Celestial Charges from Twistor Space



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A thesis submitted for
Doctor of Philosophy
Trinity 2026

Acknowledgements

It is not the destination or the journey that makes an experience great; it is the company.

First and foremost, I would like to thank my supervisor, Lionel Mason, for his support throughout my DPhil and for introducing me to twistor theory and complex geometry. I am incredibly grateful to have been given the opportunity to work on a subject that I enjoy so much.

I have been fortunate to work with many amazing people. I would like to thank my collaborators Roland Bittleston, Giuseppe Bogna, Alex Goodenbour, Emil Have, Simon Heuveline, Romain Ruzziconi, Atul Sharma, David Skinner, and Akshay Yelleshpur Srikant. In particular, I would like to thank Giuseppe, my academic older brother, who helped me enormously during my first year and always took the time to answer my questions. I would also like to give special thanks to Romain for his guidance and mentorship throughout my DPhil, and for making all of our collaborations so enjoyable.

My time in the mathematical physics group has been incredibly special, entirely because of the people in it. Thank you to Joseph, Sujay, Carmen, and Horia for making me feel so welcome during my first year and for always being there to answer my questions. Thank you to Andrea and Enrico for being the best party hosts our group could have asked for, and to Palash for being a great friend and always being willing to talk about D&D with me. Thank you to Daniel and Alice for being wonderful office-mates and colleagues. I am also extremely grateful to Niklas and Chris (KC) for being such great friends and for always taking the time to help me.

I have had the great pleasure of being part of the Happy Hour crew during my time as a DPhil student. Thank you to Marta, Tabea, Chris, Nico, Pieter, Tim, Niklas, and Maria for making it such a fun experience and for answering the age-old question: "How many mathematicians does it take to fill a fridge?"

Thank you so much to Maria for being such an amazing friend. From Happy Hour and climbing to all the biscuit breaks, these are memories that I will always cherish. Thank you also to Matteo for being the best buddy I could have asked for. Thank you to Athan and Basil for making my Oxford experience so amazing and fun.

I would like to thank my family, and particularly my parents, for their endless support and constant love. I would not be where I am today without them, and I will always be grateful.

Finally, I would like to thank my love, Holly. You have made my life so much richer and more fulfilling. From the endless laughter to our lovely trips together, I cannot think of anyone else with whom I would rather spend my time, and I am excited for all the time we have ahead of us.

Abstract

The celestial $Lw_{1+\infty}$ and S algebra symmetries in asymptotically flat spacetimes arose as a reinterpretation of collinear limits of the S-matrix in gravity and Yang-Mills, respectively, as OPEs in celestial holography. On twistor space, they have a natural geometric interpretation as holomorphic gauge transformations of complex structures and holomorphic bundles. Using this framework, we provide a first-principle derivation of the canonical generators associated with these symmetries starting from the twistor actions for self-dual gravity and Yang-Mills on asymptotic twistor space. We express these charges as surface integrals over the celestial sphere in terms of spacetime data at null infinity. The connection between twistor space and spacetime expressions at null infinity is achieved via an integral formula for the asymptotic equations of motion. Using the integral formula, we show how these symmetries act on the asymptotic data and how these are symmetries of gravity and Yang-Mills from a phase space perspective by showing the invariance of the asymptotic equations of motion. In the gauge theory case, we use the charges coming from the twistor action to construct conserved currents at spatial infinity, connecting to the twisted holography approach. We further our analysis by considering how these symmetries translate to the bulk of the self-dual theory. We find that the generators of the symmetries give solutions to higher-valence twistor equations, and the charges are built through the integrable hierarchy of the self-dual theory. We also extend some of our results to the case of a non-vanishing cosmological constant.

Statement of Originality

This thesis presents work developed in the following publications, towards which I contributed substantially.

Chapter 3 is based on

[1] A. Kmec, L. Mason, R. Ruzziconi, and A. Yelleshpur Srikant, *Celestial $Lw_{1+\infty}$ charges from a twistor action*, JHEP **10** (2024) 250, arXiv:2407.04028.

[2] R. Bittleston, G. Bogna, S. Heuveline, A. Kmec, L. Mason, and D. Skinner, *On AdS_4 deformations of celestial symmetries*, JHEP **07** (2024) 010, arXiv:2403.18011.

Chapter 4 based on

[3] A. Kmec, L. Mason, R. Ruzziconi, and A. Sharma, *S-algebra in Gauge Theory: Twistor, Spacetime and Holographic Perspectives*, Class. Quant. Grav. **42** (2025) no. 19 195008, arXiv:2506.0188.

Chapter 5 is based on

[4] A. Kmec, L. Mason, and R. Ruzziconi, *Quasi-Local Celestial Charges and Multipoles*, arXiv:2604.13362.

Chapter 2 is loosely based on the review sections of the publications listed above, as well as on material from my earlier papers and notes.

For the jointly authored publications listed above, my principal contributions consisted of performing the computations, formulating the arguments, and drafting the manuscript.

AI Declaration

I confirm that I have read the University and MPLS Division guidance on generative AI and have complied with it in preparing this thesis.

The original mathematical contributions of this thesis, including the problem formulation, computations, proofs, results, and conclusions, are my own work except where standard scholarly references and co-authorship statements indicate otherwise. I did not use generative AI to produce mathematical arguments, proofs, calculations, or conclusions.

I used ChatGPT in a limited capacity during the preparation of the literature-review material in the introduction and the review chapter. For selected topics, I used ChatGPT to help identify relevant terminology, themes, connections between areas, and possible literature to consult. The majority of the literature-review text was written by me, drawing on my previous papers, notes, and my own reading of the relevant literature.

No confidential, unpublished third-party, personal, or restricted material was entered into ChatGPT.

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Chapter 1

Introduction

The holographic principle has provided one of the most influential frameworks for investigating quantum gravity [1, 2], suggesting that the physics within a gravitational spacetime may be encoded by a theory defined on its boundary. Its best-understood realisation is the AdS/CFT correspondence [3, 4, 5], which relates a gravitational theory in an asymptotically AdS spacetime to a conformal field theory living on its timelike boundary. This correspondence has generated many insights into quantum gravity and strongly coupled quantum field theories [6]. Nevertheless, asymptotically AdS spacetimes do not describe isolated gravitating systems. Such systems, including scattering processes and sources emitting gravitational radiation, are instead naturally modelled by asymptotically flat spacetimes [7]. This motivates the search for a corresponding holographic description of flat-space gravity.

Attempts to understand asymptotically flat spacetimes from a boundary perspective go back to the work of Newman and Penrose [8, 9, 10]. Although these early investigations did not produce a complete holographic dictionary, they uncovered geometric structures that continue to play an important role in the subject. Modern approaches to flat-space holography have developed principally into Carrollian [11, 12] and celestial holography [13, 14, 15], which seek to encode bulk gravitational physics in theories associated, respectively, with null infinity and the celestial sphere.

In the celestial holography program, there have been different approaches to understanding how observables in the bulk are encoded in data on the boundary. Three complementary strands of this programme are particularly relevant to the present work: asymptotic symmetries, celestial amplitudes and soft theorems, and twistorial approaches.

Asymptotic Symmetries: The asymptotic-symmetry approach begins by constructing the gravitational phase space at null infinity. This is done by fixing a gauge, imposing asymptotic boundary conditions, and determining the residual gauge transformations that preserve both [16, 17]. The asymptotic symmetry group is defined as the residual gauge transformations that preserve the boundary conditions with associated non-vanishing charge [18, 19]. The fact that the charge does not vanish means that the action of the symmetry is non-trivial on the asymptotic phase space. For gravity, when one considers the Bondi gauge, the asymptotic fields include the radiative shear together with quantities such as the mass and angular-momentum aspects [16, 17]. The original analysis of Bondi, van der Burg, Metzner, and Sachs leads to the BMS group consisting of the semidirect product of the Lorentz group and an infinite-dimensional abelian group called supertranslations [16, 20, 17]. Depending on the boundary conditions and regularity assumptions imposed on the celestial sphere, this group may be enlarged to include local conformal transformations [21], smooth sphere diffeomorphisms [22], and Weyl transformations [23]. This analysis can become very subtle as gravitational charges can be non-integrable because radiation crosses null infinity, and their algebra can involve field-dependent generators and brackets. An interesting feature of these charges is that they are generally not conserved along null infinity due to gravitational radiation. In other words, the charges evaluated at different cuts of null infinity are related by flux-balance laws. The celebrated Bondi mass loss formula is the primary example that demonstrates that outgoing gravitational radiation causes the Bondi mass to decrease [16, 17].

The canonical charge associated with an asymptotic symmetry typically decomposes into soft and hard contributions. The soft contribution to the charge is linear in the fields, whereas the hard contribution has non-linear field contributions and acts on the in and out states of the scattering process. A conservation law equating charges on the future corner of past null infinity $\mathscr{I}^-_+$ and the past corner of future null infinity $\mathscr{I}^+_-$ gives a Ward identity for the gravitational S-matrix. The Ward identity for supertranslations is equivalent to Weinberg's leading soft graviton theorem, while suitable extensions of the BMS symmetry reproduce the subleading soft graviton theorem [24, 25, 26, 22]. A crucial ingredient in this correspondence is the matching of asymptotic data between past and future null infinity through spatial infinity. Appropriate antipodal matching conditions [27, 28] identify the angular direction of an incoming particle at $\mathscr{I}^-$ with the corresponding outgoing direction at $\mathscr{I}^+$. These conditions turn separate relations on the two components of null infinity into conservation laws for the full scattering problem. Their precise formulation depends on the behaviour of the fields near spatial infinity and can be particularly subtle.

The asymptotic symmetries approach lends itself well to the extrapolated dictionary of the usual AdS/CFT correspondence. The asymptotic symmetry group on an asymptotically AdS spacetime gives the conformal group of the boundary. At the same time, the expectation value of the stress-energy tensor for the boundary CFT is encoded in a subleading order of the Fefferman-Graham expansion of the metric [29, 30, 31]. From the perspective of Carrollian holography, the bulk four-dimensional theory is described by a putative three-dimensional theory living on null infinity which has BMS symmetry [32, 11]. The analogy with AdS/CFT is nevertheless incomplete. The boundary of an asymptotically AdS spacetime is timelike and carries a non-degenerate conformal metric, whereas null infinity is equipped with a conformal Carrollian structure. Radiation can cross null infinity, so boundary charges obey flux-balance laws rather than ordinary conservation equations. One can obtain non-conserved charges in AdS by choosing leaky boundary conditions [33, 34].

Moreover, the asymptotic expansion contains radiative and Coulombic sectors whose relations are constrained by the Einstein equations and the Bianchi identities.

Amplitudes and Soft Theorems: The central object in the amplitudes approach to celestial holography is the celestial amplitude, which is derived from the S-matrix by casting it into boost eigenstates. The process begins with the observation that the Lorentz group in four-dimensional Minkowski spacetime acts as the global conformal group on the celestial sphere [35]. A massless momentum may be parametrised by an energy ω and a point $(z, \bar{z})$ on this sphere,

$$p^\mu = \omega q^\mu(z, \bar{z}) = \epsilon\omega(1 + z\bar{z}, z + \bar{z}, -i(z - \bar{z}), 1 - z\bar{z}) \,, \tag{1.1}$$

where $\epsilon = \pm 1$ distinguishes outgoing and incoming states. Passing from momentum eigenstates to boost eigenstates is achieved via Mellin transformation. In this context, an n-point scattering amplitude in momentum space $\mathcal{A}_n$ is a function of $\{\omega_i, J_i, z_i, \bar{z}_i, \epsilon_i\}_{i=1,\dots,n}$ for each particle, where J_i denotes the particle helicities. The momentum space amplitude is mapped to a celestial amplitude via the integral formula [36, 15]

$$\mathcal{M}_n\left(\{\Delta_i, J_i, z_i, \bar{z}_i, \epsilon_i\}_{i=1,\dots,n}\right) = \prod_{i=1}^{n} \int_0^\infty \mathrm{d}\omega_i \, \omega_i^{\Delta_i - 1} \mathcal{A}_n\left(\{\omega_i, J_i, z_i, \bar{z}_i, \epsilon_i\}_{i=1,\dots,n}\right) . \tag{1.2}$$

Under a Lorentz transformation, z undergoes a Möbius transformation while a z-dependent conformal factor rescales ω. The Mellin measure converts this energy rescaling into the conformal dimension Δ, while the little-group weight supplies the two-dimensional spin J. The celestial amplitude can be shown to transform covariantly as a correlation function of two-dimensional conformal primary operators with weights $h_i = \frac{\Delta_i + J_i}{2}$, $\bar{h}_i = \frac{\Delta_i - J_i}{2}$, such that the celestial amplitude is given by

$$\mathcal{M}_n\left(\{\Delta_i, J_i, z_i, \bar{z}_i, \epsilon_i\}_{i=1,\dots,n}\right) = \langle \mathcal{O}^{\epsilon_1}_{J_1,\Delta_1}(z_1, \bar{z}_1) \dots \mathcal{O}^{\epsilon_n}_{J_n,\Delta_n}(z_n, \bar{z}_n) \rangle \,. \tag{1.3}$$

In this sense, the celestial transform re-organises the four-dimensional S-matrix into objects possessing the kinematical covariance of correlators in a two-dimensional conformal field theory.

In the celestial basis, soft limits acquire a natural interpretation. For the momentum space amplitude, the insertion of a low-energy photon, gluon, or graviton is governed by a universal soft theorem [37, 38, 27]. Schematically, and most directly at tree level, the soft expansion of the amplitude is given by

$$\mathcal{A}_{n+1}(q^\mu, \{p_i^\mu\}_{i=1,\cdots n}) \sim \left[\frac{1}{\omega} S^{(0)} + S^{(1)} + \omega S^{(2)} + \cdots\right] \mathcal{A}_n(\{p_i^\mu\}_{i=1,\cdots n}) \,, \tag{1.4}$$

when $\omega \to 0$. After the Mellin transform, the soft expansion is encoded in poles at special values of the conformal dimension. The residues at these poles define conformally soft operators whose insertions obey two-dimensional Ward identities [39, 40]

$$\langle \mathcal{O}^{(r)}_{soft} \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \lim_{\Delta \to 1-r} (\Delta + r - 1) \mathcal{M}_{n+1} = \tilde{S}^{(r)} \mathcal{M}_n \tag{1.5}$$

where Δ is the conformal dimension corresponding to the soft ω, $\tilde{S}^{(r)}$ is the Mellin transformed $S^{(r)}$ appearing in the soft theorem, and the soft operator $\mathcal{O}^{(r)}_{soft}$ is given by

$$\mathcal{O}^{(r)}_{soft}(z, \bar{z}) = \lim_{\Delta \to 1-r} (\Delta + r - 1) \mathcal{O}^{\epsilon}_{J,\Delta}(z, \bar{z}) \tag{1.6}$$

The leading soft photon and gluon theorems give rise to celestial current algebras, while the leading soft graviton theorem is associated with supertranslation symmetry. The subleading soft graviton theorem is related to superrotations and to a stress-tensor-like operator on the celestial sphere. Thus, universal infrared factorisation properties of the four-dimensional S-matrix become symmetry constraints on the putative celestial theory [40, 39, 13].

By considering collinear limits of the amplitudes in momentum space, the algebra

of soft operators can be extracted. The collinear limit of an amplitude is given by

$$\mathcal{A}_n\left(p_1,\ldots,p_n\right) \xrightarrow{p_1||p_2} \text{Split}(p_1,p_2,p_1+p_2)\,\mathcal{A}_{n-1}\left(p_1+p_2,p_3,\ldots,p_n\right)\,, \tag{1.7}$$

where $\text{Split}(p_1,p_2,p_1+p_2)$ is called the splitting function for the amplitude. On the celestial sphere, the collinear limit corresponds to two points approaching each other, motivating an OPE structure. The corresponding OPE coefficients are Mellin transforms of the splitting functions and, at tree level, often take the form of Euler beta functions [40]. By considering collinear limits and soft limits, an infinite tower of currents corresponding to the modes of the soft operators appear, including the $Lw_{1+\infty}$ algebra[1] in gravity [41, 42]

$$[w^p_{m,a}, w^q_{n,b}] = (m(q-1)-n(p-1))w^{p+q-2}_{m+n,a+b}\,, \tag{1.8}$$

where $w^p_{m,a}$ are modes of the soft operators corresponding to positive helicity gravitons. Here $p = 1, \frac{3}{2}, 2, \ldots$ labels the conformally soft level and is therefore related to the order in the energetic soft expansion, $|m| \leq p-1$, and $a \in \mathbb{Z}$ is the loop index. In gauge theory, the analogous algebra is the S algebra [41, 43]

$$[S^a_{p,m,l}, S^b_{q,n,k}] = f^{ab}_c S^c_{p+q,m+n,l+k}\,, \tag{1.9}$$

where $S^a_{k,l,m}$ are the modes of the soft operators for positive helicity gluons for the gauge group with structure constant f^{ab}_c. Therefore, the amplitudes perspective provides a direct route from scattering data to the spectrum, OPEs, and symmetries expected of a celestial CFT. At the same time, there are several conceptual challenges: celestial correlators generally have a continuous spectrum, possess distribu-

[1] More precisely, the celestial algebra for gravity is given by $LHam(\mathbb{C}^2)$, the loop algebra of the canonical transformations on $\mathbb{C}^2$ equipped with its standard holomorphic Poisson bracket. This celestial algebra is often referred to as the $Lw_{1+\infty}$ algebra, and we will use both terminologies in this thesis.

tional support inherited from momentum conservation, and need not satisfy the usual assumptions of locality or unitarity familiar from conventional Euclidean CFTs. Loop corrections and non-universal subleading soft factors all require a careful treatment [44, 45, 46].

Twistor Approaches: The third perspective, and the one considered in this thesis, makes the infinite symmetry algebras appearing in celestial holography geometric. For gravity, the $Lw_{1+\infty}$ algebra can be interpreted as the loop algebra of Hamiltonian transformations of the fibres of twistor space. In the self-dual sector, these transformations act geometrically as holomorphic Poisson diffeomorphisms, and their action on spacetime fields follows from the non-linear graviton correspondence [10, 8, 47]. For Yang-Mills, Ward's twistor construction [48, 49] provides the analogous statement: they are local holomorphic gauge transformations of holomorphic bundles on twistor space.

Twistor theory has had many applications to scattering amplitudes, and through this connection, has brought insights to celestial holography as well. Initiated by Witten, scattering amplitudes for $\mathcal{N} = 4$ super Yang-Mills were computed from the topological B-model on twistor space [50]. This subsequently sparked the development of the ambitwistor string, a chiral sigma model whose vertex-operator correlation functions reproduced scattering amplitudes and explained the origins of the scattering equations and the CHY formula [51, 52]. In the context of celestial holography, residues of the vertex operators for the ambitwistor string give the currents that generate the asymptotic symmetries, and the worldsheet OPE reproduces the celestial OPE [39, 53]. In the case of gravity, the twistor sigma model makes the geometric connection of Poisson diffeomorphisms to the algebra of soft operators particularly transparent, as the model's classical equations of motion describe deformed holomorphic curves in twistor space and thereby encode the incidence relations of the associated self-dual spacetime geometry [47]. These worldsheet models indicated

that celestial amplitudes had a natural twistorial origin as correlation functions of 2d theories.

Twisted holography [54] provides a more systematic framework for these observations. On twistor space, the open and closed sectors of the topological B-model are described by holomorphic field theories. Introducing branes supported on holomorphic curves can produce two-dimensional chiral algebras whose operator products are constrained by consistency with the bulk theory. Koszul duality supplies a systematic method for determining the algebra of defect operators and their couplings to bulk fields [55, 56]. In favourable examples, correlators of these chiral algebras encode four-dimensional form factors and scattering data [49]. Because the construction is formulated at the level of an action, it also provides access to quantum corrections and anomalies [57, 58]. However, the resulting spacetime theories are restricted by the requirement that anomalies cancel [59, 60], and the framework should be regarded as a controlled holomorphic sector rather than a complete holographic dual of general asymptotically flat gravity.

These three perspectives describe overlapping aspects of the celestial holography programme in different languages. The asymptotic-symmetry perspective formulates them as transformations and charges acting on the gravitational phase space at null infinity; celestial amplitudes reveal them through the soft and collinear limits of the S-matrix; and twistor theory realizes them geometrically as holomorphic gauge transformations or Hamiltonian diffeomorphisms preserving the twistor-space Poisson structure. The central aim of this thesis is to clarify the relation between the asymptotic symmetry description and the twistorial realization of these infinite-dimensional symmetry algebras.

In this thesis, we will construct the action of both the $Lw_{1+\infty}$ and the S algebra on the gravitational and Yang-Mills phase space on null infinity. This work provides a first-principles derivation of the charges found in [61, 62, 63] and their recursive structure found in [64, 65, 66, 67, 68]. To do this, we will begin by constructing the

twistor space for null infinity, called the asymptotic twistor space. Geometrically, this will be the blow-up of the usual twistor space along a distinguished holomorphic line that corresponds to spatial infinity. The radiative asymptotic data will be encoded in the geometry of gauge bundles or the deformation of complex structure on the asymptotic twistor space, while the Coulombic data will correspond to cohomology classes that can be translated to null infinity via integral transforms. Importantly, these integral transforms satisfy the evolution equations of the Coulombic data simply by requiring the twistor data live in cohomology.

For symmetry considerations, we use the twistor actions for self-dual gravity and self-dual Yang-Mills whose fields encode the asymptotic data, and, on-shell, satisfy the necessary cohomological and deformation theoretic criteria. After constructing the Noether charges associated to gauge transformations of the respective action, the $Lw_{1+\infty}$ and S algebras correspond to the residual gauge transformations that preserve the gauge used to encode radiative data that have non-vanishing Noether charge. Using the relationship between twistor and asymptotic data, we find that the infinite symmetry algebras have non-local and non-linear action on the fields at null infinity. The non-local action of these symmetries makes the direct application of the standard covariant phase space methods on spacetime obscure. We find that the $Lw_{1+\infty}$ and S algebra symmetries are symmetries of the evolution equations on the leading-order phase space data, which clarifies in what sense these are symmetries on spacetime.

In the context of gravity, we perform the analysis in the bulk of a self-dual spacetime where the symmetries preserve the self-dual geometry. The charges are also shown to be connected to quasi-local mass and multipole moment charge expressions. Furthermore, we also present some partial results of extending these considerations to spacetimes with a non-vanishing cosmological constant. In the context of Yang-Mills, rather than constructing charges on a finite cut of null infinity that satisfy flux-balance laws, we construct conserved currents that live at spatial infinity to make contact with

twisted holography.

The organization of the thesis is as follows. Chapter 2 will review all the necessary background. This includes a review of the Newman–Penrose (NP) and Geroch–Held–Penrose (GHP) formalisms and of self-dual geometry. We then move to a review of twistor theory. We first focus on the flat twistor geometry and discuss the Penrose transform. We then move on to how non-linear field equations are encoded on twistor space for both gravity and Yang-Mills. In Chapter 3, we first construct the geometry of asymptotic twistor space and consider the action for self-dual gravity to construct the $Lw_{1+\infty}$ symmetries on null infinity. We find the gauge parameters become field-dependent, and we must make use of various techniques in the covariant phase space method. We conclude the chapter with some partial results when there is a non-vanishing cosmological constant. In Chapter 4, we present the analogous analysis in Yang-Mills but in a more manifestly gauge-invariant formulation to be able to relate the discussion to conserved currents at spatial infinity. We include a discussion of boundary conditions that allow the twistor action from the bulk twistor space to extend to the asymptotic twistor space. Chapter 5 extends the $Lw_{1+\infty}$ analysis to the bulk of a self-dual spacetime and constructs the charges that generate these symmetries. We check that these charges reproduce the expected results at null infinity and extend the analysis to a null hypersurface at finite distance. We also present a discussion of how $Lw_{1+\infty}$ sits in the integrable system that underlies self-dual gravity by going to Plebanski's second heavenly gauge. Finally, in Chapter 6, we give concluding remarks and discuss several possible extensions of this work.

The appendices include some technical material: Appendix A includes further details of the NP and GHP formalism that were used in this thesis, in particular Chapter 5. Appendix B provides a review of sheaf theory and sheaf cohomology that is often used in twistor theory and the Penrose transform. This includes a discussion of the Čech–Dolbeault isomorphism that is used throughout this thesis.

Chapter 2

Spacetime and Twistor Theory

In this chapter, we will give an introduction to twistor theory in both gravity and Yang-Mills theory. The spinor notation in 4d that will be used throughout the thesis will be reviewed, and how the fields in each theory are represented as spinors. We will also define the self-dual sub-sectors of both Yang-Mills and gravity, as well as anti-self-dual perturbations on these backgrounds. We will then move to the twistor theory that encodes the field content of self-dual theories. We first explain how flat space is described using twistor theory and then move to the case of curved backgrounds in the case of gravity. We also explain how Yang-Mills can be encoded in holomorphic gauge bundles on twistor space.

2.1 Spacetime

In this work, we will work with complexified spacetimes, where the underlying manifold $\mathcal{M}$ is a complex 4-dimensional space with a holomorphic metric g. We will assume $\mathcal{M}$ has a spin structure and therefore utilize the isomorphism of the (complexified) holonomy group

$$SO(4,\mathbb{C}) = SL(2,\mathbb{C}) \times SL(2,\mathbb{C})/\mathbb{Z}_2. \tag{2.1}$$

The isomorphism allows us to decompose the complexified tangent bundle into a product of spin bundles

$$TM_{\mathbb{C}} = \mathbb{S} \otimes \mathbb{S}'. \tag{2.2}$$

where $\mathbb{S}$ and $\mathbb{S}'$ are the anti-self-dual (ASD) and self-dual (SD) spin bundles, respectively. Each spin bundle has a natural symplectic structure given by the Levi-Civita symbol such that raising and lowering indices are given by $\omega^\alpha = \epsilon^{\alpha\beta}\omega_\beta$ and $\omega_\alpha = \omega^\beta \epsilon_{\beta\alpha}$ encoding the duals to each spin bundle. The natural inner product is given by $\langle \omega\lambda \rangle = \omega^\alpha \lambda_\alpha$ and $[\tilde{\omega}\tilde{\lambda}] = \tilde{\omega}^{\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha}}$. In practice, the isomorphism boils down to applying the Van de Waerden symbols on a vector index to turn it into a matrix

$$V^{\alpha\dot{\alpha}} = \sigma_a^{\alpha\dot{\alpha}} V^a = \frac{1}{\sqrt{2}} \begin{pmatrix} V^0 + V^3 & V^1 + iV^2 \\ V^1 - iV^2 & V^0 - V^3 \end{pmatrix}, \tag{2.3}$$

where $\alpha, \dot{\alpha} = 0, 1$ and the norm $2\text{det}(V^{\alpha\dot{\alpha}}) = \eta_{ab}V^a V^b = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}V^{\alpha\dot{\alpha}}V^{\beta\dot{\beta}}$.

2.1.1 Yang-Mills

We first consider Yang-Mills theory in Euclidean space $\mathbb{R}^4$ where the fields are valued in the Lie algebra $\mathfrak{g}$. The Yang-Mills action is given by

$$S_{YM}[A] = \frac{1}{g_{YM}^2} \int_{\mathbb{R}^4} \text{Tr}(F \wedge *F)\,, \tag{2.4}$$

where $*$ is the Hodge star and the 2-form $F = \frac{1}{2}F_{ab}dx^a \wedge dx^b$ is the field strength. Using the anti-symmetry of the field strength, we can write

$$F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b] = F_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} + \tilde{F}_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta}\,, \tag{2.5}$$

where we used abstract index notation to relate the left and right-hand sides. The field strength decomposes into two parts, $F_{\alpha\beta}$ and $\tilde{F}_{\dot{\alpha}\dot{\beta}}$, which are the ASD and SD

parts of F_{ab} under the Hodge star, respectively.

The equations of motion of the Yang-Mills action are given by

$$\partial^a F_{ab} + [A^a, F_{ab}] = 0\,, \tag{2.6}$$

and the gauge fields A that satisfy the equations of motion, which give a finite action, are called instantons. The special subclass of instantons that minimises the Yang-Mills action are the SD solutions. These are gauge fields A whose ASD part of the field strength $F_{\alpha\beta}$ vanishes. These field configurations solve the equations of motion through the Bianchi identity on the field strength. To see how the action is minimised, consider the second Chern class, also known as the θ-term, associated to the gauge bundle

$$S_\theta = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} F \wedge F\,. \tag{2.7}$$

Using the decomposition of the field strength into SD and ASD parts, one can prove the following inequality

$$S_{YM} \geq \frac{8\pi^2}{g_{YM}^2} S_\theta\,, \tag{2.8}$$

where equality holds when the solution is SD. From this inequality, we see that the Yang-Mills action is minimized when the instanton is SD.

When the gauge field is SD, there exist two forms of matrix-valued potentials whose derivation we now review. In double null coordinates $x^a = (u, w, z, \bar{z})$, the SD condition is given by

$$\begin{aligned} \partial_z A_u - \partial_u A_z + [A_z, A_u] &= 0\,, \\ \partial_w A_u - \partial_u A_w + \partial_{\bar{z}} A_z - \partial_z A_{\bar{z}} + [A_w, A_u] + [A_{\bar{z}}, A_z] &= 0\,, \\ \partial_{\bar{z}} A_w - \partial_w A_{\bar{z}} + [A_{\bar{z}}, A_w] &= 0\,. \end{aligned} \tag{2.9}$$

The two potential formulations can be readily found by choosing a convenient gauge for A_a. The first and last equations in (2.9) can be thought of as consistency

conditions for the existence of solutions to the following PDEs

$$\begin{aligned} \partial_u h + A_u h = 0 \qquad &, \qquad \partial_z h + A_z h = 0\,, \\ \partial_w \tilde{h} + A_w \tilde{h} = 0 \qquad &, \qquad \partial_{\bar{z}} \tilde{h} + A_{\bar{z}} \tilde{h} = 0\,. \end{aligned} \tag{2.10}$$

where h and $\tilde{h}$ are two matrix-valued functions. Therefore, a gauge can always be found such that $A_z = 0 = A_u$, i.e.

$$A = A_{\bar{z}} \mathrm{d}\bar{z} + A_w \mathrm{d}w\,, \tag{2.11}$$

via the gauge transformation $A \to h^{-1}Ah + h^{-1}\mathrm{d}h$. The SD Yang-Mills equations (2.9) now become

$$\partial_{\bar{z}} A_w - \partial_w A_{\bar{z}} + [A_{\bar{z}}, A_w] = 0 \qquad , \qquad \partial_u A_w + \partial_z A_{\bar{z}} = 0\,. \tag{2.12}$$

The second equation can be taken as a consistency condition for the existence of the K-matrix, such that

$$A = \partial_u K \mathrm{d}\bar{z} - \partial_z K \mathrm{d}w \qquad , \qquad \partial_{\bar{z}}\partial_z K + \partial_w \partial_u K + [\partial_u K, \partial_z K] = 0\,. \tag{2.13}$$

Another formulation of SD Yang-Mills is via Yang's J-matrix, which instead manifestly solves the first equation of (2.12):

$$A = J^{-1}\partial_{\bar{z}} J \mathrm{d}\bar{z} + J^{-1}\partial_w J \mathrm{d}w \qquad \text{with} \qquad J = \tilde{h}^{-1} h\,. \tag{2.14}$$

This solves the first and last equation of (2.9), and one is left with Yang's equation

$$\partial_u \left(J^{-1}\partial_w J \right) + \partial_z \left(J^{-1}\partial_{\bar{z}} J \right) = 0\,. \tag{2.15}$$

2.1.2 Gravity

We now consider Einstein gravity where the fundamental field is the metric on some underlying curved manifold $\mathcal{M}$. In a co-tetrad basis, the metric is given by

$$g = \eta_{ab}\theta^a\theta^b = \epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\theta^{\alpha\dot\alpha}\theta^{\beta\dot\beta}\,, \tag{2.16}$$

where the co-tetrad satisfy $e_{\alpha\dot\alpha}\lrcorner\,\theta^{\beta\dot\beta} = \delta^\beta_\alpha\delta^{\dot\beta}_{\dot\alpha}$ with $e_{\alpha\dot\alpha}$ being the tetrad and $\lrcorner$ denoting contraction of a vector field. In the torsion free case, the tetrad $\theta^{\alpha\dot\alpha}$ satisfy Cartan's first structure equation

$$d\theta^{\alpha\dot\alpha} + \Gamma^\alpha{}_\beta \wedge \theta^{\beta\dot\alpha} + \tilde\Gamma^{\dot\alpha}{}_{\dot\beta} \wedge \theta^{\alpha\dot\beta} = 0\,, \tag{2.17}$$

where $\Gamma_{\alpha\beta}$ and $\tilde\Gamma_{\dot\alpha\dot\beta}$ are the spin connection 1-forms on the ASD and SD spin bundles, respectively. The connection 1-forms are symmetric in their indices $\Gamma_{\alpha\beta} = \Gamma_{\beta\alpha}$. Using the spin connections, the ASD and SD Riemann 2-forms are given by

$$R_{\alpha\beta} = d\Gamma_{\alpha\beta} + \Gamma_\alpha{}^\gamma \wedge \Gamma_{\gamma\beta} \qquad \tilde R_{\dot\alpha\dot\beta} = d\tilde\Gamma_{\dot\alpha\dot\beta} + \tilde\Gamma_{\dot\alpha}{}^{\dot\gamma} \wedge \tilde\Gamma_{\dot\gamma\dot\beta}\,. \tag{2.18}$$

These make up the total Riemann tensor through $R_{ab} = R_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} + \tilde R_{\dot\alpha\dot\beta}\epsilon_{\alpha\beta}$, where we have used abstract index notation. The main benefit of using spinor notation is that there emerges a natural basis of SD and ASD 2-forms on the 4-manifold $\mathcal{M}$. As in the flat space case, we consider the decomposition of the 2-form

$$\theta^{\alpha\dot\alpha} \wedge \theta^{\beta\dot\beta} = \epsilon^{\dot\alpha\dot\beta}\Sigma^{\alpha\beta} + \epsilon^{\alpha\beta}\tilde\Sigma^{\dot\alpha\dot\beta}, \tag{2.19}$$

where we have defined $\Sigma^{\alpha\beta} = \frac{1}{2}\theta^{\alpha\dot\alpha}\wedge\theta^\beta{}_{\dot\alpha}$ and $\tilde\Sigma^{\dot\alpha\dot\beta} = \frac{1}{2}\theta^{\alpha\dot\alpha}\wedge\theta_\alpha{}^{\dot\beta}$. Thus, a generic 2-form F is given by

$$F = F_{\alpha\beta}\Sigma^{\alpha\beta} + \tilde F_{\dot\alpha\dot\beta}\tilde\Sigma^{\dot\alpha\dot\beta}\,, \tag{2.20}$$

where $F_{\alpha\beta}$ is the ASD part and $\tilde{F}_{\dot{\alpha}\dot{\beta}}$ is the SD part under the Hodge star. Cartan's first structure equation can be decomposed into two equations on the two sectors of 2-forms

$$d\Sigma^{\alpha\beta} = 2\Gamma^{(\alpha}{}_{\gamma} \wedge \Sigma^{\beta)\gamma} \quad , \quad d\tilde{\Sigma}^{\dot{\alpha}\dot{\beta}} = 2\tilde{\Gamma}^{(\dot{\alpha}}{}_{\dot{\gamma}} \wedge \tilde{\Sigma}^{\dot{\beta})\dot{\gamma}}. \tag{2.21}$$

Using the various symmetries of the Riemann tensor, it can also be expanded in the basis of ASD and SD 2-forms

$$\begin{aligned} R_{\alpha\beta} &= \psi_{\alpha\beta\gamma\sigma}\Sigma^{\gamma\sigma} + \Phi_{\alpha\beta\dot{\alpha}\dot{\beta}}\tilde{\Sigma}^{\dot{\alpha}\dot{\beta}} + \frac{R}{12}\Sigma_{\alpha\beta}, \\ \tilde{R}_{\dot{\alpha}\dot{\beta}} &= \tilde{\psi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\sigma}}\tilde{\Sigma}^{\dot{\gamma}\dot{\sigma}} + \Phi_{\alpha\beta\dot{\alpha}\dot{\beta}}\Sigma^{\alpha\beta} + \frac{R}{12}\tilde{\Sigma}_{\dot{\alpha}\dot{\beta}}, \end{aligned} \tag{2.22}$$

where $\psi_{\alpha\beta\gamma\sigma}$ and $\tilde{\psi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\sigma}}$ are a fully symmetric spinors and $\Phi_{\alpha\beta\dot{\alpha}\dot{\beta}} = \Phi_{\beta\alpha\dot{\beta}\dot{\alpha}}$. One can show that the various quantities appearing are related to standard objects in Riemannian geometry

$$C_{abcd} = \psi_{\alpha\beta\gamma\sigma}\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\dot{\gamma}\dot{\sigma}} + \tilde{\psi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\sigma}}\epsilon_{\alpha\beta}\epsilon_{\gamma\sigma} \qquad R_{ab} - \frac{1}{4}Rg_{ab} = -2\Phi_{\alpha\beta\dot{\alpha}\dot{\beta}}\,, \tag{2.23}$$

where C_{abcd} is the Weyl tensor, and we have used abstract index notation $a = \alpha\dot{\alpha}$. One can further show that the second Bianchi identity then gives the equations of motion

$$\nabla^{\sigma}{}_{\dot{\alpha}}\psi_{\alpha\beta\gamma\sigma} = \nabla_{(\alpha}{}^{\dot{\beta}}\Phi_{\beta\gamma)\dot{\alpha}\dot{\beta}} \quad , \quad \nabla^{a}\Phi_{ab} + \frac{1}{8}\nabla_{b}R = 0. \tag{2.24}$$

A useful identity involving the commutator of the covariant derivatives on an ASD spinor is given by

$$[\nabla_{\alpha\dot{\alpha}}, \nabla_{\beta\dot{\beta}}]\lambda^{\gamma} = \left[\epsilon_{\dot{\alpha}\dot{\beta}}\left(\psi_{\alpha\beta\sigma}{}^{\gamma} - \frac{R}{12}\epsilon_{\sigma(\alpha}\delta^{\gamma}_{\beta)}\right) + \epsilon_{\alpha\beta}\Phi^{\gamma}{}_{\sigma\dot{\alpha}\dot{\beta}}\right]\lambda^{\sigma}\,. \tag{2.25}$$

An interesting sub-sector of full (complexified) gravity is the SD theory, where the ASD part of the Weyl tensor vanishes $\psi_{\alpha\beta\gamma\delta} = 0$. We first focus on the vacuum Einstein equations with vanishing cosmological constant, hence $\Phi_{\alpha\beta\dot{\alpha}\dot{\beta}} = 0 = R$. In

this case, the ASD part of the Riemann 2-form is given by

$$R_{\alpha\beta} = \psi_{\alpha\beta\gamma\delta}\Sigma^{\gamma\delta} = 0 \quad \text{if self-dual.} \tag{2.26}$$

In the SD case, the ASD spin bundle is trivial, and hence, locally, the ASD spin connections are pure gauge and can be set to zero. Considering the structure equations (2.21), we find a triplet of metric-compatible closed 2-forms

$$d\Sigma^{\alpha\beta} = 0\,. \tag{2.27}$$

These 2-forms can be used to construct a triplet of Kähler 2-forms whose complex structures satisfy the quaternion algebra, resulting in the manifold being hyperkähler. In [69], two local forms of the ASD Ricci-flat condition were given. The first manifests the Kähler structure of the metric

$$g = \frac{\partial^2\Omega}{\partial w^{\dot\alpha}\partial\tilde w^{\dot\beta}} dw^{\dot\alpha} d\tilde w^{\dot\beta} \qquad , \qquad \frac{1}{2}\frac{\partial^2\Omega}{\partial w^{\dot\alpha}\partial\tilde w^{\dot\beta}}\frac{\partial^2\Omega}{\partial w_{\dot\alpha}\partial\tilde w_{\dot\beta}} = 1\,, \tag{2.28}$$

where $(w^{\dot\alpha}, \tilde w^{\dot\alpha})$ are local coordinates, and if appropriate reality conditions are imposed, they are complex conjugates. This is Plebanski's first heavenly system, and the potential Ω satisfies the first heavenly equation if the spacetime is hyperkähler. The second formulation is given by

$$g = 2d\tilde x^{\dot\alpha} dx_{\dot\alpha} - 2\frac{\partial^2\Theta}{\partial x^{\dot\alpha}\partial x^{\dot\beta}} d\tilde x^{\dot\alpha} d\tilde x^{\dot\beta} \quad , \quad \frac{\partial^2\Theta}{\partial\tilde x^{\dot\alpha}\partial x_{\dot\alpha}} + \frac{1}{2}\frac{\partial^2\Theta}{\partial x^{\dot\alpha}\partial x^{\dot\beta}}\frac{\partial^2\Theta}{\partial x_{\dot\alpha}\partial x_{\dot\beta}} = 0\,, \tag{2.29}$$

where $(x^{\dot\alpha}, \tilde x^{\dot\alpha})$ are local coordinates. Similarly, this is Plebanski's second heavenly system, and the potential Θ satisfies the second heavenly equation. The first and second heavenly equations are just two different realisations of the same underlying hyperkähler structure.

The ASD part of the Riemann tensor vanishing results in the dotted spin bundle

being trivial, and hence the holonomy of the manifold being special, as is expected for a hyperkähler manifold. The other class of special holonomy manifolds in four dimensions is the quaternion-Kähler manifolds, based on the classification of Berger [70]. Since the holonomy group in 4d is (locally) the same as that of quaternion-Kähler manifolds, restricting to a special holonomy group is not a sufficient condition. The necessary condition for a quaternion-Kähler manifold is provided by looking at the SD Einstein condition, where the ASD part of the Riemann tensor is given by

$$R_{\alpha\beta} = \frac{R}{12}\Sigma_{\alpha\beta}\,, \tag{2.30}$$

and the Ricci scalar R is a constant, which can be seen from (2.24). As was the case for hyperkähler manifolds, there also exists a local form of the SD Einstein condition due to Przanowski [71].

The SD condition also has an interesting topological implication. Consider $\mathcal{M}$, in Euclidean signature for the moment, being compact and without boundary. The Hirzebruch signature theorem [72] relates the difference between the number of SD and ASD forms in the middle cohomology group to Pontryagin classes, and in four dimensions is given by

$$\mathrm{sgn}(\mathcal{M}) = \frac{-1}{24\pi^2}\int_{\mathcal{M}} R^{ab}\wedge R_{ab} = \frac{1}{12\pi^2}\int_{\mathcal{M}} \mathrm{Vol}\left(\tilde{\psi}^{\dot\alpha\dot\beta\dot\gamma\dot\delta}\tilde{\psi}_{\dot\alpha\dot\beta\dot\gamma\dot\delta} - \psi^{\alpha\beta\gamma\delta}\psi_{\alpha\beta\gamma\delta}\right)\,. \tag{2.31}$$

The other interesting topological invariant is the Euler characteristic, which, through the Chern–Gauss–Bonnet theorem [73], is given by

$$\begin{aligned}\chi(\mathcal{M}) &= \frac{1}{32\pi^2}\int_{\mathcal{M}} \epsilon_{abcd}R^{ab}\wedge R^{cd} \\ &= \frac{1}{8\pi^2}\int_{\mathcal{M}} \mathrm{Vol}\left(\psi^{\alpha\beta\gamma\delta}\psi_{\alpha\beta\gamma\delta} + \tilde{\psi}^{\dot\alpha\dot\beta\dot\gamma\dot\delta}\tilde{\psi}_{\dot\alpha\dot\beta\dot\gamma\dot\delta} - 2\Phi^{ab}\Phi_{ab} + \frac{1}{24}R^2\right)\,.\end{aligned} \tag{2.32}$$

By comparing the expressions for the signature and the Euler characteristic, the

analogue of Equation 2.8 can be written as

$$\chi(\mathcal{M}) \geq \frac{3}{2}\text{sgn}(\mathcal{M}) \tag{2.33}$$

where equality holds when $\mathcal{M}$ is SD.

Finally, the SD condition also arises in the context of higher spin field equations. First, consider the valence n zero-rest-mass equation (z.r.m.)

$$\nabla^{\alpha_1}_{\dot{\alpha}}\phi_{\alpha_1\alpha_2\cdots\alpha_n} = 0\,, \tag{2.34}$$

where $\phi_{\alpha_1\cdots\alpha_n}$ is a fully symmetric spin field. An example of a z.r.m. field is the ASD Weyl spinor when the trace-free part of the Ricci tensor vanishes (2.24) or the ASD field strength in an abelian gauge theory. The problem with these equations is that they are overdetermined: there are $n+1$ components and $2n$ equations, so one must check their consistency if $n > 1$. To be consistent, the solutions to (2.34) must satisfy the Buchdahl condition [74]

$$\psi^{\alpha_1\alpha_2\alpha_3}{}_{(\beta_3}\phi_{\beta_4\cdots\beta_n)\alpha_1\alpha_2\alpha_3} = 0\,, \tag{2.35}$$

which can be derived using (2.25). For abelian Einstein-Yang-Mills, this condition is empty, and hence the equations are consistent on a curved background. For gravity with vanishing trace-free Ricci curvature, this may seem to cause issues, but the Weyl spinor satisfies the Buchdahl condition due to the algebraic relation

$$\psi^{\alpha_1\alpha_2\alpha_3}{}_{\beta}\psi_{\gamma\alpha_1\alpha_2\alpha_3} = \left[\psi_0\psi_4 - 4\psi_1\psi_3 + 3(\psi_2)^2\right]\epsilon_{\gamma\beta}. \tag{2.36}$$

However, for higher spin z.r.m. fields, no such natural geometric relations exist, and the Buchdahl conditions further over-determine the system unless one works on a SD background.

2.1.3 Spacetime Actions

For both the theories of Yang-Mills and gravity, action principles exist that make the SD condition manifest. Starting with Yang-Mills, the standard action with the topological θ-term is given by

$$S_{YM}[A] = -\frac{1}{2g_{YM}^2}\int_{\mathbb{M}} \mathrm{Tr}(F \wedge *F) + \frac{\theta}{8\pi^2}\int_{\mathbb{M}} \mathrm{Tr}(F \wedge F)\,, \tag{2.37}$$

The θ-term, being topological, will not contribute to the equations of motion as it is a total derivative term $F \wedge F = d(\mathrm{CS}_3(A))$ where $\mathrm{CS}_3(A)$ is the Chern-Simons 3-form. By choosing a special value for $\theta = \frac{8\pi^2}{g_{YM}^2}$, the Yang-Mills action becomes chiral and only involves ASD spin fields

$$S_{YM}[A] = -\frac{2}{g_{YM}^2}\int_{\mathbb{M}} d^4x \mathrm{Tr}(F^{\alpha\beta}F_{\alpha\beta})\,. \tag{2.38}$$

The choice of working with the ASD spin bundles was made by the sign of the θ-term, and one may ask if the equations of motion of the SD part of the field strength have been lost in this process. By considering the Bianchi identity on the full field strength and the ASD part of the equation of motion, the equations of motion on the SD field strength can be recovered.

To make a SD limit manifest, we introduce a Lagrange multiplier field $B_{\alpha\beta}(x)$ and appropriately re-scale the fields[1] in the theory to give the following action [75]

$$S[A,B] = \int_{\mathbb{M}} d^4x \mathrm{Tr}\left(B^{\alpha\beta}F_{\alpha\beta} - \frac{\kappa}{4}B^{\alpha\beta}B_{\alpha\beta}\right)\,, \tag{2.39}$$

where the equations of motion, in terms of the rescaled fields, are given by

$$F_{\alpha\beta} = \partial_{(\alpha}{}^{\dot\alpha}A_{\beta)\dot\alpha} + \frac{g_3}{2}[A_{(\alpha}{}^{\dot\alpha}, A_{\beta)\dot\alpha}] = \frac{\kappa}{2}B_{\alpha\beta} \qquad \partial^\alpha_{\dot\alpha}B_{\alpha\beta} + g_3[A^\alpha_{\dot\alpha}, B_{\alpha\beta}] = 0\,. \tag{2.40}$$

[1]Defining $g_{YM} = \kappa g_3$, rescaling $A_{\alpha\dot\alpha} \to g_3 A_{\alpha\dot\alpha}$ and $B_{\alpha\beta} \to -\frac{g_3\kappa^2}{2}B_{\alpha\beta}$ gives the desired action.

When $\kappa = 0$, the theory becomes SD and the Lagrange multiplier field $B_{\alpha\beta}$ becomes an ASD perturbation to the field strength on a SD background. In the SD case, the coupling constant g_3 controls the perturbative expansion from which amplitudes can be built. In fact, due to its classical integrability, all tree-level amplitudes, except for the 3-point vertex, vanish, and the quantum theory is 1-loop exact; see [76] for a review.

We now focus on the case of the vacuum Einstein equations, where a similar action principle can be derived. The starting point is the first-order action principle given by the Palatini action [77], which has the form

$$S_{PH}[\Gamma, \theta] = \frac{1}{8\pi G} \int_{\mathcal{M}} P_{abcd} R^{ab} \wedge \theta^c \wedge \theta^d \,. \tag{2.41}$$

The tensor P_{abcd} contains two terms, the second of which corresponds to the Holst term [78] and is proportional to a constant λ

$$P_{abcd} = \frac{1}{2}\epsilon_{abcd} + i\lambda\eta_{a[c}\eta_{d]b} \,. \tag{2.42}$$

When $\lambda \neq \pm 1$ this tensor is invertible in the sense that there exists a P^{-1}_{abcd} such that $P^{abef}P^{-1}_{efcd} = \delta^{ab}_{cd}$ where δ^{ab}_{cd} is the generalised Kronecker delta. In this case, the derivation of the equations of motion follows along similar lines to the case where the Holst term is absent. In fact, the Holst term vanishes completely if one goes partially on-shell and imposes metric compatibility on the connection. Our particular interest is in the case when $\lambda = \pm 1$ as this will give a chiral version of Einstein gravity. This can be made explicit by using spinor indices, where the action splits into ASD and SD parts

$$S_{PH}[\Gamma, \theta] = \frac{i}{4\pi G} \int_{\mathcal{M}} (\lambda + 1) R_{\alpha\beta} \wedge \Sigma^{\alpha\beta} + (\lambda - 1)\tilde{R}_{\dot{\alpha}\dot{\beta}} \wedge \tilde{\Sigma}^{\dot{\alpha}\dot{\beta}} \,. \tag{2.43}$$

Consider the case of $\lambda = 1$, the action has a form similar to a BF action; however,

the crucial difference is that the field $\Sigma^{\alpha\beta}$ is not a fundamental field in the variational problem. One can uplift $\Sigma^{\alpha\beta}$ to a fundamental field at the cost of a Lagrange multiplier term that imposes the simplicity constraint [79]

$$\Sigma^{(\alpha\beta} \wedge \Sigma^{\gamma\delta)} = 0 \quad \Leftrightarrow \quad \Sigma^{\alpha\beta} = \frac{1}{2}\theta^{\alpha\dot{\alpha}} \wedge \theta^{\beta}{}_{\dot{\alpha}} \,. \tag{2.44}$$

Introducing the Lagrange multiplier, the resulting action is the Plebanski action for complexified gravity [80]

$$S[\Sigma, \Gamma] = \frac{i}{2\pi G} \int_{\mathcal{M}} \Sigma^{\alpha\beta} \wedge R_{\alpha\beta} - \frac{1}{2}\psi_{\alpha\beta\gamma\delta}\Sigma^{\alpha\beta} \wedge \Sigma^{\gamma\delta} \,, \tag{2.45}$$

where $\psi_{\alpha\beta\gamma\delta}$ is a fully symmetric spinor and on-shell equals the ASD Weyl spinor. As in Yang-Mills, using the second Bianchi identity shows that the SD equations of motion follow from the ASD ones. To obtain an action describing SD gravity, one imposes the simplicity constraint, and rescaling the fields, we obtain

$$S[\theta, \Gamma] = \frac{i}{2\pi G} \int_{\mathcal{M}} \Sigma^{\alpha\beta} \wedge (d\Gamma_{\alpha\beta} + \kappa\Gamma_{\alpha}{}^{\gamma} \wedge \Gamma_{\beta\gamma}) \,, \tag{2.46}$$

By taking the limit $\kappa = 0$, we obtain an action describing SD gravity along with $\Gamma_{\alpha\beta}$ acting as a Lagrange multiplier field and no longer as the ASD spin connection.

