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# *Singular Flows with Time-Varying Weights*

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## Abstract

We study the mean field limit for singular dynamics with time evolving weights. Our results are an extension of the work of Serfaty [14] and Bresch-Jabin-Wang [8], which consider singular Coulomb flows with weights which are constant time. The inclusion of time dependent weights necessitates the commutator estimates of [8,14], as well as a new functional inequality. The well-posedness of the mean field PDE and the associated system of trajectories is also proven.


## 1. Introduction

We aim to study the well posedness and the mean field limit of the doubly non-local transport PDE

$$\partial_t \mu - \operatorname{div}(\mu\, \mathbf{a} \star \mu) = h[\mu], \ \ \mu(0, x) = \mu_0, \tag{1.1}$$

where $\mathbf{a} : \mathbb{R}^d \to \mathbb{R}^d$ is a given vector field and $h[\mu]$ is the signed measure given by

$$h[\mu] := \mu(x) \int_{\mathbb{R}^d} S(x - y)\mu(y)dy \tag{1.2}$$

for some function $S : \mathbb{R}^d \to \mathbb{R}$. We use $\mu(x)dx$ as the notation for integrating against the measure $\mu$ independently of being absolutely continuous or not with respect to the Lebesgue measure. Moreover, we shall denote by $\mu(x)$ its density with respect to the Lebesgue measure in case it is absolutely continuous. Here, both $\mathbf{a}$ and $S$ satisfy some appropriate structural hypothesis specified in Sect. 2. The initial data $\mu_0$ belongs to $\mathcal{P}(\mathbb{R}^d)$, where $\mathcal{P}(\mathbb{R}^d)$ is the set of probability measures on $\mathbb{R}^d$. The solutions $\mu(t, \cdot) \in \mathcal{P}(\mathbb{R}^d)$ sought after are curves of probability measures, and thus the source term $h[\mu]$ has to have zero average.

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The existence and uniqueness theory of PDEs of the type (1.1) or variants thereof has been handled in several works: in [23] the existence and uniqueness of measure valued solutions has been established for non-negative source terms $h[\mu]$ which satisfy a boundeness and Lipschitz condition with respect to the Wasserstein distance. In [24], a variant of the Wasserstein distance for signed measures has been introduced and applied in order to remove the positivity assumption on $h$. The non-positivity assumption is important if one wishes to study models in which $h[\mu]$ has zero mean. Both of these results do not cover the important case where $h[\mu]$ is given by the formula (1.2), since such source terms typically satisfy the boundeness and Lipschitz conditions required only locally. This obstacle has been successfully overcome in [16], in which the derivation of the PDE (1.1) as a mean field limit is also studied.

Let us denote by $\mathbb{R}^{dN} \setminus \triangle_N$ the set of non-overlapping configurations, that is, the set of $\mathbf{x}_N := (x_1, \ldots, x_N) \in \mathbb{R}^{dN}$ such that $x_i \neq x_j$ for all $i \neq j$, and let us denote by $\mathbb{M}^N$ the set of admissible weights, that is, the set of $\mathbf{m}_N := (m_1, \ldots, m_N)$ such that $m_i \geq 0$, with $\sum_{i=1}^N m_i = N$. We can define the empirical measure associated with this configuration as

$$\mu_N := \frac{1}{N} \overset{[}{i} = 1]N \sum m_i \delta_{x_i}. \tag{1.3}$$

The following system of ODEs governs the particle dynamics:

$$\begin{cases} \dot{x}_i^N(t) = -\frac{1}{N} \overset{[}{j} = 1]N \sum m_j^N(t) \mathbf{a}(x_i^N(t) - x_j^N(t)), \ x_i^N(0) = x_i^{0,N} \\ \dot{m}_i^N(t) = \frac{1}{N} \overset{[}{j} = 1]N \sum m_i^N(t) m_j^N(t) S(x_i^N(t) - x_j^N(t)), \ m_i^N(0) = m_i^{0,N}. \end{cases} \tag{1.4}$$

Subject to several technical assumptions on the functions $\mathbf{a}$ and $S$, the ODE system (1.4) has a globally well defined flow $(\mathbf{x}_N(t), \mathbf{m}_N(t))$, see [2, Theorem 3], where $\mathbf{x}_N(t) := (x_1^N(t), \ldots, x_N^N(t)) \in \mathbb{R}^{dN}$ and $\mathbf{m}_N(t) := (m_1^N(t), \ldots, m_N^N(t)) \in \mathbb{M}^N$. We will denote by $\mu_N(t)$ the empirical measure associated to the system (1.4), with a slight abuse of notation, as the empirical measure given by (1.3) associated to the configuration $(\mathbf{x}_N(t), \mathbf{m}_N(t))$. The empirical measure is readily checked to be a distributional solution of the PDE (1.1). We say that (1.1) is the mean field limit of the interacting particle system (1.4) if

$$W_1(\mu_N(0), \mu_0) \underset{N \to \infty}{\to} 0 \Longrightarrow W_1(\mu_N(t), \mu(t)) \underset{N \to \infty}{\to} 0, \ t \in [0, T],$$

where $\mu(t)$ is the solution to the PDE (1.1) with initial data $\mu_0$ at time $t$ and $W_1$ is the Wasserstein (or Monge-Kantorovich) distance. Before we proceed with the mathematical discussion, let us say a few words about the real life phenomena that the above models idealizes. Given $N$ agents interacting pairwise via a given interaction $\mathbf{a}$ with time evolving opinions $(x_1^N(t), \ldots, x_N^N(t))$ and weights of influence $(m_1^N(t), \ldots, m_N^N(t))$ the system (1.4) describes the evolution both of the opinions and the weights of influence in time. As it can be seen from (1.4), the $i$-th opinion and the $i$-th weight evolve according to a law which takes into consideration all the other involved opinions and influences.

Opinion dynamic models enter into the realm of consensus or synchronization models. An archetypical example is the Kuramoto model for synchronization and its variants [1,11]. Similar models appear also in crowd dynamics and have received mathematical treatment in [20,24]. The book [27] and the surveys [9,12] gives an exhaustive account of various different models describing collective behavior in the context of alignment in velocity instead of opinion dynamics. These modelling ideas have also been used in global optimization methods, coined as consensus-based optimization, see [28] for a recent review of the subject. The scientific literature seems to be so far mostly limited to treat continuous interaction kernels $\mathbf{a}$, although Hegselmann-Krause-type opinion models may also include jump discontinuities, which correspond to threshold effects in the interaction between the agents, see the discussion in [10]. To the best of our knowledge, the blow-up singularities which are considered in the present work have not yet been studied for when the weights evolve in time.

Viewed as a system of opinion dynamics, the mean field limit can be interpreted as a large population limit. This limit has been proven in [16], with an approach which resembles the celebrated Dobrushin theorem [13]. In [2] a weak version of the mean field limit has been recovered through a graph limit approach. By "weak" we mean that the result is limited to a special choice of initial configurations (at the microscopic level) and a special choice of initial data for the Cauchy problem (1.1). Among other works which consider the graph limit and its link with the mean field limit we mention [7,22] as well as [3] for a general overview. The case where the weights form a non-symmetric matrix, a scenario which arises for instance in Neuron dynamics, is also very interesting and has been analyzed in [19]. All of these results considered measure valued solutions, and typically one has to require that the functions $\mathbf{a}$ and $S$ at least satisfy some Lipschitz continuity. However, the case where the function $\mathbf{a}$ admits discontinuities is also relevant and requires a different mathematical treatment. In the recent work [5], the well posedness and mean field limit has been established for the one dimensional attractive Coulomb kernel, which corresponds to the choice $\mathbf{a}(x) = \mathrm{sgn}(x)$ in one dimension. The argument proposed there relies on recasting the limit PDE as a Burgers type equation only valid in one dimension, and invoking Kruzkhov theory of conservation laws in order to prove stability estimates for the resulting Burgers equation. It should be mentioned that singularities emerging from $S$ are also relevant and have been considered in [6,20]. These singularities arise naturally in models of pairwise competition and positional influence which describe voting systems.

The purpose of this work is twofold: to study the well posedness and mean field limit for (1.1) for arbitrary dimension $d \geq 3$ in the case where $\mathbf{a}$ exhibits a Coulomb type singularity, thereby addressing a question left open in [5]. More precisely, we assume the following hypotheses (**H1**) on the functions $\mathbf{a}$ and $S$ in (1.1):

- (**H1**-i): $\mathbf{a} = \mathbb{J}\nabla V$, with $V(x)$ the $d-$dimensional repulsive Coulomb interaction ($d \geq 3$), $V(x) = -\frac{|x|^{2-d}}{2-d}$. Here $\mathbb{J}$ is either a $d \times d$ anti-symmetric matrix or the identity matrix.
- (**H1**-ii): $S \in \mathcal{S}(\mathbb{R}^d)$ is an odd ($S(-x) = -S(x)$) function.

Note that $\mathrm{div}(\mathbb{J}\nabla V) = 0$ when $\mathbb{J}$ is anti-symmetric and $\mathrm{div}(\mathbb{J}\nabla V) = -\delta_0$ when $\mathbb{J}$ is the identity. In comparison to [5], the difficulties which must be addressed in the new singular settings are reflected on several levels: First, one has to justify why (1.4) has a globally well defined flow. This result is classical when the weights are time independent and follows from the observation that initial separation of opinions is propagated in time, i.e.,

$$x_i^{0,N} \neq x_j^{0,N} \Longrightarrow \left| x_i^N(t) - x_j^N(t) \right| > 0.$$

The inclusion of weights which evolve in time necessitates imposing a condition on $S$ which would prevent a Ricatti type blow-up, and this is why we impose the parity condition $S(x) = -S(-x)$. The main difficulty for obtaining the well-posedness at the macroscopic level lies in the fact that when handling discontinuous $V$ there is no reason to expect that the term $\mathbb{J}\nabla V \star \mu$ verifies a Lipschitz condition on the space of probability *measures* with respect to the Wasserstein distance. This fact eliminates the possibility of proving the stability estimate with respect to the Wasserstein distance. We are therefore led to restrict the solutions considered to more well behaved function spaces, namely $L^p$ or Sobolev spaces. Thirdly, since the stability estimate at our disposal is restricted to $L^p$ spaces, we are unable to apply it directly for the empirical measure in order to obtain the long time convergence. The most difficult part of this work boils down to overcoming this issue. Our main idea is to make use of the functional inequalities discovered in [8,14] as well as proving new functional inequalities which are necessary due to the inclusion of a source term. The functional inequalities in [14] reflect a particularly exciting development in the theory of mean field limits, as they allow for the first time to rigorously identify the mean field limit of Coulomb flows and have already found numerous applications both in classical mean field limits [18,21,25] as well as in quantum many body systems [17,26]. Before stating our two main results, namely the well posedness and the mean field limit, we introduce the following hypotheses:

(**H2**) The initial data $\mu_0 \in \mathcal{P}(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)$ is such that there is some $R > 0$ with $\mathrm{supp}(\mu_0) \subset B(0, R)$.

**Theorem 1.1.** *Let assumptions (**H1**)–(**H2**) hold. Then, there exists a unique solution to the problem* (1.1) *with initial data* $\mu_0$*. Moreover, this solution satisfies* $\mu \in C\left([0, T]; L^p(\mathbb{R}^d)\right)$ *for all* $1 \leq p \leq \infty$ *and* $\mu \in L^\infty\left([0, T]; W^{1,p}(\mathbb{R}^d)\right)$ *for all* $1 \leq p < \infty$*. Furthermore,* $\mu(t, \cdot)$ *is compactly supported and its support satisfies*

$$\mathrm{supp}(\mu(t, \cdot)) \subset B(0, \overline{R}), \text{ for all } t \in [0, T],$$

*where* $\overline{R} = \overline{R}(\|\mu_0\|_{L^\infty}, R, T, \|S\|_{L^\infty})$.

**Theorem 1.2.** *Let assumptions (**H1**)-(**H2**) hold and let* $\mu(t, \cdot)$ *be the unique solution with initial data* $\mu_0$ *ensured via Theorem* 1.1*. Let* $\mathbf{m}_N^0 \in \mathbb{M}^N$ *such that* $m_i^{0,N} \leq M$ *for some* $M > 0$ *and* $\mathbf{x}_N^0 \in \mathbb{R}^{dN} \backslash \triangle_N$*. Denote by* $(\mathbf{x}_N(t), \mathbf{m}_N(t)) \in C([0, T]; \mathbb{R}^{dN} \times \mathbb{R}^N)$ *the solution to the system of ODEs (2.1) with initial data* $(\mathbf{x}_N^0, \mathbf{m}_N^0)$ *ensured via Theorem* 2.2*. Then*

$$\mu_N(t,\cdot) \underset{N\to\infty}{\to} \mu(t,\cdot) \textit{ in the weak sense locally uniformly in time,}$$

*provided that* $\mathcal{E}_N(0) \underset{N\to\infty}{\to} 0$ *where* $\mathcal{E}_N(t)$ *is defined by*

$$\mathcal{E}_N(t) := \int_{x\neq y} V(x-y)(\mu_N(t,\cdot)-\mu(t,\cdot))^{\otimes 2} dxdy.$$

Several parts of this work could be easily generalized to Riesz interactions, however not all main results in Theorems 1.1 and 1.2 follow in a direct manner, as discussed at the end of this work. Obtaining a full generalization of the previous theorem to Riesz interactions is an interesting open problem, in particular in the most singular range.

The paper unfolds as follows: In Sect. 2 we recap basic structural properties of the model and state the main results. Afterwards, we include an important discussion in which we elaborate on the novelty of our method. Section 3 is devoted to proving the existence of solutions to (1.1) and the stability estimate, from which the uniqueness follows at once. Section 4 is devoted to proving the global well-posedness of the ODE (1.4). Finally, Sect. 5 is devoted to modifying the modulated energy approach introduced in [14] and extended in [8], thus yielding the mean field limit.

## 2. Preliminaries

### *2.1. The Equation for the Trajectories*

The dynamics that we consider are governed by the following system of $(d+1)\times N$ ODEs:

$$\begin{cases} \dot{x}_i^N(t) = -\frac{1}{N} \overset{[}{j} = 1]N\sum m_j^N(t)\mathbb{J}\nabla V(x_i^N(t)-x_j^N(t)),\ x_i^N(0)=x_i^{0,N} \\ \dot{m}_i^N(t) = \frac{1}{N} \overset{[}{j} = 1]N\sum m_i^N(t)m_j^N(t)S(x_i^N(t)-x_j^N(t)),\ m_i^N(0)=m_i^{0,N} \end{cases} \tag{2.1}$$

The notation is as follows: the unknowns are $x_i^N \in \mathbb{R}^d$ and $m_i^N \in \mathbb{R}$ are referred to as opinions and weights respectively, and are supplemented with initial data $x_i^{0,N}, m_i^{0,N}$.

**Definition 2.1.** We say that $(\mathbf{x}_N(t), \mathbf{m}_N(t)) \in C([0,T]; \mathbb{R}^{dN}\times\mathbb{R}^N)$ is a solution to the system of ODEs (2.1) on $[0,T)$ $(T\leq\infty)$ if

$$\min_{i\neq j}\left|x_i^N(t)-x_j^N(t)\right| > 0,\ t\in[0,T)$$

and for all $t\in[0,T)$ and all $1\leq i\leq N$ it holds that

$$\begin{cases} x_i^N(t) = x_i^{0,N} - \frac{1}{N} \; \overset{[}{j}= 1]N\sum \displaystyle\int_0^t m_j^N(\tau)\mathbb{J}\nabla V(x_i^N(\tau) - x_j^N(\tau))d\tau \\ m_i^N(t) = m_i^{0,N} + \frac{1}{N} \; \overset{[}{j}= 1]N\sum \displaystyle\int_0^t m_i^N(\tau)m_j^N(\tau)S(x_i^N(\tau) - x_j^N(\tau))d\tau. \end{cases}$$

We say that $(\mathbf{x}_N(t), \mathbf{m}_N(t))$ is a solution to the system of ODEs (2.1) with maximal life span $T > 0$ if it is a solution on $[0, T)$ to (2.1), but is not a solution on $[0, T]$.

The following theorem establishes that there is a well-defined flow for the system of ODEs (2.1), the proof is postponed to Sect. 4:

**Theorem 2.2.** *Let hypothesis (**H1**) hold. Suppose that for any $N \in \mathbb{N}$ it holds that $\mathbf{x}_N^0 \in \mathbb{R}^{dN} \backslash \triangle_N$, $\mathbf{m}_N^0 \in \mathbb{M}_N$ and there is some $M > 0$ such that for all $1 \leq i \leq N$ it holds that*

$$0 \leq m_i^{0,N} \leq M.$$

*Then, the system of ODEs (2.1) has a unique global solution $(\mathbf{x}_N(t), \mathbf{m}_N(t)) \in C^1([0, \infty); \mathbb{R}^{dN} \times \mathbb{R}^N)$ solution with initial data $(\mathbf{x}_N^0, \mathbf{m}_N^0)$. In particular $x_i^N(t) \neq x_j^N(t)$ for all $t \in [0, \infty)$ and all $i \neq j$.*

*Remark 2.3.* Since $V$ is the Coulomb interaction there is some constant $\mathbf{C} > 0$ such that

$$\left\| \nabla^2 V \star \mu \right\|_2 \leq \mathbf{C} \left\| \mu \right\|_2$$

and

$$0 \leq \widehat{V}(x) \leq \frac{\mathbf{C}}{|x|^2},$$

where $\widehat{V}$ is the Fourier transform of $V$. The first inequality is the Calderon-Zygmund inequality (see [15, Theorem 4.12]), and the second inequality is due to the fact that $-\Delta V = \delta_0$.

*Remark 2.4.* The assumption that $S$ is odd implies that any solution $(\mathbf{x}_N(t), \mathbf{m}_N(t))$ of the system of ODEs (2.1) has $m_i^N(t) \geq 0$ for all $t \in [0, T]$ and conserves the total weight, i.e.

$$\frac{1}{N} \; \overset{[}{i}= 1]N \sum m_i^N(t) = 1$$

for all $t \in [0, T]$. In addition there is a constant $\overline{M} = \overline{M}(M, \|S\|_\infty)$ such that for all $t \in [0, T]$ it holds that $m_i^N(t) \leq \overline{M}, \; i = 1, \ldots, N$.

### *2.2. The Mean Field Equation*

Recall that the mean field limit PDE that is expected to be derived from (2.1) is

$$\partial_t \mu - \operatorname{div}(\mu \mathbb{J} \nabla V \star \mu) = h[\mu], \ \mu(0, x) = \mu_0. \tag{2.2}$$

We introduce the notation $\mathbf{S}[\mu] := \int_{\mathbb{R}^d} S(x-y)\mu(y)dy$ and $\mathbf{A}[\mu] := \mathbb{J}\nabla V \star \mu$. We endow the space $C([0,T]; \mathcal{P}(\mathbb{R}^d))$ with the metric

$$D(\mu, \nu) := \sup_{t \in [0,T]} W_1(\mu(t,\cdot), \nu(t,\cdot))$$

where $W_1$ is the Wasserstein distance (see [29]). Our notion of weak solution is as follows:

**Definition 2.5.** Assume $V$ and $S$ satisfy (**H1**) and let $\mu_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. A time dependent $\mu \in C([0,T]; L^1(\mathbb{R}^d)) \cap L^\infty(0,T; L^\infty(\mathbb{R}^d))$ is said to be a weak solution of (2.2) if for all $\varphi \in C_0^\infty(\mathbb{R}^d)$, it holds that

$$\frac{d}{dt}\int_{\mathbb{R}^d} \varphi(x)\mu(t,x)dx + \int_{\mathbb{R}^d} \nabla\varphi(x)\mathbf{A}[\mu](t,x)\mu(t,x)dx = \int_{\mathbb{R}^d} h[\mu](t,x)\varphi(x)dx, \ \mu(0,x) = \mu_0,$$

in the distributional sense.

Notice that the second term in the definition of weak solution to (2.2) makes sense by a separation into short and long range terms in $\nabla V \star \mu$ using that $\mu(t,\cdot) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ for all $t \in [0,T]$.

*Remark 2.6.* As remarked in [16], an equivalent definition to Definition 2.5 is to demand that for each $\varphi \in C^\infty((0,T) \times \mathbb{R}^d))$ it holds that

$$\int_0^T \int_{\mathbb{R}^d} [\partial_t \varphi(t,x)\mu(t,x) + \nabla\varphi(t,x)\mathbf{A}[\mu](t,x)\mu(t,x) - \varphi(t,x)h[\mu](t,x)]\, dxdt = 0.$$

*Remark 2.7.* In case $\nabla V$ is Lipschitz, the definition of weak solution to (2.2) can be done assuming only $\mu_0 \in L^1(\mathbb{R}^d)$.

We will frequently need several structural properties of the source term which are not influenced by the regularity imposed on $S$. We summarize these properties in the following result:

**Proposition 2.8.** [16, Proposition 9] *For all* $\mu \in \mathcal{P}(\mathbb{R}^d)$*, it holds that* $\operatorname{supp}(h[\mu]) = \operatorname{supp}(\mu)$*,* $|h[\mu]| \le \|S\|_\infty \mu$*, and*

$$\int_{\mathbb{R}^d} h[\mu](x)dx = 0.$$

### 2.3. *Comments on the Method*

The most technically involved part of this work is the derivation of the PDE (2.2) as a mean field limit from the dynamics (2.1). Therefore we turn the reader's attention to several key points in the proof. First, let us consider the case where $S = 0$. In this case the PDE (2.2) is homogeneous, and the weights are constant in time - for simplicity we can take $m_i^N = \frac{1}{N}$ for all $1 \leq i \leq N$. The method in [14] rests upon a clever renormalization argument of a weak strong stability principle for the limit PDE. Let us introduce first the weak strong stability principle at the level of the limit PDE followed by a brief outline of the renormalization procedure carried out in [14].

**The weak strong stability principle**. Given solutions $\mu_1(t, \cdot)$, $\mu_2(t, \cdot)$ to the PDE (2.2), consider the modulated energy

$$\mathcal{E}(t) := \int_{\mathbb{R}^d} (\mu_1 - \mu_2)(t, x) V \star (\mu_1 - \mu_2)(t, x) dx.$$

A standard calculation reveals that

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(t) \leq & - \int_{\mathbb{R}^d \times \mathbb{R}^d} (\mathbb{J}\nabla V \star \mu_2(t, x) - \mathbb{J}\nabla V \star \mu_2(t, y)) \\ & \nabla V(x-y) (\mu_1(t, \cdot) - \mu_2(t, \cdot))^{\otimes 2} \, dxdy \\ & \leq 2 \sup_{t \in [0,T]} \left\| \nabla^2 V \star \mu_2(t, \cdot) \right\|_\infty \int_{\mathbb{R}^d} |\nabla V \star (\mu_1(t, \cdot) - \mu_2(t, \cdot))|^2 \, dx \\ & \leq 2 \sup_{t \in [0,T]} \left\| \nabla^2 V \star \mu_2(t, \cdot) \right\|_\infty \mathcal{E}(t), \end{aligned} \tag{2.3}$$

where we used the identity

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} V(x-y)\mu(x)\mu(y) dxdy = \int_{\mathbb{R}^d \times \mathbb{R}^d} |\nabla V \star \mu|^2 (x) dx \tag{2.4}$$

for any $\mu \in \dot{H}^{-1}(\mathbb{R}^d)$, $d \geq 3$, which can be readily seen in Fourier. Assuming that $\mu_2(0, \cdot) \in W^{1,p}(\mathbb{R}^d)$ (for $p > 1$ sufficiently large) and that $\mu_2$ enjoys propagation of Sobolev regularity, it can be shown that $\sup_{t \in [0,T]} \left\| \nabla^2 V \star \mu_2(t, \cdot) \right\|_\infty \lesssim \|\mu_2(0, \cdot)\|_{W^{1,p}}$. Thus the inequality (2.3) yields the stability estimate

$$\mathcal{E}(t) \leq e^{Ct} \mathcal{E}(0)$$

for some constant $C = C(\|\mu_2(0, \cdot)\|_{W^{1,p}})$. It is remarkable that this argument necessitates only regularity on one of the solutions involved.

