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ARTICLE

Graphical sequences and plane trees

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Abstract

Balister, the second author, Groenland, Johnston, and Scott recently showed that there are asymptotically $C4^n/n^{3/4}$ many unordered sequences that occur as degree sequences of graphs with n vertices. Combining limit theory for infinitely divisible distributions with a new connection between a class of random walk trajectories and a subset counting formula from additive number theory, we describe C in terms of Walkup's number of rooted plane trees. The bijection is related to an instance of the Lévy–Khintchine formula. Our main result complements a result of Stanley, that ordered graphical sequences are related to quasi-forests.

Keywords: asymptotic enumeration; degree sequence; graphical sequence; infinite divisibility; Lévy–Khintchine formula; random walk; renewal theory

2020 MSC Codes: Primary: 05A16, 05C30; Secondary: 05A15, 05A17, 11P21, 51M20, 52B05, 60E07, 60G50, 60K05

1. Introduction

A sequence $d_1 \le \cdots \le d_n$ is *graphical* if there is a graph on n vertices with this degree sequence. The criteria under which sequences are graphical are well-known; see Havel [11], Hakimi [10], and Erdős and Gallai [7]. Recently, Balister, the second author, Groenland, Johnston, and Scott [2] showed that the number $\mathcal{G}_n$ of such sequences satisfies $n^{3/4}\mathcal{G}_n/4^n \to C$. The constant C is expressed as a certain random walk probability ρ; see (4) below.

In this work, we show that C can be expressed in terms of the number $\mathcal{T}_n$ of rooted unlabelled plane trees with n edges. Note that, in counting such trees, two trees are equivalent if one can be transformed into the other by cyclically permuting its subtrees about the root; see Figure 1 below. Using Pólya's enumeration theorem, Walkup [19] showed that

$$\mathcal{T}_n = \frac{1}{n}\sum_{d|n}\binom{2d-1}{d}\phi(n/d), \tag{1}$$

where $\phi(n)$ is Euler's totient function. Our main result reveals the following connection between plane trees and graphical sequences.

Theorem 1.1. *As $n \to \infty$, we have that*

$$\frac{n^{3/4}}{4^n}\mathcal{G}_n \to \frac{\Gamma(3/4)}{2^{5/2}\pi}e^{\omega}, \tag{2}$$

where

$$\omega = \sum_{k=1}^{\infty}\frac{\mathcal{T}_k}{k4^k}. \tag{3}$$

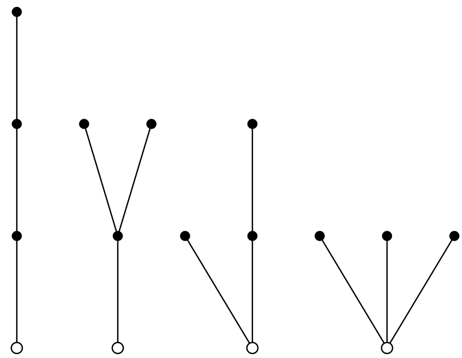

Figure 1. The $\mathcal{T}_3 = 4$ rooted unlabelled plane trees with 3 edges.

Above, as usual, $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ is the gamma function. We call ω *Walkup's constant*.

The formula (1) can be used to numerically approximate the limiting constant on the right-hand side in (2); see (6) below.

The proof of Theorem 1.1 involves an interplay between additive number theory, random walks, and renewal theory. The connection with number theory arises through a subset counting formula by von Sterneck (see, e.g., [1, 16]), which shows that $\mathcal{T}_n$ as in (1) coincides with the number of submultisets of $\{0, 1, \ldots, n-1\}$ of size n that sum to 0 mod n. As we will see, this will allow us to relate plane trees to lattice paths (random walk trajectories). The connection between random walks and graphical sequences was observed in [2] (similar to the lattice path representation of tournament score sequences observed by Erdős and Moser; see, e.g., Moon [14]). In studying the random walks related to graphical sequences, we will use limit theory developed by Hawkes and Jenkins [12] (cf. Embrechts and Hawkes [6]) for infinitely divisible sequences (of which renewal sequences are a special case) related to the Lévy–Khintchine formula from the theory of Lévy processes. See Sections 1.2 and 2 below for more details.

1.1 Degree sequences, via random walks

As already mentioned, Erdős and Moser observed a connection between random walks and degree sequences in the context of graph tournaments. A similar connection with random walks is used in [2] to show that

$$\frac{n^{3/4}}{4^n} \mathcal{G}_n \to \frac{\Gamma(3/4)}{2^{5/2}\pi} \frac{1}{\sqrt{1-\rho}}, \tag{4}$$

where ρ is defined as follows.

Let $(Y_k, k \geq 0)$ be a lazy simple symmetric random walk, started at $Y_0 = 0$, with increments $Y_{k+1} - Y_k$ equal to ± 1 with probability 1/4, and 0 with probability 1/2. Let

$$\tau = \inf\{k \geq 1 : Y_k = 0,\ A_k \leq 0\},$$

where $A_k = \sum_{i=1}^k Y_i$ is the area after k steps. Then

$$\rho = \mathbf{P}(A_\tau = 0). \tag{5}$$

In proving Theorem 1.1, we obtain the following description of ρ, in terms of Walkup's constant ω.

Theorem 1.2. *We have that* $\rho = 1 - e^{-2\omega}$.

We note that Theorem 1.1 follows by combining Theorem 1.2 with (4).

1.2 Proof overview

There is a natural representation of the lazy simple symmetric random walk $(Y_k, 0 \le k \le n)$ in terms of a simple symmetric random walk $(X_k, 0 \le k \le 2n)$ of twice the length. In this representation, the area process is $A_k = \sum_{i=1}^k Y_i = \frac{1}{2}\sum_{i=1}^k X_{2i}$. This relationship has a geometric interpretation; see the *diamond area* discussed in Section 3 below.

