



# Department of Economics Discussion Paper Series

## Multiproduct Cost Passthrough: Edgeworth's Paradox Revisited

Mark Armstrong and John Vickers

Number 967  
March, 2022

# Multiproduct Cost Passthrough: Edgeworth's Paradox Revisited

Mark Armstrong and John Vickers\*

March 2022

## Abstract

Edgeworth's paradox of taxation occurs when an increase in the unit cost of a product causes a multiproduct monopolist to *reduce* prices. We give simple illustrations of the paradox, we show how it can arise with uniform pricing, and we give an analysis of the case of linear marginal cost and demand conditions. We show how the matrix of cost-passthrough terms must be similar to a positive definite matrix. When the firm supplies two substitute products we show how the paradox always occurs with a suitable choice of cost function. We then show a connection between Ramsey pricing and the paradox in a form relating to consumer surplus, and use it to find further examples where consumer surplus increases with cost.

**JEL codes:** D42, H22, L12

**Keywords:** Multiproduct pricing, Edgeworth's paradox of taxation, cost passthrough, price discrimination, Ramsey pricing.

## 1 Introduction

Recent analyses of the rate of passthrough from cost to price as an economic tool have focussed on single-product firms: see, for example, Weyl and Fabinger (2013), and Miklos-Thal and Shaffer (2021). However, the subject of *multi*-product cost passthrough has a long history. In a remarkable article on the pure theory of monopoly published in 1897, Edgeworth demonstrated his paradox of taxation – that a tax on (or cost increase of) one product supplied by a multiproduct monopolist could lead to a reduction in the prices charged by the monopolist, including the price of the more costly product.<sup>1</sup> This finding,

---

\*Department of Economics and All Souls College, University of Oxford. We are grateful to Jidong Zhou for helpful comments. Armstrong thanks the European Research Council for financial support from Advanced Grant 833849.

<sup>1</sup>In Italian, in the *Giornale degli Economisti*. The article appears in English with some modifications in Edgeworth (1925).

controversial at the time, was established in more detail by Hotelling (1932), who gave some illustrations of the phenomenon that were easier to comprehend than Edgeworth's.

In particular, Edgeworth (1925, pages 132-4) showed for the two-product case that the second-order condition for profit maximization was compatible with consumer surplus increasing with a tax on one product. He then provided a numerical example in which both prices decreased with a tax on one of the products. The example has zero costs but Edgeworth notes that the conclusion is strengthened when there are costs of production, 'for then we have more functions at our disposal with which to manipulate a favourable example'. He goes on to illustrate with rail fares: a tax on first-class tickets might lower both first- and second-class fares, though the number of first-class travellers will nonetheless decline, as the reduced second-class fare predominates.

As a preliminary comment it is worth noting why the pricing paradox cannot arise in the single-product case.<sup>2</sup> If the cost of supplying product  $k$  increases, the monopolist will want to reduce the amount of product  $k$  that it supplies. (The revealed preference argument that confirms this is set out in section 2 below.) If  $k$  is the only product, and demand as a function of price slopes down, then the price of product  $k$  must go up, and consumer surplus decreases.<sup>3</sup> But if  $k$  is not the only product, the monopolist might adjust the prices of other products. If products are substitutes, reduced supply of  $k$  will normally induce the monopolist to supply more of product  $j$ . The latter effect will bear down on prices, including the price of  $k$ , offsetting at least partially the effect of reduced supply of  $k$ . The paradox arises when the price effect of expanded supply of  $j$  more than offsets the reduced supply of  $k$  – a phenomenon compatible with standard demand theory in the multiproduct case, just as Edgeworth observed.

Having set out the model in section 2, we analyse Edgeworth's paradox in section 3. There, we provide a simple illustration with inelastic demand, we derive a simple condition for the paradox when the monopolist is restricted to *uniform* pricing. We provide a general analysis of the case with linear marginal cost and demand conditions, and show the possibility of the paradox and range of possibilities for cost-passthrough in this case. In particular, we show that a matrix might be a possible matrix of cost-passthrough terms if and only it is similar to a positive definite matrix, and show how the eigenvectors of

---

<sup>2</sup>A related paradox can however occur if there are multiple inputs to the production of a single product. In particular, an increase in the price of one input can cause the marginal cost (though never the total cost) of the output to fall, which induces the firm to set a lower price for its product.

<sup>3</sup>The same is true if the prices of products other than  $k$  are held fixed.

this matrix can be used to diagonalize the firm's profit and revenue functions. We provide a general analysis of the two-product case, and show for given demand with substitutes that there are *always* cost conditions that give rise to the paradox. In section 4 we turn to a weaker version of Edgeworth's paradox that we call the surplus paradox—i.e., that *consumer surplus* increases as the cost of one product rises. We show that there are always cost conditions that give rise to this paradox even without products being substitutes. We also derive a connection between Ramsey pricing and the surplus paradox. This paradox cannot happen in the cost and demand conditions featured in section III of Armstrong and Vickers (2018), hereafter abbreviated to AV, but is quite possible more generally. An implication of the Ramsey connection is that the surplus paradox can be found where the profit-maximizing supply of some product exceeds its supply with marginal cost pricing. This insight gives a way to find further examples of the surplus paradox.