**Renormalization**. We now define the renormalized modulated energy associated to $\mu$ and $\mu_N$ as

$$\mathcal{E}_N(\mu, \mu_N) := \int_{x \neq y} V(x-y) (\mu_N(\cdot) - \mu(\cdot))^{\otimes 2} \, dxdy. \tag{2.5}$$

Given a solution $\mu(t,\cdot)$ to the PDE (2.2) and a solution $\mathbf{x}_N(t)$ to the system of ODEs (2.1), we define the renormalized modulated energy associated to them as

$$\mathcal{E}_N(t) := \mathcal{E}_N(\mu(t,\cdot), \mu_N(t,\cdot)). \tag{2.6}$$

The definitions (2.5) and (2.6) extend in the obvious manner in the case of weighted empirical measures. A standard calculation reveals that

$$\frac{d}{dt}\mathcal{E}_N(t) \leq -\int_{x\neq y} (\mathbb{J}\nabla V \star \mu(t,x) - \mathbb{J}\nabla V \star \mu(t,y)) \nabla V(x-y) (\mu_N(t,\cdot) - \mu(t,\cdot))^{\otimes 2}\, dxdy.$$

Note the removal of the diagonal in the integrals above, which is necessary due to the singularity of $V$ at the origin. The functional inequality which allows us to close the estimate, and which reflects the outstanding novelty of [14], is that for any given configuration $\mathbf{x}_N \in \mathbb{R}^{dN}\setminus\triangle_N$, bounded probability density $\mu$ and bounded Lipschitz vector field $u$ there holds the inequality

$$\left|\int_{x\neq y} (u(x)-u(y))\, \nabla V(x-y)\, (\mu_N-\mu)^{\otimes 2}\, dxdy\right| \leq C\int_{x\neq y} V(x-y) (\mu_N-\mu)^{\otimes 2}\, dxdy + o_N(1), \tag{2.7}$$

where $C > 0$ is some harmless constant. We will refer to (2.7) as the commutator estimate, see [26]. The representation (2.4) is inapplicable to the renormalized energy $\mathcal{E}_N(t)$ due to the removal of the diagonal. A key observation of [14] is that the identity (2.4) admits a renormalized version. More precisely, denote by $\delta_x^{(\eta)}$ the uniform measure of mass 1 on $\partial B(x,\eta)$. Given $\boldsymbol{\eta} = (\eta_1,\ldots,\eta_N)$ set $\mu_N^{(\boldsymbol{\eta})} := \frac{1}{N} \overset{[}{i}= 1]N\sum \delta_{x_i}^{(\eta_i)}$. Then, letting $r_i := \frac{1}{4}\min_{i\neq j}|x_i - x_j|$ and $\eta_i \leq r_i$, Serfaty proves the identity

$$\mathcal{E}_N(\mu,\mu_N) = \frac{1}{c_d}\left(\int_{\mathbb{R}^d\times\mathbb{R}^d} \left|\nabla V \star (\mu_N^{(\eta)} - \mu)\right|^2 (x) dx - \frac{c_d}{N} \overset{[}{i}= 1]N\sum V(\eta_i)\right) + \text{negligible terms}. \tag{2.8}$$

**The weak strong stability principle with source term**. Coming back to the more general scenario at stake, where a source term $h[\mu]$ is included, it can be shown by direct calculation that in this case $\mathcal{E}(t)$ verifies

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) \leq &-\int_{\mathbb{R}^d\times\mathbb{R}^d} (\mathbb{J}\nabla V\star\mu(t,x) - \mathbb{J}\nabla V\star\mu(t,y))\, \nabla V(x-y)\, (\mu_1(t,\cdot)-\mu_2(t,\cdot))^{\otimes 2}\, dxdy \\ &+ 2\int_{\mathbb{R}^d\times\mathbb{R}^d} V(x-y) \left(h\left[\mu_1\right](t,x) - h\left[\mu_2\right](t,x)\right)(\mu_1(t,y)-\mu_2(t,y))\, dxdy := \mathcal{D}^1(t) + \mathcal{D}^2(t).\end{aligned} \tag{2.9}$$

The first term $\mathcal{D}^1$ in the right hand side of (2.9) can be handled precisely as before. Thus, the inclusion of the source term is manifested through the second term $\mathcal{D}^2$. In this context, our key observation is that $h$ is a Lipschitz operator on $\dot{H}^{-1}(\mathbb{R}^d)$, namely we will show (see Corollary 3.7) an inequality of the form

$$\|h[\mu_1] - h[\mu_2]\|_{\dot{H}^{-1}(\mathbb{R}^d)} \leq C\, \|\mu_1 - \mu_2\|_{\dot{H}^{-1}(\mathbb{R}^d)}\,. \tag{2.10}$$

Equipped with Inequality (2.10) a bound on $\mathcal{D}^2(t)$ by means of $\mathcal{E}(t)$ can be obtained. Indeed, appealing to Fourier we see that

$$\begin{aligned}\mathcal{D}^2(t) &= 2\int_{\mathbb{R}^d} V\star(h[\mu_1]-h[\mu_2])(t,x)(\mu_1(t,x)-\mu_2(t,x))\,dx\\&= 2\int_{\mathbb{R}^d}\frac{\widehat{(h[\mu_1]-h[\mu_2])}(t,x)}{|x|}\frac{\widehat{(\mu_1-\mu_2)}(t,x)}{|x|}dx\\&\le 2\|h[\mu_1(t,\cdot)]-h[\mu_2(t,\cdot)]\|_{\dot{H}^{-1}(\mathbb{R}^d)}\|\mu_1(t,\cdot)-\mu_2(t,\cdot)\|_{\dot{H}^{-1}(\mathbb{R}^d)},\end{aligned}$$

by Cauchy-Schwarz. This combined with (2.10), shows that

$$\mathcal{D}^2(t)\lesssim\|\mu_1(t,\cdot)-\mu_2(t,\cdot)\|^2_{\dot{H}^{-1}(\mathbb{R}^d)}=\mathcal{E}(t).$$

which in turn closes the Grönwall estimate.

**Renormalization with source term**. We will show by a direct calculation (see Theorem 5.1) the inequality

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_N(t) \le &-\int_{x\neq y}(\mathbb{J}\nabla V\star\mu(t,x)-\mathbb{J}\nabla V\star\mu(t,y))\nabla V(x-y)\left(\mu_N(t,\cdot)-\mu(t,\cdot)\right)^{\otimes 2}dxdy\\&+2\int_{x\neq y}V(x-y)\left(h\left[\mu_N(t,\cdot)\right](x)-h\left[\mu(t,\cdot)\right](x)\right)\left(\mu_N(t,y)-\mu(t,y)\right)dxdy\\:=&\mathcal{D}^1_N(t)+\mathcal{D}^2_N(t).\end{aligned}\tag{2.11}$$

The first term $\mathcal{D}^1_N$ in the right hand side of Inequality (2.11) can be handled precisely via the commutator estimate (2.7) in the same manner outlined before. Thus, the main challenge is to renormalize the argument outlined in the previous paragraph in order to control $\mathcal{D}^2_N(t)$ by means of $\mathcal{E}_N(t)$. A first possibility is to attempt proving an identity in the spirit of (2.8) for

$$\int_{x\neq y}V(x-y)(h[\mu_N]-h[\mu])(x)(h[\mu_N]-h[\mu])(y)dxdy.$$

However, it is not clear if the fine cancellations which produce this formula can be extended in this manner, since the operation $\mu\mapsto h[\mu]$ is non linear. We therefore pursue a different path inspired by the work of Bresch-Jabin-Wang [8], which avoids making use of this representation. The merit of this approach is that it does not necessitate any algebraic identities, but only inequalities. With this approach we will prove a functional inequality of the form

$$\begin{aligned}&\int_{x\neq y}V(x-y)(\mu_N-\mu)(x)(h[\mu_N]-h[\mu])\\&\quad(y)dxdy\le C\int_{x\neq y}V(x-y)(\mu_N-\mu)^{\otimes 2}dxdy+o_N(1),\end{aligned}\tag{2.12}$$

for some constant $C>0$.

## 3. The Well Posedness of the Mean Field Equation

### 3.1. Existence

We start by recalling existence and uniqueness for the limit PDE as established in [16].

**Proposition 3.1.** ([16, Theorem 2]) *Assume that $\nabla V$ is Lipschitz, let $S \in W^{1,\infty}(\mathbb{R}^d)$ be odd and let the initial data $\mu_0$ satisfy (**H2**). Then, there exist a unique solution $\mu \in C([0,T];\mathcal{P}(\mathbb{R}^d))$ of (2.2) in the sense of Definition 2.5 and Remark 2.7. Moreover, $\mu(t,\cdot)$ is compactly supported for all $t \in [0,T]$.*

*Remark 3.2.* If one further assumes $V \in \mathcal{S}(\mathbb{R}^d)$, $S \in \mathcal{S}(\mathbb{R}^d)$ and $\mu_0 \in C_0^\infty(\mathbb{R}^d)$ in Proposition 3.1, then it is possible to show that the solution $\mu$ provided by Proposition 3.1 is a smooth compactly supported function. Consequently, subject to these assumption the solution is classical.

The main theorem of this section concerns the existence of a solution to (2.2) and is based on a stability argument using suitably mollified problems. An important component of the proof is the propagation of Sobolev norms. Sobolev regularity is important in order to be able to use the commutator estimate (2.7) against Lipschitz vector fields. In this context, we will need the following near-boundedness of the Calderon-Zygmund operator on $L^\infty$.

**Proposition 3.3.** ([4, Proposition 7.7]) *Let $s > 0$ and let $a \in [1,\infty)$ and $b \in [1,\infty]$. Then, there is some $C > 0$ such that*

$$\left\|\nabla^2 V \star \mu\right\|_\infty \le C\left(\min\left(\|\nabla V \star \mu\|_b, \|\mu\|_a\right) + \|\mu\|_\infty\left(1+\left|\log\|\mu\|_\infty\right|\right)\right.$$
$$\left.\log\left(\max\left(e, \|\mu\|_\infty\right) + \|\mu\|_{C^{0,s}}\right)\right.$$

**Theorem 3.4.** *Assume that $S$ and $V$ are as in (**H1**) and let the initial data $\mu_0$ satisfy (**H2**). Then, there exist a solution in the sense of Definition 2.5 to (2.2) with initial data $\mu_0$. Moreover, this solution satisfies $\mu \in C\left([0,T];L^p(\mathbb{R}^d)\right)$ for all $1 \le p \le \infty$ and $\mu \in L^\infty\left([0,T];W^{1,p}(\mathbb{R}^d)\right)$ for all $1 \le p < \infty$. Furthermore, $\mu(t,\cdot)$ is compactly supported and its support satisfies*

$$\operatorname{supp}(\mu(t,\cdot)) \subset B(0,\overline{R}),\ \textit{for all } t \in [0,T],$$

*where $\overline{R} = \overline{R}(\|\mu_0\|_{L^\infty}, R, T, \|S\|_{L^\infty})$.*

*Proof.* **Step 1. Propagation of $L^p$ norms**. Let $\chi_\varepsilon$ be a standard mollifier and let $V_\varepsilon := \chi_\varepsilon \star V$ and $\mathbf{A}_\varepsilon[\mu] := \mathbb{J}\nabla V_\varepsilon \star \mu$. Let $\mu_\varepsilon$ be the solution of the mollified equation

$$\partial_t \mu_\varepsilon - \operatorname{div}\left(\mu_\varepsilon \mathbf{A}_\varepsilon[\mu_\varepsilon]\right) = h\left[\mu_\varepsilon\right],\ \mu_\varepsilon(0,x) = \chi_\varepsilon \star \mu_0. \tag{3.1}$$

This solution is classical and is ensured via Proposition 3.1 and Remark 3.2. We wish to quantify the growth, size of support and regularity of this solution. We compute the time derivative of $\|\mu_\varepsilon(t,\cdot)\|_p^p$.

$$\begin{aligned}\frac{d}{dt}\|\mu_\varepsilon(t,\cdot)\|_p^p &= p\int_{\mathbb{R}^d}\mu_\varepsilon^{p-1}(t,x)\partial_t\mu_\varepsilon(t,x)dx\\ &= p\int_{\mathbb{R}^d}\mu_\varepsilon^{p-1}(t,x)\mathrm{div}\left(\mu_\varepsilon(t,x)\mathbf{A}_\varepsilon[\mu_\varepsilon](t,x)\right)dx\\ &\quad + p\int_{\mathbb{R}^d}\mu_\varepsilon^{p-1}(t,x)h\left[\mu_\varepsilon\right](t,x)dx := J_1 + J_2.\end{aligned}$$

To bound $J_1$, note that integration by parts yields

$$\begin{aligned}J_1 &= p\int_{\mathbb{R}^d}\mu_\varepsilon^{p-1}(t,x)\nabla\mu_\varepsilon(t,x)\mathbf{A}_\varepsilon[\mu_\varepsilon](t,x)dx + p\int_{\mathbb{R}^d}\mu_\varepsilon^{p}(t,x)\mathrm{div}\left(\mathbf{A}_\varepsilon[\mu_\varepsilon](t,x)\right)dx\\ &= (p-1)\int_{\mathbb{R}^d}\mu_\varepsilon^{p}(t,x)\mathrm{div}\left(\mathbb{J}\nabla V_\varepsilon\star\mu_\varepsilon(t,x)\right)dx.\end{aligned} \tag{3.2}$$

Note that $\mathrm{div}(\mathbb{J}\nabla V_\varepsilon\star\mu_\varepsilon)=0$ if $\mathbb{J}$ is antisymmetric and $\mathrm{div}(\mathbb{J}\nabla V_\varepsilon\star\mu_\varepsilon)=-\chi_\varepsilon\star\mu_\varepsilon\le 0$ if $\mathbb{J}$ is the identity. Thus, (3.2) proves that $J_1\le 0$. Furthermore, we have

$$\begin{aligned}|J_2| \le p\textstyle\int_{\mathbb{R}^d}\mu_\varepsilon^p(t,x)\left|\mathbf{S}[\mu_\varepsilon]\right|(t,x)dx &\le p\left\|\mathbf{S}[\mu_\varepsilon](t,\cdot)\right\|_\infty\left\|\mu_\varepsilon(t,\cdot)\right\|_p^p\\ &\le p\left\|S\right\|_\infty\left\|\mu_\varepsilon(t,\cdot)\right\|_p^p.\end{aligned}$$

Therefore, we conclude that

$$\frac{d}{dt}\left\|\mu_\varepsilon(t,\cdot)\right\|_p^p \le p\left\|S\right\|_\infty\left\|\mu_\varepsilon(t,\cdot)\right\|_p^p,$$

which entails

$$\left\|\mu_\varepsilon(t,\cdot)\right\|_p^p \le e^{p\|S\|_\infty t}\left\|\mu_\varepsilon(0,\cdot)\right\|_p^p = e^{p\|S\|_\infty t}\left\|\chi_\varepsilon\star\mu_0\right\|_p^p \le e^{p\|S\|_\infty t}\left\|\mu_0\right\|_p^p,$$

or

$$\left\|\mu_\varepsilon(t,\cdot)\right\|_p \le e^{\|S\|_\infty t}\left\|\mu_0\right\|_p,\ 1\le p<\infty.$$

Since according to Proposition 3.1 $\mu_\varepsilon(t,\cdot)$ is compactly supported we can pass to the limit as $p\to\infty$ in the last inequality in order to deduce

$$\left\|\mu_\varepsilon(t,\cdot)\right\|_p \le e^{\|S\|_\infty t}\left\|\mu_0\right\|_p\ 1\le p\le\infty. \tag{3.3}$$

**Step 2. Propagation of support**. Denote by $R_\varepsilon(t)$ the size of the support of $\mu_\varepsilon(t,\cdot)$, i.e. $R_\varepsilon(t):=|\mathrm{supp}(\mu_\varepsilon(t,\cdot))|$. For each $\eta>0$ consider the function

$$\varphi_\eta(t,x):=\frac{1}{\eta+\mu_\varepsilon(t,x)}.$$

Multiplying (3.1) by $\varphi_\eta$ and integrating on $[0,t]\times\mathbb{R}^d$ we arrive at the equation

$$\begin{aligned}\int_{\mathbb{R}^d}\mu_\varepsilon(t,x)\varphi_\eta(t,x) &= \int_{\mathbb{R}^d}\mu_\varepsilon(0,x)\varphi_\eta(0,x)+\int_0^t\int_{\mathbb{R}^d}\partial_\tau\varphi_\eta(\tau,x)\mu_\varepsilon(\tau,x)dxd\tau\\ &\quad+\int_0^t\int_{\mathbb{R}^d}\varphi_\eta(\tau,x)\mathrm{div}(\mu_\varepsilon(\tau,x)\mathbf{A}_\varepsilon[\mu_\varepsilon](\tau,x))dxd\tau\\ &\quad+\int_0^t\int_{\mathbb{R}^d}\varphi_\eta(\tau,x)h[\mu_\varepsilon](\tau,x)dxd\tau. \end{aligned}\tag{3.4}$$

We proceed by manipulating the inner integrals in the right-hand side of (3.4).

$$\begin{aligned}\int_{\mathbb{R}^d}\partial_\tau\varphi_\eta(\tau,x)\mu_\varepsilon(\tau,x)dx &= -\int_{\mathbb{R}^d}\frac{\mu_\varepsilon(\tau,x)\partial_\tau\mu_\varepsilon(\tau,x)}{(\eta+\mu_\varepsilon(\tau,x))^2}dx\\ &= -\int_{\mathbb{R}^d}\frac{\mu_\varepsilon(\tau,x)\mathrm{div}(\mu_\varepsilon(\tau,x)\mathbf{A}_\varepsilon[\mu_\varepsilon](\tau,x))}{(\eta+\mu_\varepsilon(\tau,x))^2}dx\\ &\quad-\int_{\mathbb{R}^d}\frac{\mu_\varepsilon(\tau,x)h[\mu_\varepsilon](\tau,x)}{(\eta+\mu_\varepsilon(\tau,x))^2}dx\\ &= -\int_{\mathbb{R}^d}\frac{\mu_\varepsilon(\tau,x)\nabla\mu_\varepsilon(\tau,x)\mathbf{A}_\varepsilon[\mu_\varepsilon](\tau,x)}{(\eta+\mu_\varepsilon(\tau,x))^2}dx\\ &\quad-\int_{\mathbb{R}^d}\frac{\mu_\varepsilon^2(\tau,x)\mathrm{div}(\mathbf{A}_\varepsilon[\mu_\varepsilon](\tau,x))}{(\eta+\mu_\varepsilon(\tau,x))^2}dx\\ &\quad-\int_{\mathbb{R}^d}\frac{\mu_\varepsilon(\tau,x)h[\mu_\varepsilon](\tau,x)}{(\eta+\mu_\varepsilon(\tau,x))^2}dx. \end{aligned}\tag{3.5}$$

Integrating by parts the first integral in the right-hand side of (3.5) we get

$$\begin{aligned}&\int_{\mathbb{R}^d}\frac{\mu_\varepsilon(\tau,x)\nabla\mu_\varepsilon(\tau,x)\mathbf{A}_\varepsilon[\mu_\varepsilon](\tau,x)}{(\eta+\mu_\varepsilon(\tau,x))^2}dx\\ &= -\int_{\mathbb{R}^d}\mathbf{A}_\varepsilon[\mu_\varepsilon](\tau,x)\nabla\left(\frac{1}{\eta+\mu_\varepsilon(\tau,x)}\right)\mu_\varepsilon(\tau,x)dx\\ &= \int_{\mathbb{R}^d}\mathrm{div}(\mathbf{A}_\varepsilon[\mu_\varepsilon])(\tau,x)\frac{\mu_\varepsilon(\tau,x)}{\eta+\mu_\varepsilon(\tau,x)}dx\\ &\quad+\int_{\mathbb{R}^d}\mathbf{A}_\varepsilon[\mu_\varepsilon](\tau,x)\nabla\log\left(\eta+\mu_\varepsilon(\tau,x)\right)dx. \end{aligned}\tag{3.6}$$

As for the second integral in (3.4), we observe the identity

$$\begin{aligned}\int_{\mathbb{R}^d}\varphi_\eta(\tau,x)\mathrm{div}(\mu_\varepsilon(\tau,x)\mathbf{A}_\varepsilon[\mu_\varepsilon](\tau,x))dx &= \int_{\mathbb{R}^d}\mathbf{A}_\varepsilon[\mu_\varepsilon](\tau,x)\nabla\log\left(\eta+\mu_\varepsilon(\tau,x)\right)dx\\ &\quad+\int_{\mathbb{R}^d}\mathrm{div}(\mathbf{A}_\varepsilon[\mu_\varepsilon](\tau,x))\frac{\mu_\varepsilon(\tau,x)}{\mu_\varepsilon(\tau,x)+\eta}dx. \end{aligned}\tag{3.7}$$

The combination of Eqs. (3.5), (3.6) and (3.7) yields

$$\begin{aligned}&\int_{\mathbb{R}^d}\partial_\tau\varphi_\eta(\tau,x)\mu_\varepsilon(\tau,x)dx+\int_{\mathbb{R}^d}\varphi_\eta(\tau,x)\mathrm{div}(\mu_\varepsilon(\tau,x)\mathbf{A}_\varepsilon[\mu_\varepsilon](\tau,x))dx+\int_{\mathbb{R}^d}\varphi_\eta(\tau,x)h[\mu_\varepsilon](\tau,x)dx\\ &= -\int_{\mathbb{R}^d}\frac{\mu_\varepsilon^2(\tau,x)\mathrm{div}(\mathbf{A}_\varepsilon[\mu_\varepsilon](\tau,x))}{(\eta+\mu_\varepsilon(\tau,x))^2}dx-\int_{\mathbb{R}^d}\frac{\mu_\varepsilon(\tau,x)h[\mu_\varepsilon](\tau,x)}{(\eta+\mu_\varepsilon(\tau,x))^2}dx+\int_{\mathbb{R}^d}\varphi_\eta(\tau,x)h[\mu_\varepsilon](\tau,x)dx\\ &:= I_1+I_2+I_3. \end{aligned}$$

Using that $\operatorname{div}(\mathbf{A}_\varepsilon[\mu_\varepsilon]) = -\chi_\varepsilon \star \mu_\varepsilon$ if $\mathbb{J} = \mathrm{Id}$ and $\operatorname{div}(\mathbf{A}_\varepsilon[\mu_\varepsilon]) = 0$ if $\mathbb{J}$ is antisymmetric we see that

$$\begin{aligned} |I_1| &\le \|\chi_\varepsilon \star \mu_\varepsilon\|_\infty \int_{\mathbb{R}^d} \mu_\varepsilon^2(\tau,x)\varphi_\eta^2(\tau,x)dx \\ &\le \sup_{s\in[0,T]} \|\mu_\varepsilon(s,\cdot)\|_\infty \int_{\mathbb{R}^d} \mu_\varepsilon^2(\tau,x)\varphi_\eta^2(\tau,x)dx \\ &\le e^{T\|S\|_\infty} \|\mu_0\|_\infty \int_{\mathbb{R}^d} \mu_\varepsilon^2(\tau,x)\varphi_\eta^2(\tau,x), \end{aligned} \tag{3.8}$$

where in the last inequality we used (3.3). Furthermore, we have the inequalities

$$|I_2| \le \int_{\mathbb{R}^d} \frac{\mu_\varepsilon^2(\tau,x)\,|S\star\mu_\varepsilon|\,(\tau,x)}{(\eta+\mu_\varepsilon(\tau,x))^2}dx \le \|S\|_\infty \int_{\mathbb{R}^d} \mu_\varepsilon^2(\tau,x)\varphi_\eta^2(\tau,x)dx \tag{3.9}$$

and

$$|I_3| \le \int_{\mathbb{R}^d} \varphi_\eta(\tau,x)\mu_\varepsilon(\tau,x)\,|S\star\mu_\varepsilon|\,(\tau,x)dx \le \|S\|_\infty \int_{\mathbb{R}^d} \mu_\varepsilon(\tau,x)\varphi_\eta(\tau,x)dx. \tag{3.10}$$

Putting $\overline{M} = \max\left\{2\,\|S\|_\infty, e^{\|S\|_\infty T}\,\|\mu_0\|_\infty\right\}$ and substituting (3.8), (3.9) and (3.10) in (3.4) we find that

$$\begin{aligned} \int_{\mathbb{R}^d} \mu_\varepsilon(t,x)\varphi_\eta(t,x) &\le \int_{\mathbb{R}^d} \mu_\varepsilon(0,x)\varphi_\eta(0,x) \\ &+ 2\overline{M}\left(\int_0^t \int_{\mathbb{R}^d} \varphi_\eta^2(\tau,x)\mu_\varepsilon^2(\tau,x)dxd\tau \right. \\ &\left. + \int_0^t \int_{\mathbb{R}^d} \varphi_\eta(\tau,x)\mu_\varepsilon(\tau,x)dxd\tau\right). \end{aligned}$$

Since $\varphi_\eta(t,x)\mu_\varepsilon(t,x) \underset{\eta\to 0}{\to} \mathbf{1}_{|\cdot|\le R_\varepsilon(t)}(x)$, passing to the limit in the last inequality as $\eta \to 0$ we obtain

$$R_\varepsilon^d(t) \le R_\varepsilon^d(0) + 2\overline{M}\int_0^t R_\varepsilon^d(\tau)d\tau.$$

Hence, by Grönwall's inequality we deduce that $R_\varepsilon(t) \le R_\varepsilon(0)e^{\frac{2\overline{M}t}{d}} \le (R+\varepsilon)e^{\frac{2\overline{M}T}{d}}$. To conclude, we have proved

$$R_\varepsilon(t) \le \overline{R},\ t\in[0,T] \tag{3.11}$$

for some $\overline{R} = \overline{R}(\|\mu_0\|_{L^\infty}, R, T, \|S\|_{L^\infty})$.

