

Zeros of the Riemann Zeta-Function on the Critical Line

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It was shown by Selberg [3] that the Riemann Zeta-function has at least $cT \log T$ zeros on the critical line up to height T , for some positive absolute constant c . Indeed Selberg's method counts only zeros of odd order, and counts each such zero once only, regardless of its multiplicity. With this in mind we shall write $\hat{\gamma}_i$ for the distinct ordinates of zeros of $\zeta(s)$ on the critical line of odd multiplicity. We shall number the points $\hat{\gamma}_i$ so that $0 < \hat{\gamma}_1 < \hat{\gamma}_2 < \dots$. The purpose of the present note is to extract a little more from Selberg's argument, by obtaining further information on the distribution of the $\hat{\gamma}_i$. This is given in the following result.

Theorem *For any constant $\mu \in (0, 2)$ we have*

$$\sum_{\hat{\gamma}_i \leq T} (\hat{\gamma}_{i+1} - \hat{\gamma}_i)^\mu \ll_\mu T(\log T)^{1-\mu}.$$

In particular, if $f(T)$ is any function which tends to infinity with T , then “almost all” intervals $[T, T + f(T)(\log T)^{-1}]$ contain a point $\hat{\gamma}_i$.

Clearly this result includes Selberg's. Moreover it is apparent that the second statement of the theorem follows from the first. We also remark that, if one merely sums over ordinates γ_i of the zeros in the usual sense, not restricting to those zeros which are on the critical line, then one has

$$\sum_{\gamma_i \leq T} (\gamma_{i+1} - \gamma_i)^\mu \ll_\mu T(\log T)^{1-\mu}$$

for any $\mu > 0$, as was shown by Fujii [1].

In giving the proof of our result we shall refer to the version of Selberg's argument presented by Titchmarsh [4: §§10.9-10.22]. The proof uses a “mollifier”

$$\phi(s) = \sum_{\nu \leq X} \beta_\nu \nu^{-s},$$

in which the numbers β_ν are defined in terms of the coefficients α_ν in the expansion

$$\zeta(s)^{-1/2} = \sum_{\nu=1}^{\infty} \alpha_\nu \nu^{-s} \quad \sigma > 1.$$

Titchmarsh takes

$$\beta_\nu = \alpha_\nu \left(1 - \frac{\log \nu}{\log X}\right),$$

but for our purpose the choice

$$\beta_\nu = \begin{cases} \alpha_\nu, & \nu \leq X^{1/2}, \\ 2\alpha_\nu \frac{\log X/\nu}{\log X}, & X^{1/2} \leq \nu \leq X, \end{cases}$$

is required. One then defines

$$F(t) = \frac{\Xi(t)}{t^2 + \frac{1}{4}} \left| \phi\left(\frac{1}{2} + it\right) \right|^2 e^{(\frac{1}{4}\pi - \frac{1}{2}\delta)t},$$

where δ is small and positive. In fact we shall take $\delta = T^{-1}$, where $[T, 2T]$ is the interval in which we are looking for zeros. With the above definition of $F(t)$ it follows (Titchmarsh [4: Lemma 10.17]) that

$$\int_{-\infty}^{\infty} \left| \int_t^{t+h} F(u) du \right|^2 dt \ll \frac{h}{\delta^{1/2} \log X}. \quad (1)$$

This is subject to the conditions $X = \delta^{-c}$ and $h = (a \log X)^{-1}$, where a, c are positive and satisfy $(a+2)c \leq \frac{1}{4}$. Titchmarsh takes a to be constant, but this is unnecessary. We shall set $c = \frac{1}{16}$ so that any value $a \in (0, 2)$ is permissible. We see from (1) that

$$\int_T^{2T} \left| \int_{t-h}^{t+h} F(u) du \right|^2 dt \ll \frac{hT^{1/2}}{\log T}, \quad (2)$$

on changing h into $2h$ and substituting $t-h$ for t . The bound (2) is subject to the conditions $X = T^{1/16}$ and $h = (a' \log T)^{-1}$, where $a' \in (0, \frac{1}{4})$.

The proof of (1), and hence of (2), depends on the definition of β_ν , so we must check that our modification does not materially alter the estimates. It is only Lemma 10.12 of Titchmarsh [4] which needs any change. It is shown that

$$\sum_{\kappa \leq X/d} \alpha_\kappa \kappa^{\theta-1} \log \frac{X}{d\kappa} \ll \left(\frac{X}{d}\right)^\theta \left(\log \frac{X}{d}\right)^{1/2} \prod_{p|d} (1+p^{-1})^{1/2} \quad (3)$$

uniformly for $0 < \theta \leq \frac{1}{2}$, where κ is restricted to integers coprime to ρ . We shall require a corresponding estimate in which the function

$$f(X, d, \kappa) = \begin{cases} \log X/d\kappa, & d\kappa \leq X, \\ 0, & d\kappa \geq X, \end{cases}$$

on the left is replaced by

$$g(X, d, \kappa) = \begin{cases} \log X, & d\kappa \leq X^{1/2}, \\ 2 \log X/d\kappa, & X^{1/2} \leq d\kappa \leq X, \\ 0, & d\kappa \geq X. \end{cases}$$

However, since

$$g(X, d, \kappa) = 2f(X, d, \kappa) - 2f(X^{1/2}, d, \kappa),$$

one sees that (3) remains true with g in place of f .

