Multipartite Entanglement Detection in Bosons

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We propose a simple quantum network to detect multipartite entangled states of bosons and show how to implement this network for neutral atoms stored in an optical lattice. We investigate the special properties of cluster states, multipartite entangled states, and superpositions of distinct macroscopic quantum states that can be identified by the network.

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Quantum entanglement is a very important physical resource in quantum information processing tasks, such as quantum cryptography, teleportation, quantum frequency standards or quantum-enhanced positioning [1]. Since the implementation of any of these tasks requires precise knowledge on the entangled states being used, the development of "measurement tools" for the characterization and detection of entanglement in physical systems is of great practical importance.

While the creation of multipartite entanglement has been recently achieved in a controlled way in experiments with either Mott insulating states of neutral bosonic atoms in optical lattices [2,3] or ion traps [4], its unambiguous detection has so far not been possible [2] in these setups. The usual experimental methods to detect entanglement are based on the violation of Bell-type inequalities [5], which are known to be quite inefficient in the sense that they leave many entangled states undetected [6]. Alternatively, one can perform a complete state tomography of the system [7], but this method requires the preparation of an exponentially large number of copies of the state and it is redundant, since not all parameters of the density operator are relevant for the entanglement detection.

In this Letter, we propose a simple quantum network to detect multipartite entangled states through an entanglement test more powerful than the Bell-Clauser-Horne-Shimony-Holt inequalities for all possible settings [8], albeit less powerful than full state tomography. The network is realized by coupling two identically prepared 1D rows of N previously entangled qubits via pairwise beam splitters (BS), as shown in Fig. 1. We also show how to implement this network in an optical lattice or array of magnetic microtraps loaded with atoms in a Mott insulating state with filling factor one [9]. Each of the atoms has two long lived internal states a and b which represent the qubit. The pairwise BS can be implemented by decreasing the horizontal barrier between the two rows of atoms.

We start by introducing the set of inequalities used by our network for the detection of multipartite entanglement. The information-theoretic approach to separability of bipartite quantum systems leads to a set of entropic inequalities satisfied by all separable bipartite states [8]. We extend these inequalities to separable multipartite states by considering a state $\rho_{123\cdots N}$ of N subsystems. If $\rho_{123\cdots N}$ is separable, then we can write it as

$$\rho_{123\cdots N} = \sum_{\ell} C_\ell \rho_1^\ell \otimes \rho_2^\ell \otimes \rho_3^\ell \otimes \cdots \otimes \rho_N^\ell, \tag{1}$$

where ρ_j^ℓ is a state of subsystem j, and $\sum_\ell C_\ell = 1$. The purity $\mathrm{Tr}(\rho_{123\cdots n}^2)$ of $\rho_{123\cdots n}$, where $n \in 1, 2, \ldots, N$, is smaller or equal than the purity of any of its reduced density operators. For example,

$$\mathrm{Tr}(\rho_{123\cdots n}^2) \leq \mathrm{Tr}(\rho_{123\cdots n-1}^2) \leq \mathrm{Tr}(\rho_{123\cdots n-2}^2) \cdots \leq \mathrm{Tr}(\rho_{12}^2) \leq \mathrm{Tr}(\rho_1^2) \leq 1. \tag{2}$$

This set of nonlinear inequalities provides a set of necessary conditions for separability; i.e., if for any state ρ any of these inequalities is violated, then ρ is entangled. For the case where $\rho_{123\cdots N}$ is separable and pure, we have that $\mathrm{Tr}(\rho_{123\cdots n}^2) = 1$ and the inequalities become equalities.

In order to test Eq. (2) we need to be able to determine the nonlinear functional $\mathrm{Tr}(\rho^2)$, where ρ is any of the different reduced density operators of $\rho_{123\cdots N}$. The direct estimation of this functional has been addressed in [10], where the value of $\mathrm{Tr}(\rho^2)$ is determined by measuring the expectation value, on state $\rho \otimes \rho$, of the symmetric and antisymmetric projectors.

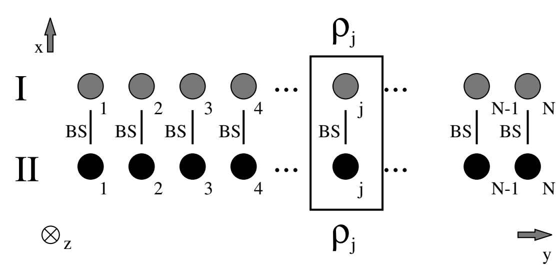

FIG. 1. Network of BS acting on pairs of identical bosons. The two rows of N atoms, labeled I and II, respectively, are identical, and the state of each of the rows is $\rho_{123\cdots N}$. The total state of the system is $\rho_{123\cdots N} \otimes \rho_{123\cdots N}$.