**Step 3. Propagation of Sobolev norms**. Expanding the divergence in (3.1), we get

$$\partial_t\mu_\varepsilon - \nabla\mu_\varepsilon\mathbb{J}\nabla V_\varepsilon\star\mu_\varepsilon - \mu_\varepsilon\operatorname{div}(\mathbb{J}\nabla V_\varepsilon\star\mu_\varepsilon) = h\,[\mu_\varepsilon]\,.$$

Taking its $j$-th derivative, we obtain

$$\partial_t \partial_{x_j} \mu_\varepsilon - \nabla \partial_{x_j} \mu_\varepsilon \mathbb{J} \nabla V_\varepsilon \star \mu_\varepsilon - \partial_{x_j} (\mathbb{J} \nabla V_\varepsilon \star \mu_\varepsilon) \nabla \mu_\varepsilon - \partial_{x_j} \mu_\varepsilon \mathrm{div}(\mathbb{J} \nabla V_\varepsilon \star \mu_\varepsilon) \\ - \mu_\varepsilon \mathrm{div}(\mathbb{J} \nabla V_\varepsilon \star \partial_{x_j} \mu_\varepsilon) = R_{j,\varepsilon}(t, x), \tag{3.12}$$

with $R_{j,\varepsilon}(t, x) := \partial_{x_j} h [\mu_\varepsilon]$. Set $v_\varepsilon(t, x) := \mathbb{J} \nabla V_\varepsilon \star \mu_\varepsilon$ and $\omega_{j,\varepsilon} := \partial_{x_j} \mu_\varepsilon$, so that the Eq. (3.12) reads

$$\partial_t \omega_{j,\varepsilon} - \nabla \omega_{j,\varepsilon} v_\varepsilon - \sum_k \partial_{x_j} v_\varepsilon^k \omega_{k,\varepsilon} - \omega_{j,\varepsilon} \mathrm{div}(v_\varepsilon) - \mu_\varepsilon \mathrm{div}(\partial_{x_j} v_\varepsilon) = R_{j,\varepsilon}(t, x).$$

We compute the time derivative of $\overset{[}{j} = 1]d \sum \left\| \omega_{j,\varepsilon}(t, \cdot) \right\|_p^p$ as follows:

$$f racddt \sum_j \left\| \omega_{j,\varepsilon}(t, \cdot) \right\|_p^p = p \sum_j \int_{\mathbb{R}^d} \left| \omega_{j,\varepsilon} \right|^{p-1} (t, x) \partial_t \left| \omega_{j,\varepsilon} \right| (t, x) dx$$

$$= p \sum_j \int_{\mathbb{R}^d} \left| \omega_{j,\varepsilon} \right|^{p-1} (t, x) \mathrm{sgn}(\omega_{j,\varepsilon}(t, x))$$

$$\times \left( \nabla \omega_{j,\varepsilon} v_\varepsilon + \sum_k (\partial_{x_j} v_\varepsilon^k) \omega_{k,\varepsilon} + \omega_{j,\varepsilon} \mathrm{div}(v_\varepsilon) + \mu_\varepsilon \mathrm{div}(\partial_{x_j} v_\varepsilon) \right) (t, x) dx$$

$$+ p \sum_j \int_{\mathbb{R}^d} \left| \omega_{j,\varepsilon} \right|^{p-1} (t, x) \mathrm{sgn}(\omega_{j,\varepsilon}(t, x)) R_{j,\varepsilon}(t, x) dx.$$

The right-hand side of the last equation is bounded by

$$\sum_j \int_{\mathbb{R}^d} \left| \omega_{j,\varepsilon} \right|^p (t, x) \left| \mathrm{div}(v_\varepsilon) \right| (t, x) dx + p \sum_{j,k} \int_{\mathbb{R}^d} \left| \omega_{j,\varepsilon} \right|^{p-1} (t, x) \left| \partial_{x_j} v_\varepsilon^k \right| (t, x) \left| \omega_{k,\varepsilon} \right| (t, x) dx$$

$$+ p \sum_j \int_{\mathbb{R}^d} \left| \omega_{j,\varepsilon} \right|^p (t, x) \left| \mathrm{div}(v_\varepsilon) \right| (t, x) dx + p \sum_j \int_{\mathbb{R}^d} \left| \omega_{j,\varepsilon} \right|^{p-1} (t, x) \left| \mu_\varepsilon \partial_{x_j} \mathrm{div}(v_\varepsilon) \right| (t, x)$$

$$+ p \sum_j \int_{\mathbb{R}^d} \left| \omega_{j,\varepsilon} \right|^{p-1} (t, x) \left| R_{j,\varepsilon} \right| (t, x) dx$$

$$\leq (p + 1) \sum_j \int_{\mathbb{R}^d} \left| \omega_{j,\varepsilon} \right|^p \left| \mathrm{div}(v_\varepsilon) \right| (t, x) dx + p \sum_{j,k} \int_{\mathbb{R}^d} \left| \omega_{j,\varepsilon} \right|^{p-1} (t, x) \left| \partial_{x_j} v_\varepsilon^k \right| (t, x) \left| \omega_{k,\varepsilon} \right| (t, x) dx$$

$$+ pC(T, \|S\|_\infty, \|\mu_0\|_\infty) \sum_j \int_{\mathbb{R}^d} \left| \omega_{j,\varepsilon} \right|^p (t, x) dx + p \sum_j \int_{\mathbb{R}^d} \left| \omega_{j,\varepsilon} \right|^{p-1} (t, x) \left| R_{j,\varepsilon} \right| (t, x) dx.$$

In the last inequality we invoked (3.3), according to which there holds the estimate

$$\|\mathrm{div}(v_\varepsilon)(t, \cdot)\|_\infty \leq \|\mathrm{div}(\mathbb{J} \nabla V \star \mu_\varepsilon)(t, \cdot)\|_\infty \leq \|\mu_\varepsilon(t, \cdot)\|_\infty \leq C$$

and thus

$$\int_{\mathbb{R}^d} \left| \omega_{j,\varepsilon} \right|^p (t, x) \left| \mathrm{div}(v_\varepsilon) \right| (t, x) dx \leq C \int_{\mathbb{R}^d} \left| \omega_{j,\varepsilon} \right|^p (t, x) dx, \tag{3.13}$$

where $C = C\left(\|\mu_0\|_\infty, \|S\|_\infty, T\right)$. Invoking Proposition 3.3, (3.3) and Young's inequality for products yields

$$
\begin{aligned}
& p\int_{\mathbb{R}^d} \left|\omega_{j,\varepsilon}\right|^{p-1}(t,x)\left|\partial_{x_j} v_\varepsilon^k\right|(t,x)\left|\omega_{k,\varepsilon}\right|(t,x)dx \\
& \leq \left\|\partial_{x_j} v_\varepsilon^k(t,\cdot)\right\|_\infty p\int_{\mathbb{R}^d}\left|\omega_{j,\varepsilon}\right|^{p-1}(t,x)\left|\omega_{k,\varepsilon}\right|(t,x)dx \\
& \leq C\left(d, \|\mu_\varepsilon(t,\cdot)\|_\infty\right)\log\left(\max\left(e, \|\mu_\varepsilon(t,\cdot)\|_\infty\right) + \|\mu_\varepsilon(t,\cdot)\|_{C^{0,s}}\right) \\
& \quad \times p\left(\frac{p-1}{p}\int_{\mathbb{R}^d}\left|\omega_{j,\varepsilon}(t,x)\right|^p dx + \frac{1}{p}\int_{\mathbb{R}^d}\left|\omega_{k,\varepsilon}(t,x)\right|^p dx\right) \\
& \leq C\log\left(C + \|\mu_\varepsilon(t,\cdot)\|_{C^{0,s}}^p\right) [j=1]d\sum\left\|\omega_{j,\varepsilon}(t,\cdot)\right\|_p^p,
\end{aligned}
$$

where $C = C\left(p, \|S\|_\infty, T, d, \|\mu_0\|_\infty\right) > e$. Choosing $s = 1 - \frac{d}{p}$ we have by Morrey's inequality

$$
\begin{aligned}
\|\mu_\varepsilon(t,\cdot)\|_{C^{0,s}}^p \lesssim \|\mu_\varepsilon(t,\cdot)\|_{W^{1,p}(\mathbb{R}^d)}^p \leq C\left(T, \|S\|_\infty, \|\mu_0\|_\infty, d, p\right) \\
\left(1 + [j=1]d\sum\left\|\omega_{j,\varepsilon}(t,\cdot)\right\|_p^p\right).
\end{aligned}
$$

Hence, we have proven

$$
\begin{aligned}
& p\int_{\mathbb{R}^d}\left|\omega_{j,\varepsilon}\right|^{p-1}(t,x)\left|\partial_{x_j} v_\varepsilon^k\right|(t,x)\left|\omega_{k,\varepsilon}\right|(t,x)dx \\
& \leq C\log\left(C + C\,[j=1]d\sum\left\|\omega_{j,\varepsilon}(t,\cdot)\right\|_p^p\right)[j=1]d\sum\left\|\omega_{j,\varepsilon}(t,\cdot)\right\|_p^p.
\end{aligned} \tag{3.14}
$$

Thanks to Hölder's inequality, it holds that

$$
\begin{aligned}
& p\int_{\mathbb{R}^d}\left|\omega_{j,\varepsilon}\right|^{p-1}(t,x)\left|R_{j,\varepsilon}\right|(t,x)dx \\
& = p\int_{\mathbb{R}^d}\left|\omega_{j,\varepsilon}\right|^{p-1}(t,x)\left|\omega_{j,\varepsilon}\right|(t,x)\left|\mathbf{S}[\mu_\varepsilon]\right|(t,x)dx + p\int_{\mathbb{R}^d}\left|\omega_{j,\varepsilon}\right|^{p-1}(t,x)\mu_\varepsilon(t,x)\left|\partial_{x_j}\mathbf{S}[\mu_\varepsilon]\right|(t,x)dx \\
& \leq p\|S\|_\infty\int_{\mathbb{R}^d}\left|\omega_{j,\varepsilon}\right|^p(t,x)dx + p\left(\int_{\mathbb{R}^d}\left|\omega_{j,\varepsilon}\right|^p(t,x)dx\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^d}\left|\mu_\varepsilon\partial_{x_j}S\star\mu_\varepsilon\right|^p(t,x)dx\right)^{\frac{1}{p}} \\
& \leq p\|S\|_\infty\int_{\mathbb{R}^d}\left|\omega_{j,\varepsilon}\right|^p(t,x)dx + C(p, \|\mu_0\|_\infty, T, \|S\|_{W^{1,\infty}})\left(\int_{\mathbb{R}^d}\left|\omega_{j,\varepsilon}\right|^p(t,x)dx\right)^{\frac{p-1}{p}} \\
& \leq C\left(\left\|\omega_{j,\varepsilon}(t,\cdot)\right\|_p^p + 1\right),
\end{aligned} \tag{3.15}
$$

where $C = C\left(T, \|S\|_{W^{1,\infty}}, \|\mu_0\|_\infty, p\right)$. Put

$$
\Omega(t) := \sum_{j=1}^d\left\|\omega_{j,\varepsilon}(t,\cdot)\right\|_p^p.
$$

Gathering (3.13), (3.14) and (3.15), we arrive at the inequality

$$\frac{d}{dt}\Omega(t) \le C \log(1+\Omega(t))\,(1+\Omega(t)), \qquad \text{or equivalently}$$
$$\frac{d}{dt} \log\log(1+\Omega(t)) \le C,$$

for a suitable constant $C = C\left(\|\mu_0\|_\infty, \|S\|_{W^{1,\infty}}, T, d, p\right)$. Solving the above inequality yields

$$\|\nabla\mu_\varepsilon(t,\cdot)\|_p^p = \Omega(t) \le C\,(1+\Omega(0))^{e^{Ct}} \le C\left(1+\|\nabla\mu_0\|_p^p\right)^{e^{Ct}}, \;\; 1 \le p < \infty. \tag{3.16}$$

**Step 4. Compactness and extraction of a solution**. We start by showing that $\mu_\varepsilon \in C([0,T]; L^p(\mathbb{R}^d))$ uniformly in $\varepsilon$. By (3.1) we have

$$\|\mu_\varepsilon(t,\cdot) - \mu_\varepsilon(s,\cdot)\|_p \le |t-s| \sup_{\tau\in[0,T]} \left(\|\mathrm{div}(\mu_\varepsilon(\tau,\cdot)\mathbf{A}_\varepsilon[\mu_\varepsilon](\tau,\cdot))\|_p + \|h[\mu_\varepsilon](\tau,\cdot)\|_p\right).$$

We expand

$$\mathrm{div}(\mu_\varepsilon(\tau,\cdot)\mathbf{A}_\varepsilon[\mu_\varepsilon](\tau,\cdot)) = \nabla\mu_\varepsilon(\tau,\cdot)\mathbf{A}_\varepsilon[\mu_\varepsilon](\tau,\cdot) + \mu_\varepsilon(\tau,\cdot)\mathrm{div}(\mathbf{A}_\varepsilon[\mu_\varepsilon](\tau,\cdot)).$$

In what follows $C = C\left(\|\mu_0\|_{W^{1,\infty}}, \|S\|_{W^{1,\infty}}, T, d, p\right)$ is a constant which may change from line to line. According to (3.16) there holds the estimate

$$\|\nabla\mu_\varepsilon(\tau,\cdot)\|_p \le C$$

while according to inequality (3.3) we have

$$\|\mathbf{A}_\varepsilon[\mu_\varepsilon](\tau,\cdot)\|_\infty \le \|\mathbb{J}\nabla V \star \mu_\varepsilon(\tau,\cdot)\|_\infty \le C. \tag{3.17}$$

Thus, we get the estimate

$$\|\nabla\mu_\varepsilon(\tau,\cdot)\mathbf{A}_\varepsilon[\mu_\varepsilon](\tau,\cdot)\|_p \le \|\nabla\mu_\varepsilon(\tau,\cdot)\|_p\,\|\mathbf{A}_\varepsilon[\mu_\varepsilon](\tau,\cdot)\|_\infty \le C.$$

From the same consideration

$$\|\mu_\varepsilon(\tau,\cdot))\|_p \le C$$

and

$$\|\mathrm{div}(\mathbf{A}_\varepsilon[\mu_\varepsilon](\tau,\cdot))\|_\infty \le \|\chi_\varepsilon \star \mu_\varepsilon(\tau,\cdot)\|_\infty \le \|\mu_\varepsilon(\tau,\cdot)\|_\infty \le C.$$

Hence

$$\sup_{\tau\in[0,T]} \|\mathrm{div}(\mu_\varepsilon(\tau,\cdot)\mathbf{A}_\varepsilon[\mu_\varepsilon](\tau,\cdot))\|_p \le C. \tag{3.18}$$

In addition, notice the bound

$$\sup_{\tau\in[0,T]} \|h[\mu_\varepsilon](\tau,\cdot)\|_p \le \|S\|_\infty \sup_{\tau\in[0,T]} \|\mu_\varepsilon(\tau,\cdot)\|_p \le C. \tag{3.19}$$

Therefore, from (3.18) and (3.19) we obtain

$$\|\mu_\varepsilon(t,\cdot)-\mu_\varepsilon(s,\cdot)\|_p \le C\,|t-s|\,. \tag{3.20}$$

By the theorem of Arzela-Ascoli and (3.20) we may extract a subsequence $\mu_{\varepsilon_m}$ and some

$$\mu \in C([0,T];L^p(\mathbb{R}^d))$$

such that

$$\sup_{t\in[0,T]}\left\|\mu_{\varepsilon_m}(t,\cdot)-\mu(t,\cdot)\right\|_p \underset{m\to\infty}{\to} 0.$$

By (3.11), $\mu(t,\cdot)$ has compact support and $\mathrm{supp}(\mu(t,\cdot))\subset\overline{R}$ and by (3.16) and the Banach-Alaoglu theorem $\mu\in L^\infty([0,T];W^{1,p}(\mathbb{R}^d))$.We proceed by showing that $\mu$ is a weak solution to the equation (2.2). To achieve this it suffices to check that

$$\begin{aligned}\sup_{t\in[0,T]}\left\|\mathbf{A}_{\varepsilon_m}[\mu_{\varepsilon_m}](t,\cdot)\mu_{\varepsilon_m}(t,\cdot)-\mathbf{A}[\mu](t,\cdot)\mu(t,\cdot)\right\|_1 &\underset{m\to\infty}{\to} 0,\\ \sup_{t\in[0,T]}\left\|h[\mu_{\varepsilon_m}](t,\cdot)-h[\mu](t,\cdot)\right\|_1 &\underset{m\to\infty}{\to} 0.\end{aligned} \tag{3.21}$$

By the triangle inequality,

$$\begin{aligned}&\left\|\mathbf{A}_{\varepsilon_m}[\mu_{\varepsilon_m}](t,\cdot)\mu_{\varepsilon_m}(t,\cdot)-\mathbf{A}[\mu](t,\cdot)\mu(t,\cdot)\right\|_1 \le \left\|\mathbf{A}_{\varepsilon_m}[\mu_{\varepsilon_m}](t,\cdot)(\mu_{\varepsilon_m}(t,\cdot)-\mu(t,\cdot))\right\|_1\\ &+\left\|\mu(t,\cdot)(\mathbf{A}_{\varepsilon_m}[\mu_{\varepsilon_m}](t,\cdot)-\mathbf{A}[\mu](t,\cdot))\right\|_1 := A_1+A_2.\end{aligned}$$

By (3.17) we see that

$$A_1 \le C\left\|\mu_{\varepsilon_m}(t,\cdot)-\mu(t,\cdot)\right\|_1. \tag{3.22}$$

Furthermore, we have

$$\begin{aligned}A_2 &\le \left\|\mu(t,\cdot)(\mathbf{A}_{\varepsilon_m}[\mu_{\varepsilon_m}](t,\cdot)-\mathbf{A}_{\varepsilon_m}[\mu](t,\cdot))\right\|_1+\left\|\mu(t,\cdot)(\mathbf{A}_{\varepsilon_m}[\mu](t,\cdot)-\mathbf{A}[\mu](t,\cdot))\right\|_1\\ &\le C\left(\left\|\mathbf{A}_{\varepsilon_m}[\mu_{\varepsilon_m}](t,\cdot)-\mathbf{A}_{\varepsilon_m}[\mu](t,\cdot)\right\|_2+\left\|\mathbf{A}_{\varepsilon_m}[\mu](t,\cdot)-\mathbf{A}[\mu](t,\cdot)\right\|_2\right)\\ &\le C\left(\left\|\mathbb{J}\nabla V\star(\mu_{\varepsilon_m}-\mu)(t,\cdot)\right\|_2+\left\|\chi_{\varepsilon_m}\star\mathbf{A}[\mu](t,\cdot)-\mathbf{A}[\mu](t,\cdot)\right\|_2\right).\end{aligned}$$

Clearly by passing to Fourier we have

$$\left\|\mathbb{J}\nabla V\star(\mu_{\varepsilon_m}-\mu)(t,\cdot)\right\|_2 \le C\left\|\mu_{\varepsilon_m}(t,\cdot)-\mu(t,\cdot)\right\|_{\dot{H}^{-1}(\mathbb{R}^d)} \le C\left\|\mu_{\varepsilon_m}(t,\cdot)-\mu(t,\cdot)\right\|_p.$$

In addition, by (3.16) one has

$$\begin{aligned}&\left\|\chi_{\varepsilon_m}\star\mathbf{A}[\mu](t,\cdot)-\mathbf{A}[\mu](t,\cdot)\right\|_2 \le C\left\|\chi_{\varepsilon_m}\star\mu(t,\cdot)-\mu(t,\cdot)\right\|_{\dot{H}^{-1}(\mathbb{R}^d)}\\ &\le C\left\|\chi_{\varepsilon_m}\star\mu(t,\cdot)-\mu(t,\cdot)\right\|_p \le C\varepsilon_m\left\|\nabla\mu(t,\cdot)\right\|_p \le C\varepsilon_m,\end{aligned}$$

from which we get

$$A_2 \le C\left(\|\mu_\varepsilon(t,\cdot)-\mu(t,\cdot)\|_p+\varepsilon_m\right). \tag{3.23}$$

Inequalities (3.22) and (3.23) entail the first convergence in (3.21). In addition, we estimate

$$\begin{aligned}\left\|h[\mu_{\varepsilon_m}](t,\cdot)-h[\mu](t,\cdot)\right\|_1 &\le \left\|(\mu_{\varepsilon_m}-\mu)(t,\cdot)S\star\mu_{\varepsilon_m}(t,\cdot)\right\|_1+\left\|\mu(t,\cdot)(S\star\mu_{\varepsilon_m}-S\star\mu)(t,\cdot)\right\|_1\\ &\le \|S\|_\infty\left\|\mu_{\varepsilon_m}(t,\cdot)-\mu(t,\cdot)\right\|_1+\|S\|_\infty\left\|\mu_{\varepsilon_m}(t,\cdot)-\mu(t,\cdot)\right\|_1\\ &=2\|S\|_\infty\left\|\mu_{\varepsilon_m}(t,\cdot)-\mu(t,\cdot)\right\|_1,\end{aligned}$$

which establishes the second convergence in (3.21), thereby proving that $\mu$ is a weak solution in the sense of Definition 2.5. □

*Remark 3.5.* For what concerns the existence proof as stated in Theorem 3.4, it suffices to require that $S$ is bounded. However, for the uniqueness and the mean-field limit more regularity is needed in order to ensure sufficient decay of the Fourier transform of $S$. Still, we make no claim to the optimality of the assumption $S\in\mathcal{S}(\mathbb{R}^d)$.

## *3.2. Uniqueness*

To prove uniqueness, we study the evolution of the interaction energy

$$\mathcal{E}(t)=\int_{\mathbb{R}^d}(\mu_1-\mu_2)(t,x)(V\star(\mu_1-\mu_2))(t,x)dx=\|\mu_1(t,\cdot)-\mu_2(t,\cdot)\|^2_{\dot H^{-1}(\mathbb{R}^d)}.$$

We start by proving that $h[\mu]$ is Lipschitz with respect to the $\dot H^{-1}(\mathbb{R}^d)$ norm. In fact we have the following more general Lemma, from which the Lipschitz continuity of $h[\mu]$ with respect to $\dot H^{-1}(\mathbb{R}^d)$ will follow. In the next lemma, we show some estimates involving a given kernel $W$. The next lemma will eventually be used for either the Coulomb interaction $V$ or some regularized kernels thereof in the next sections.

**Lemma 3.6.** *Let $S$ be as in (**H1**). Suppose further that*

- *$\mu,\nu\in\mathcal{S}'(\mathbb{R}^d)$ are tempered distributions such that $\widehat{\mu},\widehat{\nu}\in L^\infty(\mathbb{R}^d)$.*
- *$W\in L^1_{\text{loc}}(\mathbb{R}^d)$ is such that*

$$\int_{\mathbb{R}^d}|W|(x)\left|\widehat{\mu-\nu}\right|^2(x)dx<\infty.$$

- *There is some $c>0$ such that for all $x\in\mathbb{R}^d$ it holds that $\left|\widehat{S}\right|\star|W|(x)\le c\,|W|(x)$.*

*Then, it holds that*

$$\int_{\mathbb{R}^d}|W|(x)\left|\widehat{h[\mu]-h[\nu]}\right|^2(x)dx\le C\int_{\mathbb{R}^d}|W|(x)\left|\widehat{\mu-\nu}\right|^2(x)dx$$

*where $C=C\left(c,\left\|\widehat{S}\right\|_1,\|\widehat{\mu}\|_\infty,\|\widehat{\nu}\|_\infty\right)$.*

*Proof.* We have

$$
\begin{aligned}
I &:= \int_{\mathbb{R}^d} |W|(x) \left| h[\widehat{\mu] - h}[\nu] \right|^2 (x)dx = \int_{\mathbb{R}^d} |W|(x)(\mu S \star \widehat{\mu - \nu} S \star \nu)(x) \left( h[\widehat{\mu] - h}[\nu] \right)(x)dx \\
&= \int_{\mathbb{R}^d} |W|(x)((\mu - \widehat{\nu)(S} \star \nu))(x) \left( h[\widehat{\mu] - h}[\nu] \right)(x)dx \\
&\quad + \int_{\mathbb{R}^d} |W|(x)(\mu(S \star \widehat{\mu - S} \star \nu))(x) \left( h[\widehat{\mu] - h}[\nu] \right)(x)dx \\
&:= I_1 + I_2.
\end{aligned} \tag{3.24}
$$

The first integral in the right-hand side of (3.24) is controlled as follows

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}^d} |W|(x)((\mu - \widehat{\nu)(S} \star \nu))(x) \left( h[\widehat{\mu] - h}[\nu] \right)(x)dx \\
&= \int_{\mathbb{R}^d} |W|(x)(\widehat{\mu - \nu}) \star (\widehat{S}\widehat{\nu})(x) \left( h[\widehat{\mu] - h}[\nu] \right)(x)dx \\
&= \int_{\mathbb{R}^d \times \mathbb{R}^d} |W|(x)(\widehat{\mu - \nu})(y)(\widehat{S}\widehat{\nu})(x-y) \left( h[\widehat{\mu] - h}[\nu] \right)(x)dydx \\
&\leq \|\widehat{\nu}\|_\infty \int_{\mathbb{R}^d \times \mathbb{R}^d} |W|(x) \left| \widehat{\mu - \nu} \right|(y) \left| \widehat{S} \right|(x-y) \left| h[\widehat{\mu] - h}[\nu] \right|(x)dydx.
\end{aligned} \tag{3.25}
$$

Thanks to the assumption $\left|\widehat{S}\right| \star |W|(x) \leq c\,|W|(x)$ and since $\widehat{S}$ is even we have

$$
\begin{aligned}
\int_{\mathbb{R}^d \times \mathbb{R}^d} \left| \widehat{\mu - \nu} \right|^2 (y) \left| \widehat{S} \right|(x-y)|W|(x)dxdy &= \int_{\mathbb{R}^d} \left| \widehat{\mu - \nu} \right|^2 (y) \left| \widehat{S} \right| \star |W|(y)dy \\
&\leq c \int_{\mathbb{R}^d} \left| \widehat{\mu - \nu} \right|^2 (y)|W|(y)dy.
\end{aligned}
$$

Furthermore, $\widehat{S} \in L^1(\mathbb{R}^d)$ shows that

$$
\int_{\mathbb{R}^d \times \mathbb{R}^d} \left| h[\widehat{\mu] - h}[\nu] \right|^2 (x) \left| \widehat{S} \right|(x-y)|W|(x)dxdy = \left\| \widehat{S} \right\|_1 \int_{\mathbb{R}^d} \left| h[\widehat{\mu] - h}[\nu] \right|^2 (x)|W|(x)dx.
$$

Collecting the last two estimates and applying Cauchy-Schwarz to (3.25), we conclude that

$$
I_1 \leq C \left( \int_{\mathbb{R}^d} \left| \widehat{\mu - \nu} \right|^2 (y)|W|(y)dy \right)^{1/2} I^{1/2}
$$

where $C = C\left(c, \left\|\widehat{S}\right\|_1, \|\widehat{\nu}\|_\infty\right)$. As for the second integral in (3.24)

$$
\begin{aligned}
I_2 &= \int_{\mathbb{R}^d} |W|(x)((\widehat{S}(\widehat{\mu - \nu})) \star \widehat{\mu})(x) \left( h[\widehat{\mu] - h}[\nu] \right)(x)dx \\
&= \int_{\mathbb{R}^d \times \mathbb{R}^d} |W|(x)\widehat{S}(x-y)(\widehat{\mu - \nu})(x-y)\widehat{\mu}(y) \left( h[\widehat{\mu] - h}[\nu] \right)(x)dxdy
\end{aligned}
$$

and by the same considerations as before, we deduce that

$$I_2 \leq C\left(c, \left\|\widehat{S}\right\|_1, \|\widehat{\mu}\|_\infty\right)\left(\int_{\mathbb{R}^d}\left|\widehat{\mu-\nu}\right|^2(y)\,|W|\,(y)dy\right)^{1/2} I^{\frac{1}{2}},$$

from which we deduce that

$$\int_{\mathbb{R}^d}|W|\,(x)\left|\widehat{h\,[\mu]-h\,[\nu]}\right|^2(x)dx \leq C\int_{\mathbb{R}^d}|W|\,(x)\left|\widehat{\mu-\nu}\right|^2(x)dx,$$

where $C = C\left(c, \left\|\widehat{S}\right\|_1, \|\widehat{\mu}\|_\infty, \|\widehat{\nu}\|_\infty\right)$. □

As a corollary, we obtain continuity of $h[\mu]$ with respect to the $\dot{H}^1(\mathbb{R}^d)$ norm.