We shall also require the estimate

$$\int_{-\infty}^{\infty} |F(t)|^2 dt \ll \frac{\log 1/\delta}{\delta^{1/2} \log X} \ll T^{1/2} \quad (4)$$

given by Lemma 10.18 of Titchmarsh [4]. The proof of this requires no modification.

We now establish a lower bound for

$$\int_{t-h}^{t+h} |F(u)| du = J(t),$$

say, on the interval $T \leq t \leq 2T$. Titchmarsh does this only on average, while we shall, in effect, obtain a lower bound for “almost all” t . We begin by choosing a large constant integer K , and writing

$$w(z) = \left(\frac{\sin z}{z}\right)^{2K},$$

so that

$$\int_{-\infty}^{\infty} e^{i\lambda t} w(t) dt = 0 \quad \text{for } |\lambda| \geq 2K.$$

We now consider the integral

$$\int_{1/2-i\infty}^{1/2+i\infty} \zeta(s+it) \phi(s+it)^2 w\left(\frac{s-\frac{1}{2}}{i\Delta}\right) ds = I, \quad (5)$$

say, where

$$\frac{1}{\log T} \leq \Delta \leq T^{3/4}. \quad (6)$$

The integral will converge if K is chosen large enough. We now move the line of integration to $\sigma = 2$, producing a residue

$$\phi(1)^2 w\left(\frac{\frac{1}{2} - it}{i\Delta}\right) \ll (\log T)^2 \left(\frac{e^{1/2\Delta}}{|t|/\Delta}\right)^{2K} \ll (\log T)^{-2}.$$

On the line $\sigma = 2$ we may integrate termwise. We have

$$\int_{2-i\infty}^{2+i\infty} n^{-s} w\left(\frac{s - \frac{1}{2}}{i\Delta}\right) ds = i\Delta \int_{-\infty}^{\infty} n^{-1/2 - it\Delta} w(t) dt$$

on moving the line of integration back to $\sigma = 1/2$, so that terms for which $\Delta \log n \geq 2K$ make no contribution. Since

$$\zeta(s) \phi(s)^2 = \sum_{n=1}^{\infty} a_n n^{-s}$$

with $a_1 = 1$ and $a_n = 0$ for $2 \leq n \leq X^{1/2}$, we now see that

$$I = i\Delta C_K + O\left(\frac{1}{\log^2 T}\right),$$

where

$$C_K = \int_{-\infty}^{\infty} w(t) dt > 0,$$

providing that

$$\Delta \geq \frac{64K}{\log T}. \quad (7)$$

At this point we observe that if $h \leq T^{3/4}$ then

$$\begin{aligned} T^{1/4} J(t) &\gg \int_{t-h}^{t+h} \left| \zeta\left(\frac{1}{2} + iu\right) \phi\left(\frac{1}{2} + iu\right)^2 \right| du \\ &\geq \int_{t-h}^{t+h} \left| \zeta\left(\frac{1}{2} + iu\right) \phi\left(\frac{1}{2} + iu\right)^2 \right| w\left(\frac{u}{\Delta}\right) du \\ &\geq I + O\left\{ \int_{|u| \geq h} \left| \zeta\left(\frac{1}{2} + i(u+t)\right) \phi\left(\frac{1}{2} + i(u+t)\right)^2 \right| \frac{du}{(|u|/\Delta)^{2K}} \right\}, \end{aligned}$$

whence

$$T^{1/4} J(t) + \int_{|u| \geq h} \left| \zeta\left(\frac{1}{2} + i(u+t)\right) \phi\left(\frac{1}{2} + i(u+t)\right)^2 \right| \frac{du}{(|u|/\Delta)^{2K}} \gg \Delta.$$

Since

$$\zeta\left(\frac{1}{2} + i(u+t)\right) \phi\left(\frac{1}{2} + i(u+t)\right)^2 \ll (T + |u|)^{1/4} X$$

for $T \leq t \leq 2T$, it follows that the range $|u| \geq T/2$ will contribute only $O(\Delta/T)$, say. Here we use the facts that $\Delta \leq T^{3/4}$, by (6), and that K is sufficiently large. Moreover

$$\begin{aligned} \int_{h \leq |u| \leq T/2} |\zeta(\frac{1}{2} + i(u+t))\phi(\frac{1}{2} + i(u+t))^2|(|u|/\Delta)^{-2K} du \\ \ll T^{1/4} \int_{h \leq |u| \leq T/2} |F(t+u)|(|u|/\Delta)^{-2K} du \\ = T^{1/4} K(t), \end{aligned}$$

say. It follows that

$$J(t) + K(t) \gg T^{-1/4} \Delta. \quad (8)$$

We now observe that

$$\int_T^{2T} K(t) dt = \int_{h \leq |u| \leq T/2} (|u|/\Delta)^{-2K} \left\{ \int_{T+u}^{2T+u} |F(v)| dv \right\} du,$$

and Cauchy's inequality, in conjunction with (4) yields

$$\int_{T+u}^{2T+u} |F(v)| dv \ll T^{1/2} \left\{ \int_{T/2}^{5T/2} |F(v)|^2 dv \right\}^{1/2} \ll T^{3/4}.$$