## 2 The model and output reduction result

A monopolist supplies  $n \geq 2$  products. The price and quantity of product  $k$  are denoted by  $p_k$  and  $x_k$  respectively, and  $p$  and  $x$  denote the price and quantity vectors. Total output is  $X \equiv \sum_k x_k$ . As in AV, gross consumer utility  $u(x)$  is assumed to be strictly concave; the inverse demand function is given by  $p(x) = \nabla u(x)$ , the vector of partial derivatives of  $u$ ; and revenue is given by  $r(x) \equiv p(x) \cdot x$ . Consumer surplus as a function of quantities is given by the function  $s(x) \equiv u(x) - r(x)$ . Profit is  $\pi(x) = r(x) - c(x)$ , where the  $c(x)$  is the cost function. In general the monopolist maximizes the weighted sum  $\phi(x) = \pi(x) + \alpha s(x)$ , with  $\alpha \in [0, 1]$ . Profit maximization corresponds to  $\alpha = 0$  and other  $\alpha$  correspond to Ramsey pricing.

Suppose that the set of feasible quantity vectors lies in some set  $\mathfrak{X} \subset \mathbb{R}_+^n$ , and that initially  $\phi(x)$  is maximized by  $x^0 \in \mathfrak{X}$ . (In many cases it makes sense that  $\mathfrak{X} = \mathbb{R}_+^n$ , but, as with the uniform price analysis below, there are situations where the set of quantities is restricted.) Compare the situation when product  $k$  has a per-unit cost increase (or tax) of  $t_k > 0$ , and let  $x^* \in \mathfrak{X}$  then maximize  $\phi(x) - t_k x_k$ . By revealed preference we have

$$\phi(x^0) \geq \phi(x^*) \text{ and } \phi(x^*) - t_k x_k^* \geq \phi(x^0) - t_k x_k^0 .$$

Combining these inequalities we deduce that  $t_k(x_k^0 - x_k^*) \geq 0$ , confirming that the cost increase causes supply of product  $k$  to fall, at least weakly. Moreover, if  $\phi(x)$  is differen-

tiable,  $\mathfrak{X} = \mathbb{R}_+^n$  and  $x_k^0 > 0$ , then the supply of product  $k$  decreases *strictly*.<sup>4</sup> For if not, i.e., if  $x_k^0 = x_k^*$ , then  $x^*$  also maximizes  $\phi$ , and we would have the contradiction that

$$\frac{\partial}{\partial x_k} \phi(x^*) = 0 \text{ and } \left. \frac{\partial}{\partial x_k} [\phi(x) - t_k x_k] \right|_{x=x^*} = 0 \Rightarrow \frac{\partial}{\partial x_k} \phi(x^*) = t_k .$$

A recurring theme in this paper, explored more systematically in section 4, is that the Edgeworth paradox is associated with *over*-provision of the relevant product by the monopolist, relative to efficiency. Intuitively, if the monopolist supplies too much of a product, a tax on the supply of that product—which reduces its supply—may well lead to better outcomes for consumers. The fall in  $x_k$  will by itself tend to increase  $p_k$  but other  $x_j$  will adjust. Edgeworth’s paradox occurs when their adjustment both outweighs the upward effect on  $p_k$  of the fall in  $x_k$ , and causes  $p_j$  to decrease too. The next section finds a variety of ways in which that can happen.

## 3 Edgeworth’s Paradox

### 3.1 Inelastic demands

Both Edgeworth (1925, page 132) and Hotelling (1932, page 612) provide examples of the Edgeworth paradox. In each case, the demand functions are quite complex and are valid only over a restricted range of prices and quantities.<sup>5</sup> More importantly, the economic forces underlying their examples seem opaque. The following approach gives rise to the paradox in perhaps a more economically intuitive manner.

Consider a setting where  $u(x)$  takes the form

$$u(x) = \max\{u_A(x), u_B(x)\} ,$$

where  $u_A$  and  $u_B$  are distinct sub-utility functions. Suppose that the demand system associated with  $u_B$  involves lower profit-maximizing prices than  $A$ , but also a lower supply of product  $k$  than  $A$ . Then a tax  $t_k$  on product  $k$  tilts the firm in favour of inducing the consumer to use  $B$  relative to  $A$ , in which case all prices might fall as  $t_k$  increases.

For example, suppose that there are two products, that demand system  $A$  has inelastic demand product  $i = 1, 2$  equal to  $x_i^A$  so long as its price for the product does not exceed

<sup>4</sup>The following argument is in the spirit of Edlin and Shannon (1998).