**Corollary 3.7.** *Suppose that $\mu, \nu \in L^1 \cap L^\infty(\mathbb{R}^d)$ and that $S$ is as in (**H1**). Then, we have*

$$\|h[\mu]-h[\nu]\|_{\dot{H}^{-1}(\mathbb{R}^d)} \leq C\,\|\mu-\nu\|_{\dot{H}^{-1}(\mathbb{R}^d)}$$

*where $C = C\left(\left\|(1+|\cdot|^2)\widehat{S}\right\|_{L^1\cap L^\infty}, \|\widehat{\mu}\|_\infty, \|\widehat{\nu}\|_\infty\right)$.*

*Proof.* We take $W = \widehat{V} = \frac{1}{|x|^2}$ in Lemma 3.6. We need to check the condition

$$\left|\widehat{S}\right| \star W(x) \leq cW(x).$$

Indeed, we have

$$\begin{aligned}|x|^2\left(\left|\widehat{S}\right|\star\frac{1}{|\cdot|^2}\right)(x) &= |x|^2\int_{\mathbb{R}^d}\frac{\left|\widehat{S}\right|(y)}{|x-y|^2}dy \leq 2\int_{\mathbb{R}^d}\frac{\left|\widehat{S}\right|(y)\,|x-y|^2}{|x-y|^2}dy + 2\left\|(|\cdot|^2\left|\widehat{S}\right|)\star\frac{1}{|\cdot|^2}\right\|_\infty\\ &= 2\left\|\widehat{S}\right\|_1 + 2\left\|(|\cdot|^2\left|\widehat{S}\right|)\star\frac{\mathbf{1}_{\leq 1}}{|\cdot|^2}\right\|_\infty + 2\left\|(|\cdot|^2\left|\widehat{S}\right|)\star\frac{\mathbf{1}_{\geq 1}}{|\cdot|^2}\right\|_\infty.\end{aligned}$$

By Young's inequality, for any $q < \frac{d}{2}$, we have the estimate

$$\left\|(|\cdot|^2\left|\widehat{S}\right|)\star\frac{\mathbf{1}_{\leq 1}}{|\cdot|^2}\right\|_\infty \leq \left\|\frac{\mathbf{1}_{\leq 1}}{|\cdot|^2}\right\|_q\left\||\cdot|^2\,\widehat{S}\right\|_{q'} \lesssim_{d,q}\left\||\cdot|^2\,\widehat{S}\right\|_{q'} \leq \left\|(1+|\cdot|^2)\widehat{S}\right\|_{L^1\cap L^\infty}$$

and for any $p > \frac{d}{2}$, we have the estimate

$$\left\|(|\cdot|^2\left|\widehat{S}\right|)\star\frac{\mathbf{1}_{\geq 1}}{|\cdot|^2}\right\|_\infty \leq \left\|\frac{\mathbf{1}_{\geq 1}}{|\cdot|^2}\right\|_p\left\||\cdot|^2\,\widehat{S}\right\|_{p'} \lesssim_{d,q}\left\||\cdot|^2\,\widehat{S}\right\|_{p'} \leq \left\|(1+|\cdot|^2)\widehat{S}\right\|_{L^1\cap L^\infty}.$$

Therefore, one finds that

$$\left|\widehat{S}\right|\star\frac{1}{|\cdot|^2} \leq \frac{c\left(\left\|(1+|\cdot|^2)\widehat{S}\right\|_{L^1\cap L^\infty}\right)}{|x|^2}.$$

The announced result follows now by Lemma 3.6. □

**Theorem 3.8.** *Let hypothesis (**H1**) hold. Let $\mu^1(t,\cdot)$ and $\mu^2(t,\cdot)$ be weak solutions to (2.2) with initial data $\mu_0^1$, $\mu_0^2$ satisfying (**H2**). Then, the $\dot{H}^{-1}(\mathbb{R}^d)$-stability holds, i.e.*

$$\mathcal{E}(t) \le e^{Ct}\mathcal{E}(0),$$

*where*

$$C = C\left(\left\|(1+|\cdot|^2)\widehat{S}\right\|_{L^1\cap L^\infty}, \left\|\mu_0^1\right\|_\infty, T\right).$$

*In particular, uniqueness of weak solutions to (2.2) is obtained.*

*Proof.* Recall that $\operatorname{div}(\mathbb{J}\nabla V) = 0$ when $\mathbb{J}$ is anti-symmetric and $\operatorname{div}(\mathbb{J}\nabla V) = -\delta_0$ when $\mathbb{J}$ is the identity. We compute that

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(t) =& -2\int_{\mathbb{R}^d}\left(\mu^1\mathbb{J}\nabla V\star\mu^1 - \mu^2\mathbb{J}\nabla V\star\mu^2\right)(t,x)\nabla V\star\left(\mu^1-\mu^2\right)(t,x)dx\\
&+2\int_{\mathbb{R}^d}\left(h\left[\mu^1\right]-h\left[\mu^2\right]\right)(t,x)V\star\left(\mu^1-\mu^2\right)(t,x)dx\\
:=&\mathcal{D}^1(t)+\mathcal{D}^2(t).
\end{aligned}$$

**Step 1**. **Estimate on $\mathcal{D}^1(t)$**. We can rewrite $\mathcal{D}^1(t)$ as

$$\begin{aligned}
\mathcal{D}^1(t) =& -2\int_{\mathbb{R}^d}\mathbb{J}\nabla V\star\mu^1\left(\mu^1-\mu^2\right)(t,x)\nabla V\star\left(\mu^1-\mu^2\right)(t,x)dx\\
&-2\int_{\mathbb{R}^d}\mu^2\left(\mathbb{J}\nabla V\star(\mu^1-\mu^2)\right)(t,x)\nabla V\star\left(\mu^1-\mu^2\right)(t,x)dx\\
\le& -2\int_{\mathbb{R}^d}\mathbb{J}\nabla V\star\mu^1\left(\mu^1-\mu^2\right)(t,x)\nabla V\star\left(\mu^1-\mu^2\right)(t,x)dx,
\end{aligned}$$

where in the last inequality we used $\mathbb{J}\nabla V\star\mu\nabla V\star\mu\ge 0$. Therefore, we deduce

$$\begin{aligned}
\mathcal{D}^1(t) \le& -2\int_{\mathbb{R}^d}\mathbb{J}\nabla V\star\mu^1\operatorname{div}\left(\nabla V\star\left(\mu^1-\mu^2\right)\right)(t,x)\nabla V\star\left(\mu^1-\mu^2\right)(t,x)dx\\
=& -\int_{\mathbb{R}^d}\mathbb{J}\nabla V\star\mu^1(t,x)\nabla\left(\left|\nabla V\star\left(\mu^1-\mu^2\right)\right|^2\right)(t,x)dx\\
=& \int_{\mathbb{R}^d}\operatorname{div}(\mathbb{J}\nabla V\star\mu^1)(t,x)\left|\nabla V\star\left(\mu^1-\mu^2\right)\right|^2(t,x)dx\\
\le& 2\sup_{t\in[0,T]}\left\|\mu^1(t,\cdot)\right\|_\infty\mathcal{E}(t) \le C(\|S\|_\infty, \left\|\mu_0^1\right\|_\infty, T)\mathcal{E}(t).
\end{aligned}$$

Hence, we conclude

$$\mathcal{D}^1(t) \le C(\|S\|_\infty, \left\|\mu_0^1\right\|_\infty, T)\mathcal{E}(t). \tag{3.26}$$

**Step 2**. **Estimate on** $\mathcal{D}^2(t)$. By Plancherel's theorem we have

$$\begin{aligned}\mathcal{D}_2(t) &= 2\int_{\mathbb{R}^d} (\widehat{h\left[\mu^1\right]-h\left[\mu^2\right]})(t,x)\widehat{V}(x)\big(\widehat{\mu^1-\mu^2}\big)(t,x)dx\\ &\leq \mathbf{C}\int_{\mathbb{R}^d} \frac{\left|\widehat{h\left[\mu^1\right]-h\left[\mu^2\right]}\right|}{|x|}(t,x)\frac{\left|\widehat{\mu^1-\mu^2}\right|}{|x|}(t,x)dx\\ &\leq \mathbf{C}\left(\int_{\mathbb{R}^d} \frac{\left|\widehat{h\left[\mu^1\right]-h\left[\mu^2\right]}\right|^2}{|x|^2}(t,x)dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^d} \frac{\left|\widehat{\mu^1-\mu^2}\right|^2}{|x|^2}(t,x)dx\right)^{\frac{1}{2}}\\ &\leq \mathbf{C}\left\|h\left[\mu^1\right](t,\cdot)-h\left[\mu^2\right](t,\cdot)\right\|_{\dot{H}^{-1}(\mathbb{R}^d)}\left\|\mu^1(t,\cdot)-\mu^2(t,\cdot)\right\|_{\dot{H}^{-1}(\mathbb{R}^d)}.\end{aligned} \tag{3.27}$$

By Corollary 3.7 we have

$$\left\|h\left[\mu^1\right](t,\cdot)-h\left[\mu^2\right](t,\cdot)\right\|_{\dot{H}^{-1}(\mathbb{R}^d)}\leq C\left\|\mu^1(t,\cdot)-\mu^2(t,\cdot)\right\|_{\dot{H}^{-1}(\mathbb{R}^d)}=C\sqrt{\mathcal{E}(t)},$$

where $C = C\left(\|(1+|\cdot|^2)\widehat{S}\|_{L^\infty\cap L^1}\right)$, so that by (3.27) we get

$$\mathcal{D}^2(t) \leq C\mathcal{E}(t).$$

To conclude, (3.26) and (3.27) show that

$$\frac{d}{dt}\mathcal{E}(t) \leq C\left(\left\|(1+|\cdot|^2)\widehat{S}\right\|_{L^\infty\cap L^1}, \left\|\mu_0^1\right\|_\infty, T\right)\mathcal{E}(t),$$

and therefore $\mathcal{E}(t) \leq e^{Ct}\mathcal{E}(0)$. □

## 4. Well Posedness for the ODE System

In this section, we explain how to prove the existence of a well defined flow for the system

$$\begin{cases}\dot{x}_i^N(t) = -\frac{1}{N}\ [\ j=1]N\sum m_j^N(t)\mathbb{J}\nabla V(x_i^N(t)-x_j^N(t)),\ x_i^N(0)=x_i^{0,N}\\ \dot{m}_i^N(t) = \frac{1}{N}\ [\ j=1]N\sum m_i^N(t)m_j^N(t)S(x_i^N(t)-x_j^N(t)),\ m_i^N(0)=m_i^{0,N}.\end{cases} \tag{4.1}$$

We will adapt the proof about the well definition of the flow for time independent weights Riesz potentials as in [25, Section 3.2] to the time dependent setting. Consider the weighted interaction energy given by

$$\mathcal{H}_N(t):=\sum_{i\neq j} m_i(t)m_j(t)V(x_i(t)-x_j(t)).$$

We start by proving short time existence.

**Theorem 4.1.** *Let hypothesis (**H1**) hold. Assume that*

$$\forall i \neq j : x_i^{0,N} \neq x_j^{0,N}$$

*and there is some $M > 0$ such that for all $N \in \mathbb{N}$ and $1 \leq i \leq N$ one has*

$$0 \leq m_i^{0,N} \leq M.$$

*Then, there is some $T_* > 0$ such that the system (4.1) has a unique $C^1([0, T_*); \mathbb{R}^{dN} \times \mathbb{R}^N)$ solution.*

*Proof.* Let $\Psi \in C_0^\infty(\mathbb{R})$ be such that $\Psi(r) = 1$ for $|r| \leq 1$ and $\Psi(r) = 0$ for $|r| \geq 2$. Let $\psi \in C_0^\infty(\mathbb{R}^d)$ be the radial function defined by $\psi(x) := \Psi(|x|)$. Let

$$\psi_\varepsilon(x) := \Psi\left(\frac{|x|^2}{\varepsilon^2}\right),$$

and $V_\varepsilon(x) := (1 - \psi_\varepsilon(x))\, V(x)$. In what follows we omit the superscript $N$ from $x_{i,\varepsilon}^N$ and $m_{i,\varepsilon}^N$. Consider the regularized system

$$\begin{cases} \dot{x}_{i,\varepsilon}(t) = -\frac{1}{N} \overset{[}{j} = 1]N \sum m_{j,\varepsilon}(t) \mathbb{J} \nabla V_\varepsilon(x_{i,\varepsilon}(t) - x_{j,\varepsilon}(t)), \ x_{i,\varepsilon}(0) = x_i^0 \\ \dot{m}_{i,\varepsilon}(t) = \frac{1}{N} \overset{[}{j} = 1]N \sum m_{i,\varepsilon}(t) m_{j,\varepsilon}(t) S(x_{i,\varepsilon}(t) - x_{j,\varepsilon}(t)), \ m_{i,\varepsilon}(0) = m_i^0. \end{cases} \tag{4.2}$$

By [2, Theorem 3], system (4.2) has a unique global solution $(x_{i,\varepsilon}(t), m_{i,\varepsilon}(t))$ on $[0, \infty)$. We study the evolution of $\left|x_{i,\varepsilon}(t) - x_i^0\right|$ given by

$$\left|x_{i,\varepsilon}(t) - x_i^0\right| \leq \int_0^t \left| \frac{1}{N} \overset{[}{j} = 1]N \sum m_{j,\varepsilon}(s) \cdot \mathbb{J} \nabla V_\varepsilon(x_{j,\varepsilon}(s) - x_{i,\varepsilon}(s)) \right| ds\,.$$

We can now estimate the $L^\infty$ bound of the velocity field using that

$$\nabla V_\varepsilon(x) = -2\Psi'\left(\frac{|x|^2}{\varepsilon^2}\right) \frac{x}{\varepsilon^2} V(x) + (1 - \psi_\varepsilon(x))\, \nabla V(x).$$

Actually, writing $k = d - 2$ notice that

$$|(1 - \psi_\varepsilon(x))\, \nabla V(x)| \lesssim \frac{1}{\varepsilon^{k+1}} \qquad \text{and} \qquad \left| 2\Psi'\left(\frac{|x|^2}{\varepsilon^2}\right) \frac{x}{\varepsilon^2} V(x) \right| \lesssim \frac{1}{\varepsilon^{k+1}},$$

hence

$$|\nabla V_\varepsilon(x)| \lesssim \frac{1}{\varepsilon^{k+1}}.$$

So, owing to Remark 2.4, we infer the inequality

$$\left|x_{i,\varepsilon}(t) - x_i^0\right| \lesssim \frac{1}{\varepsilon^{k+1} N} \int_0^t \overset{[}{j} = 1]N \sum m_{j,\varepsilon}(s)\, ds = \frac{T}{\varepsilon^{k+1}}. \tag{4.3}$$

Thanks to Inequality (4.3), we can bound the seperation in time $t$ as follows

$$\left|x_{i,\varepsilon}(t)-x_{j,\varepsilon}(t)\right| \geq \left|x_i^0-x_j^0\right|-\left|x_i^0-x_{i,\varepsilon}(t)\right|-\left|x_j^0-x_{j,\varepsilon}(t)\right| \geq \left|x_i^0-x_j^0\right|-\frac{2C_{k,d}T}{\varepsilon^{k+1}}.$$

If we choose $\varepsilon_0 = \frac{1}{16}\min\limits_{i\neq j}\left|x_i^0 - x_j^0\right|$ and $T > 0$ such that

$$\frac{2C_{k,d}T}{\varepsilon_0^{k+1}} \leq \frac{\min\limits_{i\neq j}\left|x_i^0 - x_j^0\right|}{2},$$

then for each $t \in [0, T)$ we get the bound

$$\left|x_{i,\varepsilon_0}(t) - x_{j,\varepsilon_0}(t)\right| \geq \frac{\left|x_i^0 - x_j^0\right|}{2}. \tag{4.4}$$

It follows from (4.4) that for each $t \in [0, T]$ it holds that $\nabla V_\varepsilon(x_{i,\varepsilon}(t) - x_{j,\varepsilon}(t)) = \nabla V(x_{i,\varepsilon}(t) - x_{j,\varepsilon}(t))$, and therefore for each $t \in [0, T]$

$$\begin{cases} \dot{x}_{i,\varepsilon_0}(t) = -\frac{1}{N}\ \overset{[}{j}= 1]N\sum m_{j,\varepsilon_0}(t)\mathbb{J}\nabla V(x_{i,\varepsilon_0}(t) - x_{j,\varepsilon_0}(t)),\ x_{i,\varepsilon_0}(0) = x_i^0 \\ \dot{m}_{i,\varepsilon_0}(t) = \frac{1}{N}\ \overset{[}{j}= 1]N\sum m_{i,\varepsilon_0}(t)m_{j,\varepsilon_0}(t)S(x_{i,\varepsilon_0}(t) - x_{j,\varepsilon_0}(t)),\ m_{i,\varepsilon_0}(0) = m_i^0. \end{cases}$$

□

Next, we claim that the quantity $\mathcal{H}_N(t)$ is propagated in time, and already here the oddness of $S$ is essential.

**Lemma 4.2.** *Let* $(\mathbf{x}_N(t), \mathbf{m}_N(t))$ *be a solution to the system of ODEs* (4.1) *with initial data* $(\mathbf{x}_N(0), \mathbf{m}_N(0))$*. Then, the interaction energy satisfies*

$$\mathcal{H}_N(t) \leq e^{2\|S\|_\infty t}\mathcal{H}_N(0),\ \ for\ all\ t \in [0, T).$$

*Proof.* We compute the time derivative as follows

$$\begin{aligned} \frac{d}{dt}\mathcal{H}_N(t) &= \frac{d}{dt}\sum_{i\neq j} m_i(t)m_j(t)V(x_i(t) - x_j(t)) \\ &= \sum_{i\neq j}\dot{m}_i(t)m_j(t)V(x_i(t) - x_j(t)) + \sum_{i\neq j}m_i(t)\dot{m}_j(t)V(x_i(t) - x_j(t)) \\ &\quad + \sum_{i\neq j}m_i(t)m_j(t)\nabla V(x_i(t) - x_j(t)) \cdot (\dot{x}_i(t) - \dot{x}_j(t)). \end{aligned} \tag{4.5}$$

The first 2 terms in the right hand side of (4.5) are paired together as follows.

$$\begin{aligned} &\sum_{i\neq j}\dot{m}_i(t)m_j(t)V(x_i(t) - x_j(t)) \\ &\quad = \frac{1}{N}\sum_{i\neq j}\sum_k m_i(t)m_k(t)m_j(t)S(x_k(t) - x_i(t))V(x_i(t) - x_j(t)), \end{aligned}$$

while

$$\sum_{i\neq j} m_i(t)\dot{m}_j(t)V(x_i(t)-x_j(t)) = \frac{1}{N}\sum_{i\neq j}\sum_k m_i(t)m_j(t)m_k(t)S(x_k(t)-x_j(t))V(x_i(t)-x_j(t))$$
$$= \frac{1}{N}\sum_{i\neq j}\sum_k m_i(t)m_j(t)m_k(t)S(x_k(t)-x_i(t))V(x_j(t)-x_i(t)).$$

So, since $V$ is even, the first 2 terms in (4.5) add up to

$$\frac{2}{N}\sum_{i\neq j,k} m_i(t)m_j(t)m_k(t)S(x_k(t)-x_i(t))V(x_j(t)-x_i(t))$$
$$\leq \frac{2\,\|S\|_\infty}{N}\sum_{i\neq j,k} m_i(t)m_j(t)m_k(t)V(x_j(t)-x_i(t)) = 2\,\|S\|_\infty\,\mathcal{H}_N(t), \tag{4.6}$$

where in the last equation we used conservation of the total weight (see Remark 2.4). The third sum in the right hand side of (4.5) writes

$$\sum_{i\neq j} m_i(t)m_j(t)\nabla V(x_i(t)-x_j(t))\cdot\left(\frac{1}{N}\sum_{k\neq j} m_k(t)\mathbb{J}\nabla V(x_j(t)-x_k(t)) - \frac{1}{N}\sum_{k\neq i} m_k(t)\mathbb{J}\nabla V(x_i(t)-x_k(t))\right)$$
$$= -\frac{2}{N}\widehat{\sum_{i,j,k}} m_i(t)m_j(t)m_k(t)\nabla V(x_i(t)-x_j(t))\cdot\mathbb{J}\nabla V(x_i(t)-x_k(t))$$

where the equality is obtained by swapping the indices $i$ and $j$ in the first term and the hat sum refers to a sum over the set of indices $\{(i, j, k)$ such that $i \neq j, k \neq i\}$. We finally notice that we can rewrite this term as follows

$$\widehat{\sum_{i,j,k}} m_i(t)m_j(t)m_k(t)\nabla V(x_i(t)-x_j(t))\cdot\mathbb{J}\nabla V(x_i(t)-x_k(t))$$
$$= -\frac{2}{N}\sum_i m_i(t)\left(\sum_{j\neq i} m_j(t)\nabla V(x_i(t)-x_j(t))\right)\cdot\mathbb{J}\left(\sum_{k\neq i} m_k(t)\nabla V(x_i(t)-x_k(t))\right)$$
$$= -\frac{2}{N}\sum_i m_i(t)\left(\sum_{j\neq i} m_j(t)\nabla V(x_i(t)-x_j(t))\right)\cdot\mathbb{J}\left(\sum_{j\neq i} m_j(t)\nabla V(x_i(t)-x_j(t))\right). \tag{4.7}$$

If $\mathbb{J}$ is anti-symmetric then the right-hand side of (4.7) is 0 while if $\mathbb{J}$ is the identity matrix the right hand side of (4.7) is

$$-\frac{2}{N}\sum_i m_i(t)\left|\sum_{j\neq i} m_j(t)\nabla V(x_i(t)-x_j(t))\right|^2 \leq 0.$$

Therefore, we conclude that the third sum in the right hand side of (4.5) satisfies

$$\sum_{i\neq j} m_i(t)m_j(t)\nabla V(x_i(t)-x_j(t))\cdot(\dot{x}_i(t)-\dot{x}_j(t)) \leq 0. \tag{4.8}$$

Together with (4.6) this concludes the proof. □

As a corollary from propagation of energy, we get the following lower bound on the minimal separation of the opinions, which will be needed in order to prove that maximal lifespan solutions are in fact global. We are now well positioned to prove Theorem 2.2, i.e. the existence of a globally well-defined flow.

*Proof of Theorem 2.2.* **Step 1**. **Estimate on the separation**. Set $k{:=}d-2$. We claim that if $(\mathbf{x}_N(t), \mathbf{m}_N(t))$ is a solution to the system (4.1), then there holds the estimate

$$\min_{i\neq j}\inf_{t\in[0,T)}\left|x_i(t)-x_j(t)\right| \geq \min\left\{1, \frac{1}{e^{2\|S\|_\infty T}\mathcal{H}_N(0)}\right\}^{1/k}. \tag{4.9}$$

If $i \neq j$ are such that $\left|x_i(t)-x_j(t)\right| \leq 1$ then by Lemma 4.2 we have

$$\frac{1}{\left|x_i(t)-x_j(t)\right|^k} \leq \mathcal{H}_N(t) \leq e^{2\|S\|_\infty t}\mathcal{H}_N(0),$$

so that

$$\left|x_i(t)-x_j(t)\right|^k \geq \frac{1}{e^{2\|S\|_\infty t}\mathcal{H}_N(0)}.$$

It follows that for all $i \neq j$ one has

$$\left|x_i(t)-x_j(t)\right| \geq \min\left\{1, \frac{1}{e^{2\|S\|_\infty T}\mathcal{H}_N(0)}\right\}^{1/k},$$

which establishes (4.9).