If $X_{2n} = 0$ and $A_k \ge 0$ at all times $0 \le k \le n$ for which $X_{2k} = 0$, we call $(X_k, 0 \le k \le 2n)$ a *graphical walk* of length $2n$. A *graphical bridge* is a graphical walk whose area returns to 0 at the end of its trajectory, and a *graphical meander* is a graphical walk whose area never returns to 0. Times k when $X_{2k} = A_k = 0$ play the role of renewal times, since initially $X_0 = A_0 = 0$. Hence a graphical walk can be decomposed into a series of graphical bridges and one final graphical meander. That is, the sequence $(\mathcal{W}_n, n \ge 1)$, where $\mathcal{W}_n$ is the number of graphical walks of length $2n$, is a *delayed renewal sequence*, where the length of the final meander can be understood as the 'delay'.

The term $\Gamma(3/4)/2^{5/2}\pi$ in (4) is related to the number of graphical meanders. In this work, to describe the other term $1/\sqrt{1-\rho}$ appearing in (4), we analyse the *renewal sequence* (without a delay) $(\mathcal{B}_n, n \ge 1)$, where $\mathcal{B}_n$ is the number of graphical bridges of length $2n$. See Sections 2.1 and 2.2 below for more on renewal sequences, and Section 2.3 for more on their connection to the enumeration of $\mathcal{G}_n$.

To study the asymptotics of $\mathcal{B}_n$, we use the asymptotic transference theory, developed by Hawkes and Jenkins [12] (cf. Embrechts and Hawkes [6]), for infinitely divisible distributions. We call this method the *Lévy–Khintchine method* (see Section 2 below), as it is related to the Lévy–Khintchine transform, which associates each such distribution with a corresponding Lévy process, via another measure called its Lévy measure. We show that the Lévy–Khintchine transform $\mathcal{B}_n^*$ of $\mathcal{B}_n$ is related to $\mathcal{T}_n$. In fact, $\mathcal{B}_n^* = 2\mathcal{T}_n$. More specifically, we find that $p_n = e^{-2\omega}\mathcal{B}^n/4^n$ is an infinitely divisible probability distribution with Lévy measure $\nu_n = 2\mathcal{T}_n/n4^n$. The measure ν_n is regularly varying, and it then follows by [12] that $p_n \sim \nu_n$.

Our proof that $\mathcal{B}_n^* = 2\mathcal{T}_n$ uses a characterisation of the Lévy–Khintchine transform of renewal sequences; see Lemma 2.1 below. The proof also involves some combinatorial arguments: roughly speaking, by interleaving the steps of certain lattice paths, we relate the set of cyclical shifts of graphical bridges and the set of rooted plane trees. Our bijective arguments are geometric, using the diamond area; see Figs. 2, 3, 4, 5 and 6.

Since $\mathcal{B}_n$ is a renewal sequence, ν_n is also the harmonic renewal measure associated with the sequence $\mathcal{B}_n^{(1)}$ of *irreducible* graphical bridges. Using this, it follows that $e^{-2\omega}\nu_n/p_n$ is the harmonic moment $\mathbf{E}[1/\mathcal{I}_n]$ of the number $\mathcal{I}_n$ of irreducible graphical bridges in a uniformly random graphical bridge of length n and therefore $\mathbf{E}[1/\mathcal{I}_n] \to e^{-2\omega}$ (since $p_n \sim \nu_n$).

Moreover, it can be shown that $\mathcal{I}_n$ converges to a negative binomial with parameters $r = 2$ and $p = 1 - \rho$. Intuitively, renewal times will only occur near the start and end of the bridge. The numbers of such times on either side of the bridge are approximately independent and geometric with $p = 1 - \rho$. Therefore, $\mathbf{E}[1/\mathcal{I}_n] \to 1 - \rho$, and so we find that $1 - \rho = e^{-2\omega}$, yielding Theorem 1.2. Hence, $1/\sqrt{1-\rho} = e^{\omega}$, and so Theorem 1.1 follows by (4).

1.3 Ordered graphical sequences

The polytope D_n of degree sequences was introduced by Koren [13] and studied by Peled and Srinivasan [15]. Stanley [18, Corollary 3.4] found a formula for the number of lattice points in $D_n \cap \mathbf{Z}^n$ with even sum, expressed in terms of quasi-forests. Such points are associated with *ordered* graphical sequences $d_1, \ldots, d_n$. On the other hand, Theorem 1.1 relates the number of *unordered* graphical sequences $d_1 \le \cdots \le d_n$ and Walkup's plane trees.

1.4 Numerics

Finally, let us note that the expression for C in Theorem 1.1 can be used to approximate $\mathcal{G}_n$, for large n. Observing that the formula (1) for $\mathcal{T}_n$ is dominated by the $d = n$ term, one can show that

$$\frac{n^{3/4}}{4^n}\mathcal{G}_n \to C \approx 0.09909408323748874536144934 0935. \tag{6}$$

See the discussion of this work in [2] for more details.

2. The Lévy–Khintchine method

A random variable X is *infinitely divisible* if, for all $n \geq 1$, there are independent and identically distributed X_i such that $\sum_{i=1}^{n} X_i$ and X are equal in distribution (see, e.g., [8]).