2.1.4 NP and GHP formalisms

The traditional tetrad formalism is set up such that the tetrad is orthonormal and the metric is that of Minkowski. To use the spinor decomposition (2.2) to our advantage, it is convenient to use the Newman-Penrose (NP) formalism [81]. We first notice that under the map (2.3), null vectors turn into matrices that have vanishing determinants. Therefore, a generic null vector can be written as

$$V^{\alpha\dot{\alpha}} = \lambda^{\alpha}\tilde{\lambda}^{\dot{\alpha}}, \tag{2.47}$$

where the residual little group transformation is captured by the automorphism $\lambda \to r\lambda$ and $\tilde{\lambda} \to r^{-1}\tilde{\lambda}$ for some $r \in \mathbb{C}^*$. We therefore wish to use a null tetrad which has the internal metric

$$\eta_{ab} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{2.48}$$

This tells us that the tetrad

$$e^a_\mu = (l_\mu, n_\mu, m_\mu, \bar{m}_\mu), \tag{2.49}$$

has the normalization

$$\begin{gathered} l_\mu l^\mu = n_\mu n^\mu = m_\mu m^\mu = \bar{m}_\mu \bar{m}^\mu = l_\mu m^\mu = l_\mu \bar{m}^\mu = n_\mu m^\mu = n_\mu \bar{m}^\mu = 0, \\ l_\mu n^\mu = -m_\mu \bar{m}^\mu = 1, \end{gathered} \tag{2.50}$$

and the metric is given by

$$g_{\mu\nu} = l_\mu n_\nu + l_\nu n_\mu - m_\mu \bar{m}_\nu - m_\nu \bar{m}_\mu. \tag{2.51}$$

The translation of the null tetrad in the spinor notation is provided by choosing a spin basis (the simplest example being $\iota^\alpha = \bar{\iota}^{\dot{\alpha}} = (1, 0)$ and $o^\alpha = \bar{o}^{\dot{\alpha}} = (0, -1)$) such that the inner product $\langle \iota o \rangle = [\bar{\iota}\bar{o}] = 1$. Therefore, in abstract index notation, we can make the identification

$$l_\mu = o_\alpha \bar{o}_{\dot{\alpha}} \quad n_\mu = \iota_\alpha \bar{\iota}_{\dot{\alpha}} \quad m_\mu = o_\alpha \bar{\iota}_{\dot{\alpha}} \quad \bar{m}_\mu = \iota_\alpha \bar{o}_{\dot{\alpha}}, \tag{2.52}$$

which solves the normalization conditions (2.50). We can further define the components of the covariant derivative in the null tetrad

$$D = o^\alpha \bar{o}^{\dot{\alpha}} \nabla_{\alpha\dot{\alpha}}, \quad \Delta = \iota^\alpha \bar{\iota}^{\dot{\alpha}} \nabla_{\alpha\dot{\alpha}}, \quad \delta = o^\alpha \bar{\iota}^{\dot{\alpha}} \nabla_{\alpha\dot{\alpha}}, \quad \bar{\delta} = \iota^\alpha \bar{o}^{\dot{\alpha}} \nabla_{\alpha\dot{\alpha}}. \tag{2.53}$$

The NP formalism we have described in the notations of spinors is known as the GHP (Geroch–Held–Penrose) formalism [82] and is particularly useful in the study of null 2-surfaces and Penrose's quasi-local momentum and angular momentum. However, in the definition of the dyad (ι, o), the only requirement was that the inner product satisfy $\langle \iota o \rangle = 1$ to solve the normalization (2.50). Therefore, there is a scaling ambiguity

$$(\iota^\alpha, o^\alpha) \to (r(x)^{-1}\iota^\alpha, r(x) o^\alpha), \tag{2.54}$$

where $r(x)$ is an arbitrary non-vanishing complex function of spacetime. This allows for the definition of scalar fields of weights $\{p, q\}$ to have the transformation under the scaling

$$\eta_{p,q} \to r^p \bar{r}^q \eta_{p,q}, \tag{2.55}$$

and we define the spin weight as $s = \frac{p-q}{2}$ and the boost weight $w = \frac{p+q}{2}$.

The next task is to build the complex spin coefficients, which are linear combinations of various components of the spin connection 1-form Γ_{ab}. The other basic objects in the NP formalism are the components of the Weyl tensor displayed in Equation (2.58) along with the components of $\Phi_{\alpha\beta\dot{\alpha}\dot{\beta}}$, which give information on the matter content. In NP formalism, the spin coefficients expressed in the null tetrad

(2.52) are given by

$$\begin{aligned}
\alpha &= \iota^\alpha \iota^\beta \bar{o}^{\dot\beta} \nabla_{\beta\dot\beta} o_\alpha, \quad & \lambda &= \iota^\alpha \iota^\beta \bar{o}^{\dot\beta} \nabla_{\beta\dot\beta} \iota_\alpha, \quad & \rho &= o^\alpha \iota^\beta \bar{o}^{\dot\beta} \nabla_{\beta\dot\beta} o_\alpha, \\
\beta &= \iota^\alpha o^\beta \bar{\iota}^{\dot\beta} \nabla_{\beta\dot\beta} o_\alpha, \quad & \mu &= \iota^\alpha o^\beta \bar{\iota}^{\dot\beta} \nabla_{\beta\dot\beta} \iota_\alpha, \quad & \sigma &= o^\alpha o^\beta \bar{\iota}^{\dot\beta} \nabla_{\beta\dot\beta} o_\alpha, \\
\gamma &= \iota^\alpha \iota^\beta \bar{\iota}^{\dot\beta} \nabla_{\beta\dot\beta} o_\alpha, \quad & \nu &= \iota^\alpha \iota^\beta \bar{\iota}^{\dot\beta} \nabla_{\beta\dot\beta} \iota_\alpha, \quad & \tau &= o^\alpha \iota^\beta \bar{\iota}^{\dot\beta} \nabla_{\beta\dot\beta} o_\alpha, \\
\epsilon &= \iota^\alpha o^\beta \bar{o}^{\dot\beta} \nabla_{\beta\dot\beta} o_\alpha, \quad & \pi &= \iota^\alpha o^\beta \bar{o}^{\dot\beta} \nabla_{\beta\dot\beta} \iota_\alpha, \quad & \kappa &= o^\alpha o^\beta \bar{o}^{\dot\beta} \nabla_{\beta\dot\beta} o_\alpha.
\end{aligned} \tag{2.56}$$

These spin coefficients only capture the ASD part of the full spin connection, and the others come from interchanging the barred and unbarred spinors in the spin basis. An important spin coefficient that appears in asymptotically flat spacetimes and null geodesic congruences is the shear σ, as it captures the radiative data and null infinity. By looking at all the NP scalars, the only ones that do not scale homogeneously are $\{\alpha, \beta, \gamma, \epsilon\}$. This fact can be used to construct differential operators that send a scalar η of weight $\{p, q\}$ to scalars of a new weight

$$\begin{aligned}
\text{þ} &= D - p\epsilon - q\bar{\epsilon}, \qquad & \text{ð} &= \bar{\delta} - p\alpha - q\bar{\beta}, \\
\text{þ}' &= \Delta - p\gamma - q\bar{\gamma}, \qquad & \bar{\text{ð}} &= \delta - p\beta - q\bar{\alpha}.
\end{aligned} \tag{2.57}$$

The components of the Weyl tensor read as

$$\begin{aligned}
\psi_0 &= C_{\mu\nu\rho\sigma} l^\mu m^\nu l^\rho m^\sigma = \psi_{\alpha\beta\gamma\delta} o^\alpha o^\beta o^\gamma o^\delta, \\
\psi_1 &= C_{\mu\nu\rho\sigma} l^\mu n^\nu l^\rho m^\sigma = \psi_{\alpha\beta\gamma\delta} \iota^\alpha o^\beta o^\gamma o^\delta, \\
\psi_2 &= \frac{1}{2}(C_{\mu\nu\rho\sigma} l^\mu n^\nu l^\rho n^\sigma - C_{\mu\nu\rho\sigma} l^\mu n^\nu m^\rho \bar{m}^\sigma) = \psi_{\alpha\beta\gamma\delta} \iota^\alpha \iota^\beta o^\gamma o^\delta, \\
\psi_3 &= C_{\mu\nu\rho\sigma} l^\mu n^\nu n^\rho \bar{m}^\sigma = \psi_{\alpha\beta\gamma\delta} \iota^\alpha \iota^\beta \iota^\gamma o^\delta, \\
\psi_4 &= C_{\mu\nu\rho\sigma} n^\mu \bar{m}^\nu n^\rho \bar{m}^\sigma = \psi_{\alpha\beta\gamma\delta} \iota^\alpha \iota^\beta \iota^\gamma \iota^\delta,
\end{aligned} \tag{2.58}$$

which we can see are simply obtained from the Weyl spinor decomposition

$$\psi_{\alpha\beta\gamma\delta} = \psi_0\, \iota_\alpha\iota_\beta\iota_\gamma\iota_\delta - 4\psi_1\, o_{(\alpha}\iota_\beta\iota_\gamma\iota_{\delta)} + 6\psi_2\, o_{(\alpha}o_\beta\iota_\gamma\iota_{\delta)} - 4\psi_3\, o_{(\alpha}o_\beta o_\gamma\iota_{\delta)} + \psi_4\, o_\alpha o_\beta o_\gamma o_\delta, \tag{2.59}$$

and similarly for the SD part of the curvature $\tilde{\psi}_{\dot\alpha\dot\beta\dot\gamma\dot\delta}$. One can further define the components of $\Phi_{\alpha\beta\dot\alpha\dot\beta}$ which enter the equations for the spin coefficients and the second Bianchi identity (2.24). For completeness, see Appendix A for all these details or see [74].

In the coming sections, we will also be interested in studying field equations that are Weyl covariant. To this end, one can go further and define operators that transform homogeneously under Weyl rescaling of the metric

$$\hat{g} = \Omega^2 g\,, \tag{2.60}$$

where $\Omega = 0$ defines infinity. In spinor notation, the Levi-Civita symbols will transform

$$\hat{\epsilon}_{\alpha\beta} = \Omega\epsilon_{\alpha\beta} \qquad \hat{\epsilon}_{\dot\alpha\dot\beta} = \Omega\epsilon_{\dot\alpha\dot\beta}. \tag{2.61}$$

The basis of spinors we use to define our null tetrad will also be rescaled via

$$\begin{aligned} \hat{o}_\alpha = o_\alpha \quad &, \quad \hat{\iota}_\alpha = \Omega\iota_\alpha, \\ \hat{o}^\alpha = \Omega^{-1}o^\alpha \quad &, \quad \hat{\iota}^\alpha = \iota^\alpha, \end{aligned} \tag{2.62}$$

such that the inner product $\langle\hat{\iota}\hat{o}\rangle = \langle\iota o\rangle = 1$ is invariant [83]. Under (2.62), a Weyl-weighted scalar transforms as

$$\hat{\eta}^W_{p,q} = \Omega^W \eta^W_{p,q}. \tag{2.63}$$

Using the spin coefficients which do not have a definite Weyl weight, we define the

covariant operators

$$\begin{aligned} \text{Þ}_{\mathcal{C}} &= \text{Þ} + W\rho, & \eth_{\mathcal{C}} &= \eth - (W+p)\pi, \\ \text{Þ}'_{\mathcal{C}} &= \text{Þ}' - (W+p+q)\mu, & \bar{\eth}_{\mathcal{C}} &= \bar{\eth} + (W+q)\tau. \end{aligned} \tag{2.64}$$

To demonstrate the compactness of the notation, consider the z.r.m. field (2.34). If, under Weyl rescaling, the field transforms as $\hat{\phi}_{\alpha_1\cdots\alpha_n} = \Omega^{-1}\phi_{\alpha_1\cdots\alpha_n}$, the z.r.m equation is invariant as one can show

$$\hat{\nabla}^{\alpha_1}{}_{\dot{\alpha}}\hat{\phi}_{\alpha_1\cdots\alpha_n} = \Omega^{-2}\nabla^{\alpha_1}{}_{\dot{\alpha}}\phi_{\alpha_1\cdots\alpha_n} = 0, \tag{2.65}$$

In this case, we should expect the z.r.m. equation to be expressed in terms of the Weyl covariant operators (2.64). If we define the components of the z.r.m. field as

$$\phi_m = \phi_{\alpha_1\cdots\alpha_n}\iota^{\alpha_1}\cdots\iota^{\alpha_m}o^{\alpha_{m+1}}\cdots o^{\alpha_n}\,, \tag{2.66}$$

then the component ϕ_n will have Weyl weight $W = n-m-1$ and the z.r.m. equations can be compactly expressed as

$$\begin{aligned} \text{Þ}'_{\mathcal{C}}\phi_m - \bar{\eth}_{\mathcal{C}}\phi_{m+1} &= (n-m-1)\sigma\phi_{m+2} + m\nu\phi_{m-1} \\ \bar{\eth}_{\mathcal{C}}\phi_m - \text{Þ}_{\mathcal{C}}\phi_{m+1} &= (n-m-1)\kappa\phi_{m+2} + m\lambda\phi_{m-1}\,. \end{aligned} \tag{2.67}$$

If one considers the vacuum Einstein equations, which can be found in Appendix A, the Weyl spinor satisfies a valence 4 z.r.m. equation. However, since the transformation law for the Weyl tensor is $\hat{C}_{abcd} = \Omega^2 C_{abcd}$, the Weyl spinor is unchanged $\hat{\psi}_{\alpha\beta\gamma\delta} = \psi_{\alpha\beta\gamma\delta}$. Therefore, the Weyl spinor does not have the correct Weyl weight for its equation of motion to be Weyl invariant, which should be expected since Einstein gravity is not Weyl invariant.

2.2 Twistors

While initially formulated by Roger Penrose to find a new approach to quantum gravity, twistor theory has seen many applications not only in geometry and integrable systems but also in perturbative QFT. The real power of twistor theory is in its ability to describe the integrable sectors of various theories of interest, such as Yang-Mills and gravity. In the QFT setting, this enables the computation of objects such as scattering amplitudes around non-linear backgrounds. On the mathematical side, twistor theory builds a bridge between differential geometry data on the manifold of interest, the physical space, and the algebraic geometry data on the twistor space. The Penrose transform is the main example, as it relates elements of cohomology on twistor space to solutions of linear PDEs on the physical space. The extension to non-linear integrable PDEs requires the construction of vector bundles and deformations of complex structures on the twistor space. Here we will try to present the mathematical foundations that make twistor theory so interesting.

2.2.1 Vectors and Forms on $\mathbb{CP}^n$

Due to the projective nature of twistor space, it is worth discussing the tangent and cotangent space on $\mathbb{CP}^n$, which is defined as

$$\mathbb{CP}^n = \{(Z^0, \cdots, Z^n) \in \mathbb{C}^{n+1} | (Z^0, \cdots, Z^n) \sim (\lambda Z^0, \cdots, \lambda Z^n) : \lambda \in \mathbb{C}^*\}, \quad (2.68)$$

where the Z^A are called homogeneous coordinates. We wish to use these coordinates rather than choosing a patch U_i where $Z^i \neq 0$ and the affine coordinates are $z_j = Z^j/Z^i$ such that $i \neq j$ to keep our formula agnostic to the choice of the patch. To construct the tangent space, we consider the space of all derivations on functions

$$f : \mathbb{CP}^n \to \mathbb{C}, \quad (2.69)$$

which in homogeneous coordinates can be viewed as

$$f : \mathbb{C}^{n+1} \to \mathbb{C} \quad , \quad f(\lambda Z) = f(Z), \tag{2.70}$$

a function with zero homogeneity. Therefore the homogeneity vector field $\Upsilon = Z^A \partial_A$ which generates rescaling will not appear, put another way

$$T^{1,0}_Z \mathbb{CP}^n \otimes \mathcal{O}(m, \bar{m}) \cong T^{1,0}_Z \mathbb{C}^{n+1} / \langle \Upsilon \rangle, \tag{2.71}$$

where $\langle \Upsilon \rangle \subset T^{1,0}_Z \mathbb{C}^{n+1}$ and the twisting of the line bundle $\mathcal{O}(m, \bar{m})$ appears as our vector fields may have a non-zero homogeneity. Therefore to get sections of $T^{1,0}_Z \mathbb{CP}^n$ we must multiply our vector fields by sections of $\mathcal{O}(-m, -\bar{m})$. The cotangent bundle is realized by $\omega \in \Omega^{1,0}(\mathbb{C}^{n+1})$ such that $\Upsilon \lrcorner\, \omega = 0$. More generally sections of $\Omega^{p,q}(\mathbb{CP}^n) \otimes \mathcal{O}(m, \bar{m})$ are given by the subspace of $\Omega^{p,q}(\mathbb{C}^{n+1})$ such that [84]

$$\begin{gathered} \Upsilon \lrcorner\, \omega = \bar{\Upsilon} \lrcorner\, \omega = 0, \\ \mathcal{L}_\Upsilon \omega = m\omega \quad \mathcal{L}_{\bar{\Upsilon}} \omega = \bar{m}\omega, \end{gathered} \tag{2.72}$$

where Cartan's magic formula $\mathcal{L}_V \omega = d(V \lrcorner\, \omega) + V \lrcorner\, d\omega$ can be used. The next task is to define the exterior derivative on $\mathbb{CP}^n$ from the one on $\mathbb{C}^{n+1}$, as these two objects do not coincide due to the presence of the line bundle. By definition of a connection on a vector bundle, we expand the exterior derivative acting on forms ω valued in $\mathcal{O}(m, \bar{m})$ on $\mathbb{CP}^n$ as

$$d_{\mathbb{CP}^n} \omega = d\omega - m\theta \wedge \omega - \bar{m}\bar{\theta} \wedge \omega, \tag{2.73}$$

where $d = dZ^A \frac{\partial}{\partial Z^A} + d\bar{Z}^{\bar{A}} \frac{\partial}{\partial \bar{Z}^{\bar{A}}}$ and $\theta, \bar{\theta}$ are the connections for the vector bundle. By requiring that $d_{\mathbb{CP}^n} \omega$ is a form under the conditions (2.72), we find that

$$\Upsilon \lrcorner\, \theta = 1 \quad , \quad \bar{\Upsilon} \lrcorner\, \bar{\theta} = 1, \tag{2.74}$$

and all other contractions are zero. This means that the connection 1-forms must lie in the image of the homogeneity vector fields, and therefore can be built in the process of building the cotangent space. As an example, let us take the canonical bundle of $\mathbb{CP}^n$, which is defined as $K_{\mathbb{CP}^n} = \bigwedge^n T^{*1,0}\mathbb{CP}^n$. Taking an n-form on $\mathbb{C}^{n+1}$

$$\omega = \sum_{A=0}^{n} f^A(Z) dZ^0 \wedge \cdots \wedge d\hat{Z}^A \wedge \cdots \wedge dZ^n, \tag{2.75}$$

where $d\hat{Z}^A$ corresponds to the omission. Now taking the contraction with the homogeneity vector field

$$\Upsilon \lrcorner\, \omega = \sum_{A,B=0}^{n} (-1)^B f^A(Z) Z^B dZ^0 \wedge \cdots \wedge d\hat{Z}^A \wedge \cdots \wedge d\hat{Z}^B \wedge \cdots \wedge dZ^n, \tag{2.76}$$

which vanishes if $f^A(Z) = (-1)^A Z^A g(Z)$ by antisymmetry. Therefore we can compactly write the forms in $K_{\mathbb{CP}^n}$ as

$$\omega = g(Z) \epsilon_{A_0 \cdots A_n} Z^{A_0} dZ^{A_1} \wedge \cdots \wedge dZ^{A_n}, \tag{2.77}$$

with $g(Z)$ being a section of $\mathcal{O}(-n-1)$ and therefore $K_{\mathbb{CP}^n} \cong \mathcal{O}(-n-1)$. Another example is provided by the simplest case of $\mathbb{CP}^1$, which sees much use in twistor theory. The unique sections of $T^{1,0}\mathbb{CP}^1 \otimes \mathcal{O}(-2)$ and $T^{*1,0}\mathbb{CP}^1 \otimes \mathcal{O}(2)$ are given by

$$\partial_0 = \left[\frac{a^\alpha}{\langle a\lambda \rangle} \frac{\partial}{\partial \lambda^\alpha} \right] \qquad e^0 = D\lambda = \langle \lambda d\lambda \rangle, \tag{2.78}$$

where $[a] \in \mathbb{CP}^1$ and $[\lambda] \neq [a]$ is a reference spinor. The definition of ∂_0 is independent of the spinor a due to the equivalence class and so can be defined globally. The exterior derivative acting on forms valued in $\mathcal{O}(m)$ with an explicit example of θ as defined in (2.73) is given by

$$d_{\mathbb{CP}^1} = d - m \frac{\langle a d\lambda \rangle}{\langle a\lambda \rangle} \wedge\ , \qquad \theta = \frac{\langle a d\lambda \rangle}{\langle a\lambda \rangle}\,. \tag{2.79}$$

2.2.2 Geometry of Flat Twistor Space

Rather than starting from space-time and constructing twistor space, we go in the other direction and see how space-time is an emergent object from twistor geometry.

Definition 2.2.1 (Twistor Space)**.** A twistor space for a manifold M is a complex manifold Z along with a double fibration

$$\begin{array}{ccc} & F & \\ {\scriptstyle \pi_1}\swarrow & & \searrow{\scriptstyle \pi_2} \\ M & & Z \end{array} \tag{2.80}$$

where F is called the correspondence space (also known as a universal family) for the moduli space M of embedding complex manifolds L holomorphically into Z.

This definition probes the complex geometry of Z by considering holomorphic embeddings of a complex manifold. The moduli space of all possible holomorphic embeddings will correspond to our physical space M, and the correspondence space locally will have the product structure $M \times L$.

To understand the correspondence better, we introduce the embeddings that define the moduli space problem for embedding spheres into a complex manifold Z

$$\begin{aligned} \iota_x : \mathbb{CP}^1_x &\to Z \\ [\sigma_\alpha] &\mapsto Z^A = F^A(x, [\sigma_\alpha]), \end{aligned} \tag{2.81}$$

with σ_α being homogeneous coordinates on $\mathbb{CP}^1$, such that the embedding is holomorphic; in other words, it needs to satisfy the defining equation for embedding (pseudo-)holomorphic curves into an (almost) complex manifold [85]

$$d\iota_x + J \cdot d\iota_x \cdot j = 0, \tag{2.82}$$

where J and j are the (almost) complex structures of Z and $\mathbb{CP}^1$ respectively. This

then leads to various isomorphisms of tangent and cotangent bundles, such as

$$T^{0,1}Z|_X = T^{0,1}\mathbb{CP}^1, \tag{2.83}$$

where $|_X$ denotes the pullback of the form to a holomorphic curve defined by ι_x. On a more practical note, this can be recast as a PDE problem by using the standard (0,1) vector field on $\mathbb{CP}^1$ and requiring

$$\bar{\partial}_0 \lrcorner\, \partial Z^A|_X = \left[[\tilde{a}\bar{\sigma}]\ \tilde{a}^{\dot{\alpha}} \frac{\partial}{\partial \bar{\sigma}^{\dot{\alpha}}} \right] \lrcorner\, \partial Z^A|_X = 0, \tag{2.84}$$

where ∂Z^A are a basis for $(T^{1,0})^* Z$. Many known constructions of twistor spaces come from deformations of the so-called flat twistor space. This is the twistor space of complexified Minkowski $\mathbb{CM}$, which is denoted $\mathbb{PT}$ and is the total space of

$$\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{CP}^1, \tag{2.85}$$

where $\mathcal{O}(1)$ is the dual to the tautological line bundle $\mathcal{O}(-1)$ over $\mathbb{CP}^1$. $\mathbb{PT}$ can be provided with homogenous coordinates $Z^A = (\mu^{\dot{\alpha}}, \lambda_\alpha)$ with $Z^A \sim tZ^A$, where λ_α are the coordinates on the base and $\mu^{\dot{\alpha}}$ on the fibers. Since $[\lambda_\alpha] \in \mathbb{CP}^1$, these components must be non-vanishing, which results in the identification

$$\mathbb{PT} = \mathbb{CP}^3 - \mathbb{CP}^1, \tag{2.86}$$

where the removed $\mathbb{CP}^1$ corresponds to $\lambda_\alpha = 0$. This removed $\mathbb{CP}^1$ will correspond to the point at infinity in our spacetime, as we will show shortly. To find the holomorphic sections, we use a generalisation of Liouville's theorem

Theorem 1. A holomorphic function on $\mathbb{C}^2$ homogenous of degree $n > 0$ is of the form

$$f(\lambda_\alpha) = \phi^{\alpha_1 \cdots \alpha_n} \lambda_{\alpha_1} \cdots \lambda_{\alpha_n}, \tag{2.87}$$

where $\phi^{\alpha_1 \cdots \alpha_n}$ is symmetric.

We now wish to apply this theorem to the flat twistor space. To do so we must consider the PDE problem (2.84). The standard complex structure of $\mathbb{CP}^3$

$$\bar{\partial} = d\bar{Z}^{\bar{A}} \frac{\partial}{\partial \bar{Z}^{\bar{A}}} = d\bar{\lambda}_{\dot{\alpha}} \frac{\partial}{\partial \bar{\lambda}_{\dot{\alpha}}} + d\bar{\mu}^{\alpha} \frac{\partial}{\partial \bar{\mu}^{\alpha}}, \tag{2.88}$$

which means that the basis for $(T^{1,0})^* \mathbb{PT}$ is $\partial Z^A = \{D\lambda, d\mu^{\dot{\alpha}}\}$. Since the flat twistor space is the total space of a vector bundle over $\mathbb{CP}^1$, the fibration preserves the coordinate $\sigma_\alpha = \lambda_\alpha$, so only $D\lambda$ appears in the basis. Since the $\mu^{\dot{\alpha}}$ coordinates are sections of $\mathcal{O}(1)$, we can use Theorem 1 and (2.84) to write down the holomorphic sections of the fibration known as incidence relations

$$\mu^{\dot{\alpha}} = x^{\alpha\dot{\alpha}} \lambda_\alpha. \tag{2.89}$$

The $x^{\alpha\dot{\alpha}}$ are the moduli parameters for the solution space of the PDE we solved. To understand the moduli parameters further, we require some extra machinery.

In twistor theory, we wish to consider the moduli space of holomorphic embeddings of $\mathbb{CP}^1$ into twistor space Z. There are several theorems due to Kodaira which we summarize here that help us.

Let L be a complex submanifold of a complex manifold Z. We first define the normal bundle $N(L) \to L$ to be $\sqcup_{z \in L} N_z(L)$ where $N_z(L) = T_z Z / T_z L$ is the quotient vector space.

Definition 2.2.2. (Complete Analytic Family of Compact Submanifolds) Let Z be a complex manifold of dimension $r + d$. The pair (F, M) is called a complete analytic family of compact submanifolds of Z of dimensions d if

1. F is a complex submanifold of $F \times M$ of co-dimension r such that $\forall x \in M$ the intersection $L_x \times x := F \cap (Z \times x)$ is a compact submanifold of $Z \times x$ of dimension d.

2. $\exists$ an isomorphism

$$T_xM \simeq H^0(L_x, N(L_x)), \tag{2.90}$$

where L_x is a submanifold of Z and $N(L_x)$ is the normal bundle of L_x in Z.

The existence of such objects is given by the following theorem

Theorem 2. (Kodaira [86]) Let L be a dimension d complex compact submanifold of a complex manifold Z, and $N(L)$ the normal bundle of L in Z. If $H^1(L, N) = 0$ then there exists a complete analytic family of compact submanifolds (F, M) such that $L = L_x$ for some $x \in M$

The space M is known as the moduli space of holomorphic curves in the space Z, and its dimension is found simply by considering the dimension of $H^0(L, N(L))$. Therefore, an understanding of the compact submanifold L is crucial. In our context, we will take $L = \mathbb{CP}^1$ as this is the most basic example and all the cohomologies are known. Any vector bundle of dimension n can be decomposed as a direct sum of line bundles $\mathcal{O}(m_1) \oplus \cdots \oplus \mathcal{O}(m_n)$ for some integers m_i via the Birkhoff–Grothendieck theorem [87] and cohomology classes valued in these vector bundles are known to be

$$\begin{aligned} H^0(\mathbb{CP}^1, \mathcal{O}(m)) &= \mathbb{C}^{m+1} \;\; m \geq 0, \\ H^1(\mathbb{CP}^1, \mathcal{O}(-m-2)) &= \mathbb{C}^{m+1} \;\; m \geq 0, \end{aligned} \tag{2.91}$$

and zero otherwise. Often the twistor space will be the total space of these vector bundles, and by construction, the normal bundle to $\mathbb{CP}^1$ is the vector bundle itself. The correspondence space F appearing in the double fibration will often be the projective spin bundle $\mathbb{PS}$, which locally is given by $M \times \mathbb{CP}^1$. Therefore, for the particular case of flat twistor space, the maps in the double fibration are given by

$$\begin{aligned} &\pi_1 : (x, [\lambda_\alpha]) \mapsto x, \\ &\pi_2 : (x, [\lambda_\alpha]) \mapsto [\mu^{\dot{\alpha}} = F^{\dot{\alpha}}(x, \lambda), \lambda_\alpha]. \end{aligned} \tag{2.92}$$

Since, $H^0(\mathbb{CP}^1, \mathcal{O}(1) \oplus \mathcal{O}(1)) = \mathbb{C}^4$ our moduli space is four dimensional. We parameterize this moduli space in a way that matches our spinor decomposition of vectors in a four-dimensional space-time

$$x^{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + x^3 & x^1 - \mathrm{i}x^2 \\ x^1 + \mathrm{i}x^2 & x^0 - x^3 \end{pmatrix}. \tag{2.93}$$

In this way, we are led to identify the moduli space as complexified Minkowski space $\mathbb{CM}$. Using the incidence relations (2.89), we can find the relationship between twistor space and space-time. We can already see there is a non-local relationship between spacetime and twistor space, as a point x corresponds to a complex line in twistor space. We can investigate this relationship further by considering two different sections of the fibration parameterised by $\{x^{\alpha\dot{\alpha}}, y^{\alpha\dot{\alpha}}\}$ that intersect at a point $Z^A = (\mu^{\dot{\alpha}}, \lambda_\alpha)$ and take the difference of their incidence relations

$$(x - y)^{\alpha\dot{\alpha}} \lambda_\alpha = 0. \tag{2.94}$$

This requires that $(x-y)^{\alpha\dot{\alpha}} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}$ for some generic spinor $\tilde{\lambda}^{\dot{\alpha}}$. Therefore, the points x and y must be null-separated if their corresponding lines in twistor space intersect. Furthermore, the point Z^A in twistor space is a null 2-plane called an α-plane in spacetime, as $\tilde{\lambda}^{\dot{\alpha}}$ has two components. While here we have taken the twistor space as our starting point, one could equivalently start from spacetime, and by considering the space of all null planes emanating from a point, one can construct the twistor space. To be more precise, from the spacetime perspective, twistor space $\mathbb{PT}$ is the space of all α-planes in complexified Minkowski space. Since the α-planes are surface-forming, there exists an integrable distribution of vector fields tangent to the surface given by

$$L_{\dot{\alpha}} = \lambda^\alpha \partial_{\alpha\dot{\alpha}}\,. \tag{2.95}$$

Due to the presence of λ_α, there exists a $\mathbb{CP}^1$s worth of α-planes through each point in $\mathbb{CM}$. Therefore, to obtain twistor space, we move to the correspondence space and quotient by the span of $L_{\dot\alpha}$, effectively identifying all points on the α-surface as one. Functions and forms that live on the correspondence space descend to $\mathbb{PT}$ if they are Lie dragged by the distribution $L_{\dot\alpha}$. An important example is the incidence relations, since $L_{\dot\alpha}\mu^{\dot\beta} = 0$, which is an alternative way to construct the twistor correspondence.

2.2.3 Symmetries of Flat Twistor Space

Another interesting feature of twistor space is its connection to the conformal structure of spacetime. From the incidence relation, we can see how the conformal group acts on a twistor $(\mu^{\dot\alpha}, \lambda_\alpha)$. The list of conformal transformations is given by

$$\begin{aligned}
&\text{Translations}: \quad T_a : x^\mu \mapsto x^\mu + a^\mu \\
&\text{Lorentz Boost\textbackslash Rotation}: L : x^\mu \mapsto L^\mu_{\ \nu}\, x^\nu \quad \text{with } L^\alpha_{\ \mu} L^\beta_{\ \nu} \eta_{\alpha\beta} = \eta_{\mu\nu} \\
&\text{Dilatation}: D : x^\mu \mapsto \Omega x^\mu \\
&\text{Special Conformal}: S_b = I T_{-b} I : x^\mu \mapsto \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2}
\end{aligned}$$

Where $I : x^\mu \mapsto \frac{x^\mu}{x^2}$ is the inversion operator. We can now consider the action of the conformal operators on the incidence relation

$$\begin{aligned}
T_a: &\ \ \mu^{\dot\alpha} = (x^{\alpha\dot\alpha} + a^{\alpha\dot\alpha})\lambda_\alpha \quad &\Rightarrow \quad &\boxed{\mu'^{\dot\alpha} = \mu^{\dot\alpha} + a^{\alpha\dot\alpha}\lambda_\alpha \ ,\ \lambda'_\alpha = \lambda_\alpha} \\
L: &\ \ \mu^{\dot\alpha} = L^\alpha_{\ \beta}\, x^{\beta\dot\beta}\, (L^\dagger)^{\ \dot\alpha}_{\dot\beta} \lambda_\alpha \quad &\Rightarrow \quad &\boxed{\mu'^{\dot\alpha} = (L^\dagger)^{\ \dot\alpha}_{\dot\beta}\, \mu^{\dot\beta} \ ,\ \lambda'_\alpha = L^\beta_{\ \alpha}\lambda_\beta} \\
D: &\ \ \mu^{\dot\alpha} = \Omega\, x^{\alpha\dot\alpha}\lambda_\alpha \quad &\Rightarrow \quad &\boxed{\mu'^{\dot\alpha} = \Omega\, \mu^{\dot\alpha} \ ,\ \lambda'_\alpha = \lambda_\alpha} \\
I: &\ \ \mu^{\dot\alpha} = \frac{x^{\alpha\dot\alpha}}{x^2}\lambda_\alpha \quad &\Rightarrow \quad &\lambda_\alpha = x_{\alpha\dot\alpha}\mu^{\dot\alpha} \ \Rightarrow\ \mu^{\dot\alpha} \leftrightarrow \lambda_\alpha \\
S: &\ \ \mu^{\dot\alpha} = I T_{-b} I(x)^{\alpha\dot\alpha}\lambda_\alpha \quad &\Rightarrow \quad &\mu^{\dot\alpha} = x^{\alpha\dot\alpha}(\lambda_\alpha - b_{\alpha\dot\alpha}\mu^{\dot\alpha}) \\
& &\Rightarrow \quad &\boxed{\mu'^{\dot\alpha} = \mu^{\dot\alpha} \ ,\ \lambda'_\alpha = \lambda_\alpha - b_{\alpha\dot\alpha}\mu^{\dot\alpha}}
\end{aligned}$$

All of the conformal transformations act linearly on the twistor variables and, therefore, we can organize them into a 4×4 matrix

$$Z'^A = \left(\begin{array}{c|c} SL(2,\mathbb{C}) & \text{Special Conformal} \\ \hline \text{Translations} & SL(2,\mathbb{C}) \ + \Omega\mathbb{1} \end{array}\right) \begin{pmatrix} \lambda_\alpha \\ \mu^{\dot{\alpha}} \end{pmatrix} = M^A_{\ B} Z^B. \tag{2.96}$$

The matrix $M^A_{\ B}$ at the moment is an element of $GL(4,\mathbb{C})$. However, since we are working in projective space $\mathbb{CP}^3$ and $GL(4,\mathbb{C}) = SL(4,\mathbb{C}) \rtimes \mathbb{C}^*$, we can restrict $M^A_{\ B} \in SL(4,\mathbb{C})$. Therefore, the conformal group that acts on spacetime acts as $SL(4,\mathbb{C})$ on twistors. At the moment, we have been working purely in the complexified setting in which signature has no meaning. We must impose a reality structure on our twistor space such that $2\text{det}(x^{\alpha\dot{\alpha}}) = \eta_{\mu\nu}x^\mu x^\nu$ gives an inner product of definite signature. From the form of the matrix $x^{\alpha\dot{\alpha}}$, Lorentzian signature requires a Hermitian matrix $x^{\alpha\dot{\alpha}} = (x^\dagger)^{\alpha\dot{\alpha}}$. To see how this condition manifests itself on twistor space, it is convenient to factor out an i in the incidence relations such that $\mu^{\dot{\alpha}} = ix^{\alpha\dot{\alpha}}\lambda_\alpha$. Complex conjugation on twistors changes the index structure as the Weyl representation in four dimensions is not self-conjugate, so $\bar{Z}^{\bar{A}} = (\bar{\lambda}_{\dot{\alpha}}, \bar{\mu}^\alpha)$ which now satisfy the relations $\bar{\mu}^\alpha = -i(x^\dagger)^{\alpha\dot{\alpha}}\bar{\lambda}_{\dot{\alpha}}$. The conjugate twistors are called dual twistors, and there is a natural inner product

$$Z \cdot \bar{Z} = \Sigma_{A\bar{A}} Z^A \bar{Z}^{\bar{A}} = \langle \bar{\mu}\lambda \rangle + [\mu\bar{\lambda}]. \tag{2.97}$$

The inner product $\Sigma_{A\bar{A}}$ is $SU(2,2)$ invariant and, therefore, its introduction can be understood as picking the Lorentzian conformal group out of $SL(4,\mathbb{C})$. Inserting the incidence relations gives

$$Z \cdot \bar{Z} = i(x - x^\dagger)^{\alpha\dot{\alpha}}\lambda_\alpha\bar{\lambda}_{\dot{\alpha}}, \tag{2.98}$$

which vanishes if $x \in \mathbb{M}$. Therefore, the slice of twistor space corresponding to Minkowski space is given by

$$\mathbb{PN} = \{Z \in \mathbb{PT} | Z \cdot \bar{Z} = 0\}. \tag{2.99}$$

Now a point in $\mathbb{PN}$ corresponds to a real null geodesic in $\mathbb{M}$ because the reality conditions fix $(x-y)^{\alpha\dot{\alpha}} = \lambda^{\alpha}\bar{\lambda}^{\dot{\alpha}}$ where $\lambda^{\alpha}\bar{\lambda}^{\dot{\alpha}}$ is the null vector tangent to the geodesic. This can be understood geometrically as the α-plane intersecting the $\mathbb{M}$ slice.

To understand the removed $\mathbb{CP}^1$ in the construction of $\mathbb{PT}$, we first consider an inversion on the incidence relations $\lambda_{\alpha} = \frac{x_{\alpha\dot{\alpha}}}{x^2}\mu^{\dot{\alpha}}$. The removed line corresponded to $\lambda_{\alpha} = 0$, but since we work in projective space $\mu^{\dot{\alpha}} \neq 0$, and, therefore, components of x must go to infinity. In this way, the removed $\mathbb{CP}^1$ corresponds to space-like infinity i^0 and intersecting lines will correspond to the null cone emanating from i^0, which is null infinity $\mathscr{I}$.

2.2.4 Penrose Transform

Generally speaking, the Penrose transform is an isomorphism that relates cohomology classes on twistor space to solutions of PDEs on spacetime via the double fibration. There are many examples [88] and the proofs use spectral sequence arguments for the Čech cohomology groups. Often, one seeks an integral transform that captures this isomorphism, in the form of contour integrals for the Čech picture or volume integrals for the Dolbeaut picture. The most basic of these Penrose transforms are for flat twistor space [89]

$$H^{0,1}(\mathbb{PT}, \mathcal{O}(-n-2)) \cong \{\partial_{\dot{\alpha}}^{\alpha_1}\phi_{\alpha_1\cdots\alpha_n} = 0\}\,. \tag{2.100}$$

We will mainly concern ourselves with the Dolbeault form of the integral formulae. The integral form of the isomorphism is given explicitly by

$$\phi_{\alpha_1\cdots\alpha_n} = \frac{1}{2\pi i}\int_X D\lambda \wedge \lambda_{\alpha_1}\cdots\lambda_{\alpha_n} f|_X\,, \tag{2.101}$$

where f is an element of the respective cohomology class and the restriction $f|_X$ restricts the form to a holomorphic curve and hence only contains forms that exist on $\mathbb{CP}^1$. One can show the integral formula satisfies the z.r.m. equation by simply noting that forms from twistor space must be Lie dragged by the vector field $L_{\dot\alpha} = \lambda^\alpha \partial_{\alpha\dot\alpha}$ and hence

$$\mathcal{L}_{L_{\dot\alpha}} f = L_{\dot\alpha} \lrcorner\, df|_X + d L_{\dot\alpha} \lrcorner\, f|_X = L_{\dot\alpha} f|_X = 0\,. \tag{2.102}$$

The reason f lives in cohomology is that we can shift $f \to f + \bar\partial g$ without changing the spacetime field, where g is a smooth function on twistor space. Since the restriction of the Dolbeault operator on twistor space $\bar\partial|_X$ gives the Dolbeault operator on the sphere, one finds an exact form inside the integral and by Stokes' theorem gives no contribution. One can find an analogous integral transform for a dotted z.r.m. field, see [90] for a review, and we will consider Penrose transforms in the context of gauge theory and gravity.

2.2.5 Gauge Theory

To encode gauge fields on twistor space, the Ward correspondence provides a construction that gives an SD Yang-Mills field from geometric data on twistor space.

Theorem (Ward [48]) *There is a one-to-one correspondence between:*

1. *SD gauge fields on $\mathbb{CM}$ up to gauge transformations valued in $GL(r,\mathbb{C})$, and*
2. *rank-r holomorphic vector bundles $E \to \mathbb{PT}$ with $E|_X$ topologically trivial for every twistor line $X \cong \mathbb{CP}^1$ corresponding to $x \in \mathbb{CM}$.*

In the Dolbeault presentation, the holomorphicity of the Ward bundle $E \to \mathbb{PT}$ corresponds to the integrability $\bar{D}^2 = 0$ of the Dolbeault operator $\bar{D}$ given by

$$\bar{D} = \bar{\partial} + a \,, \tag{2.103}$$

where a is a $(0,1)$-form on $\mathbb{PT}$. The integrability condition translates to the following non-linear equation

$$\bar{\partial} a + \frac{1}{2}[a, a] = 0 \,, \tag{2.104}$$

where $[\cdot, \cdot]$ is the Lie bracket for the chosen gauge group. On generic $X = \mathbb{CP}^1$s corresponding to points $x \in \mathbb{CM}$, the Ward bundle is assumed to be trivial (this will generically be the case for a holomorphic bundle that is a topologically trivial bundle on the appropriate region). Thus there will exist holomorphic frames $F(x, \lambda) : E|_X \to \mathbb{C}^r$, where r is the rank of E, such that [48, 91]

$$\bar{\partial}|_X F + a|_X F = 0 \,. \tag{2.105}$$

In other words, the connection a is pure gauge on each twistor line. Using the fact that $\lambda^\alpha \partial_{\alpha\dot{\alpha}} a|_X = 0$ since twistor space forms are Lie dragged by the Lax operator (2.95), one can show

$$\bar{\partial}|_X \left(F^{-1} \lambda^\alpha \partial_{\alpha\dot{\alpha}} F \right) = 0 \,, \tag{2.106}$$

Therefore, by Theorem 1

$$F^{-1} \lambda^\alpha \partial_{\alpha\dot{\alpha}} F = \lambda^\alpha A_{\alpha\dot{\alpha}} \,. \tag{2.107}$$

where $A_{\alpha\dot{\alpha}}$ is a function of spacetime only and will be shown to be the gauge field. This may be translated into a system of PDE's

$$D_{\dot{\alpha}} F^{-1} = \lambda^\alpha (\partial_{\alpha\dot{\alpha}} + A_{\alpha\dot{\alpha}}) F^{-1} = 0 \,. \tag{2.108}$$

Since this is a system of overdetermined PDEs, there must exist an integrability condition to make the system consistent. This consistency condition is given by

$$[D_{\dot\alpha}, D_{\dot\beta}] = 0 \qquad \Rightarrow \qquad F_{\alpha\beta} = \partial_{(\alpha}{}^{\dot\alpha} A_{\beta)\dot\alpha} + \frac{1}{2}[A_{(\alpha}{}^{\dot\alpha}, A_{\beta)\dot\alpha}] = 0\,, \tag{2.109}$$

giving the SD Yang-Mills equations.

Around the SD background, the Penrose transform gives perturbations to the SD and ASD fields in spacetime using twistor data. The abstract form of the Penrose transform in this case is given by

$$\begin{aligned} b \in H^{0,1}_{\bar D}(\mathbb{PT}, \mathcal{O}(-4)\otimes \mathrm{End}(E)) &\cong \{B_{\alpha\beta} \in \Omega^2(\mathbb{M}) | D^{\alpha\dot\alpha} B_{\alpha\beta} = 0\}\,, \\ \delta a \in H^{0,1}_{\bar D}(\mathbb{PT}, \mathcal{O}\otimes \mathrm{End}(E)) &\cong \{a_{\alpha\dot\alpha} \in \Omega^1(\mathbb{M}) | f_{\alpha\beta} = D_{(\alpha}{}^{\dot\alpha} a_{\beta)\dot\alpha} = 0\}\,, \end{aligned} \tag{2.110}$$

where $\mathcal{O}$ denotes the sheaf of holomorphic functions on $\mathbb{PT}$, i.e. weightless holomorphic functions. Concretely, we can obtain $B_{\alpha\beta}$ from a Penrose integral formula,

$$B_{\alpha\beta}(x) = \frac{1}{2\pi i}\int_X D\lambda \wedge \lambda_\alpha \lambda_\beta\, F^{-1} b|_X F\,. \tag{2.111}$$

The appearance of the holomorphic frame ensures that the holomorphic vector bundle E is trivialised on X. Using the properties of the frame F and the fact that the form $b|_X$ is Lie dragged by the operator (2.95), one can show that the integral formula satisfies the equation for an ASD perturbation to the field strength

$$D^{\alpha\dot\alpha} B_{\alpha\beta} = \frac{1}{2\pi i}\int_X D\lambda \wedge \lambda_\beta \left(D^{\dot\alpha} F^{-1} b|_x F + F^{-1}\lambda_\alpha \partial^{\alpha\dot\alpha} b|_X + F^{-1} b|_X D^{\dot\alpha} F\right) = 0\,. \tag{2.112}$$

For the case of an SD perturbation, we can construct the perturbed gauge field directly. Using the frame to trivialise the vector bundle, the form $F^{-1}\delta a|_X F \in H^{0,1}(\mathbb{CP}^1, \mathcal{O})$ since the frame translates $\bar D$ exact terms on twistor space to $\bar\partial|_X$ on the

$\mathbb{CP}^1$. Since this cohomology group is trivial, the representative must be exact

$$F^{-1}\delta a|_X F = \bar{\partial} k\,. \tag{2.113}$$

A short calculation following the same logic as the ASD perturbation case shows that

$$D_{\dot{\alpha}}\left(F^{-1}\delta a|_X F\right) = F^{-1}\lambda^{\alpha}\partial_{\alpha\dot{\alpha}}\delta a|_X F = 0\,. \tag{2.114}$$

Since the operator $D_{\dot{\alpha}}$ is holomorphic, the combination $D_{\dot{\alpha}}k$ is holomorphic and, therefore, by Liouville's theorem, is given by

$$D_{\dot{\alpha}}k = \lambda^{\alpha}a_{\alpha\dot{\alpha}}\,. \tag{2.115}$$

By applying the commutator $[D_{\dot{\alpha}}, D_{\dot{\beta}}]$ to the function k, the field $a_{\alpha\dot{\alpha}}$ is indeed a SD perturbation

$$D_{(\alpha}{}^{\dot{\alpha}}a_{\beta)\dot{\alpha}} = 0\,. \tag{2.116}$$

2.2.6 Deformation Theory and the Non-Linear Graviton

Having obtained the twistor definitions for flat spacetimes, Penrose's non-linear gravitation theorem [10, 8] allows us to extend these considerations to curved SD spacetimes by deforming the complex structure. The complex structure of $\mathbb{PT}$ is a splitting of holomorphic and anti-holomorphic coordinates such that the exterior derivative splits as $d = \partial + \bar{\partial}$, where

$$\partial = dZ^A\frac{\partial}{\partial Z^A} \qquad \bar{\partial} = d\bar{Z}^A\frac{\partial}{\partial \bar{Z}^A}. \tag{2.117}$$

A change in the Dolbeault operator can represent a deformation of a complex structure

$$\bar{\partial} \longrightarrow \bar{\nabla} = \bar{\partial} + V, \qquad V = V^A_{\ B}\, d\bar{Z}^B \otimes \frac{\partial}{\partial Z^A}\,. \tag{2.118}$$

such that $V \in \Omega^{0,1}(\mathbb{PT}, T^{1,0}\mathbb{PT})$. A general deformation only defines an almost complex structure on twistor space, as one must consider global issues. The deformed Dolbeault operator $\bar{\nabla}$ must satisfy an integrability condition, $\bar{\nabla}^2 = 0$, to be a complex structure. This condition gives the Maurer-Cartan equation on the deformation

$$\bar{\partial} V + \frac{1}{2}[V, V]_{SN} = 0\,, \tag{2.119}$$

where $[\cdot, \cdot]_{SN}$ is the Schouten–Nijenhuis bracket defined as the commutator between the vector fields combined with the wedge product on $(0, \cdot)$-forms

$$[V, W]_{SN} = (V^A \wedge \partial_A W^B - (-1)^{pq} W^A \wedge \partial_A V^B)\partial_B\,. \tag{2.120}$$

Finally, the connection to the Newlander-Nirenberg theorem can be found by considering the $(1, 0)$ forms ∂Z^A on the deformed space Z. These forms satisfy the condition $\bar{\nabla} \lrcorner\, \partial Z^A = 0$, and subsequently are given by

$$\partial Z^A = dZ^A - V^A, \tag{2.121}$$

where we have used $V = V^A \partial_A$. These forms are holomorphic with respect to the new complex structure and span the space of $(1, 0)$-forms of the non-projective space. The Newlander-Nirenberg theorem [92] asserts that if there is a complex manifold, it has an integrable distribution of holomorphic vector fields $[T^{1,0}Z, T^{1,0}Z] \subset T^{1,0}Z$. This can be re-phrased as saying that the forms ∂Z^A span a closed differential ideal [93]. This can be checked by

$$d\partial Z^A = \partial_B V^A \wedge \partial Z^B - (\bar{\partial} V^A + V^B \wedge \partial_B V^A) = \partial_B V^A \wedge \partial Z^B - (\bar{\partial} V + \frac{1}{2}[V, V]_{SN})^A. \tag{2.122}$$

We see once again the appearance of the so-called torsion $T = (\bar{\partial} V + \frac{1}{2}[V, V]_{SN})^A \partial_A$ as an obstruction for the manifold Z to be complex. The new complex manifold built

using this complex structure is denoted $Z = \mathcal{PT}$. In general, such complex structure deformations provide twistor spaces for spacetimes with SD conformal structure, i.e., they only guarantee that $\psi_{\alpha\beta\gamma\delta} = 0$. One must add extra structures on the deformed twistor space to give a further curvature restriction on spacetime. For the cases of interest in this thesis, namely the Ricci-flat and Einstein cases, there are two key structures: a Poisson structure and a distinguished holomorphic 1-form. The first key feature is that of the Poisson structure, which is given by

$$\{\omega, \eta\} = I^{AB} \mathcal{L}_{\partial_A} \omega \wedge \mathcal{L}_{\partial_B} \eta \,, \tag{2.123}$$

for ω, η are generic forms, and we have also introduced the object called the infinity twistor I^{AB}, as depending on its choice, it can change the conformal structure of the spacetime. We require that this Poisson structure be holomorphic with respect to $\bar{\nabla}$. This can be checked by considering

$$\bar{\nabla}\{\omega, \eta\} = \{\bar{\nabla}\omega, \eta\} + (-1)^p \{\omega, \bar{\nabla}\eta\} + 2I^{A[B} \mathcal{L}_{\partial_A} V^{C]} \wedge \mathcal{L}_{\partial_B} \omega \wedge \mathcal{L}_{\partial_C} \eta. \tag{2.124}$$

We find that $I^{A[B} \mathcal{L}_{\partial_A} V^{C]} = 0$ for the Poisson structure to be holomorphic; this condition is equivalent to requiring that V^A be Hamiltonian. Requiring the vector field part of $V = V^A \partial_A$ be Hamiltonian with respect to this Poisson structure results in

$$\{f, h\} = V \lrcorner \, df \quad \rightarrow \quad V^A = I^{AB} \mathcal{L}_{\partial_B} h = I^{AB} \frac{\partial h}{\partial Z^B}, \tag{2.125}$$

where the Hamiltonian $h \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(2))$ and $f \in C^\infty(\mathbb{PT})$. One can explicitly check that this form of V satisfies the requirement that the Poisson structure be holomorphic. Using the Hamiltonian deformation of complex structure, the condition for this deformation to be integral is now given by

$$\bar{\partial} h + \frac{1}{2}\{h, h\} = 0. \tag{2.126}$$

The holomorphic curve condition (2.84) written in terms of the Hamiltonian is given by

$$\bar{\partial}_0 F^A = I^{AB} \bar{\partial}_0 \lrcorner \, \mathcal{L}_{\partial_B} h|_X \, . \tag{2.127}$$

The functions F^A define the map from the correspondence space $F = \mathbb{PS} = M \times \mathbb{CP}^1$, which is the projective spin bundle over spacetime, and the deformed twistor space $\mathcal{PT}$. The $\mathbb{CP}^1$ from the spacetime perspective has the interpretation of parametrising the α-surface, which is the generalisation of α-planes in the curved case, through a point on M. For there to exist a surface in M, there must exist an integrable distribution; in this case, the distribution is generated by

$$L_{\dot{\alpha}} = \lambda^\alpha \left(\nabla_{\alpha\dot{\alpha}} - \Gamma_{\alpha\dot{\alpha}\beta}{}^{\gamma} \lambda^\beta \frac{\partial}{\partial \lambda^\gamma} \right) \tag{2.128}$$

These operators are called Lax operators, and they have deep connections to the integrable nature of the twistor correspondence. One can show that the Lax operators generate an integrable distribution if and only if the spacetime is SD

$$[L_{\dot{\alpha}}, L_{\dot{\beta}}] = 0 \Leftrightarrow \psi_{\alpha\beta\gamma\delta} = 0 \tag{2.129}$$

From this perspective, one can define the twistor space for an SD structure on M to be the space of α-surfaces. In other words, twistor space arises as the quotient of the correspondence space by the Lax operator. As was the case in the flat space case, functions and forms that descend to twistor space from the correspondence space are Lie dragged by the Lax operator, an example of which are the functions defining the map $F \to \mathcal{PT}$

$$L_{\dot{\alpha}} F^A = 0 \tag{2.130}$$

The second feature, which picks out curvature restrictions, is a distinguished holo-

morphic 1-form, which is defined to be

$$\tau_{\mathcal{PT}} = I_{AB} Z^A \partial Z^B \tag{2.131}$$

where $I_{AB} = \frac{1}{2}\epsilon_{ABCD} I^{AB}$ is the dual infinity twistor. This distinguished 1-form has the property that under pullback to the correspondence space, there is an explicit form to which it maps. Secondly, the vanishing locus of the form defines the points at infinity on the boundary of the spacetime M

$$\tau_{\mathcal{PT}}|_X = 0 \quad \text{for} \quad x \to \infty \tag{2.132}$$

The first construction is of SD spacetime, which has vanishing Ricci curvature. The non-linear graviton theorem captures the key ingredients

Theorem 3. (Penrose [8]) There is a one-to-one correspondence between solutions (M, g) to the SD Ricci-flat equations and three-dimensional complex manifolds $\mathcal{PT}$ with the following structure:

1. A projection $\mu : \mathcal{PT} \to \mathbb{CP}^1$,
2. A four-parameter family of sections of μ with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$,
3. A degenerate Poisson structure I preserving the fibration of μ with values in the pull-back from $\mathbb{CP}^1$ of $\mathcal{O}(2)$.