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Let us first consider the simple scenario of one pair of identical bosons, in state $\rho_j \otimes \rho_j$, impinging on the BS (as depicted in Fig. 1). After applying the BS, we write the purification of the state $\rho'_j = \mathcal{U}_{\rm BS}\rho_j \otimes \rho_j \mathcal{U}^\dagger_{\rm BS}$, where $\mathcal{U}_{\rm BS}$ is the unitary time evolution operator of the BS, in the triplet/singlet basis. The operator $\mathcal{U}_{\rm BS}$ transforms $a(b)^{(j)}_{\rm I,II} \rightarrow [a(b)^{(j)}_{\rm I,II} - ia(b)^{(j)}_{\rm II,I}]\sqrt{2}$, where $a^{(j)}_l$ and $b^{(j)}_l$ are the bosonic destruction operators for particles in row $l =$ I, II, site number $j = 1, \ldots, N$ and internal state a and b, respectively. The triplet basis elements are $(\alpha^{(j)\dagger}_{\rm I}\beta^{(j)\dagger}_{\rm I} + \alpha^{(j)\dagger}_{\rm II}\beta^{(j)\dagger}_{\rm II})|{\rm vac}\rangle$, $\alpha, \beta \in a, b$, with respective prefactors $c^{(j)}_{\alpha\beta}$, and the singlet element is $(a^{(j)\dagger}_{\rm I}b^{(j)\dagger}_{\rm II} - a^{(j)\dagger}_{\rm II}b^{(j)\dagger}_{\rm I})\times |{\rm vac}\rangle$, with prefactor $c^{(j)}_{-}$, and $|{\rm vac}\rangle$ the vacuum state. The symmetric/antisymmetric components of $\rho_j \otimes \rho_j$ are spanned by the triplet/singlet basis elements. Note that in the triplet basis states both bosons are in the same spatial mode, while in the singlet basis state each boson occupies a different spatial mode. Hence, measuring the probability of finding two bosons in either the same spatial mode or different spatial modes, after the BS, is tantamount to measuring the probabilities $P^i_\pm$ of projecting the state $\rho_j \otimes \rho_j$ on its symmetric or antisymmetric subspaces. These probabilities are given by the expectation value of the symmetric/antisymmetric projectors $S_\pm$, defined by $S_\pm = (\mathbb{I} \pm V)/2$, where $\mathbb{I}$ is the identity operator, and V is the swap operator, with $V\alpha^{(j)\dagger}_{\rm I}\beta^{(j)\dagger}_{\rm II}|{\rm vac}\rangle = \beta^{(j)\dagger}_{\rm I}\alpha^{(j)\dagger}_{\rm II}|{\rm vac}\rangle$, $\forall \alpha^{(j)\dagger}_{\rm I}, \beta^{(j)\dagger}_{\rm II}$. Therefore we have

$$P^j_\pm = \tfrac{1}{2}{\rm Tr}[(\mathbb{I} \pm V)\rho_j \otimes \rho_j] = \tfrac{1}{2} \pm \tfrac{1}{2}{\rm Tr}(\rho_j^2), \tag{3}$$

where ("$-$") "$+$" stands for projection on the (anti)symmetric subspace. Thus, after we let $\rho_j \otimes \rho_j$ go through the pairwise BS, we can determine the purity of ρ_j by measuring the projections of ρ'_j on the symmetric and antisymmetric subspaces. For pairs of polarization entangled photons, a scenario similar to this one is currently being implemented experimentally [11].