<sup>5</sup>For instance, Edgeworth’s example had the prices  $p_1$  and  $p_2$  being related to quantities  $x_1$  and  $x_2$  as (approximately)  $p_1 = 1.6053 - .2x_1 - \frac{2}{3}(x_1 - .96)^{\frac{3}{2}} - \frac{1}{2}x_2$  and  $p_2 = 3.918 - 2\sqrt{x_2} - .6975 - \frac{1}{2}x_1$ . Production was assumed costless, and the profit-maximizing quantities were  $x_1 = x_2 = 1$ .

1, while system  $B$  has inelastic demand for product  $i$  equal to  $x_i^B$  so long as its price does not exceed  $v < 1$ . In particular, suppose that  $x_1^A = x_2^B = 2$  and  $x_1^B = x_2^A = 1$ , and that unit costs are  $c_1 \geq 0$  and  $c_2 = 0$ , so system  $A$  involves more of the costly product. (With constant unit costs, as here, cost level  $c_k$  is equivalent to tax  $t_k$ , and we use the two terms interchangeably.) For small  $c_1$  the profit-maximizing strategy is to extract all consumer surplus in system  $A$  by setting  $p_1 = p_2 = 1$  to obtain profit of  $3 - 2c_1$ . Suppose however that that  $9(1 - v) < c_1 < v$  (which requires  $v > \frac{9}{10}$ ). Then the firm makes more profit by inducing the consumer to choose system  $B$  by offering prices  $p_1 = v$  and  $p_2 = v - 3(1 - v)$ . Such prices give consumer surplus of  $6(1 - v)$  in both systems  $A$  and  $B$ , so there is no incentive for the consumer to deviate to  $A$ , while the firm obtains profit of

$$(p_1 - c_1) + 2p_2 = (v - c_1) + 2[v - 3(1 - v)] = 9v - c_1 - 6 ,$$

which exceeds  $(3 - 2c_1)$  when  $c_1 > 9(1 - v)$ . So if  $c_1$  rises above  $9(1 - v)$ , the firm shifts from using  $A$  to using  $B$  because  $B$  involves less of the costly product. Both prices fall as a result, and we have the Edgeworth paradox.

Notice in this example that if  $3(1 - v) < c_1 < 9(1 - v)$ , the profit-maximizing firm prefers system  $A$  despite the fact that  $B$  involves higher total welfare, so the monopoly supply of product 1 then exceeds its efficient level. This is an instance of the close connection between monopolistic over-supply of a product and the Edgeworth paradox.

### 3.2 Uniform pricing

We now find a simple condition for Edgeworth's paradox to arise when the firm is constrained to set the same price for each of its products. For instance, regulation or social norms might require a restaurant to set the same price for dinner regardless of the day of the week. In this situation it is more convenient to work with demand functions rather than inverse demand functions. Here,  $\mathfrak{X}$  is the set of quantity vectors  $x$  traced out by the path  $x(P, \dots, P)$  as the scalar price  $P$  varies. Let  $P^0$  and  $P^*$  be the uniform prices that maximize the firm's objective before and after the product  $k$  cost increase. As the cost increase causes  $x_k$  to fall weakly,  $x_k(P^0, \dots, P^0) \geq x_k(P^*, \dots, P^*)$ , and the following result is immediate:

**Proposition 1** *When uniform pricing is required, the uniform price (weakly) decreases with the cost of product  $k$  if  $x_k(P, \dots, P)$  strictly increases with  $P$ .*

With two products,  $x_k(P, P)$  can increase with  $P$  only if products are substitutes. Revealed preference shows that the uniform price  $P$  that maximizes the Ramsey objective  $\phi$  decreases with the weight on consumer surplus,  $\alpha$ , and in particular that the profit-maximizing uniform price exceed the welfare-maximizing uniform price. Therefore, when  $x_k$  increases with  $P$ , the firm chooses to supply too much product  $k$  relative to the welfare-maximizing case.

In any demand system total output  $X$  falls with the uniform price  $P$ . In particular, in the two-product case if  $x_1$  increases with  $P$  then  $x_2$  must fall with  $P$ . Thus if demand for one product rises with  $P$ , the set  $\mathfrak{X}$  of feasible quantities is a *downward*-sloping curve in  $\mathbb{R}_+^2$ . In the differentiable case, the condition that  $x_k$  rises with  $P$  is equivalent to total quantity  $X$  increasing with  $p_k$ . This is because by Slutsky symmetry

$$\frac{d}{dP}x_k(P, \dots, P) = \sum_j \frac{\partial x_k}{\partial p_j} = \sum_j \frac{\partial x_j}{\partial p_k} = \frac{\partial}{\partial p_k}X(P, \dots, P) .$$

This condition cannot occur with a situation with standard discrete choice and unit demands, since in that case the number of consumers who buy something (which is  $X$ ) decreases if any price rises. However, more generally it is possible that total quantity rises when a price rises, as the following example illustrates.