The Poisson structure in this case is degenerate and only has derivatives along the fibers of the map μ, and hence is given by $I^{AB} = \text{diag}(\epsilon^{\dot{\alpha}\dot{\beta}}, 0)$ and the dual infinity twistor is $I_{AB} = \text{diag}(0, \epsilon^{\alpha\beta})$. The other main feature of the theorem is that the fibration over $\mathbb{CP}^1$ is preserved, and so the form $D\lambda$ stays as a basis element. Put another way, the coordinate λ_α stays as a holomorphic coordinate. To see this more concretely, consider the $(1,0)$-forms given by $\{D\lambda, \partial\mu^{\dot{\alpha}} = d\mu^{\dot{\alpha}} + \epsilon^{\dot{\alpha}\dot{\beta}}\partial_{\dot{\beta}} h\}$. Using this

basis, the holomorphic sections can be found by using (2.84)

$$\left[[\tilde{a}\bar{\sigma}]\ \tilde{a}^{\dot{\alpha}}\frac{\partial}{\partial\bar{\sigma}^{\dot{\alpha}}}\right]\lrcorner\ D\lambda = 0|_X \quad , \quad \left[[\tilde{a}\bar{\sigma}]\ \tilde{a}^{\dot{\alpha}}\frac{\partial}{\partial\bar{\sigma}^{\dot{\alpha}}}\right]\lrcorner\ \partial\mu^{\dot{\alpha}}|_X = 0. \tag{2.133}$$

The first is solved, up to an $GL(2,\mathbb{C})$ transformation, by $\lambda_\alpha = \sigma_\alpha$; we see the particular infinity twistor was chosen to preserve the base of the fibration as required by the theorem. Using the dual twistor, the distinguished 1-form $\tau_{\mathcal{PT}} = D\lambda$ equals the unique holomorphic 1-form on $\mathbb{CP}^1$ in the correspondence space under the pullback. Furthermore, we confirm that infinity in spacetime corresponds to $\lambda_\alpha = 0$ in twistor space, since $D\lambda$ vanishes on this locus.

The second gives a PDE that defines holomorphic curves in twistor space

$$\bar{\partial}_0 F^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\partial}_0\lrcorner\ \mathcal{L}_{\partial_{\dot{\beta}}}h|_X. \tag{2.134}$$

and on the spin bundle, these functions must be annihilated by the Lax operators $L_{\dot{\alpha}}F^{\dot{\beta}} = 0$. To reconstruct the metric, we consider the 2-form valued in $\mathcal{O}(2)$

$$\Sigma = \frac{1}{2}\partial\mu^{\dot{\alpha}}\wedge\partial\mu_{\dot{\alpha}}. \tag{2.135}$$

Since the deformation is integrable, the deformed twistor space is a complex manifold, and hence, using (2.122), one can show

$$D\lambda\wedge d\Sigma = 0, \tag{2.136}$$

so Σ is closed up to $D\lambda$ terms. Next, consider the Lie derivative of the pull-back

$$\mathcal{L}_{\bar{\partial}_0}(D\lambda\wedge\pi_2^*\Sigma) = D\lambda\wedge\mathcal{L}_{\bar{\partial}_0}(\pi_2^*\Sigma) = D\lambda\wedge[\bar{\partial}_0\lrcorner\ \pi_2^*d\Sigma + d(\bar{\partial}_0\lrcorner\ \pi_2^*\Sigma)] = 0, \tag{2.137}$$

where the first term vanishes because of the closure of Σ up to $D\lambda$ terms and the

second term vanished because of (2.134) as this states that under pull-back $\partial\mu^{\dot{\alpha}}$ has no component along the $(0,1)$ of the $\mathbb{CP}^1$. Therefore, since Σ is valued in $\mathcal{O}(2)$ we can write

$$\pi_2^*\Sigma = \lambda_\alpha\lambda_\beta\Sigma^{\alpha\beta}(x) \mod D\lambda. \tag{2.138}$$

The $\Sigma^{\alpha\beta}(x)$ satisfies $d_x\Sigma^{\alpha\beta} = 0$ on spacetime due to the closure of Σ, and further more $\Sigma\wedge\Sigma = 0$ corresponds to the simplicity constraint $\Sigma^{(\alpha\beta}\wedge\Sigma^{\gamma\sigma)} = 0$ which means it can be decomposed into a tetrad $\Sigma^{\alpha\beta} = \frac{1}{2}\theta^{\alpha\dot{\alpha}}\wedge\theta^{\beta}{}_{\dot{\alpha}}$ [79]. These are exactly the conditions for a self-dual spacetime with a vanishing ASD Riemann curvature.

One can access a spacetime with a non-vanishing cosmological constant by choosing a different infinity twistor that is non-degenerate. The relevant twistor correspondence is given by Ward's extension of the non-linear graviton [94]

Theorem 4 (Ward '80)**.** There is a one-to-one correspondence between

- complexified space-times with vanishing trace-free Ricci tensor and non-vanishing scalar curvature,
- deformed twistor spaces containing a rational curve with normal bundle $\mathcal{O}(1)\oplus\mathcal{O}(1)$ and a non-degenerate holomorphic contact structure.

The contact structure is specified by a holomorphic $(1,0)$-form $\tau_{\mathcal{PT}}$ taking values in $\mathcal{O}(2)$ and such that $\tau_{\mathcal{PT}}\wedge \mathrm{d}\tau_{\mathcal{PT}} \neq 0$ and is proportional to the holomorphic top form. If the scalar curvature is taken to zero, the contact structure becomes degenerate, and the twistor space admits a holomorphic $\mathbb{CP}^1$ fibration, as per standard non-linear graviton.

The new Poisson structure is non-degenerate, and a standard representation is given by

$$I_\Lambda^{AB} = \begin{pmatrix} \varepsilon^{\dot{\alpha}\dot{\beta}} & 0 \\ 0 & \Lambda\,\varepsilon_{\alpha\beta} \end{pmatrix}, \qquad I^\Lambda_{AB} = \begin{pmatrix} \Lambda\,\varepsilon_{\dot{\alpha}\dot{\beta}} & 0 \\ 0 & \varepsilon^{\alpha\beta} \end{pmatrix}. \tag{2.139}$$

We can make the contact structure explicit by considering

$$\begin{aligned} d\tau_{\mathcal{PT}} &= I_{AB} dZ^A \wedge dZ^B - 2\Lambda dh \\ &= I_{AB} \partial Z^A \wedge \partial Z^B + 2\Lambda \partial Z^A \wedge \partial_A h + \Lambda\{h,h\} - 2\Lambda \partial h - 2\Lambda \bar{\partial} h \\ &= \Sigma_\Lambda - 2\Lambda T \,, \end{aligned} \tag{2.140}$$

where we have defined the 2-form $\Sigma_\Lambda = I_{AB} \partial Z^A \wedge \partial Z^B$. From this we see that only when the torsion vanishes can we obtain a holomorphic contact structure. In this case, the contact structure condition becomes

$$\tau_{\mathcal{PT}} \wedge d\tau_{\mathcal{PT}} = \tau_{\mathcal{PT}} \wedge \Sigma_\Lambda = I_{AB} I_{CD} Z^A \partial Z^B \wedge \partial Z^C \wedge \partial Z^D = 2\Lambda \Omega \,, \tag{2.141}$$

where $\Omega = \frac{1}{4!} \epsilon_{ABCD} Z^A \partial Z^B \wedge \partial Z^C \wedge \partial Z^D$ is the holomorphic volume form on twistor space.

Under the pullback, one can show [95] that the holomorphic 1-form equals the following 1-form on the projective spin bundle

$$\pi_2^* \tau_{\mathcal{PT}} = \tau_{\mathbb{PS}} = \sigma^\alpha d\sigma_\alpha + \Gamma^{\alpha\beta} \sigma_\alpha \sigma_\beta \tag{2.142}$$

This form plays a special role in the Euclidean signature twistor correspondence where $\mathbb{PS} \simeq \mathcal{PT}$ and $\tau_{\mathbb{PS}}$ is one of the $(1,0)$-forms in the Atiyah-Hitchin-Singer complex structure [96]. One can perform the same calculation on the spin bundle, giving a relation to the basis of ASD 2-forms

$$\tau_{\mathbb{PS}} \wedge \mathrm{d}\tau_{\mathbb{PS}} = \tau_{\mathbb{PS}} \wedge R_{\alpha\beta} \sigma^\alpha \sigma^\beta = \frac{\Lambda}{3} \tau_{\mathbb{PS}} \wedge \Sigma_{\alpha\beta} \sigma^\alpha \sigma^\beta \,. \tag{2.143}$$

By noting that the exterior derivative commutes with the pullback, we obtain a

relationship between twistor data and the metric components

$$\pi_2^*\Sigma_\Lambda = \frac{3}{\Lambda}\sigma_\alpha\sigma_\beta\Sigma^{\alpha\beta}(x) \mod \tau_{\mathbb{PS}}\,. \tag{2.144}$$

To conclude, we consider perturbations of the background in the SD and ASD directions. On the curved SD background, one can construct an ASD perturbation on twistor space using the Penrose transform. In particular, there exists an isomorphism

$$g \in H^{0,1}(\mathcal{PT}, \mathcal{O}(-6)) \simeq \{\phi_{\alpha\beta\gamma\delta} \text{ on } \mathcal{M} | \nabla^{\alpha\dot\alpha}\phi_{\alpha\beta\gamma\delta} = 0\}. \tag{2.145}$$

The correspondence can be made explicit through the use of an integral formula over the Riemann sphere $X \simeq \mathbb{CP}^1$

$$\phi_{\alpha\beta\gamma\delta} = \frac{1}{2\pi i}\int_X D\lambda \wedge \lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\delta\, g|_X. \tag{2.146}$$

The ASD perturbation only gives the perturbations of the ASD Weyl spinor. Purely ASD perturbations of the metric are obstructed by the SD background curvature [97]. However, for the SD perturbations, one can construct the perturbation to the metric explicitly via a potential. The relevant cohomology class giving the SD perturbations is

$$\delta h \in H^{0,1}(\mathcal{PT}, \mathcal{O}(2)) \simeq \{\delta g_{ab} \text{ on } \mathcal{M} | \psi_{\alpha\beta\gamma\delta} = 0\}. \tag{2.147}$$

If one restricts the perturbation to the holomorphic curve X, the form $\delta h|_X \in H^{0,1}(\mathbb{CP}^1, \mathcal{O}(2))$ since all $(0,1)$-forms are closed under the Dolbeaut operator on $\mathbb{CP}^1$ and the definition of the holomorphic curves (2.82). Since $H^{0,1}(\mathbb{CP}^1, \mathcal{O}(2))$ is trivial, the restriction of the SD perturbation is exact

$$\delta h|_X = \bar\partial|_X j\,, \tag{2.148}$$

where $j = j(x, \lambda)$ is a function on $\mathbb{PS}$ with homogeneity 2. Using the fact that $\delta h|_X$

is a $(0,1)$-form on $\mathbb{CP}^1$ and its Lie derivative with respect to the Lax operator must vanish, it follows that

$$\mathcal{L}_{L_{\dot\alpha}}\delta h|_X = L_{\dot\alpha}\lrcorner \mathrm{d}\delta h|_X + \mathrm{d}L_{\dot\alpha}\lrcorner\delta h|_X = L_{\dot\alpha}\delta h|_X = 0\,. \tag{2.149}$$

Therefore, due the holomorphicity of $L_{\dot\alpha}$, the function $L_{\dot\alpha}j$ is globally holomorphic of weight 3 and by Liouville's theorem can be decomposed

$$L_{\dot\alpha}j = \lambda^\alpha\lambda^\beta\lambda^\gamma\varphi_{\alpha\beta\gamma\dot\alpha}\,, \tag{2.150}$$

where $\varphi_{\alpha\beta\gamma\dot\alpha}$ is a spin field on $\mathcal{M}$. By considering $[L_{\dot\alpha}, L_{\dot\beta}]j$, it can be shown that the spin field satisfies

$$\nabla_{(\alpha}^{\ \dot\alpha}\varphi_{\beta\gamma\delta)\dot\alpha} = 0. \tag{2.151}$$

This is a potential for a perturbation to the metric

$$\delta g_{\alpha\dot\alpha\beta\dot\beta} = \nabla^\gamma{}_{(\dot\alpha}\varphi_{\dot\beta)\alpha\beta\gamma}\,, \tag{2.152}$$

which gives a vanishing linearised ASD Weyl spinor. We further note $\varphi_{\alpha\beta\gamma\dot\alpha}$ is a potential modulo gauge description for SD perturbation, as (2.151) has a gauge symmetry on a SD background given by $\varphi_{\alpha\beta\gamma\dot\alpha} \sim \varphi_{\alpha\beta\gamma\dot\alpha} + \nabla_{\dot\alpha(\alpha}\omega_{\beta\gamma)}$. At the level of the twistor data, this gauge symmetry can be seen from the ambiguity of defining j as $j \sim j+\omega$ where ω is a holomorphic weight 2 function that would give the same $\delta h|_X$. Acting on ω with the Lax operators reproduces the gauge symmetry. As a result, one can choose a gauge in which all the undotted indices point in the same direction $\varphi_{\alpha\beta\gamma\dot\alpha} = \iota_\alpha\iota_\beta\iota_\gamma\varphi_{\dot\alpha}$ for some reference spinor ι_α as an example.

We now consider the description of solutions to the twistor equation using twistor space data. A solution to the valence n twistor equation (5.19) corresponds to ele-

ments of cohomology

$$\xi_n \in H^0(\mathcal{PT}, \mathcal{O}(n)) \simeq \left\{ \xi^{\alpha_1 \cdots \alpha_n} \text{ on } \mathcal{M} | \nabla_{\dot{\alpha}}^{(\alpha_0} \xi^{\alpha_1 \cdots \alpha_{s+1})} = 0 \right\}. \tag{2.153}$$

When restricted to a holomorphic curve $\xi_n|_X = \xi_n(F^A, \bar{F}^{\bar{A}})$. A short calculation using the chain rule shows

$$\bar{\partial}_0 \xi_n|_X = \bar{\partial}_0 F^A \partial_A \xi_n|_X + \bar{\partial}_0 \bar{F}^{\bar{A}} \partial_{\bar{A}} \xi_n|_X = \bar{\partial}_0 \lrcorner (\bar{\nabla} \xi_n)|_X \,, \tag{2.154}$$

where we have used (2.127). Therefore, if ξ_n is holomorphic with respect to the deformed complex structure, $\bar{\nabla} \xi_s = 0$, then on each twistor line $\xi_s|_X$ is holomorphic in σ_α with homogenous weight n. Therefore,

$$\xi_n|_X = \xi^{\alpha_1 \cdots \alpha_n} \sigma_{\alpha_1} \cdots \sigma_{\alpha_n}. \tag{2.155}$$

Since $\xi_n|_X$ originated from a function on twistor space, it must be constant along the distribution generated by the Lax operator, and, therefore, gives the twistor equation

$$L_{\dot{\alpha}} \xi_n|_X = \nabla^{(\alpha_0}{}_{\dot{\alpha}} \xi^{\alpha_1 \cdots \alpha_n)} \sigma_{\alpha_0} \cdots \sigma_{\alpha_n} = 0. \tag{2.156}$$

However, on a curved twistor space, such ξ_n generically do not exist, and only special twistor spaces admit sections of $H^0(\mathcal{PT}, \mathcal{O}(n))$ [98, 99, 100] corresponding to non-constant Killing spinors. Since the twistor equation is an overdetermined system even on an SD background, the existence of such functions on $\mathcal{PT}$ is not guaranteed beyond homogeneous polynomials on the $\mathbb{CP}^1$ base. In Chapter 5 we will consider a modification of the twistor equation related to residual gauge transformations.

Chapter 3

Charges at Infinity: Gravity

In this chapter, we present a first-principle derivation of the surface charges associated with the $Lw_{1+\infty}$ symmetries. To do so, we start from the action describing self-dual gravity on twistor space [101], where the action of the celestial symmetries becomes local and has a natural geometric interpretation as Poisson diffeomorphisms [102, 47]. At null infinity, this self-dual phase space agrees with that of conventional gravity. In this framework, the charge algebra for these symmetries is easily derived, by contrast to the spacetime computations, which are quite cumbersome and were initially only done at soft and quadratic order [61, 62, 63, 64, 65, 66, 67, 68]. Our approach involves several salient features of the covariant phase space methods, such as a treatment of non-integrability and the use of a modified bracket [19, 103, 33, 34, 104, 105, 106, 107]. Using the Bramson-Curtis-Tod (BCT) integral formula [108, 109, 110], we relate our twistor space expressions for the symplectic structure and the charges in terms of spacetime data. The recursion relations of [61, 62] derive naturally from the integral formula and the on-shell conditions for self-duality implied by the twistor action. In particular, this demonstrates that the truncation of the Bianchi identities postulated in [61, 62, 63] corresponds to ASD perturbations in the self-dual sector of gravity and their hierarchy; essentially, they correspond to the asymptotic expansion of Newman's $\mathcal{H}$-space constructed from data at null infinity [9].

The self-duality equations are well-known to be an integrable system [111, 112, 93] and the recursion relations of [61, 62, 63] can be understood to be those arising from the associated recursion operators for the self-duality equations as described in [99] for gravity and [112] for Yang-Mills. These recursion relations extend the self-duality equations to an integrable hierarchy, and here its definition is adapted to data at null infinity. In particular, we see that $Lw_{1+\infty}$ arises by application of these recursion operators to the BMS group.

We also check the consistency of the action of the symmetries on twistor space with the one expected on spacetime and show the invariance of the recursion relations. This clarifies in which sense $Lw_{1+\infty}$ is a symmetry of the gravitational solution space. Finally, we compare our results with those of [61, 62] and almost find perfect agreement. The differences occur in the expression of the surface charges for superrotation and higher spin charges. In particular, our derivation offers a new and unambiguous proposal for the expression of the angular momentum at $\mathscr{I}$ in the presence of radiation.

3.1 Twistor space action

In this section, we introduce the twistor space action describing self-dual gravity [101] with an ASD perturbation and how it captures the important features of the non-linear graviton theorem. The twistor space action is a Poisson BF theory explicitly given by

$$S[g,h] = \frac{1}{2\pi i}\int_{\mathbb{PT}} g \wedge \left(\bar{\partial} h + \frac{1}{2}\{h,h\}\right) \wedge D^3Z\,, \tag{3.1}$$

where $D^3Z \in \Omega^{3,0}(\mathbb{PT}, \mathcal{O}(4))$ is the holomorphic top form, and $g \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(-6))$ and $h \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(2))$ are the off-shell twistor data for respectively the ASD and SD parts of the dynamical gravitational field. The action is invariant under gauge

transformations

$$\begin{aligned}\delta_{(\xi,\phi)}h &= \bar{\nabla}\xi\,,\\ \delta_{(\xi,\phi)}g &= \{g,\xi\}+\bar{\nabla}\phi\,,\end{aligned} \tag{3.2}$$

where $(\xi,\phi)\in\mathcal{O}(2)\oplus\mathcal{O}(-6)$ are smooth and arbitrary gauge parameters. In these transformations, we are using the bracket (2.123) as well as the deformed complex structure introduced in (2.118). The equations of motion of the action (3.1) are given by

$$\begin{aligned}\bar{\nabla}g = \bar{\partial}g+\{h,g\} &= 0\,,\\ \bar{\partial}h+\frac{1}{2}\{h,h\} &= 0\,.\end{aligned} \tag{3.3}$$

One can check explicitly that the equations of motion are invariant under the gauge transformations (3.2) although it also follows from the corresponding invariance of the Lagrangian. On-shell, it follows from (3.3) that the field g is $\bar{\nabla}$-closed and h satisfies the integrability condition (2.126). In particular, we note that the linearized version of this second equation is $\bar{\nabla}\delta h=0$. Moreover, from (3.2), the fields are defined up to $\bar{\nabla}$-exact terms. In other words, the on-shell field g and linearized perturbation δh around h modulo gauge transformations define elements of the cohomology groups for the deformed complex structure denoted by $H^{0,1}_{\bar{\nabla}}(\mathbb{PT},\mathcal{O}(-6))$ and $H^{0,1}_{\bar{\nabla}}(\mathbb{PT},\mathcal{O}(2))$.

3.2 Asymptotic twistor space and the BCT integral formula

In this section, we will describe the geometry of twistor space at null infinity and the definition of asymptotic twistor space. We then discuss how the asymptotic data is encoded in the asymptotic twistor space, introduce an integral transform that solves the asymptotic equations of motion, and extend it to include higher spin-weighted

fields.

3.2.1 Null Infinity and the Homogeneous formalism

The appearance of weighted scalars in the NP formalism suggests that the homogeneous coordinates on the sphere in Bondi coordinates can be used to carry the weight [113]. The homogeneous $\mathscr{I}$ formalism starts with a homogenous Bondi coordinate system such that $(r, u, \zeta_\alpha, \bar{\zeta}_{\dot{\alpha}}) \sim (|b|^{-2}r, |b|^2 u, b\zeta_\alpha, \bar{b}\bar{\zeta}_{\dot{\alpha}})$ for $b \in \mathbb{C}^*$. The affine coordinates on the celestial sphere are recovered by going to a local patch where $z = \zeta_1/\zeta_0$. By choosing a vector $T^{\alpha\dot{\alpha}}$, we can write the coordinates in a compact form

$$x^{\alpha\dot{\alpha}} = \frac{uT^{\alpha\dot{\alpha}}}{\langle\hat{\zeta}\zeta\rangle} + r\zeta^\alpha\bar{\zeta}^{\dot{\alpha}}, \tag{3.4}$$

where $\hat{\zeta}_\alpha$ is given by

$$\hat{\zeta}_\alpha = T_{\alpha\dot{\alpha}}\bar{\zeta}^{\dot{\alpha}}, \tag{3.5}$$

We note that, depending on the choice of $T^{\alpha\dot{\alpha}}$, the metric one obtains in these coordinates is either going to have the round-sphere metric or the complex plane as its space-like component. For the round-sphere metric this will be $T^{\alpha\dot{\alpha}} = \mathrm{diag}(1,1)$ while for the complex plane $T^{\alpha\dot{\alpha}} = \mathrm{diag}(1,0)$.

In this formalism, a $\{p, q\}$ scalar in the NP formalism is translated to a section of the line bundle $\mathcal{O}_{\mathscr{I}}(p, q)$ over $\mathbb{CP}^1$, which is a function such that

$$f(|b|^{-2}r, |b|^2 u, b\zeta_\alpha, \bar{b}\bar{\zeta}_{\dot{\alpha}}) = b^p\bar{b}^q f(r, u, \zeta_\alpha, \bar{\zeta}_{\dot{\alpha}}) \qquad b \in \mathbb{C}^*. \tag{3.6}$$

with the spin and boost weights [74] defined through

$$s = \frac{p-q}{2}, \qquad w = \frac{p+q}{2}. \tag{3.7}$$

Using the language of line bundles, null infinity can be viewed as the total space of

$\mathcal{O}_{\mathscr{I}}(1,1)_{\mathbb{R}} \to \mathbb{CP}^1$. To relate the homogeneous formalism to the NP formalism, it is natural to introduce

$$\iota_\alpha = \zeta_\alpha \quad , \quad o_\alpha = \frac{\hat{\zeta}_\alpha}{\langle \zeta \hat{\zeta} \rangle}, \tag{3.8}$$

which satisfies the normalization $\langle \iota o \rangle = 1$ and the homogeneity gives the correct scaling needed in the NP formalism. The metric is given by the asymptotic expansion following the Bondi-Sachs gauge fixing [114]

$$\begin{aligned} g = 2dudr - r^2 \left(D\zeta D\bar{\zeta} - \frac{\bar{\sigma}^0}{r} D\zeta^2 - \frac{\sigma^0}{r} D\bar{\zeta}^2 \right) + \left(\frac{T^2}{\langle \hat{\zeta} \zeta \rangle^2} - 2\frac{m_B}{r} \right) du^2 \\ - du(\eth \sigma^0 D\bar{\zeta} + \bar{\eth} \bar{\sigma}^0 D\zeta) + \mathcal{O}(r^{-1}). \end{aligned} \tag{3.9}$$

The differential operators appearing in the asymptotic expansion are dual vector fields on the Riemann sphere to the forms $D\zeta, D\bar{\zeta}$, which define the Edth operator and its complex conjugate

$$\eth = \frac{\hat{\zeta}_\alpha}{\langle \hat{\zeta} \zeta \rangle} \frac{\partial}{\partial \zeta_\alpha} \quad , \quad \bar{\eth} = \frac{\zeta_\alpha}{\langle \zeta \hat{\zeta} \rangle} \frac{\partial}{\partial \hat{\zeta}_\alpha}, \tag{3.10}$$

which can act on sections of $\mathcal{O}_{\mathscr{I}}(m,n)$. The action of the Edth operators changes the weights

$$\begin{aligned} \eth &: \mathcal{O}_{\mathscr{I}}(m,n) \to \mathcal{O}_{\mathscr{I}}(m-2,n), \\ \bar{\eth} &: \mathcal{O}_{\mathscr{I}}(m,n) \to \mathcal{O}_{\mathscr{I}}(m,n-2). \end{aligned} \tag{3.11}$$

The fields appearing in the leading-order expansion are the asymptotic shear σ^0 and its complex conjugate, along with the Bondi mass aspect m_B.

It is important to note that there is a distinction between the coordinates used in the Bondi-Sachs formalism and those in the homogeneous formalism. The relationship

between the homogeneous and non-homogeneous coordinates is given by

$$u_B = \frac{u}{\langle \hat{\zeta} \zeta \rangle} \quad , \quad r_B = r \langle \zeta \hat{\zeta} \rangle . \tag{3.12}$$

At this point, one may ask why introduce the homogeneous formalism at all. For one, spin and boost weights are found simply by counting the homogeneous weights of functions. Secondly, it is a fully Lorentz covariant formalism as the only objects that transform are the ζ_α coordinates by an $SL(2,\mathbb{C})$ matrix. The coordinate u_B transforms in the expected way

$$u'_B = \frac{1 + z\bar{z}}{(az+b)(\bar{a}\bar{z}+\bar{b}) + (cz+d)(\bar{c}\bar{z}+\bar{d})} \; u_B = \frac{\langle \hat{\zeta} \zeta \rangle}{\langle \hat{\zeta}' \zeta' \rangle} \; u_B . \tag{3.13}$$

The asymptotic symmetry analysis of the Bondi gauge shows that the residual gauge transformations are the BMS transformations, which act on the non-homogeneous coordinates

$$\delta_{(T,Y,\bar{Y})} u_B = \frac{T}{\langle \hat{\zeta} \zeta \rangle} + \frac{u_B}{2} (\eth Y + \bar{\eth} \bar{Y}) \quad \delta_{(T,Y,\bar{Y})} z = \frac{Y}{\zeta_0^2} \, , \tag{3.14}$$

with $T = T(\zeta, \bar{\zeta}) \in \mathcal{O}(1,1)$ the supertranslations and $Y(\zeta) \in \mathcal{O}(2,0)$, $\bar{Y}(\bar{\zeta}) \in \mathcal{O}(0,2)$ the superrotations.

Peeling and recursion relations. See chapter 9 of [115] for full details of the following. In asymptotically flat spacetimes, the large r behaviour of the Weyl tensor in a neighbourhood of $\mathscr{I}$ is dictated by the peeling theorem

$$\psi_n = \frac{\psi_n^0}{r^{5-n}} + \mathcal{O}(r^{n-6}) \, , \qquad n = 0, 1, 2, 3, 4 \, . \tag{3.15}$$

We have the following relation with the asymptotic shear

$$\psi_4^0 = \ddot{\bar{\sigma}}^0 , \qquad \psi_3^0 = \bar{\eth} \dot{\bar{\sigma}}^0 , \qquad \psi_2^0 - \bar{\psi}_2^0 = \eth^2 \sigma^0 - \bar{\eth}^2 \bar{\sigma}^0 + \bar{\sigma}^0 \dot{\sigma}^0 - \sigma^0 \dot{\bar{\sigma}}^0 \, , \tag{3.16}$$

where the Bondi mass aspect is given by $m_B = \psi_2^0 + \bar{\eth}^2\bar{\sigma}^0 + \sigma^0\dot{\bar{\sigma}}^0$ and is fully real. The asymptotic Bianchi identities on the Weyl spinor imply the following time evolution equations

$$\partial_u \psi_n^0 = \bar{\eth}\psi_{n+1}^0 + (3-n)\sigma^0\psi_{n+2}^0, \qquad n = 0,1,2,3\,. \tag{3.17}$$

Using the notations of [61, 62], we denote

$$\mathcal{Q}_s = \psi_{2-s}^0, \qquad s = -1,0,1,2\,, \tag{3.18}$$

where s denotes the spin-weight of these Newman-Penrose scalars under Lorentz transformations on the celestial sphere; this suggests a continuation of the series in s which can be motivated as follows.

One can show that the subleading orders in r of ψ_1, ψ_2, ψ_3, ψ_4 are completely fixed by the Bianchi identities once the leading orders ψ_1^0, ψ_2^0, ψ_3^0, ψ_4^0 are given. By contrast, there is no Bianchi identity fixing subleading orders in ψ_0 in r in terms of ψ_0^0, and so its expansion in r involves an infinite tower of independent data. Following the examples found in [62], we write the expansion of Ψ_0 as

$$\psi_0 = \frac{\psi_0^0}{r^5} + \sum_{s=3}^{\infty} \frac{1}{r^{s+3}} \frac{(-1)^s}{(s-2)!} \left(\eth^{s-2}\mathcal{Q}_s + \Phi_{s-2}\right)\,, \tag{3.19}$$

where this splitting of the sub-leading terms is chosen so that the evolution equations in u can be cast into the form

$$\partial_u \mathcal{Q}_s = \bar{\eth}\mathcal{Q}_{s-1} + (s+1)\sigma^0\mathcal{Q}_{s-2}\,, \tag{3.20}$$

and the infinitesimal extended BMS transformations [116, 117] are

$$\delta_{(T,Y,\bar{Y})}\mathcal{Q}_s = \left(f\partial_u + Y\eth + \bar{Y}\bar{\eth} + \frac{3-s}{2}\eth Y + \frac{3+s}{2}\bar{\eth}\bar{Y}\right)\mathcal{Q}_s + (s+2)\mathcal{Q}_{s-1}\bar{\eth}f\,, \tag{3.21}$$

where $f = T + \frac{u}{2}(\eth Y + \bar{\eth}\bar{Y})$. The terms Φ_{s-2} in (3.19) can be shown to be expressible as

non-local expressions of the $\mathcal{Q}_s$'s appearing in lower order [62]. It would be interesting to develop a proof of this splitting in full generality, including on asymptotically AdS backgrounds. Neglecting these terms would yield a truncation of the Bianchi identities for $s \geq 4$ in the full theory of gravity [61]. As we will see in Chapter 5, these components will have a different interpretation if the theory in the bulk is SD.

We see that the recursion relations (3.20) are valid for any spin $s = -1, 0, 1, 2, 3 \ldots$ and naturally extend (3.17), as do the transformation laws (3.21) under BMS. As we shall explain, both the recursion relations of the $\mathcal{Q}_s$ and the transformation laws will arise naturally from twistor space via a natural extension of a twistor integral formula.

3.2.2 Twistors at $\mathscr{I}$

The Penrose transform (2.101) allows us to relate fields on twistor space to fields on space-time. However, those formulae are adapted to representing the fields in the bulk space-time, whereas here in this section, we will use formulae adapted to null infinity originally introduced by Curtis [108, 118] and later discussed by Bramson and Tod in [109, 110] for g and by Sparling [119, 91] for h. In doing so, we introduce the notion of asymptotic twistor space and homogeneous coordinates at $\mathscr{I}$. Asymptotic twistor space encodes the gravitational data at $\mathscr{I}$ into a deformed complex structure on twistor space and makes contact between Newman's $\mathcal{H}$-space construction and Penrose's nonlinear graviton construction [120].

To understand the geometry of twistor space at $\mathscr{I}$, we first consider the flat space incidence relations. By inverting the incidence relations and taking $r \to \infty$, we obtain the following

$$\begin{aligned} \lambda_\alpha &= \frac{2x_{\alpha\dot{\alpha}}}{x^{\beta\dot{\beta}}x_{\beta\dot{\beta}}}\mu^{\dot{\alpha}} \\ &\overset{\mathrm{r}\to\infty}{=} \frac{[\mu\bar{\zeta}]}{u_B\langle\hat{\zeta}\zeta\rangle}\zeta_\alpha \equiv n\zeta_\alpha\,, \qquad n := \frac{[\mu\bar{\zeta}]}{u_B\langle\hat{\zeta}\zeta\rangle}\,. \end{aligned} \tag{3.22}$$

These are the incidence relations for null infinity, and they state that the λ_α coor-

dinate on twistor space becomes proportional to the celestial sphere coordinate ζ_α on spacetime. Importantly, by taking $u_B \to \pm\infty$, we obtain the expected result of $\lambda_\alpha = 0$ for spatial or future timelike infinity. Geometrically, these incidence relations can be described by a blow-up of twistor space along a special complex submanifold. In our case, the manifold we are considering is $\mathbb{CP}^3$, and we wish to blow up the $\mathbb{CP}^1$ corresponding to $\lambda_\alpha = 0$. The blow-up is given by

$$\mathrm{Bl}_{\mathbb{CP}^1}(\mathbb{CP}^3) = \{([\zeta_\alpha], [Z^A]) \in \mathbb{CP}^1 \times \mathbb{CP}^3 | \langle \zeta \lambda \rangle = 0\}. \tag{3.23}$$

The exceptional divisor is given by $E = \mathbb{CP}^1 \times \mathbb{CP}^1$ with homogeneous coordinate $(\mu^{\dot\alpha}, \zeta_\alpha)$. The condition $\langle \zeta \lambda \rangle = 0$ tells us that on the blow-up $\lambda_\alpha = n\zeta_\alpha$, which is exactly the incidence relations when $r \to \infty$. Therefore, we call the blow-up space the asymptotic twistor space $\overline{\mathbb{PT}}$ for null infinity $\mathscr{I}$. The homogeneous formalism is well-suited for its relationship to twistor space, which we describe now. There is a natural projection from the twistor space with coordinates $Z^A = (\mu^{\dot\alpha}, \lambda_\alpha)$ to null infinity

$$p : \overline{\mathbb{PT}} \longrightarrow \mathscr{I}_{\mathbb{C}} \qquad (\mu^{\dot\alpha}, \lambda_\alpha, \zeta_\alpha) \longmapsto \left(u_B = \frac{[\mu\bar\lambda]}{\langle \hat\lambda \lambda \rangle}, \zeta_\alpha, \bar\zeta_{\dot\alpha} \right), \tag{3.24}$$

where $\mathcal{I}_{\mathbb{C}} \cong \mathbb{C} \times \mathbb{CP}^1$ is a partial complexification where $u = [\mu\bar\lambda]$ becomes complex, but the ζ_α is identified with the celestial sphere coordinate. Under the projection map (3.24), we decompose the twistor coordinate $\mu^{\dot\alpha}$ into the image and kernel of p

$$\mu^{\dot\alpha} = \frac{u\hat{\bar\lambda}^{\dot\alpha}}{\langle \hat\lambda \lambda \rangle} + q\bar\lambda^{\dot\alpha} \quad , \quad q = \frac{[\mu\hat{\bar\lambda}]}{\langle \lambda \hat\lambda \rangle}, \tag{3.25}$$

where $q \in \mathcal{O}(1,-1)$ is the fiber coordinate over the projection to $\mathscr{I}$. Note that the fibers are $\mathbb{CP}^1$ since $q = \infty$ is included due to us working on $\overline{\mathbb{PT}}$ where this corresponds to $n = 0$. We make an important distinction between the Edth operator on twistor

space and the one on $\mathscr{I}$

$$\begin{aligned}\partial_0 &= \frac{\hat{\lambda}_\alpha}{\langle\hat{\lambda}\lambda\rangle}\frac{\partial}{\partial\lambda_\alpha}\bigg|_\mu = \eth - \bar{q}\partial_{\bar{u}} + \frac{u}{\langle\hat{\lambda}\lambda\rangle^2}\partial_q\,,\\ \bar{\partial}_0 &= \frac{\lambda_\alpha}{\langle\lambda\hat{\lambda}\rangle}\frac{\partial}{\partial\hat{\lambda}_\alpha}\bigg|_\mu = \bar{\eth} - q\partial_u + \frac{\bar{u}}{\langle\hat{\lambda}\lambda\rangle^2}\partial_{\bar{q}}\,.\end{aligned} \tag{3.26}$$

In particular, one can check that $\bar{\eth}u = 0$, but $\bar{\partial}_0 u \neq 0$. For notational convenience, we will continue to use the coordinate λ_α instead of the celestial sphere coordinate ζ_α, and functions on $\mathscr{I}$ will be functions of $u, \lambda_\alpha, \bar{\lambda}_{\dot{\alpha}}$ and valued in the line bundle on twistor space. To translate to the celestial sphere coordinate and the associated line bundle, all one must do is multiply by a sufficient power of n and $\bar{n}$ to obtain zero weights in the twistor space line bundle. In other words given a function $f(u, \lambda, \bar{\lambda})$ valued in $\mathcal{O}(n, m)$, we can turn it into a section of $\mathcal{O}_{\mathscr{I}}(n, m)$ simply by $f(u_B, \zeta, \bar{\zeta}) = n^{-n}\bar{n}^{-m}f(u, \lambda, \bar{\lambda})$. This distinction will only be important when discussing boundary conditions at ι^0 as the powers of n determine the falloffs.

To encode asymptotic data of null infinity, we consider the deformation of the complex structure to be given by

$$V^{\dot{\alpha}} = \sigma^0\bar{\lambda}^{\dot{\alpha}}D\bar{\lambda}, \tag{3.27}$$

where for the moment σ^0 is some function on twistor space and all other components of V vanish. Inserting this complex structure into the integrability condition (2.126) we get

$$\frac{\partial\sigma^0}{\partial\bar{\mu}^\alpha} = 0 \quad , \quad \bar{\lambda}_{\dot{\alpha}}\frac{\partial\sigma^0}{\partial\bar{\lambda}_{\dot{\alpha}}} = -3\sigma^0. \tag{3.28}$$

Furthermore, the condition for this V to be Hamiltonian $\mathcal{L}_{\dot{\alpha}}V^{\dot{\alpha}} = 0$ gives

$$\bar{\lambda}^{\dot{\alpha}}\frac{\partial\sigma^0}{\partial\mu^{\dot{\alpha}}} = 0, \tag{3.29}$$

so we see that that $\sigma^0 = \sigma^0([\mu\bar{\lambda}], \lambda_\alpha, \bar{\lambda}_{\dot{\alpha}})$ is a section of $\mathcal{O}(1,-3)$, which are the weights of the asymptotic shear. With this complex structure, the holomorphic sections of the fibration satisfy

$$\bar{\partial}_0 F^{\dot{\alpha}} = \bar{\lambda}^{\dot{\alpha}}\sigma^0 \quad \Longrightarrow \quad \frac{\partial F^{\dot{\alpha}}}{\partial \bar{\lambda}_{\dot{\beta}}} = \sigma^0 \bar{\lambda}^{\dot{\alpha}} \bar{\lambda}^{\dot{\beta}}. \tag{3.30}$$

If we define the projection of the holomorphic line as $Z(x, \lambda_\alpha, \bar{\lambda}_{\dot{\alpha}}) = [F\bar{\lambda}]$ where x labels the moduli space of solutions, we find the following equation

$$\frac{\partial^2 Z}{\partial \bar{\lambda}^{\dot{\alpha}} \partial \bar{\lambda}^{\dot{\beta}}} = \sigma^0(Z, \lambda_\alpha, \bar{\lambda}_{\dot{\alpha}}) \bar{\lambda}_{\dot{\alpha}} \bar{\lambda}_{\dot{\beta}} \quad \Longrightarrow \quad \bar{\eth}^2 Z = \sigma^0(Z, \lambda_\alpha, \bar{\lambda}_{\dot{\alpha}}). \tag{3.31}$$

These are called the Good Cut equations defined by Newman [121, 122, 9] as the cuts of $\mathscr{I}$ solving these PDEs have vanishing shear. The moduli space, the space of all Good Cuts, was called $\mathcal{H}$-space and by construction is a self-dual spacetime that is tautologically holographic as it is fully determined by data living on the boundary. These spacetimes that arise in this way are called radiative, as the shear captures the radiative aspects of the spacetime, and this is the only non-vanishing piece of asymptotic data [113].

In this set-up, the Hamiltonian generating deformations of the complex structure on $\overline{\mathbb{PT}}$ introduced in (2.125) can be constructed from the asymptotic shear $\sigma^0(u, \lambda, \bar{\lambda})$ as

$$h(\mu, \lambda, \bar{\lambda}) = \int^u \bar{\sigma}(u', \lambda, \bar{\lambda}) du' D\bar{\lambda}\,. \tag{3.32}$$

We will also use the notation $h = h_0 D\bar{\lambda}$.

The BMS transformation on the $\mathscr{I}$ coordinates can be uplifted to twistor space by the following infinitesimal variations. The supertranslations are given by

$$\delta_{\mathrm{ST}} \mu^{\dot{\alpha}} = \frac{\partial T}{\partial \bar{\lambda}_{\dot{\alpha}}} \quad , \quad \delta_{\mathrm{ST}} \lambda^\alpha = 0\,, \tag{3.33}$$

where T is valued in $\mathcal{O}(1,1)$. For superrotations, we take

$$\delta_{\rm SR}\mu^{\dot\alpha} = \frac{1}{2}\frac{\partial^2 \bar{Y}}{\partial\bar\lambda_{\dot\alpha}\partial\bar\lambda^{\dot\beta}}\mu^{\dot\beta} \quad , \quad \delta_{\rm SR}\lambda^{\alpha} = \frac{1}{2}\frac{\partial^2 Y}{\partial\lambda_{\alpha}\partial\lambda^{\beta}}\lambda^{\beta}\,, \tag{3.34}$$

where $Y, \bar{Y}$ are valued in $\mathcal{O}(2,0)$ and $\mathcal{O}(0,2)$ respectively. All the variations can also be extended to $\bar\lambda_{\dot\alpha}$ and $\bar\mu^{\alpha}$ by complex conjugation. Furthermore, the BMS generators $\{T, Y, \bar{Y}\}$ are only functions of λ_α and $\bar\lambda_{\dot\alpha}$.

BCT integral formula and its extension. The original BCT integral formula [118, 110, 108] relates twistor space cohomology classes in $H^1(\mathbb{PT}, \mathcal{O}(-6))$ in the Čech representation to solutions to the asymptotic Bianchi identities (3.17). Here, we reformulate this integral in the Dolbeault framework to connect more directly to the ingredient g in the twistor action (3.1). In particular, it extends directly to give the $\mathcal{Q}_s$, providing their origin in Noether's theory. The two pictures are completely equivalent on shell, see for example [123, 124].

The integral formula is given by

$$\mathcal{Q}_s = \frac{1}{2\pi i}\int q^{s+2}\mathrm{d}q\wedge g|_{X_{(u,\lambda,\bar\lambda)}} = \frac{1}{2\pi i}\int q^{s+2} g_1 \mathrm{d}q\mathrm{d}\bar{q}\,, \tag{3.35}$$

where $X_{(u,\lambda,\bar\lambda)}$ is the fiber over $\mathscr{I}$ for the projection (3.24), and where $g|_{X_{(u,\lambda,\bar\lambda)}} = g_1\mathrm{d}\bar{q}$, with $g_1 \in \mathcal{O}(-5,-1)$, denotes the component of the form g along the fiber. Here, the integration takes place on the fiber with coordinates $(q,\bar{q})$ as introduced in (3.25). From now on, we will also assume the following holomorphicity condition with respect to u:

$$\hat\lambda^{\alpha}\frac{\partial}{\partial\bar\mu^{\alpha}}\lrcorner\, g = 0\,. \tag{3.36}$$

As discussed at the beginning of this section, this ensures that, on shell, we can analytically continue our twistor data from $\mathscr{I}$ to $\mathscr{I}_{\mathbb{C}}$. In the next section, we will re-interpret this condition as a gauge-fixing condition on twistor space.

Using the BCT formula (3.35), we now show that the closure condition $\bar{\nabla} g = 0$, which corresponds to the equation of motion for g given in (3.3), is equivalent to the recursion relations

$$\partial_u \mathcal{Q}_s = \bar{\eth} \mathcal{Q}_{s-1} + (s+1)\sigma^0 \mathcal{Q}_{s-2}, \qquad s = -1, 0, 1, 2 \dots \tag{3.37}$$

discussed in the previous paragraph. This justifies the identification of the left-hand side of (3.35) with the spin s momenta discussed in the previous paragraph. Using the BCT integral formula, the recursion relations impose a constraint equation on g_1

$$\begin{aligned} \partial_u \mathcal{Q}_s - \bar{\eth} \mathcal{Q}_{s-1} - (s+1)\sigma^0 \mathcal{Q}_{s-2} &= \frac{1}{2\pi i} \int \left[q^{s+2} \partial_u g_1 - q^{s+1} \bar{\eth} g_1 - (s+1)\sigma^0 q^s g_1 \right] \mathrm{d}q \mathrm{d}\bar{q} \\ &= \frac{1}{2\pi i} \int q^{s+1} [q \partial_u g_1 - \bar{\eth} g_1 + \sigma^0 \partial_q g_1] \mathrm{d}q \mathrm{d}\bar{q} \,, \end{aligned} \tag{3.38}$$

where we have integrated by parts and dropped the boundary terms as the fibers are taken to have no boundary. The LHS is zero if the integrand is a total derivative; hence the condition on g_1 is

$$q \partial_u g_1 - \bar{\eth} g_1 + \sigma^0 \partial_q g_1 - \frac{1}{\langle \hat{\lambda} \lambda \rangle^2} \partial_{\bar{q}} g_0 = 0 \,, \tag{3.39}$$

where $g_0 \in \mathcal{O}(-4)$ is an arbitrary function which does no depend on $\bar{u}$, and the $\langle \hat{\lambda} \lambda \rangle$ factor was inserted for future convenience. This constraint equation arises naturally from the on-shell condition $\bar{\nabla} g = 0$ for g in an appropriate basis of $(0,1)$-forms on twistor space. This is adapted to the projection over $\mathscr{I}$ by first introducing the variables

$$\bar{\mathfrak{q}} = \frac{\langle \bar{\mu} \hat{\lambda} \rangle}{\langle \lambda \hat{\lambda} \rangle^2} = \frac{\bar{q}}{\langle \lambda \hat{\lambda} \rangle} \quad , \quad \bar{u}_B = \frac{\bar{u}}{\langle \hat{\lambda} \lambda \rangle}. \tag{3.40}$$

These have no anti-holomorphic homogeneity weight (and holomorphic weights -2

and 0 respectively) and so we have

$$\begin{aligned}\bar{\partial}\bar{\mathfrak{q}} &= \frac{\langle \mathrm{d}\bar{\mu}\hat{\lambda}\rangle}{\langle\lambda\hat{\lambda}\rangle^2} - \bar{\mathfrak{q}}\frac{\langle \mathrm{d}\hat{\lambda}\lambda\rangle}{\langle\hat{\lambda}\lambda\rangle} + \bar{u}_B\bar{e}^0\,,\\ \bar{\partial}\bar{u}_B &= \frac{\langle \mathrm{d}\bar{\mu}\lambda\rangle}{\langle\hat{\lambda}\lambda\rangle} - \bar{u}_B\frac{\langle \mathrm{d}\hat{\lambda}\lambda\rangle}{\langle\hat{\lambda}\lambda\rangle}\,,\\ \bar{e}^0 &= \frac{\mathrm{D}\bar{\lambda}}{\langle\lambda\hat{\lambda}\rangle^2},\end{aligned} \tag{3.41}$$

where these have vanishing contraction $\bar{\Upsilon}\lrcorner\cdot = 0$; they are all necessarily $\bar{\partial}$-closed. Using these coordinates and forms, we can construct the exterior derivative and forms on the projective space. The exterior derivative acting on forms valued in $\mathcal{O}(m,n)$ is given by

$$d_{\mathbb{CP}^1} = d - m\theta\wedge - n\bar{\theta}\wedge\,. \tag{3.42}$$

On twistor space, the basis of 1-forms and the connection for the line bundle are given by

$$\Omega^1(\mathbb{PT}) = \mathrm{Span}\left\{\mathrm{D}\lambda, \partial\mathfrak{q}, \partial u_B, \bar{e}^0, \bar{\partial}\bar{\mathfrak{q}}, \bar{\partial}\bar{u}_B\right\} \quad , \quad \theta = \frac{\langle\hat{\lambda}\mathrm{d}\lambda\rangle}{\langle\hat{\lambda}\lambda\rangle} \quad , \quad \bar{\theta} = \frac{\langle\mathrm{d}\hat{\lambda}\lambda\rangle}{\langle\hat{\lambda}\lambda\rangle}\,. \tag{3.43}$$

Furthermore, the holomorphic top form on twistor space can be decomposed into the coordinates of the fibration over $\mathscr{I}$

$$\begin{aligned}\mathrm{D}^3Z &= \frac{1}{4!}\epsilon_{ABCD}Z^A\mathrm{d}Z^B\wedge\mathrm{d}Z^C\wedge\mathrm{d}Z^D\\ &= \frac{1}{2}\mathrm{D}\lambda\wedge\mathrm{d}\mu^{\dot{\alpha}}\wedge\mathrm{d}\mu_{\dot{\alpha}} + \frac{1}{2}[\mu\mathrm{d}\mu]\wedge\mathrm{d}\lambda^{\alpha}\wedge\mathrm{d}\lambda_{\alpha}\\ &= \mathrm{d}_{\mathbb{CP}^1}u\wedge\mathrm{d}_{\mathbb{CP}^1}q\wedge\mathrm{D}\lambda + q\mathrm{d}_{\mathbb{CP}^1}q\wedge\mathrm{D}\bar{\lambda}\wedge\mathrm{D}\lambda\\ &= \langle\lambda\hat{\lambda}\rangle\mathrm{d}u_B\wedge\mathrm{d}_{\mathbb{CP}^1}q\wedge\mathrm{D}\lambda + q\mathrm{d}_{\mathbb{CP}^1}q\wedge\mathrm{D}\bar{\lambda}\wedge\mathrm{D}\lambda\,.\end{aligned} \tag{3.44}$$

Note that since the decomposition of $\mu^{\dot{\alpha}}$ does not preserve the complex structure, the presence of a $\mathrm{D}\bar{\lambda}$ form is to be expected. We also note that $\mathrm{d}_{\mathbb{CP}^1}\bar{q} = \langle\lambda\hat{\lambda}\rangle\bar{\partial}\bar{\mathfrak{q}}$. For notational convenience, we drop the $\mathbb{CP}^1$ subscript. We expand our generic (0,1)-form

g in this basis

$$g = g_0 \bar{e}^0 + g_{\bar{\mathfrak{q}}} \bar{\partial}\bar{\mathfrak{q}} + g_{\bar{u}_B} \bar{\partial}\bar{u}_B \,. \tag{3.45}$$

Now we impose the axial gauge condition (3.36)

$$\hat{\lambda}^\alpha \frac{\partial}{\partial \bar{\mu}^\alpha} \lrcorner\, g = 0 \quad \Longrightarrow \quad g_{\bar{u}_B} = 0 \,, \tag{3.46}$$

which will result in holomorphicity in u of the other components when g is on-shell. In this gauge, we find

$$\bar{\partial} g = \bar{\partial} g_0 \wedge \bar{e}^0 + \bar{\partial} g_{\bar{\mathfrak{q}}} \wedge \bar{\partial}\bar{\mathfrak{q}} \,. \tag{3.47}$$

For a general section F of $\mathcal{O}(n)$, $\bar{\partial} F$ can be expanded in the basis (3.41) as

$$\begin{aligned} \bar{\partial} F &= \frac{\partial F}{\partial \bar{\lambda}_{\dot{\alpha}}} \mathrm{d}\bar{\lambda}_{\dot{\alpha}} + \frac{\partial F}{\partial \bar{\mu}^\alpha} \mathrm{d}\bar{\mu}^\alpha \\ &= -\bar{\partial}_0 F \, \mathrm{D}\bar{\lambda} + \bar{\lambda}_{\dot{\alpha}} \frac{\partial F}{\partial \bar{\lambda}_{\dot{\alpha}}} \frac{\langle \mathrm{d}\hat{\lambda} \lambda \rangle}{\langle \hat{\lambda} \lambda \rangle} + \partial_{\bar{u}_B} F \frac{\langle \mathrm{d}\bar{\mu} \lambda \rangle}{\langle \hat{\lambda} \lambda \rangle} + \partial_{\bar{\mathfrak{q}}} F \frac{\langle \mathrm{d}\bar{\mu} \hat{\lambda} \rangle}{\langle \lambda \hat{\lambda} \rangle^2} \\ &= \left(\frac{\bar{u}}{\langle \hat{\lambda} \lambda \rangle^2} \partial_{\bar{q}} F - \bar{\partial}_0 F \right) \mathrm{D}\bar{\lambda} + \bar{\Upsilon} F \frac{\langle \mathrm{d}\hat{\lambda} \lambda \rangle}{\langle \hat{\lambda} \lambda \rangle} + \partial_{\bar{\mathfrak{q}}} F \bar{\partial}\bar{\mathfrak{q}} + \partial_{\bar{u}_B} F \bar{\partial}\bar{u}_B \\ &= \left(q \partial_u F - \bar{\eth} F \right) \mathrm{D}\bar{\lambda} + \partial_{\bar{\mathfrak{q}}} F \bar{\partial}\bar{\mathfrak{q}} + \partial_{\bar{u}_B} F \bar{\partial}\bar{u}_B \,, \end{aligned} \tag{3.48}$$

where we have used $\bar{\Upsilon} F = 0$ and the push-forward of the Edth operator from twistor space to $\mathscr{I}$ in (3.26). The non-linear term in the complex structure is given by

$$\begin{aligned} \{h, g\} &= \epsilon^{\dot{\beta}\dot{\alpha}} \mathcal{L}_{\partial_{\dot{\alpha}}} h \wedge \mathcal{L}_{\partial_{\dot{\beta}}} g \\ &= \sigma^0 \bar{\lambda}^{\dot{\alpha}} \frac{\partial g_{\bar{\mathfrak{q}}}}{\partial \mu^{\dot{\alpha}}} \mathrm{D}\bar{\lambda} \wedge \bar{\partial}\bar{\mathfrak{q}} \\ &= \sigma^0 \partial_q g_{\bar{\mathfrak{q}}} \mathrm{D}\bar{\lambda} \wedge \bar{\partial}\bar{\mathfrak{q}} \,. \end{aligned} \tag{3.49}$$

Therefore, the $\bar{\nabla}$-closure condition in the basis (3.41) is

$$\bar{\nabla} g = \left[q \partial_u g_{\bar{\mathfrak{q}}} - \bar{\eth} g_{\bar{\mathfrak{q}}} + \sigma^0 \partial_q g_{\bar{\mathfrak{q}}} + \frac{1}{\langle \hat{\lambda} \lambda \rangle} \partial_{\bar{q}} g_0 \right] \mathrm{D}\bar{\lambda} \wedge \bar{\partial}\bar{\mathfrak{q}} + \partial_{\bar{u}_B} g_{\bar{\mathfrak{q}}} \, \bar{\partial}\bar{u}_B \wedge \bar{\partial}\bar{\mathfrak{q}} + \partial_{\bar{u}_B} g_0 \, \bar{\partial}\bar{u}_B \wedge \bar{e}^0 . \tag{3.50}$$

The on-shell condition for g then gives the conditions that the components satisfy

$$\partial_{\bar{u}} g_1 = 0 = \partial_{\bar{u}} g_0 \,, \tag{3.51}$$

$$q\partial_u g_1 - \bar{\eth} g_1 + \sigma^0 \partial_q g_1 - \frac{1}{\langle \hat{\lambda} \lambda \rangle^2} \partial_{\bar{q}} g_0 = 0 \,, \tag{3.52}$$

where we have made the redefinition $g_{\bar{\mathfrak{q}}} = g_1 \langle \lambda \hat{\lambda} \rangle$ to obtain the corrected weighted function in the integral formula. The first conditions give holomorphicity in the u-coordinate and are a direct consequence of the axial gauge choice (3.36). The second gives the condition that if g is $\bar{\nabla}$-closed, the BCT integral formula gives solutions to (3.37) as desired.