Suppose that a positive sequence $(1 = a_0, a_1, \dots)$ is summable, so that it is proportional to a probability distribution $(\pi_n, n \geq 0)$ on the non-negative integers $n \geq 0$. As is well-known (see, e.g., [12]) such a $(\pi_n, n \geq 0)$ is infinitely divisible if and only if, for some non-negative sequence $(a_1^*, a_2^*, \dots)$, we have that

$$\sum_{n=0}^{\infty} a_n x^n = \exp\left(\sum_{k=1}^{\infty} \frac{a_k^*}{k} x^k\right). \tag{7}$$

In this case, we call $(a_0, a_1, \dots)$ an *infinitely divisible sequence*. By taking logs and differentiating with respect to x on both sides of (7) and then comparing coefficients, we see that (7) is equivalent to the recursion

$$n a_n = \sum_{i=1}^{n} a_i^* a_{n-i}, \qquad n \geq 1. \tag{8}$$

Since, as discussed in [6], (7) is a special case of the Lévy–Khintchine formula, we call a_n^* the *Lévy–Khintchine transform* of a_n. We note that, in fact, $(\nu_n = a_n^*/n, n \geq 1)$ is the *Lévy measure* (see, e.g., [4]) associated with the *Lévy process* $(\mathcal{L}_t : t \geq 0)$, for which the law of the process $\mathcal{L}_1$ at time $t = 1$ is described by the infinitely divisible probability measure

$$\pi_n = a_n \exp\left(-\sum_{k=1}^{\infty} \frac{a_k^*}{k}\right), \qquad n \geq 0.$$

In combinatorial settings, it is convenient to consider $a_n = A_n/\alpha^n$, where A_n enumerates a class of objects of size n that has exponential growth rate α. If a_n is infinitely divisible, then A_n and $A_n^* = \alpha^n a_n^*$ also satisfy (7) and (8), and naturally we also call A_n^* the Lévy–Khintchine transform of A_n.

A positive sequence $\vartheta(n)$ is *regularly varying*, with *index* γ, if

$$\lim_{n\to\infty} \frac{\vartheta(\lfloor xn \rfloor)}{\vartheta(n)} = x^{\gamma}, \qquad \forall x > 0.$$

One of our main tools for studying the asymptotics of sequences with a Lévy–Khintchine transform is a result by Hawkes and Jenkins [12], which shows that, if a_n^* is regularly varying with some index $\gamma < 0$, then

$$a_n \sim \frac{a_n^*}{n} \exp\left(\sum_{k=1}^{\infty} \frac{a_k^*}{k}\right). \tag{9}$$

In other words, in terms of the underlying Lévy process, the associated infinitely divisible probability measure π_n and its corresponding Lévy measure ν_n are asymptotically equivalent, $\pi_n \sim \nu_n$ as $n \to \infty$. This is related to the well-known *one big jump principle* in Lévy process theory.

2.1 Renewal sequences

A subfamily of sequences $(A_n, n \geq 0)$ that have a Lévy–Khintchine transform is the class of *renewal sequences*. See, e.g., Feller [8, Section XIII] for background information on renewal theory.

Definition 2.1. We call a sequence $(A_n, n \geq 0)$ a renewal sequence if its generating function $A(x) = \sum_{n\geq 1} A_n x^n$ can be written as

$$A(x) = \frac{1}{1 - A^{(1)}(x)},$$

where $A^{(1)}(x) = \sum_{n\geq 0} A_n^{(1)} x^n$ is the generating function of some other sequence $(A_n^{(1)}, n \geq 0)$.

In combinatorics, such a sequence often arises when A_n counts the number of some type of structures of length n, each of which can be decomposed into a series of irreducible parts. In this context, $A_n^{(1)}$ counts the number of irreducible structures of length n.

It is well known (see, e.g., [12, p. 66]) that renewal sequences are infinitely divisible. In our recent work, we show that, if $(A_n, n \geq 0)$ is a renewal sequence then the Lévy–Khintchine transform A_n^* of A_n takes a special form.

Lemma 2.1 [3, Lemma 2]. *If $(A_n, n \geq 0)$ is a renewal sequence then*

(1) the Lévy–Khintchine transform A_n^* of A_n is the number of pairs (X, m), where X is a structure of length n and $0 \leq m < \ell$, where $\ell = \ell(X)$ is the length of the first irreducible part of X, and

(2) we have that

$$\frac{A_n^*}{nA_n} = \mathbf{E}\left[\frac{1}{\mathcal{I}_n}\right], \tag{10}$$

where $\mathcal{I}_n$ is the number of irreducible parts in a uniformly random structure of length n.

We note that, in terms of the underlying Lévy process, the formula (10) says that, when $(A_n, n \geq 0)$ is a renewal sequence with growth rate α, the Lévy measure $\nu_n = A_n^*/n\alpha^n$ is the *harmonic renewal measure* (see, e.g., [9]) associated with the measure $\mu_n = A_n^{(1)}/\alpha^n$.

Remark 2.1. With Lemma 2.1 in hand, one can use (9) to obtain the asymptotics of a renewal sequence $(A_n, n \geq 1)$ using its Lévy–Khintchine transform A_n^*, provided that there exists an $\alpha > 0$ for which A_n^*/α^n is regularly varying with some index $\gamma < 0$. In our experience, the sequence A_n^* tends to be more tractable than A_n itself, and this lies at the core of the method.

2.2 Delayed renewal sequences

A related family of sequences $(A_n, n \geq 0)$ is the class of *delayed renewal sequences*. Such sequences arise when A_n enumerates structures of length n that can be decomposed into exactly one special part (the 'delay') and a sequence of irreducible parts. In this case, the generating function A of $(A_n, n \geq 0)$ satisfies

$$A(x) = \frac{D(x)}{1 - A^{(1)}(x)}, \tag{11}$$

where $D(x)$ is the generating function of the delay and $A^{(1)}(x)$ is the generating function of the irreducible structures. Note that the geometric series

$$\frac{1}{1 - A^{(1)}(x)} = \sum_{k=0}^{\infty} (A^{(1)}(x))^k$$

takes into account structures with any number k of irreducible parts.

Although A_n itself may not necessarily admit a Lévy–Khintchine transform, we illustrate in this work that the asymptotic growth of A_n can still be understood by studying the delay and its renewal structure separately. We believe that this method will be useful in enumerating many other combinatorial sequences of interest.