Suppose there are two products with the linear demands

$$x_1 = 1 - p_1 + \frac{3}{2}p_2 , \quad x_2 = 3 - 4p_2 + \frac{3}{2}p_1 ,$$

and constant unit costs of  $c_1 \geq 0$  and 0 respectively for products 1 and 2. (Since  $(\frac{3}{2})^2 < 4$  this demand system corresponds to a concave utility function  $u(x_1, x_2)$ .) With uniform price  $P$ , demand for product 1 is the increasing function  $x_1 = 1 + \frac{P}{2}$ , and so the condition in Proposition 1 is met. Profit is

$$(4 - 2P)P - (1 + \frac{1}{2}P)c_1 ,$$

and the profit-maximizing price

$$\hat{P} = 1 - \frac{1}{8}c_1 \tag{1}$$

decreases with  $c_1$ . Without the restriction to uniform pricing, however, there is no Edgeworth paradox in this example. (We discuss the case of linear demand more generally in the next section.) Indeed, the profit-maximizing prices without the uniform price constraint are

$$p_1 = \frac{17}{7} + \frac{1}{2}c_1 \quad \text{and} \quad p_2 = \frac{9}{7} ,$$

which are *both* higher than  $\hat{P}$  in (1), and as usual with linear demand and constant unit costs there is no cross-cost passthrough from  $c_1$  to  $p_2$ .

Another reason why the firm might be constrained to use uniform pricing is if consumers view the products as *perfect* substitutes, and for cost reasons the firm wishes to supply quantities of all products. That is, consumer utility is just a function of total output,  $U(X)$ , and define

$$C(X) \equiv \min_{x \geq 0} : c(x) \text{ such that } \sum_j x_j = X \quad (2)$$

as the least cost way to supply total output  $X$  with the cost function  $c(x)$ . Suppose now that a small tax  $t_1$  is introduced on the sale of product 1. It is clear that the cost of supplying  $X$ ,  $C(X)$ , weakly increases due to  $t_1$ , but it does not follow that marginal cost  $C'(X)$  also increases, and marginal cost is what matters for output  $X$  and price  $P = U'(X)$ . Indeed, since the envelope theorem implies that

$$\frac{\partial}{\partial t_1} C(X) = x_1(X) ,$$

where  $x_1(X)$  is the quantity of product 1 that solves (2), it follows from the symmetry of cross-derivatives that

$$\frac{\partial}{\partial t_1} C'(X) = \frac{d}{dX} \left( \frac{\partial}{\partial t_1} C(X) \right) = \frac{d}{dX} x_1(X) .$$

Thus marginal cost  $C'(X)$  decreases with  $t_1$  if and only if the quantity of product 1 used to supply total quantity  $X$  falls with  $X$ , so that product 1 is akin to an inferior input. In this case, since marginal cost falls with  $t_1$  it follows that  $X$  rises with  $t_1$  and hence  $P = U'(X)$  falls with  $t_1$ . In other words, the Edgeworth paradox is observed.

With two products,  $x_1(X)$  falls with  $X$  if the function  $c(x_1, X - x_1)$  is submodular in  $(x_1, X)$ , and when  $c$  is differentiable this requirement corresponds to  $c_{22} < c_{12}$ . For example, with quadratic costs

$$c(x) = c_1 x_1 + c_2 x_2 + d_{11} x_1^2 + d_{22} x_2^2 + 2d x_1 x_2 \quad (3)$$

the condition  $c_{22} < c_{12}$  is  $d > d_{22}$ . (For the problem of minimizing  $c(x_1, X - x_1)$  to be convex in  $x_1$  in this example, we require  $2d < d_{11} + d_{22}$ .)



### 3.3 Linear marginal costs and demands

Hotelling (1932, section 7) gives an example with linear marginal cost and demand in which Edgeworth's paradox arises with marginal cost pricing.<sup>6</sup> We now give a general analysis of the case where marginal costs and demands vary linearly with quantities. Specifically, suppose that the  $n$  products have linear inverse demands

$$p(x) = a - Bx \quad (4)$$

where  $a$  is a vector of positive constants and  $B$  is a (symmetric) positive-definite matrix.<sup>7</sup> Thus the firm's revenue function is

$$r(x) = a^T x - x^T B x ,$$

where ' $T$ ' stands for transpose, and  $a^T x$  is the inner product  $\sum_i a_i x_i$ . Suppose the firm's cost function is

$$c(x) = c^T x + x^T D x ,$$

where  $c$  is a vector of positive constants  $c_k$ , one per product, and  $D$  is again symmetric (though not necessarily positive-definite). Let  $M \equiv 2(B + D)$  so that the firm's profit is

$$\pi = (a - c)^T x - \frac{1}{2} x^T M x .$$

To ensure that profit is concave in quantities, suppose that  $M$  is positive-definite. Assuming an interior solution, the first-order condition for profit-maximizing quantities is

$$Mx = a - c \Rightarrow x = M^{-1}(a - c) , \quad (5)$$

which from (4) in turn implies optimal prices are given by

$$p = (I - \Gamma)a + \Gamma c , \quad (6)$$

where  $I$  is the identity matrix and  $\Gamma \equiv BM^{-1}$ . Thus,  $\Gamma$  is the matrix of cost passthrough terms, so that  $\partial p_i / \partial c_j = \gamma_{ij}$ , the  $ij^{th}$  element of  $\Gamma$ . Thus an increase in  $c_k$  (e.g., due to a new tax  $t_k > 0$  on that product) will reduce all prices if the  $k$ th column of  $\Gamma$  has all negative entries.