The integral formula (3.35) can be inverted to give an explicit form of g which is only $\bar{\nabla}$-closed if the recursion relations (3.37) are satisfied. The BCT representative is given by

$$g = -\sum_{n=0}^{\infty} \partial_u \mathcal{Q}_{n-2} \frac{\bar{q}^n}{(1+q\bar{q})^{n+1}} \mathrm{D}\bar{\lambda} + \sum_{n=0}^{\infty} \mathcal{Q}_{n-2} \frac{(n+1)\bar{q}^n}{(1+q\bar{q})^{n+2}} \mathrm{d}\bar{q} \,. \tag{3.53}$$

An interesting feature of this representative is that when it is restricted to a line $X_{(u,\lambda,\bar{\lambda})}$, one can show

$$g|_{X_{(u,\lambda,\bar{\lambda})}} = \bar{\partial}|_{X_{(u,\lambda,\bar{\lambda})}} \left(\sum_{n=0}^{\infty} \frac{\mathcal{Q}_{n-2} \bar{q}^{n+1}}{(1+q\bar{q})^{n+1}} \right) \,, \tag{3.54}$$

seemingly making it exact on the fibers, and by Stokes' theorem, expect the BCT integral to give a vanishing contribution as the sphere has no boundary. However, the functions of q and $\bar{q}$ in the parentheses are not functions on the sphere fibers, as they do not satisfy the following property

$$\int_{\mathbb{CP}^1} d\left(f(q,\bar{q}) dq\right) = 0 \;\text{ if }\; f(q,\bar{q}) \to \frac{1}{q\bar{q}} \text{ as } q \to \infty \,, \tag{3.55}$$

where in the case at hand $f(q,\bar{q}) = q^n \bar{q}^n / (1+q\bar{q})^n$. Therefore, we will obtain non-

vanishing contour integrals at $q = \infty$ and hence pick out the $\mathcal{Q}_{n-2}$ in the expansion.

BMS Transformations Here we demonstrate that we can recover the BMS transformation of the components of the Weyl spinor using only the integral transform, the weights naturally present in the construction, and the BMS transformations uplifted to asymptotic twistor space (3.33) and (3.34). All variations are taken on-shell and therefore $\partial_{\bar{u}} g_1 = 0$. We begin by using the BCT integral formula and inserting the infinitesimal coordinate transformations. Starting with supertranslations

$$\begin{aligned}
\delta_{\mathrm{ST}} \mathcal{Q}_s &= \int q^{s+2} \left(\frac{\partial g_1}{\partial \mu^{\dot{\alpha}}} \delta_{\mathrm{ST}} \mu^{\dot{\alpha}} + \frac{\partial g_1}{\partial \bar{\mu}^{\alpha}} \delta_{\mathrm{ST}} \bar{\mu}^{\alpha} \right) \mathrm{d}q \mathrm{d}\bar{q} \\
&= \int q^{s+2} \left(\frac{\partial g_1}{\partial \mu^{\dot{\alpha}}} \frac{\partial T}{\partial \bar{\lambda}_{\dot{\alpha}}} + \partial_{\bar{q}} g_1 \bar{\eth} \bar{T} \right) \mathrm{d}q \mathrm{d}\bar{q} \\
&= \int q^{s+2} \left(T \partial_u g_1 - \partial_q g_1 \bar{\eth} T \right) \mathrm{d}q \mathrm{d}\bar{q} \\
&= T \partial_u \mathcal{Q}_s + (4-s) \mathcal{Q}_{s-1} \bar{\eth} T \,,
\end{aligned} \tag{3.56}$$

where integration by parts and the homogeneity of T have been used. Now moving on to superrotations

$$\delta_{\mathrm{SR}} \mathcal{Q}_s = \int q^{s+2} \left(\frac{\partial g_1}{\partial \mu^{\dot{\alpha}}} \delta_{\mathrm{SR}} \mu^{\dot{\alpha}} + \frac{\partial g_1}{\partial \bar{\lambda}^{\dot{\alpha}}} \delta_{\mathrm{SR}} \bar{\lambda}^{\dot{\alpha}} + \frac{\partial g_1}{\partial \bar{\mu}^{\alpha}} \delta_{\mathrm{SR}} \bar{\mu}^{\alpha} + \frac{\partial g_1}{\partial \lambda^{\alpha}} \delta_{\mathrm{SR}} \lambda^{\alpha} \right) \mathrm{d}q \mathrm{d}\bar{q} \,. \tag{3.57}$$

Taking this term by term, starting with the dotted spin bundle terms

$$\begin{aligned}
\int q^{s+2} \frac{\partial g_1}{\partial \mu^{\dot{\alpha}}} \delta_{\mathrm{SR}} \mu^{\dot{\alpha}} \mathrm{d}q \mathrm{d}\bar{q} &= \frac{1}{2} \int q^{s+2} \epsilon^{\dot{\alpha}\dot{\beta}} \frac{\partial g_1}{\partial \mu^{\dot{\beta}}} \left(u \bar{\eth} \left(\frac{\partial \bar{Y}}{\partial \bar{\lambda}^{\dot{\alpha}}} \right) + q \frac{\partial \bar{Y}}{\partial \bar{\lambda}^{\dot{\alpha}}} \right) \mathrm{d}q \mathrm{d}\bar{q} \\
&= \frac{1}{2} \int q^{s+2} \left[\partial_u g_1 \left(u \bar{\eth} \bar{Y} + 2q \bar{Y} \right) - \partial_q g_1 \left(u \bar{\eth}^2 \bar{Y} + q \bar{\eth} \bar{Y} \right) \right] \mathrm{d}q \mathrm{d}\bar{q} \\
&= \frac{u}{2} \bar{\eth} \bar{Y} \partial_u \mathcal{Q}_s + \bar{Y} \partial_u \mathcal{Q}_{s+1} + \frac{s+3}{2} \bar{\eth} \bar{Y} \mathcal{Q}_s + \frac{u}{2} (s+2) \mathcal{Q}_{s-1} \bar{\eth}^2 \bar{Y} \,,
\end{aligned} \tag{3.58}$$

where we only use the homogeneity of $\bar{Y}$ and integration by parts. The next term gives

$$\begin{aligned}\int q^{s+2}\frac{\partial g_1}{\partial\bar{\lambda}^{\dot{\alpha}}}\delta_{\rm SR}\bar{\lambda}^{\dot{\alpha}}\mathrm{d}q\mathrm{d}\bar{q} &= \frac{1}{2}\int q^{s+2}\epsilon^{\dot{\alpha}\dot{\beta}}\frac{\partial g_1}{\partial\bar{\lambda}^{\dot{\beta}}}\frac{\partial\bar{Y}}{\partial\bar{\lambda}^{\dot{\alpha}}}\mathrm{d}q\mathrm{d}\bar{q}\\ &= \frac{1}{2}\int q^{s+2}\left[2\bar{Y}\bar{\partial}_0 g_1 - \bar{\lambda}^{\dot{\alpha}}\frac{\partial g_1}{\partial\bar{\lambda}^{\dot{\alpha}}}\bar{\eth}\bar{Y}\right]\mathrm{d}q\mathrm{d}\bar{q}\\ &= \frac{1}{2}\int q^{s+2}\left[2\bar{Y}\bar{\partial}_0 g_1 + (g_1 + \bar{u}\partial_{\bar{u}}g_1 + \bar{q}\partial_{\bar{q}}g_1)\bar{\eth}\bar{Y}\right]\mathrm{d}q\mathrm{d}\bar{q} \qquad (3.59)\\ &= \bar{Y}\int q^{s+2}\left[\bar{\eth} - q\partial_u + \frac{\bar{u}}{\langle\hat{\lambda}\lambda\rangle^2}\partial_{\bar{q}}\right]g_1\mathrm{d}q\mathrm{d}\bar{q}\\ &= \bar{Y}\bar{\eth}\mathcal{Q}_s - \bar{Y}\partial_u\mathcal{Q}_{s+1}\,,\end{aligned}$$

where we have used the fact that g_1 is a section of $\mathcal{O}(-5,-1)$ and on-shell $\partial_{\bar{u}}g_1 = 0$. Now looking at the undotted transformations

$$\int q^{s+2}\frac{\partial g_1}{\partial\bar{\mu}^{\alpha}}\delta_{\rm SR}\bar{\mu}^{\alpha}\mathrm{d}q\mathrm{d}\bar{q} = -\frac{1}{2}\int q^{s+2}\partial_{\bar{q}}g_1\left(\bar{u}\bar{\eth}^2 Y + \bar{q}\bar{\eth}Y\right)\mathrm{d}q\mathrm{d}\bar{q} = \frac{1}{2}\eth Y\,\mathcal{Q}_s\,, \qquad (3.60)$$

where we have used the homogeneity of Y and integration by parts. The final term gives us

$$\begin{aligned}\int q^{s+2}\frac{\partial g_1}{\partial\lambda^{\alpha}}\delta_{\rm SR}\lambda^{\alpha}\mathrm{d}q\mathrm{d}\bar{q} &= \frac{1}{2}\int q^{s+2}\left[2Y\partial_0 g_1 - \lambda^{\alpha}\frac{\partial g_1}{\partial\lambda^{\alpha}}\eth Y\right]\mathrm{d}q\mathrm{d}\bar{q}\\ &= \frac{1}{2}\int q^{s+2}\left[\eth Y(u\partial_u g_1 + (2-s)f) + 2Y\left(\eth g_1 + \frac{u}{\langle\hat{\lambda}\lambda\rangle^2}\partial_q f\right)\right]\mathrm{d}q\mathrm{d}\bar{q}\\ &= \frac{u}{2}\eth Y\partial_u\mathcal{Q}_s + \frac{2-s}{2}\eth Y\,\mathcal{Q}_s + Y\eth\mathcal{Q}_s - (s+2)\frac{u}{\langle\hat{\lambda}\lambda\rangle^2}Y\mathcal{Q}_{s-1}\\ &= \frac{u}{2}\eth Y\partial_u\mathcal{Q}_s + \frac{2-s}{2}\eth Y\,\mathcal{Q}_s + Y\eth\mathcal{Q}_s + \frac{u}{2}(s+2)\mathcal{Q}_{s-1}[\bar{\eth},\eth]Y\,,\end{aligned} \qquad (3.61)$$

where we have used the homogeneity of g_1 and $\partial_{\bar{u}} g_1 = 0$. A useful commutator between the $\eth$ and $\bar{\eth}$ operators was used

$$[\eth, \bar{\eth}] = \frac{1}{\langle \hat{\lambda} \lambda \rangle^2} \left[\lambda_\alpha \frac{\partial}{\partial \lambda_\alpha} - \bar{\lambda}_{\dot{\alpha}} \frac{\partial}{\partial \bar{\lambda}_{\dot{\alpha}}} \right] . \tag{3.62}$$

Putting all of this together, along with the assumption that $\bar{\eth} Y = 0$, the full infinitesimal BMS transformations give

$$\delta_{(T,Y,\bar{Y})} \mathcal{Q}_s = \left(f \partial_u + Y \eth + \bar{Y} \bar{\eth} + \frac{3-s}{2} \eth Y + \frac{3+s}{2} \bar{\eth} \bar{Y} \right) \mathcal{Q}_s + (s+2) \mathcal{Q}_{s-1} \bar{\eth} f \, , \tag{3.63}$$

where $f = T + \frac{u}{2} \left(\eth Y + \bar{\eth} \bar{Y} \right)$.

3.3 $Lw_{1+\infty}$ symmetry algebra

In [47], the $Lw_{1+\infty}$ symmetry algebra was geometrically realized as the algebra of Hamiltonians for local holomorphic Poisson diffeomorphisms on the region of twistor space that excludes $\lambda_0 = 0$ and $\lambda_1 = 0$. These are the local symmetries of twistor space in the Čech description of Penrose's original nonlinear graviton construction [8]. In that discussion, the Hamiltonians for the diffeomorphisms also play a role as Čech representatives of $H^1(\mathbb{PT}, \mathcal{O}(2))$ which also define infinitesimal deformations and hence generic self-dual gravitons via the Penrose transform.

Here we take a different approach, deriving the symmetries from those of the self-dual twistor action. Being a Dolbeault description, these are smooth rather than holomorphic functions, essentially being functions of the six real coordinates of twistor space, rather than the three complex variables as in the previous paragraph. So we perform various gauge fixings to bring the gauge freedom down to functions of three rather than six variables. In this way, we will arrive at $Lw_{1+\infty}$.

Thus, in this section, we perform a residual gauge symmetry analysis on twistor space (see e.g. [125] for a review). We start from the solution space associated with

(3.3) and impose a natural gauge fixing. We then derive the subset of gauge transformations (3.2) that preserve this gauge fixing and show that they form the $Lw_{1+\infty}$ algebra after using an appropriate bracket. We also compute the transformation of the solution space under the residual gauge transformations and deduce the action of $Lw_{1+\infty}$ symmetries on spacetime using the BCT integral formula (3.35). As in the Čech description of [47], we will see that these gauge transformations correspond to an element $\delta h \in H^{0,1}_{\bar{\nabla}}(\mathbb{PT}, \mathcal{O}(2))$ defining a self-dual graviton, now in a Dolbeault representation with the gauge transformation being obtained via an integral formula.

Residual gauge symmetries. In the first instance, we will choose the following gauge fixing conditions on the forms (g, h) on asymptotic twistor space:

$$\hat{\lambda}^\alpha \frac{\partial}{\partial \bar{\mu}^\alpha} \lrcorner\, g = 0\,, \tag{3.64}$$

$$\frac{\partial}{\partial \bar{\mu}^\alpha} \lrcorner\, h = 0\,. \tag{3.65}$$

These choices are analogous to those in [84] adapted to a fibration of twistor space over Euclidean signature space-time. Here we use the same general idea, but for the fibration of twistor space over $\mathscr{I}_{\mathbb{C}}$. As discussed above after (3.36), the first condition gives the on-shell components of g independence of $\bar{u}$ that follows from the analytic extension of the fields to complex u in $\mathscr{I}_{\mathbb{C}}$. For homogeneity ≥ -1 the cohomology up the $\mathbb{CP}^1$ fibers vanishes and so a gauge can be found so that h is trivial up the fibers in the $\partial_{\bar{q}}$ direction. However, $H^1(\mathbb{CP}^1, \mathcal{O}(n)) \neq 0$ for $n < -1$ so we cannot ask this of g. The condition on h restricts us to the solution space in which we can hope to interpret the $\mathrm{D}\bar{\lambda}$ component as coming from the shear as in (3.32) although we will see that it is not quite sufficient. The gauge conditions restrict our gauge

transformations in the following way

$$\hat{\lambda}^\alpha \frac{\partial}{\partial \bar{\mu}^\alpha} \lrcorner\, \delta_{(\xi,\phi)} g = 0 \quad \Longrightarrow \quad \partial_{\bar{u}} \phi = 0\,, \tag{3.66}$$

$$\frac{\partial}{\partial \bar{\mu}^\alpha} \lrcorner\, \delta_{(\xi,\phi)} h = 0 \quad \Longrightarrow \quad \frac{\partial \xi}{\partial \bar{\mu}^\alpha} = 0\,, \tag{3.67}$$

where we used (3.2). From this we see that the ξ are independent of $\bar{\mu}^\alpha$ while being arbitrary in $\mu^{\dot{\alpha}}, \lambda_\alpha, \bar{\lambda}_{\dot{\alpha}}$ so, taking homogeneities into account, we are now reduced to gauge freedom depending on four variables rather than the three we are aiming for.

In the following, we will therefore also impose the further gauge condition

$$\mathcal{L}_{\partial_q} h = 0\,. \tag{3.68}$$

Together with (3.65), this ensures the compatibility with (3.32) where the shear is seen as a function defined on $\mathscr{I}_{\mathbb{C}}$ uplifted to twistor space, and therefore does not depend on q; the residual gauge freedoms at this point will be the supertranslations and this gauge choice is tantamount to performing the Penrose transform to a potential modulo gauge for the self-dual field. Using (3.2), this additional gauge condition (3.68) implies

$$\partial_q (q\partial_u - \bar{\eth} + \sigma^0 \partial_q)\xi = 0\,. \tag{3.69}$$

We can solve this formally by expanding ξ as a finite polynomial in q

$$\xi_s = \sum_{n=0}^{s+1} \xi_{s,n} q^n\,, \tag{3.70}$$

where $\xi_{s,n} = \xi_{s,n}(u, \lambda, \bar{\lambda})$ and the gauge condition (3.69) gives the following recursion relations

$$\partial_u \xi_{s,n-1} - \bar{\eth}\xi_{s,n} + (n+1)\sigma^0 \xi_{s,n+1} = 0\,, \qquad \xi_{s,s+1} := \tau_s(\lambda, \bar{\lambda})\,. \tag{3.71}$$

These recursion relations can be solved by an integral formula similar to (3.35)

$$\xi_{s,n} = \frac{1}{2\pi i}\int q^{-n-1}\mathrm{d}q \wedge \delta h_s|_{X_{(u,\lambda,\bar{\lambda})}}\,, \tag{3.72}$$

where $\delta h_s \in H^{0,1}_{\bar{\nabla}}(\mathbb{PT}, \mathcal{O}(2))$ is an infinitesimal deformation around the background h. In this way, the local Hamiltonian vector field generated by ξ_s under the Poisson bracket (2.123) corresponds to the infinitesimal deformation δh_s around the curved background h. The point here is that, like the holomorphic Poisson diffeomorphisms of [47], ξ_s is generically singular on twistor space whereas δh_s, which is gauge equivalent to $\bar{\nabla}\xi_s$, will need to be global and smooth to represent a Dolbeault class. However, the integral formula is somewhat implicit, and it will be instructive to solve the recursion relations (3.71) more directly.

To proceed, we consider a polynomial of order $s+1$ in $\mu^{\dot{\alpha}}$

$$\tilde{\xi}_s = \frac{1}{(s+1)!}\frac{\partial^{s+1}\tau_s}{\partial\bar{\lambda}^{\dot{\alpha}_1}\cdots\partial\bar{\lambda}^{\dot{\alpha}_{s+1}}}\mu^{\dot{\alpha}_1}\cdots\mu^{\dot{\alpha}_{s+1}} = \sum_{n=-1}^{s}\frac{u^{s-n}q^{n+1}}{(s-n)!}\bar{\eth}^{s-n}\tau_s\,, \tag{3.73}$$

where $\tau_s = \tau_s(\lambda_\alpha, \bar{\lambda}_{\dot{\alpha}})$ and is valued in $\mathcal{O}(1-s, 1+s)$ and $s = \{-1, 0, 1, \dots\}$. When $\bar{\eth}^{s+2}\tau_s = 0$ these coincide with the twistor space $Lw_{1+\infty}$ generators given in [47] and this ansatz will be consistent with the wedge condition, but we will work more generally here following [61]. This ansatz preserves the two first gauge conditions (3.64) and (3.65), i.e. it satisfies (3.67). Furthermore, $\tilde{\xi}_s$ solves (3.69) in the absence of shear. Acting with this parameter on the curved background given by h, gives

$$\delta_{\tilde{\xi}_s} h = \bar{\nabla}\tilde{\xi}_s = \left[\sigma^0\sum_{n=0}^{s}\frac{n+1}{(s-n)!}u^{s-n}q^n\bar{\eth}^{s-n}\tau_s - \frac{u^{s+1}}{(s+1)!}\bar{\eth}^{s+2}\tau_s\right]\mathrm{D}\bar{\lambda}\,, \tag{3.74}$$

where we used $\mathcal{L}_{\partial_u}h = \sigma^0 D\bar{\lambda}$ (3.32) to make the shear appear. As we can see this gives a polynomial in q which is not compatible with the gauge condition (3.68). Another

way of saying this is that $\tilde{\xi}_s$ does not solve (3.69). The first few examples are

$$\begin{aligned}
\bar{\nabla}\tilde{\xi}_{-1} &= -\bar{\partial}\tau_{-1}\mathrm{D}\bar{\lambda}\,,\\
\bar{\nabla}\tilde{\xi}_{0} &= \left[\tau_0\sigma^0 - u\bar{\partial}^2\tau_0\right]\mathrm{D}\bar{\lambda}\,,\\
\bar{\nabla}\tilde{\xi}_{1} &= \left[(u\bar{\partial}\tau_1 + 2q\tau_1)\sigma^0 - \frac{u^2}{2}\bar{\partial}^3\tau_1\right]\mathrm{D}\bar{\lambda}\,,\\
\bar{\nabla}\tilde{\xi}_{2} &= \left[\left(\frac{u^2}{2}\bar{\partial}^2\tau_2 + 2uq\bar{\partial}\tau_2 + 3q^2\tau_2\right)\sigma^0 - \frac{u^3}{6}\bar{\partial}^4\tau_2\right]\mathrm{D}\bar{\lambda}\,,\\
\bar{\nabla}\tilde{\xi}_{3} &= \left[\left(\frac{u^3}{6}\bar{\partial}^3\tau_3 + u^2q\bar{\partial}^2\tau_3 + 3uq^2\bar{\partial}\tau_3 + 4q^3\tau_3\right)\bar{\sigma} - \frac{u^4}{24}\bar{\partial}^5\tau_3\right]\mathrm{D}\bar{\lambda}\,,
\end{aligned} \tag{3.75}$$

and the issue arises for $s \geq 1$. To remedy this, we use the linear dependence of q in $\bar{\nabla}$ to cancel the highest order in q terms by adding an $\mathcal{O}(2)$ function f_s to $\tilde{\xi}_s$ of a lower order in q:

$$\xi_s = \tilde{\xi}_s + f_s\,. \tag{3.76}$$

We must now do this order by order. For $s = -1, 0, 1, 2, 3$, this gives the field-depended parameters

$$\begin{aligned}
f_{-1} &= 0\,,\\
f_0 &= 0\,,\\
f_1 &= -2\tau_1\partial_u^{-1}\sigma^0\,,\\
f_2 &= -3q\tau_2\partial_u^{-1}\sigma^0 - 3\bar{\partial}(\tau_2\partial_u^{-2}\sigma^0) - 2\bar{\partial}\tau_2\partial_u^{-1}(u\sigma^0)\,,\\
f_3 &= -4q^2\tau_3\partial_u^{-1}\sigma^0 - 3q\bar{\partial}\tau_3\partial_u^{-1}(u\sigma^0) - 4q\bar{\partial}(\tau_3\partial_u^{-2}\sigma^0) - \bar{\partial}^2\tau_3\partial_u^{-1}(u^2\sigma^0)\\
&\qquad + 8\tau_3\partial_u^{-1}(\sigma^0\partial_u^{-1}\sigma^0) - 3\bar{\partial}(\bar{\partial}\tau_3\partial_u^{-2}(u\sigma^0)) - 4\bar{\partial}^2(\tau_3\partial_u^{-3}\sigma^0)\,.
\end{aligned} \tag{3.77}$$

For the symmetries generated by ϕ, we use the facts discussed in (3.55) and the gauge fixing condition (3.64). The symmetry parameter ϕ that gives a non-vanishing

transformation law on $\mathcal{Q}_s$ and preserves the representative (3.53) is given by

$$\phi_s = \sum_{n=0}^{s+1} \phi_{s,-n-3} q^n + \sum_{n=0}^{\infty} \phi_{n-2} \frac{\bar{q}^{n+1}}{(1+q\bar{q})^{n+1}} \,, \tag{3.78}$$

such that

$$\partial_u \phi_{s,n} = \bar{\eth} \phi_{s,n-1} + (n+1)\sigma^0 \phi_{s,n-2} \qquad \phi_{s,-s-4} := \rho_s(\lambda, \bar{\lambda}) \,, \tag{3.79}$$

where $\rho_s(\lambda, \bar{\lambda})$ is valued in $\mathcal{O}(-6-s-1, s+1)$ and the index $s = \{-1, 0, 1, \dots\}$. A useful way to pick out the coefficients ϕ_s is by using a contour integral centred at $q = \infty$. In particular

$$\phi_{s,n} = \frac{1}{2\pi i} \oint_{C_\infty} dq \, q^{n+2} \phi_s \,. \tag{3.80}$$

Importantly, the contour integral satisfies the recursion relations

$$\partial_u \phi_{s,n} - \bar{\eth} \phi_{s,n-1} - (n+1)\sigma^0 \phi_{s,n-2} = \frac{-1}{2\pi i} \oint_{C_\infty} dq \, q^{n+1} \sum_{m=0}^{\infty} \partial_u \phi_{s,n-2} \frac{\bar{q}^m}{(1+q\bar{q})^{m+1}} = 0 \,, \tag{3.81}$$

where the last equality is checked explicitly using the parametrisation of $q = Re^{i\theta}$ and taking $R \to \infty$.

We note that the expansion in q is intimately related to the Bondi radial expansion in r, as we will see in Chapter 5, the coefficients in ξ and ϕ will satisfy two towers of recursion relations, one of which will evolve them radially into the bulk.

Asymptotic symmetry algebra. We now compute the asymptotic symmetry algebra formed by the residual gauge transformations on twistor space. We will mainly focus on the $\xi \in \Omega^{0,0}(\mathbb{PT}, \mathcal{O}(2))$, as these generators satisfy the $Lw_{1+\infty}$ while the ϕ will generate a module over this algebra. The requirement (3.68) is technically demanding as it yields field dependence in the residual gauge transformations, see (3.77). However, this feature often arises in the gauge-fixing approach to asymptotic

symmetries [126, 127, 128, 129, 130], and tools have been developed over the years to deal with these subtleties. In particular, the residual gauge transformations form an algebra, provided one uses a modified Lie bracket [126] which takes into account the field dependence of the parameters. In our case, we define the bracket

$$\{\xi_s, \xi_{s'}\}_* = \{\xi_s, \xi_{s'}\} + \delta_{\xi_s}\xi_{s'} - \delta_{\xi_{s'}}\xi_s \,, \tag{3.82}$$

where the first term on the right-hand side is the Poisson bracket on twistor space (2.123). The two additional terms take into account the field dependence in ξ and ensure that the bracket closes. Denoting $\xi_s \equiv \xi(\tau_s)$ the expression (3.76) with (3.73)-(3.77), one can show that

$$\{\xi(\tau_s), \xi(\tau_{s'})\}_* = \xi((s'+1)\tau_{s'}\bar{\eth}\tau_s - (s+1)\tau_s\bar{\eth}\tau_{s'}) \,. \tag{3.83}$$

The gauge parameter on the right-hand side is again of the form (3.76) with (3.73)-(3.77). To prove this claim, we first check if the modified bracket gives a function on twistor space that satisfies (3.68), as this is the defining equation. First consider

$$\begin{aligned} \bar{\nabla}\{\xi_s, \xi_{s'}\}_* &= \{\bar{\nabla}\xi_s, \xi_{s'}\} + \bar{\nabla}\delta_{\xi_s}\xi_{s'} - (s \leftrightarrow s') \\ &= \delta_{\xi_s}\bar{\nabla}\xi_{s'} - \delta_{\xi_{s'}}\bar{\nabla}\xi_s \\ &= [\delta_{\xi_s}, \delta_{\xi_{s'}}]h \,, \end{aligned} \tag{3.84}$$

from which we conclude that $\mathcal{L}_{\partial_q}\bar{\nabla}\{\xi_s, \xi_{s'}\}_* = 0$ since the Lie derivative is unaffected by the variations on the fields. To determine the function $\tau_{s''}(\lambda, \bar{\lambda})$ which seeds the recursion of the new generator, we must look at the field-independent part of the modified bracket, which corresponds to the highest power of q. The highest order term is given by

$$\{\xi_s, \xi_{s'}\}_* = \left((s'+1)\tau_{s'}\bar{\eth}\tau_s - (s+1)\tau_s\bar{\eth}\tau_{s'}\right) q^{s+s'} + \cdots \,, \tag{3.85}$$

which concludes the proof of (3.83), and we conclude that the symmetry algebra of the residual gauge diffeomorphisms ξ is given by the $Lw_{1+\infty}$ algebra. Notice that, in this approach, there is no wedge condition on the parameters, which is in agreement with the phase space discussions on spacetime in [61, 62].[1]

The second bracket we can construct is between the ϕ and ξ generators. By counting weights, the new generator must be valued in $\mathcal{O}(-6)$ and, naturally, one expects it to be another ϕ generator. As these are field-dependent generators, we define the modified bracket

$$\{\xi_s, \phi_{s'}\}_* = \{\xi_s, \phi_{s'}\} + \delta_{\xi_s}\phi_{s'}\,, \tag{3.86}$$

where only the variation due to ξ_s appears as the only field dependence appear through the σ^0. If we denote $\phi_s = \phi(\rho_s)$, the modified bracket gives a new generator

$$\{\xi(\tau_s), \phi(\rho_{s'})\}_* = \phi((s'+1)\rho_{s'}\bar{\eth}\tau_s - (s+1)\tau_s\bar{\eth}\rho_{s'})\,, \tag{3.87}$$

where the components of the generator on the right-hand side will satisfy the recursion relations (3.79). To prove this claim, consider using the contour integral in (3.80) on the modified bracket

$$\phi_{s'',n} = \frac{1}{2\pi i}\oint_{C_\infty} dq q^{n+2}\{\xi(\tau_s), \phi(\rho_{s'})\}_*\,. \tag{3.88}$$

[1] We note that this algebra only becomes a Lie algebra if we restrict to the wedge, as the Jacobi identity would not be satisfied. This is because $Lw_{1+\infty}$ has no Lie subalgebra keeping only $s \geq -1$. On the spacetime phase-space, one can also realize the generalized BMS algebra. We thank Nicolas Cresto and Laurent Freidel for pointing out this subtlety.

We now must check if the recursion relations are satisfied

$$\begin{aligned}\partial_u \phi_{s'',n} - \bar{\eth}\phi_{s'',n-1} - (n+1)\sigma^0 \phi_{s'',n-2} &= \frac{1}{2\pi i}\oint_{C_\infty} dq q^{n+1}\left[\bar{\nabla}\{\xi_s, \phi_{s'}\}_*\right]_{D\bar{\lambda}} \\ &= \frac{1}{2\pi i}\oint_{C_\infty} dq q^{n+1}\left[\{\xi_s, \bar{\nabla}\phi_{s'}\}_* + \delta_{\xi_s}\bar{\nabla}\phi_{s'}\right]_{D\bar{\lambda}} \\ &= 0\,,\end{aligned} \tag{3.89}$$

where $[\cdots]_{D\bar{\lambda}}$ corresponds to taking the $D\bar{\lambda}$ part of the form. The last equality holds because, under the parametrisation $q = Re^{i\theta}$, the only non-vanishing θ integrals will have integrands of the form $q^{n-1}\bar{q}^n/(1+q\bar{q})^{n+1}$ and when $R \to \infty$ the integrals will vanish at an order R^{-2}. Therefore, the new modes $\phi_{s'',n}$ will also satisfy the recursion relations. To find the field independent function $\rho_{s''}(\lambda, \bar{\lambda})$ that seeds the recursion relations, notice the lowest power of q that gives a non-zero mode is

$$\begin{aligned}\phi_{s'',-s-s'-3} &= \frac{1}{2\pi i}\oint_{C_\infty} \frac{dq}{q^{s+s'+1}}\{\xi_s, \phi_{s'}\}_* \\ &= (s'+1)\rho_{s'}\bar{\eth}\tau_s - (s+1)\tau_s\bar{\eth}\rho_{s'}\,,\end{aligned} \tag{3.90}$$

where a higher power of q in the denominator would give zero. One can also ask if there is a bracket between two ϕ generators, but by weight counting, there is no generator of the resulting weight. Furthermore, the charge algebra will only involve the two brackets we have discussed so far.

Action of $Lw_{1+\infty}$. Let us now compute the action of the above residual gauge transformations on the solution space, and deduce the action of $Lw_{1+\infty}$ on spacetime. Here we focus on the residual gauge transformations associated with $\xi \in \Omega^{0,0}(\mathbb{PT}, \mathcal{O}(2))$ symmetries. The starting point for the transformations of the fields is (3.2), together with the explicit form of the gauge parameters (3.76) discussed in the previous section.

Denoting $\delta_{\xi_s} \equiv \delta_s$, the transformation of h reads as

$$
\begin{aligned}
\delta_{-1} h &= -\bar{\eth}\tau_{-1}\mathrm{D}\bar{\lambda}\,,\\
\delta_0 h &= (\tau_0\sigma^0 - u\bar{\eth}^2\tau_0)\mathrm{D}\bar{\lambda}\,,\\
\delta_1 h &= \left(2\bar{\eth}(\tau_1\partial_u^{-1}\sigma^0) + u\sigma^0\bar{\eth}\tau_1 - \frac{u^2}{2}\bar{\eth}^3\tau_1\right)\mathrm{D}\bar{\lambda}\,,\\
\delta_2 h &= \left(\frac{u^2}{2}\sigma^0\bar{\eth}^2\tau_2 + 2\bar{\eth}(\bar{\eth}\tau_2\partial_u^{-1}(u\sigma^0)) + 3\bar{\eth}^2(\tau_2\partial_u^{-2}\sigma^0) - 3\tau_2\sigma^0\partial_u^{-1}\sigma^0 - \frac{u^3}{6}\bar{\eth}^4\tau_2\right)\mathrm{D}\bar{\lambda}\,,\\
\delta_3 h &= \Bigg(\frac{u^3}{6}\sigma^0\bar{\eth}^3\tau_3 + \bar{\eth}(\bar{\eth}^2\tau_3\partial_u^{-1}(u^2\sigma^0)) + 3\bar{\eth}^2(\bar{\eth}\tau_3\partial_u^{-2}(u\sigma^0)) - 3\sigma^0\bar{\eth}\tau_3\partial_u^{-1}(u\sigma^0)\\
&\qquad +4\bar{\eth}^3(\tau_3\partial_u^{-3}\sigma^0) - 8\bar{\eth}(\tau_3\partial_u^{-1}(\sigma^0\partial_u^{-1}\sigma^0)) - 4\sigma^0\bar{\eth}(\tau_3\partial_u^{-2}\sigma^0) - \frac{u^4}{24}\bar{\eth}^5\tau_3\Bigg)\mathrm{D}\bar{\lambda}\,,
\end{aligned}
\tag{3.91}
$$

and so on. Notice, in particular, that the spin $s = -1$ symmetry is acting non-trivially on h. Using (3.32), we can easily deduce the action on the asymptotic shear as $\delta_s\sigma^0 = \partial_u\delta_s h_0$. We display them explicitly for completeness:

$$
\begin{aligned}
\delta_{-1}\bar{\sigma} &= 0\,,\\
\delta_0\bar{\sigma} &= \tau_0\partial_u\sigma^0 - \bar{\eth}^2\tau_0\,,\\
\delta_1\bar{\sigma} &= \left(u\bar{\eth}\tau_1\partial_u + 2\tau_1\bar{\eth} + 3\bar{\eth}\tau_1\right)\sigma^0 - u\bar{\eth}^3\tau_1\,,\\
\delta_2\bar{\sigma} &= \frac{u^2}{2}(\bar{\eth}^2\tau_2\partial_u\sigma^0 - \bar{\eth}^4\tau_2) + u(3\bar{\eth}^2\tau_2\sigma^0 + 2\bar{\eth}\tau_2\bar{\eth}\sigma^0)\\
&\qquad\qquad + 3(\bar{\eth}^2 - \partial_u\sigma^0)(\tau_2\partial_u^{-1}\sigma^0) - 3\tau_2(\sigma^0)^2\,,\\
\delta_3\bar{\sigma} &= \frac{u^3}{6}(\bar{\eth}^3\tau_3\partial_u\sigma^0 - \bar{\eth}^5\tau_3) + \frac{u^2}{2}(3\bar{\eth}^3\tau_3\sigma^0 + 2\bar{\eth}^2\tau_3\bar{\eth}\sigma^0)\\
&\quad + 3u\left((\bar{\eth}^2 - \partial_u\sigma^0)(\bar{\eth}\tau_3\partial_u^{-1}\sigma^0) - \tau_3(\sigma^0)^2\right) + 4\bar{\eth}^3(\tau_3\partial_u^{-2}\sigma^0) - 3\bar{\eth}^2(\bar{\eth}\tau_3\partial_u^{-2}\sigma^0)\\
&\quad - \bar{\eth}\tau_3\partial_u\sigma^0\partial_u^{-2}\sigma^0 - 4\tau_3\partial_u\sigma^0\partial_u^{-2}\bar{\eth}\sigma^0 - 12\sigma^0\bar{\eth}(\tau_3\partial_u^{-1}\sigma^0) - 8\tau_3\bar{\eth}\sigma\partial_u^{-1}\sigma^0\,,
\end{aligned}
\tag{3.92}
$$

and so on. These transformations reproduce correctly those of [61, 62]. In particular, the terms linear in the shear agree with the transformations deduced from twistor space in [131]. The spin $s = -1$ symmetry does not act on the shear, which explains

why it was invisible in the phase space approach on spacetime [61, 62]. The $s = 0, 1$ transformations reproduce the expected transformations under supertranslations T and superrotations $\bar{Y}$ by identifying $T = \tau_0$ and $\bar{Y} = 2\tau_1$ [116, 117].

The transformations of the spin s momenta $\mathcal{Q}_s$ introduced above can be computed similarly, using the BCT integral formula (3.35). Starting from

$$\delta_{s'} \mathcal{Q}_s = \frac{1}{2\pi i} \int q^{s+2} \mathrm{d}q \wedge \{g, \xi_{s'}\}|_{X_{(u,\lambda,\bar{\lambda})}} , \tag{3.93}$$

and using again (3.2), we find explicitly, on-shell,

$$\begin{aligned}
\delta_{-1} \mathcal{Q}_s &= 0 \,, \\
\delta_0 \mathcal{Q}_s &= \tau_0 \partial_u \mathcal{Q}_s + (s+2) \mathcal{Q}_{s-1} \bar{\eth} \tau_0 \,, \\
\delta_1 \mathcal{Q}_s &= \left(u \bar{\eth} \tau_1 \partial_u + 2\tau_1 \bar{\eth} + (s+3) \bar{\eth} \tau_1 \right) \mathcal{Q}_s + u(s+2) \mathcal{Q}_{s-1} \bar{\eth}^2 \tau_1 \,, \\
\delta_2 \mathcal{Q}_s &= \frac{u^2}{2} \left(\bar{\eth}^2 \tau_2 \partial_u \mathcal{Q}_s + (s+2) \mathcal{Q}_{s-1} \bar{\eth}^3 \tau_2 \right) + u \left(2 \bar{\eth} \tau_s \bar{\eth} \mathcal{Q}_s + (s+3) \mathcal{Q}_s \bar{\eth}^2 \tau_2 \right) \\
&\quad + 3\tau_2 \bar{\eth} \mathcal{Q}_{s+1} + (s+4) \bar{\eth} \tau_2 \mathcal{Q}_{s+1} - 3 \left(\partial_u \mathcal{Q}_s + (s+2) \mathcal{Q}_{s-1} \bar{\eth} \right) \left(\tau_2 \partial_u^{-1} \sigma^0 \right) \,, \\
\delta_3 \mathcal{Q}_s &= \frac{u^3}{6} \left(\bar{\eth}^3 \tau_3 \partial_u \mathcal{Q}_s + (s+2) \mathcal{Q}_{s-1} \bar{\eth}^4 \tau_3 \right) + \frac{u^2}{2} \left(2 \bar{\eth}^2 \tau_3 \bar{\eth} \mathcal{Q}_s + (s+3) \bar{\eth}^3 \tau_3 \mathcal{Q}_s \right) \\
&\quad + u \left(3 \bar{\eth} \tau_3 \bar{\eth} \mathcal{Q}_{s+1} + (s+4) \bar{\eth}^2 \tau_3 \mathcal{Q}_{s+1} - 3 \left(\partial_u \mathcal{Q}_s + (s+2) \mathcal{Q}_{s-1} \bar{\eth} \right) \left(\bar{\eth} \tau_3 \partial_u^{-1} \sigma^0 \right) \right) \\
&\quad + 4\tau_3 \bar{\eth} \mathcal{Q}_{s+2} + (s+5) \bar{\eth} \tau_3 \mathcal{Q}_{s+2} - 4 \left(2 \bar{\eth} \mathcal{Q}_s + (s+3) \mathcal{Q}_s \bar{\eth} \right) \left(\tau_3 \partial_u^{-1} \sigma^0 \right) \\
&\quad - \left(\partial_u \mathcal{Q}_s + (s+2) \mathcal{Q}_{s-1} \bar{\eth} \right) \left(\bar{\eth} \tau_3 \partial_u^{-2} \sigma^0 + 4\tau_3 \partial_u^{-2} \bar{\eth} \sigma^0 \right) \,.
\end{aligned} \tag{3.94}$$

We note for both the transformation of σ^0 and $\mathcal{Q}_s$ there is a recursive structure. In particular, the expressions with a non-zero power of u are built from a lower spin transformation of the same object. The recursive nature is made manifest by expressing a general spin s variation in terms of the $\xi_{s,n}$ that satisfy recursion relations

$$\begin{aligned}
\delta_s \sigma^0 &= \partial_u \left(\sigma^0 \xi_{s,1} - \bar{\eth} \xi_{s,0} \right) \,, \\
\delta_{s'} \mathcal{Q}_s &= \sum_{n=0}^{s'+1} \left(n \xi_{s',n} \partial_u \mathcal{Q}_{s+n-1} + (s+n+2) \mathcal{Q}_{s+n-1} \partial_u \xi_{s',n} \right) .
\end{aligned} \tag{3.95}$$

Again, the symmetry $s = -1$ does not act on these objects, and the cases $s = 0, 1$ reproduce the part of the BMS symmetries associated with T and $\bar{Y}$. To our knowledge, this is the first time the transformations (3.94) are deduced systematically for all s. This illustrates the power of the formula (3.35).

The variations generated by the ϕ_s can be derived using the integral formula and the action on the field g on twistor space

$$\delta_{\phi_{s'}} \mathcal{Q}_s = \frac{1}{2\pi i} \int q^{s+2} dq \wedge \bar{\nabla} \phi_{s'}|_{X_{(u,\lambda,\bar{\lambda})}} = \phi_{s',s} \tag{3.96}$$

For simplicity, the first examples of these transformations as they act on the $\mathcal{Q}_{-2}$ field are given by

$$\begin{aligned}
\delta_{\phi_{-1}} \mathcal{Q}_{-2} &= u \bar{\eth} \rho_{-1} \,, \\
\delta_{\phi_0} \mathcal{Q}_{-2} &= \frac{u^2}{2} \bar{\eth}^2 \rho_0 - \rho_0 \partial_u^{-1} \sigma^0 \,, \\
\delta_{\phi_1} \mathcal{Q}_{-2} &= \frac{u^3}{6} \bar{\eth}^3 \rho_1 - 2\bar{\eth}(\rho_1 \partial_u^{-2} \sigma^0) - \bar{\eth} \rho_1 \partial_u^{-1} (u \sigma^0) \,, \\
\delta_{\phi_2} \mathcal{Q}_{-2} &= \frac{u^4}{24} \bar{\eth}^4 \rho_2 - 3\bar{\eth}^2(\rho_2 \partial_u^{-3} \sigma^0) - 2\bar{\eth}(\bar{\eth} \rho_2 \partial_u^{-2} (u\sigma^0)) \\
&\qquad - \frac{1}{2} \bar{\eth}^2 \rho_2 \partial_u^{-1} (u^2 \sigma^0) + 3 \rho_2 \partial_u^{-1} (\sigma^0 \partial_u^{-1} \sigma^0) \,.
\end{aligned} \tag{3.97}$$

To obtain the variations on the higher $\mathcal{Q}_s$, one must push each variation through the recursion relations (3.79).

Finally, let us mention that the action of the residual gauge transformations on the solution space forms a representation of the $Lw_{1+\infty}$ symmetries:

$$[\delta_\xi, \delta_{\xi'}] = \delta_{\{\xi, \xi'\}_*} \,. \tag{3.98}$$

This can be checked explicitly on all the fields using the following relations on twistor space:

$$[\delta_\xi, \delta_{\xi'}] g = \{g, \{\xi, \xi'\}_*\} \,, \qquad [\delta_\xi, \delta_{\xi'}] h = \bar{\nabla} \{\xi, \xi'\}_* \,, \tag{3.99}$$

where the Jacobi identity on the twistor space bracket (2.123) was used. These relations also ensure that the modified bracket (3.82) satisfies the Jacobi identity.

Invariance of the recursion relations under $Lw_{1+\infty}$. We now make the following important observation: the recursion relations (3.37) are invariant under the action of the $Lw_{1+\infty}$. This is a direct consequence of the fact that the equation of motion for g in (3.3) is invariant under gauge transformations (3.2), combined with the fact that the recursion relations (3.37) are deduced from the closure condition of g through the BCT integral formula (3.35). As a consistency check, one can also verify the invariance of (3.37) explicitly using the spacetime transformations found in (3.92) and (3.94). More explicitly, for each s, one can show the recursion relations are invariant under the higher spin transformations by lifting to twistor space. We only focus on the transformations generated by ξ, as the invariance under ϕ follows trivially since σ^0 is unchanged and the $\mathcal{Q}_s$ are shifted to an object that also satisfies the recursion relations. We first rewrite the recursion relations using the BCT integral formula

$$\begin{aligned}\delta_{s'}\left(\partial_u \mathcal{Q}_s - \bar{\eth}\mathcal{Q}_{s-1} - (s+1)\sigma^0\mathcal{Q}_{s-2}\right) &= \frac{1}{2\pi i}\delta_{s'}\int q^{s+1}\left[q\partial_u g_1 - \bar{\eth}g_1 + \sigma^0\partial_q g_1\right]\mathrm{d}q\mathrm{d}\bar{q} \\ &= \frac{1}{2\pi i\langle\lambda\hat{\lambda}\rangle}\int q^{s+1}\delta_{s'}\left[\bar{\nabla}g_{\bar{\mathfrak{q}}}\right]_{\mathrm{D}\bar{\lambda}}\mathrm{d}q\mathrm{d}\bar{q}\,,\end{aligned} \tag{3.100}$$

where $\left[\bar{\nabla}g_{\bar{\mathfrak{q}}}\right]_{\mathrm{D}\bar{\lambda}}$ denotes the $\mathrm{D}\bar{\lambda}$ component of the form. The variation of the form is given by

$$\begin{aligned}\delta_{s'}\bar{\nabla}g_{\bar{\mathfrak{q}}} &= \bar{\partial}\delta_{s'}g_{\bar{\mathfrak{q}}} + \{\delta_{s'}h, g_{\bar{\mathfrak{q}}}\} + \{h, \delta_{s'}g_{\bar{\mathfrak{q}}}\} \\ &= \bar{\partial}\{g_{\bar{\mathfrak{q}}}, \xi_{s'}\} + \{\bar{\nabla}\xi_{s'}, g_{\bar{\mathfrak{q}}}\} + \{h, \{g_{\bar{\mathfrak{q}}}, \xi_{s'}\}\} \\ &= \{\bar{\partial}g_{\bar{\mathfrak{q}}}, \xi_{s'}\} + \{\{h, \xi_{s'}\}, g_{\bar{\mathfrak{q}}}\} + \{\{\xi_{s'}, g_{\bar{\mathfrak{q}}}\}, h\} \\ &= \frac{1}{\langle\lambda\hat{\lambda}\rangle}\{\partial_{\bar{q}}g_0, \xi_{s'}\}\mathrm{D}\bar{\lambda} + \{\partial_{\bar{\mathfrak{q}}}g_{\bar{\mathfrak{q}}}, \xi_{s'}\}\bar{\partial}\bar{\mathfrak{q}}\,,\end{aligned} \tag{3.101}$$

where we have used the equations of motion for $g_{\bar{\mathfrak{q}}}$ and the Jacobi identity for (2.123). Finally, using the restriction on the gauge parameters (3.67) due to the gauge conditions, we find

$$\delta_{s'}\left(\partial_u \mathcal{Q}_s - \bar{\eth}\mathcal{Q}_{s-1} - (s+1)\sigma^0\mathcal{Q}_{s-2}\right) = \frac{1}{2\pi i\langle\hat{\lambda}\lambda\rangle^2}\int \partial_{\bar{q}}\left(q^{s+1}\{g_0,\xi_{s'}\}\right)\mathrm{d}q\mathrm{d}\bar{q} = 0\,, \tag{3.102}$$

a total boundary term that vanishes in the integral. Therefore, the higher spin variations preserve the recursion relations

$$\partial_u\delta_{s'}\mathcal{Q}_s = \bar{\eth}\delta_{s'}\mathcal{Q}_{s-1} + (s+1)\delta_{s'}\sigma^0\mathcal{Q}_{s-2} + (s+1)\sigma^0\delta_{s'}\mathcal{Q}_{s-2}\,, \tag{3.103}$$

as desired. This clarifies in which sense $Lw_{1+\infty}$ are symmetries from a spacetime perspective.

Wedge condition At this stage, as mentioned below (3.83), the wedge condition does not seem to appear in our analysis. However, if one requires the radiative phase space to obey the Schwartzian falloffs as in [61, 62, 132],

$$\lim_{u\to\pm\infty}\sigma^0 = \mathcal{O}(u^{-s-\epsilon})\,, \tag{3.104}$$

the $Lw_{1+\infty}$ algebra reduces to its wedge subalgebra, $Lw^{\wedge}_{1+\infty}$, defined by the commutation relations

$$[\tau_s,\tau_{s'}] = \epsilon^{\dot{\beta}\dot{\alpha}}\frac{\partial\tau_s}{\partial\bar{\lambda}^{\dot{\alpha}}}\frac{\partial\tau_{s'}}{\partial\bar{\lambda}^{\dot{\beta}}} = (s'+1)\tau_{s'}\bar{\eth}\tau_s - (s+1)\tau_s\bar{\eth}\tau_{s'}\,, \tag{3.105}$$

together with the wedge condition on the parameters $\tau_s(\lambda,\bar{\lambda})$

$$\bar{\eth}^{s+2}\tau_s = 0\,. \tag{3.106}$$

Indeed, preserving (3.104) under the action of $Lw_{1+\infty}$ implies $\delta_s\sigma^0|_{\mathscr{I}_\pm} = 0$, which requires setting all the non-homogeneous terms in the transformation written in (3.92) to 0, leading to (3.106). Notice that the field-dependent contributions in (3.76) vanish at $\mathscr{I}^+_\pm$ if we impose the falloffs (3.104). Hence, we have $\xi_s|_{\mathscr{I}^+_\pm} = \tilde{\xi}_s|_{\mathscr{I}^+_\pm}$. Interestingly, using the bracket (2.123), the parameters $\tilde{\xi}_s$ defined in (3.73) with the wedge condition (3.106) form the $Lw^\wedge_{1+\infty}$ algebra

$$\{\tilde{\xi}(\tau_s), \tilde{\xi}(\tau_{s'})\} = \tilde{\xi}((s'+1)\tau_{s'}\bar{\eth}\tau_s - (s+1)\tau_s\bar{\eth}\tau_{s'})|_\wedge\,. \tag{3.107}$$

The explicit computations concerning the algebra can be found in Appendix B of [133]. To connect with the geometric interpretation of $Lw^\wedge_{1+\infty}$ symmetries discussed in [47, 134], we can expand $\tilde{\xi}_s$ in the wedge as

$$\tilde{\xi}_s(\mu^{\dot{\alpha}}, \lambda_\alpha) = \sum_{|m|\leq p-1}\sum_{a\in\mathbb{Z}} \tau^p_{m,a} w^p_{m,a}\,, \qquad w^p_{m,a} := \frac{(\mu^{\dot{0}})^{p+m-1}(\mu^{\dot{1}})^{p-m-1}}{2\lambda_0^{p-a-2}\lambda_1^{p+a-2}}\,, \tag{3.108}$$

where we define $s = 2p-3$ and $\tau^p_{m,a}$ are constants. Using these definitions, the algebra (3.107) gives the well-known structure constants

$$\{w^p_{m,a}, w^q_{n,b}\} = (m(q-1) - n(p-1))w^{p+q-2}_{m+n,a+b}\,. \tag{3.109}$$

As defined, the $\tilde{\xi}$ can be viewed as Lie algebra homomorphisms, and one can show that the τ_s have the following expansion

$$\tau_s(\lambda_\alpha, \bar{\lambda}_{\dot{\alpha}}) = \sum_{|m|\leq p-1}\sum_{a\in\mathbb{Z}} \tau^p_{m,a} \tilde{w}^p_{m,a}\,, \qquad \tilde{w}^p_{m,a} := \frac{(\bar{\lambda}^{\dot{0}})^{p+m-1}(\bar{\lambda}^{\dot{1}})^{p-m-1}}{2\lambda_0^{p-a-2}\lambda_1^{p+a-2}}\,, \tag{3.110}$$

where the $\tilde{w}^p_{m,a}$ have the same structure constants under the bracket (3.105). In the following, we will proceed without imposing the wedge condition restriction (3.106), and keep the full $Lw_{1+\infty}$ symmetries on the phase space.

