VARIATIONAL APPROXIMATION OF FLUX IN CONFORMING FINITE ELEMENT METHODS FOR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS: A MODEL PROBLEM

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ABSTRACT. We consider the approximation of elliptic boundary value problems by conforming finite element methods. A model problem, the Poisson equation with Dirichlet boundary conditions, is used to examine the convergence behavior of flux defined on an internal boundary which splits the domain in two. A variational definition of flux, designed to satisfy local conservation laws, is shown to lead to improved rates of convergence.

1. INTRODUCTION

In mathematical modeling of physical phenomena one frequently encounters instances when the primary quantity of interest is not the analytical solution to the underlying partial differential equation but a linear functional of the analytical solution to the equation; in such cases, solving the differential equation considered is only an intermediate stage in the process of computing the main quantity of concern. For example, in fluid dynamics one may be interested in calculating the lift and drag coefficients of a body immersed in a viscous incompressible fluid whose flow is governed by the Navier-Stokes equations. The lift and drag coefficients are defined as integrals, over the boundary of the body, of the stress tensor components normal and tangential to the flow, respectively. Similarly, in elasticity theory, the quantities of prime interest, such as the stress intensity factor, or the moments of a shell or a plate, are derived quantities.

A further aspect of *measurement problems* of this kind is that, frequently, the functional under consideration may be expressed in various forms which are mutually equivalent at the continuous level but result in very different approximations under discretization. Thus it

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is important to select the appropriate representation of the functional before formulating its discretization. This basic idea has been widely exploited in structural mechanics [1, 2, 3, 4] and heat conduction [14] to post-process finite element approximations, and more recently also in the field of computational fluid dynamics in the context of *a posteriori* error estimation for lift and drag computations (cf. [9]).

In this paper we shall be concerned with the finite element approximation of one particular functional: the diffusive flux over an interface. Our aim is to show that the natural "variational definition" of the discrete flux can provide a high order of accuracy in suitably defined dual norms. In doing so, we shall not aim at generality. On the contrary, we will try to present the main idea on the simplest possible problem, in order to stress what we believe to be the crucial points and instruments, avoiding technicalities as much as possible. We believe however that more general results are true, and the difficulty in their proof is mainly of a technical nature. The related problem of error analysis of the diffusive flux approximation over the entire boundary of the computational domain has been considered by Barrett and Elliott in [5].

The paper is structured as follows. The model problem is described in Section 2, together with the definition of the proposed approximation of the flux, while Section 3 is devoted to the proof of our error estimate.

Throughout the paper we shall use the usual notation for Sobolev spaces and their norms and seminorms. See, for example, [7], [11].

2. The model problem and the main result

2.1. **The geometry of the problem.** Let Ω be a rectangle in the plane with boundary $\partial\Omega$. We shall suppose that Ω is split into two disjoint open subdomains Ω_1 and Ω_2 by a straight line Γ. We do not assume Γ to be parallel to one of the edges of Ω; however, for the sake of simplicity of the notation we shall require that Γ has equation $x = 0$.

2.2. **The model problem.** For a given f, smooth enough, we consider the following problem:

$$\begin{cases} \text{find } u \in H_0^1(\Omega) \text{ such that} \\ -\Delta\, u = f \text{ in } \Omega. \end{cases} \tag{2.1}$$

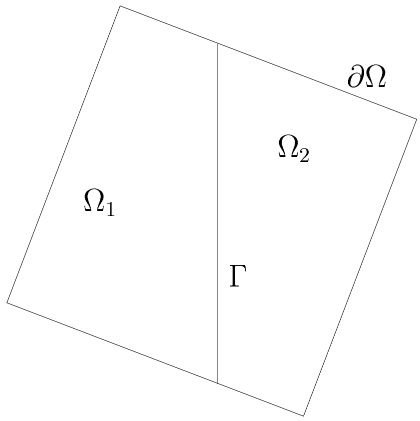

FIGURE 1. The domain Ω and the internal boundary Γ.

It is well known that (2.1) has a unique solution. We set $V := H_0^1(\Omega)$, and for u and v in V we let

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x \, \mathrm{d}y. \tag{2.2}$$

Hence, the variational formulation of (2.1) is

$$\begin{cases} \text{find } u \in V \text{ such that} \\ a(u,v) = (f,v) \quad \forall v \in V, \end{cases} \tag{2.3}$$

where, as usual, $(\cdot,\cdot)$ represents the inner product in $L^2(\Omega)$.

