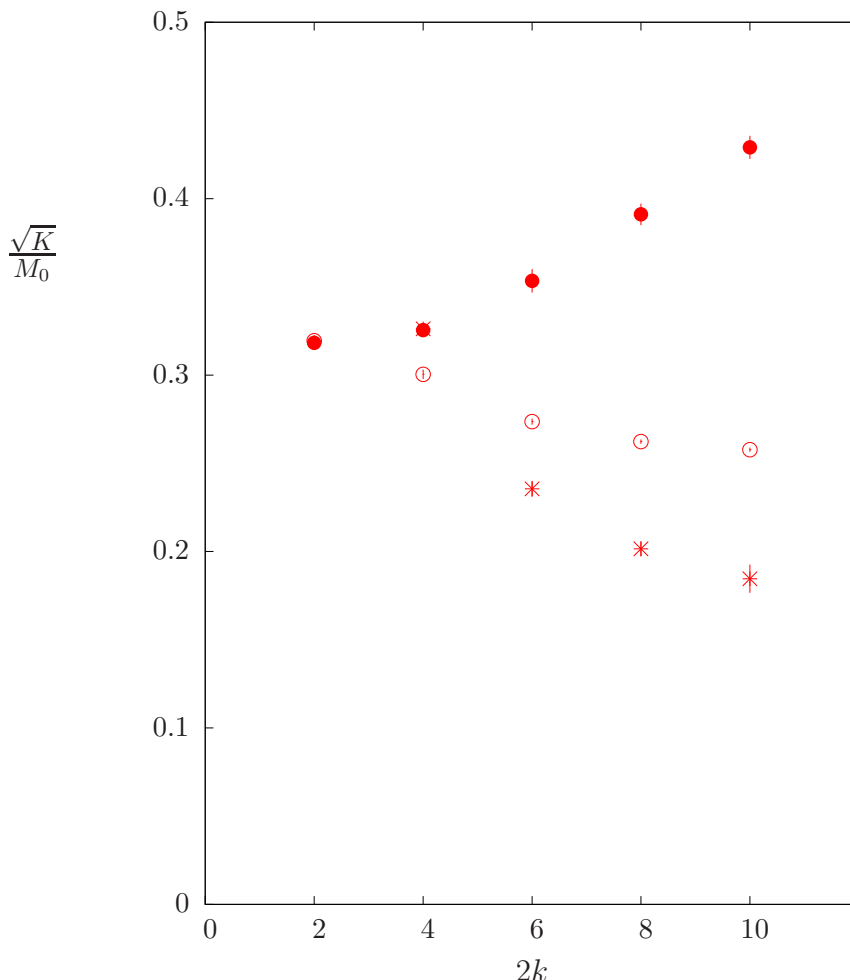


Figure 11. String tensions in $\text{SO}(2k)$ from the traces, \circ , and Pfaffians, \bullet , of flux loop operators in units of the mass gap. Also the Pfaffian string tension in units of the lightest Pfaffian glueball, \star .

7 Conclusions

In this paper we showed that (generalised) Pfaffian operators play an essential role in completing the glueball spectrum calculations of $\text{SO}(2k)$ gauge theories. In particular for low k where some of the $\text{SO}(2k)$ gauge theories have the same Lie algebras as some $\text{SU}(N)$ gauge theories, the Pfaffians provide the half of the spectrum that is missing when we use only traces of loops for our operator basis. Thus they provide the counterparts in $\text{SO}(6)$ of the $C = -$ particles in $\text{SU}(4)$, and similarly in $\text{SO}(2)$ the counterparts of the $C = -$ particles in $\text{U}(1)$. And they also provide the doubling of the spectrum in $\text{SO}(4)$ that one might expect given that it has the same Lie algebra as $\text{SU}(2) \times \text{SU}(2)$. As k grows we could identify a linearly growing component to the mass, which is no surprise given the fact that the Pfaffian of $\text{SO}(2k)$ contains pieces of the product of k adjoint fields. The linear growth of the lightest Pfaffian particle in units of the mass gap was $M_{\text{Pf}}/M_{0++} \simeq 0.06 + 0.47k$ which means that the mass increases by $\sim 0.5M_{0++}$ as the Pfaffian operator length increases by one adjoint field. It is intriguing that the same energy gap (about one half of the mass gap) arises in other contexts, for example in the massive excitation of the winding flux

tube [25, 26]. In any case the fact that the masses of the Pfaffian particles increase with k means that they decouple as $k \rightarrow \infty$ and so do not upset the expected equality of the $\text{SO}(\infty)$ and $\text{SU}(\infty)$ mass spectra.

We also investigated the states that couple to Pfaffians of the flux loop operators whose trace projects onto confining flux tubes that wind around a spatial circle (of our periodic lattice). We argued that these will represent some kind of flux tube in $\text{SO}(4k+2)$ gauge theories but not necessarily in $\text{SO}(4k)$ gauge theories. For $\text{SO}(2)$ we saw, rather trivially, that the Pfaffian of flux loops corresponds to the $C = -$ flux loop of $\text{U}(1)$, complementing the trace that corresponds to the $C = +$ flux loop of $\text{U}(1)$. Our calculations show that for $\text{SO}(6)$ the energy increases nearly linearly with length as one expects for a confining flux tube and it corresponds essentially to the symmetric $k = 2$ flux tube of $\text{SU}(4)$, complementing the trace that corresponds to the antisymmetric $k = 2$ flux tube of $\text{SU}(4)$. This confirms our expectations for the $\text{SO}(4k+2)$ series. $\text{SO}(4)$ is the first of the $\text{SO}(4k)$ series and here we found that the Pfaffian of flux loop operators had a visible but very weak overlap onto a particle like state as well as a much larger overlap onto a string state. The features of the latter are ambiguous within the modest accuracy of our calculations. We also performed similar calculations in $\text{SO}(8)$ which showed no sign of a particle-like state which suggests some kind of large- k decoupling. Together with our $\text{SO}(10)$ calculations, all this showed that the $\text{SO}(2k)$ Pfaffian string tension increases nearly linearly with k at larger k , so that just like the Pfaffian particles these Pfaffian strings will decouple as $k \rightarrow \infty$.

Finally we emphasise that the present study has been very much an exploratory one, although it has already succeeded in answering the question that originally motivated it. At the trivial level this means that in various cases the calculations need greater accuracy and smaller errors in order to be completely convincing and/or useful. Less trivially we have identified and discussed in section 6 a number of questions that need to be addressed and a number of further calculations that need to be performed.

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