---

<sup>6</sup>In Hotelling's example the Edgeworth paradox arises not only with marginal cost pricing but with the Ramsey objective  $\phi = \pi + \alpha s$  for all  $\alpha \in (0, 1]$ . But with pure monopoly ( $\alpha = 0$ ) the price of product 1 does not vary with its unit cost in his example.

<sup>7</sup>From now on, we adopt the convention that a positive definite matrix is symmetric.

If the cost function is linear (i.e., if  $D = 0$ ), then  $\Gamma$  is the diagonal matrix  $\frac{1}{2}I$  and (6) simplifies to

$$p = \frac{1}{2}(a + c) ,$$

in which case there are no cross-cost effects on prices, and the paradox cannot occur.

Turn next to the case with a quadratic cost function, as in (3). Suppose for simplicity there are two products, and write  $b = b_{12} = b_{21}$  and  $d = d_{12} = d_{21}$ . We have

$$\Gamma = \frac{2}{\det M} \begin{pmatrix} b_{11}(b_{22} + d_{22}) - b(b + d) & bd_{11} - db_{11} \\ bd_{22} - db_{22} & b_{22}(b_{11} + d_{11}) - b(b + d) \end{pmatrix} \quad (7)$$

where  $\det M > 0$  is the determinant of  $M$ . Thus an increase in  $c_1$  will reduce both prices if

$$b_{11}(b_{22} + d_{22}) - b(b + d) < 0 \quad (8)$$

and

$$bd_{22} - db_{22} < 0 . \quad (9)$$

Given that  $b_{11} > 0$  and  $b_{22} + d_{22} > 0$ , condition (8) implies that  $b(b + d)$  needs to be positive, so there must be cross effects in demand. In fact, the two conditions (8) and (9) together require  $b > 0$  and hence that  $b + d > 0$ . To see this, suppose in contrast that  $b < 0$ , in which case (8) can be written

$$0 > -\frac{b_{11}}{b}(b_{22} + d_{22}) + (b + d) > -\frac{b}{b_{22}}(b_{22} + d_{22}) + (b + d) = d - \frac{bd_{22}}{b_{22}} ,$$

which contradicts (9). (In the second inequality we used the fact that  $b_{11}b_{22} > b^2$ .) Thus if the Edgeworth paradox occurs we must have products being substitutes in demand ( $b > 0$ ) and substitutes in the profit function too (in the sense that  $b + d > 0$ ). Indeed, given that  $b > 0$  and  $b + d > 0$ , once the own-cost effect (8) is negative then the cross-cost effect (9) is automatically negative as well. We summarise this discussion as follows:

**Proposition 2** *Suppose there are two products and marginal cost and demand is linear such that products are substitutes in demand (i.e.,  $b > 0$ ) and profit (i.e.,  $b + d > 0$ ). Then the Edgeworth paradox holds if and only if (8) is satisfied.*

A further observation is that (9) together with  $b > 0$  implies that either  $d_{22}$  is strictly negative or  $d$  is strictly positive.<sup>8</sup> The former means that product 2's marginal cost falls

---

<sup>8</sup>In particular,  $d_{22} < d$ , which was the condition in the previous section to obtain the Edgeworth paradox with perfect substitutes.

when the firm supplies more of that product, while the latter implies that product 2's marginal cost falls when the firm supplies less of product 1. As we explore more fully in the next section, one way to obtain Edgeworth's paradox is when the reduction in  $x_1$  due to the tax induces a particularly large expansion in  $x_2$ , and having product 2's marginal cost fall with a reduction in  $x_1$  or an increase in  $x_2$  are two ways to achieve this effect.