We therefore see that

$$\int_T^{2T} K(t) dt \ll \frac{hT^{3/4}}{(h/\Delta)^{2K}}, \quad (9)$$

since

$$\int_{h \leq |u| \leq T/2} (|u|/\Delta)^{-2K} du \ll \frac{h}{(h/\Delta)^{2K}}.$$

We shall write (8) as

$$J(t) + K(t) \geq CT^{-1/4} \Delta,$$

and define

$$R_h = \{t \in [T, 2T] : J(t) \leq \frac{C}{2} T^{-1/4} \Delta\}.$$

Then $K(t) \gg T^{-1/4} \Delta$ on R_h , whence

$$T^{-1/4} \Delta \text{mes}(R_h) \ll \frac{hT^{3/4}}{(h/\Delta)^{2K}},$$

by (9). It follows that

$$\text{mes}(R_h) \ll \frac{T}{(h/\Delta)^{2K-1}}. \quad (10)$$

We can now complete the proof of the theorem. We consider the set S_h of $t \in [T, 2T]$ for which the interval $t - h \leq u \leq t + h$ contains no sign change of the function $F(u)$, or equivalently no zero of odd order of $\zeta(\frac{1}{2} + iu)$. Thus

$$J(t) = \left| \int_{t-h}^{t+h} F(u) du \right|$$

for $t \in S_h$. If $t \in S_h \setminus R_h$ then $J(t) \geq \frac{C}{2} T^{-1/4} \Delta$, whence (2) yields

$$\left(\frac{C}{2} T^{-1/4} \Delta \right)^2 \text{mes}(S_h \setminus R_h) \ll \frac{h T^{1/2}}{\log T}.$$

Thus

$$\text{mes}(S_h \setminus R_h) \ll \frac{h T}{\Delta^2 \log T}.$$

On choosing

$$\Delta = h(h \log T)^{-1/(2K+1)}$$

we therefore deduce from (10) that

$$\text{mes}(S_h) \ll T(h \log T)^{-1+2/(2K+1)}.$$

Our choice of Δ will satisfy the conditions (6) and (7) if $h = (a' \log T)^{-1}$ with $h \leq T^{3/4}$ and $0 < a' \leq a'(K)$, say.

We are now ready to estimate

$$\sum_{\hat{\gamma}_i} (\hat{\gamma}_{i+1} - \hat{\gamma}_i)^\mu$$

for $T \leq \hat{\gamma}_i \leq 2T$. We shall choose K to be a fixed integer such that

$$\frac{2}{2K+1} < 2 - \mu.$$

According to a result of Hardy and Littlewood [2] we have $\hat{\gamma}_{i+1} - \hat{\gamma}_i \ll T^\theta$ for any $\theta > \frac{1}{4}$, so that it suffices to prove that

$$\sum_{\hat{\gamma}_i} (\hat{\gamma}_{i+1} - \hat{\gamma}_i)^\mu \ll T(\log T)^{1-\mu} \quad (11)$$

for $T \leq \hat{\gamma}_i < \hat{\gamma}_{i+1} \leq 2T$. Moreover, summands for which

$$\hat{\gamma}_{i+1} - \hat{\gamma}_i \leq \frac{8}{a'(K) \log T}$$

clearly make a satisfactory contribution. We shall classify the remaining terms according to the value of $h = 2^H$, $H = 1, 2, \dots$, for which

$$4h < \hat{\gamma}_{i+1} - \hat{\gamma}_i \leq 8h.$$

Thus we may assume that

$$\frac{1}{a'(K) \log T} \leq h \leq T^{3/4}.$$

Since $t \in S_h$ for $\hat{\gamma}_i + h \leq t \leq \hat{\gamma}_i + 2h$, we see that the number of points $\hat{\gamma}_i$ corresponding to any such h is at most

$$h^{-1} \text{mes}(S_h) \ll h^{-2+2/(2K+1)} T (\log T)^{-1+2/(2K+1)}.$$

The corresponding contribution to (11) is therefore

$$\ll h^{\mu-2+2/(2K+1)} T (\log T)^{-1+2/(2K+1)},$$

and summing over $h = 2^H \gg (\log T)^{-1}$ yields the required result.

References

- [1] Fujii, A.: On the distribution of the zeros of the Riemann Zeta function in short intervals. Bull. Amer. Math. Soc. **81**, 139-142 (1975)
- [2] Hardy, G.H., Littlewood, J.E.: Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes. Acta Math. **41**, 119-196 (1918)
- [3] Selberg, A.: On the zeros of Riemann's zeta-function. Skr. Norske Vid. Akad. Oslo no. **10**, (1942)
- [4] Titchmarsh, E.C.: The theory of the Riemann Zeta-function, 2nd Edition, revised by D.R. Heath-Brown. Oxford: Clarendon Press 1986