2.3 Application to $\mathcal{G}_n$

By the lattice path representation of graphical sequences in [2], we have that $\mathcal{G}_n = \mathcal{W}_n$, where $\mathcal{W}_n$ counts the number of *graphical walks* $(X_k, 0 \le k \le 2n)$ of length $2n$, such that (1) $X_0 = X_{2n} = 0$ and (2) $A_k \ge 0$ for all $0 \le k \le n$. We let $\mathcal{M}_n$ denote the number of *graphical meanders* such that (1) holds and also (2') $A_k > 0$ for all $0 < k \le n$. We also let $\mathcal{B}_n$ be the number of *graphical bridges* such that (1) and (2) hold and also (3) $A_n = 0$. Note that every graphical walk can be uniquely expressed in terms of a graphical bridge and a graphical meander. In other words, if $\mathcal{W}(x) = \sum_{n\ge1} \mathcal{W}_n x^n$, $\mathcal{M}(x) = \sum_{n\ge1} \mathcal{M}_n x^n$ and $\mathcal{B}(x) = \sum_{n\ge1} \mathcal{B}_n x^n$ are the corresponding generating functions, then

$$\mathcal{W}(x) = \mathcal{M}(x)\mathcal{B}(x) \tag{12}$$

The sequence $(\mathcal{B}_n, n \ge 1)$ is a renewal sequence (since $X_0 = A_0 = 0$ is the initial condition of a graphical walk). In other words, any graphical bridge can be decomposed into a series of *irreducible* graphical bridges. We let $\mathcal{B}_n^{(1)}$ denote the number of irreducible graphical bridges of length $2n$, for which (1) and (3) hold and also (2") $A_k > 0$ for all $0 < k < n$. Then

$$\mathcal{W}(x) = \frac{\mathcal{M}(x)}{1 - \mathcal{B}^{(1)}(x)}, \tag{13}$$

where $\mathcal{B}^{(1)}(x) = \sum_{n\ge1} \mathcal{B}_n^{(1)} x^n$. In this sense (see (11) above), the sequence $(\mathcal{G}_n, n \ge 1)$ is a delayed renewal sequence.

However, as we will see, in studying the asymptotics of $\mathcal{G}_n$ via (13), the contribution from $\mathcal{M}(x)$ is already accounted for by the factor $\Gamma(3/4)/2^{5/2}\pi$ in (2) and (4) above. Indeed, as it turns out (see Remark 3.1 below), the factor $1/\sqrt{1-\rho}$ in (4) is related only to $\mathcal{B}(x) = 1/(1 - \mathcal{B}^{(1)}(x))$ in (12) and (13). We will deduce Theorem 1.1 from (4) and Theorem 1.2, which gives a formula for ρ in terms of Walkup's constant ω. In studying the asymptotics of $\mathcal{B}_n$, we will apply the Lévy–Khintchine method, as described above. More specifically, we consider $a_n = \mathcal{B}_n/4^n$, and its Lévy–Khintchine transform a_n^*, and show that the limit in (9) equals $1 - \rho$. Finally, we use Lemma 2.1 and a bijective argument to prove that $a_n^* = 2\mathcal{T}_n/4^n$, leading to the appearance of $\mathcal{T}_n$ and ω in Theorems 1.1 and 1.2. We note that the two parts of Lemma 2.1 above will each play essential roles in our arguments. Lemma 2.1(1) allows us to approach the asymptotics of $\mathcal{B}_n$ combinatorially, and Lemma 2.1(2) opens the door to probability theory.

3. Graphical bridges

Suppose that $X = (X_0, \ldots, X_{2n})$ is a walk with ±1 increments.

Definition 3.1. We call

$$\sigma(X) = \frac{1}{2}\sum_{i=1}^{n} X_{2i} \tag{14}$$

the diamond area of X.

In fact, $\sigma(X)$ is the usual area of the *lazy version* $\Lambda(X)$ of X, defined as follows. Let $\Delta_i = X_i - X_{i-1}$, for $i \ge 1$, be the increments of X. The increments Δ_i' of $\Lambda(X)$ are the averages $\Delta_i' = (\Delta_{2i} + \Delta_{2i-1})/2$, for $i \ge 1$. Informally, we look at the steps of X as a series of pairs: then to obtain $\Lambda(X)$, we convert each (up,up) pair to one up step, each (down,down) pair to one down step, and each (up,down) or (down,up) pair to one lazy step. Note that, if X has increments ±1 with probability

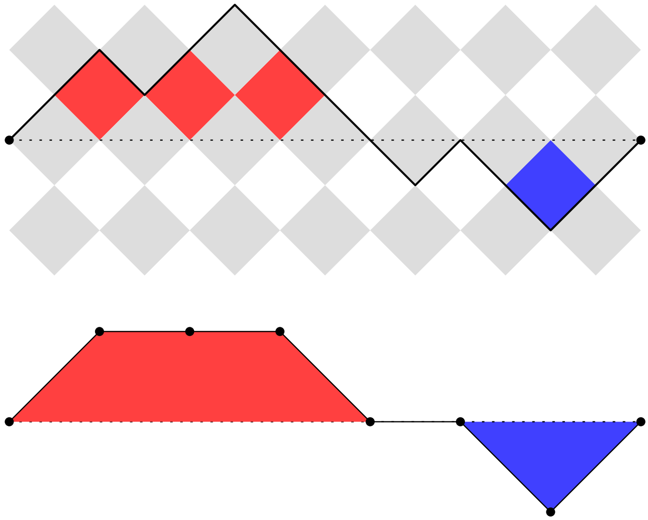

Figure 2. A bridge B with diamond area $\sigma(B) = 2$, which equals the signed area enclosed by B and the x-axis, minus the area in grey. In the figure, the rotated squares all represent area 1. We also depict its lazy version $\Lambda(B)$, whose regular integral equals $\sigma(B)$.

1/2, then $\Lambda(X)$ has increments ± 1 with probability 1/4, and 0 with probability 1/2. The reason for the name 'diamond area' is that $\sigma(X)$ is the signed area under the walk that lies outside a rotated checkerboard of diamonds, as in Figure 2.

For walks $X = (X_0, X_1, \dots)$, we let $X^{(k)} = (X_0, X_1, \dots, X_k)$.