2.3. **The decomposition and the discrete problem.** Let $\mathcal{T}_h$ be a decomposition of Ω into triangles which is compatible with the splitting of Ω into Ω_1 and Ω_2 (this obviously means that each triangle is a subset of one of the two subdomains.) For k an integer ≥ 1 we consider the space V_h^k defined as

$$V_h^k := \{v_h \in C^0(\overline{\Omega}) \cap H_0^1(\Omega),\ v_{h|T} \in \ P_k \ \forall T \in \mathcal{T}_h\},$$

that is the usual finite element space of continuous piecewise polynomials of degree k over the decomposition $\mathcal{T}_h$ which obey the homogeneous Dirichlet boundary condition on $\partial\Omega$.

In tandem with (2.3) consider the following discrete problem:

$$\begin{cases} \text{find } u_h \in V_h^k \text{ such that} \\ a(u_h,v_h) = (f,v_h) \quad \forall v_h \in V_h^k. \end{cases} \tag{2.4}$$

It is well known (see, for instance, [7]) that (2.4) has a unique solution u_h, and that the following error estimates hold, whenever the analytical solution u has the necessary regularity:

$$||u - u_h||_{r,\Omega} \leq C\, h^{k+1-r} |u|_{k+1,\Omega}, \quad r = 0, 1. \tag{2.5}$$

2.4. **The discrete flux, and the statement of the main result.** First of all, in analogy with (2.2) we introduce

$$a_i(u\,, v) := \int_{\Omega_i} \nabla u \cdot \nabla v \, \mathrm{d}\Omega \tag{2.6}$$

for $i = 1, 2$. We define now the continuous flux from Ω_1 into Ω_2 through the interface Γ as

$$F_u := (\frac{\partial u}{\partial x})_{|\Gamma} \tag{2.7}$$

where we assumed that Ω_1 is to the *left* of Γ, that is

$$\Omega_1 = \{(x, y) \in \Omega,\ x < 0\}.$$

Definition (2.7) is to be understood in the pointwise sense (or a.e. if u is not smooth enough.) Here, however, we shall be more interested in the flux in the distributional sense. Therefore, we notice that for every $\varphi \in \mathcal{D}(\Gamma) = C_0^\infty(\Gamma)$ we have

$$\langle F_u, \varphi \rangle = \int_\Gamma \frac{\partial u}{\partial x} \varphi \, \mathrm{d}s, \tag{2.8}$$

where $\langle\, , \,\rangle$ is the duality pairing between $\mathcal{D}(\Gamma)$ and its dual. By Green's formula we have

$$\langle F_u, \varphi \rangle = \int_\Gamma \frac{\partial u}{\partial x} \varphi \, \mathrm{d}s \, = a_1(u\,, \tilde\varphi) \, - \int_{\Omega_1} f \tilde\varphi \, \mathrm{d}\Omega \tag{2.9}$$

for every $\tilde\varphi$ in $H^1(\Omega_1)$ that has trace equal to φ on Γ and equal to zero on the rest of $\partial\Omega_1$.

In turn, the *discrete flux* F_{u_h} will be defined as a linear mapping acting from the space

$$\Sigma_h = (V_h^k)_{|\Gamma} \tag{2.10}$$

into $\mathbb{R}$. More precisely, in agreement with (2.9) and with [12], [10], we set, for every $\varphi_h \in \Sigma_h$:

$$\langle F_{u_h}, \varphi_h \rangle = a_1(u_h\,, \tilde\varphi_h) \, - \int_{\Omega_1} f \tilde\varphi_h \, \mathrm{d}\Omega \tag{2.11}$$

where $\tilde{\varphi}_h$ is any function in V_h^k whose trace on Γ coincides with φ_h. Definitions such as (2.11) are fundamental to the development of local conservation laws; see [10].