3.4 Charge algebra

In this section, we construct the symplectic structure on twistor space, from which we deduce the charges associated with the $Lw_{1+\infty}$ symmetries discussed in the previous section. We show that these charges form a representation of these symmetries, provided one uses an appropriate bracket. We then translate these results to spacetime expressions and compare them with previous expressions discussed in the literature.

Charges on twistor space. To find the symplectic structure we use the covariant phase space formalism [135, 136, 137, 138]. The variation of the twistor action (3.1) yields

$$\begin{aligned}\delta S =& \frac{1}{2\pi i}\int_{\mathbb{PT}} \mathrm{D}^3 Z \wedge \left[\left(\bar\partial h + \frac{1}{2}\{h,h\}\right)\wedge \delta g + \bar\nabla g \wedge \delta h\right] \\ &+ \frac{1}{2\pi i}\int_{\mathbb{PT}} \mathrm{D}^3 Z \wedge \frac{\partial}{\partial \mu^{\dot\alpha}}\left(g \wedge \epsilon^{\dot\alpha\dot\beta}\frac{\partial h}{\partial \mu^{\dot\beta}} \wedge \delta h\right) + \frac{1}{2\pi i}\int_{\mathbb{PT}} \mathrm{D}^3 Z \wedge \bar\partial(g\wedge \delta h)\,.\end{aligned} \tag{3.111}$$

The first term on the right-hand side corresponds to the equations of motion (3.3). The second term vanishes due to the gauge fixing condition (3.65). The last term is a total derivative term from which the presymplectic potential can be extracted:

$$\boldsymbol{\theta}[\delta] = \frac{1}{2\pi i}\int_\Sigma \theta[\delta] = \frac{1}{2\pi i}\int_\Sigma \delta h \wedge g \wedge D^3 Z\,. \tag{3.112}$$

Here the integration takes place over the co-dimension 1-surface $\Sigma = \{Z \in \mathbb{PT} | u_B - \bar u_B = \text{constant}\}$.[2] Taking one more variation of Θ, we obtain the presymplectic form

$$\boldsymbol{\omega}[\delta,\delta'] = \frac{1}{2\pi i}\int_\Sigma \omega[\delta,\delta'] = \frac{1}{2\pi i}\int_\Sigma \delta h \wedge \delta g \wedge D^3 Z\,, \tag{3.113}$$

with $\omega[\delta,\delta'] = \delta'\theta[\delta] - \delta\theta[\delta']$, the presymplectic current.

The charges are then obtained by contracting the presymplectic form with the

[2]The space of null twistors is the hypersurface $PN = \{Z \in \mathbb{PT} | u_B - \bar u_B = 0\}$.

residual gauge transformations obtained in the previous section. Performing this contraction, we have [136, 137, 138]

$$\delta_\alpha \lrcorner\, \omega = dk_\alpha, \qquad \not{\delta} H_\xi = \frac{1}{2\pi i}\int_{\partial\Sigma} k_\xi\,, \tag{3.114}$$

where $\alpha = (\xi, \phi)$ and k_ξ is a co-dimension 2 form on twistor space, which can be integrated over the boundary of $\Sigma = \{Z \in \mathbb{PT} | u_B - \bar{u}_B = 0\}$. The resulting charges are non-integrable (i.e. non δ-exact on the solution space) and can be split as

$$\not{\delta} H_\alpha = \delta H_\alpha - \Xi_\alpha[\delta]\,, \tag{3.115}$$

where the integrable and non-integrable pieces for the ξ variation are given by

$$H_\xi = \frac{1}{2\pi i}\int_{\partial\Sigma} D^3Z \wedge \xi g\,, \qquad \Xi_\xi[\delta] = H_{\delta\xi} + \frac{1}{2\pi i}\int_{\partial\Sigma} \underline{\xi} \lrcorner\, \theta[\delta]\,. \tag{3.116}$$

where $\underline{\xi} = \{\xi, \cdot\,\}$ is the Hamiltonian vector field generated by ξ. We note that the part of the boundary integral that is non-vanishing is more precisely given by $H_\xi|^{u_B=\infty}_{u_B=-\infty}$. The non-integrable piece Ξ_ξ vanishes when $u_B \to \pm\infty$ for $s = 0, 1$ if we take the same falloffs in u_B as in [139, 140], and vanishes for all s if we restrict ourselves to the Schwartzian falloffs for the shear (3.104), as considered in [61, 62]. At finite u, the non-integrability of the charges appears for $s \geq 1$. This might have been expected, as it is well known that the charges associated to the BMS symmetries in asymptotically flat spacetimes are already non-integrable [141, 18, 19, 142, 103, 143, 144, 23]. It is, however, instructive to derive the surface charge expressions at a finite value of u_B, which will coincide with the surface charges integrated on a finite cut of $\mathscr{I}$ after translation to spacetime.

For the ϕ variation, the two pieces are given by

$$H_\phi = \frac{1}{2\pi i}\int_{\partial\Sigma} D^3Z \wedge \phi h\,, \qquad \Xi_\phi[\delta] = H_{\delta\phi}\,. \tag{3.117}$$

The boundary of Σ that is being integrated over is a contour integral around $q = \infty$ along the fibers since ϕ was chosen to give non-trivial variations. The integrable/non-integrable split adopted here corresponds to the Noetherian split discussed, e.g. in [106]. We see that the charges associated with ξ and ϕ are non-vanishing, hence confirming that the $Lw_{1+\infty}$ are indeed large gauge transformations on twistor space. In particular, the charge associated with $s = -1$ does not vanish, which means that the corresponding symmetry has a non-trivial action on phase space.

Several tools have been developed to deal with this non-integrability, including a modified bracket [19, 103, 33, 34, 104, 105, 106, 107] defined as

$$\{H_\alpha, H_{\alpha'}\}_\star = \delta_\alpha H_{\alpha'} - \Xi_\alpha[\delta_{\alpha'}]\,. \tag{3.118}$$

Using this bracket, one can easily show that

$$\begin{aligned} \{H_\xi, H_{\xi'}\}_\star &= H_{\{\xi,\xi'\}_*}\,, \\ \{H_\xi, H_\phi\}_\star &= -\{H_\phi, H_\xi\}_\star = H_{\{\xi,\phi\}_*} + K_{\xi,\phi}\,, \\ \{H_\phi, H_{\phi'}\}_\star &= 0\,, \end{aligned} \tag{3.119}$$

where the bracket for vector fields was given in (3.82). The charge bracket between the ϕ and ξ generators picks up a cocyle given by

$$K_{\xi,\phi} = -K_{\phi,\xi} = \frac{1}{2\pi i}\int_{\partial\Sigma} D^3Z \wedge (\phi\bar{\partial}\xi + \{\xi h, \phi\})\,. \tag{3.120}$$

This field-dependent cocycle is anti-symmetric and satisfies the generalised cocycle condition [19]

$$K_{\{\alpha_1,\alpha_2\}_*,\alpha_3} - \delta_{\alpha_3}K_{\alpha_1,\alpha_2} + \text{cyclic}(1,2,3) = 0\,. \tag{3.121}$$

These conditions are required for the charge bracket (3.118) to be a Poisson bracket. Hence, the charges form a representation of the $Lw_{1+\infty}$ symmetries on twistor space.

The consistency of this bracket with Jacobi identities follows directly from the consistency of the bracket (3.82) and the cocycle condition. Notice that restricting (3.119) for the ξ generators to $s = 0, 1$ reproduces the BMS flux algebra, in agreement with [139, 140]. Finally, notice that a different split between integrable and non-integrable pieces in (3.115)-(3.116),

$$H_\alpha \to H'_\alpha = H_\alpha + N_\alpha, \qquad \Xi_\alpha \to \Xi'_\alpha = \Xi_\alpha + \delta N_\alpha \,, \tag{3.122}$$

would not affect $\not{\delta} H_\xi$, but it would produce a (trivial) field-dependent 2-cocycle on the right-hand side of (3.119):

$$\{H'_\alpha, H'_{\alpha'}\}_\star = H'_{\{\alpha,\alpha'\}_*} + K'_{\alpha,\alpha'}, \qquad K'_{\alpha,\alpha'} = K_{\alpha,\alpha'} + \delta_\alpha N_{\alpha'} - \delta_{\alpha'} N_\alpha - N_{\{\alpha,\alpha'\}_*} \,. \tag{3.123}$$

This justifies a posteriori why the natural split chosen in (3.115)-(3.116) is the most appropriate.

Charges on spacetime. We now translate the symplectic structure and charges obtained from first principles on twistor space to spacetime expressions using the dictionary discussed in section 3.2. Starting from the symplectic structure (3.113) and using the BCT integral formula (3.35), we find

$$\Omega = \frac{1}{2\pi i} \int_\Sigma \delta h_0 \delta g \wedge \mathrm{d}q \wedge \mathrm{d}u \wedge \mathrm{D}\lambda \wedge \mathrm{D}\bar{\lambda} = - \int_{\mathscr{I}} \delta h_0 \delta \psi_4^0 \mathrm{d}u \mathrm{D}\lambda \mathrm{D}\bar{\lambda} \,. \tag{3.124}$$

Hence, on the radiative phase space coming from twistor space, the natural symplectic pairing is between h_0 and ψ_4^0. The standard Ashtekar-Streubel symplectic structure on the radiative phase space [141] can be obtained by integrating by parts with respect to u, using (3.16) and (3.32), and assuming strong fall-offs on the news to get rid of the boundary term:

$$\Omega = \int_{\mathscr{I}} \delta \sigma^0 \delta \dot{\bar{\sigma}}^0 \mathrm{d}u \mathrm{D}\lambda \mathrm{D}\bar{\lambda} \,. \tag{3.125}$$

In this case, the symplectic pairing is between the shear and the news. The virtue of choosing the symplectic pairing (h_0, ψ_4^0) instead of $(\sigma^0, \dot{\bar{\sigma}}^0)$ is that it allows us to see the non-trivial action of the $s = -1$ symmetry on phase space (compare (3.91) with (3.92)).

Let us now focus on the integrable part of the charges (3.116) associated with the $Lw_{1+\infty}$ symmetries. Again, using (3.35), as well as the explicit expressions for the spin s parameters (3.76) together with (3.73)-(3.77), we find explicit expressions for the surface charges at a cut $\mathcal{S} = \{(u, z, \bar{z}) \in \mathscr{I} | u = \text{constant}\}$ of null infinity:

$$
\begin{aligned}
H_{-1} &= \int_{\mathcal{S}} \tau_{-1} \mathcal{Q}_{-1} \mathrm{D}\lambda \mathrm{D}\bar{\lambda}\,, \\
H_0 &= \int_{\mathcal{S}} \tau_0 (\mathcal{Q}_0 - u\bar{\eth}\mathcal{Q}_{-1}) \mathrm{D}\lambda \mathrm{D}\bar{\lambda}\,, \\
H_1 &= \int_{\mathcal{S}} \tau_1 \Big(\mathcal{Q}_1 - u\bar{\eth}\mathcal{Q}_0 + \frac{u^2}{2}\bar{\eth}^2 \mathcal{Q}_{-1} - 2\mathcal{Q}_{-1}\partial_u^{-1}\sigma^0 \Big) \mathrm{D}\lambda \mathrm{D}\bar{\lambda}\,, \\
H_2 &= \int_{\mathcal{S}} \tau_2 \Big(\mathcal{Q}_2 - u\bar{\eth}\mathcal{Q}_1 + \frac{u^2}{2}\bar{\eth}^2 \mathcal{Q}_0 - \frac{u^3}{6}\bar{\eth}^3 \mathcal{Q}_{-1} \\
&\qquad\qquad - 3\mathcal{Q}_0 \partial_u^{-1}\sigma^0 + 3\bar{\eth}\mathcal{Q}_{-1}\partial_u^{-2}\sigma^0 + 2\bar{\eth}\left(\mathcal{Q}_{-1}\partial_u^{-1}(u\sigma^0)\right) \Big) \mathrm{D}\lambda \mathrm{D}\bar{\lambda}\,, \\
H_3 &= \int_{\mathcal{S}} \tau_3 \Big(\mathcal{Q}_3 - u\bar{\eth}\mathcal{Q}_2 + \frac{u^2}{2}\bar{\eth}^2 \mathcal{Q}_1 - \frac{u^3}{6}\bar{\eth}^3 \mathcal{Q}_0 + \frac{u^4}{24}\bar{\eth}^4 \mathcal{Q}_{-1} - 4\mathcal{Q}_1 \partial_u^{-1}\sigma^0 \\
&\qquad\qquad + 3\bar{\eth}\left(\mathcal{Q}_0 \partial_u^{-1}(u\sigma^0)\right) + 4\bar{\eth}\mathcal{Q}_0 \partial_u^{-2}\sigma^0 + 8\mathcal{Q}_{-1}\partial_u^{-1}(\sigma^0 \partial_u^{-1}\sigma^0) \\
&\qquad\qquad - 3\bar{\eth}\left(\bar{\eth}\mathcal{Q}_{-1}\partial_u^{-2}(u\sigma^0)\right) - \bar{\eth}^2\left(\mathcal{Q}_{-1}\partial_u^{-1}(u^2\sigma^0)\right) - 4\bar{\eth}^2 \mathcal{Q}_{-1}\partial_u^{-3}\sigma^0 \Big) \mathrm{D}\lambda \mathrm{D}\bar{\lambda}\,.
\end{aligned}
\tag{3.126}
$$

Using the recursion relations, a general expression for the charges valid for any spin s can be presented:

$$
H_s = \sum_{n=-1}^{s} \int_{\mathcal{S}} \xi_{s,n+1}[\tau_s, \sigma^0] \mathcal{Q}_n \mathrm{D}\lambda \mathrm{D}\bar{\lambda}\,, \tag{3.127}
$$

where as before $\xi_{s,n}$ are depended on the coordinates of $\mathcal{S}$, the transformation parameter τ_s, and the shear σ^0.

Let us make a couple of comments concerning these surface charge expressions. First, we observe that these charges are very similar to those constructed heuristically

in [61, 62]. By contrast with these works, we find a non-vanishing spin $s = -1$ charge on spacetime, involving the momentum $\mathcal{Q}_{-1} = \psi_3^0$. This is consistent with the fact that the spin $s = -1$ acts non-trivially on the phase space variable h. Notice that this charge would disappear if the wedge condition (3.106) was imposed, see (3.16) (in that case, the symmetry $s = -1$ would be a central term). We find a similar contribution $\frac{(-u)^{s+1}}{(s+1)!}\mathcal{Q}_{-1}$, for all spin s charge, which differs from [61, 62] and which would disappear on the wedge. Moreover, we find some terms $\sim \partial_u^{-1}\bar{\sigma}$ of difference compared to [61, 62]. Despite these differences, one can verify that our charges (3.126) satisfy the two following criteria:

(i) The charges are conserved in time in the absence of radiation, i.e.

$$\psi_4^0 = 0 \quad \Longrightarrow \quad \frac{d}{du}H_s = 0\,. \tag{3.128}$$

Notice that the non-radiative condition we are considering here is weaker than that in [61, 62] where the additional condition $\psi_3^0 = 0$ is imposed, but it matches exactly the one proposed earlier in [145]. This makes a lot of sense regarding our construction: the canonical variable ψ_4^0 controls the radiation which spoils the conservation of the $Lw_{1+\infty}$ charges constructed out of $\mathcal{Q}_s$, with $s = -1, 0, 1, 2, \ldots$ The explicit expressions

of the local fluxes $\mathcal{F}_s = \partial_u H_s$ read as

$$\begin{aligned}
\mathcal{F}_{-1} &= \int_{\mathcal{S}} \tau_{-1} \bar{\eth} \psi_4^0 \mathrm{D}\lambda \mathrm{D}\bar{\lambda}\,, \\
\mathcal{F}_0 &= \int_{\mathcal{S}} \tau_0 \left(\sigma^0 \psi_4^0 - u \bar{\eth}^2 \psi_4^0\right) \mathrm{D}\lambda \mathrm{D}\bar{\lambda}\,, \\
\mathcal{F}_1 &= \int_{\mathcal{S}} \tau_1 \left(\frac{u^2}{2} \bar{\eth}^3 \psi_4^0 - u \bar{\eth}(\sigma^0 \psi_4^0) - 2 \bar{\eth} \psi_4^0 \partial_u^{-1} \sigma^0\right) \mathrm{D}\lambda \mathrm{D}\bar{\lambda}\,, \\
\mathcal{F}_2 &= \int_{\mathcal{S}} \tau_2 \left(\frac{u^2}{2} \bar{\eth}^2(\bar{\sigma} \psi_4^0) - \frac{u^3}{6} \bar{\eth}^4 \psi_4^0 - 3 \Psi_4^0 \sigma^0 \partial_u^{-1} \sigma^0 + 2 \bar{\eth}\left(\bar{\eth} \psi_4^0 \partial_u^{-1}(u \sigma^0)\right)\right) \mathrm{D}\lambda \mathrm{D}\bar{\lambda}\,, \\
\mathcal{F}_3 &= \int_{\mathcal{S}} \tau_3 \left(\frac{u^4}{24} \bar{\eth}^5 \psi_4^0 - \frac{u^3}{6} \bar{\eth}^3(\sigma^0 \psi_4^0) - \bar{\eth}^2\left(\bar{\eth} \psi_4^0 \partial_u^{-1}(u^2 \sigma^0)\right) - 3 \bar{\eth}\left(\bar{\eth}^2 \psi_4^0 \partial_u^{-2}(u \sigma^0)\right)\right. \\
&\qquad + 3 \bar{\eth}\left(\psi_4^0 \sigma^0 \partial_u^{-1}(u \sigma^0)\right) + 4 \bar{\eth}(\sigma^0 \psi_4^0) \partial_u^{-2} \sigma^0 + 8 \bar{\eth} \psi_4^0 \partial_u^{-1}(\sigma^0 \partial_u^{-1} \sigma^0) \\
&\qquad \left. - 4 \bar{\eth}^3 \psi_4^0 \partial_u^{-3} \sigma^0\right) \mathrm{D}\lambda \mathrm{D}\bar{\lambda}\,.
\end{aligned} \tag{3.129}$$

One can further show that the fluxes have a compact expression in terms of the $\xi_{s,n}$

$$\mathcal{F}_s = \int_{\mathcal{S}} \psi_4^0 \left(\sigma^0 \xi_{s,1} - \bar{\eth} \xi_{s,0}\right) \mathrm{D}\lambda \mathrm{D}\bar{\lambda}\,, \tag{3.130}$$

showing that the charges H_s satisfy (3.128).

(*ii*) One can easily show that the integrated fluxes $\int_{-\infty}^{+\infty} du \mathcal{F}_s$ are exactly the same as those considered in [61, 62], provided one assumes the same falloffs in u on the shear given in (3.104) allowing for integration by parts. In particular, these fluxes are consistent with the action of the $Lw_{1+\infty}$ symmetries on the radiative phase space.

Let us emphasize that the phase space analysis on twistor space also provides new proposals for the more standard BMS charges. Introducing the supertranslation $T = \tau_0$ and superrotation $\bar{Y} = 2\tau_1$ parameters, and using the correspondence (3.18),

we find the BMS surface charges:[3]

$$\begin{aligned} H_0 &= \int_{\mathcal{S}} T(\psi_2^0 - u\bar{\eth}\psi_3^0)\mathrm{D}\lambda\mathrm{D}\bar{\lambda}\,, \\ H_1 &= \frac{1}{2}\int_{\mathcal{S}} \bar{Y}\left(\psi_1^0 - u\bar{\eth}\psi_2^0 + \frac{u^2}{2}\bar{\eth}^2\psi_3^0 - 2\psi_3^0\partial_u^{-1}\sigma^0\right)\mathrm{D}\lambda\mathrm{D}\bar{\lambda}\,. \end{aligned} \tag{3.131}$$

Notably, we find new expressions for the mass and angular momentum in general relativity, involving the field h, which is natural from a twistor space perspective. These charges have the virtue of being fully compatible with the whole tower of $Lw_{1+\infty}$ symmetries in asymptotically flat spacetime.

The charges for the ϕ generators when translated to spacetime are flux integrals

$$H_{\phi_s} = \int \phi_{s,-2}\,\partial_u^{-1}\sigma^0 du D\lambda D\bar{\lambda}\,. \tag{3.132}$$

Considering the action of the ϕ symmetries on the $\mathcal{Q}_s$, the action of these charges is simply a shift of $\mathcal{Q}_s$ by $\phi_{s',s}$ as we expect from the formulae derived from twistor space.

3.5 An Extension with $\Lambda \neq 0$

In this section, we give some partial results for the case of a non-vanishing cosmological constant, which extend the ideas developed in [146]. We will focus on linear theory, where the bulk is pure complexified AdS, and we work in a Bondi gauge [147]

$$g_{AdS} = -\frac{r^2}{l^2}du^2 - 2dudr + 2r^2 dzd\bar{z}\,, \tag{3.133}$$

where we use the complex plane metric for the spacelike component. We have introduced the AdS length $l^2 = -\frac{3}{\Lambda}$ for notational convenience. If we conformally rescale

[3] In our conventions, $4\pi G = 1$.

the metric with $1/r^2$ to bring infinity to a finite location, we obtain

$$\hat{g}_{AdS} = -\frac{1}{l^2}du^2 + 2dudw + 2dzd\bar{z}\,, \tag{3.134}$$

where $w = r^{-1}$. By taking $w \to 0$, one obtain the boundary metric

$$g_{\partial AdS} = -\frac{1}{l^2}du^2 + 2dzd\bar{z} \tag{3.135}$$

which has a flat space limit $l \to \infty$, yielding a degenerate metric as one would expect. We also note that in these coordinates we use the complex plane metric for the spacetime component of the metric rather than the sphere.

The asymptotic twistor space in this case is no longer given by the blow-up of the $\lambda_\alpha = 0$ line, as was the case for null infinity in the asymptotically flat case. The reason is that the boundary of AdS is no longer null, and as such doesn't have the twistor space realisation of being the space of lines intersecting a line in twistor space. Furthermore, the locus $\lambda_\alpha = 0$ no longer corresponds to the point at infinity and is instead replaced by the criteria given in (2.132). Therefore, we need a different way to construct an asymptotic twistor space and the projection map developed for the asymptotically flat case. To do this, we consider the incidence relations using the conformally rescaled metric, where the point at infinity is located at the origin

$$\omega^{\dot{\alpha}} = x^{\alpha\dot{\alpha}}q_\alpha \qquad , \qquad x^{\alpha\dot{\alpha}} = \begin{pmatrix} u & z \\ -\bar{z} & w - \frac{u}{2l^2} \end{pmatrix} \tag{3.136}$$

where $q_\alpha = (1, q)$ and $\omega^{\dot{\alpha}}$ are the twistor coordinates of the twistor space corresponding to the spacetime with metric $\hat{g}$. By setting $w = 0$, we find the incidence relations at infinity

$$\omega^{\dot{0}} = u - \bar{z}q \qquad \omega^{\dot{1}} = z - \frac{u}{2l^2}q \tag{3.137}$$

Now, to match with null infinity in the $\Lambda = 0$ limit, we re-label the coordinates and

work in the $\lambda_0 \neq 0$ patch

$$\nu^{\dot{\alpha}} = \frac{\mu^{\dot{\alpha}}}{\lambda_0} = \begin{pmatrix} u - \bar{z}q \\ q \end{pmatrix} \qquad \lambda = \frac{\lambda_1}{\lambda_0} = z - \frac{u}{2l^2}q\,. \tag{3.138}$$

As we can see, in the flat space limit $l \to \infty$, we obtain the flat space projection. Now, in the AdS case, we find that the $\nu^{\dot{\alpha}}$ coordinate remains unchanged, while the λ_α coordinate is no longer identified as the celestial sphere coordinate. The fact that we had to re-label the twistor coordinates should be expected, as we performed a partial inversion of the coordinates on spacetime, and from Section 2.2.3, inversions swap the coordinates in the incidence relations.

We now consider the field g corresponding to an ASD perturbation and its equation of motion on a conformally flat background $h = 0$,

$$\bar{\partial} g = 0\,. \tag{3.139}$$

We first decompose g in the basis of $(0,1)$-forms

$$g = g_{\bar{\lambda}} d\bar{\lambda} + g_0 d\bar{\nu}^0 + g_1 d\bar{\nu}^1\,, \tag{3.140}$$

along with the Dolbeault operator

$$\bar{\partial} = d\bar{\lambda}\partial_{\bar{\lambda}} + d\bar{\nu}^\alpha \partial_{\bar{\nu}^\alpha}\,. \tag{3.141}$$

In terms of the spacetime coordinates, the anti-holomorphic vector fields are given by

$$\partial_{\bar{\nu}^0} = \partial_{\bar{u}} + \frac{\bar{q}}{2l^2}\partial_{\bar{\lambda}} \qquad \partial_{\bar{\nu}^1} = \partial_{\bar{q}} + \frac{\bar{u}}{2l^2}\partial_{\bar{\lambda}} + z\partial_{\bar{\nu}^0}\,, \tag{3.142}$$

where

$$\left[1 - \left(\frac{q\bar{q}}{2l^2}\right)^2\right]\partial_{\bar{\lambda}} = \partial_{\bar{z}} + q\partial_u + \frac{q^2}{2l^2}\partial_z + \frac{\bar{q}q^2}{2l^2}\partial_{\bar{u}}\,. \tag{3.143}$$

The holomorphic vector fields can simply be obtained by complex conjugation and will be used in the expression for the Poisson bracket. Using the same definition integral formula, we define the $\mathcal{Q}_s$ as

$$\mathcal{Q}_s = \frac{1}{2\pi i}\int q^{s+2} dq \wedge g|_{X_{(u,z,\bar{z})}} \,. \tag{3.144}$$

The restriction to the fibers of g is given by

$$g|_X = f d\bar{q} \qquad f = g_1 - z g_0 - \frac{\bar{u}}{2l^2} g_{\bar{\lambda}} \,. \tag{3.145}$$

We note that we could have used a similar gauge choice for g as was the case in the null infinity case, namely $\partial_{\bar{u}} \lrcorner g = 0$, but we keep this discussion more general. Using the equations of motion for g in the spacetime coordinates, the function f on-shell satisfies

$$\begin{aligned} \partial_{\bar{u}} f &= \partial_{\bar{q}} \left(g_0 - \frac{\bar{q}}{2l^2} g_{\bar{\lambda}} \right) \\ (\partial_{\bar{z}} + q\partial_u + \frac{q^2}{2l^2}\partial_z) f &= \partial_{\bar{q}} \left(g_{\bar{\lambda}} - \frac{\bar{q} q^2}{2l^2} g_0 \right) \end{aligned} \tag{3.146}$$

Using the integral formula and dropping total derivative terms as we integrate over a sphere, we find

$$\partial_{\bar{u}} \mathcal{Q}_n = 0 \qquad \partial_u \mathcal{Q}_{n+1} + \partial_{\bar{z}} \mathcal{Q}_n - \frac{\Lambda}{6} \partial_z \mathcal{Q}_{n+2} = 0 \tag{3.147}$$

where we have used $l^2 = -\frac{3}{\Lambda}$. The first simply tells us that $\mathcal{Q}_s$ is holomorphic in u and can be viewed as the analytic continuation of a field on the spacetime with reality conditions. The second equation is the analogue of the recursion relations (3.37) with a cosmological constant and is the higher spin extension of the asymptotic Bianchi identities in the case of a constant boundary metric [148, 149]. The non-linear terms present in the full asymptotic Bianchi identities vanish in this case, including the

important shear terms σ^0, due to the fact that we work with a constant boundary metric.

If we consider two functions $f \in \mathcal{O}(n)$ and $g \in \mathcal{O}(m)$ the Poisson bracket using the infinity twistor (2.139) becomes

$$\{f,g\}_\Lambda = \partial_q f\left(\partial_u + \frac{q}{l^2}\partial_\lambda\right) g + \frac{m}{2l^2} g\partial_\lambda f - \partial_q g\left(\partial_u + \frac{q}{l^2}\partial_\lambda\right) f - \frac{n}{2l^2} f\partial_\lambda g \tag{3.148}$$

Now, using this, we can consider the action of ξ on the functions $\mathcal{Q}_n$, with $\delta_\xi g = \{\xi, g\}$ interpreted as a symplectomorphism. In this case $\bar\partial\xi = 0$, and so a basis of generators is given by $w^p_{m,a}$ explicitly given in (3.108). Using the bracket and the equations of motion for g and the gauge condition for ξ, the variation is given by

$$\delta_\xi \mathcal{Q}_n = \frac{1}{2\pi i}\int q^{n+2} d^2q \left[\partial_q\xi\left(\partial_u + \frac{q}{l^2}\partial_z\right) f - \frac{3}{l^2} f\partial_z\xi - \partial_q f\left(\partial_u + \frac{q}{l^2}\partial_z\right)\xi - \frac{1}{l^2}\xi\partial_z f\right]. \tag{3.149}$$

This formula can be readily evaluated to get the variations on spacetime. The algebra of the variations can readily be obtained by considering the commutator and using the Jacobi identity on the bracket

$$\begin{aligned}
[\delta_\xi, \delta_{\xi'}]\mathcal{Q}_s &= \frac{1}{2\pi i}\int q^{s+2} dq \wedge (\{\{g,\xi\},\xi'\}_\Lambda - \{\{g,\xi'\},\xi\}_\Lambda)\,|_{X_{u,z,\bar z}} \\
&= \frac{1}{2\pi i}\int q^{s+2} dq \wedge \{g,\{\xi,\xi'\}\}_\Lambda \\
&= \delta_{\{\xi,\xi'\}_\Lambda}\mathcal{Q}_s\,,
\end{aligned} \tag{3.150}$$

which gives the expected result that the variations are a representation of the Poisson bracket between the generators. By using the basis for the symmetry generators (3.108), the Λ-deformed algebra $\mathrm{Ham}_\Lambda(\mathbb{C}^2)$ is given by [150, 146]

$$\{w^p_{m,a}, w^q_{n,b}\}_\Lambda = (m(q-1)-n(p-1))w^{p+q-2}_{m+n,a+b} - \Lambda(a(q-2)-b(p-2))w^{p+q-1}_{m+n,a+b}\,, \tag{3.151}$$

Chapter 4

Charges at Infinity: Yang-Mills

In this chapter, we present the analogous analysis in Yang-Mills theory, with some key differences. We first consider the twistor action for SD Yang-Mills and construct the associated charges and charge algebra. We will do this in a gauge-invariant manner, which will involve using a holomorphic frame that trivialises the Ward bundle on twistor space on every twistor line. In doing so, we will be able to construct the hierarchy corresponding to the SD field, which could not be made manifest in the gravity case because we chose a special gauge. The holomorphic frame will contain the SD hierarchy, which will be made explicit using contour integral formulae. Finally, by using a different Cauchy surface Σ, we construct the conserved currents at i^0, linking Celestial currents found in top-down and bottom-up approaches. We note that in this chapter, we will use the complex plane model for the boundary metric.

4.1 Twistor space action

In this section, we introduce the twistor space action for SD Yang-Mills with an ASD perturbation [50]. The action is a holomorphic BF theory on twistor space

$$S[a,b] = \int_{\mathbb{PT}} \mathrm{D}^3 Z \wedge \mathrm{Tr}\, b \wedge \left(\bar{\partial} a + \frac{1}{2}[a,a]\right) . \tag{4.1}$$

where $a \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O})$ and $b \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(-4))$ are the off-shell twistor data for the SD background and ASD perturbation, respectively. The action is invariant under gauge transformations

$$\begin{aligned} \delta_{(\xi,\phi)} a &= \bar{D}\xi \,, \\ \delta_{(\xi,\phi)} b &= [b, \xi] + \bar{D}\phi \,, \end{aligned} \tag{4.2}$$

where $(\xi, \phi) \in \mathcal{O} \oplus \mathcal{O}(-4)$ are smooth and arbitrary gauge parameters[1]. In these transformations, we are using the Lie bracket for the Lie algebra $\mathfrak{g}$ as well as the Dolbeault operator on the Ward bundle introduced in (2.103). The equations of motion of the action are given by

$$\begin{aligned} \bar{D}b = \bar{\partial} b + [a, b] = 0 \,, \\ \bar{\partial} a + \frac{1}{2}[a, a] = 0 \,. \end{aligned} \tag{4.3}$$

One can check explicitly that the equations of motion are invariant under the gauge transformations (4.2) although it also follows from the corresponding invariance of the Lagrangian. On-shell, it follows from (4.3) that the field b is $\bar{D}$-closed, and a satisfies the integrability condition (2.104). In particular, we note that the linearized version of this second equation is $\bar{D}\delta a = 0$. Moreover, from (3.2), the fields are defined up to $\bar{D}$-exact terms. In other words, the on-shell field b and linearized perturbation δa around a modulo gauge transformations define elements of the cohomology groups for the deformed complex structure denoted by $H^{0,1}_{\bar{D}}(\mathbb{PT}, \mathcal{O}(-4) \otimes \mathrm{End}(E))$ and $H^{0,1}_{\bar{D}}(\mathbb{PT}, \mathcal{O} \otimes \mathrm{End}(E))$.

The S-algebra constitutes the asymptotic symmetries of SD Yang-Mills theory. Before closing this section, let us also review the symplectic form of the twistorial theory but in the case of the Yang-Mills action, which we will use to construct Noether charges of such asymptotic symmetries.

[1]We note that we denote the gauge parameter associated to the SD sector ξ and to the ASD sector ϕ for notational convenience in both the gravity and gauge theory case.

The presymplectic potential is obtained as the boundary term in the on-shell variation of the action (4.1). Along an on-shell variation δ around a point (a, b) in phase space, it is explicitly given by

$$\boldsymbol{\theta} = \int_\Sigma \mathrm{D}^3 Z \wedge \mathrm{Tr}\, b \wedge \delta a \,, \tag{4.4}$$

where Σ is a real-codimension-1 hypersurface in twistor space that will be specified in Sections 4.2 and 4.3. The presymplectic form is obtained by taking one more phase space variation,

$$\boldsymbol{\omega} = \int_\Sigma \mathrm{D}^3 Z \wedge \mathrm{Tr}(\delta b \wedge \delta a) \,. \tag{4.5}$$

The symplectic form would then be obtained by performing symplectic reduction by the gauge symmetries. But $\boldsymbol{\omega}$ is all that we will ever need.

Standard Noether arguments give boundary charge integrals associated with gauge transformations (4.2) generated by $\alpha = (\xi, \phi)$. The gauge variations act as vector fields on phase space,

$$\delta_\alpha = \delta_\alpha a \, \frac{\delta}{\delta a} + \delta_\alpha b \, \frac{\delta}{\delta b} \,. \tag{4.6}$$

Contracting the presymplectic form with these vector fields yields real-codimension-1 integrands that are total derivatives when the equations of motion are imposed. This gives rise to Noether charges that are codimension-2 boundary integrals [151, 137, 138],

$$\begin{aligned} \not\delta H_\alpha = \delta_\alpha \lrcorner\, \boldsymbol{\omega} = \int_\Sigma \mathrm{d} k_\alpha = \int_{\partial\Sigma} k_\alpha \,, \\ k_\alpha = \mathrm{D}^3 Z \wedge \mathrm{Tr}\,(\xi\, \delta b + \phi\, \delta a) \,. \end{aligned} \tag{4.7}$$

Here k_α is a 4-form on twistor space which can be integrated over $\partial\Sigma$.

The notation $\not\delta$ allows for the possibility that α depends on the fields a, b (perhaps due to gauge fixing),[2] so we must subtract off their gauge variation. The resulting

[2]For instance, our asymptotic symmetry generators will get related to soft gluon wavefunctions

charges can thus be written as

$$\not\delta H_{\xi,\phi} := \delta H_{\xi,\phi} - H_{\delta\xi,\delta\phi}\,, \qquad H_{\xi,\phi} = \int_{\partial\Sigma} \mathrm{D}^3 Z \wedge \mathrm{Tr}\,(\xi\, b + \phi\, a)\,. \tag{4.8}$$

With the subtraction of $\delta\alpha$ implied by $\not\delta$, the variation is generally *not* δ-exact. The charges become integrable, i.e., δ-exact, if the gauge parameters are field independent, $\delta\xi = 0 = \delta\phi$. But generally, this is the case only around a trivial background $a = 0$.

For generic non-integrable expressions, the Barnich-Troessaert bracket [19]

$$\{H_\alpha, H_{\alpha'}\}_\star = \delta_\alpha H_{\alpha'} - H_{\delta_{\alpha'}\alpha}\,, \tag{4.9}$$

can be used to write the charge algebra. We find

$$\{H_\alpha, H_{\alpha'}\}_\star = H_{[\alpha,\alpha']_\star} + K_{\alpha,\alpha'}\,. \tag{4.10}$$

The modified bracket [126] on the right-hand side takes into account the possible field dependence of the symmetry parameters ξ and ϕ and reads explicitly as

$$[\alpha,\alpha']_\star = ([\xi,\xi'] + \delta_\alpha\xi' - \delta_{\alpha'}\xi\,, [\phi,\xi'] + [\xi,\phi'] + \delta_\alpha\phi' - \delta_{\alpha'}\phi)\,. \tag{4.11}$$

The (possibly field-dependent) cocycle in the right-hand side is given by

$$K_{\alpha,\alpha'} = \int_{\partial\Sigma} \mathrm{D}^3 Z \wedge \mathrm{Tr}\left(\xi'\bar\partial\phi + \phi'\bar\partial\xi\right)\,. \tag{4.12}$$

It satisfies the generalized cocycle condition [19]

$$K_{[\alpha_1,\alpha_2]_*,\alpha_3} - \delta_{\alpha_3} K_{\alpha_1,\alpha_2} + \mathrm{cyclic}(1,2,3) = 0\,, \tag{4.13}$$

which extends the standard cocycle condition to the field-dependent case. The BRST

around generic backgrounds (a, b), so they will have to depend nonlinearly on a.

interpretation of field-dependent cocycles has been discussed in [152, 153].

Another subtlety associated with these charges is that even though the integrand of H_ξ is adjoint-valued, the integrand of H_ϕ is $\phi\, a$, which isn't adjoint-valued because it contains a connection. Because of this, it does not appear to obey any gauge-covariant conservation in nonlinear theory. A simple way around this is to replace a by any Lie derivative $\mathcal{L}_\ell a$ of a along a holomorphic vector field ℓ on $\mathbb{PT}$.[3] In later sections, we will often choose to project twistor space onto null infinity and take ℓ to be ∂_{u_B}, the generator of Bondi time translations.

This defines a new type of charge that we will label $\tilde{H}_\phi$. In summary, we will take the definition of our boundary charges to be

$$\begin{aligned} H_\xi &= \int_{\partial\Sigma} \mathrm{D}^3 Z \wedge \mathrm{Tr}\, \xi\, b\,, \\ \tilde{H}_\phi &= \int_{\partial\Sigma} \mathrm{D}^3 Z \wedge \mathrm{Tr}\, \phi\, \mathcal{L}_\ell a\,. \end{aligned} \tag{4.14}$$

As usual, variations of a connection transform in the adjoint. Explicitly, under $a \mapsto a + \bar{D}\xi$, we get $\mathcal{L}_\ell a \mapsto \mathcal{L}_\ell a + [\mathcal{L}_\ell a, \xi] + \bar{D}(\mathcal{L}_\ell \xi)$, so $\mathcal{L}_\ell a$ transforms just like b. The resulting charge $\tilde{H}_\phi$ will be seen to be invariant under small gauge transformations. We can also obtain $\tilde{H}_\phi$ by applying a field-dependent parameter redefinition $\phi \mapsto -\mathcal{L}_\ell \phi$ (sometimes referred to as a "change of slicing", see e.g. [128, 117, 129, 105, 130, 154, 155]). Another way to think of this to consider the parameter $\phi = \mathcal{L}_l \phi'$ and integrate by parts such that the derivative acts only on the gauge field a.

These charges are conserved in the following sense. Let $\bar{m}$ denote a $(0,1)$-vector field normal to $\partial\Sigma$. Acting with $\bar{m}$ on the charges produces (we will see many explicit examples of this general computation in the text)

$$\begin{aligned} \bar{m}(H_\xi) &= \int_{\partial\Sigma} \mathrm{D}^3 Z \wedge \bar{m} \lrcorner \mathrm{Tr}(\bar{D}\xi \wedge b + \xi\, \bar{D} b)\,, \\ \bar{m}(\tilde{H}_\phi) &= \int_{\partial\Sigma} \mathrm{D}^3 Z \wedge \bar{m} \lrcorner \mathrm{Tr}(\bar{D}\phi \wedge \mathcal{L}_\ell a + \phi\, \bar{D} \mathcal{L}_\ell a)\,. \end{aligned} \tag{4.15}$$

[3] Holomorphic here means that ℓ is a $(1,0)$-vector that commutes with the $(0,1)$-vectors $\partial/\partial \bar{Z}^{\bar{A}}$.

On-shell, b satisfies $\bar{D}b = 0$. Similarly, acting on $\bar{D}^2 = 0$ with $\mathcal{L}_\ell$ shows that $\mathcal{L}_\ell a$ satisfies $\bar{D}(\mathcal{L}_\ell a) = 0$, which reconfirms that it transforms in the adjoint (up to $\bar{D}$-exact terms as seen above). So the fluxes along $\bar{m}$ reduce to

$$\begin{aligned} \bar{m}(H_\xi) &= \int_{\partial\Sigma} \mathrm{D}^3 Z \wedge \bar{m} \lrcorner \mathrm{Tr}(\bar{D}\xi \wedge b)\,, \\ \bar{m}(\tilde{H}_\phi) &= \int_{\partial\Sigma} \mathrm{D}^3 Z \wedge \bar{m} \lrcorner \mathrm{Tr}(\bar{D}\phi \wedge \mathcal{L}_\ell a)\,. \end{aligned} \tag{4.16}$$

These will vanish if $\bar{D}\xi = \bar{D}\phi = 0$ holds in the region of interest, which will be a strong version of our definition of large gauge parameters.

We will also encounter weaker versions of this definition where, instead of vanishing identically, $\bar{D}\xi$ and $\bar{D}\phi$ reduce to gauge transformations of radiative data on null infinity. From the twistor perspective, these will be residual gauge symmetries of radiative gauge. In this case, we will find that the charges are conserved if radiation vanishes. If the radiation doesn't vanish, it will give us flux-balance laws. We return to these considerations in Sections 4.2 and 4.3.

The algebra of H_ξ and $\tilde{H}_\phi$ can be computed by noting that $\tilde{H}_\phi$ generates the gauge transformations

$$\tilde{\delta}_\phi a = 0\,, \qquad \tilde{\delta}_\phi b = -\bar{D}(\mathcal{L}_\ell \phi)\,. \tag{4.17}$$

Using the definition (4.9), a short calculation shows that

$$\begin{aligned} \{H_\xi, \tilde{H}_\phi\}_\star &= \tilde{H}_{[\xi,\phi]_\star} + \tilde{K}_{\xi,\phi}\,, \\ \{\tilde{H}_\phi, \tilde{H}_{\phi'}\}_\star &= 0\,, \end{aligned} \tag{4.18}$$

where $[\xi, \phi]_\star = [\xi, \phi] + \delta_\xi \phi$ as before, and the new field-dependent cocycle can be written as

$$\tilde{K}_{\alpha,\alpha'} = \int_{\partial\Sigma} \mathrm{D}^3 Z \wedge \mathrm{Tr}\left(\mathcal{L}_\ell \xi' \bar{D}\phi - \mathcal{L}_\ell \xi \bar{D}\phi'\right)\,, \tag{4.19}$$

after some integration by parts. It satisfies the generalized cocycle condition

$$\tilde{K}_{[\alpha_1,\alpha_2]_\star,\alpha_3} - \tilde{\delta}_{\alpha_3}\tilde{K}_{\alpha_1,\alpha_2} + \text{cyclic}(1,2,3) = 0\,. \tag{4.20}$$

In short, the algebra of the adjoint-valued charges is also governed by the same Barnich-Troessaert bracket (4.9) together with the modified bracket $[\alpha,\alpha']_\star$ of the gauge parameters (4.11).

Boundary conditions on E. We have presented $\overline{\mathbb{PT}}$ as a compact space by embedding it as a projective subvariety of $\mathbb{P}^3\times\mathbb{P}^1$. Let $\mathcal{O}(r)\otimes\mathcal{O}(s)\to\mathbb{P}^3\times\mathbb{P}^1$ denote the line bundle whose sections carry weight r in Z^A and s in π_α, and let $\mathcal{O}(r)\otimes\mathcal{O}(s)|_X$ denote its restriction to any subvariety $X\subset\mathbb{P}^3\times\mathbb{P}^1$. Eg., the canonical bundle of $\overline{\mathbb{PT}}$ is $K_{\overline{\mathbb{PT}}}=\mathcal{O}(-3)\otimes\mathcal{O}(-1)|_{\overline{\mathbb{PT}}}$.

The volume form on $\mathbb{PT}$ was $\Omega_{\mathbb{PT}}=\mathrm{D}^3Z$ and was valued in $\mathcal{O}(4)\otimes\mathcal{O}(0)_{\mathbb{PT}}$. The volume form on $\overline{\mathbb{PT}}$ is

$$\Omega_{\overline{\mathbb{PT}}} = \operatorname*{Res}_{\langle\lambda\pi\rangle=0}\frac{\mathrm{D}^3Z\wedge\mathrm{D}\pi}{\langle\lambda\pi\rangle} \tag{4.21}$$

and is indeed valued in $\mathcal{O}(3)\otimes\mathcal{O}(1)|_{\overline{\mathbb{PT}}}$, the dual of $K_{\overline{\mathbb{PT}}}$. On the variety $\langle\lambda\pi\rangle=0$, one can use the coordinate n normal to $E=\{\lambda_\alpha=0\}$. A computation of residues shows that the volume form on $\mathbb{PT}$ is proportional to that on $\overline{\mathbb{PT}}$ with proportionality factor n,

$$\Omega_{\mathbb{PT}} = n\,\Omega_{\overline{\mathbb{PT}}}\,. \tag{4.22}$$

Since n is a section of $\mathcal{O}(1)\otimes\mathcal{O}(-1)|_{\overline{\mathbb{PT}}}$, and $\Omega_{\overline{\mathbb{PT}}}$ a section of $\mathcal{O}(3)\otimes\mathcal{O}(1)|_{\overline{\mathbb{PT}}}$, their product comes out a section of $\mathcal{O}(4)\otimes\mathcal{O}(0)|_{\overline{\mathbb{PT}}}$ exactly as desired.

To study the behavior of the SD Yang-Mills twistor action (4.1) near the boundary E, we want to extend it to an action over $\overline{\mathbb{PT}}$. To do this, we follow [54] and make sure that the terms inside the Lagrangian are well defined over the locus $n=0$ and that the variational principle is well defined. To do this, we first consider the interaction

vertex $\mathrm{Tr}\, b[a,a]$. Since $\Omega_{\mathbb{PT}} \sim n$ the interaction term could in principle go like n^{-1} near E and still be well defined. The second criterion is to have a well defined variational principle; the boundary term must vanish

$$\oint_{n=0} n dn \mathrm{Tr}(b\delta a)\,. \tag{4.23}$$

Naively, there are many choices of boundary conditions that satisfy these criteria. We will consider two choices: one adapted to constructing charges on a finite cut of null infinity, and the other adapted to currents on spatial infinity.

The first choice will result in using Schwartzian falloffs on the radiative data, which is encoded in the gauge field a. This means that a vanishes at $n = 0$ faster than any polynomial in $1/n$ grows; this then allows b to have any polynomial growth around $n = 0$, which is related to the Bondi coordinate u_B via (3.22). The second choice makes all soft gluons overleading, and as a result will give currents at spatial infinity. This condition will result in the following behaviour around $n = 0$

$$a \sim O(n)\,, \quad b \sim O(n^{-2}) \quad \text{as } n \to 0\,. \tag{4.24}$$

That is, a needs to have a first-order zero along E, while b can at most have a second-order pole along E.

The $n \to 0$ behavior can also be correlated with the $u_B \to \infty$ behavior, giving a Carrollian version of our boundary conditions. To do this, we note the relationship between the Bondi coordinate u_B and n in (3.22). The boundary divisor is approached as $u_B, q \to \infty$ with q/u_B held fixed. From this, we deduce the boundary conditions

$$a \sim O(u_B^{-1})\,, \quad b \sim O(u_B^2) \quad \text{as } u_B \to \infty\,, \tag{4.25}$$

that hold along null infinity.

4.2 Gauge Theory Charges on Null Infinity

In this section, we perform the analogous analysis to that performed in gravity to construct charges on a finite cut of $\mathscr{I}$. To do this, we will consider Schwartzian falloffs on the radiative data, which will be encoded in the gauge field a, in other words

$$\lim_{n\to 0} a = 0\,, \tag{4.26}$$

faster than any polynomial in n. This then implies that the field a vanishes in the limit of large u_B and q. This choice is not the only choice, as we will see in the next section.

4.2.1 Holomorphic Frame and Large Gauge Transformations

As we saw in Chapter 2.2.5, to translate twistor data in gauge theory to spacetime fields, we must consider the holomorphic frame that trivializes the Ward bundle on twistor lines. We start by considering the frame in the bulk and pushing it to null infinity

$$f(u_B, z, \bar{z}, q, \bar{q}) = \lim_{r\to\infty} F(x^{\alpha\dot{\alpha}}, q, \bar{q})\,. \tag{4.27}$$

On $\overline{\mathbb{PT}}$ in the basis (3.41), we expand the various forms

$$b = b_{\bar{i}}\theta^{\bar{i}} \quad , \quad a = a_{\bar{i}}\theta^{\bar{i}}\,. \tag{4.28}$$

which translates to the equations of motion being given by

$$\begin{aligned} \bar{\partial}a + \frac{1}{2}[a,a] = 0 \qquad &\Rightarrow \qquad e_{\bar{i}}a_{\bar{j}} - e_{\bar{j}}a_{\bar{i}} + [a_{\bar{i}}, a_{\bar{j}}] = 0\,, \\ \bar{\partial}b + [a,b] = 0 \qquad &\Rightarrow \qquad e_{\bar{i}}b_{\bar{j}} - e_{\bar{j}}b_{\bar{i}} + [a_{\bar{i}}, b_{\bar{j}}] - [a_{\bar{j}}, b_{\bar{i}}] = 0\,, \end{aligned} \tag{4.29}$$

where $e_{\bar{i}} = (\partial_{\bar{z}} + q\partial_{u_B}\,, \partial_{\bar{q}}\,, \partial_{\bar{u}})$.

Using the equations of motion for the a-field, we can find an identity for the frame.

Consider

$$\begin{aligned}\partial_{\bar{q}}\left(f^{-1}(e_{\bar{z}}+a_{\bar{z}})f\right) &= \partial_{\bar{q}}f^{-1}(e_{\bar{z}}+a_{\bar{z}})f + f^{-1}(e_{\bar{z}}+a_{\bar{z}})\partial_{\bar{q}}f + f^{-1}\partial_{\bar{q}}a_{\bar{z}}f \\ &= f^{-1}(\partial_{\bar{q}}a_{\bar{z}} - \partial_{\bar{z}}a_{\bar{q}} + [a_{\bar{q}}, a_{\bar{z}}])f \\ &= 0\,.\end{aligned} \tag{4.30}$$

via the equations of motion and $[e_{\bar{z}}, \partial_{\bar{q}}] = 0$. Therefore, $f^{-1}(e_{\bar{z}} + a_{\bar{z}})f$ is globally holomorphic on the asymptotic twistor lines. So Liouville's theorem tells us that it must be independent of q,

$$e_{\bar{z}}f + a_{\bar{z}}f = f\bar{\mathcal{A}}\,. \tag{4.31}$$

The variation of the frame depends on the variation of the a field, which can be realized by the formula

$$f^{-1}\delta f = \frac{1}{2\pi i}\int_L \frac{\mathrm{d}q'}{q-q'} \wedge f^{-1}\delta a|_L f\,, \tag{4.32}$$

where we have used the boundary condition $f|_{q=\infty} = 1$. If we insert a gauge transformation δ_ξ and use Cauchy's theorem,[4] we find

$$\begin{aligned}\delta_\xi f &= -\xi f + \frac{f}{2\pi i}\oint_{C_\infty}\frac{dq'}{q'-q}f^{-1}\xi f \\ &= -\xi f + f\tilde{\xi}\,,\end{aligned} \tag{4.33}$$

where

$$\tilde{\xi} = \sum_{n=0}^{\infty}\frac{q^n}{2\pi i}\oint_{C_\infty}\frac{dw}{w^{n+1}}f^{-1}\xi f\,. \tag{4.34}$$

By considering the action of the variation on the frame in (4.31), we find that this

[4]For a disk $\Delta \subset \mathbb{C}$ and $h \in C^\infty(\bar{\Delta})$, we have

$$h(q) = \frac{1}{2\pi i}\oint_{\partial\Delta}\frac{h(q')dq'}{q'-q} + \frac{1}{2\pi i}\int_\Delta \frac{\partial h(q')}{\partial\bar{q}'}\frac{\mathrm{d}q'\wedge\mathrm{d}\bar{q}'}{q'-q}\,.$$

induces a variation on $\bar{\mathcal{A}}$

$$\delta_\xi \bar{\mathcal{A}} = q\partial_{u_B}\tilde{\xi} + \partial_{\bar{z}}\tilde{\xi} + [\bar{\mathcal{A}}, \tilde{\xi}]\,. \tag{4.35}$$

Since $\bar{\mathcal{A}} = \bar{\mathcal{A}}(u_B, z, \bar{z})$ is q-independent (which is crucial for the relation with space-time) implies restrictions on the gauge transformation ξ. We note that this is a choice imposed on the gauge field a, and in the rest of this chapter we will work in the radiative gauge

$$f^{-1}(\bar{\partial} + a)f = \bar{\mathcal{A}}d\bar{z}\,, \tag{4.36}$$

which states that our gauge field a is gauge equivalent to the radiative data on $\mathscr{I}$ pulled-back from twistor space $p^*(A|_{\mathscr{I}})$ via the projection (3.24).