*Patterns of cost passthrough:* Moving beyond the focus on the Edgeworth paradox, a natural question is what patterns of cost-passthrough are possible in this framework with linear marginal costs and demands. Expression (6) shows that the matrix of cost passthrough terms is  $\Gamma = BM^{-1}$ . Since the two matrices  $B$  and  $M^{-1}$  can be freely chosen subject only to the constraints that they both be positive-definite, the only constraint on the matrix of cost passthrough terms  $\Gamma$  is that it be the product of two positive definite matrices.<sup>9</sup> Such a matrix need not be symmetric—see (7) for instance—and we might have  $\partial p_i/\partial c_j \neq \partial p_j/\partial c_i$ . (By contrast, we always have optimal quantities in (5) satisfying  $\partial x_i/\partial c_j = \partial x_j/\partial c_i$ .) Balantine (1968, Theorem 2) shows that a matrix  $\Gamma$  is the product of two positive definite matrices if and only if it is similar to a positive definite matrix.<sup>10</sup> Thus, we have the following immediate result:

**Proposition 3** *With linear demands and marginal costs,  $\Gamma$  is a feasible cost passthrough matrix if and only if it is similar to a positive definite matrix.*

This result holds more generally. If demands and costs are smooth functions, then at the firm's optimal set of quantities its inverse demand function are locally linear (with cross-quantity effects given by a matrix  $B$  which is positive definite) and where its profit function is local quadratic (where the second-order condition for optimality implies the matrix of second derivatives is (minus) a positive definite matrix  $M$ ). Thus, for small changes in unit costs, the matrix of cost passthrough terms is similar to a positive definite matrix.

Proposition 3 implies that the trace of  $\Gamma$  is positive (i.e., on average the own-cost passthrough terms are positive) and the determinant of  $\Gamma$  is positive (so in broad terms own-cost passthrough terms dominate cross-cost terms). The condition that  $\Gamma$  be similar

---

<sup>9</sup>If  $M$  is positive-definite then so is  $M^{-1}$ . Given the linear demand matrix  $B$ , one can construct any positive-definite matrix  $M$  by choosing the cost matrix  $D = \frac{1}{2}M - B$ .

<sup>10</sup>Two matrices  $A$  and  $B$  are said to be similar if they are related as  $A = Z^{-1}BZ$  for some invertible matrix  $Z$ . Similar matrices have the same determinant, trace, and eigenvalues.

to a positive definite matrix is almost the same as requiring that all eigenvalues of  $\Gamma$  are real and positive. If  $\Gamma$  is similar to a positive definite matrix, then it has the same (positive) eigenvalues as the positive definite matrix. Conversely, if  $\Gamma$  is a matrix with *distinct* positive eigenvalues then it is similar to a (positive definite) diagonal matrix with entries given by its eigenvalues.<sup>11</sup>

Thus, if  $\Gamma$  has distinct positive eigenvalues then it is a feasible matrix of cost passthrough terms. With two products,  $\Gamma$  has two distinct positive eigenvalues when the characteristic equation  $\det(\Gamma - \lambda I) = 0$  has distinct positive roots for  $\lambda$ , i.e., when

$$\gamma_{11}\gamma_{22} > \gamma_{12}\gamma_{21} \text{ and } \gamma_{11} + \gamma_{22} > 2\sqrt{\gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21}}. \quad (10)$$

If the Edgeworth paradox arises for  $c_1$  (i.e., if  $\gamma_{11} < 0$  and  $\gamma_{21} < 0$ ), then (10) requires that  $\gamma_{22} > 0$  and  $\gamma_{12} > 0$  so that an increase in the other cost must cause both prices to rise. Asymmetry in cross-cost passthrough is crucial for the paradox: if  $\gamma_{12}$  and  $\gamma_{21}$  merely have the same sign, then (10) implies that both own-cost passthrough terms are positive. Another kind of paradox occurs if a given price is a decreasing function of both costs, e.g., if  $\gamma_{11} < 0$  and  $\gamma_{12} < 0$ . Clearly this is possible, and can be achieved by swapping  $\gamma_{12}$  with  $\gamma_{21}$  in any instance of the Edgeworth paradox. In such a situation, the other price will increase with both costs. In fact, if  $\gamma_{11} < 0$  then (10) implies that either  $\gamma_{12} < 0$  or  $\gamma_{21} < 0$  (but not both).

As well as being similar to a positive definite matrix, the passthrough matrix  $\Gamma$  has additional interesting properties. Given demand and profit matrices  $B$  and  $M$ , both positive definite, there exists an invertible matrix  $\hat{Z}$  such that  $\hat{Z}^T M \hat{Z} = I$  and  $\hat{Z}^T B \hat{Z} = \Lambda$ , a diagonal matrix with positive entries. (See, for instance, Strang (2006, p. 361).) It is easier to work with the inverse matrix  $Z \equiv \hat{Z}^{-1}$ . Thus, with the linear change of variables given by  $y = Zx$ , we have  $x^T M x = y^T \hat{Z}^T M \hat{Z} y = y^T y$ , and the firm's profit becomes additively separable in the variables  $y_1, \dots, y_n$ . Likewise, consumer surplus, which is  $s = \frac{1}{2} x^T B x$  is equal to the separable expression  $\frac{1}{2} y^T \Lambda y$  when expressed in terms of the  $y$  quantities. Since  $M = Z^T Z$ , the matrix  $Z$  is a “square root” of  $M$ . Then  $\Gamma$  can be written as

$$\Gamma = B M^{-1} = [Z^T \Lambda Z] [Z^{-1} (Z^T)^{-1}] = Z^T \Lambda (Z^T)^{-1},$$

and hence

$$\Lambda = (Z^T)^{-1} \Gamma Z^T$$

---

<sup>11</sup>If  $\Gamma$  has repeated eigenvalues then it may or may not be diagonalizable, and in the latter case it is not similar to a positive definite matrix.