For a given $\varphi \in \mathcal{D}(\Gamma)$ we consider now its interpolant $\varphi^I \in \Sigma_h$. Our goal is to estimate the *error in the flux*; thus we consider

$$\langle F_u - F_{u_h}, \varphi^I \rangle. \tag{2.12}$$

Our main result, to be proved in the next section, is that there exists a constant C, independent of u, φ, and h, such that

$$|\langle F_u - F_{u_h}, \varphi^I \rangle| \leq C h^{2k} \, |u|_{k+1,\Omega} \, \|\varphi\|_{k+1/2,\Gamma}. \tag{2.13}$$

Remark. The estimate (2.13) is essentially an error estimate in the space $H^{-k-1/2}(\Gamma)$. As usual, we pay for the increase in the order of convergence by the weakness of the norm. On the other hand, it is known in similar situations that estimates of this type (that is, with high order in dual spaces) are the *crucial ingredient* for proving that suitable *postprocessings* of the discrete solution converge, with the same high order, in more reasonable spaces, for instance in $L^2(\Gamma)$. These postprocessors are typically constructed through suitable local averages that are generally rather inexpensive to compute. We refer, for instance, to the classical papers from the Cornell school (see for instance [6], and [13]), and to the more recent approach of [8].

Remark. One may also consider the possibility when Ω is separated into the subdomains Ω_1 and Ω_2 by a general smooth curve Γ (instead of a straight line as assumed here). However, for $k > 1$, this necessitates the use of *isoparametric elements*, compatible with Γ, in our decomposition. Moreover, the presence of a curved interface Γ would require the use of an *approximate curve* Γ_h for the definition of the approximate flux. It is clear that the additional technical complications to deal with this situation would be considerable.

3. The proof of the error estimate

3.1. **The construction of ψ.** For a given $\varphi \in \mathcal{D}(\Gamma)$ we construct now a suitable lifting $\psi \in \mathcal{D}(\overline{\Omega}_1)$ such that

$$\psi \text{ vanishes on a strip around } \partial\Omega_1 \setminus \Gamma, \tag{3.1}$$

$$\frac{\partial^s \psi}{\partial x^s} = (-1)^{s/2} \varphi^{(s)} \text{ on } \Gamma, \text{ for } s \text{ even}, \ 0 \leq s \leq k, \tag{3.2}$$

$$\frac{\partial^s \psi}{\partial x^s} = 0 \text{ on } \Gamma, \text{ for } s \text{ odd}, \ 0 \le s \le k, \tag{3.3}$$

and there exists a constant C, independent of φ, such that

$$||\psi||_{k+1,\Omega_1} \le C\,||\varphi||_{k+1/2,\Gamma}. \tag{3.4}$$

Here and below the $\varphi^{(s)}$ denotes the derivative of φ of order s with respect to the variable y along Γ. Notice that the boundary conditions (3.2), (3.3) are compatible with the existence of a continuous lifting (that is, of one satisfying (3.4)). For instance, we can extend (by zero) the function φ to the whole line $x = 0$, find the lifting in the half-plane $\{x < 0\}$ according to [11] (Chapter 1, Theorem 7.5) and then apply a suitable cut-off.

We also point out explicitly that, thanks to our construction,

$$\Delta\psi \in H_0^{k-1}(\Omega_1). \tag{3.5}$$

Indeed, for $k = 2$ we have

$$\Delta\psi = \psi_{xx} + \psi_{yy} = \psi_{xx} + \varphi^{(2)} = 0 \text{ on } \Gamma.$$

If $k = 3$ we also have

$$(\Delta\psi)_x = \psi_{xxx} + \psi_{yyx} = \psi_{xxx} + (\psi_x)_{yy} = 0 \text{ on } \Gamma,$$

and for $k = 4$ we can add

$$(\Delta\psi)_{xx} = \psi_{xxxx} + \psi_{yyxx} = \psi_{xxxx} + (\psi_{xx})_{yy} = \psi_{xxxx} - \varphi^{(2)}_{yy} = 0 \text{ on } \Gamma,$$

and so on.

Remark. The construction of ψ is feasible in more general geometries (and for more general operators) than the one considered here. The relevant properties, as we shall see, are (3.4) and (3.5). However, the construction would be technically much more complicated.