Let us now study the constraints on ξ_s implied by the use of the frame. The transformation law of the frame f defines $\tilde{\xi}_s$ through

$$\begin{aligned}
\delta_s f &= -\xi_s f + \frac{f}{2\pi i}\oint_{C_\infty} \frac{\mathrm{d}q'}{q'-q} f^{-1}\xi_s f \\
&= -\xi_s f + f\sum_{n=0}^{\infty} \frac{q^n}{2\pi i}\oint_{C_\infty} \frac{\mathrm{d}q'}{(q')^{n+1}} f^{-1}\xi_s f \\
&= -\xi_s f + f\tilde{\xi}_s\,,
\end{aligned} \tag{4.37}$$

where

$$\tilde{\xi}_s = \sum_{n=0}^{\infty} \tilde{\xi}_{s,n} q^n \qquad , \qquad \tilde{\xi}_{s,n} = \frac{1}{2\pi i}\oint_{C_\infty} \frac{\mathrm{d}q}{q^{n+1}} f^{-1}\xi_s f\,. \tag{4.38}$$

The variation of $\bar{\mathcal{A}}$ in (4.35) imposes restrictions on the gauge parameter ξ_s. Since $\bar{\mathcal{A}}$ is q independent we obtain the recursions relations on $\tilde{\xi}_{s,n}$

$$\partial_{u_B}\tilde{\xi}_{s,n-1} + \partial_{\bar{z}}\tilde{\xi}_{s,n} + [\bar{\mathcal{A}}, \tilde{\xi}_{s,n}] = 0\,. \tag{4.39}$$

By using the contour integral (4.38) and the boundary condition (4.26), we obtain

the condition

$$q\partial_{u_B}\xi_s + \partial_{\bar{z}}\xi_s = e_{\bar{z}}\xi_s = 0\,. \tag{4.40}$$

Since $\bar{\mathcal{A}}$ is $\bar{u}_B$ independent we also obtain

$$\partial_{\bar{u}_B}\xi_s = 0\,. \tag{4.41}$$

We can satisfy these conditions by considering the ξ_s that satisfy $\bar{\partial}\xi_s = 0$ away from $q = \infty$ and we can expand it in a convenient basis

$$\xi_s = \frac{1}{s!}\frac{\partial^s \rho_s(\lambda,\bar{\lambda})}{\partial\bar{\lambda}^{\dot{\alpha}_1}\cdots\partial\bar{\lambda}^{\dot{\alpha}_s}}\mu^{\dot{\alpha}_1}\cdots\mu^{\dot{\alpha}_s} = \sum_{n=0}^{s}\frac{(-1)^n}{(s-n)!}u_B^{s-n}q^n\partial_{\bar{z}}^{s-n}\rho_s(z,\bar{z})\,, \tag{4.42}$$

where $\rho_s(t\lambda,\bar{t}\bar{\lambda}) = t^{-s}\bar{t}^s\rho_s(\lambda,\bar{\lambda})$ for $t\in\mathbb{C}^\times$ and the parameters satisfy the wedge condition

$$\partial_{\bar{z}}^{s+1}\rho_s = 0\,. \tag{4.43}$$

These are exactly the generators of the S-algebra discussed in [49]. By solving the recursion relations (4.39), the first few examples of $\tilde{\xi}_s$ are given by

$$\begin{aligned}
\tilde{\xi}_0 &= \rho_0\,,\\
\tilde{\xi}_1 &= \rho_1 q - u_B\partial_{\bar{z}}\rho_1 - [\partial_{u_B}^{-1}\bar{\mathcal{A}},\rho_1]\,,\\
\tilde{\xi}_2 &= \rho_2 q^2 - u_B q\partial_{\bar{z}}\rho_2 - q[\partial_{u_B}^{-1}\bar{\mathcal{A}},\rho_2] + \frac{u_B^2}{2}\partial_{\bar{z}}^2\rho_2\\
&\qquad + \partial_{\bar{z}}[\partial_{u_B}^{-2}\bar{\mathcal{A}},\rho_2] + [\partial_{u_B}^{-1}(u_B\bar{\mathcal{A}}),\partial_{\bar{z}}\rho_2] + \partial_{u_B}^{-1}[\bar{\mathcal{A}},[\partial_{u_B}^{-1}\bar{\mathcal{A}},\rho_2]]\,.
\end{aligned} \tag{4.44}$$

Another way to compute the $\tilde{\xi}_s$ is by explicitly evaluating the contour integrals (4.38). To do this, we must find a formula for the frame f around $q=\infty$. We first note that the holomorphic frame satisfies

$$\partial_{\bar{q}}f + a|_L f = 0 \quad\to\quad f = 1 + \int_L \frac{dq'}{q'-q}\wedge a|_L f\,, \tag{4.45}$$

where the second is an iterative formula for the frame. If we expand this around $q = \infty$ we find the following series

$$f = 1 + \sum_{n=1}^{\infty} \frac{f_n}{q^n} \,. \tag{4.46}$$

To find out what the coefficients f_n are, we use the boundary condition (4.26) and the identity (4.31) around $q = \infty$, giving

$$\partial_{u_B} f_{n+1} + \partial_{\bar{z}} f_n - f_n \bar{\mathcal{A}} = 0 \,, \qquad f_1 = \partial_{u_B}^{-1} \bar{\mathcal{A}} \,. \tag{4.47}$$

Solving these equations recursively and evaluating the contour integrals at $q = \infty$ will give the $\tilde{\xi}_s$. If we define $\xi_s = \xi(\rho_s)$, then the generators satisfy the S-algebra

$$[\xi(\rho_s), \xi(\rho_{s'})] = \xi([\rho_s, \rho_{s'}]) \,, \tag{4.48}$$

where the wedge condition is used. To obtain the more familiar form of the S-algebra choose $\rho_s(z, \bar{z}) = (-1)^l \bar{z}^k z^m T^a$ with $s = k + l$ which gives

$$\xi^a_{k,l,m} = (u_B - \bar{z}q)^k q^l z^m T^a = \frac{(\mu^{\dot{0}})^k (\mu^{\dot{1}})^l}{\lambda_0^{k+l+m} \lambda_1^{-m}} T^a \,, \tag{4.49}$$

which gives the commutator

$$[\xi^a_{k,l,m}, \xi^b_{k',l',m'}] = f^{ab}{}_c \xi^c_{k+k',l+l',m+m'} \,, \tag{4.50}$$

Hence we have shown that the large gauge transformations associated with ξ coincide with the S-algebra. A similar discussion holds for the gauge parameter ϕ. We find the following expression for the large gauge transformations:

$$\phi_s = \frac{1}{s!} \frac{\partial^s \chi_s(\lambda, \bar{\lambda})}{\partial \bar{\lambda}^{\dot{\alpha}_1} \cdots \partial \bar{\lambda}^{\dot{\alpha}_s}} \mu^{\dot{\alpha}_1} \cdots \mu^{\dot{\alpha}_s} = \frac{1}{n^4} \sum_{m=0}^{s} \frac{(-1)^m}{(s-m)!} u_B^{s-m} q^m \partial_{\bar{z}}^{s-m} \chi_s(z, \bar{z}) \,, \tag{4.51}$$

where $\chi_s(t\lambda, \bar{t}\bar{\lambda}) = t^{-s-4}\bar{t}^s\chi_s(\lambda,\bar{\lambda})$ for $t \in \mathbb{C}^\times$ and satisfies the wedge condition $\partial_{\bar{z}}^{s+1}\chi_s = 0$. Defining $\phi_s = \phi(\chi_s)$ we obtain the algebra

$$[\xi(\rho_s), \phi(\chi_{s'})] = \phi\left([\rho_s, \chi_{s'}]\right) . \tag{4.52}$$

Choosing $\chi_s(z,\bar{z}) = (-1)^l \bar{z}^k z^m T^a$ with $s = k + l$ gives

$$\phi^a_{k,l,m} = \frac{(u_B - \bar{z}q)^k q^l z^m T^a}{n^4} = \frac{(\mu^{\dot{0}})^k (\mu^{\dot{1}})^l}{\lambda_0^{k+l+m+4}\lambda_1^{-m}} T^a , \tag{4.53}$$

which gives the commutator

$$[\xi^a_{k,l,m}, \phi^b_{k',l',m'}] = f^{ab}{}_c \phi^c_{k+k',l+l',m+m'} . \tag{4.54}$$

The commutator between the ϕ's projects to zero, and we find that the algebra of large gauge transformations is given by a semi-direct sum of the S-algebra (4.50) acting on an abelian algebra as in (4.54).

The parameters of the large gauge transformations are field independent ($\delta\xi = 0 = \delta\phi$) and holomorphic, so that the charges discussed in Section 4.1 are integrable:

$$H_{\xi_s,\phi_s} = \int_{\partial\Sigma} \mathrm{D}^3 Z \wedge \mathrm{Tr}(\xi_s b + \phi_s a) . \tag{4.55}$$

The charge algebra (4.10) simplifies to

$$\{H_{\xi,\phi}\, , H_{\xi',\phi'}\} = H_{[\xi,\xi'],[\phi,\xi']+[\xi,\phi']} + K_{(\xi,\phi),(\xi',\phi')} , \tag{4.56}$$

and the cocycle has a contribution from the singular points of the generators. Here we used the usual bracket $\{H_{\xi,\phi}, \cdot\} = \delta_{\xi,\phi}$.

Finally, it is also useful to compute the algebra satisfied by $\tilde{\xi}_s$ introduced in (4.38), as they will be the natural parameters for the translation to spacetime discussed in

Section 4.2.2. Because of their field dependence in terms of the frame, one uses the algebroid bracket [156]

$$[\tilde{\xi}_s, \tilde{\xi}_{s'}]_* = [\tilde{\xi}_s, \tilde{\xi}_{s'}] + \delta_{\xi_s}\tilde{\xi}_{s'} - \delta_{\xi_{s'}}\tilde{\xi}_s \,, \tag{4.57}$$

Consider

$$\left(q\partial_u + \partial_{\bar{z}} + [\bar{\mathcal{A}}, \cdot\,]\right)[\tilde{\xi}_s, \tilde{\xi}_{s'}]_* = [\delta_{\xi_s}, \delta_{\xi_{s'}}]\bar{\mathcal{A}} \,, \tag{4.58}$$

which tells us that $[\tilde{\xi}_s, \tilde{\xi}_{s'}]_*$ is another $\tilde{\xi}_{s''}$ constructed out of a $\xi_{s''}$. To identify $\xi_{s''}$ we consider

$$[\delta_{\xi_s}, \delta_{\xi_{s'}}]a = \bar{\partial}[\xi_s, \xi_{s'}] + [a, [\xi_s, \xi_{s'}]] \,, \tag{4.59}$$

which gives us the expected result

$$[\delta_{\xi_s}, \delta_{\xi_{s'}}] = \delta_{[\xi_s, \xi_{s'}]} \,. \tag{4.60}$$

If we write $\xi_s = \xi(\rho_s)$, then inside the wedge

$$[\xi(\rho_s), \xi(\rho_{s'})] = \xi([\rho_s, \rho_{s'}]) \,, \tag{4.61}$$

and similarly $\tilde{\xi}_s = \tilde{\xi}(\rho_s)$ but with the algebroid bracket

$$[\tilde{\xi}(\rho_s), \tilde{\xi}(\rho_{s'})]_* = \tilde{\xi}([\rho_s, \rho_{s'}]), \,. \tag{4.62}$$

4.2.2 From twistor space to spacetime

We now wish to make contact with the asymptotic data at null infinity using the twistor data we have discussed; we will follow the bulk gauge choice found in [114]. In a radial and asymptotic retarded gauge, the gauge field restricted to null infinity is given by

$$A|_{\mathscr{I}} = \mathcal{A}dz + \bar{\mathcal{A}}d\bar{z} \,. \tag{4.63}$$

As we have already mentioned, the $\bar{\mathcal{A}}$ already appears in the construction of the holomorphic frame that trivialises the Ward bundle for the asymptotic twistor space. The gauge-invariant asymptotic data is found by considering the field strength and its associated peeling theorem. As was the case in gravity, we consider the NP quantities built from the field strength. To do this, consider the ASD part of the field strength and decompose it in a dyad of spinors

$$F_{\alpha\beta} = F_2 o_\alpha o_\beta + 2F_1 o_{(\alpha}\iota_{\beta)} + F_0 \iota_\alpha \iota_\beta \,. \tag{4.64}$$

The peeling theorem for the components of the field strength is given by

$$F_n = \frac{R_{1-n}}{r^{3-n}} + \mathcal{O}(r^{n-4}), \qquad n \in \{0,1,2\}\,, \tag{4.65}$$

where the radiative data is encoded in R_{-1} and is given by

$$R_{-1} = \partial_{u_B}\mathcal{A}\,. \tag{4.66}$$

The field equations at leading order are given by

$$\partial_{u_B} R_s + \partial_{\bar{z}} R_{n-1} + [\bar{\mathcal{A}}, R_{n-1}] = 0\,. \tag{4.67}$$

We will consider the extension of these equations for $n \geq -1$, which was conjectured in [63] to encode a portion of the sub-leading expansion of F_0. We will also define the recursion relations associated to the anti-holomorphic part of the gauge field $\bar{\mathcal{A}}$ by

$$\partial_{u_B} \tilde{R}_s + \partial_{\bar{z}} \tilde{R}_{n-1} + [\bar{\mathcal{A}}, \tilde{R}_{n-1}] = 0\,, \tag{4.68}$$

where the seed of the recursion is given by $\tilde{R}_{-1} = \bar{\mathcal{A}}$. It can be shown that these recursion relations are related to the integrable hierarchy of SD Yang-Mills in the

bulk [157].

The Yang-Mills extension of the BCT integral formulae allows us to relate the twistor date (b, a) with spacetime data $(R_s, \tilde{R}_s)$:

$$\begin{aligned} R_s(u_B, z, \bar{z}) &= \frac{1}{2\pi i} \int_L q^{s+1} \mathrm{d}q \wedge f^{-1} b|_L f \,, \\ \tilde{R}_s(u_B, z, \bar{z}) &= \frac{1}{2\pi i} \int_L q^{s+1} \mathrm{d}q \wedge f^{-1} \partial_{u_B} a|_L f \,, \end{aligned} \tag{4.69}$$

where $|_L$ denotes the restriction of the from to the fibers over $\mathscr{I}$. These integral formulae solve the recursion relations (4.67) and (4.68), respectively. We have explicitly:

$$\begin{aligned} \partial_{u_B} R_s + \partial_{\bar{z}} R_{s-1} + [\bar{\mathcal{A}}, R_{s-1}] &= \frac{1}{2\pi i} \int_L q^s \mathrm{d}q \wedge \left(e_{\bar{z}} \left[f^{-1} b|_L f \right] + [\bar{\mathcal{A}}, f^{-1} b|_L f] \right) \\ &= \frac{1}{2\pi i} \int_L q^s \mathrm{d}q \wedge \mathrm{d}\bar{q}\, f^{-1} \left(e_{\bar{z}} b_{\bar{q}} + [a_{\bar{z}}, b_{\bar{q}}] \right) f \\ &= \frac{1}{2\pi i} \int_L q^s \mathrm{d}q \wedge \mathrm{d}\bar{q}\, f^{-1} \left(\partial_{\bar{q}} b_{\bar{z}} + [a_{\bar{q}}, b_{\bar{z}}] \right) f \\ &= \frac{1}{2\pi i} \int_L q^s \mathrm{d}q \wedge \bar{\partial}|_L \left(f^{-1} b_{\bar{z}} f \right) \\ &= 0 \,, \end{aligned} \tag{4.70}$$

where the equations of motion for the b field in (4.3) have been used. Similarly, we have

$$\begin{aligned} \partial_{u_B} \tilde{R}_s + \partial_{\bar{z}} \tilde{R}_{s-1} + [\bar{\mathcal{A}}, \tilde{R}_{s-1}] &= \frac{1}{2\pi i} \int_L q^s \mathrm{d}q \wedge \left[e_{\bar{z}} \left(f^{-1} \partial_{u_B} a|_L f \right) + [\bar{\mathcal{A}}, f^{-1} \partial_{u_B} a|_L f] \right] \\ &= \frac{1}{2\pi i} \int_L q^s \mathrm{d}q \wedge \mathrm{d}\bar{q} \left[f^{-1} e_{\bar{z}} \partial_{u_B} a_{\bar{q}} f + f^{-1} [a_{\bar{z}}, \partial_{u_B} a_{\bar{q}}] f \right] \\ &= \frac{1}{2\pi i} \int_L q^s \mathrm{d}q \wedge \mathrm{d}\bar{q} \left[f^{-1} \partial_{\bar{q}} \partial_{u_B} a_{\bar{z}} f + f^{-1} [a_{\bar{q}}, \partial_{u_B} a_{\bar{z}}] f \right] \\ &= \frac{1}{2\pi i} \int_L q^s \mathrm{d}q \wedge \bar{\partial}|_L \left(f^{-1} \partial_{u_B} a_{\bar{z}} f \right) \\ &= 0 \,, \end{aligned} \tag{4.71}$$

after using the equations of motion for the a field in (4.3).

The variations of the charge aspects R_s and $\tilde{R}_s$ can be computed using (4.69) and (4.2) and the large gauge transformations derived in Section 4.2.1. We find

$$\begin{aligned}\delta_{\xi_s,\phi_l} R_{s'} &= \sum_{m=0}^{s} [R_{s'+m}, \tilde{\xi}_{s,m}] - \omega_{s'}(\phi_l)\,, \\ \delta_{\xi_s} \tilde{R}_{s'} &= \sum_{m=0}^{s} [\tilde{R}_{s'+m}, \tilde{\xi}_{s,m}] - \tilde{\omega}_{s'}(\xi_s)\,,\end{aligned} \tag{4.72}$$

where the shift symmetry

$$\omega_s(\phi_{s'}) = \frac{1}{2\pi i}\oint_{C_\infty} q^{s+1} \mathrm{d}q\, f^{-1}\phi_{s'} f \qquad , \qquad \tilde{\omega}_s(\xi_{s'}) = \frac{1}{2\pi i}\oint_{C_\infty} q^{s'+1} \mathrm{d}q\, f^{-1}\partial_{u_B}\xi_s f\,, \tag{4.73}$$

also satisfy the recursion relations (4.67) and (4.68):

$$\begin{aligned}\partial_{u_B}\omega_s(\phi_{s'}) + \partial_{\bar{z}}\omega_{s-1}(\phi_{s'}) + [\bar{\mathcal{A}}, \omega_{s-1}(\phi_{s'})] = 0\,, \\ \partial_{u_B}\tilde{\omega}_s(\xi_{s'}) + \partial_{\bar{z}}\tilde{\omega}_{s-1}(\xi_{s'}) + [\bar{\mathcal{A}}, \tilde{\omega}_{s-1}(\xi_{s'})] = 0\,.\end{aligned} \tag{4.74}$$

Therefore, the large gauge transformations on twistor space derived in Section 4.2.1 correspond to the symmetries of the spacetime recursion relations. We display a few explicit examples of these transformations for the lowest spins: The transformations for the R_s coming from the ξ_s

$$\begin{aligned}\delta_{\xi_0} R_s &= [R_s, \rho_0]\,, \\ \delta_{\xi_1} R_s &= [R_{s+1}, \rho_1] - u_B[R_s, \partial_{\bar{z}}\rho_1] - [R_s, [\partial_{u_B}^{-1}\bar{\mathcal{A}}, \rho_1]]\,, \\ \delta_{\xi_2} R_s &= [R_{s+2}, \rho_2] - u_B[R_{s+1}, \partial_{\bar{z}}\rho_2] - [R_{s+1}, [\partial_{u_B}^{-1}\bar{\mathcal{A}}, \rho_2]] + \frac{u_B^2}{2}[R_s, \partial_{\bar{z}}^2\rho_2] \\ &\quad + [R_s, \partial_{\bar{z}}[\partial_{u_B}^{-2}\bar{\mathcal{A}}, \rho_2]] + [R_s, [\partial_{u_B}^{-1}(u_B\bar{\mathcal{A}}), \partial_{\bar{z}}\rho_2]] + [R_s, \partial_{u_B}^{-2}[\bar{\mathcal{A}}, [\partial_{u_B}^{-1}\bar{\mathcal{A}}, \rho_2]]]\,,\end{aligned} \tag{4.75}$$

while for the ϕ_s transformation, the first few examples are given by

$$\delta_{\phi_0} R_{-1} = [\partial_{u_B}^{-1}\bar{\mathcal{A}}, \chi_0] \qquad , \qquad \delta_{\phi_0} R_0 = [\chi_0, \partial_{\bar{z}}\partial_{u_B}^{-2}\bar{\mathcal{A}}] - \partial_{u_B}^{-1}[\bar{\mathcal{A}}, [\partial_{u_B}^{-1}\bar{\mathcal{A}}, \chi_0]]\,. \tag{4.76}$$

The variations for ϕ_s with $s > 0$ can be recursively derived by using the result from ϕ_0

$$\omega_{s'}(\phi_s) = \sum_{m=0}^{s} \frac{(-1)^m}{(s-m)!} u_B^{s-m} \omega_{s'+m}(\partial_{\bar{z}}^{s-m} \chi_s) \,. \tag{4.77}$$

Similarly, we can construct the $\tilde{\omega}_s(\xi_{s'})$

$$\tilde{\omega}_s(\xi_{s'}) = -\sum_{m=0}^{s'} \frac{(-1)^m}{(s'-m)!} u_B^{s'-m} \omega_{s+m-1} \left(\partial_{\bar{z}}^{s'+1-m} \rho_{s'} \right) \,, \tag{4.78}$$

Importantly, the transformation law for $\tilde{R}_{-1}$ matches the transformation of $\bar{\mathcal{A}}$ which is given by

$$\delta_{\xi_s} \bar{\mathcal{A}} = \partial_{\bar{z}} \tilde{\xi}_{s,0} + [\bar{\mathcal{A}}, \tilde{\xi}_{s,0}] \,. \tag{4.79}$$

The first few examples are given by

$$\begin{aligned}
&\delta_{\xi_0} \bar{\mathcal{A}} = \partial_{\bar{z}} \rho_0 + [\bar{\mathcal{A}}, \rho_0] \,, \\
&\delta_{\xi_1} \bar{\mathcal{A}} = -u_B \left(\partial_{\bar{z}}^2 \rho_1 + [\bar{\mathcal{A}}, \partial_{\bar{z}} \rho_1] \right) - \partial_{\bar{z}} [\partial_{u_B}^{-1} \bar{\mathcal{A}}, \rho_1] - [\bar{\mathcal{A}}, [\partial_{u_B}^{-1} \bar{\mathcal{A}}, \rho_1]] \,, \\
&\delta_{\xi_2} \bar{\mathcal{A}} = \frac{u_B^2}{2} \left(\partial_{\bar{z}}^3 \rho_2 + [\bar{\mathcal{A}}, \partial_{\bar{z}}^2 \rho_2] \right) + \partial_{\bar{z}}^2 [\partial_{u_B}^{-2} \bar{\mathcal{A}}, \rho_2] + [\bar{\mathcal{A}}, \partial_{\bar{z}} [\partial_{u_B}^{-1} \bar{\mathcal{A}}, \rho_2]] + \partial_{\bar{z}} [\partial_{u_B}^{-1} (u_B \bar{\mathcal{A}}), \partial_{\bar{z}} \rho_2] \\
&\qquad + \partial_{\bar{z}} \partial_{u_B}^{-1} [\bar{\mathcal{A}}, [\partial_{u_B}^{-1} \bar{\mathcal{A}}, \rho_2]] + [\bar{\mathcal{A}}, [\partial_{u_B}^{-1} (u_B \bar{\mathcal{A}}), \partial_{\bar{z}} \rho_2]] + [\bar{\mathcal{A}}, \partial_{u_B}^{-1} [\bar{\mathcal{A}}, [\partial_{u_B}^{-1} \bar{\mathcal{A}}, \rho_2]]] \,.
\end{aligned} \tag{4.80}$$

To discuss the translation of the phase space from twistor space to spacetime at $\mathscr{I}$, a convenient choice of codimension 1 hypersurface is $\Sigma = \{u_B = \bar{u}_B\}$. The surface charges will then be integrated over the codimension 2 surface $\partial\Sigma = \{u_B = \bar{u}_B = \text{constant}\}$. The charges associated with the large gauge transformations on twistor space (4.55) cannot be directly translated to spacetime using the above BCT integral formulae. To circumvent this issue, we explicitly go to the gauge where $a = \bar{\mathcal{A}} d\bar{z}$ such that $f = 1$. In this case, the charges become non-integrable, and our gauge

parameters become field-dependent. The integrable pieces read as

$$H_{\tilde{\xi}} = \int_{\partial\Sigma} \mathrm{D}^3 Z \wedge \mathrm{Tr}(\tilde{\xi} b), \qquad H_{\tilde{\phi}} = \int_{\partial\Sigma} \mathrm{D}^3 Z \wedge \mathrm{Tr}(\tilde{\phi} \bar{\mathcal{A}} d\bar{z})\,. \tag{4.81}$$

These expressions allow us to directly apply the BCT integral formulae (4.69) in the case of $f = 1$ and obtain spacetime expressions. We note the field-dependent parameter $\tilde{\phi}_s$ is built from the $\omega_s(\phi_{s'})$ that were considered in (4.72) and the charge $H_{\tilde{\phi}_s}$ performs a contour integral on the q-sphere. We find one spin s charge at a finite cut $u =$ constant of $\mathscr{I}$ and the other is a integral over all of $\mathscr{I}$

$$H_{\tilde{\xi}_s}(u_B) = \sum_{n=0}^{s} \int_{\mathbb{C}} \mathrm{d}z\mathrm{d}\bar{z}\, \mathrm{Tr}\left(\tilde{\xi}_{s,n} R_n\right) \qquad , \qquad H_{\tilde{\phi}_s} = \int_{\mathscr{I}} \mathrm{d}u_B \mathrm{d}z\mathrm{d}\bar{z}\, \mathrm{Tr}\left(\omega_{-1}(\phi_s)\bar{\mathcal{A}}\right)\,. \tag{4.82}$$

We provide below some explicit expressions for the lowest spins:

$$\begin{aligned}
H_0(u_B) &= \int_{\mathbb{C}} \mathrm{d}z\mathrm{d}\bar{z}\mathrm{Tr}(\rho_0 R_0)\,, \\
H_1(u_B) &= \int_{\mathbb{C}} \mathrm{d}z\mathrm{d}\bar{z}\mathrm{Tr}(\rho_1 R_1 - u_B \partial_{\bar{z}} \rho_1 R_0 - [\partial_{u_B}^{-1}\bar{\mathcal{A}}, \rho_1] R_0)\,, \\
H_2(u_B) &= \int_{\mathbb{C}} \mathrm{d}z\mathrm{d}\bar{z}\mathrm{Tr}\Big(\rho_2 R_2 - u_B \partial_{\bar{z}}\rho_2 R_1 + \frac{u_B^2}{2}\partial_{\bar{z}}^2 \rho_2 R_0 - [\partial_{u_B}^{-1}\bar{\mathcal{A}}, \rho_2] R_1 \\
&\qquad + \partial_{\bar{z}}[\partial_{u_B}^{-2}\bar{\mathcal{A}}, \rho_2] R_0 + [\partial_{u_B}^{-1}(u_B\bar{\mathcal{A}}), \partial_{\bar{z}}\rho_2] R_0 + \partial_{u_B}^{-1}[\bar{\mathcal{A}}, [\partial_{u_B}^{-1}\bar{\mathcal{A}}, \rho_2]] R_0\Big)\,,
\end{aligned} \tag{4.83}$$

while the first few examples of the $\tilde{H}_s$ charges are given by:

$$
\begin{aligned}
H_{\phi_0} &= \int_{\mathscr{I}} \mathrm{d}u_B \mathrm{d}z \mathrm{d}\bar{z} \mathrm{Tr}\left([\chi_0, \partial_{u_B}^{-1}\bar{\mathcal{A}}]\bar{\mathcal{A}}\right) \,, \\
H_{\phi_0} &= \int_{\mathscr{I}} \mathrm{d}u_B \mathrm{d}z \mathrm{d}\bar{z} \mathrm{Tr}\left(\partial_{\bar{z}}[\partial_{u_B}^{-2}\bar{\mathcal{A}}, \chi_1]\bar{\mathcal{A}} + [\partial_{u_B}^{-1}(u_B\bar{\mathcal{A}}), \partial_{\bar{z}}\chi_1]\bar{\mathcal{A}} + \partial_{u_B}^{-1}[\bar{\mathcal{A}}, [\partial_{u_B}^{-1}\bar{\mathcal{A}}, \chi_1]]\bar{\mathcal{A}}\right) \,, \\
H_{\phi_2} &= \int_{\mathscr{I}} \mathrm{d}u_B \mathrm{d}z \mathrm{d}\bar{z} \mathrm{Tr}\left(\partial_{\bar{z}}^2[\chi_2, \partial_{u_B}^{-3}\bar{\mathcal{A}}]\bar{\mathcal{A}} - \partial_{u_B}^{-1}[\bar{\mathcal{A}}, \partial_{\bar{z}}[\partial_{u_B}^{-2}\bar{\mathcal{A}}, \chi_2]]\bar{\mathcal{A}} + \partial_{\bar{z}}[\partial_{\bar{z}}\chi_2, \partial_{u_B}^{-2}(u_B\bar{\mathcal{A}})]\bar{\mathcal{A}}\right. \\
&\qquad + \frac{1}{2}[\partial_{\bar{z}}^2\chi_2, \partial_{u_B}^{-1}(u_B^2\bar{\mathcal{A}})]\bar{\mathcal{A}} - \partial_{\bar{z}}\partial_{u_B}^{-2}[\bar{\mathcal{A}}, [\partial_{u_B}^{-1}\bar{\mathcal{A}}, \chi_2]]\bar{\mathcal{A}} \\
&\qquad \left. - \partial_{u_B}^{-1}[\bar{\mathcal{A}}, \partial_{u_B}^{-1}[\bar{\mathcal{A}}, [\partial_{u_B}^{-1}\bar{\mathcal{A}}, \chi_2]]]\bar{\mathcal{A}} - \partial_{u_B}^{-1}[\bar{\mathcal{A}}, [\partial_{u_B}^{-1}(u_B\bar{\mathcal{A}}), \partial_{\bar{z}}\chi_2]]\bar{\mathcal{A}}\right) \,.
\end{aligned}
\tag{4.84}
$$

4.3 Holography

In this section, we relate our previous considerations to the top-down holographic approaches of [49, 59]. The charges provided in [59] were written in the linearized case. We extend this analysis by providing the non-linear expressions.

We will be using the falloffs (4.25) in this section. The falloffs of the fields at $\mathscr{I}$ can be deduced from the falloffs of the fields on twistor space

$$
R_s \sim \mathcal{O}(u_B^s) \,, \qquad \tilde{R}_s \sim \mathcal{O}(u_B^s) \,. \tag{4.85}
$$

Note that the fields a, b in the BCT integrals are the fields in affine coordinates and are rescaled by an appropriate factor of λ_0 to make them weightless inside the integral formula. In this section we will also make use of the modified charge (4.14) to construct conserved currents at spatial infinity.

4.3.1 Large gauge transformations

The large gauge transformations in [49] are defined as gauge transformations on twistor space (4.2) that preserve the background, up to two points on the celestial sphere. In the linearized theory, this implies the following condition on the ξ

parameter:

$$\bar{\partial}\xi_s = 0 \qquad \text{on } \mathbb{PT} - \{z = 0, \infty\}\,, \tag{4.86}$$

i.e., the SD background is unchanged except for the singular points $\{z = 0, \infty\}$. Furthermore, the ϕ_s parameter is similarly given by

$$\bar{\partial}\phi_s = 0 \qquad \text{on } \mathbb{PT} - \{z = 0, \infty\}\,. \tag{4.87}$$

The solutions to these equations are given by

$$\begin{aligned} \xi^a_{k,l,m} &= (u - \bar{z}q)^k q^l z^m T^a = \frac{(\mu^{\dot{0}})^k (\mu^{\dot{1}})^l}{\lambda_0^{k+l+m} \lambda_1^{-m}} T^a\,, \\ \phi^a_{k,l,m} &= \frac{(u - \bar{z}q)^k q^l z^m T^a}{n^4} = \frac{(\mu^{\dot{0}})^k (\mu^{\dot{1}})^l}{\lambda_0^{k+l+m+4} \lambda_1^{-m}} T^a\,. \end{aligned} \tag{4.88}$$

In the non-linear theory, preserving the background yields

$$\bar{D}\xi_s = 0 \qquad , \qquad \bar{D}\phi_s = 0\,, \tag{4.89}$$

on $\mathbb{PT} - \{z = 0, \infty\}$. We can use the frame to solve these relations

$$\xi_s = f\tilde{\xi}_s f^{-1} \qquad , \qquad \phi_s = f\tilde{\phi}_s f^{-1}\,, \tag{4.90}$$

where $\tilde{\xi}_s = \tilde{\xi}_s(u_B, q, z, \bar{z})$ and $\tilde{\phi}_s = \tilde{\phi}_s(u_B, q, z, \bar{z})$ and satisfy

$$q\partial_{u_B}\tilde{\xi}_s + \partial_{\bar{z}}\tilde{\xi}_s + [\bar{\mathcal{A}}, \tilde{\xi}_s] = 0 \qquad , \qquad q\partial_{u_B}\tilde{\phi} + \partial_{\bar{z}}\tilde{\phi}_s + [\bar{\mathcal{A}}, \tilde{\phi}_s] = 0\,. \tag{4.91}$$

These can be solved using the linearised results along with a Wilson loop which satisfies

$$q\partial_{u_B}W + \partial_{\bar{z}}W + \bar{\mathcal{A}}W = 0 \qquad , \qquad q\partial_{u_B}W^{-1} + \partial_{\bar{z}}W^{-1} - W^{-1}\bar{\mathcal{A}} = 0\,. \tag{4.92}$$

We now solve (4.91) by dressing the linear results

$$\tilde{\xi}_{k,l} = (u_B - \bar{z}q)^k q^l W \rho(z) W^{-1} \qquad , \qquad \tilde{\phi}_{k,l} = (u_B - \bar{z}q)^k q^l W \chi(z) W^{-1} \,, \tag{4.93}$$

where $\rho(z), \chi(z) \in \mathfrak{g} \times \mathbb{C}[z, z^{-1}]$. Using the Cauchy kernel on $\mathbb{C}$, we can write the generators as an infinite sum of integrals

$$\tilde{\xi}_{k,l} = (u_B - \bar{z}q)^k q^l \left[\rho(z) + \sum_{n=1}^{\infty} (-1)^n \int_{\mathbb{C}^n} \frac{\mathrm{d}^2 z_n \cdots \mathrm{d}^2 z_1}{(z_n - z_{n-1}) \cdots (z_1 - z)} E_1 \cdots E_n \rho(z) \right] , \tag{4.94}$$

where we have introduced the operator $E_n = q\partial_{u_B} + [\bar{\mathcal{A}}(u, z_n, \bar{z}_n), \cdot\,]$. A similar expansion can be written for the $\tilde{\phi}_{k,l}$ generator.

Notice that, at a linearised level, the definition of the large gauge transformations (4.86) and (4.87) agrees with the one considered in Section 4.2.1. However, in the non-linear case, the realisation of the symmetries via (4.89) is different from the one presented in Section 4.2.1. Therefore, as we will see, the non-linear terms appearing in the associated charge expressions will differ accordingly.

4.3.2 Conserved holographic currents

Again, we will start with the linearised setting where the equations of motion for the a and b fields are given by

$$\bar{\partial} b = 0 = \bar{\partial} a \,, \tag{4.95}$$

and the symmetries of the fields are given by

$$\delta_\xi a = \bar{\partial} \xi \qquad , \qquad \tilde{\delta}_\phi b = \bar{\partial} \partial_{u_B} \phi \,. \tag{4.96}$$

In this case, the frame $f = 1$ and the linearised version of the recursion relations (4.67) and (4.68) are solved by

$$\begin{aligned} \partial_{u_B} R_s + \partial_{\bar{z}} R_{s-1} = 0 \qquad &\Rightarrow \qquad R_s = \frac{1}{2\pi i} \int_L q^{s+1} \mathrm{d}q \wedge b|_L \,, \\ \partial_{u_B} \tilde{R}_s + \partial_{\bar{z}} \tilde{R}_{s-1} = 0 \qquad &\Rightarrow \qquad \tilde{R}_s = \frac{1}{2\pi i} \int_L q^{s+1} \mathrm{d}q \wedge \partial_{u_B} a|_L \,. \end{aligned} \tag{4.97}$$

Consider the Cauchy slice $\Sigma = \{|z| = 1\}$, which allows u_B to be a complex coordinate and, upon going to $\partial\Sigma$, gives a contour integral. In the following, we consider the integrand of the contour integral in z. Now consider the charges [59]

$$\begin{aligned} J^a[k,l] &= \frac{1}{2\pi i} \oint_{u_B=\infty} \mathrm{d}u_B \int_L \mathrm{d}q \wedge q^l (u_B - \bar{z}q)^k b^a|_L \,, \\ \tilde{J}^a[k,l] &= \frac{1}{2\pi i} \oint_{u_B=\infty} \mathrm{d}u_B \int_L \mathrm{d}q \wedge q^l (u_B - \bar{z}q)^k \partial_{u_B} a^a|_L \,. \end{aligned} \tag{4.98}$$

These are related to the charge expressions (4.82) by considering their flux and taking contour integrals in u_B. Using the linearised BCT integral formulae (4.97), we obtain

$$\begin{aligned} J^a[k,l] &= \sum_{m=0}^{k} \frac{k!}{m!(k-m)!} \oint_{u_B=\infty} \mathrm{d}u_B \, u_B^{k-m} (-\bar{z})^m R^a_{l+m-1} \,, \\ \tilde{J}^a[k,l] &= \sum_{m=0}^{k} \frac{k!}{m!(k-m)!} \oint_{u_B=\infty} \mathrm{d}u_B \, u_B^{k-m} (-\bar{z})^m \tilde{R}^a_{l+m-1} \,. \end{aligned} \tag{4.99}$$

One can show that these currents are holomorphic when the linearised recursion relations are satisfied

$$\begin{aligned} \partial_{\bar{z}} J^a[k,l] &= \sum_{m=0}^{k} \frac{k!}{m!(k-m)!} \oint_{u_B=\infty} \mathrm{d}u_B \, u_B^{k-m} (-\bar{z})^m \left[\partial_{u_B} R^a_{l+m} + \partial_{\bar{z}} R^a_{l+m-1} \right] = 0 \,, \\ \partial_{\bar{z}} \tilde{J}^a[k,l] &= \sum_{m=0}^{k} \frac{k!}{m!(k-m)!} \oint_{u_B=\infty} \mathrm{d}u_B \, u_B^{k-m} (-\bar{z})^m \left[\partial_{u_B} \tilde{R}^a_{l+m} + \partial_{\bar{z}} \tilde{R}^a_{l+m-1} \right] = 0 \,. \end{aligned} \tag{4.100}$$

We now wish to consider the non-linear case. Consider the currents as in the

linear case but now dressed in the frames as in (4.93)

$$\begin{aligned} J[k,l] &= \oint_{u_B=\infty} \mathrm{d}u_B \int \mathrm{d}q \wedge \mathrm{Tr}\left(\tilde{\xi}_{k,l}\, f^{-1} b|_L f\right) , \\ \tilde{J}[k,l] &= \oint_{uB=\infty} \mathrm{d}u_B \int \mathrm{d}q \wedge \mathrm{Tr}\left(\tilde{\phi}_{k,l}\, f^{-1}\partial_{u_B} a|_L f\right) , \end{aligned} \tag{4.101}$$

where $\tilde{\xi}_{k,l} = \tilde{\xi}_{k,l}(u_B, q, z, \bar{z})$ and $\tilde{\phi}_{k,l} = \tilde{\phi}_{k,l}(u_B, q, z, \bar{z})$. For these currents to be holomorphic

$$\begin{aligned} \partial_{\bar{z}} J[k,l] &= \oint_{u_B=\infty} \mathrm{d}u_B \int \mathrm{d}q \wedge \mathrm{Tr}\left(\left(q\partial_{u_B}\tilde{\xi}_{k,l} + \partial_{\bar{z}}\tilde{\xi}_{k,l} + [\bar{\mathcal{A}}, \tilde{\xi}_{k,l}]\right)\, f^{-1} b|_L f\right) = 0 , \\ \partial_{\bar{z}} \tilde{J}[k,l] &= \oint_{u_B=\infty} \mathrm{d}u_B \int \mathrm{d}q \wedge \mathrm{Tr}\left(\left(q\partial_{u_B}\tilde{\phi}_{k,l} + \partial_{\bar{z}}\tilde{\phi}_{k,l} + [\bar{\mathcal{A}}, \tilde{\phi}_{k,l}]\right)\, f^{-1}\partial_{u_B} a|_L f\right) = 0 , \end{aligned} \tag{4.102}$$

which vanishes if (4.91) is satisfied. Just like in linear theory, the currents are holomorphic on the supports of the equations of motion. Since the equations of motion on twistor space give the recursion relations (4.67) and (4.68) on spacetime, the holographic currents are conserved if the recursion is satisfied. To see this more explicitly, we use the BCT integral formulae (4.69) to translate the currents to spacetime. The first few examples are given by

$$\begin{aligned} \tilde{J}[0,0] &= \oint_{u_B=\infty} \mathrm{d}u_B \mathrm{Tr}(\chi \tilde{R}_{-1}) , \\ \tilde{J}[0,1] &= \oint_{u_B=\infty} \mathrm{d}u_B \mathrm{Tr}\left(\chi\tilde{R}_0 - \sum_{n=0}^{\infty}(-1)^n [\partial_{u_B}^n \partial_{\bar{z}}^{-(n+1)}\bar{\mathcal{A}}, \chi]\tilde{R}_n\right) , \\ \tilde{J}[1,0] &= \oint_{u_B=\infty} \mathrm{d}u_B \mathrm{Tr}\left[u_B\left(\chi\tilde{R}_{-1} - \sum_{n=0}^{\infty}(-1)^n [\partial_{u_B}^n \partial_{\bar{z}}^{-(n+1)}\bar{\mathcal{A}}, \chi]\tilde{R}_{n-1}\right)\right. \\ &\qquad \left. -\bar{z}\left(\chi\tilde{R}_0 - \sum_{n=0}^{\infty}(-1)^n [\partial_{u_B}^n \partial_{\bar{z}}^{-(n+1)}\bar{\mathcal{A}}, \chi]\tilde{R}_n\right)\right] , \end{aligned} \tag{4.103}$$

where we have defined

$$\partial_{\bar{z}}^{-n}\bar{\mathcal{A}}(u_B, z, \bar{z}) = \int_{\mathbb{C}^n} \frac{\mathrm{d}^2 z_n \cdots \mathrm{d}^2 z_1}{(z_n - z_{n-1})\cdots(z_1 - z)} \bar{\mathcal{A}}(u_B, z_n, \bar{z}_n) , \tag{4.104}$$

which satisfies $\partial_{\bar{z}}\partial_{\bar{z}}^{-n}\bar{\mathcal{A}} = \partial_{\bar{z}}^{-(n-1)}\bar{\mathcal{A}}$. Similar expressions hold for the $J[k,l]$ but involve the R_n. We can show that these currents are conserved by using recursion relations and falloffs (4.85).

Using the symplectic structure, we find that the algebra satisfied by the $J, \tilde{J}$ is given by

$$\begin{aligned}
\{J[k,l][\rho], J[k',l'][\rho']\} &= J[k+k', l+l'][[\rho,\rho']] \\
\{J[k,l][\rho], \tilde{J}[k',l'][\chi']\} &= \tilde{J}[k+k', l+l'][[\rho,\chi']] \\
\{\tilde{J}[k,l][\chi], \tilde{J}[k',l'][\chi']\} &= 0
\end{aligned} \tag{4.105}$$

We conclude this chapter by noting that we have presented these results in the context of null infinity using Bondi coordinates. These are not the only coordinates one can choose for the asymptotic twistor space; in fact, to make contact with Burns holography [59] and Koszul duality [56, 54] one needs to work in different coordinates where the results are more illuminating [157].

Chapter 5

Gravitational Charges in the Bulk

In this section, we introduce Penrose's quasi-local mass [158] and its Hamiltonian origins [159], which will serve as the basis of our discussion. We extend Penrose's quasi-local mass definition to include higher-spin charges associated with the celestial $Lw_{1+\infty}$ symmetries and present formulae that provide explicit expressions that can be computed on finite 2-surfaces, given a choice of null hypersurface. We also present a first-principles phase-space derivation of these charges, starting from a twistor space action for self-dual gravity that can be identified with the standard gravitational asymptotic phase space at null infinity, performing a large gauge transformation analysis and using the Penrose transform to connect with the corresponding spacetime expressions. We also formulate the spacetime analysis in the Plebanski gauge and relate the celestial symmetries to the integrability of self-dual gravity in the case of a self-dual background. Finally, in the linearized theory, using [118], we relate our quasi-local charge expressions to the Geroch–Hansen definition of multipoles [160, 161, 162], thereby providing a concrete physical interpretation of these celestial charges.

5.1 Penrose's quasi-local mass formula

Due to the equivalence principle, there is no well-defined local gravitational energy-momentum density in general relativity. This motivates assigning energy–momentum to a finite region, or equivalently to a closed spacelike two-surface bounding it. Numerous quasi-local constructions have been proposed, including the Hawking mass [163], Brown–York energy [164], and the Liu–Yau mass [165]; a comprehensive review may be found in [166]. In this work, we focus on Penrose's twistor-based construction [158]. Unlike definitions that assign only a scalar mass, Penrose's construction associates a kinematic twistor with the two-surface, encoding quasi-local energy–momentum and angular momentum. It is particularly natural for our purposes because it is formulated using twistor geometry and is therefore closely connected with twistor-action living on self-dual backgrounds.

Penrose defines the quasi-local mass associated with a spacetime region bounded by the compact space-like 2-surface $\mathcal{S}$ to be [158, 167, 168, 83]:

$$H[f] = \oint_{\mathcal{S}} *R_{ab} f^{ab}. \tag{5.1}$$

Here $*$ is the Hodge star, and R_{ab} is the Riemann curvature 2-form, which can be expanded in a co-tetrad basis θ^a and further decomposed into anti-self-dual (ASD) and self-dual (SD) parts

$$R_{ab} := R_{abcd}\, \theta^c \wedge \theta^d = R_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} + \tilde R_{\dot\alpha\dot\beta}\epsilon_{\alpha\beta}. \tag{5.2}$$

The charge $H[f]$ is conserved if the integrated 2-form is closed

$$d\left(*R_{ab} f^{ab}\right) = 0\,. \tag{5.3}$$

This occurs when the spacetime satisfies the vacuum Einstein equations and f^{ab}

satisfies the conformal Killing-Yano equation

$$\nabla_a f_{bc} = \nabla_{[a} f_{bc]} - \frac{2}{3} g_{a[b} \nabla^d f_{c]d}. \tag{5.4}$$

In terms of spinors, we can decompose f^{ab} as a sum of an ASD and SD part

$$f^{ab} = \xi^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} + \tilde{\xi}^{\dot{\alpha}\dot{\beta}} \epsilon^{\alpha\beta}\,, \tag{5.5}$$

using the antisymmetry of f^{ab}. With Minkowski reality conditions, $\tilde{\bar{\xi}}_{\dot{\alpha}\dot{\beta}} = \bar{\xi}_{\dot{\alpha}\dot{\beta}}$, but for now, we keep the discussion complex. The equation (5.4) on f^{ab} becomes the valence-2 twistor equations on $\xi^{\alpha\beta}$ and $\tilde{\xi}^{\dot{\alpha}\dot{\beta}}$

$$\nabla^{(\alpha}_{\ \dot{\alpha}} \xi^{\beta\gamma)} = 0, \qquad \nabla_{\alpha}^{\ (\dot{\alpha}} \tilde{\xi}^{\dot{\beta}\dot{\gamma})} = 0. \tag{5.6}$$

We can ignore the $\tilde{\xi}^{\dot{\alpha}\dot{\beta}}$ term and take the real part later if desired.[1] If we decompose the Riemann curvature 2-form using spinors and assume Ricci flatness, we can write

$$R_{\alpha\beta} = \psi_{\alpha\beta\gamma\delta} \Sigma^{\gamma\delta}, \qquad \Sigma^{\alpha\beta} := \theta^{(\alpha}_{\dot{\alpha}} \wedge \theta^{\beta)\dot{\alpha}},$$

in terms of the ASD Weyl spinor $\psi_{\alpha\beta\gamma\delta}$ which satisfies the vacuum Bianchi identity

$$\nabla^{\dot{\alpha}\alpha} \psi_{\alpha\beta\gamma\delta} = 0. \tag{5.7}$$

The charge (5.1) becomes the real part of

$$H[\xi] = \oint_{\mathcal{S}} \psi_{\alpha\beta\gamma\delta} \xi^{\alpha\beta} \Sigma^{\gamma\delta}\,, \tag{5.8}$$

which can now be seen to be conserved as a consequence of (5.6) and (5.7). Such conserved charges correspond to solutions to the twistor equation. On a general

[1] In fact, the imaginary part will often automatically vanish [115] chapter 6.

curved background, solutions $\xi^{\alpha\beta}$ and, in turn, f^{ab} do not exist in much the same way that Killing vectors generally do not exist. Indeed, given a solution to the Killing-Yano equation, one can construct a complex Killing vector K^a by defining

$$K^{\dot{\alpha}(\alpha}\epsilon^{\beta)\gamma} = \nabla^{\dot{\alpha}\gamma}\xi^{\alpha\beta}\,, \qquad \Im K_a := \epsilon_{abcd}\nabla^b f^{cd}\,, \qquad \Re K^a = \nabla_b f^{ab}\,, \tag{5.9}$$

and a calculation shows that K_a satisfies Killing's equations as a consequence of (5.6), see Section 6.4 of [83]. In the linearized case, there are, in general, 20 solutions to (5.4), while only half of these give non-vanishing Killing vectors that generate the real Poincaré group.

Following [158], any solution to (5.6) can be decomposed as a symmetric product of solutions to the valence 1 twistor equation

$$\nabla^{(\alpha}_{\dot{\alpha}}\xi^{\beta)} = 0 \quad \text{with} \quad \xi^\alpha = \xi_0\,\iota^\alpha + \xi_1 o^\alpha. \tag{5.10}$$

There are six equations for only two functions ξ_0 and ξ_1, resulting in an overdetermined system. We can consider instead the twistor equation involving only derivatives tangential to $\mathcal{S}$ given by

$$\eth_{\mathcal{C}}\xi_1 + \xi_0\lambda = 0, \qquad \bar{\eth}_{\mathcal{C}}\xi_0 - \xi_1\sigma = 0. \tag{5.11}$$

Using the Atiyah-Singer index theorem, if $\mathcal{S}$ is topologically a sphere, there exist at least four independent solutions and generically just four. Having established existence, the space of all solutions to these equations is called the 2-surface twistor space. However, due to the curvature of spacetime, the 2-surface twistor equations are incompatible with the rest of the valence one twistor equation on a general background, resulting in the charges no longer being conserved. Non-conserved charges are a general feature of curved spacetimes. In fact, at null infinity, the non-conservation of BMS charges is a result of gravitational radiation.

The alternative to the definition (5.11) is that exploited in [169, 170]. This takes holomorphic or anti-holomorphic local twistors on the 2-surface. In vacuum, the holomorphic case leads to the equations

$$\bar{\eth}\xi^{\alpha} + io^{\alpha}\iota^{\dot{\alpha}}\chi_{\dot{\alpha}} = 0\,, \qquad \bar{\eth}\chi_{\dot{\alpha}} = 0\,. \tag{5.12}$$

This definition yields the Bondi definitions on large 2-surfaces going out to future null infinity. The anti-holomorphic version diverges in this limit when there is radiation, but yields the Bondi definitions as spheres are taken out to $\mathscr{I}^{-}$. Both definitions lead to well-defined mass, energy-momentum and angular momentum in general, with a positivity theorem for the energy. These issues remain open for the Penrose definition.

Putting these ingredients together, the quasi-local charge becomes

$$H[\xi] = \oint_{\mathcal{S}} R_{\alpha\beta}\xi^{\alpha}\xi'^{\beta} = \oint_{\mathcal{S}} dS\left(\psi_3\xi_0\xi_0' - \psi_2\left(\xi_0\xi_1' + \xi_1\xi_0'\right) + \psi_1\xi_1\xi_1'\right)\,, \tag{5.13}$$

where dS is the volume form on the 2-surface $\mathcal{S}$, and $\xi^{\alpha}, \xi'^{\alpha}$ are generally two different solutions to the 2-surface twistor equations. At this stage, we are left with a complex charge and need to identify the mass and angular momentum. To proceed, one introduces an inner product on the 2-surface twistor space and requires it to be constant in order to select a suitable reality structure. Although such an inner product is not constant on general curved backgrounds, Penrose's definition of quasi-local mass nevertheless succeeds in capturing mass and energy appropriately in several important examples [171, 172, 173], suggesting that it contains essential physical insight, even if further refinements are needed in the general case.