Definition 3.2. A bridge $B = (B_0, \dots, B_{2n})$ of length $2n$ with increments ± 1 is graphical if $\sigma(B) = 0$ and $\sigma_{2k} = \sigma(B^{(2k)}) \geq 0$, for all $1 \leq k < n$. A graphical bridge B is irreducible if $\sigma_{2k} > 0$, for all $1 \leq k < n$. We let $\mathcal{B}_n$ be the number of graphical bridges of length $2n$. See Figure 3.

3.1 Relation to ρ

Recall that ρ is defined, in Section 1.1 above, in terms of a lazy random walk. We can rephrase ρ in terms of a standard, simple symmetric random walk (X_n) with ± 1 increments, started from $X_0 = 0$, using its diamond area. Specifically, let $\sigma_{2k} = \sigma(X^{(2k)})$ denote the diamond area accumulated after $2k$ steps. Since, as discussed above, the diamond area of X is the usual area of its lazy version $\Lambda(X)$, it follows that

$$\rho = \mathbf{P}(\sigma_{2\kappa} = 0), \qquad \kappa = \inf\{k \geq 1 : X_{2k} = 0,\ \sigma_{2k} \leq 0\}.$$

To see how ρ relates to our problem, we observe that a bridge $B = (B_0, \dots, B_{2n})$ (with ± 1 increments) is graphical if and only if $\sigma(B) = 0$ and $\sigma_{2k} = \sigma(B^{(2k)}) \geq 0$, for all times $2k$ for which $B_{2k} = 0$, since σ is monotone between such *diagnostic times*. Then, roughly speaking, ρ is the probability that, the first time a random walk trajectory is in danger of not satisfying $\sigma_{2k} \geq 0$, at such a diagnostic time $2k$, the condition is in fact satisfied with equality $\sigma_{2k} = 0$.

Consider now a random walk $X = (X_0, X_1, \dots)$. If $X_{2k} = \sigma_{2k} = 0$ at the first diagnostic time $2k < 2n$, then $(X_0, \dots, X_{2n})$ is a graphical bridge if and only if $(X_{2k}, \dots, X_{2n})$ is a graphical bridge. Therefore, times at which $X_{2k} = \sigma_{2k} = 0$ are of crucial importance, as they decompose a graphical bridge into its irreducible parts and correspond to *renewal times* in the analysis.

Remark 3.1. As mentioned at the end of Section 2.3 above, the probability ρ is only related to graphical bridges. Indeed, this can be seen directly from the definition of ρ in (5) above, since ρ is the probability that, at time τ, the walk $Y = (Y_k, 0 \leq k \leq \tau)$ is the lazy version $Y = \Lambda(B)$ of a graphical bridge $B = (B_0, \dots, B_{2\tau})$ of length 2τ.

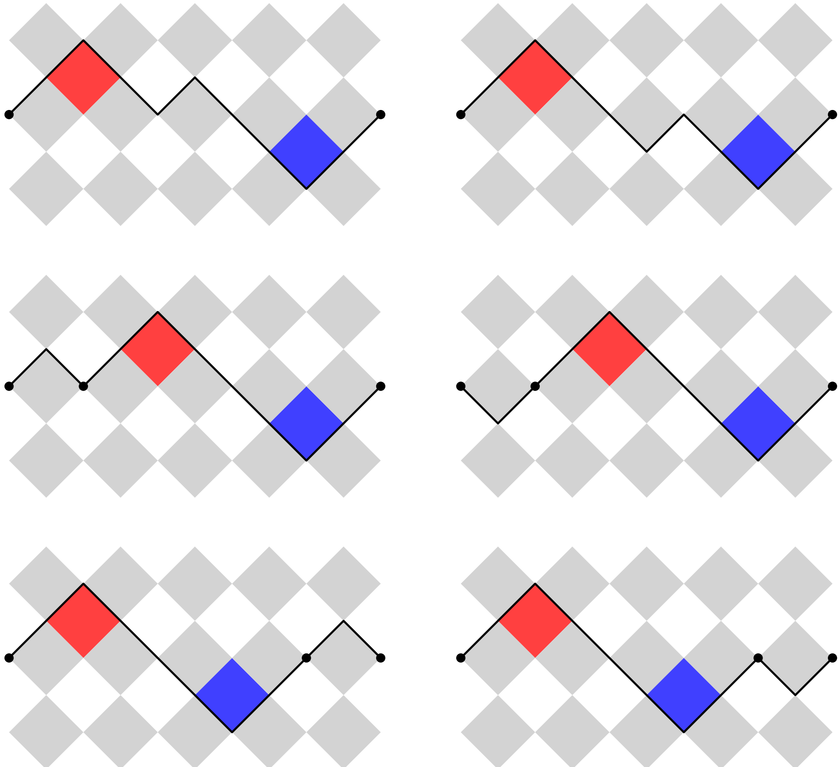

Figure 3. There are $\mathcal{B}_5 = 38$ graphical bridges of length 10. Of these, 32 stay within the string of grey diamonds centred along the x-axis. The other 6 are depicted above. The top two are irreducible. All others have two irreducible parts. Irreducible parts are separated by solid dots.

Definition 3.3. We let $\mathcal{I}_n$ denote the number of irreducible parts in a uniformly random graphical bridge of length $2n$.

The following fact will play a central role in our arguments.

Lemma 3.1. *We have that*

$$\mathcal{I}_n \xrightarrow{d} 1 + \mathcal{X}, \tag{15}$$

where $\mathcal{X}$ is a negative binomial with parameters $r = 2$ and $p = 1 - \rho$.

This result follows by (the proof of) [5, Proposition 21]. The appearance of the negative binomial random variable can, intuitively, be explained as follows: with high probability, renewal times occur only very close to the start and end of the bridge. Moreover, the number of renewal times on either side are approximately geometric and independent.