3.2. **The construction of χ.** We can now proceed to the construction of the new auxiliary function χ. We set

$$p = -\Delta\psi \quad \text{in } \Omega_1, \qquad \text{and} \qquad p = 0 \quad \text{in } \Omega_2, \tag{3.6}$$

and we define χ as the solution of the following problem:

$$\begin{cases} \text{find } \chi \in H_0^1(\Omega) \text{ such that} \\ -\Delta\,\chi = p \text{ in } \Omega. \end{cases} \tag{3.7}$$

Now, from (3.5) we easily deduce that $p \in H^{k-1}(\Omega)$; moreover,

$$||p||_{k-1,\Omega} \le \ ||p||_{k-1,\Omega_1} \le \ C\,||\psi||_{k+1,\Omega_1}. \tag{3.8}$$

We also recall that ψ in (3.1) vanishes in a strip near $\partial\Omega_1 \setminus \Gamma$. Hence p has compact support in Ω, and, in particular, it belongs to $H_0^{k-1}(\Omega)$. Therefore there exists a constant (that we, again, call C), independent of ψ, such that

$$||\chi||_{k+1,\Omega} \leq C\, ||p||_{k-1,\Omega}. \tag{3.9}$$

The regularity result (3.9) is well known, and its proof can be obtained by the standard technique of reflecting the problem in an *odd* way a suitable number of times, and then using the internal regularity results for the problem in the enlarged domain. The crucial point is that the odd reflections of p (which is the Laplacian of χ) are still in H^{k-1} of the enlarged domain. This is true thanks to (3.5).

Using (3.9), (3.8), and (3.4) we then have immediately:

$$||\chi||_{k+1,\Omega} \leq C\, ||\varphi||_{k+1/2,\Gamma}. \tag{3.10}$$

Remark. As we can see, in order to have an auxiliary function χ satisfying (3.7) and (3.10) we could just assume that Ω is sufficiently smooth, provided that p $(= -\Delta\psi)$ satisfies (3.5) and (3.8).

3.3. **The error estimates.** As stated in (2.13), we wish to estimate the error in the finite element approximation of the flux:

$$\langle F_u - F_{u_h}, \varphi^I \rangle. \tag{3.11}$$

It is immediate to see that, taking as $\tilde{\varphi}^I$ the interpolant ψ^I of ψ (in V_h^k), and using (2.9) and (2.11) we get

$$\begin{aligned} \langle F_u - F_{u_h}, \varphi^I \rangle &= a_1(u - u_h, \tilde{\varphi}^I) = a_1(u - u_h, \psi^I) \\ &= a_1(u - u_h, \psi^I - \psi) + a_1(u - u_h, \psi) \\ &= I + II. \end{aligned} \tag{3.12}$$

The estimate of I follows easily from (2.6), (2.5), usual interpolation estimates, and (3.4):

$$I \leq C\, h^{2k}\, |u|_{k+1,\Omega}\, |\psi|_{k+1,\Omega_1} \leq C h^{2k}\, |u|_{k+1,\Omega}\, ||\varphi||_{k+1/2,\Gamma}. \tag{3.13}$$

The estimate of II is also easy: using (2.6), integrating by parts, using (3.6), (3.3) and then (3.7) we obtain first

$$\begin{aligned} II &= -\int_{\Omega_1} (u - u_h)\, \Delta\psi \,\mathrm{d}\Omega = \int_{\Omega_1} (u - u_h)\, p \,\mathrm{d}\Omega \\ &= -\int_{\Omega} (u - u_h) \Delta\chi \,\mathrm{d}\Omega = a(u - u_h, \chi). \end{aligned} \tag{3.14}$$

Then, using (3.14), choosing χ^I as the usual interpolant of χ in V_h^k, and using Galerkin orthogonality, (2.5), interpolation estimates, and (3.10), we obtain

$$
\begin{aligned}
II &= a(u-u_h,\chi) = a(u-u_h,\chi-\chi^I) \\
&\le Ch^{2k}\,|u|_{k+1,\Omega}\,|\chi|_{k+1,\Omega} \le Ch^{2k}\,|u|_{k+1,\Omega}\,\|\varphi\|_{k+1/2,\Gamma}.
\end{aligned}
\tag{3.15}
$$

Now, from (3.12), (3.13), and (3.15) we easily conclude the proof of the desired estimate (2.13).

Remark. In the particular case of $k = 1$ we see that the crucial properties (3.5), (3.4), and hence (3.10) can easily be obtained under much more general assumptions. It is then clear that the extension of our result to more general problems and geometries, for linear elements, is trivial. The technical difficulties would arise only for $k > 1$.

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