To relate this definition to Hamiltonian and Noether arguments, first note that the twistor equation (5.10) allows us to define $\chi_{\dot{\alpha}}$ via the covariant exterior derivative

$$D\xi^{\alpha} := \theta^{\beta\dot{\alpha}}\nabla_{\beta\dot{\alpha}}\xi^{\alpha} = \theta^{\beta\dot{\alpha}}\chi_{\dot{\alpha}}\delta_{\beta}{}^{\alpha} = \chi_{\dot{\alpha}}\theta^{\alpha\dot{\alpha}}\,. \tag{5.14}$$

With this, one can construct what in flat space would be real Killing vectors

$$K^{\alpha\dot{\alpha}} = \mathrm{Re}\left(\xi^{\alpha}\chi^{\dot{\alpha}}\right) , \qquad \tilde{K}^{\alpha\dot{\alpha}} = \mathrm{Im}\left(\xi^{\alpha}\chi^{\dot{\alpha}}\right) \ . \tag{5.15}$$

Defining $\chi_{\dot{\alpha}}$ on a 2-surface by the components of (5.14) tangent to the 2-surface, one can use the curvature identity $D^2\xi^{\alpha} = R^{\alpha}_{\ \beta}\xi^{\beta}$ for covariant exterior derivative D to rewrite the charge with $\xi'^{\alpha} = \xi^{\alpha}$ for simplicity as

$$H[\xi] = \oint_S \xi_{\alpha} D^2\xi^{\alpha} = \oint_S \chi_{\dot{\alpha}}\theta^{\alpha\dot{\alpha}} \wedge D\xi_{\alpha} \, , \tag{5.16}$$

where we have used (5.14) restricted to the 2-surface. The right-hand side is the Witten-Nestor 2-form used in the Witten positive energy proof [174, 83]. It is the boundary term for the total Hamiltonian generating diffeomorphisms in the direction K^a, here used quasi-locally rather than asymptotically, see [159, 175] for more details. This is automatically real for real K^a. Thus the issue of too many charges and identifying which charges should correspond to momenta, or angular momenta, reduces to the issues of constructing real K^a and identifying their nature as putative translations or rotations. In general, for 2-surface twistors this issue has not been resolved without assumptions on the space-time and/or 2-surface. However, more progress can be made using local twistors as in [169, 170] where closely related definitions are provided based on (anti-) holomorphic local twistors and these definitions lead to definitions of real 4-momenta with proofs of positivity of energy and so on.

5.2 Higher-spin extension

We now introduce a higher-spin generalization of Penrose's quasi-local mass formula. The form of (5.8) motivates a higher-spin generalization

$$H_s = \oint_{\mathcal{S}} \phi_{\alpha_1\cdots\alpha_{s+1}\beta\gamma}\xi^{\alpha_1\cdots\alpha_{s+1}}\Sigma^{\beta\gamma}. \tag{5.17}$$

Here $\phi_{\alpha_1\cdots\alpha_{s+3}}$ is a valence $s+3$ zero-rest-mass (z.r.m.) field required to satisfy

$$\nabla^{\alpha_1}_{\dot\alpha}\phi_{\alpha_1\cdots\alpha_{s+1}\beta\gamma}=0, \tag{5.18}$$

which is the higher-spin generalization of (5.7). In addition, conservation of the charge H_s would require $\xi^{\alpha_1\cdots\alpha_{s+1}}$ to be a valence $s+1$ twistor satisfying

$$\nabla^{(\alpha_0}_{\dot\alpha}\xi^{\alpha_1\cdots\alpha_{s+1})}=0, \tag{5.19}$$

hence generalizing the first of (5.6). With these equations satisfied, the charge H_s is easily seen to be conserved and independent of the 2-surface $\mathcal{S}$ as

$$\nabla^{\beta}_{\dot\beta}(\phi_{\alpha_1\cdots\alpha_{s+1}\beta\gamma}\xi^{\alpha_1\cdots\alpha_{s+1}})=0\,, \tag{5.20}$$

follows directly from these equations. Thus these charges are exactly conserved. On the one hand, this formula indicates that charges of higher spin fields should be thought of as dual to higher valence Killing spinors. On the other, given a procedure to generate a higher spin field from the gravitational data, this will be shown to be related to multipole formulae and the $Lw_{1+\infty}$ charges in Section 5.6 and Section 5.2.1.

5.2.1 Twistor action and charges

For convenience, we reproduce the action describing SD gravity with an ASD perturbation [101]

$$S[g,h]=\int_{\mathbb{PT}}D^3Z\wedge g\wedge\left(\bar\partial h+\frac{1}{2}\{h,h\}\right). \tag{5.21}$$

The equations of motion for this action are given by

$$\bar\partial h+\frac{1}{2}\{h,h\}=0 \qquad\qquad \bar\partial g+\{h,g\}=0. \tag{5.22}$$

The first is the integrability condition for h to be the Hamiltonian generating complex structure deformations of $\mathbb{PT}$. The second states that g is $\bar{\partial}$-closed $(0,1)$-form for the deformed complex structure. Along with the equations of motion, the symmetries of this action are given by

$$\delta_\xi h = \bar{\nabla}\xi, \qquad \delta_{\xi,\chi} g = \{g,\xi\} + \bar{\nabla}\chi. \tag{5.23}$$

The ξ symmetry describes symplectomorphisms on twistor space, while the χ symmetry makes g valued in Dolbeault cohomology.

Using standard phase-space methods, e.g. [137, 18, 138, 176], the twistor action (5.21) yields the symplectic potential

$$\theta[\delta] = \Omega \wedge \delta h \wedge g\,, \tag{5.24}$$

where Ω is the holomorphic volume form on twistor space in the deformed complex structure. The symplectic current arises by taking a further variation to give

$$\begin{aligned} \omega[\delta_1,\delta_2] &= \delta_1\theta[\delta_2] - \delta_2\theta[\delta_1] \\ &= \delta_1\Omega \wedge \delta_2 h \wedge g + \Omega \wedge \delta_2 h \wedge \delta_1 g - (1 \leftrightarrow 2). \end{aligned} \tag{5.25}$$

On shell, the symmetries contracted into the symplectic current give exact terms

$$\begin{aligned} \delta_\chi \lrcorner\, \omega &= d\left(\Omega \wedge \chi \delta h\right), \\ \delta_\xi \lrcorner\, \omega &= d\left(\Omega \wedge \xi \delta g - D\lambda \wedge \theta^{\dot\alpha} \wedge \mathcal{L}_{\partial_{\dot\alpha}}(\xi \delta h) \wedge g\right). \end{aligned} \tag{5.26}$$

We now integrate over a co-dimension 1 hypersurface Σ on twistor space, resulting in charge expressions that are integrals over the boundary of Σ. In general, the contraction of the symmetries will not give a δ-exact term but will split into an

integrable and non-integrable part

$$\delta_\alpha \lrcorner \int_\Sigma \omega = \delta H_\alpha - \Xi_\alpha[\delta]\,, \tag{5.27}$$

where $\alpha = (\xi, \chi)$ and the charges decompose into two parts $H_\alpha = (H_\xi, \tilde{H}_\chi)$. The integrable part gives the charges for each symmetry as integrals over $\partial\Sigma$

$$H_\xi = \frac{1}{2\pi i}\int_{\partial\Sigma} \Omega \wedge \xi g, \qquad H_\chi = \frac{1}{2\pi i}\int_{\partial\Sigma} \Omega \wedge \chi h. \tag{5.28}$$

The non-integrable part of the charges is given by

$$\begin{aligned} \Xi_\xi[\delta] &= \int_{\partial\Sigma} \delta\Omega \wedge \xi g + \Omega \wedge \delta\xi g + D\lambda \wedge \theta^{\dot\alpha} \wedge \mathcal{L}_{\partial_{\dot\alpha}}(\xi\delta h) \wedge g, \\ \Xi_\chi[\delta] &= \int_{\partial\Sigma} \delta\Omega \wedge \chi h + \Omega \wedge \delta\chi h. \end{aligned} \tag{5.29}$$

From these expressions, we observe that the non-integrability of charges comes not only from the possibility of the gauge parameters being field dependent, but also because we work on a curved spacetime background. This is in contrast to what occurs in the gauge theory case, where only the variation of the gauge parameters appears [157].

In the presence of non-integrable charges, a new charge bracket must be defined [19] that utilises the non-integrable part

$$\{H_\alpha, H_{\alpha'}\}_\star = \delta_\alpha H_{\alpha'} - \Xi_\alpha[\delta_{\alpha'}]. \tag{5.30}$$

The charge algebra is given by

$$\begin{aligned} \{H_\xi, H_{\xi'}\} &= H_{\{\xi,\xi'\}_*}, \\ \{H_\xi, \tilde{H}_\chi\} &= -\{\tilde{H}_\chi, H_\xi\} = \tilde{H}_{\{\xi,\chi\}_*} + K_{\xi,\chi}, \\ \{\tilde{H}_\chi, \tilde{H}_{\chi'}\} &= 0, \end{aligned} \tag{5.31}$$

where the modified bracket for the gauge parameters is given by [156, 126]

$$\begin{aligned}\{\xi,\xi'\}_* &= \{\xi,\xi'\} + \delta_\xi\xi' - \delta_{\xi'}\xi,\\ \{\xi,\chi\}_* &= \{\xi,\chi\} + \delta_\xi\chi\,.\end{aligned} \tag{5.32}$$

The charge algebra picks up a field-dependent cocycle given by

$$K_{\xi,\chi} = \int_{\partial\Sigma} \Omega\wedge\chi\bar\partial\xi + D\lambda\wedge\theta^{\dot\alpha}\mathcal{L}_{\partial_{\dot\alpha}}\xi\wedge\bar\nabla(\chi h). \tag{5.33}$$

5.2.2 Large gauge transformations

We will now choose a gauge in which to work, such that the SD perturbations, when restricted to a holomorphic curve $\delta h|_X = \bar\partial|_X j$ satisfy

$$L_{\dot\alpha} j = \langle \iota\lambda\rangle^3\varphi_{\dot\alpha}. \tag{5.34}$$

This gauge choice uses the redundancy for the potential for the metric perturbation on spacetime $\varphi_{\alpha\beta\gamma\dot\alpha} = \iota_\alpha\iota_\beta\iota_\gamma\varphi_{\dot\alpha}$. From the action, the variation of the background h is given explicitly by

$$\delta h|_X = \bar\nabla\xi|_X = \bar\partial|_X\xi|_X. \tag{5.35}$$

Non-trivial SD perturbations correspond to ξ which are not global on twistor space. The ξ that generate such SD perturbations were considered in [47] in the framework of Čech cohomology. On twistor space, these generators are explicitly given by

$$\check\xi_s = \sum_{|m|\le p-1}\sum_{a\in\mathbb{Z}} \tau^p_{m,a} w^p_{m,a}, \qquad w^p_{m,a} = \frac{\left(\mu^{\dot 0}\right)^{p+m-1}\left(\mu^{\dot 1}\right)^{p-m-1}}{\lambda_0^{2p-4-a}\lambda_1^a}\,, \tag{5.36}$$

where $s = 2p-3$. The generators $w^p_{m,a}$ are holomorphic on the patch of twistor space where $\lambda_1\neq 0$ and $\lambda_0\neq 0$ and they satisfy the algebra $LHam(\mathbb{C}^2)$ (also referred to as

$Lw_{1+\infty}$, see Footnote 1), given explicitly as

$$\{w^p_{m,a}, w^q_{n,b}\} = 2(m(q-1) - n(p-1))w^{p+q-2}_{m+n,a+b}. \tag{5.37}$$

As we are working in a Dolbeault framework, we wish to translate the holomorphic sections of $\mathcal{O}(2)$ $\check{\xi}_s$ to their smooth counterparts ξ_s also of weight two. To achieve this, we utilize bump functions. On $\mathbb{CP}^1$, there exists an open cover $\{U_+, U_-\}$ where $U_- = \{\lambda_0 \neq 0\}$ and $U_+ = \{\lambda_1 \neq 0\}$ describe the two hemispheres. We can construct a partition of unity $\{f_-, f_+\}$ where $f_\pm$ are functions having compact support in $U_\pm$ and satisfy $f_- + f_+ = 1$. When we pull back $\check{\xi}_s$ to the spin bundle, in general, it will consist of two terms

$$\check{\xi}_s|_X = \check{\xi}^+_s - \check{\xi}^-_s \,, \tag{5.38}$$

where $\check{\xi}^-_s$ carries singular terms in λ_0 and $\check{\xi}^+_s$ carries singular terms in λ_1. In principle, one can also have terms that are regular everywhere and hence holomorphic. Where one puts these regular terms will determine which gauge one works in for the $\varphi_{\alpha\beta\gamma\dot{\alpha}}$. The smooth function $\xi_s|_X$ is defined as

$$\xi_s|_X = \check{\xi}^+_s f_+ + \check{\xi}^-_s f_-. \tag{5.39}$$

Therefore, the perturbation to h is given by

$$\delta h|_X = \bar{\partial}|_X \left(\check{\xi}^+_s f_+ + \check{\xi}^-_s f_-\right). \tag{5.40}$$

Now applying the Lax operator on $\xi_s|_X$ and using the fact that $L_{\dot{\alpha}}\check{\xi}_s|_X = 0$, we find

$$L_{\dot{\alpha}}\xi_s|_X = L_{\dot{\alpha}}\check{\xi}^-_s = \langle \iota\lambda\rangle^3 \varphi_{\dot{\alpha}}. \tag{5.41}$$

We therefore work in a patch where $\lambda_\alpha \neq \iota_\alpha$, i.e. $\lambda_0 \neq 0$, and allow for arbitrarily high poles when $\lambda_\alpha = \iota_\alpha$. We also place all the regular parts in $\check{\xi}^-_s$ which explicitly

can be expand as

$$\check{\xi}_s^- = \frac{\xi^{\alpha_1\cdots\alpha_{s+1}}\lambda_{\alpha_1}\cdots\lambda_{\alpha_{s+1}}}{\langle\iota\lambda\rangle^{s-1}}\,, \tag{5.42}$$

where, for the moment, we ignore the loop part of the algebra. We note that we retain even the non-singular terms for values of $s = -1, 0, 1$, as these can be understood on space-time as diffeomorphisms preserving the gauge choice of the metric (see Section 5.4). In other words, the resulting variations of the metric end up being local while the singular gauge parameters give non-local variations. The SD perturbation is given by

$$L_{\dot\alpha}\check{\xi}_s^- = \frac{\lambda_{\alpha_0}\cdots\lambda_{\alpha_{s+1}}\nabla^{\alpha_0}_{\ \dot\alpha}\xi^{\alpha_1\cdots\alpha_{s+1}}}{\langle\iota\lambda\rangle^{s-1}}. \tag{5.43}$$

To preserve the gauge choice (5.34), the field $\xi^{\alpha_1\cdots\alpha_{s+1}}$ must satisfy

$$\iota_{\alpha_0}\nabla^{(\alpha_0}_{\ \dot\alpha}\xi^{\alpha_1\cdots\alpha_{s+1})} = 0, \tag{5.44}$$

which results in the expression for the potential for the SD metric perturbation to be

$$\varphi_{\dot\alpha} = o_{\alpha_0}\cdots o_{\alpha_{s+1}}\nabla^{\alpha_0}_{\ \dot\alpha}\xi^{\alpha_1\cdots\alpha_{s+1}}. \tag{5.45}$$

In the GHP formalism, the equations (5.44) can be decomposed into two towers of equations called dual recursion relations

$$\begin{aligned}
&\text{þ}'_{\mathcal{C}}\xi_{s,m} - \bar{\eth}_{\mathcal{C}}\xi_{s,m+1} + (m+2)\sigma\xi_{s,m+2} + (3-m)\nu\xi_{s,m-1} = 0,\\
&\eth_{\mathcal{C}}\xi_{s,m} - \text{þ}_{\mathcal{C}}\xi_{s,m+1} + (m+2)\kappa\xi_{s,m+2} + (3-m)\lambda\xi_{s,m-1} = 0,
\end{aligned} \tag{5.46}$$

where the index $m = 0, \dots, s+1$[2]. The components appearing in the dual recursion relations are components of $\xi^{\alpha_1 \cdots \alpha_{s+1}}$ explicitly given by

$$\xi^{\alpha_1 \cdots \alpha_{s+1}} = \sum_{m=0}^{s+1} \xi_{s,m} o^{(\alpha_1} \cdots o^{\alpha_m} \iota^{\alpha_{m+1}} \cdots \iota^{\alpha_{s+1})}. \tag{5.47}$$

The two equations that appear in the SD perturbation $\varphi_{\dot{\alpha}}$ are given by

$$\varphi_{\dot{\alpha}} = \bar{o}_{\dot{\alpha}} \left(\bar{\eth}_{\mathcal{C}} \xi_{s,0} - \sigma \xi_{s,1} \right) - \bar{\iota}_{\dot{\alpha}} \left(\text{þ}_{\mathcal{C}} \xi_{s,0} - \kappa \xi_{s,1} \right) . \tag{5.48}$$

As discussed in Section 5.2.1, the potential $\varphi_{\alpha\beta\gamma\dot{\alpha}}$ generates a SD perturbation to the metric if (2.151) is satisfied. To check this, denote $\varphi_{\dot{\alpha}} = \varphi_0 \bar{o}_{\dot{\alpha}} - \varphi_1 \bar{\iota}_{\dot{\alpha}}$, after which (2.151) gives

$$\begin{gathered} \nu\varphi_1 - \lambda\varphi_0 = 0, \qquad \bar{\eth}\varphi_1 + \bar{\rho}\varphi_0 - \text{þ}\varphi_0 + \bar{\pi}\varphi_1 = 0, \\ \text{þ}'\varphi_1 + \bar{\tau}\varphi_0 - \eth\varphi_0 + \bar{\mu}\varphi_1 + 3\pi\varphi_0 - 3\mu\varphi_1 = 0. \end{gathered} \tag{5.49}$$

The first equation requires us to set $\lambda = 0 = \nu$, which is a consequence of choosing to work in the gauge $\varphi_{\alpha\beta\gamma\dot{\alpha}} = \iota_\alpha \iota_\beta \iota_\gamma \varphi_{\dot{\alpha}}$. This means that the intrinsic shear λ of the null hypersurface generated by n vanishes and, as such, is foliated by α-planes. Furthermore, $\nu = 0$ in the real Minkowski setting results in the fact that n is geodesic and as such generates a shear-free null geodesic congruence. In more general signatures this needs to be supplemented by $\bar{\nu} = 0$. The last two equations are identically zero when the explicit expressions for φ_0, φ_1 appearing in (5.48) are used, assuming the space-time is SD, the vanishing of the spin coefficients λ, ν, and the $\xi_{s,m}$ satisfy the dual recursion relations (5.46). The fact that the $\xi_{s,m}$ satisfy (2.151) requires the use of various spin coefficient identities and commutators of GHP operators, see Appendix A, making the calculation non-trivial. This calculation becomes significantly simpler

[2]We note that the covariant derivative appearing in (5.43) include the fact that the spin field $\xi^{\alpha_1 \cdots \alpha_{s+1}}$ has a GHP weight $(1-s, 0)$ since the twistor function ξ_s has no weight. This results in $\xi_{s,m}$ having weight $(2-2m, 0)$ and conformal weigth $W = m$.

when a special gauge is used; see Section 5.4. The general metric perturbation can be written explicitly as

$$\begin{aligned}\delta g_{\alpha\dot{\alpha}\beta\dot{\beta}} = \iota_\alpha \iota_\beta \Big[\bar{o}_{\dot{\alpha}} \bar{o}_{\dot{\beta}} \left(\bar{\nu}\varphi_1 - \text{Þ}'\varphi_0 - \mu\varphi_0 \right) + \bar{\iota}_{(\dot{\alpha}} \bar{o}_{\dot{\beta})} \left(\eth\varphi_0 + 2\mu\varphi_1 - \bar{\mu}\varphi_1 - \pi\varphi_0 \right) \\ - \iota_{\dot{\alpha}} \iota_{\dot{\beta}} \left(\eth\varphi_1 + \pi\varphi_1 + \bar{\sigma}\varphi_0 \right) \Big] \, .\end{aligned} \tag{5.50}$$

The loop part of the algebra only changes the number of spinor indices. If we pick out a mode from the loop, we obtain

$$\check{\xi}^{-}_{s,a} = \frac{\xi^{\alpha_1 \cdots \alpha_{s+a+1}} \lambda_{\alpha_1} \cdots \lambda_{\alpha_{s+a+1}}}{\langle \iota \lambda \rangle^{s+a-1}}. \tag{5.51}$$

For values of $a < -s-1$, the field will vanish and hence have no action on spacetime. While the loop affects the spin indices for a given value of s, it does not change the growth of the field as

$$\xi^{\alpha_1 \cdots \alpha_{s+a+1}} \sim O\left(x^{s+1}\right). \tag{5.52}$$

Therefore, the loop in practice labels how many more times one must solve the dual recursion relations (5.46) while keeping the same seed solution with boundary conditions labeled by s. In what follows, we will suppress the loop part of the algebra, as many of the considerations follow along similar lines. See Appendix C of [177] for further discussion of the asymptotic growth of the SD perturbation in linear theory.

5.3 Quasi-local higher-spin charges

In this section, we translate the above spacetime results using Penrose's transform. We recover the general covariant higher-spin expression heuristically constructed in Section 5.1, hence providing a first-principles derivation of the quasi-local celestial charges. We also discuss the charges associated with the other graviton helicity. We then specialize the discussion to null infinity to connect with [61, 62, 133], and to a finite-distance null hypersurface to connect with [178].

5.3.1 Covariant spacetime expressions

To begin, we can translate the charge containing the ASD perturbation H_s to spacetime by defining a higher-spin field, as the higher $Lw_{1+\infty}$ generators act as spin-raising operators. This requires extra boundary conditions to be imposed on g so as to give a well-defined integral formula. The Penrose transform for this field is given by

$$\phi_{\alpha_1\cdots\alpha_{s+3}} = \frac{1}{2\pi i}\int_X \frac{D\lambda}{\langle \iota\lambda\rangle^{s-1}} \wedge \lambda_{\alpha_1}\cdots\lambda_{\alpha_{s+3}} g|_X. \tag{5.53}$$

To make the projective weights work[3], we have had to introduce a singularity of order $s-1$ when $\lambda_\alpha = \iota_\alpha$. As a consequence, g must vanish to sufficiently high order at $\lambda_\alpha = \iota_\alpha$ to make the integral well-defined. Therefore, to capture arbitrarily large values of s, one can use Schwartzian falloffs to g at this point. By considering the components of this z.r.m field, one finds

$$Q_n = \frac{1}{2\pi i}\int_X q^{n+2} dq \wedge g'|_X, \qquad q = \frac{\langle o\lambda\rangle}{\langle \iota\lambda\rangle}, \tag{5.55}$$

where $g' = \langle \iota\lambda\rangle^6 g$. As we will see in Section 7, the components Q_n of the higher valence z.r.m. field are built from the repeated action of the recursion operator. The exact relationship between the Q_n and contractions of the z.r.m. field is given by

$$Q_n = \phi_{\alpha_1\cdots\alpha_{s+3}} o^{\alpha_1}\cdots o^{\alpha_{n+2}} \iota^{\alpha_{n+3}}\cdots\iota^{\alpha_{s+3}}. \tag{5.56}$$

In the GHP formalism, the z.r.m equations can be decomposed into equations living on a null hypersurface $\mathcal{N}$ and equations propagating the field away from the null

[3]We note that by requiring the projective weight to work, the field $\phi_{\alpha_1\cdots\alpha_{s+3}}$ obtains a GHP weight of $(s-1,0)$ while the components Q_n have a GHP weight of $(2n,0)$ and conformal weight $W=-n-3$. Furthermore, due to the shift in weights, the z.r.m. equation changes to

$$\nabla^{\alpha_1}_{\dot\alpha}\phi_{\alpha_1\cdots\alpha_{s+3}} + (s-1)\Gamma_{\alpha_0\dot\alpha\alpha_1\beta}\,\iota^\beta \phi^{\alpha_0\alpha_1}{}_{\alpha_2\cdots\alpha_{s+3}} = 0. \tag{5.54}$$

hypersurface. These are given respectively by

$$\begin{aligned}\text{þ}'_{\mathcal{C}}Q_m - \bar{\eth}_{\mathcal{C}}Q_{m-1} &= (2-m)\nu Q_{m+1} + (m+1)\sigma Q_{m-2},\\ \eth_{\mathcal{C}}Q_m - \text{þ}_{\mathcal{C}}Q_{m-1} &= (2-m)\lambda Q_{m+1} + (m+1)\kappa Q_{m-2}.\end{aligned} \tag{5.57}$$

To translate the charge H_s to spacetime, we use the pullback of the map $p : \mathbb{PS} \to \mathcal{PT}$ from the spin bundle to twistor space and apply all the definitions

$$\begin{aligned}H_s &= \frac{1}{2\pi i}\int_{\partial\Sigma} \Omega_Z \wedge \xi_s g\\ &= \frac{1}{2\pi i}\int_{\mathbb{CP}^1\times\mathcal{S}} D\lambda \wedge \Sigma^{\alpha\beta} \wedge \lambda_\alpha\lambda_\beta \frac{\xi^{\alpha_1\cdots\alpha_{s+1}}\lambda_{\alpha_1}\cdots\lambda_{\alpha_{s+1}}}{\langle\iota\lambda\rangle^{s-1}} g|_X\\ &= \int_{\mathcal{S}} \phi_{\alpha_1\cdots\alpha_{s+1}\alpha\beta}\xi^{\alpha_1\cdots\alpha_{s+1}}\Sigma^{\alpha\beta}.\end{aligned} \tag{5.58}$$

When pulled back to the spin bundle $p^*(\Sigma) = \mathbb{CP}^1 \times \mathscr{N}$ and hence as $g|_X$ is a global form on $\mathbb{CP}^1$, the only boundary contribution arises from $\partial\mathscr{N} = \mathcal{S}$ where the surface $\mathcal{S}$ is a slice of the null hypersurface $\mathscr{N}$. Notice that the last line of (5.58) reproduces the heuristic expression given in (5.17). We can further reduce this charge into components using the expansion of the SD two-form

$$\Sigma^{\alpha\beta} = o^\alpha o^\beta \boldsymbol{n}\wedge\bar{\boldsymbol{m}} - \iota^{(\alpha}o^{\beta)}\boldsymbol{m}\wedge\bar{\boldsymbol{m}} + \iota^{(\alpha}o^{\beta)}\boldsymbol{l}\wedge\boldsymbol{n} + \iota^\alpha\iota^\beta\boldsymbol{m}\wedge\boldsymbol{l}\,. \tag{5.59}$$

where $\{\boldsymbol{l},\boldsymbol{n},\boldsymbol{m},\bar{\boldsymbol{m}}\}$ is the co-tetrad basis. The null hypersurface can be defined by the condition that $\boldsymbol{n}|_{\mathcal{N}} = 0$ such that the vector field n^μ is the generator of the null direction. Therefore, on $\mathcal{S}$, the only non-zero elements of the co-tetrad basis are $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$. The charge in terms of spin fields and components is given

$$H_s = \int_{\mathcal{S}} \phi_{\alpha_1\cdots\alpha_{s+1}\alpha\beta}\xi^{\alpha_1\cdots\alpha_{s+1}}\Sigma^{\alpha\beta} = \sum_{m=0}^{s+1}\int_{\mathcal{S}} \boldsymbol{m}\wedge\bar{\boldsymbol{m}}\,\xi_{s,m}Q_{m-1}. \tag{5.60}$$

We can also consider the flux, which is defined as the difference between two cuts $\mathcal{S}$ and $\mathcal{S}'$ of $\mathcal{N}$

$$\begin{aligned} F_s &= \int_{\mathcal{S}} - \int_{\mathcal{S}'} \left[\phi_{\alpha_1\cdots\alpha_{s+1}\alpha\beta}\xi^{\alpha_1\cdots\alpha_{s+1}}\Sigma^{\alpha\beta}\right] \\ &= \int_{\mathcal{N}} d\left[\phi_{\alpha_1\cdots\alpha_{s+1}\alpha\beta}\xi^{\alpha_1\cdots\alpha_{s+1}}\Sigma^{\alpha\beta}\right] \\ &= -\frac{2}{3}\int_{\mathcal{N}} \left(\nabla^{\alpha}_{\ \dot{\alpha}}\phi_{\alpha_1\cdots\alpha_{s+1}\alpha\beta}\xi^{\alpha_1\cdots\alpha_{s+1}} + \phi_{\alpha_1\cdots\alpha_{s+1}\alpha\beta}\nabla^{\alpha}_{\ \dot{\alpha}}\xi^{\alpha_1\ldots\alpha_{s+1}}\right)\theta_{\gamma}^{\ \dot{\alpha}}\wedge\Sigma^{\gamma\beta}, \end{aligned} \tag{5.61}$$

where Stokes' theorem was used. We see that if the hypersurfaces in question were general, the conservation of the charges would require the full z.r.m. equation and twistor equation to hold. On a SD background, the z.r.m. equation holds for arbitrarily high spin, and with the choice of gauge parameter satisfying (5.44), we find a non-zero flux

$$F_s = \int_{\mathcal{N}} \boldsymbol{l}\wedge\boldsymbol{m}\wedge\bar{\boldsymbol{m}}\, Q_{-2}\left(\sigma\xi_{s,1} - \bar{\eth}_{\mathcal{C}}\xi_{s,0}\right). \tag{5.62}$$

As we can see, the vanishing of the flux is obstructed by Q_{-2} and if it were to vanish the charges H_s would be conserved on the null hypersurface. We note that by choosing a $\mathcal{N}$ the choice of the vector field n^{μ} is fixed, as it generates the null direction. Therefore, the residual Lorentz transformation of the tetrad does not change the definition of Q_{-2} and therefore the condition for conservation of the charge $Q_{-2}=0$ is preserved under these transformations.

Having translated the charges H_s, one can ask to translate the charges $\tilde{H}_s$ associated with the other symmetry. The obstruction to translating the charge is in how the non-linear field h translates to spacetime. Only when working in special gauges is there an explicit formula for h in terms of spacetime fields. We will discuss the Plebanski gauge in Section 7, where explicit calculations can be done. In the present case, we wish to work in a generic gauge on spacetime, so we consider

$$\int_{\Sigma}\delta_{\chi}\lrcorner\,\omega = \int_{\Sigma} d\left(\Omega\wedge\chi\delta h\right). \tag{5.63}$$

To translate the charge to spacetime, we can work in a gauge (Woodhouse gauge) where δh only has spacetime components on the spin bundle. Hence, if χ is a global function on twistor space, the charge will vanish as there would be no boundary contribution, signaling that these transformations on the phase space are small gauge transformations. To obtain non-vanishing charges, such that they generate a non-trivial action on the phase space, the gauge parameter χ must be singular on twistor space, such that

$$\chi_{s,n} = \frac{1}{2\pi i} \oint_\Gamma q^{n+2} dq \, \chi'_s|_X \,, \tag{5.64}$$

with $\chi' = \langle \iota \lambda \rangle^6 \chi$ giving non-vanishing contour integrals. This line of reasoning amounts to identifying χ_s as the Čech representative for the cohomology $H^1(\mathcal{PT}, \mathcal{O}(-6))$. The index s labels a truncation such that $\chi_{s,n} = 0$ when $n < -s-3$ with $s \geq -1$. Therefore, we can translate the contracted symplectic structure to spacetime

$$\int_\Sigma \delta_\chi \lrcorner \, \omega = \int_{\mathscr{N}} \boldsymbol{l} \wedge \boldsymbol{m} \wedge \bar{\boldsymbol{m}} \, \chi_{s,-2} \, \varphi_0 = \delta \tilde{H}_s - \Xi_{\chi_s}[\delta]. \tag{5.65}$$

The difficulty now is to extract the δ inside the φ_0 by identifying how the SD perturbation affects the components of the metric. In general, this is difficult, but in certain circumstances, this is possible, as will be discussed in the next sections.

Using the same representative for the SD perturbation on the spin bundle, the symplectic potential can also be translated to spacetime

$$\int \theta[\delta] = \int_{\mathscr{N}} \boldsymbol{l} \wedge \boldsymbol{m} \wedge \bar{\boldsymbol{m}} \, \varphi_{\alpha\beta\gamma\dot{\alpha}} \bar{\iota}^{\dot{\alpha}} \iota_\delta \phi^{\alpha\beta\gamma\delta} = \int_{\mathscr{N}} \boldsymbol{l} \wedge \boldsymbol{m} \wedge \bar{\boldsymbol{m}} \, \varphi_0 Q_{-2}. \tag{5.66}$$

So far, the ξ_s symmetry on the SD metric has been discussed. To write down its action on the ASD higher-spin field, we use (5.55) and the explicit expansion of the ξ_s parameter when pulled back to the spin bundle (5.42). Using the distinguished ι_α

spinor, we find

$$\delta_{\xi_s}\phi_{\alpha_1\cdots\alpha_{s'+3}} = \iota^{\alpha}\iota^{\beta}\nabla_{\beta}{}^{\dot{\alpha}}\phi_{\alpha_1\cdots\alpha_{s'+3}\beta_1\cdots\beta_{s+1}}\nabla_{\alpha\dot{\alpha}}\xi^{\beta_1\cdots\beta_{s+1}}. \tag{5.67}$$

An interesting feature, which will be made more explicit, is that the ξ_s symmetry naturally introduces higher spin objects.

5.3.2 From null infinity...

The discussion so far has been general, keeping the gauge on spacetime arbitrary. We now wish to push the generic null hypersurface $\mathscr{N}$ to null infinity $\mathscr{I}$. To this end, it is convenient to work in the Newman-Unti (NU) gauge [179]. We find complete agreement with [133].

In the coordinate system $x^{\mu} = (r, u, x^A)$ where $x^A = (z, \bar{z})$, the tetrad in NU gauge is given by

$$l = \frac{\partial}{\partial r}, \qquad n = \frac{\partial}{\partial u} + U\frac{\partial}{\partial r} + X^A\frac{\partial}{\partial x^A}, \qquad m = \omega\frac{\partial}{\partial r} + L^A\frac{\partial}{\partial x^A}, \tag{5.68}$$

along with the constraint on the spin coefficients given by

$$\pi = \epsilon = \kappa = 0, \qquad \tau = \bar{\alpha} + \beta, \qquad \rho = \bar{\rho}. \tag{5.69}$$

We note that the complex conjugate is seen to be independent in general in the complexified setting. In what follows, we further require that the falloffs of the spin coefficients be satisfied

$$\mu = O\left(r^{-2}\right), \qquad \lambda = O\left(r^{-2}\right), \qquad \nu = O\left(r^{-1}\right). \tag{5.70}$$

On shell, along with the SD constraint, this results in these spin coefficients vanishing $\mu = \lambda = \nu = 0$. The vanishing of these spin coefficients is crucial for the metric

perturbation to stay in the SD sector. In other words, the equation (2.151) is only satisfied if these spin coefficients vanish. The metric has the expansion

$$ds^2 = -2dudr + 2r^2 dzd\bar{z} + r\left(\sigma^0 d\bar{z}^2 + \bar{\sigma}^0 dz^2\right) + O(r^0). \tag{5.71}$$

The details can be found in [179]. The SD perturbation expanded around null infinity will have a fall-off of the form

$$\varphi_0 = \frac{\partial_u^{-1}\delta\sigma^0}{r} + O\left(r^{-2}\right) , \tag{5.72}$$

where the leading contribution can be identified with the perturbation of the asymptotic shear. Furthermore, since $\phi_{\alpha_1\cdots\alpha_{s+3}}$ is a z.r.m. field with conformal weight $W = -1$, it will have a peeling theorem, such that the components have the fall-offs given by

$$Q_n = \frac{\mathcal{Q}_n}{r^{n+3}} + O\left(r^{-n-4}\right) . \tag{5.73}$$

Using these fall-offs, the symplectic potential at null infinity is given by

$$\int \theta[\delta] = \int_{\mathscr{I}} dudzd\bar{z}\, \mathcal{Q}_{-2}\partial_u^{-1}\delta\sigma^0 , \tag{5.74}$$

which matches the result found in Chapter 3 and [133]. Furthermore, using the NU gauge and the peeling theorem, the asymptotic z.r.m. equation along $\mathscr{I}$ on the leading-order components gives

$$\partial_u \mathcal{Q}_m = \bar{\eth}^0 \mathcal{Q}_{m-1} + (m+1)\sigma^0 \mathcal{Q}_{m-2} , \tag{5.75}$$

where the $\bar{\eth}$ operator appearing is the leading-order contribution given by

$$\bar{\eth} = \frac{\bar{\eth}^0}{r} + O\left(r^{-2}\right) \qquad \bar{\eth}^0 = \delta^0 - p\beta^0 - q\bar{\alpha}^0. \tag{5.76}$$

The second tower of equations equates the subleading orders of the field to the overleading parts. Turning to the field $\xi^{\alpha_1\cdots\alpha_{s+1}}$, the components are overleading such that

$$\xi_{s,m} = r^m \xi^0_{s,m} + O\left(r^{m-1}\right) , \tag{5.77}$$

where the dual recursion relations (5.46) at leading order are given by

$$\partial_u \xi^0_{s,m} - \bar{\eth}^0 \xi^0_{s,m+1} + (m+2)\sigma^0 \xi^0_{s,m+2} = 0. \tag{5.78}$$

The second tower of equations relates the subleading orders to the overleading orders as before, together with the condition $\eth^0 \xi^0_{s,s+1} = 0$.

Similarly, the spacetime fields generated by χ_s satisfy the recursion relation

$$\partial_u \chi^0_{s,m} = \bar{\eth} \chi^0_{s,m-1} + (m+1)\sigma^0 \chi^0_{s,m-2} \,, \tag{5.79}$$

along with $\eth^0 \chi_{s,-s-3} = 0$. The variation generated by the ξ_s and χ_s generators at null infinity is given by

$$\begin{aligned} \delta_{\xi(\tau_s),\chi(\rho_s)} \mathcal{Q}_n &= \chi^0_{s,n} + \sum_{m=0}^{s+1} m \xi^0_{s,m} \partial_u \mathcal{Q}_{n+m-1} + (n+m+2)\mathcal{Q}_{n+m-1} \partial_u \xi^0_{s,n}, \\ \delta_{\xi(\tau_s)} \sigma^0 &= \partial_u \left(\sigma^0 \xi^0_{s,1} - \bar{\eth}^0 \xi^0_{s,0} \right). \end{aligned} \tag{5.80}$$

The explicit variations generated by the $\xi(\tau_s)$ can be found in Chapter 3. The first few examples of the χ generators are given by

$$\begin{aligned} \delta_{\chi(\rho_{-1})} \mathcal{Q}_{-2} &= \rho_{-1}, \\ \delta_{\chi(\rho_0)} \mathcal{Q}_{-2} &= u \bar{\eth} \rho_0, \\ \delta_{\chi(\rho_1)} \mathcal{Q}_{-2} &= \frac{u^2}{2} \bar{\eth}^2 \rho_1 - \rho_1 \partial_u^{-1} \sigma^0, \end{aligned} \tag{5.81}$$

where the expressions for the higher $\mathcal{Q}_n$ are derived through the use of their recursion relations.

The charge H_s can now be pushed to null infinity, where one obtains a finite result; the exact form is given explicitly by

$$H_s = \sum_{m=0}^{s+1} \int_{S^2} \mathrm{vol}(S^2) \xi^0_{s,m} \mathcal{Q}_{m-1}, \tag{5.82}$$

$$\tilde{H}_s = \int du dz d\bar{z}\, \chi^0_{s,-2}(\rho_s) \partial_u^{-1} \sigma^0. \tag{5.83}$$

5.3.3 ...to a bulk null hypersurface

Following [178, 180], it is also convenient to work with a NU tetrad around a generic null hypersurface $\mathcal{N}$ in the bulk, which satisfies (5.68) and (5.69). The null hypersurface is situated at $r = 0$. At leading order, the metric around $\mathcal{N}$ is given by

$$ds^2 = q_{AB} dx^A dx^B + r((\chi_{AB} + q_{AB}\chi) dx^A dx^B - V_1 dv^2 - P_A dx^A) + O(r^2). \tag{5.84}$$

The metric can be decomposed in the NU co-tetrad. We have $q_{AB} = m^0_{(A} \bar{m}^0_{B)}$, and the other terms in the metric expansion are related to the leading spin coefficients via

$$\begin{aligned} \chi_{AB} &= 2(\sigma^0 \bar{m}^0_A \bar{m}^0_B + \bar{\sigma}^0 m^0_A m^0_B), \\ P_A &= 2(\tau^0 \bar{m}^0_A + \bar{\tau}^0 m^0_A), \\ V_1 &= \lambda^0 - \bar{\lambda}^0 - 4\gamma^0, \\ \chi &= 4\rho^0. \end{aligned} \tag{5.85}$$

We refer to [180] for a complete dictionary between metric and NP formalism. The variation of the metric can be expanded as

$$\delta q_{AB} = m^0_{(A} \bar{m}^0_{B)} \left(\bar{m}^{0\,C} \delta m^0_C + m^{0\,C} \delta \bar{m}^0_C \right) + \bar{m}^0_A \bar{m}^0_B m^{0\,C} \delta m^0_C + m^0_A m^0_B \bar{m}^{0\,C} \delta \bar{m}^0_C \,, \tag{5.86}$$

together with

$$m^{0\,C}\delta m^0_C = -(\text{Þ}'^0 + \mu^0)\varphi^0_0. \tag{5.87}$$

This then gives the leading order relationship to the variation coming from twistor theory

$$\begin{aligned}\varphi_0 = &-(\text{Þ}'^0 + \mu^0)^{-1} m^{0\,C}\delta m^0_C \\ &- r(\text{Þ}'^0 + \mu^0)^{-1}\left(2\delta\sigma^0 + m^{0A}\delta\bar{m}^0_A - (\text{Þ}'^1 + \mu^1)\varphi^0_0 + \bar{L}^1_A m^{0A}(\text{Þ}'^0 + \mu^0)\varphi^0_0\right) + O\left(r^2\right),\end{aligned} \tag{5.88}$$

where we have assumed that $\tau^0 = \bar{\nu}^1 = 0$. We see if we impose the constraints

$$\delta m^0_A = 0, \qquad \delta\text{det}(q) = \delta q = 0\,, \tag{5.89}$$

as in [178, 180], the leading order contribution vanishes. Under this assumption, the phase space becomes subleading, and the variation is given by

$$\varphi_0 = -2r(\text{Þ}'^0 + \mu^0)^{-1}\delta\sigma^0 + O(r^2). \tag{5.90}$$

We assume the following falloffs for Q_n around $\mathcal{N}$:

$$Q_n = Q^0_n + O\left(r\right). \tag{5.91}$$

In general, the symplectic potential (5.66) admits the following expansion:

$$\int \theta[\delta] = \int \theta^0[\delta] + r\theta^1[\delta] + \mathcal{O}(r^2). \tag{5.92}$$

The leading-order phase space encoded in $\theta^0[\delta]$ provides variations of a single component of the hypersurface metric, while the subleading order encoded in $\theta^1[\delta]$ contains the variation of the shear. If the leading order is not fixed, there will be corrections involving the variations of the hypersurface metric in the subleading orders. How-

ever, if we impose the constraints (5.89), the phase space becomes subleading, i.e. $\theta^0[\delta] = 0$, and the symplectic potential simplifies to

$$\int \theta[\delta] = -2r \int dvdzd\bar{z}\, \sqrt{q} Q^0_{-2} (\text{þ}'^0 + \mu^0)^{-1} \delta\sigma^0 + O(r^2)\,. \tag{5.93}$$

Upon defining $Q^0_{-2} = -(\text{þ}' + \mu^0)\tilde{\lambda}^0$ and integrating by parts, this reproduces the subleading phase space considered in [178, 180]. The latter is formally similar to the Ashtekar-Streubel phase space at null infinity [141], and allows the heuristic construction of celestial charges as, it was done at null infinity [61, 62].

We now focus on the recursion relations. Taking into account the falloffs (5.91), the leading order recursion relations read as

$$\text{þ}'^0_{\mathcal{C}} Q^0_m = \bar{\eth}^0_{\mathcal{C}} Q^0_{m-1} + (m+1)\sigma^0 Q^0_{m-2}. \tag{5.94}$$

which correctly reproduces the recursion relations of [178, 180]. Similarly, one can expand the parameters $\xi_{s,m}$ around $\mathcal{N}$ as

$$\xi_{s,m} = \xi^0_{s,m} + r\xi^1_{s,m} + O(r^2)\,. \tag{5.95}$$

The leading order dual recursion relations read as (5.78). Again, imposing the conditions (5.89) implies $\xi^0_{s,m} = 0$ by consistency, and therefore, the subleading parameters $\xi^1_{s,m}$ will satisfy the dual recursion relations. These are the celestial symmetry parameters considered in [178, 180], where an explicit solution is provided in terms of a "dressed time" on the null hypersurface. Similarly, the parameters $\xi_{s,m}$ can be expanded around $\mathcal{N}$ as

$$\chi_{s,m} = \chi^0_{s,m} + \mathcal{O}(r)\,, \tag{5.96}$$

and the $\chi^0_{s,m}$ satisfy (5.79). Notice that these leading parameters will appear both in the leading and subleading phase space.

At leading order, the celestial charges integrated over a cut $v =$ constant of $\mathcal{N}$ are given explicitly by

$$H_s = \sum_{m=0}^{s+1} \int dz d\bar{z} \sqrt{q} \xi^0_{s,m} Q^0_{m-1}, \tag{5.97}$$

and the associated integrated fluxes are

$$F_s = \int_{\mathcal{N}} dv dz d\bar{z} \, \sqrt{q} Q^0_{-2} \left(\sigma^0 \xi^0_{s,1} - \bar{\eth}^0_{\mathcal{C}} \xi^0_{s,0} \right). \tag{5.98}$$

The dual charges are in the form of an integrated flux, but as discussed previously, only the contracted symplectic structure can be written down

$$\delta \tilde{H}_s - \Xi_{\chi_s}[\delta] = \int dv dz d\bar{z} \, \sqrt{q} \chi^0_{s,-2}(\rho_s) (\text{þ}'^0 + \mu^0)^{-1} \left(m^{0\,A} \delta m^0_A \right) . \tag{5.99}$$

Imposing the conditions (5.89), the leading phase space and charges completely trivialize. The subleading celestial charges read as

$$H_s = r \sum_{m=0}^{s+1} \int dz d\bar{z} \sqrt{q} \xi^1_{s,m} Q^0_{m-1}, \tag{5.100}$$

and the integrated fluxes read as

$$F_s = r \int_{\mathcal{N}} dv dz d\bar{z} \, \sqrt{q} Q^0_{-2} \left(\sigma^0 \xi^1_{s,1} - \bar{\eth}^0_{\mathcal{C}} \xi^1_{s,0} \right). \tag{5.101}$$

The dual charge under these assumptions can be written down explicitly and is given by

$$\tilde{H}_s = r \int dv dz d\bar{z} \, \sqrt{q} \chi^0_{s,-2}(\rho_s) (\text{þ}'^0 + \mu^0)^{-1} \sigma^0 . \tag{5.102}$$

In particular, this subleading structure is formally completely analogous of that appearing at null infinity.

5.4 Celestial symmetries of Plebanski's equations

In this section, we will use a special gauge where calculations can be made explicit, and the symmetry algebra can be computed. In the SD sector, two potential formulations exist due to Plebanski [69]. In the present case, we will consider Plebanski's second heavenly formulation.

5.4.1 Plebanski's second heavenly equations

We begin by considering the twistor action (5.21) when one chooses a special gauge for the field h, namely,

$$h = \langle \iota \lambda \rangle^3 d_{\dot{\alpha}} \Theta \, \bar{e}^{\dot{\alpha}} \,, \tag{5.103}$$

where $\bar{e}^{\dot{\alpha}}$ is the $(0,1)$-form of homogeneity -1 on the spin bundle that points along the spacetime directions only and Θ is a function of spacetime coordinates only. We have also defined the spacetime coordinates $\left(x^{\dot{\alpha}} = (x, y), \tilde{x}^{\dot{\alpha}} = (-\tilde{x}, \tilde{y})\right)$ along with their respective derivatives $d_{\dot{\alpha}} = \frac{\partial}{\partial x^{\dot{\alpha}}}$ and $\tilde{d}_{\dot{\alpha}} = \frac{\partial}{\partial \tilde{x}^{\dot{\alpha}}}$. In this gauge, the action reduces to

$$S[\Theta, Q_{-2}] = \int d^4x Q_{-2} \left(\tilde{d}^{\dot{\alpha}} d_{\dot{\alpha}} \Theta + \frac{1}{2} d_{\dot{\alpha}} d_{\dot{\beta}} \Theta d^{\dot{\alpha}} d^{\dot{\beta}} \Theta \right). \tag{5.104}$$

The equations of motion for this action are given by

$$\tilde{d}^{\dot{\alpha}} d_{\dot{\alpha}} \Theta + \frac{1}{2} d_{\dot{\alpha}} d_{\dot{\beta}} \Theta d^{\dot{\alpha}} d^{\dot{\beta}} \Theta = 0, \qquad \tilde{d}^{\dot{\alpha}} d_{\dot{\alpha}} Q_{-2} + d_{\dot{\alpha}} d_{\dot{\beta}} \Theta d^{\dot{\alpha}} d^{\dot{\beta}} Q_{-2} = 0. \tag{5.105}$$

The first of which is called Plebanski's second heavenly equation, while the second equation is the linearized version. In [69], Plebanski showed that the SD vacuum condition, $R_{\alpha\beta} = 0$, locally implies the existence of a potential Θ along with a coordinate system $\left(x^{\dot{\alpha}}, \tilde{x}^{\dot{\alpha}}\right)$ that generates a metric

$$\begin{aligned} g &= 2 d\tilde{x}^{\dot{\alpha}} dx_{\dot{\alpha}} - 2 d_{\dot{\alpha}} d_{\dot{\beta}} \Theta d\tilde{x}^{\dot{\alpha}} d\tilde{x}^{\dot{\beta}} \\ &= 2 dy d\tilde{x} + 2 dx d\tilde{y} - 2\Theta_{xx} d\tilde{x}^2 - 2\Theta_{yy} d\tilde{y}^2 + 4\Theta_{xy} d\tilde{x} d\tilde{y} \,, \end{aligned} \tag{5.106}$$

where the potential Θ satisfies the second heavenly equation which in explicit coordinates is given by

$$\Theta_{x\tilde{y}} + \Theta_{y\tilde{x}} + \Theta_{xx}\Theta_{yy} - \Theta_{xy}^2 = 0\,. \tag{5.107}$$

When the second heavenly equation holds, one can show that the spin connections are given by

$$\Gamma_{\alpha\beta} = 0, \qquad \tilde{\Gamma}_{\dot{\alpha}\dot{\beta}} = d_{\dot{\alpha}} d_{\dot{\beta}} d_{\dot{\gamma}} \Theta d\tilde{x}^{\dot{\gamma}}, \tag{5.108}$$

which results in the Riemman 2-forms

$$R_{\alpha\beta} = 0, \qquad \tilde{R}_{\dot{\alpha}\dot{\beta}} = d_{\dot{\alpha}} d_{\dot{\beta}} d_{\dot{\gamma}} d_{\dot{\delta}} \Theta \tilde{\Sigma}^{\dot{\gamma}\dot{\delta}}. \tag{5.109}$$

As expected, the ASD Riemann 2-form vanishes; moreover, in this gauge, the ASD spin connection is zero, making the undotted spin bundle trivial. This is the major benefit of this gauge.

The second heavenly system has a Lax representation by considering the tetrad $e_{\alpha\dot{\alpha}}$ and a spectral parameter λ_α

$$L_{\dot{\alpha}} = \lambda^\alpha e_{\alpha\dot{\alpha}} = \langle \lambda \iota \rangle \left(q d_{\dot{\alpha}} - \tilde{d}_{\dot{\alpha}} + d_{\dot{\alpha}} d_{\dot{\beta}} \Theta d^{\dot{\beta}} \right). \tag{5.110}$$

The second heavenly equations are equivalent to the commutator $[L_{\dot{\alpha}}, L_{\dot{\beta}}] = 0$ for all values of q.

Let us focus on the linearized second heavenly equations given by

$$\tilde{d}^{\dot{\alpha}} d_{\dot{\alpha}} Q_{-2} + d_{\dot{\alpha}} d_{\dot{\beta}} \Theta d^{\dot{\alpha}} d^{\dot{\beta}} Q_{-2} = 0. \tag{5.111}$$

The linearised equation plays an important role in the integrable hierarchy and the

recursion operator. First, consider manipulating the linearised equation

$$\begin{aligned} 0 &= \tilde{d}^{\dot{\alpha}} d_{\dot{\alpha}} Q_{-2} + d_{\dot{\alpha}} d_{\dot{\beta}} \Theta d^{\dot{\alpha}} d^{\dot{\beta}} Q_{-2} \\ &= d_{\dot{\alpha}} \left(\tilde{d}^{\dot{\alpha}} Q_{-2} - d^{\dot{\alpha}} d_{\dot{\beta}} \Theta d^{\dot{\beta}} Q_{-2} \right) \\ &= d_{\dot{\alpha}} d^{\dot{\alpha}} Q_{-1} \,, \end{aligned} \tag{5.112}$$

where the last line vanishes identically due to the anti-symmetry of the contraction of indices. Therefore, the linearised heavenly equation locally guarantees the existence of the relations

$$d_{\dot{\alpha}} Q_{-1} = \tilde{d}_{\dot{\alpha}} Q_{-2} - d_{\dot{\alpha}} d_{\dot{\beta}} \Theta d^{\dot{\beta}} Q_{-2}. \tag{5.113}$$

One can further check by acting with $\tilde{d}^{\dot{\alpha}}$ and using the second heavenly equation that Q_{-1} satisfies the linearised equation (5.111). One can iterate this procedure and construct an infinite tower satisfying the recursion relations

$$d_{\dot{\alpha}} Q_{n} = \tilde{d}_{\dot{\alpha}} Q_{n-1} - d_{\dot{\alpha}} d_{\dot{\beta}} \Theta d^{\dot{\beta}} Q_{n-1}. \tag{5.114}$$

With these considerations in hand, one can define the recursion operator $R : \mathcal{W} \to \mathcal{W}$ where $\mathcal{W}$ is the space of solutions to (5.111) and the map is defined formally by the recursion relations (5.114). Furthermore, in this gauge, these are exactly the high valence z.r.m. equations for a z.r.m. field with components Q_n. The action of the recursion operator on twistor space has a simple form when considering (5.55). Since $R(Q_n) = Q_{n+1}$, the action of the recursion operator on g is simply $R(g) = q \cdot g$, see [99, 100] for more details. Furthermore, since these recursion relations are satisfied by both $\xi_{s,n}$ and $\chi_{s,n}$, the recursion operator will also appear in the construction of the symmetry algebra.