so that the matrix  $Z^T$  diagonalizes  $\Gamma$ . In particular,  $\Lambda$  consists of the eigenvalues of  $\Gamma$  and  $Z^T$  consists of associated eigenvectors of  $\Gamma$  arranged in columns, i.e.,  $Z$  consists of eigenvectors of  $\Gamma$  arranged as rows. Thus, via its eigenvectors, the passthrough matrix  $\Gamma$  contains the information required to find a matrix  $Z$  such that with the change of variables  $y = Zx$  both profit and consumer surplus become separable functions of  $y_1, \dots, y_n$ .

To illustrate, consider the demand and profit matrices

$$B = \begin{pmatrix} \frac{7}{2} & 3 \\ 3 & \frac{8}{3} \end{pmatrix}, \quad M = \begin{pmatrix} 10 & 8 \\ 8 & \frac{20}{3} \end{pmatrix},$$

both of which are positive definite.<sup>12</sup> These induce the passthrough matrix

$$\Gamma = BM^{-1} = \frac{1}{4} \begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix},$$

which exhibits the Edgeworth paradox for  $c_1$ . Here,  $\Gamma$  has eigenvalues  $\lambda_1 = 1/4$  and  $\lambda_2 = 1/2$ , and respective eigenvectors that are (proportional to)  $\sqrt{\frac{2}{3}} \times (3, 2)$  and  $(2, 2)$ , so that

$$Z = \begin{pmatrix} 3\sqrt{\frac{2}{3}} & 2\sqrt{\frac{2}{3}} \\ 2 & 2 \end{pmatrix}.$$

(The particular normalization for the eigenvectors is chosen so  $Z$  is a square root of  $M$ .) Thus, with the change of variables  $y_1 = \sqrt{\frac{2}{3}}(3x_1 + 2x_2)$  and  $y_2 = 2x_1 + 2x_2$ , both profit and consumer surplus become separable in  $y_1$  and  $y_2$ .

### 3.4 Two-product analysis

The possibility of the paradox with linear quadratic costs and demands raises the question of how prevalent it is more generally. We now focus on the two-product case with a profit-maximizing firm, and show that, for any demand system with substitutes, there exists a cost function such that the paradox occurs. This validates Edgeworth's remark, quoted above, that a flexible choice of cost functions expands the scope for examples of the paradox.

Suppose then that there are two products, that utility  $u$  is differentiable and strictly concave and that the cost function is differentiable, and that the marginal cost of product

---

<sup>12</sup>For instance, these matrices arise when inverse demand and cost are given by

$$p_1 = 18 - \frac{7}{2}x_1 - 3x_2, \quad p_2 = \frac{44}{3} - \frac{8}{3}x_2 - 3x_1, \quad c(x) = \frac{1}{6}(3x_1 + 2x_2)^2,$$

in which case the firm chooses profit-maximizing quantities equal to  $x_1 = x_2 = 1$ .

1 increases by  $t_1 > 0$ . Then  $x_1$  will decrease (except in the trivial case where  $x_1$  is already zero). For given  $x_1$ , the firm's optimal choice of  $x_2$ , denoted  $x_2(x_1)$ , does not depend on  $t_1$ . Standard comparative statics results show that optimal  $x_2$  increases [decreases] with  $x_1$  according to whether  $\pi_{12}(x)$  is positive [negative], where subscripts of  $\pi(x)$  denote its partial derivatives. If both quantities fall then at least one price must rise.<sup>13</sup> So to observe the Edgeworth paradox it is necessary that optimal  $x_2$  decreases with  $x_1$ , i.e., that  $\pi_{12} < 0$ . Moreover, for  $p_1$  to fall when  $x_1$  falls and  $x_2$  rises it is necessary that

$$\frac{\partial p_1}{\partial x_2} = \frac{\partial p_2}{\partial x_1} < 0 ,$$

so that the products are substitutes in demand.