3.2 Calculating ρ

We observed that a graphical bridge can be decomposed into a series of irreducible parts, and so $(\mathcal{B}_n, n \geq 0)$ is a renewal sequence. As discussed in Section 2, this means that the generating function $\mathcal{B}(x) = \sum_n \mathcal{B}_n x^n$ can be expressed as

$$\mathcal{B}(x) = \frac{1}{1 - \mathcal{B}^{(1)}(x)},$$

where $\mathcal{B}^{(1)}(x) = \sum_n \mathcal{B}_n^{(1)} x^n$ is the generating function for the number $\mathcal{B}_n^{(1)}$ of *irreducible* graphical bridges of length $2n$.

Therefore, by Lemma 2.1(2), we have

$$\frac{\mathcal{B}_n^*}{n\mathcal{B}_n} = \mathbf{E}\left[\frac{1}{\mathcal{I}_n}\right]. \tag{16}$$

Note that $I_n \le 1$, and so $1/I_n$ is uniformly integrable. Therefore, combining (15) and (16), we find that

$$\frac{\mathcal{B}_n^*}{n\mathcal{B}_n} \to 1 - \rho. \tag{17}$$

Hence, in proving Theorem 1.2, the following is key.

Proposition 3.2. *We have that* $\mathcal{B}_n^* = 2\mathcal{T}_n$.

We prove Proposition 3.2 in the next two sections. For now, let us note that our main result follows.

Proof of Theorem **1.2.** By (1) and Proposition 3.2,

$$\frac{\mathcal{B}_n^*}{4^n} \sim \frac{1}{n4^n}\binom{2n}{n} \sim \frac{1}{\sqrt{\pi}\, n^{3/2}}$$

is regularly varying with index $\gamma = -3/2$. Therefore, by (9),

$$\frac{\mathcal{B}_n^*}{n\mathcal{B}_n} \to \exp(-2\omega). \tag{18}$$

Combining (17) and (18), we find that $\rho = 1 - \exp(-2\omega)$, as claimed. □

4. Combinatorial lemmas

In this section, we prove two combinatorial lemmas, showing that $2\mathcal{T}_n$ can be described in terms of areas α below lattice paths L and diamond areas σ under bridges B.

4.1 Lattice paths

Suppose that L is an $\uparrow, \rightarrow$ lattice path from (a_1, b_1) to (a_2, b_2), for some integers $a_1 \le a_2$ and $b_1 \le b_2$. We let $\alpha(L)$ denote the area of the region between L and the lines $x = a_2$ and $y = b_1$.

Definition 4.1. We let $\mathcal{N}_n$ be the number of lattice paths L from $(0, 0)$ to (n, n) such that $\alpha(L) \equiv 0$ mod n.

To see the relationship between $\mathcal{N}_n$ and $\mathcal{T}_n$, first note that the area of a lattice path L from $(0, 0)$ to (n, n) is

$$\alpha(L) = \sum_i u_i, \tag{19}$$

summing over the sequence $u_1 \le \cdots \le u_n$, where u_i is the number of $\uparrow$ steps before the ith $\rightarrow$ step of L. In other words, if we picture L as a bar graph, then the u_i are the heights of the bars. Hence, $\mathcal{N}_n$ is the number of submultisets of $\{0, 1, \ldots, n\}$ of size n that sum to 0 mod n. On the other hand, recall that, as was already mentioned above, $\mathcal{T}_n$ can be counted using a formula by von Sterneck.

Lemma 4.1 (von Sterneck [1, 16]). *Let $\mathcal{S}_n$ be the number of submultisets* $\{0, 1, \ldots, n-1\}$ *of size n that sum to* 0 *mod n. Then $\mathcal{S}_n = \mathcal{T}_n$.*

Using these observations, we show the following.

Lemma 4.2. *We have that* $\mathcal{N}_n = 2\mathcal{T}_n$.

Proof. Note that Lemma 4.1 allows us to instead prove that $\mathcal{N}_n = 2\mathcal{S}_n$. To this end, we first claim that $\mathcal{S}_n$ is the number of $\uparrow, \rightarrow$ lattice paths L from $(0,0)$ to $(n, n-1)$ with $\alpha(L) \equiv 0 \bmod n$. Indeed, submultisets $0 \le u_1 \le \cdots \le u_n \le n-1$ correspond to such L with $\rightarrow$ steps at heights u_i, and so $\alpha(L) = \sum_i u_i \equiv 0 \bmod n$. Next, let $\mathcal{L}_n^{\uparrow}$ (resp. $\mathcal{L}_n^{\rightarrow}$) be the number of lattice paths L from $(0,0)$ to (n, n) such that $\alpha(L) \equiv 0 \bmod n$, ending with an $\uparrow$ (resp. $\rightarrow$) step. Then $\mathcal{N}_n = \mathcal{L}_n^{\uparrow} + \mathcal{L}_n^{\rightarrow}$. To conclude, we show that $\mathcal{L}_n^{\uparrow} = \mathcal{L}_n^{\rightarrow} = \mathcal{S}_n$. Indeed, $\mathcal{L}_n^{\uparrow} = \mathcal{L}_n^{\rightarrow}$ follows by symmetry, reflecting over $x = y$, and observing that divisibility by n of the enclosed area is invariant under this reflection. Finally, $\mathcal{L}_n^{\uparrow} = \mathcal{S}_n$, since if L from $(0,0)$ to (n, n) has last step $\uparrow$, then removing this step yields a corresponding L' from $(0,0)$ to $(n, n-1)$ with the same area as L. □

4.2 Bridges

Next, we connect $\mathcal{N}_n$ to bridges.

Definition 4.2. We let $\mathcal{N}'_n$ denote the number of bridges B of length $2n$, with diamond area $\sigma(B) \equiv 0 \bmod n$.

Lemma 4.3. *We have that* $\mathcal{N}_n = \mathcal{N}'_n$.

Proof. To obtain a correspondence between bridges $B = (B_0, \ldots, B_{2n})$ of length $2n$ such that $\sigma(B) \equiv 0 \bmod n$ and lattice paths L from $(0,0)$ to (n, n) with $a(L) \equiv 0 \bmod n$, we proceed as follows.