One can also construct the recursion operator on the SD sector of the theory by simply taking a derivative of the Plebanski equation. Define $\tilde{Q}^{\dot{\alpha}}_{-2} = d^{\dot{\alpha}}\Theta$ which

satisfies the linearised equation

$$\tilde{d}^{\dot{\alpha}} d_{\dot{\alpha}} \tilde{Q}^{\dot{\beta}}_{-2} + d_{\dot{\alpha}} d_{\dot{\gamma}} \Theta d^{\dot{\alpha}} d^{\dot{\gamma}} \tilde{Q}^{\dot{\beta}}_{-2} = 0. \tag{5.115}$$

Following the same logic as above, one can construct the so-called integrable hierarchy for SD gravity

$$d_{\dot{\alpha}} \tilde{Q}^{\dot{\beta}}_{n} = \tilde{d}_{\dot{\alpha}} \tilde{Q}^{\dot{\beta}}_{n-1} - d_{\dot{\alpha}} d_{\dot{\gamma}} \Theta d^{\dot{\gamma}} \tilde{Q}^{\dot{\beta}}_{n-1}, \qquad \tilde{Q}^{\dot{\alpha}}_{-2} = d^{\dot{\alpha}} \Theta, \qquad \tilde{Q}^{\dot{\alpha}}_{-1} = \tilde{d}^{\dot{\alpha}} \Theta. \tag{5.116}$$

The higher $\tilde{Q}^{\dot{\alpha}}_{n}$ can be seen as Hamiltonian flows in higher times associated with the integrable hierarchy [99, 100, 181, 111]. There exists a precise relationship between these spacetime fields and twistor space data given by

$$\tilde{Q}^{\dot{\alpha}}_{n} = \frac{1}{2\pi i} \int_X q^{n+2} dq \wedge \epsilon^{\dot{\alpha}\dot{\beta}} \left. \frac{\partial h}{\partial \mu^{\dot{\beta}}} \right|_X . \tag{5.117}$$

Once again, the recursion operator can simply be seen as multiplication by the coordinate q on twistor space.

5.4.2 Symmetry algebra

We now focus on the explicit form of the symmetry generators in this gauge as they appear in the bulk. In this gauge, the Poisson bracket on twistor space is given by

$$\{f_1, f_2\} = \langle \iota \lambda \rangle^{-2} d^{\dot{\alpha}} f_1 d_{\dot{\alpha}} f_2. \tag{5.118}$$

The gauge parameters ξ and χ can be constructed as residual gauge transformations. Working on the spin bundle, the pullback $\xi_s|_X = \xi(\tau_s)$ is expanded as discussed in Section 3.3

$$\xi(\tau_s) = \sum_{n=0}^{s+1} \xi_{s,n}(\tau_s) q^n. \tag{5.119}$$

The coefficients of $\xi(\tau_s)$ must satisfy the recursion relations to preserve the gauge choice (5.103)

$$d_{\dot\alpha}\xi_{s,n-1} = \tilde{d}_{\dot\alpha}\xi_{s,n} - d_{\dot\alpha}d_{\dot\beta}\Theta d^{\dot\beta}\xi_{s,n}. \tag{5.120}$$

These recursion relations are truncated such that $d_{\dot\alpha}\xi_{s,s+1} = 0$ and we define the function $\xi_{s,s+1} = \tau_s(\tilde{x}^{\dot\alpha})$ depending only on half the spacetime variables. As mentioned previously, the coefficients for $\xi(\tau_s)$ are defined via the recursion operator R and as such each $\xi_{s,n}$ satisfies the linearised Plebanski equation.

We now turn to the construction of the χ_s generators. As seen in (5.64), the χ_s pulls back to give fields on spacetime via the contour integral

$$\chi_{s,n} = \frac{1}{2\pi i}\oint_\Gamma q^{n+2}dq\,\chi_s'|_X\,, \tag{5.121}$$

where $\chi_s' = \langle\iota\lambda\rangle^6\chi_s$. Since χ_s is the Čech representative, it satisfies $L_{\dot\alpha}\chi_s|_X = 0$ which translates to the recursion relations

$$d_{\dot\alpha}\chi_{s,n+1} = \tilde{d}_{\dot\alpha}\chi_{s,n} - d_{\dot\alpha}d_{\dot\beta}\Theta d^{\dot\beta}\chi_{s,n}\,, \tag{5.122}$$

with the truncation $\chi_{s,n} = 0$ for $n < -s-3$ which results in $\chi_{s,-s-3} = \rho_s(\tilde{x}^{\dot\alpha})$ only depending on half the spacetime coordinates. For future convenience, we denote $\chi_s|_X = \chi(\rho_s)$ as the pull back of the χ_s generator. These parameters form an algebra through the use of the algebroid brackets (5.32) on the field space due to the inherent field dependence coming from the recursion relations

$$\begin{aligned}\{\xi,\xi'\}_* &= \{\xi,\xi'\} + \delta_\xi\xi' - \delta_{\xi'}\xi,\\ \{\xi,\chi\}_* &= \{\xi,\chi\} + \delta_\xi\chi.\end{aligned} \tag{5.123}$$

We note that the only field dependence arises from the background metric generated by h, and as such, only δ_ξ terms appear. With these brackets, one can show that the algebra satisfied by the generators is a representation of the algebra between the

field-independent parameters τ_s and ρ_s

$$\begin{aligned}\{\xi(\tau_s),\xi(\tau_{s'})\}_* &= \xi\left(\tau_{s+s'-1} = \tilde{d}^{\dot{\alpha}}\tau_s\tilde{d}_{\dot{\alpha}}\tau_{s'}\right),\\ \{\xi(\tau_s),\chi(\rho_{s'})\}_* &= \chi\left(\rho_{s+s'-1} = \tilde{d}^{\dot{\alpha}}\tau_s\tilde{d}_{\dot{\alpha}}\rho_{s'}\right).\end{aligned} \tag{5.124}$$

We prove these brackets in Appendix 5.5. If we expand these functions in a basis

$$\tau[m,n] = \langle\iota\lambda\rangle^2\tilde{x}^m\tilde{y}^n, \qquad \rho[m,n] = \langle\iota\lambda\rangle^{-6}\tilde{x}^m\tilde{y}^n, \tag{5.125}$$

such that $m+n = s+1$. We find the algebra satisfied by these generators to be

$$\begin{aligned}\tilde{d}^{\dot{\alpha}}\tau[m,n]\tilde{d}_{\dot{\alpha}}\tau[r,l] &= (ml-nr)\tau[m+r-1,n+l-1],\\ \tilde{d}^{\dot{\alpha}}\tau[m,n]\tilde{d}_{\dot{\alpha}}\rho[r,l] &= (ml-nr)\rho[m+r-1,n+l-1],\end{aligned} \tag{5.126}$$

which is the algebra $\mathrm{Ham}(\mathbb{C}^2)$. Since the generators ξ and ϕ are linear in τ and χ respectively, the structure constant factors through, and we obtain the field-dependent algebra given by

$$\begin{aligned}\{\xi[m,n],\xi[r,l]\}_* &= (ml-nr)\xi[m+r-1,n+l-1],\\ \{\xi[m,n],\chi[r,l]\}_* &= (ml-nr)\chi[m+r-1,n+l-1],\end{aligned} \tag{5.127}$$

where $\xi[m,n] = \xi(\tau[m,n])$ and $\chi[m,n] = \chi(\rho[m,n])$. Therefore, since the charge algebra is a representation of the modified bracket, we find the charges manifest the $\mathrm{Ham}(\mathbb{C}^2)$ algebra.

By using the integral formula (5.55), the action of these symmetries on the space-time fields can be explicitly written down

$$\begin{aligned}\delta_{\xi(\tau_s),\chi(\rho_{s'})}Q_n &= \chi_{s',n} + \sum_{m=0}^{s+1} d^{\dot{\alpha}}Q_{n+m}d_{\dot{\alpha}}\xi_{s,m},\\ \delta_{\xi(\tau_s)}d_{\dot{\alpha}}\Theta &= \tilde{d}_{\dot{\alpha}}\xi_{s,0} - d_{\dot{\alpha}}d_{\dot{\beta}}\Theta d^{\dot{\beta}}\xi_{s,0}.\end{aligned} \tag{5.128}$$

One can show explicitly that these are the symmetries of the recursion relations (5.114). We now present several examples of such transformations and highlight their features. The first examples of the variation of the Plebanski potential are given by

$$\begin{aligned}
\delta_{\xi(\tau_{-1})}\Theta &= x^{\dot\alpha}\tilde{d}_{\dot\alpha}\tau_{-1},\\
\delta_{\xi(\tau_0)}\Theta &= \frac{1}{2!}x^{\dot\alpha}x^{\dot\beta}\tilde{d}_{\dot\alpha}\tilde{d}_{\dot\beta}\tau_0 - d_{\dot\alpha}\Theta\tilde{d}^{\dot\alpha}\tau_0,\\
\delta_{\xi(\tau_1)}\Theta &= \frac{1}{3!}x^{\dot\alpha}x^{\dot\beta}x^{\dot\gamma}\tilde{d}_{\dot\alpha}\tilde{d}_{\dot\beta}\tilde{d}_{\dot\gamma}\tau_1 - x^{\dot\alpha}\tilde{d}_{\dot\alpha}\tilde{d}^{\dot\beta}\tau_1 d_{\dot\beta}\Theta + \tilde{d}^{\dot\alpha}\Theta\tilde{d}_{\dot\alpha}\tau_1,\\
\delta_{\xi(\tau_2)}\Theta &= \frac{1}{4!}x^{\dot\alpha}x^{\dot\beta}x^{\dot\gamma}x^{\dot\epsilon}\tilde{d}_{\dot\alpha}\tilde{d}_{\dot\beta}\tilde{d}_{\dot\gamma}\tilde{d}_{\dot\epsilon}\tau_2 + \frac{1}{2}x^{\dot\alpha}x^{\dot\beta}\tilde{d}_{\dot\alpha}\tilde{d}_{\dot\beta}\tilde{d}_{\gamma}\tau_2 d^{\dot\gamma}\Theta\\
&\qquad + x^{\dot\alpha}\tilde{d}_{\dot\alpha}\tilde{d}_{\dot\beta}\tau_2\tilde{d}^{\dot\beta}\Theta + \frac{1}{2}d_{\dot\alpha}\Theta d_{\dot\beta}\Theta\tilde{d}^{\dot\alpha}\tilde{d}^{\dot\beta}\tau_2 + \tilde{Q}_0^{\dot\alpha}\tilde{d}_{\dot\alpha}\tau_2.
\end{aligned} \tag{5.129}$$

We note that the first three variations appear as the residual diffeomorphisms that preserve the Plebanski gauge [99, 182]. Furthermore, these residual diffeomorphisms were used in [183] to identify a $Lw_{1+\infty}$ algebra on spacetime; however, the present higher spin variations differ as can be seen in the last variation where there is a term that is non-local and defined via the integrable hierarchy equations. For higher spins, the variations continue to be non-local as one would expect. Furthermore, note that the $s = -1$ variation changes the potential, but the metric will be left unchanged. The $\xi(\tau_s)$ transformations follow a similar patterns for the Q_n where the first few examples are given by

$$\begin{aligned}
\delta_{\xi(\tau_{-1})}Q_n &= 0,\\
\delta_{\xi(\tau_0)}Q_n &= \tilde{d}_{\dot\alpha}\tau_0 d^{\dot\alpha}Q_n,\\
\delta_{\xi(\tau_1)}Q_n &= x^{\dot\beta}\tilde{d}_{\dot\alpha}\tilde{d}_{\dot\beta}\tau_1 d^{\dot\alpha}Q_n + \tilde{d}_{\dot\alpha}\tau_1\tilde{d}^{\dot\alpha}Q_n,\\
\delta_{\xi(\tau_2)}Q_n &= \left(\frac{1}{2}x^{\dot\beta}x^{\dot\gamma}\tilde{d}_{\dot\alpha}\tilde{d}_{\dot\beta}\tilde{d}_{\dot\gamma}\tau_2 + \tilde{d}_{\dot\alpha}\tilde{d}_{\dot\beta}\tau_2 d^{\dot\beta}\Theta + \tilde{d}_{\dot\beta}\tau_2 d_{\dot\alpha}\tilde{d}^{\dot\beta}\Theta\right)d^{\dot\alpha}Q_n\\
&\qquad + x^{\dot\beta}\tilde{d}_{\dot\alpha}\tilde{d}_{\dot\beta}\tau_2\tilde{d}^{\dot\alpha}Q_n + \tilde{d}_{\dot\alpha}\tau_2\tilde{d}^{\dot\alpha}Q_{n+1}.
\end{aligned} \tag{5.130}$$

We first note that the $s = -1$ variation does not change the field, and as such does not vary the physical fields, only the potential. Secondly, for $s = 0, 1$ the variation

only involves the field itself, while for higher values of s, not only does the potential appear, but also the higher spin Q_n. Therefore, to make full sense of the higher spin variations, one must include the entire tower of Q_n for all values of n. The variations associated with the $\chi(\rho_s)$ symmetry can also be written down in the same fashion, where the first examples are given by

$$\begin{aligned}
\delta_{\chi(\rho_{-1})}Q_{-2} &= \rho_{-1}, \\
\delta_{\chi(\rho_0)}Q_{-2} &= x^{\dot{\alpha}}\tilde{d}_{\dot{\alpha}}\rho_0, \\
\delta_{\chi(\rho_1)}Q_{-2} &= \frac{1}{2!}x^{\dot{\alpha}}x^{\dot{\beta}}\tilde{d}_{\dot{\alpha}}\tilde{d}_{\dot{\beta}}\rho_1 - d_{\dot{\alpha}}\Theta\tilde{d}^{\dot{\alpha}}\rho_1,
\end{aligned} \tag{5.131}$$

and to obtain the variation of the higher spin Q_n, one has to apply the recursion operator to these variations. As mentioned previously, the benefit of Plebanski gauge is the ability to write the non-linear field h in terms of explicit spacetime fields. As such, the two sets of charges are given by

$$H_s = \sum_{n=0}^{s+1}\int_{\mathcal{S}} dyd\tilde{x}\, Q_{n-1}\xi_{s,n}\ , \qquad \tilde{H}_s = \int_{\mathcal{N}} dxdyd\tilde{x}\,\chi_{s,-2}\partial_x\Theta. \tag{5.132}$$

where the null hypersurface $\mathcal{N}$ is chosen such that $\tilde{y}$ is a constant.

5.5 Proof of the algebra

Here we prove the following, as seen in Section 5.4.2

$$\begin{aligned}
\{\xi(\tau_s),\xi(\tau_{s'})\}_* &= \xi\left(\tau_{s+s'-1} = \tilde{d}^{\dot{\alpha}}\tau_s\tilde{d}_{\dot{\alpha}}\tau_{s'}\right), \\
\{\xi(\tau_s),\chi(\rho_{s'})\}_* &= \chi\left(\rho_{s+s'-1} = \tilde{d}^{\dot{\alpha}}\tau_s\tilde{d}_{\dot{\alpha}}\rho_{s'}\right),
\end{aligned} \tag{5.133}$$

when working in Plebanski gauge. To begin, we focus on the algebra between the $\xi(\tau_s)$ generators. The $\xi(\tau_s)$ are constructed through the condition

$$L_{\dot{\alpha}}\xi(\tau_s) = \langle\iota\lambda\rangle^3\delta_{\xi_s}d_{\dot{\alpha}}\Theta\,. \tag{5.134}$$

Therefore, for $\{\xi(\tau_s), \xi(\tau_{s'})\}_*$ to give another generator of the same type, it must also satisfy this condition. A short calculation shows that

$$\begin{aligned} L_{\dot\alpha}\{\xi(\tau_s), \xi(\tau_{s'})\}_* &= \{L_{\dot\alpha}\xi(\tau_s), \xi(\tau_{s'})\} + L_{\dot\alpha}\delta_{\xi(\tau_s)}\xi(\tau_{s'}) - (s \leftrightarrow s') \\ &= \langle\iota\lambda\rangle^3\{d_{\dot\alpha}\Theta, \xi(\tau_{s'})\} + L_{\dot\alpha}\delta_{\xi(\tau_s)}\xi(\tau_{s'}) - (s \leftrightarrow s') \\ &= \delta_{\xi(\tau_s)}L_{\dot\alpha}\xi(\tau_{s'}) - \delta_{\xi(\tau_{s'})}L_{\dot\alpha}\xi(\tau_s) \\ &= \langle\iota\lambda\rangle^3[\delta_{\xi(\tau_s)}, \delta_{\xi(\tau_{s'})}]d_{\dot\alpha}\Theta = \langle\iota\lambda\rangle^3\delta_{\{\xi(\tau_s),\xi(\tau_{s'})\}_*}d_{\dot\alpha}\Theta\,, \end{aligned} \tag{5.135}$$

which shows that the new $\xi(\tau_{s''})$ generator also satisfies the same recursion relations. By considering the expansion in q of the generators, the highest order term determines $\tau_{s''}$, which we find to be field independent and given by

$$\{\xi(\tau_s), \xi(\tau_{s'})\}_* = q^{s+s'}\tilde{d}^{\dot\alpha}\tau_s\tilde{d}_{\dot\alpha}\tau_{s'} + \cdots\,, \tag{5.136}$$

where we have used the recursion relations and $d_{\dot\alpha}\tau = 0$. Therefore, we find that

$$\tau_{s''} = \tau_{s+s'-1} = \tilde{d}^{\dot\alpha}\tau_s\tilde{d}_{\dot\alpha}\tau_{s'}\,, \tag{5.137}$$

proving the first claim.

Next, we consider the bracket between a ξ generator and a χ generator. The condition on the pullback of $\chi|_X = \chi(\rho)$ is that the Lax operator annihilates it $L_{\dot\alpha}\chi(\rho) = 0$. Another short calculation shows

$$\begin{aligned} L_{\dot\alpha}\{\xi(\tau_s), \chi(\rho_{s'})\}_* &= \{L_{\dot\alpha}\xi(\tau_s), \chi(\rho_{s'})\} + \{\xi(\tau_s), L_{\dot\alpha}\chi(\rho_{s'})\} + L_{\dot\alpha}\delta_{\xi(\tau_s)}\chi(\rho_{s'}) \\ &= \{\xi(\tau_s), L_{\dot\alpha}\chi(\rho_{s'})\} + \delta_{\xi(\tau_s)}L_{\dot\alpha}\chi(\rho_{s'}) \\ &= 0\,, \end{aligned} \tag{5.138}$$

where we have used $L_{\dot\alpha}\chi(\rho_{s'}) = 0$. Therefore, the modified bracket gives another $\chi(\rho_{s''})$ generator. The lowest power of q in the integral that gives a non-zero result

for $\chi(\rho_{s'})$ is given by

$$\rho_{s'} = \frac{1}{2\pi i} \oint_\Gamma \frac{dq}{q^{s'+1}} \chi(\rho_{s'}) \,, \tag{5.139}$$

which is importantly field independent. Therefore, since the highest power in $\xi(\tau_s)$ is of order $s+1$ in q and $d_{\dot\alpha}\tau_s = 0 = d_{\dot\alpha}\rho_{s'}$, the new generator of the same type is given by

$$\rho_{s''} = \frac{1}{2\pi i} \oint_\Gamma \frac{dq}{q^{s+s'}} \chi(\rho_{s''}) = \tilde{d}^{\dot\alpha}\tau_s \tilde{d}_{\dot\alpha}\rho_{s'} \,, \tag{5.140}$$

where we have used the recursion relations for both ξ and χ. Therefore, we find

$$\rho_{s''} = \rho_{s+s-1} = \tilde{d}^{\dot\alpha}\tau_s \tilde{d}_{\dot\alpha}\rho_{s'} \,, \tag{5.141}$$

proving the second bracket.

5.6 Multipole moments for stationary fields in linear theory

To obtain some intuition as to what these charge quantities measure, we first investigate them in linear theory about a flat background where our higher spin charges will be conserved. In particular, we will see that in the time-independent case they yield standard multipole moments in a form identified by Geroch [160] but reformulated into a spinor and twistor version by Curtis [118] which we now review.

To make contact with multipole moments for linear gravity, we must first generate the higher valence solution to the z.r.m. equations from the linearized metric. To relate to the Geroch, Hansen, Curtis definitions, we start by taking our linearized metric to be stationary. Thus, following [118], we assume that the linearized Weyl spinor is constant along some constant timelike vector t_a so that

$$t \cdot \nabla \psi_{\alpha\beta\gamma\delta} = 0 \,. \tag{5.142}$$

We take t^a to be a constant vector of length $t^2 = 2$ so that in a normalized spin-frame it can be expressed as the identity matrix. This will allow us to use $t_{\alpha\dot{\alpha}}$ to eliminate dotted spinor indices in favour of undotted ones. A vector projected into the three-dimensional space not spanned by t_a can be defined by

$$V^i = V^{\alpha\beta} = t^{(\alpha}_{\ \dot{\alpha}} V^{\beta)\dot{\alpha}}\,, \tag{5.143}$$

where the index $i = 1, 2, 3$. This projection can be inverted explicitly via the relation

$$V^{\alpha\dot{\alpha}} = t_\beta^{\ \dot{\alpha}} V^{\alpha\beta} + \frac{1}{2}(t \cdot V) t^{\alpha\dot{\alpha}}\,, \tag{5.144}$$

and we can define the spatial derivative by

$$D_{\alpha\beta} = t^{\dot{\alpha}}_{(\alpha} \nabla_{\beta)\dot{\alpha}}\,. \tag{5.145}$$

Given a time-independent Weyl spinor, Geroch defines a sequence of higher spin fields by

$${}_{(n)}\phi_{\alpha_1 \ldots \alpha_{2n}} = D_{\alpha_{2n}\alpha_{2n-1}} \ldots D_{\alpha_6\alpha_5} \psi_{\alpha_4\alpha_3\alpha_2\alpha_1}\,. \tag{5.146}$$

It is easy to see that this is totally symmetric and that it satisfies (5.18) for all n as a consequence of the field on $\psi_{\alpha\ldots\delta}$, and (5.142); $n = 2$ gives back the Weyl spinor. In fact, such a time-independent Weyl spinor is automatically curl-free as a consequence of the linearized Bianchi identity, and so is a second derivative of a complex potential

$$\psi_{\alpha\beta\gamma\delta} = D_{\alpha\beta} D_{\gamma\delta} \phi\,, \tag{5.147}$$

which is the real Newtonian potential in the static case, but more generally complex with a magnetic imaginary part. It is a time-independent solution to the wave equation $t \cdot \nabla\phi = 0$, $\nabla^2\phi = 0$, which of course reduces to Laplace's equation. With this,

we see

$$ {}_{(n)}\phi_{\alpha_1\ldots\alpha_{2n}} = D_{\alpha_1\alpha_2}\ldots D_{\alpha_{2n-1}\alpha_{2n}}\phi\,. \tag{5.148} $$

If we flip the dotted index of (5.19) with $t_{\alpha\dot{\alpha}}$ and symmetrize over all spinor indices with $s+1=2n$ is even we obtain

$$ D^{(\alpha\beta}\xi^{\alpha_1\ldots\alpha_{2n})} = 0\,. \tag{5.149} $$

Identifying symmetric pairs of spinor indices with 3-vector indices as above, we obtain $\xi^{i_1\ldots i_n}$ symmetric and trace-free over all its indices, whose trace-free symmetrized derivative vanishes so that it satisfies the 3-dimensional conformal Killing tensor equation

$$ D^{(j}\xi^{i_1\cdots i_n)} = g_{3d}^{(j\,i_1}K^{i_2\cdots i_n)}\,, \tag{5.150} $$

for some $K^{i_2\ldots i_n}$. Here the three dimensional metric is given by $g_{3d}^{ij} = g_{3d}^{\alpha\beta\gamma\delta} = \epsilon^{\alpha(\gamma|}\epsilon^{\beta|\delta)}$ when written in terms of spinor indices. With these definitions, we see that (5.17) reduces to the Geroch definition of multipole moments as given and applied in [160, 161, 162]

$$ H_s = \oint_{\mathcal{S}} \xi^{i_1\cdots i_n} D_j D_{i_1}\cdots D_{i_n}\phi\, dS^j\,. \tag{5.151} $$

To conclude, we note that while the form of the multiple formula is identical for our charges and Geroch-Hansen multipole moments, the recursion relations used to build the higher valence z.r.m. field are different and therefore will give expressions with different numerical coefficients. At null infinity the component of ${}_{(n)}\phi_{\alpha_1\ldots\alpha_{2n}}$ are proportional to $\mathcal{Q}_n$ up to a numerical constant, see Section 3.2 in [177] for more details.

Chapter 6

Conclusion

In this work, we have shown that twistor space provides the natural framework to understand the $Lw_{1+\infty}$ and S algebra symmetries and their associated charges. The crucial advantage of twistor space compared to space-time is that the action of the symmetries becomes local and geometric, emerging from the gauge symmetries of the twistor space action. This allows for the application of standard covariant phase space methods. This analysis allowed us to better understand the organization of the solutions of gravity and Yang-Mills in asymptotically flat spacetimes. Furthermore, using the same methods, we were able to make connections to multipole moments, quasi-local charges, and integrable systems in the bulk using the Penrose transform and the non-linear graviton constructions. Several questions remain to be understood.

Non-vanishing Cosmological Constant: In chapter 3, we presented some partial results of how the analysis could be extended to spacetimes with a non-vanishing cosmological constant. The main difficulty is that both the geometry of the asymptotic twistor space and the nature of the asymptotic data have changed. The construction of the asymptotic twistor space can be traced back to developments of the $\mathcal{H}$-space [120], and luckily there exists an $\mathcal{H}$-space for the case of a non-vanishing cosmological constant [184]. In LeBrun's construction, he proves that for geodesically

convex 3-manifolds $\partial\mathcal{M}$ with a holomorphic metric, the space of null geodesics is a 3-complex-dimensional manifold also called ambitwistor space $\mathbb{A}$. Ambitwistor space is defined as follows, see [51] for a review: first, define the space of complex null directions $T^*_N\partial M = \{(x,p) \in T^*\partial M | g^{-1}(p,p) = 0\}$ and then the ambitwistor space is the quotient space $\mathbb{A} = T^*_N\partial M/D_0$ where flows along D_0 generate null geodesics

$$D_0 = p^\mu \left(\frac{\partial}{\partial x^\mu} + \Gamma^\rho_{\mu\nu} p_\rho \frac{\partial}{\partial p_\nu} \right) , \tag{6.1}$$

where $\Gamma^\rho_{\mu\nu}$ is the Levi-Civita connection for the 3-metric. The cotangent bundle has a natural symplectic form $\omega := dp_\mu \wedge dx^\mu$ or a symplectic potential $\tau := p_\mu dx^\mu$ such that $\omega = d\tau$. As it turns out, D_0 is the Hamiltonian vector field corresponding to the Hamiltonian $f = -\frac{1}{2} g^{\mu\nu} p_\mu p_\nu$ with respect to ω, i.e. $D_0 \lrcorner\, \omega = df$. Furthermore, one can show,

$$\mathcal{L}_{D_0}\tau - \frac{1}{2} d\left(g^{\mu\nu} p_\mu p_\nu\right) = 0, \quad D_0 \lrcorner\, \tau = g^{\mu\nu} p_\mu p_\nu \,, \tag{6.2}$$

so that on restriction to the null tangent space, $\mathcal{L}_{D_0}\tau = 0$, $D_0 \lrcorner \tau = 0$ and τ descends to $\mathbb{A}$. One can further go to the projective ambitwistor space $\mathbb{PA} = \mathbb{A}/p \cdot \partial_p$ where we quotient by the Euler vector field on the cotangent bundle.

The null condition $p^2 = 0$ defines a quadric in the cotangent space, and for the case of a 3-manifold this quadric is a $\mathbb{CP}^1$. The set of geodesics that pass through a point $x \in \partial M$ defines a rational curve $L_x \subset \mathbb{PA}$ (this is defined by the $p^2 = 0$) which turns out to have normal bundle $N = \mathcal{O}(1) \oplus \mathcal{O}(1)$. There is only a 3-parameter family coming from points of ∂M, but theorems of Kodaira state [86] that there is a 4-parameter family M of sections of $\mathcal{O}(1) \oplus \mathcal{O}(1)$. In other words, $\mathbb{PA}$ is also a regular twistor space that has the extra structure of the contact structure τ. We are now in a familiar situation; the theorem of Ward [98] discussed in Section 2.2.6 states there is a 1-1 correspondence between self-dual solutions to Einstein's vacuum equation with non-vanishing cosmological constant and complex 3-manifolds see and the complex 3-manifold $\mathbb{PA}$ along with a contact potential τ valued in $K^{-1/2}$ where K

is the canonical bundle over $\mathbb{PA}$ and rational curves with normal bundle $\mathcal{O}(1)\oplus\mathcal{O}(1)$. One can uniquely construct the 1-form τ such that $\tau|_X = 0$ when $x \in \partial M$ and $\tau|_X \neq 0$ when $x \in M - \partial M$

This construction only requires the 3-metric to reconstruct the bulk self-dual 4-manifold. However, the Fefferman-Graham expansion [29] teaches us that one needs both the boundary 3-metric and a trace-free tensor $T_{\mu\nu}$ satisfying the divergence-free condition to reconstruct the bulk metric. From this perspective, it is clear why we have landed in the SD sector and that the homogeneity -6 cohomology class on ambitwistor space will give the stress-energy tensor on the boundary while giving an ASD perturbation on the $\mathcal{H}$-space. These ideas can be applied to recent work on light-ray operators for CFT_3 manifesting the $\text{Ham}_\Lambda(\mathbb{C}^2)$ algebra [185] and the asymptotic symmetry analysis of Λ-BMS [33]. A more speculative direction would be to consider what ambitwistor string correlation functions on the boundary would correspond to in the bulk through the Penrose transform.

Spatial Infinity: The construction of asymptotic twistor space in Section 3 utilised the blow-up of the twistor line corresponding to spatial infinity. The result of this procedure generated an exceptional divisor $E = \mathbb{CP}^1 \times \mathbb{CP}^1$, which is the mini-twistor space [186, 187], after suitable reality conditions, for dS_3. The appearance of de Sitter space is not a surprise if one considers an expansion of the bulk metric around i^0. In Beig-Schmidt gauge [188], the expansion of a large class of asymptotically flat metrics around i^0 is given by

$$g = \left(1 + \frac{\sigma}{r}\right)^2 dr^2 + r^2 \left(\gamma_{ab} + \frac{h^{(1)}_{ab}}{r} + \cdots\right) dx^a dx^b \,, \tag{6.3}$$

where (r, x^a) with $a = 1, 2, 3$ are the coordinates. After conformal compactification, the metric around i^0 is given by the metric for Lorentzian dS_3

$$\gamma_{ab} = -dt^2 + \cosh^2(t)(d\theta^2 + \sin^2(\theta) d\phi^2) \,. \tag{6.4}$$

The Beig-Schmidt expansion plays the role at spatial infinity that the Bondi expansion plays at null infinity. Furthermore, the Beig-Schmidt gauge serves to blow-up the point at spatial infinity, and one obtains exactly dS_3 as was expected from twistor theory. In the leading-order solution space, one finds the mass aspect σ and the tensor

$$k_{ab} = h^{(1)}_{ab} + 2\sigma\gamma_{ab}\,. \tag{6.5}$$

After imposing gauge conditions on the supertranslation that give a well-defined variational principle, the Einstein equations on the leading-order data become

$$(\Box_{\mathrm{dS}_3} + 3)\,\sigma = 0\,, \quad D^a k_{ab} = 0\,, \quad (\Box_{\mathrm{dS}_3} - 3)\,k_{ab} = 0\,, \tag{6.6}$$

where D_a is the covariant derivative for dS_3 and $\Box_{\mathrm{dS}_3} = \gamma^{ab} D_a D_b$. These equations can be solved using the Penrose transform for cohomology classes on minitwistor space [189] and integral formulae adapted to this correspondence. It is an open question how these cohomology elements are related to those in the rest of the asymptotic twistor space. However, one may hope that this could give a twistorial version of the antipodal matching conditions [27, 28]. There is also a question of how the curved minitwistor space construction, which gives Einstein-Weyl spaces [187], may fit in this picture.

Beyond Self-Duality: In this work, we focused on the self-dual sector of gravity and Yang-Mills for which the twistor actions take a very simple form. Although at $\mathscr{I}$, the asymptotic solution space of self-dual gravity and Yang-Mills is identical to that of the full theory, it will start to differ in the subleading orders of the expansion. It would be interesting to understand where this difference exactly occurs and see to what extent the $Lw_{1+\infty}$ and S algebra symmetries survive beyond the self-dual sector. We remark that a twistor action for full gravity was found by Sharma in [190], but being based on reducing the twistor action for conformal gravity of [191],

it does not sit so well with the self-dual twistor action of [101] used here; see also [97, 192] for related discussion. For Yang-Mills, the twistor action for the full theory is much better understood [193].

Such symmetries are not expected to extend to the full Yang-Mills or Einstein vacuum equations in the bulk, which are not believed to be integrable. There is therefore no direct evidence that these should be symmetries of the S-matrix for gravity or Yang-Mills save that they act naturally at the boundary. Indeed, the integrability of the self-dual sector implies much more than the symmetry of the S-matrix; it implies the vanishing of all scattering amplitudes except for the 3-point $\overline{\text{MHV}}$ amplitude and the 1-loop all-positive-helicity amplitude [97]. So, the lack of integrability does not preclude some natural action on the amplitude. However, it is expected to be more complicated than the Ward identities arising from the extended BMS symmetries at low spin.

Loop Level: Finally, there is a question of how one may extend these ideas to the quantum level. It would be interesting if one could reproduce the quantum deformations to the algebras found in [57, 58] by taking the charges we have found in this thesis and considering their algebra after quantisation. The twistor actions we have used in this thesis have been discussed at the quantum level [194, 195] where they are found to be anomalous and must be coupled to additional fields to be consistent. With these considerations, one would construct the Green's functions for the $\bar{\partial}$ operator on twistor space [194, 195] and compute the OPE between the charges to find their algebra, including any quantum corrections. To make these computations more systematic, it would be fruitful to use the BV formalism, as it is well suited for quantum considerations.

Appendix A

Further Details of the NP and GHP Formalism

In this appendix, we provide the Einstein equations in the NP formalism.

The equations satisfied by the spin coefficients arise by considering the components of (2.22) and ar given by

$$
\begin{aligned}
&\text{þ}\rho - \eth\kappa = \rho^2 + \sigma\bar{\sigma} - \bar{\kappa}\tau + \pi\kappa + \phi_{00} && \bar{\eth}\nu - \text{þ}'\mu = \mu^2 + \lambda\bar{\lambda} - \bar{\nu}\pi + \tau\nu + \phi_{22}\\
&\text{þ}\sigma - \bar{\eth}\kappa = (\rho + \bar{\rho})\sigma - (\tau - \bar{\pi})\kappa + \psi_0 && \eth\nu - \text{þ}'\lambda = (\mu + \bar{\mu})\lambda + (\bar{\tau} - \pi)\nu + \psi_4\\
&\text{þ}\tau - \text{þ}'\kappa = (\tau + \bar{\pi})\rho + (\bar{\tau} + \pi)\sigma + \psi_1 && \text{þ}\nu - \text{þ}'\pi = (\pi + \bar{\tau})\mu + (\bar{\pi} + \tau)\lambda + \psi_3 + \phi_{21}\\
&\bar{\eth}\rho - \eth\sigma = (\rho - \bar{\rho})\tau + (\mu - \bar{\mu})\kappa - \psi_1 + \phi_{01} && \bar{\eth}\lambda - \eth\mu = (\mu - \bar{\mu})\pi - (\bar{\rho} - \rho)\nu - \psi_3 + \phi_{21}\\
&\bar{\eth}\tau - \text{þ}'\sigma = \mu\sigma + \bar{\lambda}\rho + \tau^2 - \nu\kappa + \phi_{02} && \text{þ}\lambda - \eth\pi = \rho\lambda + \bar{\sigma}\mu + \pi^2 - \nu\bar{\kappa} + \phi_{20}\\
&\text{þ}'\rho - \eth\tau = \kappa\nu - \rho\bar{\mu} - \sigma\lambda - \tau\bar{\tau} - \psi_2 - 2\Lambda && \bar{\eth}\pi - \text{þ}\mu = \nu\kappa - \mu\bar{\rho} - \lambda\sigma - \pi\bar{\pi} - \psi_2 - 2\Lambda
\end{aligned}
\tag{A.1}
$$

along with their SD versions. The fields ϕ_{ij} are related to contractions of the spin

field $\Phi_{\alpha\beta\dot{\alpha}\dot{\beta}}$ and are explicitly given by

$$\begin{aligned}
\phi_{00} &= \Phi_{\alpha\beta\dot{\alpha}\dot{\beta}} o^{\alpha} o^{\beta} \bar{o}^{\dot{\alpha}} \bar{o}^{\dot{\beta}} \quad \phi_{10} = \Phi_{\alpha\beta\dot{\alpha}\dot{\beta}} o^{\alpha} \iota^{\beta} \bar{o}^{\dot{\alpha}} \bar{o}^{\dot{\beta}} \quad \phi_{20} = \Phi_{\alpha\beta\dot{\alpha}\dot{\beta}} \iota^{\alpha} \iota^{\beta} \bar{o}^{\dot{\alpha}} \bar{o}^{\dot{\beta}} \\
\phi_{01} &= \Phi_{\alpha\beta\dot{\alpha}\dot{\beta}} o^{\alpha} o^{\beta} \bar{o}^{\dot{\alpha}} \bar{\iota}^{\dot{\beta}} \quad \phi_{11} = \Phi_{\alpha\beta\dot{\alpha}\dot{\beta}} o^{\alpha} \iota^{\beta} \bar{o}^{\dot{\alpha}} \bar{\iota}^{\dot{\beta}} \quad \phi_{21} = \Phi_{\alpha\beta\dot{\alpha}\dot{\beta}} \iota^{\alpha} \iota^{\beta} \bar{o}^{\dot{\alpha}} \bar{\iota}^{\dot{\beta}} \\
\phi_{02} &= \Phi_{\alpha\beta\dot{\alpha}\dot{\beta}} o^{\alpha} o^{\beta} \bar{\iota}^{\dot{\alpha}} \bar{\iota}^{\dot{\beta}} \quad \phi_{12} = \Phi_{\alpha\beta\dot{\alpha}\dot{\beta}} o^{\alpha} \iota^{\beta} \bar{\iota}^{\dot{\alpha}} \bar{\iota}^{\dot{\beta}} \quad \phi_{22} = \Phi_{\alpha\beta\dot{\alpha}\dot{\beta}} \iota^{\alpha} \iota^{\beta} \bar{\iota}^{\dot{\alpha}} \bar{\iota}^{\dot{\beta}}
\end{aligned} \tag{A.2}$$

The remaining spin coefficient identities are encoded in the commutators of the GHP operators. Given a (p, q)-scalar η the commutator relations are given by

$$\begin{aligned}
[\text{þ}, \text{þ}'] &= (\bar{\tau} + \pi)\bar{\text{ð}} + (\tau + \bar{\pi})\text{ð} - p(\pi\tau - \kappa\nu + \psi_2 + \phi_{11} - \Lambda) - q(\bar{\pi}\bar{\tau} - \bar{\kappa}\bar{\nu} + \bar{\psi}_2 + \phi_{11} - \Lambda) \\
[\text{þ}, \bar{\text{ð}}] &= \bar{\rho}\bar{\text{ð}} + \sigma\text{ð} + \bar{\pi}\text{þ} - \kappa\text{þ}' - p\left(\pi\sigma - \mu\kappa + \psi_1\right) - q(\bar{\rho}\bar{\pi} - \bar{\lambda}\bar{\kappa} + \phi_{10}) \\
[\bar{\text{ð}}, \text{ð}] &= (\mu - \bar{\mu})\text{þ} + (\rho - \bar{\rho})\text{þ}' + p(\sigma\lambda - \rho\mu + \psi_2 - \phi_{11} - \Lambda) - q(\bar{\sigma}\bar{\lambda} - \bar{\rho}\bar{\mu} + \bar{\psi}_2 - \phi_{11} - \Lambda)
\end{aligned} \tag{A.3}$$

The last three commutators can be obtained by complex conjugation and by replacing $\iota^{\alpha}, \bar{\iota}^{\dot{\alpha}} \leftrightarrow o^{\alpha}, \bar{o}^{\dot{\alpha}}$.

Finally, the second Bianchi identities in vacuum for simplicity are given by

$$\begin{aligned}
&\text{þ}'\psi_0 - \bar{\text{ð}}\psi_1 + \mu\psi_0 + 4\tau\psi_1 - 3\sigma\psi_2 = 0, \\
&\text{þ}'\psi_1 - \bar{\text{ð}}\psi_2 - \nu\psi_0 + 2\mu\psi_1 + 3\tau\psi_2 - 2\sigma\psi_3 = 0, \\
&\text{þ}'\psi_2 - \bar{\text{ð}}\psi_3 - 2\nu\psi_1 + 3\mu\psi_2 + 2\tau\psi_3 - \sigma\psi_4 = 0, \\
&\text{þ}'\psi_3 - \bar{\text{ð}}\psi_4 - 3\nu\psi_2 + 4\mu\psi_3 + \tau\psi_4 = 0, \\
&\text{þ}\psi_1 - \text{ð}\psi_0 - \pi\psi_0 - 4\rho\psi_1 + 3\kappa\psi_2 = 0, \\
&\text{þ}\psi_2 - \text{ð}\psi_1 + \lambda\psi_0 - 2\pi\psi_1 - 3\rho\psi_2 + 2\kappa\psi_3 = 0, \\
&\text{þ}\psi_3 - \text{ð}\psi_2 + 2\lambda\psi_1 - 3\pi\psi_2 - 2\rho\psi_3 + \kappa\psi_4 = 0, \\
&\text{þ}\psi_4 - \text{ð}\psi_3 + 3\lambda\psi_2 - 4\pi\psi_3 - \rho\psi_4 = 0.
\end{aligned} \tag{A.4}$$

Appendix B

Sheaf Theory and Deformation Theory

B.1 Sheaf Theory

Given a complex compact manifold Z which has a finite open cover $U = \{U_i\}$ with the patching data for a point $z_k \in U_k$ then $z_j = f_{jk}(z_k) \in U_j$. A deformation of Z is given by different gluings of the same $\{U_i\}$ by $f_{jk}(z_k, t) = f_{jk}(z_k, t_1, \cdots, t_m)$ such that $f_{jk}(z_k, 0) = f_{jk}(z_k)$ gives the original and $t = \{t_1, \cdots, t_m\}$ are deformation parameters. In this way, we get a deformation Z_t. Let us consider

$$f^A_{ik}(z_k, t) = f^A_{ij} \circ f^B_{jk}(z_k, t), \tag{B.1}$$

where $A = 1, \cdots, n$ and $n = \dim(Z)$. This then gives

$$\begin{aligned} \frac{\partial f^A_{ik}}{\partial t}(z_k, t) &= \frac{\partial f^A_{ij}}{\partial t}(z_j, t) + \frac{\partial f^A_{ij}}{\partial z^B_j}(z_j, t)\frac{\partial f^B_{jk}}{\partial t}(z_k, t) \\ &= \frac{\partial f^A_{ij}}{\partial t}(z_j, t) + \frac{\partial z^A_i}{\partial z^B_j}\frac{\partial f^B_{jk}}{\partial t}(z_k, t), \end{aligned} \tag{B.2}$$

where we have used the fact that $z_i^A = f_{ij}^A(z_j, t)$. If we define the holomorphic vector field

$$\theta_{jk}(t) = \frac{\partial f_{jk}^A}{\partial t} \frac{\partial}{\partial z_j^A}, \tag{B.3}$$

we obtain the relations

$$\theta_{jk}(t) - \theta_{ik}(t) + \theta_{ij}(t) = 0 \quad , \quad \theta_{kj}(t) = -\theta_{jk}(t). \tag{B.4}$$

The second is a special case of the first, once we restrict $i = k$ and notice that $\theta_{kk}(t) = 0$. To understand these relations we need the language of sheaf cohomology.

Definition B.1.1. (Presheaf and Sheaf) A presheaf $\mathcal{S}$ on a topological space Z of abelian groups(or rings, or vector space, or algebras, etc.) consists of abelian groups $\mathcal{S}(U)$ for all open $U \subseteq Z$ and restriction homomorphisms

$$r_{UV} : \mathcal{S}(U) \to \mathcal{S}(V), \tag{B.5}$$

for all open $V \subseteq U$ with

$$\begin{aligned} r_{WV} \circ r_{VU} &= r_{WU}, \\ r_{UU} &= \mathrm{id}. \end{aligned} \tag{B.6}$$

A presheaf $\mathcal{S}$ on Z is a sheaf if in addition

1. $\forall s \in \mathcal{S}(U)$, if $U = \bigcup U_i$ is an open cover and $r_{U_iU}(s) = 0 \ \forall i$ then $s = 0$
2. If $U = \bigcup U_i$, $s_i \in \mathcal{S}(U_i)$ with

$$r_{U_i \cap U_j U_i}(s_i) = r_{U_i \cap U_j U_j}(s_j), \tag{B.7}$$

then $\exists s \in \mathcal{S}(U)$ with $r_{U_iU}(s) = s_i$

We will call elements of $\mathcal{S}(U)$ sections of the sheaf $\mathcal{S}$. The main idea here is

that we have some algebraic object living over each patch of our topological space and local sections can be pieced together to build a global section and that a section is uniquely determined by its local behavior. We have already seen the sheaf of holomorphic functions of weight n denoted by $\mathcal{O}(n)$ but there are many more. One can now ask if $s_U \in \mathcal{S}(U)$ and $s_V \in \mathcal{S}(V)$ when is there a $S \in \mathcal{S}(Z)$ with $r_{UZ}(s) = s_U$ and $r_{VZ}(s) = s_V$. This is the question that sheaf cohomology wishes to answer. If $\mathcal{U} = \{U_\alpha\}_\alpha$ is a locally finite open cover, we write

$$U_{\alpha_0} \cap \cdots \cap U_{\alpha_p} = U_{\alpha_0 \cdots \alpha_p}, \tag{B.8}$$

and define the p-cochains

$$C^p(\mathcal{U}, \mathcal{S}) = \prod_{\alpha_0 < \cdots < \alpha_p} \mathcal{S}(U_{\alpha_0 \cdots \alpha_p}). \tag{B.9}$$

For a p-cochain $\sigma \in C^p(\mathcal{U}, \mathcal{S})$ we set

$$\sigma_{\alpha_0 \cdots \alpha_i \alpha_{i+1} \cdots \alpha_p} = -\sigma_{\alpha_0 \cdots \alpha_{i+1} \alpha_i \cdots \alpha_p}, \tag{B.10}$$

and the boundary map

$$\delta : C^p(\mathcal{U}, \mathcal{S}) \to C^{p+1}(\mathcal{U}, \mathcal{S}), \tag{B.11}$$

is given by

$$(\delta\sigma)_{\alpha_0 \cdots \alpha_{p+1}} = \sum_{j=0}^{p+1} (-1)^j r_{U_{\alpha_0 \cdots \alpha_{p+1}} U_{\alpha_0 \cdots \hat{\alpha}_j \cdots \alpha_{p+1}}} (\sigma_{\alpha_0 \cdots \hat{\alpha}_j \cdots \alpha_{p+1}}). \tag{B.12}$$

In general one can show that $\delta^2 = 0$ and allows one to define a cohomology

Definition B.1.2. (Sheaf Cohomology) Let Z be a topological space and $\mathcal{U}$ a local

finite open cover of Z and $\mathcal{S}$ a sheaf on Z. The sheaf cohomology is given by

$$H^p(\mathcal{U},\mathcal{S}) = \frac{\ker(\delta : C^p(\mathcal{U},\mathcal{S}) \to C^{p+1}(\mathcal{U},\mathcal{S}))}{\mathrm{im}(\delta : C^{p-1}(\mathcal{U},\mathcal{S}) \to C^p(\mathcal{U},\mathcal{S}))}. \tag{B.13}$$

At this point, one can complain that these cohomologies are dependent on the particular cover we choose and are not something we can assign as a topological invariant to the space Z. There are several ways of getting around this, but the most practical is that of a Leray cover $\mathcal{U}$ such that

$$H^p(\mathcal{U},\mathcal{S}) \cong H^p(Z,\mathcal{S}). \tag{B.14}$$

Where $H^p(Z,\mathcal{S}) = \limind_{\mathcal{U}} H^p(\mathcal{U},\mathcal{S})$ is called the Čech cohomology group and is the inductive limit with respect to a refinement. The Leray covers have a special property that if $U \in \mathcal{U}$ then $H^p(U,\mathcal{S}) = 0$ for $p > 0$, a proof can be found in [72]. Here we use the idea of building the global cohomology out of cohomologically trivial pieces, just as in the case of simplicial homology. Many isomorphisms relate the Čech cohomology over a particular sheaf to other well-known cohomology theories. One of particular interest is Dolbeaut's isomorphism [196, 124]

$$H^q(Z,\Omega^p(E)) \cong H^{p,q}(Z,\mathcal{E}), \tag{B.15}$$

where $E \to Z$ is a holomorphic vector bundle and $\Omega^p(E)$ the sheaf of holomorphic p-forms valued in holomorphic sections of E, and $\mathcal{E}$ is the sheaf of C^∞ sections of E . In twistor theory, there are often two pictures of either thinking in terms of Čech cohomology or Dolbeault cohomology depending on which one works best in the particular circumstance. This is seen most readily in the Penrose transform where an integral formula exists for both objects.

To be more explicit, let us consider the example of the Čech-Dolbeaut isomorphism

in the case of

$$H^1(Z,E) \cong H^{0,1}(Z,\mathcal{E})\,. \tag{B.16}$$

Now given a Leray cover $\mathcal{U} = \{U_i\}$, then an element of $H^1(Z,E)$ is given by a collection of holomorphic sections of E, $[\sigma_{ij}]$, such that $\sigma_{ij}+\sigma_{jk}+\sigma_{ki} = 0$ modulo the gauge redundancy $\sigma_{ij} \sim \sigma_{ij} + g_i - g_j$. To turn a Čech representative into a Dolbeault representative, we make use of a partition of unity f_i subordinate to the cover $\{U_i\}$. This means that f_i only has support in U_i and the condition $\sum_i f_i = 1$ holds. If we define $\sigma_i = \sum_j \sigma_{ij} f_j$, then from the cocycle condition we find

$$\sigma_i - \sigma_j = \sum_k (\sigma_{ik} - \sigma_{jk}) f_k = \sum_k \sigma_{ij} f_k = \sigma_{ij}\,. \tag{B.17}$$

This tells us that the difference $\sigma_i - \sigma_j$ is holomorphic on the overlap $U_i \cap U_j$, and therefore

$$\bar\partial\sigma_i - \bar\partial\sigma_j = \bar\partial\sigma_{ij} = 0\,. \tag{B.18}$$

We find that the $\bar\partial\sigma_i$ agree on overlaps, and as a consequence, give a globally defined $(0,1)$-form ω, which is naturally closed, and is defined patch-wise $\omega|_{U_i} = \bar\partial\sigma_i$. The gauge freedom to add $g_i - g_j$ to σ_{ij} amounts to adding $-\bar\partial(\Sigma_j g_j f_j)$ to ω.

To go the other way, we begin with a Dolbeault representative ω and solve the equation $\bar\partial\sigma_i = \omega$ in U_i, which is always possible since the cover was assumed to be Leray and, as such, every U_i is topologically trivial, so that the $\bar\partial$-Poincaré lemma can be used. Then, if we define $\sigma_{ij} = \sigma_i - \sigma_j$, it must be the case that $\bar\partial\sigma_{ij} = 0$ on the overlap $U_i \cap U_j$ since ω was a global $(0,1)$-form. The σ_{ij} built in this way are the Čech representatives of the Dolbeault form ω. The gauge redundancy to add $\bar\partial\beta$ to ω does not change the definition of σ_{ij}, but there is a redundancy in defining $\sigma_i \sim \sigma_i + g_i$ where g_i is holomorphic in U_i when we solved $\bar\partial\sigma_i = \omega$. This redundancy then results in the gauge freedom expected in Čech cohomology $\sigma_{ij} \sim \sigma_{ij} + g_i - g_j$.

Going back to (B.4), we see that this is the closure condition for a 1-cocycle and

$\theta_{jk}(t)$ are sections of the sheaf of holomorphic vector fields $\mathcal{V}_t$ living over $U_j \cap U_k$. Therefore, infinitesimal deformations of complex structures of the original complex manifold Z are given by

$$[\theta_{jk}] \in H^1(Z, \mathcal{V}) \cong H^{0,1}(Z, T^{1,0}Z). \tag{B.19}$$

Via Dolbeaut's isomorphism, we expect there to exists a $(0,1)$-form vector field that measures linear deformations of the complex structure. As explained in Section 2.2.6, deformations of complex structure can be viewed as changes to the Dolbeaut operator $\bar{\partial} \to \bar{\partial} + V(t)$ where $V(t) \in \Omega^{0,1}(Z, T^{1,0}Z)$ and is a function of the deformation parameter. For the new complex structure to be integrable, the Maurer-Cartan equation must be satisfied

$$\bar{\partial} V(t) + \frac{1}{2}[V(t), V(t)] = 0 \tag{B.20}$$

If we expand $V(t)$ and noting that $V(0) = 0$ as this value of t corresponds to the undeformed complex structure, we obtain the formal power series

$$V(t) = tV_1 + t^2 V_2 \cdots \tag{B.21}$$

Inserting this expansion into the Maurer-Cartan equation and solving order by order in t we find a hierarchy of equations that must be solved

$$\begin{aligned} \bar{\partial} V_1 &= 0 \\ \bar{\partial} V_2 + [V_1, V_1] &= 0 \\ &\vdots \\ \bar{\partial} V_k + \sum_{0<i<k} [V_i, V_{k-i}] &= 0 \end{aligned} \tag{B.22}$$

From this, we see that the first-order perturbation V_1 is determined by a $(0,1)$-form valued in the holomorphic tangent bundle and hence $V_1 \in H^{0,1}(Z, T^{1,0})$. Thus, the

precise relationship to θ_{jk} can be constructed using the Čech-Dolbeaut isomorphism

$$\bar{\partial}\theta_i = \left.\frac{\partial V}{\partial t}\right|_{t=0} \quad , \quad \theta_i = \sum_j \theta_{ij}\rho_j, \tag{B.23}$$

where θ_i is defined through the use of a partition of unity ρ_i. To get the full complex structure, one must solve the Maurer-Cartan equation order by order, but obstructions may occur at any order; see [197] for more details.

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