**Step 2**. **Long time existence and uniqueness**. Let $(\mathbf{x}_N(t), \mathbf{m}_N(t))$ be a solution to the system of ODEs (4.1) with maximal lifespan $T>0$. We claim that if $T<\infty$ then

$$\lim_{t\nearrow T}\min_{i\neq j}\left|x_i(t)-x_j(t)\right| = 0.$$

Indeed suppose that $T<\infty$ and assume on the contrary that

$$\lim_{t\nearrow T}\min_{i\neq j}\left|x_i(t)-x_j(t)\right| = \delta > 0.$$

Choose $T'<T$ sufficiently close to $T$ such that

$$\inf_{T'\leq t\leq T}\min_{i\neq j}\left|x_i(t)-x_j(t)\right| \geq \frac{\delta}{2}.$$

Assume also that $T-T'<T_\Delta$, where $T_\Delta$ is the maximal lifespan solution of the system of ODEs (4.1) with initial data $(x_i(T'), m_i(T'))$, i.e. the equation

$$\begin{cases} \dot{z}_i(t) = -\frac{1}{N}\overset{[}{j}= 1]N\sum n_j(t)\mathbb{J}\nabla V(z_i(t)-z_j(t)), \ z_i(0)=x_i(T') \\ \dot{n}_i(t) = \frac{1}{N}\overset{[}{j}= 1]N\sum n_i(t)n_j(t)S(z_i(t)-z_j(t)), \ n_i(0)=m_i(T'). \end{cases} \tag{4.10}$$

Note that by Theorem 4.1, there exists a maximal life span solution $(\mathbf{z}_N(t), \mathbf{n}_N(t))$ on $[0, T_\Delta)$ to the system of ODEs (4.10). Define

$$\mathbf{y}_N(t) := \begin{cases} \mathbf{x}_N(t) & 0 \le t \le T' \\ \mathbf{z}_N(t - T') & T' \le t < T' + T_\Delta \end{cases}, \quad \mathbf{l}_N(t) := \begin{cases} \mathbf{m}_N(t) & 0 \le t \le T' \\ \mathbf{n}_N(t - T') & T' \le t < T' + T_\Delta \end{cases}.$$

Note that $\mathbf{y}_N(t)$ is continuous and that by (4.9) we have

$$\inf_{t \in [0, T' + T_\Delta)} \min_{i \neq j} \left| y_i(t) - y_j(t) \right| > 0.$$

We leave the reader to check that $(\mathbf{y}_N(t), \mathbf{l}_N(t))$ is a solution to the system on $[0, T' + T_\Delta)$ since we are dealing with an autonomous system. This entails a contradiction to the assumption that $T$ is maximal, because $T' + T_\Delta > T$. We conclude that

$$T < \infty \Longrightarrow \lim_{t \nearrow T} \min_{i \neq j} \left| x_i(t) - x_j(t) \right| = 0.$$

In view of (4.9) it follows that $T = \infty$, as desired. □

## 5. The Mean Field Limit

In Sect. 5.1 we compute the time derivative of $\mathcal{E}_N(t)$, and in Sect. 5.2 we establish the functional inequality (2.12) which in turn leads to a Grönwall estimate on $\mathcal{E}_N(t)$.

### *5.1. Time Derivative of the Re-normalized Modulated Energy*

We recall that the re-normalized interaction energy (2.6) is defined as

$$\begin{aligned} \mathcal{E}_N(t) &= \int_{x \neq y} V(x - y) \left( \mu_N(t, \cdot) - \mu(t, \cdot) \right)^{\otimes 2} dx dy \\ &= \frac{1}{N^2} \sum_{i \neq j} m_i(t) m_j(t) V(x_i(t) - x_j(t)) \\ &\quad - \frac{2}{N} [i = 1]N \sum m_i(t) (V \star \mu)(x_i(t)) \\ &\quad + \int_{\mathbb{R}^d} \mu(t, x) (V \star \mu)(t, x) dx. \end{aligned}$$

The aim of this section is to compute the time derivative of $\mathcal{E}_N(t)$, which is given in the following result.

**Proposition 5.1.** *Let hypotheses **(H1)**–**(H2)** hold. Let $\mu(t, \cdot)$ be the unique solution to the PDE (2.2) with initial data $\mu_0$ provided by Theorem 1.1. Then, we have*

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_N(t) \le &- \int_{x \neq y} \left( \mathbb{J} \nabla V \star \mu(t, x) - \mathbb{J} \nabla V \star \mu(t, y) \right) \nabla V(x - y) \left( \mu_N(t, \cdot) - \mu(t, \cdot) \right)^{\otimes 2} dx dy \\ &+ 2 \int_{x \neq y} V(x - y) \left( h\left[ \mu_N(t, \cdot) \right](x) - h\left[ \mu(t, \cdot) \right](x) \right) \left( \mu_N(t, y) - \mu(t, y) \right) dx dy \\ :=& \mathcal{D}_N^1(t) + \mathcal{D}_N^2(t). \end{aligned}$$

*Proof.* We recall that from Theorem 2.2, we have that $x_i(t) \neq x_j(t)$ for all $t \geq 0$ and all $i \neq j$. This makes it straightforward to justify all calculations in the proof. To make the equations lighter we shall omit the time variable whenever there is no ambiguity.

**Step 1**. *Calculation of* $\frac{d}{dt}\frac{1}{N^2}\sum_{i\neq j} m_i(t)m_j(t)V(x_i(t)-x_j(t))$. We have

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{N^2}\sum_{i\neq j} m_i(t)m_j(t)V(x_i(t)-x_j(t)) \\
&\quad = \frac{1}{N^2}\sum_{i\neq j} m_i(t)m_j(t)\nabla V(x_i(t)-x_j(t))\cdot\big(\dot{x}_i(t)-\dot{x}_j(t)\big) \\
&\qquad + \frac{1}{N^2}\sum_{i\neq j}(\dot{m}_i(t)m_j(t)+m_i(t)\dot{m}_j(t))V(x_i(t)-x_j(t)) \\
&\quad := E_1 + T_1.
\end{aligned} \tag{5.1}$$

The first term in (5.1) is non-positive due to (4.8) in Lemma 4.2, i.e. $E_1 \leq 0$. The second term in (5.1) can be rewritten using symmetry as

$$\begin{aligned}
T_1 = \frac{2}{N^2}\sum_{i\neq j}\dot{m}_i m_j V(x_i-x_j) &= \frac{2}{N^3}\sum_{\substack{i,j,k\\ i\neq j}} m_i m_j m_k S(x_i-x_k)V(x_i-x_j) \\
&= 2\int_{x\neq y} V(x-y)h\left[\mu_N(t,\cdot)\right](x)\mu_N(t,y)dxdy.
\end{aligned}$$

**Step 2.** *Calculations of* $-\frac{d}{dt}\frac{2}{N}\overset{[}{i}= 1]N\sum m_i(t)(V\star\mu)(t,x_i(t))$ *and* $\frac{d}{dt}\int_{\mathbb{R}^d}(V\star\mu)(t,x)\mu(t,x)$. Let us start with

$$\begin{aligned}
&-\frac{d}{dt}\frac{2}{N}\overset{[}{i}= 1]N\sum m_i(t)V\star\mu(t,x_i(t)) \\
&= -\frac{2}{N}\overset{[}{i}= 1]N\sum \dot{m}_i V\star\mu(x_i) - \frac{2}{N}\overset{[}{i}= 1]N\sum m_i V\star\partial_t\mu(t,x_i(t)) \\
&\quad -\frac{2}{N}\overset{[}{i}= 1]N\sum m_i\nabla V\star\mu(x_i)\dot{x}_i = T_2 + E_2,
\end{aligned}$$

where we have set

$$\begin{aligned}
T_2 &:= -\frac{2}{N^2}\sum_{i,k} m_i m_k S(x_i-x_k)V\star\mu(x_i) - \frac{2}{N}\overset{[}{i}= 1]N\sum m_i V\star h\left[\mu\right](x_i), \\
E_2 &:= -\frac{2}{N}\overset{[}{i}= 1]N\sum m_i V\star(\mathrm{div}(\mu\mathbb{J}\nabla V\star\mu))(x_i) \\
&\quad -\frac{2}{N^2}\sum_{i\neq k} m_i m_k \nabla V\star\mu(x_i)\mathbb{J}\nabla V(x_k-x_i).
\end{aligned} \tag{5.2}$$

Observe that we can further write that

$$T_2 = -2\int_{\mathbb{R}^d} V\star\mu(t,x)h\left[\mu_N(t,\cdot)\right]dx - 2\int_{\mathbb{R}^d} V\star h\left[\mu\right](t,x)\mu_N(t,x)dx.$$

With similar calculations, we find that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} V\star\mu(t,x)\mu(t,x)dx &= 2\int_{\mathbb{R}^d} V\star\left(\operatorname{div}(\mu\mathbb{J}\nabla V\star\mu)+h[\mu]\right)(x)\mu(x)dx\\ &= 2\int_{\mathbb{R}^d} V\star\left(\operatorname{div}(\mu\mathbb{J}\nabla V\star\mu)\right)(x)\mu(x)dx + 2\int_{\mathbb{R}^d} V\star h[\mu](x)\mu(x)dx := E_3+T_3.\end{aligned} \tag{5.3}$$

We observe that

$$T_1+T_2+T_3=2\int_{x\neq y} V(x-y)\left(h\left[\mu_N(t,\cdot)\right](x)-h[\mu(t,\cdot)](x)\right)(\mu_N(t,y)-\mu(t,y))\,dxdy,$$

which implies that

$$\begin{aligned}&\frac{d}{dt}\frac{1}{N^2}\sum_{i\neq j} m_i(t)m_j(t)V(x_i(t)-x_j(t)) = E_1+E_2+E_3\\ &+2\int_{x\neq y} V(x-y)\left(h[\mu_N(t,\cdot)](x)-h[\mu(t,\cdot)](x)\right)(\mu_N(t,y)-\mu(t,y))\,dxdy,\end{aligned} \tag{5.4}$$

where $E_1\le 0$, $E_2$ is given by (5.2) and $E_3$ by (5.3).

**Step 3**. Collecting (5.2) and (5.3), we get

$$\begin{aligned}E_2+E_3 = &\,2\int_{\mathbb{R}^d} V\star\left(\operatorname{div}(\mu\mathbb{J}\nabla V\star\mu)\right)(x)\mu(x)dx\\ &-\frac{2}{N}\, [\, i=1]N\sum m_i V\star(\operatorname{div}(\mu\mathbb{J}\nabla V\star\mu))(x_i)\\ &-\frac{2}{N^2}\sum_{i,k} m_i m_k \nabla V\star\mu(x_i)\mathbb{J}\nabla V(x_k-x_i).\end{aligned}$$

We now work in each term on the right-hand side as follows. We first rewrite the first term as

$$\begin{aligned}2\int_{\mathbb{R}^d} V\star\left(\operatorname{div}(\mu\mathbb{J}\nabla V\star\mu)\right)(x)\mu(x)dx &= -2\int_{\mathbb{R}^d}\mu(x)(\nabla V\star\mu)(x)(\mathbb{J}\nabla V\star\mu)(x)dx\\ &= -2\int_{\mathbb{R}^d\times\mathbb{R}^d}\nabla V(x-y)\mu(x)\mu(y)(\mathbb{J}\nabla V\star\mu)(x)dxdy,\end{aligned}$$

by integration by parts, while the second term can also be rewritten as

$$\begin{aligned}-\frac{2}{N}\, [\, i=1]N\sum m_i V\star\operatorname{div}(\mu\mathbb{J}\nabla V\star\mu)(x_i) &= -2\int_{\mathbb{R}^d\times\mathbb{R}^d} V(x-y)\operatorname{div}(\mu\mathbb{J}\nabla V\star\mu)(y)\mu_N(x)dxdy\\ &= -2\int_{\mathbb{R}^d\times\mathbb{R}^d}\nabla V(x-y)(\mathbb{J}\nabla V\star\mu)(y)\mu(y)\mu_N(x)dxdy\\ &= 2\int_{\mathbb{R}^d\times\mathbb{R}^d}\nabla V(x-y)(\mathbb{J}\nabla V\star\mu)(x)\mu(x)\mu_N(y)dxdy,\end{aligned}$$

by integration by parts and symmetrization and the final term is equivalenty written as

$$-\frac{2}{N^2}\sum_{i,k} m_i m_k \mathbb{J}\nabla V\star\mu(x_i)\nabla V(x_k-x_i) = -2\int_{\mathbb{R}^d\times\mathbb{R}^d}\nabla V(x-y)\mathbb{J}\nabla V\star\mu(x)\mu_N^{\otimes 2}(dxdy).$$

Hence, we find that

$$
\begin{aligned}
E_2 + E_3 = & -2 \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla V(x-y) \mathbb{J} \nabla V \star \mu(x) \, (\mu_N - \mu)^{\otimes 2} \, (dxdy) \\
& - 2 \int_{\mathbb{R}^d} \mathbb{J} \nabla V \star \mu(x) \nabla V \star \mu(x) \mu_N(x) dx \\
\leq & -2 \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla V(x-y) \mathbb{J} \nabla V \star \mu(x) \, (\mu_N - \mu)^{\otimes 2} \, (dxdy) \\
= & - \int_{x \neq y} \left( \mathbb{J} \nabla V \star \mu(x) - \mathbb{J} \nabla V \star \mu(y) \right) \nabla V(x-y) \, (\mu_N - \mu)^{\otimes 2} \, (dxdy),
\end{aligned}
$$

because $\mathbb{J} \nabla V \star \mu(x) \nabla V \star \mu(x) \geq 0$. Hence, we conclude that

$$
E_1 + E_2 + E_3 \leq - \int_{x \neq y} \left( \mathbb{J} \nabla V \star \mu(x) - \mathbb{J} \nabla V \star \mu(y) \right) \nabla V(x-y) \, (\mu_N - \mu)^{\otimes 2} \, dxdy,
$$

so that together with (5.4), we get

$$
\begin{aligned}
\frac{d}{dt} \mathcal{E}_N(t) \leq & - \int_{x \neq y} \left( \mathbb{J} \nabla V \star \mu(x) - \mathbb{J} \nabla V \star \mu(y) \right) \nabla V(x-y) \, (\mu_N - \mu)^{\otimes 2} \, dxdy \\
& + 2 \int_{x \neq y} V(x-y) \left( h[\mu_N](x) - h[\mu](x) \right) \left( \mu_N(y) - \mu(y) \right) dxdy,
\end{aligned}
$$

as desired. □

### *5.2. Functional Inequalities*

This sub-section develops the most subtle part of this work. Indeed, re-normalizing the stability estimate from the previous section requires new arguments in comparison to [8], due to the inclusion of a source term. The crucial technical argument is found in Lemma 5.6 for which we need certain preliminary technical results. The following regularization Lemma is an adaptation of [8, Lemma 4.1], which was proved for the periodic case, to the Euclidean setting. We will designate by $\|V\|_{L^p+L^q}$ the norm of the space $L^p + L^q$ defined by

$$
\|V\|_{L^p+L^q} := \inf(\|V_1\|_p + \|V_2\|_q)
$$

where the infimum is taken over all splittings $V = V_1 + V_2$ such that $V_1 \in L^p(\mathbb{R}^d)$ and $V_2 \in L^q(\mathbb{R}^d)$.

**Lemma 5.2.** *Let* $V(x) = \frac{1}{|x|^k}$ *where* $0 < k < d$. *Let* $0 \leq \zeta \leq 1$ *be such that* $\zeta \in C^\infty(\mathbb{R}^d)$, $\zeta \equiv 0$ *on* $B_1(0)$ *and* $\zeta \equiv 1$ *on* $\mathbb{R}^d \backslash B_2(0)$. *Set* $\zeta_\delta(x) = \zeta(\frac{x}{\delta})$ *for* $0 < \delta < 1$. *Assume* $\max\{\frac{d}{2k}, 1\} < p < \frac{d}{k} < q < \infty$, *then for each* $r \in \mathbb{N}$ *there is some constant* $C = C(p, q, d, k)$ *and a* $r$*-differentiable approximation* $V_\varepsilon \in C^r(\mathbb{R}^d)$ *of* $V$ *such that*

*i.* $\widehat{V_\varepsilon} \geq 0$.
*ii.* $\|V_\varepsilon - V\|_{L^p+L^q} \leq \eta(\varepsilon)$ *with* $\eta(\varepsilon)$ *satisfying* $\eta(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

*iii.* $\|\zeta_\delta(V_\varepsilon - V)\|_{L^p+L^q} \leq C\left(\frac{\varepsilon}{\delta^{k+1-\frac{d}{q}}} + \delta^{\frac{d}{p}-k}\right)$.

*iv.* $V_\varepsilon(x) \leq V(x) + \varepsilon$.

*v.* $\|V_\varepsilon\|_\infty \leq \frac{1}{\Theta(\varepsilon)}$ *where* $\Theta(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

*Proof.* **Step 0**. Note that $V \in L^p(\mathbb{R}^d) + L^q(\mathbb{R}^d)$ for $1 < p < \frac{d}{k} < q < \infty$. We will need a further restriction $p > \frac{d}{2k}$ later in the proof. In addition, we have $\widehat{V} = \mathbf{C}\,|x|^{-\kappa}$ with $\kappa = d - k$. Consider a kernel $K^1 : \mathbb{R}^d \to \mathbb{R}$ such that

- $\int_{\mathbb{R}^d} K^1(z)dz = 1$, $\widehat{K^1} \geq 0$, and $\left|K^1(z)\right| \leq \exp\left(-|z|^2\right)$.
- There is some $C > 0$ and $r > 0$ such that

$$\frac{1}{C(1+|\xi|^r)} \leq \widehat{K^1}(\xi) \leq \frac{C}{1+|\xi|^r}.$$

Let $K^1_\delta := \frac{1}{\delta^d} K^1(\frac{x}{\delta})$. Fix $R = R(\delta)$ to be chosen later. Let $\chi \in C_0^\infty(\mathbb{R}^d)$ such that $\chi \geq 0$, $\chi \equiv 1$ on $B_{\frac{R}{2}}(0)$ and $\chi \equiv 0$ on $\mathbb{R}^d \backslash B_R(0)$. Clearly, we get

$$\left|K^1_\delta \star V(x) - V(x)\right| \leq \int_{\mathbb{R}^d} \chi(z)\left|K^1\right|(z)\,|V(x-\delta z) - V(x)|\,dz + \int_{\mathbb{R}^d} (1-\chi(z))\left|K^1\right|(z)\,|V(x-\delta z) - V(x)|\,dz. \quad (5.5)$$

Ultimately $V_\varepsilon$ will be defined by means of $K^1_{\delta(\varepsilon)} \star V$ for a well chosen function $\delta(\varepsilon)$. Estimating (5.5) will be done by distinguishing between possible intervals in which $|x|$ lies in. In this Lemma, $\lesssim$ stands for an inequality up to a constant depending only on $p, q, d, k$.

**Step 1**. The aim of this step is to prove there is some $0 < \lambda < 1$ such that

$$K^1_\delta \star V(x) \leq V(x) + C\delta^\lambda,\ \ |x| \geq \delta^\lambda \quad (5.6)$$

for all $\delta$ arbitrarily small. We assume that $|x| \geq 2R\delta$. First, note that $|x| \geq 2\delta R$ and $|z| \leq R$ implies that $|x - \delta z| \geq \delta R$, so

$$\int_{\mathbb{R}^d} \chi(z)\left|K^1\right|(z)\,|V(x-\delta z) - V(x)|\,dz \lesssim \frac{\delta R}{|x|^{k+1}}, \quad (5.7)$$

due to $|\nabla V(x)| \lesssim \frac{1}{|x|^{k+1}}$ and the mean value theorem. We split the second integral in (5.5) as

$$T_1 + T_2 := \int_{|x-\delta z| \leq \frac{3|x|}{2}} (1-\chi(z))\left|K^1\right|(z)\,|V(x-\delta z) - V(x)|\,dz + \int_{|x-\delta z| \geq \frac{3|x|}{2}} (1-\chi(z))\left|K^1\right|(z)\,|V(x-\delta z) - V(x)|\,dz.$$

The remaining of step 1 is occupied with estimating $T_1$ and $T_2$. The term $T_2$ is

$$T_2 \leq \int_{|x-\delta z| \geq \frac{3|x|}{2}} (1-\chi(z))\left|K^1\right|(z) V(x-\delta z)dz + V(x)\int_{\mathbb{R}^d}(1-\chi(z))\left|K^1\right|(z)dz$$
$$\lesssim (V(\tfrac{3x}{2}) + V(x))\int_{\mathbb{R}^d}(1-\chi(z))\left|K^1\right|(z)dz. \quad (5.8)$$

Observe that for any given $l > 0$ we have the estimate

$$\int_{\mathbb{R}^d} (1-\chi(z)) \left|K^1\right|(z)dz \leq \int_{|z|\geq \frac{R}{2}} e^{-\frac{|z|^2}{2}} dz \lesssim \frac{1}{R^l}. \tag{5.9}$$

Substituting (5.9) into (5.8) yields

$$T_2 \lesssim \frac{V(x)}{R^l}. \tag{5.10}$$

For the term $T_1$, we separate between two cases: $|x| \geq R$ first and then $|x| \leq R$. Start with $|x| \geq R$, where we have

$$T_1 \leq \int_{\mathbb{R}^d} (1-\chi(z)) \left|K^1\right|(z) \left|V(x-\delta z) - V(x)\right| dz \lesssim V(x) + \int_{\mathbb{R}^d} (1-\chi(z)) \left|K^1\right|(z) V(x-\delta z) dz.$$

We estimate

$$|x|^k \int_{\mathbb{R}^d} (1-\chi(z)) \left|K^1\right|(z) V(x-\delta z) dz \lesssim \int_{\mathbb{R}^d} (1-\chi(\tfrac{\zeta}{\delta})) \left|K^1_\delta\right|(\zeta)(|x-\zeta|^k + |\zeta|^k) V(x-\zeta) d\zeta$$
$$\lesssim \frac{1}{R^l} + V \star K(x),$$

where we have set $K(\zeta) := \left|K^1_\delta\right|(\zeta)\,|\zeta|^k$. Thanks to Young's inequality it holds that

$$\|V \star K\|_\infty \leq \|V\|_{L^p + L^q} \left(\|K\|_{p'} + \|K\|_{q'}\right) \lesssim \delta^{k-\frac{d}{p}} \left(\int_{\mathbb{R}^d} \left|K^1\right|^{p'} (\xi)\, |\xi|^{kp'} d\xi\right)^{\frac{1}{p'}}$$
$$+ \delta^{k-\frac{d}{q}} \left(\int_{\mathbb{R}^d} \left|K^1\right|^{q'} (\xi)\, |\xi|^{kq'} d\xi\right)^{\frac{1}{q'}} \lesssim \delta^{k-\frac{d}{p}} + \delta^{k-\frac{d}{q}}.$$

Therefore, we find

$$|x|^k \int_{\mathbb{R}^d} (1-\chi(z)) \left|K^1\right|(z) V(x-\delta z) dz \lesssim \frac{1}{R^l} + \delta^{k-\frac{d}{p}} + \delta^{k-\frac{d}{q}} \lesssim \frac{1}{R^l} + \delta^{k-\frac{d}{p}},$$

using $p < \frac{d}{k} < q$, and hence for $|x| \geq R$, we have

$$\int_{\mathbb{R}^d} (1-\chi(z)) \left|K^1\right|(z) V(x-\delta z) dz \lesssim \frac{V(x)}{R^l} + \frac{\delta^{k-\frac{d}{p}}}{|x|^k} \lesssim \frac{1}{R^{l+k}} + \frac{1}{\delta^{\frac{d}{p}-k} R^k}.$$

Therefore we conclude that

$$T_1 \lesssim \frac{1}{R^{l+k}} + \frac{1}{\delta^{\frac{d}{p}-k} R^k}, \text{ for } |x| \geq R. \tag{5.11}$$

If $|x| \leq R$, we estimate

$$T_1 \lesssim e^{-\frac{R^2}{2}} \int_{|x-\delta z| \leq \frac{3|x|}{2}} V(x-\delta z) dz + \frac{V(x)}{R^l} = \frac{e^{-\frac{R^2}{2}}}{\delta^d} \int_{|x-\zeta| \leq \frac{3|x|}{2}} V(x-\zeta) d\zeta + \frac{V(x)}{R^l}$$
$$= \frac{e^{-\frac{R^2}{2}}}{\delta^d} \int_{|\xi| \leq \frac{3|x|}{2}} V(\xi) d\xi + \frac{V(x)}{R^l} \lesssim \frac{e^{-\frac{R^2}{2}}}{\delta^d} |x|^{d-k} + \frac{1}{R^l\, |x|^k},$$

so that we conclude that

$$T_1 \lesssim \frac{R^{d-k} e^{-\frac{R^2}{2}}}{\delta^d} + \frac{1}{R^l |x|^k}, \text{ for } |x| \leq R. \tag{5.12}$$

Note that in the proof (5.10), (5.11) and (5.12) we did not employ the assumption $|x| \geq 2R\delta$, a fact which will be useful later on. We now gather inequalities (5.7), (5.10), (5.11) and (5.12) to obtain

$$\left| K_\delta^1 \star V(x) - V(x) \right| \lesssim \frac{\delta R}{|x|^{k+1}} + \frac{1}{R^{l+k}} + \frac{1}{\delta^{\frac{d}{p}-k} R^k} + \frac{R^{d-k} e^{-R^2}}{\delta^d} + \frac{1}{R^l |x|^k}. \tag{5.13}$$

We can pick $\lambda' \in (0, 1)$ small enough so that $2k - \frac{d}{p} - \lambda' > 0 > k - \frac{d}{p} - \lambda'$ assuming that $\frac{d}{2k} < p < \frac{d}{k}$. We observe that

$$\frac{1}{\delta^{\frac{d}{p}-k} R^k} = \delta^{\lambda'}, \ \delta R = \delta^{2-\frac{d}{kp}-\frac{\lambda'}{k}} \quad \text{by setting } R = R(\delta) = \delta^{1-\frac{d}{kp}-\frac{\lambda'}{k}}.$$

Notice that there is some $\beta' > 0$ such that the function $\delta \mapsto \frac{R^{d-k}(\delta) e^{-R^2(\delta)}}{\delta^d}$ satisfies $\frac{R^{d-k}(\delta) e^{-R^2(\delta)}}{\delta^d} \leq \delta^{\beta'}$ when $\delta \to 0$, since both $\frac{R^{d-k}(\delta)}{\delta^d}$ and $R^2(\delta)$ are negative powers of $\delta$ under our assumptions. Then, owing to (5.13) we get

$$\left| K_\delta^1 \star V(x) - V(x) \right| \lesssim \frac{\delta^\alpha}{|x|^{k+1}} + \delta^\beta + \frac{\delta^\gamma}{|x|^k},$$

for some $\alpha > 0$, $\beta > 0$, $\gamma > 0$, which implies that for a suitable choice of $\lambda > 0$ it holds that

$$K_\delta^1 \star V(x) \leq V(x) + C\delta^\lambda, \ |x| \geq \delta^\lambda,$$

as desired. Notice that we can assume that $\lambda < 1$ without loss of generality since $\delta$ will be chosen converging to zero.