We now extend the above two-boson scenario to the general situation (see Fig. 1) and consider two copies of a state of N bosons, undergoing pairwise BS. By correlating the probabilities of projecting the state of each pair of identical bosons on the symmetric/antisymmetric subspaces, we can estimate the purity of $\rho_{123\cdots N}$ and of any of its reduced density operators. As a more concrete example, let us consider the probabilities for $N = 3$. We will label the subsystems 1, 2, 3, respectively:

$$\begin{aligned} P_{\pm_1\pm_2\pm_3} &= (1/2^3){\rm Tr}\left[\prod_{i=1}^{3}(\mathbb{I} \pm_i V_i)\rho_{123} \otimes \rho_{123}\right] \\ &= \frac{1}{8}[1 \pm_1 {\rm Tr}(\rho_1^2) \pm_2 {\rm Tr}(\rho_2^2) \pm_3 {\rm Tr}(\rho_3^2) \\ &\quad \pm_{1,2} {\rm Tr}(\rho_{12}^2) \pm_{1,3} {\rm Tr}(\rho_{13}^2) \pm_{2,3} {\rm Tr}(\rho_{23}^2) \\ &\quad \pm_{1,2,3} {\rm Tr}(\rho_{123}^2)], \end{aligned} \tag{4}$$

where $\pm_{i,i'} = (\pm_i)(\pm'_i)$, $i, i' = 1, 2, 3$, and $V_{1,2,3}$ stand for the swap operator acting on subsystem 1, 2, 3. The purities related to ρ_{123} are unequivocally determined by the eight probabilities $P_{\pm_1\pm_2\pm_3}$. The expression for the probabilities in Eq. (4) can be straightforwardly extended to the states of N bosons, where we consider the expectation values of the projector $\prod_{i=1}^N(\mathbb{I} \pm_i V_i)/2$, on $\rho_{123\cdots N} \otimes \rho_{123\cdots N}$. In the N boson case, the $2^N - 1$ unknown purities will be determined by the $2^N - 1$ independent probabilities.

The implementation of this entanglement detection scheme in optical lattices and magnetic microtraps follows four steps: (i) Creation of two identical copies of the entangled state $\rho_{123\cdots N}$: Each of the two rows of bosons shown in Fig. 1 is realized by a 1D chain of entangled atoms. The entanglement can, e.g., be created by spin selective movement and controlled interactions between atoms as described in [2] or by entangling beam splitters as investigated in [12]. We assume that any hopping of atoms between the lattice sites is initially turned off and that the two chains consist of exactly one atom per lattice site [9]. (ii) Implementation of the pairwise BS: This is achieved by decreasing the potential barrier between the two rows of atoms. In an optical lattice one can decrease the corresponding laser intensities [9], while in an array of magnetic microtraps electric/magnetic fields can be switched to change the barrier height [13]. The dynamics after lowering the potential barrier is described by the Hamiltonian $H = H_{\rm BS} + H_{\rm int}$, where (cf. [9])

$$\begin{aligned} H_{\rm BS} &= \sum_{j=1}^{N} -J(a^{(j)\dagger}_{\rm I}a^{(j)}_{\rm II} + b^{(j)\dagger}_{\rm I}b^{(j)}_{\rm II} + {\rm H.c.}), \\ H_{\rm int} &= \sum_{l={\rm I,II}}\sum_{j=1}^{N}\frac{U_a}{2}a^{(j)\dagger}_l a^{(j)\dagger}_l a^{(j)}_l a^{(j)}_l \\ &\quad + \frac{U_b}{2}b^{(j)\dagger}_l b^{(j)\dagger}_l b^{(j)}_l b^{(j)}_l \\ &\quad + U_{ab}b^{(j)\dagger}_l b^{(j)}_l a^{(j)\dagger}_l a^{(j)}_l. \end{aligned} \tag{5}$$

Here $H_{\rm BS}$ describes vertical hopping of particles between the two rows with hopping matrix element J [14], and $H_{\rm int}$ gives the on-site interaction of two particles in a lattice site with internal state dependent interaction strengths U_a, U_b, and U_{ab}. For simplicity we assume that the interaction terms $H_{\rm int}$ can be neglected while the hopping is turned on; i.e., $J \gg U$, and we assume $U_a = U_b = U_{ab} = U$ [15]. Turning on the Hamiltonian $H_{\rm BS}$ for the specific time $T = \pi/(4J)$ implements the N pairwise BS. (iii) Acquisition of a relative phase between the symmetric and antisymmetric parts of the wave function: After implementing the BS we let the system evolve according to $H_{\rm int}$ for time τ. This introduces a phase $\theta = U\tau$ in each doubly occupied lattice site while it has no influence on singly occupied lattice sites. Recently, it was demonstrated in an interference experiment [16] that this phase θ allows the double occupancy sites to be distinguished from single occupancy ones. (iv) Turning off the lattice and measuring the resulting interference pattern [16]: After time τ the particles are released from the trap such that their wave function dominantly spreads along the vertical direction x (see Fig. 1). The density profile resulting from the pair of atoms j in state ρ_j' will exhibit interference terms dependent on $c_-^{(j)}$, $c_{\alpha\beta}^{(j)}$, and θ; so by varying the interaction phase θ, and noting that $P_+^j = 1 - P_-^j$, we can determine $|c_-^{(j)}|^2$. For N pairs of atoms the density profile will depend on $|c_{\alpha\beta}^{(j)}|^2$ and $|c_-^{(j)}|^2$, as well as on correlations between the density profiles of different pairs of atoms, according to Eq. (4). Measuring the density profile for the N pairs of atoms allows us to determine the different joint probabilities $P_{\pm\pm\cdots\pm}$; so by solving Eq. (4) we can detect multipartite entangled states which violate Eq. (2). We note that both the creation of the two copies of $\rho_{123\cdots N}$ and the network can be implemented with current experimental technology and do not introduce any novel or unknown sources of imperfections.