First, let $\Delta_i = B_i - B_{i-1}$ be the ith increment of B. For $1 \le k \le n$, put

$$B_k^{\text{odd}} = \sum_{i=1}^{k} \Delta_{2i-1}, \qquad B_k^{\text{even}} = \sum_{i=1}^{k} \Delta_{2i},$$

so that $B^{\text{odd}} = (B_1^{\text{odd}}, \ldots, B_n^{\text{odd}})$ and $B^{\text{even}} = (B_1^{\text{even}}, \ldots, B_n^{\text{even}})$ are the walks with the odd and even increments of B, respectively.

Next, rotate B^{odd} counterclockwise by $\pi/4$ to obtain a lattice path L_1 from $(0,0)$ to $(n-\ell, \ell)$ for some ℓ (since B^{odd} has n steps). Likewise, rotate B^{even} counterclockwise by $\pi/4$ to obtain a lattice walk L_2 from $(n-\ell, \ell)$ to (n, n) (since B is a bridge). Let L be the concatenation of L_1 and L_2. This procedure is depicted in Figure 4.

Geometric considerations (see Figure 5) imply that

$$\alpha(L) = \alpha(L_1) + \alpha(L_2) + \ell^2.$$

Furthermore, observe that, by definition,

$$\sigma(B) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{i} \left(\Delta_{2j-1} + \Delta_{2j}\right) = \frac{1}{2} \sum_{k=1}^{n} (B_k^{\text{odd}} + B_k^{\text{even}}),$$

which equals the signed area enclosed by B^{odd} and $-B^{\text{even}}$, as depicted in Figure 5. Therefore, it can be seen that

$$\sigma(B) = \alpha(L_1) - [\ell(n-\ell) - \alpha(L_2)] = \alpha(L) - \ell n.$$

Since $\alpha(L) \equiv \sigma(B) \bmod n$, this completes the proof. □

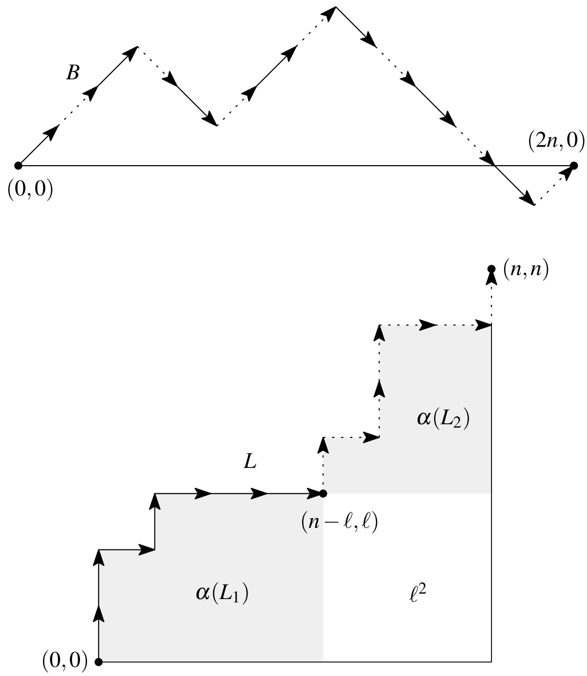

Figure 4. The bijection in Lemma 4.3.

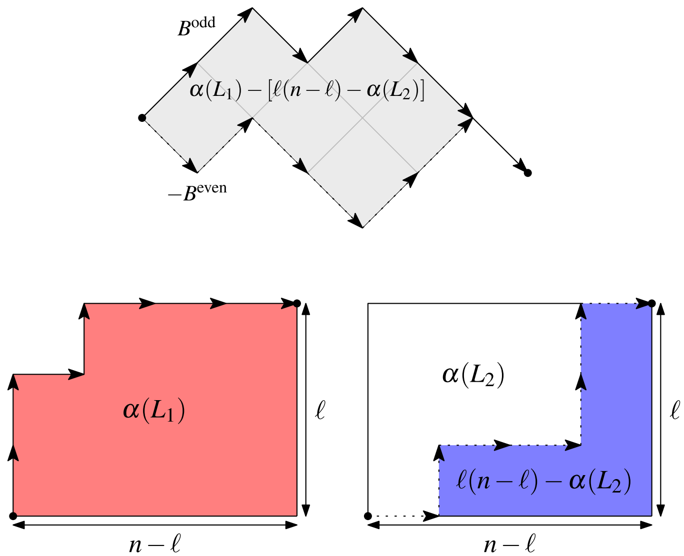

Figure 5. Calculating areas in Lemma 4.3.

5. The Lévy–Khintchine transform

We are now in a position to prove our key result Proposition 3.2, which identifies $\mathcal{B}_n^* = 2\mathcal{T}_n$ as the Lévy–Khintchine transform of $\mathcal{B}_n$.

***Proof of Proposition** 3.2.* As discussed, $(\mathcal{B}_n, n \geq 0)$ is a renewal sequence, and so $\mathcal{B}_n^*$ can be described in terms of Lemma 2.1(1). Combining this with our combinatorial results Lemmas 4.2 and 4.3 above, it follows that, to show that $\mathcal{B}_n^* = 2\mathcal{T}_n$, it suffices to find a bijection:

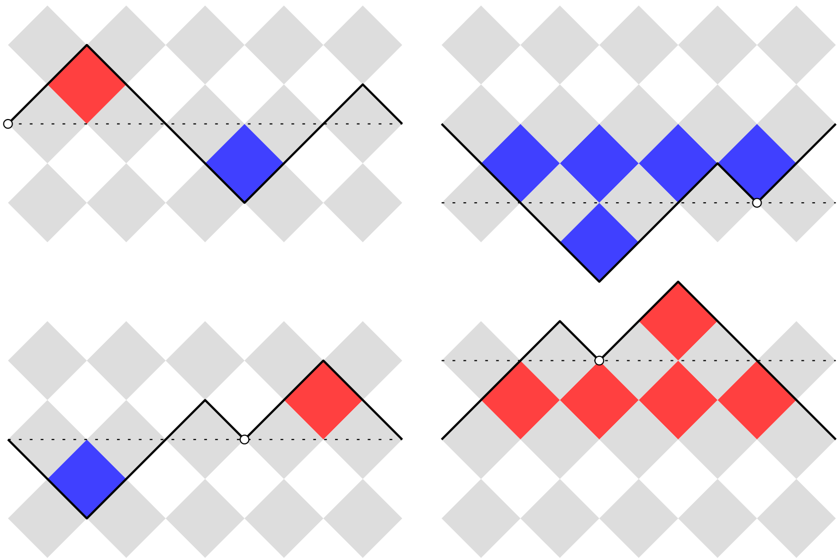

Figure 6. A graphical bridge B (top left) of length 10, with first irreducible part of length 8. The bridge B is $\phi(B,0)$. Its other shifts $\phi(B,i)$, for $1 \le i < 4$, are also depicted. All bridges have diamond area divisible by 5. To find the inverse mapping ϕ^{-1}, we shift the x-axis (see dotted line) by some factor of 2 so that the diamond area is equal to 0, and then find the rightmost point (open dot) that starts a graphical sequence. Such a point exists by Raney's lemma.

- from the set of ordered pairs (B,i), where B is a graphical bridge of length $2n$, whose first irreducible part is of length 2ℓ, and $0 \le i < \ell$
- to the set of bridges B' of length $2n$, with diamond area $\sigma(B') \equiv 0 \mod n$.

We claim that such a bijection can be constructed as follows: for each such pair (B,i), let $\phi(B,i)$ be the bridge B' obtained from B, via a cyclical shift to the right by $2i$. Note that, geometrically, B' can be obtained from B by shifting the x-axis up or down by some multiple of 2 and then starting the bridge from a particular intersection point of the new x-axis and the bridge B. Only the former operation affects the total diamond area. In fact, by (14), it can be seen that such a shift changes the area by a multiple of n. Since $\sigma(B) = 0$, it follows, again by (14), that $\sigma(B') \equiv 0 \mod n$. See Figure 6.

On the other hand, for any B' with $\sigma(B') \equiv 0 \mod n$, the inverse $\phi^{-1}(B')$ is found as follows. Select the unique shift of the x-axis for which the diamond area becomes 0, and then choose the rightmost starting point $2n - 2i$ for which the resulting bridge B is graphical; then $\phi^{-1}(B') = (B,i)$. Such a point exists by Raney's lemma[1] [17], and by choosing i minimal we ensure that $2i$ is smaller than the length of the first irreducible part of B. □

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[1] Raney proved that any sequence of integers that sums to 1 has a unique cyclic shift such that all partial sums are positive. If instead the sum is 0 (as in our case) then there are cyclic shifts (perhaps not unique) such that all partial sums are non-negative.

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References

[1] Bachmann, P. G. H. (1902) *Niedere Zahlentheorie*, vol. 2, Leipzig B.G. Teubner.
[2] Balister, P., Donderwinkel, S., Groenland, C., Johnston, T. and Scott, A. (2025) Counting graphic sequences via integrated random walks. *Trans. Amer. Math. Soc.* **378**(7) 4627–4669.
[3] Bassan, M., Donderwinkel, S. and Kolesnik, B. (2026) Tournament score sequences, Erdős–Ginzburg–Ziv numbers, and the Lévy–Khintchine method. *Electron. Commun. Probab.* **31** 1–10.
[4] Bertoin, J. (1996) *Lévy Processes*, Vol. 121 of Cambridge Tracts in Mathematics, Cambridge University Press.
[5] Donderwinkel, S. and Kolesnik, B. (2024) *Tournaments and random walks*. Preprint available at arXiv: 2403.12940.
[6] Embrechts, P. and Hawkes, J. (1982) A limit theorem for the tails of discrete infinitely divisible laws with applications to fluctuation theory. *J. Austral. Math. Soc. Ser. A* **32**(3) 412–422.
[7] Erdős, P. and Gallai, T. (1960) Gráfok előírt fokszámú pontokkal. *Mat. Lapok* **11** 264–274.
[8] Feller, W. (1968) *An Introduction to Probability Theory and its Applications*, Vol. I, 3rd edn. John Wiley & Sons, Inc..
[9] Greenwood, P., Omey, E. and Teugels, J. L., (1982) Harmonic renewal measures. *Z. Wahrsch. Verw. Gebiete* **59**(3) 391–409.
[10] Hakimi, S. L. (1962) On realizability of a set of integers as degrees of the vertices of a linear graph, I. *J. Soc. Indust. Appl. Math.* **10** 496–506.
[11] Havel, V. (1955) A remark on the existence of finite graphs. *Math. Bohem.* **80**(4) 477–480.
[12] Hawkes, J. and Jenkins, J. D. (1978) Infinitely divisible sequences. *Scand. Actuar. J.* **1978**(2) 65–76.
[13] Koren, M. (1973) Extreme degree sequences of simple graphs. *J. Comb. Theory Ser. B* **15** 213–224.
[14] Moon, J. W. (1968) *Topics on Tournaments*, Holt, Rinehart and Winston.
[15] Peled, U. N. and Srinivasan, M. K. (1989) The polytope of degree sequences. *Linear Algebra Appl.* **114**(115) 349–377.
[16] Ramanathan, K. G. (1944) *Some applications of Ramanujan's trigonometrical sum* $C_m(n)$, Proc. *Indian Acad. Sci., Sect. A* **20** 62–69.
[17] Raney, G. N. (1960) Functional composition patterns and power series reversion. *Trans. Amer. Math. Soc.* **94** 441–451.
[18] Stanley, R. P. (1991) A zonotope associated with graphical degree sequences. In *Applied Geometry and Discrete Mathematics*, Vol. 4 of DIMACS Series in Discrete Mathematics and Theoretical Computer Science, American Mathematical Society, pp. 555–570.
[19] Walkup, D. W. (1972) The number of plane trees. *Mathematika* **19** 200–204.

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