**Step 2**. In this step, we prove that

$$K_\delta^1 \star V(x) \leq C\left(V(x) + \delta^\lambda\right), \text{ for all } x \in \mathbb{R}^d. \tag{5.14}$$

Note that in (5.14) we allow the estimate to hold up to a constant (unlike inequality (5.6)). Assume first that $|x| \leq 2R\delta$. The following inequality follows by a direct calculation

$$\int_{B(0,\delta)} V(y)dy \lesssim \int_{\frac{\delta}{2} \leq |y| \leq \delta} V(y)dy, \tag{5.15}$$

for $0 < k < d$. We now split $K_\delta^1 \star V$ as

$$K_\delta^1 \star V(x) = \int_{\mathbb{R}^d} \chi\left(\frac{x-z}{\delta}\right) K_\delta^1(x-z) V(z)dz + \int_{\mathbb{R}^d} \left(1 - \chi\left(\frac{x-z}{\delta}\right)\right) K_\delta^1(x-z) V(z)dz.$$

The first term can be estimated as

$$\int_{\mathbb{R}^d} \chi\left(\frac{x-z}{\delta}\right) K_\delta^1 (x-z)\, V(z)dz \le \int_{|z|\le 3R\delta} \left|K_\delta^1\right| (x-z)\, V(z)dz = \int_{A\cup B} \left|K_\delta^1\right| (x-z)\, V(z)dz,$$

where $A = \{z \in \mathbb{R}^d : |z| \le 3R\delta, |z| \le \frac{1}{2}|x|\}$ and $B = \{z \in \mathbb{R}^d : \frac{1}{2}|x| \le |z| \le 3R\delta\}$. If $|z| \le \frac{1}{2}|x|$ then $|x-z| \ge \frac{1}{2}|x|$ so that

$$\int_A \left|K_\delta^1\right| (x-z)\, V(z)dz \le \frac{\exp\left(-\frac{|x|^2}{2\delta^2}\right)}{\delta^d} \int_A V(z)dz \le \frac{\exp(-\frac{|x|^2}{2\delta^2})}{\delta^d} \int_{|z|\le\frac{1}{2}|x|} V(z)dz$$
$$\lesssim \frac{\exp(-\frac{|x|^2}{2\delta^2})}{\delta^d} \int_{\frac{1}{4}|x|\le|z|\le\frac{1}{2}|x|} V(z)dz \lesssim \exp\left(-\frac{|x|^2}{2\delta^2}\right) \frac{|x|^d}{\delta^d} V(x),$$

where we used (5.15). Note that the function $x \mapsto \exp(-\frac{|x|^2}{2\delta^2})\frac{|x|^d}{\delta^d}$ is uniformly bounded in $\delta$. Therefore we conclude that

$$\int_A \left|K_\delta^1\right| (x-z)\, V(z)dz \lesssim V(x).$$

In addition, we obtain

$$\int_B \left|K_\delta^1\right| (x-z)\, V(z)dz \lesssim V(x) \int_{|z|\le 3R\delta} \left|K_\delta^1\right| (x-z)\, dz \lesssim V(x).$$

Thus, we have proved that

$$\int_{\mathbb{R}^d} \chi\left(\frac{x-z}{\delta}\right) \left|K_\delta^1\right| (x-z)\, V(z)dz \lesssim V(x). \tag{5.16}$$

Next we handle the truncation far from the origin. By exactly the same argument leading to (5.10), (5.11) and (5.12) we have

$$\int_{\mathbb{R}^d} \left(1-\chi\left(\frac{x-z}{\delta}\right)\right) K_\delta^1 (x-z)\, V(z)dz = \int_{\mathbb{R}^d} (1-\chi(z))K^1(z)V(x-\delta z)dz$$
$$\lesssim V(x) + \delta^\lambda + \delta^\alpha. \tag{5.17}$$

In view of (5.16) and (5.17), we deduce $K_\delta^1 \star V(x) \lesssim V(x)$ for $|x| \le 2R\delta$. We are left to treat the range $2R\delta \le |x| \le \delta^\lambda$. If $|x| \ge 2R\delta$ and $|z| \le R$ then clearly

$$\frac{|x|}{2} = |x| - \frac{|x|}{2} \le ||x| - R\delta| \le |x-\delta z| \le 2\,|x|\,.$$

Thus, we infer that

$$\int_{\mathbb{R}^d} \chi(z) \left|K^1\right| (z)V(x-\delta z)dz \lesssim V(\tfrac{x}{2}) \int_{|z|\le R} \left|K^1\right| (z)dz \lesssim V(x).$$

In addition, by step 2, we have

$$\left|\int_{\mathbb{R}^d} (1-\chi(z))K^1(z)V(x-\delta z)dz\right| \lesssim V(x) + \delta^\lambda + \delta^\alpha,$$

for $|x| \geq 2R\delta$. Together with inequality (5.6) in step 1, this establishes (5.14) by redefining $\lambda$ if needed since $\lambda < 1$ and $\delta$ will be chosen converging to zero.

**Step 3**. We claim that there is a non-decreasing function $f$ such that

$$K_\delta^1 \star V(x) \leq V(x), \ |x| \leq f(\delta). \tag{5.18}$$

In order to prove the claim, we first estimate it as

$$K_\delta^1 \star V(x) \leq \int_{|z|\leq 1} V(x-\delta z) \left|K^1\right|(z)dz + \int_{|z|\geq 1} V(x-\delta z)\left|K^1\right|(z)dz := I + J.$$

By change of variables, we have

$$I \leq \frac{1}{\delta^d} \int_{B_\delta(x)} V(z)dz,$$

and since the function $x \mapsto \int_{B_\delta(x)} V(z)dz$ attains its maximum at $x = 0$,[1] it follows that

$$I \leq \frac{1}{\delta^d} \int_{B_\delta(0)} V(z)dz = \frac{1}{(d-k)\delta^k}.$$

Hence, for all $|x| \leq \left(\frac{d-k}{2}\right)^{\frac{1}{k}} \delta$, we get

$$I \leq \frac{1}{2\,|x|^k} = \frac{1}{2} V(x).$$

In addition, we obtain

$$J \leq \frac{1}{(\delta - |x|)^k} \leq \frac{1}{2\,|x|^k},$$

for all $|x| \leq \frac{\delta}{1+2^{\frac{1}{k}}}$, so that $J \leq \frac{1}{2}V(x)$.

To conclude, the claim follows by taking $f(\delta) = \min\left\{\frac{1}{1+2^{\frac{1}{k}}}, \left(\frac{d-k}{2}\right)^{\frac{1}{k}}\right\}\delta$.

**Step 4**. **Construction of** $V_\varepsilon$. In this step we will construct $V_\varepsilon$ and then show that it satisfies the requested properties i.-iv. We construct our regularized kernel following the same strategy as in [8, Lemma 4.1].

We now need to interpolate between this inequality near the origin and the far-field inequality (5.6) by means of the inequality (5.14). This is crucially needed in

1 For each $x, y$ with $|x| < |y|$ it holds that $|x - z| \leq |y - z|$ for all $z \in B_\delta(0)$ and $\delta > 0$ sufficiently small. Consequently, given a radially decreasing function $V(z) = v(|z|)$ for each $|x| < |y|$ and $\delta > 0$ small enough it holds that

$$\int_{B_\delta(x)} V(z)dz - \int_{B_\delta(y)} V(z)dz = \int_{B_\delta(0)} (v(|x-z|) - v(|y-z|))\, dz \geq 0.$$

order to keep the constant 1 in front of $V(x)$ on the right-hand side of our statement in point iv. As mentioned the construction is motivated by [8, Lemma 4.1] proved in the torus.

Given $\varepsilon > 0$ pick $0 < \delta_1 \leq \frac{\varepsilon^{\frac{2}{\lambda}}}{C}$, where $C > 0$ is the constant in (5.14), and recursively pick $0 < \delta_{i+1}$ such that

$$\delta_{i+1} \leq \min\left\{ f(\delta_i), \delta_i^{\frac{1}{\lambda}} \right\}.$$

Let $M := \left\lfloor \frac{1}{\varepsilon} \right\rfloor$ where $\lfloor x \rfloor$ denotes the closest lower integer to $x$, and set

$$W_\varepsilon := \frac{1}{M} \overset{[}{i} = 1]M \sum K^1_{\delta_i} \star V. \tag{5.19}$$

Obviously $\delta_i$ is decreasing since $\lambda < 1$ and $\max_{1 \leq i \leq M} \delta_i \underset{\varepsilon \to 0}{\to} 0$ by construction.

If $|x| \geq \delta_1^\lambda$, (5.6) implies that

$$W_\varepsilon(x) \leq V(x) + \varepsilon.$$

If $|x| \leq f(\delta_M)$, (5.18) implies that $W_\varepsilon(x) \leq V(x)$.

If $f(\delta_M) \leq |x| \leq \delta_1^\lambda$, fix $1 \leq i \leq M$ such that $\delta_{i+1} \leq |x| \leq \delta_i$, then:

- If $j > i+1$ then $|x| \geq \delta_{i+1} \geq \delta_j^\lambda$ so that using again (5.6) we get $K^1_{\delta_j} \star V(x) \leq V(x) + \varepsilon$.
- If $j < i$ then $|x| \leq \delta_i \leq f(\delta_j)$ so that again by (5.18) we have $K^1_{\delta_j} \star V(x) \leq V(x)$.
- If $j = i$ or $j = i+1$ then according to (5.14) $K^1_{\delta_j} \star V(x) \leq C\left(V(x) + \varepsilon\right)$.

We finally define

$$V_\varepsilon(x) := \frac{1}{1 + 2C\varepsilon} W_\varepsilon(x) \tag{5.20}$$

and conclude that $V_\varepsilon(x) \leq V(x) + \varepsilon$, which establishes point iv. Notice that the definition of $M$ is crucially used for this estimate.

That $\widehat{V_\varepsilon} \geq 0$ is immediate by construction by the choice of $K^1$ and establishes point i. We are left to prove point ii,iii and v. If we split $V = V_1 + V_2$ where $V_1 \in L^p(\mathbb{R}^d)$ and $V_2 \in L^q(\mathbb{R}^d)$ then

$$\begin{aligned} \| W_\varepsilon - V \|_{L^p + L^q(\mathbb{R}^d)} &\leq \max_{1 \leq i \leq M} \left\| K_{\delta_i} \star V_1 - V_1 \right\|_{L^p(\mathbb{R}^d)} + \max_{1 \leq i \leq M} \left\| K_{\delta_i} \star V_2 - V_2 \right\|_{L^q(\mathbb{R}^d)} \\ &\leq 2 \max_{1 \leq i \leq M} \delta_i \underset{\varepsilon \to 0}{\longrightarrow} 0, \end{aligned}$$

which is assertion ii. Note further that for a given $\delta > 0$ one has

$$\begin{aligned} \left\| \zeta_\delta (K^1_{\delta_i} \star V) - \zeta_\delta V \right\|_{L^p + L^q(\mathbb{R}^d)} &\leq \left\| K^1_{\delta_i} \star (\zeta_\delta V) - \zeta_\delta V \right\|_{L^p + L^q(\mathbb{R}^d)} + \left\| \zeta_\delta (K^1_{\delta_i} \star ((1 - \zeta_\delta) V)) \right\|_{L^p + L^q(\mathbb{R}^d)} \\ &\quad + \left\| (\zeta_\delta - 1)(K^1_{\delta_i} \star (\zeta_\delta V)) \right\|_{L^p + L^q(\mathbb{R}^d)} \\ &:= I + II + III. \end{aligned}$$

It is not difficult to check that $\left\| K^1_\delta \star f - f \right\|_q \lesssim_d \delta \, \|\nabla f\|_q$ by using that the first moment of $K^1_\delta$ is of order $\delta$. Therefore, the first term is

$$I \le \left\| K^1_{\delta_i} \star \zeta_\delta V - \zeta_\delta V \right\|_{L^q(\mathbb{R}^d)} \lesssim \max_{1\le i\le M} \delta_i \, \|\nabla(\zeta_\delta V)\|_q \le \max_{1\le i\le M} \delta_i \left( \|\zeta_\delta \nabla V\|_q + \|V \nabla \zeta_\delta\|_q \right).$$

Observe that

$$\|\zeta_\delta \nabla V\|_q \lesssim \left( \int_{|\cdot|\ge\delta} \frac{1}{|x|^{q(k+1)}} dx \right)^{\frac{1}{q}} = \left( \int_\delta^\infty r^{d-1-q(k+1)} dr \right)^{\frac{1}{q}} \lesssim \frac{\delta^{\frac{d}{q}}}{\delta^{k+1}}.$$

Also, a similar calculation reveals that

$$\|V \nabla \zeta_\delta\|_q \le \|\nabla \zeta_\delta\|_\infty \, \|V\|_{L^q(B_{2\delta}\setminus B_\delta)} \lesssim \|\nabla \zeta_\delta\|_\infty \frac{\delta^{\frac{d}{q}}}{\delta^k} \le \frac{\delta^{\frac{d}{q}}}{\delta^{k+1}}, \tag{5.21}$$

hence $I \lesssim \varepsilon \delta^{\frac{d}{q}-1-k}$. Moreover

$$\begin{aligned} II &= \left\| \zeta_\delta (K^1_{\delta_i} \star ((1-\zeta_\delta)V)) \right\|_{L^p + L^q(\mathbb{R}^d)} \le \left\| K^1_{\delta_i} \star ((1-\zeta_\delta)V) \right\|_{L^p(\mathbb{R}^d)} \\ &\le \left\| K^1_{\delta_i} \right\|_{L^1(\mathbb{R}^d)} \|(1-\zeta_\delta)V\|_{L^p(\mathbb{R}^d)} \lesssim \delta^{\frac{d}{p}-k}. \end{aligned}$$

As for $III$, we estimate

$$\begin{aligned} III \le \left\| (\zeta_\delta - 1)(K^1_{\delta_i} \star \zeta_\delta V) \right\|_{L^p(\mathbb{R}^d)} &\le \|\zeta_\delta - 1\|_{L^p(\mathbb{R}^d)} \left\| K^1_{\delta_i} \star \zeta_\delta V \right\|_{L^\infty(\mathbb{R}^d)} \\ &\lesssim \delta^{\frac{d}{p}} \, \|\zeta_\delta V\|_{L^\infty(\mathbb{R}^d)} \le \delta^{\frac{d}{p}-k}. \end{aligned}$$

Therefore, we have proved

$$\left\| \zeta_\delta K_{\delta_{n_i}} \star V - \zeta_\delta V \right\|_{L^p + L^q(\mathbb{R}^d)} \lesssim \frac{\varepsilon}{\delta^{k+1-\frac{d}{q}}} + \delta^{\frac{d}{p}-k},$$

which concludes the proof of item iii. We finally have

$$\left\| K^1_\delta \star V \right\|_\infty \le \left\| K^1_\delta \right\|_{p'} \left\| \mathbf{1}_{|\cdot|\le 1} V \right\|_p + \left\| K^1_\delta \right\|_{q'} \left\| \mathbf{1}_{|\cdot|\ge 1} V \right\|_q \le \delta^{-\frac{d}{p}} + \delta^{-\frac{d}{q}}$$

so that by defining

$$\|V_\varepsilon\|_\infty \le \left( \min_{1\le i\le M} \delta_i \right)^{-\frac{d}{p}} + \left( \min_{1\le i\le M} \delta_i \right)^{-\frac{d}{q}} := \frac{1}{\Theta(\varepsilon)},$$

the statement v. follows. □

In the following Lemma we show that given a function in Fourier variable $s(\xi)$ satisfying some growth assumptions we can bound $s \star \widehat{V_\varepsilon}$ pointwise in absolute value by means of $\widehat{V_\varepsilon}$, where $V_\varepsilon$ is the approximating kernel constructed in Lemma 5.2.

**Lemma 5.3.** *Let $V_\varepsilon$ be as in Lemma 5.2. Assume that there are some $S_0 > 0$ and some $\sigma > 2d$ such that*

$$|s(\xi)| \leq \frac{S_0}{(1+|\xi|)^\sigma}. \tag{5.22}$$

*Given $r > \frac{d}{d-k}$ set $S_0' := \max\left\{\|s\|_1, \left\||\cdot|^{d-k} s\right\|_{r'}, \left\|(1+|\cdot|^{d-k})s\right\|_{L^\infty}\right\}$. Then, it holds that*

$$\left|(s \star \widehat{V}_\varepsilon)(\xi)\right| \leq C_1 \widehat{V}_\varepsilon(\xi) + C_2 \frac{o_\varepsilon(1)}{(1+|\xi|)^{\frac{\sigma}{2}}},$$

*where $C_1 = c_1(k,d)S_0'$ and $C_2 = c_2(k,\sigma,d)S_0$.*

*Proof.* We separate between two regions. Put $\overline{\delta} := \max_{1\leq i \leq M} \delta_i \leq \varepsilon$.

**Step 1**. Suppose that $|\xi| < \frac{1}{\delta}$. Note that for all $\xi \in \mathbb{R}^d$

$$\widehat{V}_\varepsilon(\xi) = \frac{1}{M(1+2C\varepsilon)} [i = 1]M \sum \widehat{K^1_{\delta_i}}(\xi)\widehat{V}(\xi) \leq \left\|\widehat{K^1}\right\|_\infty \widehat{V}(\xi). \tag{5.23}$$

Fix some $\chi \in C_0^\infty(\mathbb{R}^d)$ with $0 \leq \chi \leq 1$ and $\chi \equiv 1$ on $B_1(0)$ and $\chi \equiv 0$ on $\mathbb{R}^d \backslash B_2(0)$. In view of (5.23) we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d} \widehat{V_\varepsilon}(\xi-\zeta)s(\zeta)d\zeta\right| \\
&\leq \left\|\widehat{K^1}\right\|_\infty \int_{\mathbb{R}^d} \widehat{V}(\xi-\zeta)\,|s(\zeta)|\,d\zeta \\
&= \left\|\widehat{K^1}\right\|_\infty \left(\int_{\mathbb{R}^d} \chi(\xi-\zeta)\widehat{V}(\xi-\zeta)\,|s(\zeta)|\,d\zeta + \int_{\mathbb{R}^d} (1-\chi(\xi-\zeta))\widehat{V}(\xi-\zeta)\,|s(\zeta)|\,d\zeta\right).
\end{aligned} \tag{5.24}$$

We control each of the integrals in (5.24) separately. Thanks to (5.22) one has

$$\begin{aligned}
&|\xi|^{d-k} \int_{\mathbb{R}^d} \chi(\zeta)\widehat{V}(\zeta)\,|s(\xi-\zeta)|\,d\zeta \lesssim \int_{\mathbb{R}^d} \chi(\zeta)\widehat{V}(\zeta)\,|\xi-\zeta|^{d-k}\,|s(\xi-\zeta)|\,d\zeta \\
&+ \int_{\mathbb{R}^d} \chi(\zeta)\widehat{V}(\zeta)\,|\zeta|^{d-k}\,|s(\xi-\zeta)|\,d\zeta \\
&\lesssim \left\|(1+|\cdot|^{d-k})s\right\|_\infty \int_{\mathbb{R}^d} (1+|\zeta|^{d-k})\chi(\zeta)\widehat{V}(\zeta)d\zeta \\
&\lesssim \left\|(1+|\cdot|^{d-k})s\right\|_\infty \left\|\left(1+|\cdot|^{d-k}\right)\widehat{V}(\cdot)\right\|_{L^1(B_2(0))} \lesssim \left\|(1+|\cdot|^{d-k})s\right\|_\infty,
\end{aligned}$$

which shows that

$$\int_{\mathbb{R}^d} \chi(\xi-\zeta)\widehat{V}(\xi-\zeta)\,|s(\zeta)|\,d\zeta \lesssim \frac{\left\|(1+|\cdot|^{d-k})s\right\|_\infty}{|\xi|^{d-k}} \lesssim \left\|(1+|\cdot|^{d-k})s\right\|_\infty \widehat{V}(\xi). \tag{5.25}$$

Secondly, we can estimate the truncation far from origin as follows

$$
\begin{aligned}
&|\xi|^{d-k}\int_{\mathbb{R}^d}(1-\chi(\xi-\zeta))\widehat{V}(\xi-\zeta)\,|s(\zeta)|\,d\zeta\\
&\lesssim \int_{\mathbb{R}^d}|\xi-\zeta|^{d-k}\,(1-\chi(\xi-\zeta))\widehat{V}(\xi-\zeta)\,|s(\zeta)|\,d\zeta\\
&+\int_{\mathbb{R}^d}|\zeta|^{d-k}\,(1-\chi(\xi-\zeta))\widehat{V}(\xi-\zeta)\,|s(\zeta)|\,d\zeta\\
\lesssim &\int_{\mathbb{R}^d}|s(\zeta)|\,d\zeta+\int_{\mathbb{R}^d}|\xi-\zeta|^{d-k}\,|s(\xi-\zeta)|\,(1-\chi(\zeta))\widehat{V}(\zeta)d\zeta\\
\lesssim &\|s\|_1+\left\|(1-\chi)\widehat{V}\right\|_r\left\||\cdot|^{d-k}\,s\right\|_{r'}.
\end{aligned}\tag{5.26}
$$

Choosing $r>\frac{d}{d-k}$ ensures that $\left\|(1-\chi)\widehat{V}\right\|_r<\infty$. Therefore we conclude from (5.25), (5.26) that

$$
\int_{\mathbb{R}^d}\widehat{V_\varepsilon}(\xi-\zeta)\,|s|\,(\zeta)d\zeta\lesssim S_0'\widehat{V}(\xi).
$$

Now, utilizing the assumption that $|\xi|<\frac{1}{\delta}$ we see that $\widehat{K_{\delta_i}^{1}}(\xi)\gtrsim 1$ for all $1\le i\le M$, so that we find $\widehat{V}(\xi)\lesssim\widehat{V_\varepsilon}(\xi)$, for $|\xi|<\frac{1}{\delta}$, and thus we get

$$
\left|s\star\widehat{V_\varepsilon}(\xi)\right|\lesssim S_0'\widehat{V_\varepsilon}(\xi),\ \text{ for }\ |\xi|<\frac{1}{\delta}.\tag{5.27}
$$

**Step 2**. Suppose now $|\xi|\ge\frac{1}{\delta}$. In this case we split the integral as follows

$$
\int_{\mathbb{R}^d}\widehat{V_\varepsilon}(\xi-\zeta)s(\zeta)d\zeta=\int_{|\zeta|\ge\frac{|\xi|}{2}}\widehat{V_\varepsilon}(\xi-\zeta)s(\zeta)d\zeta+\int_{|\zeta|<\frac{|\xi|}{2}}\widehat{V_\varepsilon}(\xi-\zeta)s(\zeta)d\zeta:=I(\xi)+J(\xi).
$$

Pick a function $0\le\chi\le 1$ such that $\chi\equiv 1$ on $|\zeta|\le\frac{1}{4\delta}$ and $\chi\equiv 0$ on $|\zeta|\ge\frac{1}{2\delta}$. We start with $I(\xi)$. We can write

$$
\begin{aligned}
I(\xi)&=\int_{|\zeta|\ge\frac{|\xi|}{2}}\widehat{V_\varepsilon}(\xi-\zeta)(1-\chi(\zeta))s(\zeta)d\zeta\\
&=\int_{|\zeta|\ge\frac{|\xi|}{2}}\mathbf{1}_{|\cdot|\le 1}(\xi-\zeta)\widehat{V_\varepsilon}(\xi-\zeta)(1-\chi(\zeta))s(\zeta)d\zeta\\
&\quad+\int_{|\zeta|\ge\frac{|\xi|}{2}}\mathbf{1}_{|\cdot|\ge 1}(\xi-\zeta)\widehat{V_\varepsilon}(\xi-\zeta)(1-\chi(\zeta))s(\zeta)d\zeta\\
&:=I_1(\xi)+I_2(\xi).
\end{aligned}
$$

First, we have

$$
|I_2(\xi)|\lesssim\int_{|\zeta|\ge\frac{|\xi|}{2}}|(1-\chi(\zeta))s(\zeta)|\,d\zeta\lesssim\frac{S_0}{(1+|\xi|)^{\frac{\sigma}{2}}}\int_{|\zeta|\ge\frac{1}{2\delta}}\frac{1}{(1+|\zeta|)^{\frac{\sigma}{2}}}d\zeta\lesssim\frac{S_0 o_\varepsilon(1)}{(1+|\xi|)^{\frac{\sigma}{2}}}.\tag{5.28}
$$