We now focus on the predictions of the entanglement detection network for multipartite entangled states recently generated in optical lattices by letting atoms in the chain interact with their nearest neighbors [2]. For an arbitrary number of atoms N, the state generated is given by $(1 + e^{i\phi})/2\,|\,000\cdots0\rangle + (1 - e^{i\phi})/2\,|\,\text{Cluster State}\rangle$, where ϕ is the phase arising from the controlled interactions during the entanglement creation procedure and the precise definition of the cluster state is given in [17]. The entangled state of N atoms generated by the process described in [2] will violate the inequalities Eq. (2) $\forall N$. For $N = 2$, tracing out one atom yields a violation V of Eq. (2) for $\phi \neq \{0, \pi, 2\pi\}$; i.e., our network detects all entangled states and is thus more powerful than Bell-Clauser-Horne-Shimony-Holt inequalities, which are violated only for $\phi \in [\varphi, \pi - \varphi] \cup [\pi + \varphi, 2\pi - \varphi]$, with $\varphi \approx 0.7$. When starting with $N = 3$, tracing marginal atoms yields violations V, as shown by the dashed

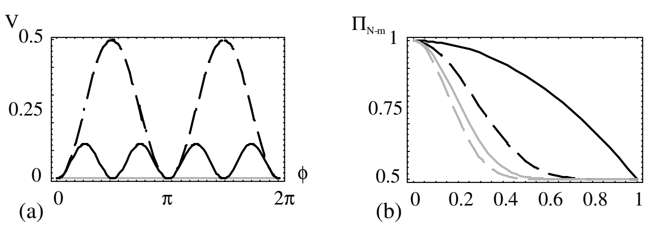

FIG. 2. In (a), we plot the violation V of the inequalities Eq. (2), $V_1 = \mathrm{Tr}(\rho_{123}^2) - \mathrm{Tr}(\rho_{12}^2)$ (dashed line), $V_2 = \mathrm{Tr}(\rho_{12}^2) - \mathrm{Tr}(\rho_1^2)$ (gray line), and $V_3 = \mathrm{Tr}(\rho_{12}^2) - \mathrm{Tr}(\rho_2^2)$ (solid line), as a function of the phase ϕ, for $N = 3$ atoms. Whenever $V > 0$, entanglement is detected by our network. In (b), we plot the purity Π_{N-m} for $m = 1$ (solid black line), $m = 7$ (dashed black line), $m = 14$ (solid gray line), and $m = 20$ (dashed gray line), as a function of ϵ, for $N = 300$ atoms.

and the solid curves in Fig. 2(a), respectively. Again, the network detects and distinguishes all multipartite entangled states created in this case. Also for $N > 3$ multiparticle entanglement is detected by tracing the marginal atoms. However, tracing further atoms does not yield additional violations of Eq. (2), as indicated by the gray line in Fig. 2(a), but the purity of the further reduced states remains constant. This is due to the characteristic of cluster states being resilient to measurements on individual subsystems [17]. In fact, the knowledge of the cluster states' purities can be further exploited to identify defects (empty sites) in the lattice and errors in the entanglement creation process [18].

Let us also note that any pure maximally entangled multipartite state can unequivocally be identified through its unique property of all reduced density operators having a purity of $1/2$. In optical lattices such maximally entangled generalized Greenberger-Horne-Zeilinger (GHZ) states that could be tested with the network can, e.g., be created by the beam splitter setups in 1D atomic pipelines as shown in [12].