Secondly, we can deduce that

$$\begin{aligned}\left(1+|\xi|^{\frac{\sigma}{2}}\right)|I_1(\xi)| &\le \left(1+|\xi|^{\frac{\sigma}{2}}\right)\int_{\mathbb{R}^d} \mathbf{1}_{|\cdot|\le 1}(\xi-\zeta)\widehat{V}_\varepsilon(\xi-\zeta)(1-\chi(\zeta))\,|s(\zeta)|\,d\zeta \\ &\lesssim \int_{\mathbb{R}^d} |\xi-\zeta|^{\frac{\sigma}{2}}\,\mathbf{1}_{|\cdot|\le 1}(\xi-\zeta)\widehat{V}_\varepsilon(\xi-\zeta)(1-\chi(\zeta))\,|s(\zeta)|\,d\zeta \\ &\quad + \int_{\mathbb{R}^d} \mathbf{1}_{|\cdot|\le 1}(\xi-\zeta)\widehat{V}_\varepsilon(\xi-\zeta)(1-\chi(\zeta))\left(1+|\zeta|^{\frac{\sigma}{2}}\right)|s(\zeta)|\,d\zeta \\ &\lesssim \int_{\mathbb{R}^d} \mathbf{1}_{|\cdot|\le 1}(\xi-\zeta)\,|\xi-\zeta|^{\frac{\sigma}{2}}\,\widehat{V}(\xi-\zeta)\,|(1-\chi(\zeta))s(\zeta)|\,d\zeta \\ &\quad + \int_{\mathbb{R}^d} \mathbf{1}_{|\cdot|\le 1}(\xi-\zeta)\widehat{V}(\xi-\zeta)\left|(1-\chi(\zeta))\left(1+|\zeta|^{\frac{\sigma}{2}}\right)s(\zeta)\right|d\zeta.\end{aligned}$$

The right-hand side of the last equation is bounded by

$$\begin{aligned}&\int_{|\zeta|\ge\frac{1}{4\delta}} |s(\zeta)|\,d\zeta + \int_{\mathbb{R}^d} \mathbf{1}_{|\cdot|\le 1}(\xi-\zeta)\widehat{V}(\xi-\zeta)\left|(1-\chi(\zeta))\left(1+|\zeta|^{\frac{\sigma}{2}}\right)s(\zeta)\right|d\zeta \\ &\lesssim S_0 o_\varepsilon(1) + \sup_{|\zeta|\ge\frac{1}{4\delta}}\left|\left(1+|\zeta|^{\frac{\sigma}{2}}\right)s(\zeta)\right| \lesssim S_0 o_\varepsilon(1). \qquad (5.29)\end{aligned}$$

The combination of (5.28) and (5.29) reveal that

$$|I(\xi)| \lesssim \frac{S_0 o_\varepsilon(1)}{(1+|\xi|)^{\frac{\sigma}{2}}}. \qquad (5.30)$$

To bound $J(\xi)$, recall that by assumption, there is some $C > 0$ and $r > 0$ such that

$$\frac{1}{C(1+|\xi|^r)} \le \widehat{K^1}(\xi) \le \frac{C}{1+|\xi|^r}.$$

As a result, for any $\zeta, \xi$ such that $|\zeta| < \frac{|\xi|}{2}$ it holds that

$$\widehat{K^1_{\delta_i}}(\xi-\zeta) = \widehat{K^1}(\delta_i(\xi-\zeta)) \lesssim \frac{1}{1+\delta_i^r\,|\xi-\zeta|^r} \lesssim \frac{1}{1+\delta_i^r\,|\xi|^r} \lesssim \widehat{K^1_{\delta_i}}(\xi).$$

Consequently, we infer that

$$\begin{aligned}\widehat{V}_\varepsilon(\xi-\zeta) &= \frac{1}{M(1+2C\varepsilon)}\,[i=1]M\sum \widehat{K^1}(\delta_i(\xi-\zeta))\widehat{V}(\xi-\zeta) \\ &\lesssim \frac{1}{M(1+2C\varepsilon)}\,[i=1]M\sum \widehat{K^1}(\delta_i\xi)\widehat{V}(\xi) = \widehat{V}_\varepsilon(\xi),\end{aligned}$$

for $|\zeta| < \frac{|\xi|}{2}$. Hence we obtain

$$|J(\xi)| \lesssim \widehat{V}_\varepsilon(\xi)\int_{\mathbb{R}^d} |s|(\zeta)d\zeta \lesssim S_0'\widehat{V}_\varepsilon(\xi). \qquad (5.31)$$

Gathering Inequalities (5.27), (5.30) and (5.31) yields the announced result. □

As a corollary we can estimate the Fourier transform of the $\delta$-truncation of $V_\varepsilon$ as follows.

**Corollary 5.4.** *Under the same assumptions and notations of Lemma 5.2, it holds that*

$$\left|\widehat{\zeta_\delta V_\varepsilon}(\xi)\right| \lesssim \widehat{V_\varepsilon}(\xi) + \frac{\delta^{d-k-\frac{\sigma}{2}} o_{\varepsilon\delta^{-1}}(1)}{(1+|\xi|)^{\frac{\sigma}{2}}}$$

*and*

$$\left|\widehat{(1-\zeta_\delta)V_\varepsilon}(\xi)\right| \lesssim \widehat{V_\varepsilon}(\xi) + \frac{\delta^{d-k-\frac{\sigma}{2}} o_{\varepsilon\delta^{-1}}(1)}{(1+|\xi|)^{\frac{\sigma}{2}}}.$$

*Proof.* Set $\chi_\delta = 1 - \zeta_\delta$. Observe that $\widehat{\zeta_\delta V_\varepsilon} = \widehat{\zeta_\delta} \star \widehat{V_\varepsilon} = \widehat{V_\varepsilon} - \widehat{\chi_\delta} \star \widehat{V_\varepsilon}$, which shows that the first inequality would follow from the second inequality. To prove the second inequality, notice that using the construction of $V_\varepsilon$ in (5.19)–(5.20), we can deduce for each $\delta > 0$ that

$$\widehat{V_\varepsilon}(\delta\xi) = \frac{1}{M(1+2C\varepsilon)} \sum_{i=1}^{d} \widehat{K^1}(\delta_i\delta\xi)\widehat{V}(\delta\xi) = \frac{\delta^{k-d}}{M(1+2C\varepsilon)} \sum_{i=1}^{d} \widehat{K^1_{\delta_i\delta}}(\xi)\widehat{V}(\xi)$$
$$= \delta^{k-d} \frac{1+2C\delta\varepsilon}{1+2C\varepsilon} \widehat{V_{\delta\varepsilon}}(\xi).$$

Note that $\widehat{\chi_\delta}(x) = \delta^d \widehat{\chi}(\delta x)$. Therefore, we get

$$(\widehat{\chi_\delta} \star \widehat{V_\varepsilon})(\delta\xi) = \int_{\mathbb{R}^d} \widehat{\chi}(\delta\xi - \zeta)\widehat{V_\varepsilon}\left(\delta^{-1}\zeta\right) d\zeta = \delta^{d-k} \frac{1+2C\delta^{-1}\varepsilon}{1+2C\varepsilon}$$
$$\int_{\mathbb{R}^d} \widehat{\chi}(\delta\xi - \zeta)\widehat{V}_{\delta^{-1}\varepsilon}(\zeta)\, d\zeta$$
$$= \delta^{d-k} \frac{1+2C\delta^{-1}\varepsilon}{1+2C\varepsilon} (\widehat{\chi} \star \widehat{V}_{\delta^{-1}\varepsilon})(\delta\xi).$$

By using Lemma 5.3 with $s = \widehat{\chi}$, we conclude that

$$\left|(\widehat{\chi_\delta} \star \widehat{V_\varepsilon})(\delta\xi)\right| \le \frac{1+2C\delta^{-1}\varepsilon}{1+2C\varepsilon} \left( C_1 \delta^{d-k} \widehat{V}_{\delta^{-1}\varepsilon}(\delta\xi) + C_2 \delta^{d-k} \frac{o_{\varepsilon\delta^{-1}}(1)}{(1+|\delta\xi|)^{\frac{\sigma}{2}}} \right)$$
$$\le C_1 \widehat{V_\varepsilon}(\xi) + C_2 \frac{\delta^{d-k-\frac{\sigma}{2}} o_{\varepsilon\delta^{-1}}(1)}{(1+|\xi|)^{\frac{\sigma}{2}}},$$

where we made use again of (5.19)–(5.20). □

The following Lemma is the deterministic version of [8, Lemma 5.1], and it is meant to provide a bound on the truncated off-diagonal interaction part in terms of the modulated energy.

**Lemma 5.5.** *Under the same assumptions and notations of Lemma 5.2, let $\mu \in L^1 \cap L^\infty(\mathbb{R}^d)$ be a probability density and let $\mu_N$ be an empirical measure defined by (1.3) associated to any $\mathbf{x}_N \in \mathbb{R}^{dN} \setminus \triangle_N$ and $\mathbf{m}_N \in \mathbb{M}^N$ such that $m_i^N \le \overline{M}$ for some $\overline{M} > 0$. Then, there is a constant $C = C(\|\mu\|_\infty, d, k, p, q)$ such that the renormalized energy (2.5) can be estimated from below as*

$$\mathcal{E}_N(\mu, \mu_N) = \int_{x \neq y} V(x-y)\,(\mu_N - \mu)^{\otimes 2}\,(dxdy) \ge -\frac{\overline{M}\,\|V_\varepsilon\|_\infty}{N} - C\left(\frac{\varepsilon}{\delta^{k+1-\frac{d}{q}}} + \delta^{\frac{d}{p}-k} + o_\varepsilon(1)\right) \tag{5.32}$$

*and*

$$\frac{1}{N^2}\sum_{i \neq j} m_i m_j (1-\zeta_\delta) V(x_i - x_j) \le C\,(\mathcal{E}_N(\mu, \mu_N) + \mathcal{O}(\varepsilon, \delta, N)) \tag{5.33}$$

*where $\mathcal{O}$ has the form*

$$\mathcal{O}(\varepsilon, \delta, N) = \frac{\varepsilon}{\delta^{k+1-\frac{d}{q}}} + \delta^{d-k-\frac{\sigma}{2}} o_{\varepsilon\delta^{-1}}(1) + \frac{\overline{M}\,\|V_\varepsilon\|_\infty}{N} + \delta^{\frac{d}{p}-k}. \tag{5.34}$$

*Proof.* **Step 1**. In this step we prove (5.32). We denote by $\lesssim$ inequality up to a constant depending only on $\|\mu\|_\infty, d, k, p, q$. We have

$$\begin{aligned}
&\int_{x\neq y} V(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy) \\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d} \zeta_\delta(x-y) V(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy) \\
&\quad + \int_{x\neq y} (1-\zeta_\delta(x-y))\,V(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy).
\end{aligned} \tag{5.35}$$

Observing that

$$\|((1-\zeta_\delta)V) \star \mu\|_\infty \lesssim \delta^{\frac{d-pk}{p}} \tag{5.36}$$

by Hölder's inequality, we see that the second integral in the right-hand side is

$$\begin{aligned}
&\int_{x\neq y} (1-\zeta_\delta(x-y))\,V(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy) \\
&= \frac{1}{N^2}\sum_{i\neq j} m_i m_j V(x_i - x_j)\big(1-\zeta_\delta(x_i-x_j)\big) \\
&- \frac{2}{N}\,[i= 1]N\sum m_i((1-\zeta_\delta)V) \star \mu(x_i) \\
&- \int_{\mathbb{R}^d} \mu(x)((1-\zeta_\delta)V) \star \mu(x)dx \\
\ge\ & \frac{1}{N^2}\sum_{i\neq j} m_i m_j V(x_i - x_j)\big(1-\zeta_\delta(x_i-x_j)\big) \\
&- C(\|\mu\|_\infty, d, k, p)\delta^{\frac{d-pk}{p}}.
\end{aligned}$$

Hence, in view of Lemma 5.2-iv, we get

$$\int_{x\neq y}(1-\zeta_\delta(x-y))V(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy)$$
$$\geq \frac{1}{N^2}\sum_{i\neq j} m_i m_j V_\varepsilon(x_i-x_j)\left(1-\zeta_\delta(x_i-x_j)\right) - C(\|\mu\|_\infty, d, k, p)\delta^{\frac{d-pk}{p}} - \varepsilon. \tag{5.37}$$

Moreover, Lemma 5.2-iii entails

$$\begin{aligned}
&\|(\zeta_\delta V)\star\mu - (\zeta_\delta V_\varepsilon)\star\mu\|_\infty \\
&\leq \|(\zeta_\delta(V_\varepsilon - V))\star\mu\|_\infty \\
&\lesssim \|\zeta_\delta(V_\varepsilon - V)\|_{L^p+L^q} \lesssim \frac{\varepsilon}{\delta^{k+1-\frac{d}{q}}} + \delta^{\frac{d}{p}-k},
\end{aligned}$$

so that we can bound from below the first integral in (5.35) as

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbb{R}^d}\zeta_\delta(x-y)V(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy) \\
&\geq \int_{\mathbb{R}^d\times\mathbb{R}^d}\zeta_\delta(x-y)V_\varepsilon(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy) \\
&\quad - C(\|\mu\|_\infty, d, k, p)\left(\frac{\varepsilon}{\delta^{k+1-\frac{d}{q}}} + \delta^{\frac{d}{p}-k}\right) - \varepsilon,
\end{aligned} \tag{5.38}$$

using again Lemma 5.2-iv. Hence summing up inequalities (5.37) and (5.38) we get

$$\begin{aligned}
&\int_{x\neq y} V(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy) \geq \\
&\quad \int_{\mathbb{R}^d\times\mathbb{R}^d}\zeta_\delta(x-y)V_\varepsilon(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy) \\
&\quad + \frac{1}{N^2}\sum_{i\neq j} m_i m_j V_\varepsilon(x_i-x_j)\left(1-\zeta_\delta(x_i-x_j)\right) \\
&\quad - C(\|\mu\|_\infty, d, k, p)\left(\frac{\varepsilon}{\delta^{k+1-\frac{d}{q}}} + \delta^{\frac{d}{p}-k} + \varepsilon\right).
\end{aligned} \tag{5.39}$$

Notice that the second term in the right-hand side can be rewritten as

$$\begin{aligned}
&\frac{1}{N^2}\sum_{i\neq j} m_i m_j V_\varepsilon(x_i-x_j)\left(1-\zeta_\delta(x_i-x_j)\right) = \int_{\mathbb{R}^d\times\mathbb{R}^d}(1-\zeta_\delta(x-y))V_\varepsilon(x-y)(\mu_N-\mu)^{\otimes 2}(dxdy) \\
&- \frac{1}{N^2}\sum_{j=1}^N m_j^2 V_\varepsilon(0) - \int_{\mathbb{R}^d}((1-\zeta_\delta)V_\varepsilon)\star\mu(x)\mu(x)dx \\
&+ \frac{2}{N}\sum_{j=1}^N m_j((1-\zeta_\delta)V_\varepsilon)\star\mu(x_j).
\end{aligned} \tag{5.40}$$

Using Lemma 5.1-ii and (5.36), we have

$$\|((1-\zeta_\delta)V_\varepsilon)\star\mu\|_\infty \le \|((1-\zeta_\delta)(V_\varepsilon - V))\star\mu\|_\infty + \|((1-\zeta_\delta)V)\star\mu\|_\infty \\ \lesssim \|V_\varepsilon - V\|_{L^p+L^q} + \delta^{\frac{d}{p}-k} \lesssim \eta(\varepsilon) + \delta^{\frac{d}{p}-k}.$$

Inserting this into (5.40) together with a trivial estimate on the second term of the right-hand side of (5.40) we obtain

$$\frac{1}{N^2}\sum_{i\neq j} m_i m_j V_\varepsilon(x_i - x_j)\left(1-\zeta_\delta(x_i-x_j)\right) \ge \int_{\mathbb{R}^d\times\mathbb{R}^d} (1-\zeta_\delta(x-y))V_\varepsilon(x-y)(\mu_N-\mu)^{\otimes 2}(dxdy) \\ - \frac{\overline{M}\,\|V_\varepsilon\|_\infty}{N} - C(\|\mu\|_\infty, d, k, p)\left(\delta^{\frac{d}{p}-k} + \eta(\varepsilon)\right). \tag{5.41}$$

We finally substitute (5.41) into (5.39) to conclude that

$$\int_{x\neq y} V(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy) \ge \int_{\mathbb{R}^d\times\mathbb{R}^d} V_\varepsilon(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy) - \frac{\overline{M}\,\|V_\varepsilon\|_\infty}{N} \\ - C(\|\mu\|_\infty, d, k, p)\left(\frac{\varepsilon}{\delta^{k+1-\frac{d}{q}}} + \delta^{\frac{d}{p}-k} + \eta(\varepsilon)\right). \tag{5.42}$$

By Lemma 5.2 we have $\widehat{V_\varepsilon} \ge 0$ so that in view of (5.42) we obtain

$$\int_{x\neq y} V(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy) \ge -\frac{\overline{M}\,\|V_\varepsilon\|_\infty}{N} \\ - C(\|\mu\|_\infty, d, k, p)\left(\frac{\varepsilon}{\delta^{k+1-\frac{d}{q}}} + \delta^{\frac{d}{p}-k} + \eta(\varepsilon)\right),$$

which is the desired inequality (5.32).

**Step 2**. We now prove (5.33). We follow the same procedure as in (5.40) reconstructing the tensor measure $(\mu_N-\mu)^{\otimes 2}$ together with (5.36) to deduce

$$\frac{1}{N^2}\sum_{i\neq j} m_i m_j (1-\zeta_\delta)(x_i-x_j)V(x_i-x_j) \le \int_{x\neq y} (1-\zeta_\delta)(x-y)V(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy) \\ + C(k,d,\|\mu\|_\infty)\delta^{\frac{d-kp}{p}} \\ = \int_{x\neq y} V(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy) \\ - \int_{x\neq y} \zeta_\delta(x-y)V(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy) \\ + C(k,d,\|\mu\|_\infty)\delta^{\frac{d-kp}{p}}, \tag{5.43}$$

Notice that there is no diagonal term with respect to (5.40) since the integration domain avoids the diagonal. Inequality (5.38) can be written as

$$
\begin{aligned}
&-\int_{x\neq y} \zeta_\delta(x-y)V(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy) \leq -\int_{\mathbb{R}^d\times\mathbb{R}^d} \zeta_\delta(x-y)V_\varepsilon(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy) \\
&+ C(\|\mu\|_\infty, d, k, p)\left(\frac{\varepsilon}{\delta^{k+1-\frac{d}{q}}} + \delta^{\frac{d}{p}-k}\right) + \varepsilon \\
&\leq \int_{\mathbb{R}^d\times\mathbb{R}^d} (1-\zeta_\delta)(x-y)V_\varepsilon(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy) \\
&+ C(\|\mu\|_\infty, d, k, p)\left(\frac{\varepsilon}{\delta^{k+1-\frac{d}{q}}} + \delta^{\frac{d}{p}-k}\right) + \varepsilon.
\end{aligned} \tag{5.44}
$$

By Corollary 5.4 we have $\widehat{(1-\zeta_\delta)V_\varepsilon} \lesssim \widehat{V_\varepsilon} + \frac{\delta^{d-k-\frac{\sigma}{2}} o_{\varepsilon\delta^{-1}}(1)}{(1+|\xi|)^{\frac{\sigma}{2}}}$ so that

$$
\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbb{R}^d} (1-\zeta_\delta)(x-y)V_\varepsilon(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy) \lesssim \int_{\mathbb{R}^d\times\mathbb{R}^d} V_\varepsilon(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy) \\
&+ \delta^{d-k-\frac{\sigma}{2}} o_{\varepsilon\delta^{-1}}(1)\int_{\mathbb{R}^d} \frac{|1-\widehat{\mu}|^2(\xi)}{(1+|\xi|)^{\frac{\sigma}{2}}}d\xi \\
&\lesssim \int_{\mathbb{R}^d\times\mathbb{R}^d} V_\varepsilon(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy) + \delta^{d-k-\frac{\sigma}{2}} o_{\varepsilon\delta^{-1}}(1) \\
&\lesssim \int_{x\neq y} V_\varepsilon(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy) \\
&+ \frac{\overline{M}\,\|V_\varepsilon\|_\infty}{N} + \delta^{d-k-\frac{\sigma}{2}} o_{\varepsilon\delta^{-1}}(1).
\end{aligned} \tag{5.45}
$$

Notice that this passage to Fourier variables is justified since we work with the regular interaction potential $V_\varepsilon$ instead of $V$. By Lemma 5.2, we obtain

$$
\|V_\varepsilon \star \mu - V \star \mu\|_\infty \lesssim \|V_\varepsilon - V\|_{L^p+L^q(\mathbb{R}^d)} = \eta(\varepsilon). \tag{5.46}
$$

Gathering (5.44), (5.45), and (5.46) we find

$$
\begin{aligned}
&-\int_{x\neq y} \zeta_\delta(x-y)V(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy) \\
&\quad\lesssim \int_{x\neq y} V_\varepsilon(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy) + \frac{\overline{M}\,\|V_\varepsilon\|_\infty}{N} \\
&\qquad + C(\|\mu\|_\infty, d, k, p)\left(\frac{\varepsilon}{\delta^{k+1-\frac{d}{q}}} + \delta^{\frac{d}{p}-k} + \delta^{d-k-\frac{\sigma}{2}} o_{\varepsilon\delta^{-1}}(1)\right) \\
&\quad\lesssim \int_{x\neq y} (V_\varepsilon - V)(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy) + \mathcal{E}_N(\mu,\mu_N) + \frac{\overline{M}\,\|V_\varepsilon\|_\infty}{N} \\
&\qquad + C(\|\mu\|_\infty, d, k, p)\left(\frac{\varepsilon}{\delta^{k+1-\frac{d}{q}}} + \delta^{\frac{d}{p}-k} + \delta^{d-k-\frac{\sigma}{2}} o_{\varepsilon\delta^{-1}}(1)\right).
\end{aligned}
$$

Moreover, Lemmas 5.2-ii and 5.2-iv using (5.46) implies that

$$
\int_{x\neq y} (V_\varepsilon - V)(x-y)\,\left(\mu_N-\mu\right)^{\otimes 2}(dxdy) \lesssim o_\varepsilon(1) + \int_{x\neq y} (V_\varepsilon - V)(x-y)\mu_N^{\otimes 2}(dxdy) \lesssim o_\varepsilon(1) + \varepsilon.
$$

Therefore, in view of (5.43) we obtain

$$\frac{1}{N^2}\sum_{i\neq j} m_i m_j (1-\zeta_\delta)V(x_i-x_j) \leq C(k,d,p,q,\|\mu\|_\infty)\left(\mathcal{E}_N(\mu,\mu_N)+\mathcal{O}(\varepsilon,\delta,N)\right)$$

with

$$\mathcal{O}(\varepsilon,\delta,N)=\frac{\varepsilon}{\delta^{k+1-\frac{d}{q}}}+\delta^{d-k-\frac{\sigma}{2}}o_{\varepsilon\delta^{-1}}(1)+\frac{\overline{M}\,\|V_\varepsilon\|_\infty}{N}+\delta^{\frac{d}{p}-k},$$

as desired. □

We are now ready to estimate $\mathcal{D}_N^2$ in (2.11) by means of $\mathcal{E}_N(\mu,\mu_N)$ in (2.6), which is the main novelty of this section.

**Lemma 5.6.** *Under the assumptions of Lemma 5.5, it holds that*

$$\mathcal{D}_N^2(\mu)=\int_{x\neq y} V(x-y)\left(\mu_N-\mu\right)(x)\left(h\left[\mu_N\right]-h\left[\mu\right]\right)(y)(dxdy) \leq C\left(\mathcal{E}_N(\mu,\mu_N)+\eta(\varepsilon)+\mathcal{O}(\varepsilon,\delta,N)\right),$$

*where $\mathcal{O}(\varepsilon,\delta,N)$ is given by (5.34).*

*Proof.* Let $\zeta_\delta$ be a function as in Lemma 5.5. We split the integral into a near and far parts from the origin as follows

$$\begin{aligned}\mathcal{D}_N^2(\mu)=&\int_{\mathbb{R}^d\times\mathbb{R}^d}(\zeta_\delta V)(x-y)\left(\mu_N-\mu\right)(x)\left(h\left[\mu_N\right]-h\left[\mu\right]\right)(y)dxdy\\ &+\int_{x\neq y}((1-\zeta_\delta)V)(x-y)\left(\mu_N-\mu\right)(x)\left(h\left[\mu_N\right]-h\left[\mu\right]\right)(y)dxdy\\ :=&I_\delta+J_\delta.\end{aligned}$$

We denote by $\lesssim$ inequality up to a constant depending on $d,k,p,q,\|S\|_\infty,\|\mu\|_\infty$.
**Step 1**. *The integral* $J_\delta$. We expand the integral as

$$\begin{aligned}J_\delta=&\int_{x\neq y}((1-\zeta_\delta)V)(x-y)\mu_N(x)h\left[\mu_N\right](y)dxdy\\ &-\frac{1}{N}\,[\,i=1]N\sum m_i\left(((1-\zeta_\delta)V)\right)\star\mu)(x_i)\left(S\star\mu_N\right)(x_i)\\ &-\frac{1}{N}\,[\,i=1]N\sum m_i\left(((1-\zeta_\delta)V)\right)\star h\left[\mu\right])(x_i)\\ &+\int_{\mathbb{R}^d}\mu(x)(((1-\zeta_\delta)V)\star h\left[\mu\right])(x)dx:=[\,i=1]4\sum J_\delta^i.\end{aligned}$$

Owing to Lemma 5.5, we have

$$\left|J_\delta^1\right|\leq\|S\|_\infty\frac{1}{N^2}\sum_{i\neq j}m_i m_j(1-\zeta_\delta)(x_i-x_j)V(x_i-x_j)\leq C\left(\mathcal{E}_N(\mu,\mu_N)+\mathcal{O}(\varepsilon,\delta,N)\right).$$

Next, using similar computations as in (5.21), we deduce the following elementary bound

$$\|((1-\zeta_\delta)V)\star\mu\|_\infty\leq\|(1-\zeta_\delta)V\|_1\,\|\mu\|_\infty\lesssim\delta^{d-k}, \tag{5.47}$$

which together with the assumption (**H1**) directly implies that $\left|J_\delta^2\right| \lesssim \delta^{d-k}$. Similarly, we have that

$$\|((1-\zeta_\delta)V) \star h\,[\mu]\|_\infty \leq \|(1-\zeta_\delta)V\|_1 \,\|h\,[\mu]\|_\infty \lesssim \delta^{d-k},$$

yielding that $\left|J_\delta^3\right| \lesssim \delta^{d-k}$ and $\left|J_\delta^4\right| \lesssim \delta^{d-k}$. Collecting these estimates, we find

$$J_\delta \lesssim \mathcal{E}_N(\mu, \mu_N) + \mathcal{O}(\varepsilon, \delta, N) + \delta^{d-k}. \tag{5.48}$$

To estimate $I_\delta$, we need to use the regularization $V_\varepsilon$ constructed in Lemma 5.2. Note the identity

$$\begin{aligned} I_\delta = &\int_{\mathbb{R}^d\times\mathbb{R}^d} \zeta_\delta(x-y)(V-V_\varepsilon)(x-y)\,(\mu_N-\mu)\,(x)\left((h\left[\mu_N\right] - \|S\|_\infty\,\mu_N)\right.\\ &\left.-(h\,[\mu]-\|S\|_\infty\,\mu))\,(y)dxdy\right.\\ &+\int_{\mathbb{R}^d\times\mathbb{R}^d} \zeta_\delta(x-y)V_\varepsilon(x-y)\,(\mu_N-\mu)\,(x)\left(h\left[\mu_N\right]-h\,[\mu]\right)(y)dxdy\\ &+\|S\|_\infty\int_{\mathbb{R}^d\times\mathbb{R}^d} \zeta_\delta(x-y)(V-V_\varepsilon)(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy) := I^1_{\delta,\varepsilon} + I^2_{\delta,\varepsilon} + I^3_{\delta,\varepsilon}. \end{aligned}$$

In what follows, we separately estimate $I^1_{\delta,\varepsilon}$, $I^2_{\delta,\varepsilon}$, $I^3_{\delta,\varepsilon}$.

**Step 2**. *The integral* $I^1_{\delta,\varepsilon}$. The term $I^1_{\delta,\varepsilon}$ is recast as

$$\begin{aligned} I^1_{\delta,\varepsilon} = &\int_{\mathbb{R}^d\times\mathbb{R}^d} \zeta_\delta(x-y)(V_\varepsilon - V)(x-y)\mu_N(x)(\|S\|_\infty\,\mu_N - h\,[\mu_N])(y)dxdy\\ &+\frac{1}{N}\,[i=1]N\sum m_i((\zeta_\delta(V_\varepsilon - V))\star(h\,[\mu]-\|S\|_\infty\,\mu))(x_i)\\ &+\frac{1}{N}\,[i=1]N\sum m_i((\zeta_\delta(V_\varepsilon - V))\star\mu)(x_i)(\|S\|_\infty - (S\star\mu_N)(x_i))\\ &+\int_{\mathbb{R}^d}((\zeta_\delta(V_\varepsilon - V))\star\mu)(x)(\|S\|_\infty\,\mu - h\,[\mu])(x)dx. \end{aligned} \tag{5.49}$$

As $\|S\|_\infty\,\mu_N - h[\mu_N]$ is a non-negative measure, by Lemma 5.2-iv, the 1st term in the right-hand side of (5.49) is bounded by

$$\varepsilon\int_{\mathbb{R}^d} \zeta_\delta(x-y)\mu_N(x)(\|S\|_\infty\,\mu_N - h\,[\mu_N])dxdy \leq 2\,\|S\|_\infty\,\varepsilon.$$

Utilizing Lemma 5.2-iii. we see that the 2nd term in the right-hand side of (5.49) is less than

$$\begin{aligned} &\|(\zeta_\delta(V_\varepsilon - V))\star(h\,[\mu]-\|S\|_\infty\,\mu)\|_\infty\\ &\leq \|\zeta_\delta(V_\varepsilon - V)\|_{L^p+L^q}\,\|h\,[\mu]-\|S\|_\infty\,\mu\|_\infty \leq 2C\,\|\mu\|_\infty\,\|S\|_\infty\left(\frac{\varepsilon}{\delta^{k+1-\frac{d}{q}}} + \delta^{\frac{d}{p}-k}\right). \end{aligned}$$

By the same token, the 3rd term in the right hand side of (5.49) is bounded by

$$2\,\|S\|_\infty\,\|\zeta_\delta(V_\varepsilon - V)\star\mu\|_\infty \leq 2\,\|S\|_\infty\,\|\mu\|_\infty\,\|\zeta_\delta(V_\varepsilon - V)\|_{L^p+L^q} \leq 2C\,\|\mu\|_\infty\,\|S\|_\infty\left(\frac{\varepsilon}{\delta^{k+1-\frac{d}{q}}} + \delta^{\frac{d}{p}-k}\right),$$

and the 4th term is dominated by

$$2\left\|S\right\|_{\infty}\left\|\mu\right\|_{\infty}\left\|\zeta_{\delta}(V_{\varepsilon}-V)\right\|_{L^{p}+L^{q}}\leq 2C\left\|\mu\right\|_{\infty}\left\|S\right\|_{\infty}\left(\frac{\varepsilon}{\delta^{k+1-\frac{d}{q}}}+\delta^{\frac{d}{p}-k}\right).$$

Gathering our inequalities, we find that

$$I^{1}_{\delta,\varepsilon}\leq C\left(\frac{\varepsilon}{\delta^{k+1-\frac{d}{q}}}+\delta^{\frac{d}{p}-k}\right)$$

where $C=C\left(\left\|S\right\|_{\infty},\left\|\mu\right\|_{\infty},d,k,p,q\right)$.

**Step 3**. *The integral* $I^{2}_{\delta,\varepsilon}$. Note first that

$$\begin{aligned}I^{2}_{\delta,\varepsilon}&=\int_{\mathbb{R}^{d}}\zeta_{\delta}V_{\varepsilon}(x-y)\left(\mu_{N}-\mu\right)(x)\left(h\left[\mu_{N}\right]-h\left[\mu\right]\right)(y)dxdy\\&=\int_{\mathbb{R}^{d}}\widehat{\zeta_{\delta}V_{\varepsilon}}(x)\left(\widehat{\mu_{N}-\mu}\right)(x)\left(\widehat{h\left[\mu_{N}\right]-h\left[\mu\right]}\right)(x)dx.\end{aligned}$$

Therefore, the Cauchy-Schwarz inequality yields

$$I^{2}_{\delta,\varepsilon}\leq\int_{\mathbb{R}^{d}}\left|\widehat{\zeta_{\delta}V_{\varepsilon}}\right|\left|\widehat{\mu_{N}-\mu}\right|^{2}(x)dx+\int_{\mathbb{R}^{d}}\left|\widehat{\zeta_{\delta}V_{\varepsilon}}\right|\left|\widehat{h\left[\mu_{N}\right]-h\left[\mu\right]}\right|^{2}(x)dx. \tag{5.50}$$

Now, we want to apply Lemma 3.6 to the second term on the right-hand side of (5.50). The first two assumptions are satisfied by the choice of $S$ in **(H1)**. We are reduced to check the third assumption in Lemma 3.6. By Corollary 5.4 it holds that

$$\left|\widehat{\zeta_{\delta}V_{\varepsilon}}\right|(\xi)\lesssim\widehat{V}_{\varepsilon}(\xi)+\frac{\delta^{d-k-\frac{\sigma}{2}}o_{\varepsilon\delta^{-1}}(1)}{(1+|\xi|)^{\frac{\sigma}{2}}}. \tag{5.51}$$

for $\sigma>2d$. By Lemma 5.3 we have

$$\left|\widehat{S}\right|\star\widehat{V}_{\varepsilon}(x)\lesssim\widehat{V}_{\varepsilon}(x)+\frac{o_{\varepsilon}(1)}{(1+|x|)^{\frac{\sigma}{2}}},$$

while by the assumption that $S\in\mathcal{S}(\mathbb{R}^{d})$ we have

$$\left|\widehat{S}\right|\star\left(\frac{1}{(1+|\cdot|)^{\frac{\sigma}{2}}}\right)\lesssim\frac{1}{(1+|x|)^{\frac{\sigma}{2}}}.$$

Thus, we conclude that

$$\left|\widehat{S}\right|\star\left(\widehat{V}_{\varepsilon}+\frac{o_{\varepsilon\delta^{-1}}(1)}{(1+|\cdot|)^{\frac{\sigma}{2}}}\right)(x)\lesssim\widehat{V}_{\varepsilon}(x)+\frac{\delta^{d-k-\frac{\sigma}{2}}o_{\varepsilon\delta^{-1}}(1)}{(1+|x|)^{\frac{\sigma}{2}}}$$

leading to the desired convolution inequality in Lemma 3.6. So, we are now entitled to apply Lemma 3.6 with $W(x) = \widehat{V}_\varepsilon(x) + \frac{\delta^{d-k-\frac{\sigma}{2}} o_{\varepsilon\delta-1}(1)}{(1+|x|)^{\frac{\sigma}{2}}}$ to obtain that

$$\int_{\mathbb{R}^d} \left|\widehat{\zeta_\delta V_\varepsilon}\right| \left|h\left[\widehat{\mu_N] - h}\,[\mu]\right|^2 (x)dx \leq C \int_{\mathbb{R}^d} \left(\widehat{V}_\varepsilon(x) + \frac{\delta^{d-k-\frac{\sigma}{2}} o_{\varepsilon\delta-1}(1)}{(1+|x|)^{\frac{\sigma}{2}}}\right) \left|\widehat{\mu_N - \mu}\right|^2 (x)dx,$$

which proves that

$$I^2_{\delta,\varepsilon} \lesssim \int_{\mathbb{R}^d} \left(\widehat{V}_\varepsilon(x) + \frac{\delta^{d-k-\frac{\sigma}{2}} o_{\varepsilon\delta-1}(1)}{(1+|x|)^{\frac{\sigma}{2}}}\right) \left|\widehat{\mu_N - \mu}\right|^2 (x)dx.$$

Hence, similar to Step 2 in Lemma 5.5, we arrive at the estimate

$$\begin{aligned} I^2_{\delta,\varepsilon} &\lesssim \int_{x\neq y} V_\varepsilon(x-y)\left(\mu_N - \mu\right)^{\otimes 2}(dxdy) + \frac{\overline{M}\,\|V_\varepsilon\|_\infty}{N} + \delta^{d-k-\frac{\sigma}{2}} o_{\varepsilon\delta-1}(1) \\ &\lesssim \int_{x\neq y} \left(V_\varepsilon(x-y) - V(x-y)\right)\left(\mu_N - \mu\right)^{\otimes 2}(dxdy) + \mathcal{E}_N(\mu,\mu_N) + \frac{\overline{M}\,\|V_\varepsilon\|_\infty}{N} + \delta^{d-k-\frac{\sigma}{2}} o_{\varepsilon\delta-1}(1). \end{aligned} \tag{5.52}$$

To finish, we estimate the first integral in the right-hand side of (5.52), which is achieved by similar considerations to those in Step 2. Expanding the terms break$(\mu_N - \mu)^{\otimes 2}(dxdy)$ and using the fourth and the second statements in Lemma 5.2, we have

$$\frac{1}{N^2}\sum_{i\neq j} m_i m_j \left(V_\varepsilon(x_i - x_j) - V(x_i - x_j)\right) \leq \varepsilon,$$

$$\left|\frac{2}{N}\, \overset{[}{i}= 1]N \sum m_i (V_\varepsilon - V) \star \mu(x_i)\right| \leq \frac{2}{N}\, \overset{[}{i}= 1]N \sum \|(V_\varepsilon - V)\star\mu\|_\infty$$
$$\leq \|\mu\|_\infty \|V_\varepsilon - V\|_{L^p+L^q} \lesssim \eta(\varepsilon),$$

and

$$\int_{\mathbb{R}^d} (V_\varepsilon - V)\star\mu(x)\mu(x)dx \leq \|(V_\varepsilon - V)\star\mu\|_\infty \leq \|\mu\|_\infty \|V_\varepsilon - V\|_{L^p+L^q} \lesssim \eta(\varepsilon).$$

Collecting terms implies that

$$\int_{x\neq y} (V_\varepsilon - V)(x-y)\left(\mu_N - \mu\right)^{\otimes 2}(dxdy) \lesssim \varepsilon + \eta(\varepsilon). \tag{5.53}$$

Thus, (5.52) and (5.53) entail

$$I^2_{\delta,\varepsilon} \leq C\left(\mathcal{E}_N(\mu,\mu_N) + \frac{\overline{M}\,\|V_\varepsilon\|_\infty}{N} + \delta^{d-k-\frac{\sigma}{2}} o_{\varepsilon\delta^{-1}}(1)\right),$$

where $C = C\left(\|S\|_\infty, \|\mu\|_\infty, d, k, p, q\right)$.

**Step 4**. We immediately have that

$$
\begin{aligned}
I^3_{\delta,\varepsilon} = \|S\|_\infty \int_{\mathbb{R}^d\times\mathbb{R}^d} \zeta_\delta(x-y)V(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy) \\
- \|S\|_\infty \int_{\mathbb{R}^d\times\mathbb{R}^d} \zeta_\delta(x-y)V_\varepsilon(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy).
\end{aligned}
$$

First, note that

$$
\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbb{R}^d} \zeta_\delta(x-y)V(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy) \\
&= \int_{x\neq y} \zeta_\delta(x-y)V(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy) \\
&= \mathcal{E}_N(\mu,\mu_N) - \frac{1}{N^2}\sum_{i\neq j} m_i m_j ((1-\zeta_\delta)V)(x_i-x_j) \\
&- \int_{\mathbb{R}^d} \mu(x)(((1-\zeta_\delta)V)\star\mu)(x)dx \\
&- \frac{1}{N} \overset{[}{i}= 1]N\sum m_i(((1-\zeta_\delta)V)\star\mu)(x_i) \\
&\leq C\left(\mathcal{E}_N(\mu,\mu_N) + \mathcal{O}(\varepsilon,\delta,N) + \delta^{d-k}\right),
\end{aligned}
$$

where the last inequality uses Lemma 5.5 to estimate the second term and the observation that $\|(1-\zeta_\delta)V\|_1 \lesssim \delta^{d-k}$ as in (5.47). Second, by Corollary 5.4 it holds that

$$
\left|\widehat{\zeta_\delta V_\varepsilon}\right| \lesssim \widehat{V_\varepsilon} + \frac{\delta^{d-k-\frac{\sigma}{2}} o_{\varepsilon\delta^{-1}}(1)}{(1+|\xi|)^{\frac{\sigma}{2}}},
$$

so that we get

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^d} \zeta_\delta V_\varepsilon(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy)\right| = \left|\int_{\mathbb{R}^d} \widehat{\zeta_\delta V_\varepsilon}(x)\left|\widehat{\mu_N-\mu}\right|^2(x)dx\right| \\
&\lesssim \int_{\mathbb{R}^d} \widehat{V_\varepsilon}(x)\left|\widehat{\mu_N-\mu}\right|^2(x)dx + \delta^{d-k-\frac{\sigma}{2}} o_{\varepsilon\delta^{-1}}(1).
\end{aligned}
$$

By (5.53) of step 3, it follows that

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^d} \zeta_\delta V_\varepsilon(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy)\right| \lesssim \int_{x\neq y} (V_\varepsilon - V)(x-y)(\mu_N-\mu)^{\otimes 2}(dxdy) \\
&+ \frac{\overline{M}\,\|V_\varepsilon\|_\infty}{N} + \mathcal{E}_N(\mu,\mu_N) + \delta^{d-k-\frac{\sigma}{2}} o_{\varepsilon\delta^{-1}}(1) \\
&\lesssim \mathcal{E}_N(\mu,\mu_N) + \varepsilon + \eta(\varepsilon) + \frac{\overline{M}\,\|V_\varepsilon\|_\infty}{N} + \delta^{d-k-\frac{\sigma}{2}} o_{\varepsilon\delta^{-1}}(1).
\end{aligned}
$$

This proves that

$$
I^3_{\delta,\varepsilon} \lesssim \mathcal{E}_N(\mu,\mu_N) + \varepsilon + \eta(\varepsilon) + \frac{\overline{M}\,\|V_\varepsilon\|_\infty}{N} + \delta^{d-k-\frac{\sigma}{2}} o_{\varepsilon\delta^{-1}}(1) + \delta^{d-k} + \mathcal{O}(\varepsilon,\delta,N).
$$

Collecting all previous terms $I^1_{\delta,\varepsilon}$, $I^2_{\delta,\varepsilon}$ and $I^3_{\delta,\varepsilon}$, we find that

$$I_\delta \lesssim \mathcal{E}_N(\mu,\mu_N) + \varepsilon + \eta(\varepsilon) + \frac{\varepsilon}{\delta^{k+1-\frac{d}{q}}} + \delta^{\frac{d}{p}-k} + \frac{\overline{M}\,\|V_\varepsilon\|_\infty}{N} \\ + \delta^{d-k-\frac{\sigma}{2}} o_{\varepsilon\delta^{-1}}(1) + \delta^{d-k} + \mathcal{O}(\varepsilon,\delta,N).$$

Combining estimate (5.54) with (5.48) and taking into account the definition of $\mathcal{O}(\varepsilon,\delta,N)$ in (5.34), we deduce that

$$\mathcal{D}^2_N(\mu) = I_\delta + J_\delta \lesssim \mathcal{E}_N(\mu,\mu_N) + \eta(\varepsilon) + \mathcal{O}(\varepsilon,\delta,N), \tag{5.54}$$

since we will eventually take $\delta \to 0$, $p > 1$ and $k - \frac{d}{q} > 0$, and thus we conclude the announced result. □

In order to prove the mean field limit, we need the asymptotic positivity, coercivity inequalities and the commutator estimate, which are the fundamental discoveries in [14].

**Proposition 5.7.** ([14, Corollary 3.5]) *Let $V(x) = \frac{1}{|x|^k}$ where $0 < k < d-1$ and let $\mu \in L^\infty(\mathbb{R}^d) \cap \mathcal{P}(\mathbb{R}^d)$. For any weights $\mathbf{m}_N \in \mathbb{M}^N$ with $m_i^N \le \overline{M}$, $i = 1,\dots,N$ and configurations $\mathbf{x}_N \in \mathbb{R}^{dN}\backslash\triangle_N$, there exist a constant $C = C(\overline{M}, \|\mu\|_\infty, k, d)$ such that the modulated energy is bounded below as*

$$\mathcal{E}_N(\mu,\mu_N) \ge -C(1+\|\mu\|_\infty)N^{\frac{k}{d}-1}$$

*and thus, asymptotically non-negative.*

**Proposition 5.8.** [14, Proposition 3.6] *Let $V(x) = \frac{1}{|x|^k}$ where $0 < k < d-1$ and let $\mu \in L^\infty(\mathbb{R}^d) \cap \mathcal{P}(\mathbb{R}^d)$. For any weights $\mathbf{m}_N \in \mathbb{M}^N$ with $m_i^N \le \overline{M}$, $i = 1,\dots,N$ and configurations $\mathbf{x}_N \in \mathbb{R}^{dN}\backslash\triangle_N$ and $0 < \alpha \le 1$ there is some $C = C(d,k,\overline{M}) > 0$ and $\lambda = \lambda(d,k) > 0$ such that for any $\varphi \in C^\infty(\mathbb{R}^d)$ it holds that*

$$\int_{\mathbb{R}^d} \varphi(x)\,(\mu_N - \mu)\,(x)(dx) \le CN^{-\lambda}\,\|\varphi\|_{C^{0,\alpha}} \\ + C\,\|\varphi\|_{\dot{H}^{\frac{d-k}{2}}} \left(\mathcal{E}_N(\mu,\mu_N) + C(1+\|\mu\|_\infty)N^{\frac{k}{d}-1}\right)^{\frac{1}{2}}.$$

**Theorem 5.9.** ([14, Proposition 1.1]) *Let the assumptions of Proposition 5.8 hold and assume further that $\mu \in C^{0,\alpha}(\mathbb{R}^d)$ for some $\alpha \in (0,1)$ and that $u : \mathbb{R}^d \to \mathbb{R}^d$ is a Lipschitz vector field. Then*

$$\left|\int_{x\neq y} (u(x)-u(y))\,\nabla V(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy)\right| \\ \le C\left(\int_{x\neq y} V(x-y)\,(\mu_N-\mu)^{\otimes 2}\,(dxdy) + N^{-\beta}\right)$$

*where $C = C\left(\overline{M}, \|u\|_{W^{1,\infty}}, \|\mu\|_{C^{0,\alpha}}, d, k\right)$ and $\beta = \beta(k,d,\alpha) > 0$.*

We are now in a good position to prove our main theorem.

*Proof of Theorem 1.2.* By Lemma 5.6 we have

$$\mathcal{D}_N^2(t) \lesssim \mathcal{E}_N(t) + \eta(\varepsilon) + \mathcal{O}(\varepsilon, \delta, N)$$

where $\mathcal{O}$ has the form

$$\mathcal{O}(\varepsilon, \delta, N) = \frac{\varepsilon}{\delta^{k+1-\frac{d}{q}}} + \delta^{d-k-\frac{\sigma}{2}} o_{\varepsilon\delta^{-1}}(1) + \frac{\overline{M}\,\|V_\varepsilon\|_\infty}{N} + \delta^{\frac{d}{p}-k}.$$

Note that $\|V_\varepsilon\|_\infty \leq \frac{1}{\Theta(\varepsilon)}$ for some $\Theta(\varepsilon) \to 0$ as $\varepsilon \to 0$ due to Lemma 5.2-v. Choose $\varepsilon = \varepsilon(N) \to 0$ as $N \to \infty$ and choose $\delta = \delta(\varepsilon)$ such that

$$\max\left\{\eta(\varepsilon(N)), \frac{1}{\Theta(\varepsilon(N))N}, \frac{\varepsilon(N)}{\delta(\varepsilon(N))^{k+1-\frac{d}{q}}}\right\} \to 0 \text{ as } N \to \infty\,.$$

Then, we get

$$\mathcal{D}_N^2(t) \lesssim \mathcal{E}_N(t) + o_N(1),$$

since $p > 1$ $k - \frac{d}{q} > 0$. By Theorem 3.4 and the assumption $\mu_0 \in W^{1,\infty}(\mathbb{R}^d)$, we have $\mathbb{J}\nabla V \star \mu \in L^\infty([0, T], W^{1,\infty}(\mathbb{R}^d))$. Applying Theorem 5.9 with $u = \mathbb{J}\nabla V \star \mu$, we infer

$$\mathcal{D}_N^1(t) \lesssim \mathcal{E}_N(t) + N^{-\beta}$$

where $C = C(\|\mu_0\|_{W^{1,\infty}}, d, k)$. By Proposition 5.1 we conclude

$$\frac{d}{dt}\mathcal{E}_N(t) \leq \mathcal{D}_N^1(t) + \mathcal{D}_N^2(t) \leq C\mathcal{E}_N(t) + o_N(1).$$

By Grönwall's inequality and Proposition 5.7, it follows that $\sup_{t\in[0,T]} \mathcal{E}_N(t) \underset{N\to\infty}{\to} 0$, which by Proposition 5.8 concludes the proof. □

*Remark 5.10.* Proposition 5.7, Proposition 5.8 and Theorem 5.9 in [14] are all stated for the specific choice $\mathbf{m}_N = (1, \ldots, 1)$, but in fact hold for arbitrary convex combinations of Diracs, as can be deduced from a careful examination of the proof. Alternatively, this can be seen from Lemma 5.5 which we proved for general weights: Proposition 5.7 follows directly from this Lemma by choosing $\varepsilon$ and $\delta$ appropriately and Theorem 5.9 follows by this Lemma by exactly the same argument outlined in [8, Corollary 5.1]. Those relations form a critical example of a larger family of functional inequalities controlling various quantities by the modulated energy. Those inequalities could often be proved in a straightforward if the empirical measure was replaced by a smooth function but cannot make for general measures, which is why we constantly need to remove the diagonal here. Roughly speaking Lemma 5.5 allows to control a neighborhood of the diagonal in the modulated energy, and this is the key step in extending functional inequalities, such as given by Propositions 5.7, 5.8 and Theorem 5.9, to the empirical measure.

*Remark 5.11.* In accordance to [8], we expect that it is possible to cover the full range of Riesz interactions (i.e. allowing $d-1 \leq k < d$) in the statement of the main theorems. However, for what concerns the existence and uniqueness of (1.1) the choice of Coulomb interaction offers some simplifications, which make the proofs more self-contained. Moreover the asymptotic positivity, coercivity and commutator estimates estimates (Proposition 5.7–Theorem 5.9) require some adjustments when $d-1 \leq k < d$ due to regularity needed in the functions in order to make sense of singular integrals showing up for the gradient of $V$. Even more, the definition of the convolution with the gradient of $V$ appearing in the main equation (1.1) will need to be considered in the principal value sense for the range $d-1 \leq k < d$ leading to fine tuning of the estimates for singular integrals in different steps in the proofs. We do not pursue this interesting generalization further in this work, although we emphasize that our main regularized potential estimate in Lemma 5.2, one of the most important technical steps in the proof of the mean field limit, holds in the full range $0 < k < d$.

*Remark 5.12.* Theorem 1.2 may be made quantitative, i.e. an explicit convergence rate of $\mathcal{E}_N(t)$ might be deduced. Note however that in order to achieve this, one has to optimize all the expressions involving ratios of $\varepsilon$ and $\delta$ (for instance (5.34) in Lemma 5.5). This would result in more technical work which we do not include in the present work.

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## Declarations

**Conflict of interest** The authors declare that they have no conflict of interest.

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