Our network can also be used to study superpositions of distinct quantum macroscopic states which are of great importance for the better understanding of fundamental aspects of quantum theory [4,19]. There have been several proposals on how to create macroscopic superpositions in systems ranging from superconductors [20], Bose-Einstein condensates (BECs) [21] to optomechanical setups [22]. In the case of BECs, the macroscopic superpositions are multipartite entangled states of the form

$$|\psi\rangle = \frac{1}{\sqrt{K}}(|\phi_1\rangle^{\otimes N} + |\phi_2\rangle^{\otimes N}), \tag{6}$$

where $K = 2 + \langle\phi_1|\phi_2\rangle^N + \langle\phi_2|\phi_1\rangle^N$ and we define a parameter ϵ by the overlap $\epsilon^2 = 1 - |\langle\phi_1|\phi_2\rangle|^2$. Recently, a measure based on ϵ for the effective size S of such superpositions of distinct macroscopic quantum states was introduced [23]. It compares states of the form $|\psi\rangle$ with generalized GHZ states of N atoms, $(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})/\sqrt{2}$, where $\epsilon = 1$ for a generalized GHZ state. The effective size S of the state $|\psi\rangle$ is given by $S = N\epsilon^2$ [23].

We can determine S from the measurement of the purity of any reduced density operator of Eq. (6). We derive an explicit formula for the purity $\Pi_{N-m} = \mathrm{Tr}(\rho_{N-m}^2)$, where ρ_{N-m} is the density operator $\rho_N = |\psi\rangle\langle\psi|$ reduced by m subsystems. We find

$$\Pi_{N-m} = \frac{1 + \gamma^m + \gamma^N + 4\gamma^{N/2} + \gamma^{N-m}}{2(1 + \gamma^{N/2})^2}, \tag{7}$$

with $\gamma = 1 - \epsilon^2 = |\langle\phi_1|\phi_2\rangle|^2$.

Suppose we create two identical BECs, each in state ρ_N, wait for a time t_c to let their density operators be inelastically reduced via single particle loss processes to $\rho_{N-m} \otimes \rho_{N-m'}$, and then let the two BECs go through a BS-like transformation. As an aside we note that the reduced density operators emerging from multiparticle collisions not only depend on ϵ but also on $|\phi_1\rangle$, $|\phi_2\rangle$ and thus could be used to gain further insight into the properties of the state. The BS can be implemented either through collisional interactions between the atoms in two arms of a spatial interferometer [24], or by first turning both BECs into Mott insulator states [9] trapping them in an optical lattice and then switching on H_{BS}. We consider only the latter method since it corresponds more directly to the situation of Fig. 1. The loss processes which reduce the density operators ρ_N are stochastic so, in general, $m \neq m'$, which means that only $N - n$, where $n = \max\{m, m'\}$, pairs of atoms will undergo pairwise BS in the lattice. Since only density profiles of pairs of vertical sites with two atoms contribute to the interference pattern, measuring the collective density profile will determine $\mathrm{Tr}(\rho_{N-n}^2)$. Plots of different Π_{N-n} for an initial number of $N = 300$ atoms as a function of ϵ are presented in Fig. 2(b). The dependence of these curves on N is very weak, but for constant ϵ the values of Π_{N-n} quickly tend towards $1/2$ as n increases. Therefore from measuring the density profile the determination of ϵ from Π_{N-n} is best done for small $n \sim 15$. For a given particle loss rate, the average value of n after time t_c will be known and ϵ can be found by averaging over several runs of the experiment performed under identical initial conditions. We note that this measurement is considerably simpler than those in the previous entanglement detection schemes, since we do not require the ability to distinguish between individual pairs of bosons but need only to find the overall probability of projecting on the symmetric and antisymmetric subspaces. If the experimental setup allows one to determine the number of pairwise beam splitters $N - n$, the measurement can be performed in one run, and inelastic processes are not necessary if one can measure the collective density profile associated with a subset of the pairs of atoms.

We have presented and investigated a simple quantum network that detects multipartite entanglement, requiring only two identical copies of the quantum state and the pairwise BS between the constituents of each copy. We have shown how the network can be implemented in optical lattices and magnetic microtraps using current technology. As examples of its power, we have applied the network to detect entanglement and imperfections in cluster states and have shown that it also can be used to characterize macroscopic superposition states.

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