Optimal  $x_2$  given  $x_1$  satisfies the first-order condition  $\pi_2(x_1, x_2(x_1)) = 0$ , and differentiating this condition yields

$$x_2'(x_1) = -\frac{\pi_{12}}{\pi_{22}} ,$$

where the second-order condition requires  $\pi_{22} < 0$ . It follows that the net impact on  $p_1$  of the reduction in  $x_1$  and consequent increase in  $x_2$  is negative if

$$\frac{\partial p_1}{\partial x_2} \frac{\pi_{12}}{\pi_{22}} - \frac{\partial p_1}{\partial x_1} < 0 , \quad (11)$$

while the impact on  $p_2$  is negative if

$$\frac{\partial p_2}{\partial x_2} \frac{\pi_{12}}{\pi_{22}} - \frac{\partial p_2}{\partial x_1} < 0 . \quad (12)$$

Condition (11) can be written as

$$\frac{\pi_{12}}{\pi_{22}} > \frac{\partial p_1 / \partial x_1}{\partial p_1 / \partial x_2} . \quad (13)$$

Since the concavity of utility implies that

$$\frac{\partial p_1}{\partial x_1} \frac{\partial p_2}{\partial x_2} \geq \frac{\partial p_1}{\partial x_2} \frac{\partial p_2}{\partial x_1} ,$$

it follows that if the own-cost effect on price (13) is negative then so is the cross-cost effect (12), and so the paradox occurs simply when (13) holds. Condition (13) implies that either  $\frac{\partial p_1 / \partial x_2}{\partial p_1 / \partial x_1} > 1$  or  $\pi_{12} / \pi_{22} > 1$ , i.e., that cross-effects are dominant for either inverse

---

<sup>13</sup>If  $\hat{x} \neq x$  then the strict concavity of  $u$  implies that  $u(\hat{x}) < u(x) + (\hat{x} - x) \cdot p(x)$  and  $u(x) < u(\hat{x}) + (x - \hat{x}) \cdot p(\hat{x})$ , and adding these implies  $(\hat{x} - x) \cdot (p(\hat{x}) - p(x)) < 0$ . Therefore, if  $\hat{x}_k \leq x_k$  for each  $k$  with at least one strict inequality then  $p_i(\hat{x}) > p_i(x)$  for some product  $i$ .

demand  $p_1$  or for profit  $\pi$  (or both). Note that the concavity of  $u$  and  $\pi$  implies that the corresponding condition to (13) for product 2 cannot hold simultaneously, i.e., that the Edgeworth paradox can occur for at most one of the two products.

Condition (13) involves both the demand function and the profit function. However, one can choose the demand function and the profit function independently, by means of a suitable choice of cost function. Indeed, for any given revenue function  $r(x)$ , if one chooses the cost function  $c(x) = r(x) - \pi(x)$  one can implement the profit function  $\pi(x)$ . This means for any given demand system involving substitutes, one can construct a cost function which leads to the Edgeworth paradox.

The method is as follows. Take a demand system  $p_1(x)$  and  $p_2(x)$  with  $\partial p_1/\partial x_2 = \partial p_2/\partial x_1 < 0$ , and let  $r(x) \equiv x_1 p_1(x) + x_2 p_2(x)$  be the associated revenue function. Pick a quantity vector  $x^*$  such that  $r$  is increasing in  $x_1$  and  $x_2$  at  $x^*$ . Then find a (concave) profit function  $\pi$  such that  $\pi$  is maximized at  $x = x^*$  and where (13) holds when evaluated at  $x = x^*$ . For any negative constants  $\pi_{11}$ ,  $\pi_{22}$  and  $\pi_{12}$  such that  $\pi_{11}\pi_{22} > \pi_{12}^2$ , the quadratic function

$$\pi(x) = \frac{1}{2}\pi_{11}x_1^2 + \frac{1}{2}\pi_{22}x_2^2 + \pi_{12}x_1x_2 - (\pi_{11}x_1^* + \pi_{12}x_2^*)x_1 - (\pi_{22}x_2^* + \pi_{12}x_1^*)x_2 \quad (14)$$

is concave, is maximized at  $x = x^*$  (with positive profits), and has second derivatives equal to the constant terms  $\pi_{11}$ ,  $\pi_{22}$  and  $\pi_{12}$ . Since these constant terms can be freely chosen (subject to the requirement  $\pi_{11}\pi_{22} > \pi_{12}^2$ ), we can always find such a profit function where  $\pi_{12}/\pi_{22}$  is large enough to satisfy (13). The final step is then to choose the cost function  $c(x) = r(x) - \pi(x)$ , which implements the profit function. Since  $r(x)$  is increasing at  $x = x^*$ , so is  $c(x)$ . With such a cost function, a small tax  $t_1 > 0$  on product 1 will induce the firm to reduce both its prices.

We summarise this discussion in the following result.

**Proposition 4** *For any differentiable two-product demand system with substitute products, there exists a cost function such that Edgeworth's paradox arises.*

## 4 The Surplus Paradox

For Edgeworth's paradox to occur, all prices must fall when the cost of one product rises. Consider instead the *surplus paradox* that he mentioned, where consumer surplus increases

when the cost of one product rises.<sup>14</sup> Clearly the surplus paradox occurs whenever Edgeworth’s paradox does, but it can occur more generally and with less in the way of “manipulation” needed.

#### 4.1 Further two-product analysis

When utility  $u(x)$  is smooth and strictly concave, the surplus function  $s(x)$  must strictly increase with at least one  $x_k$ .<sup>15</sup> However, when (some) products are complements it is possible that  $s(x)$  decreases with other quantities. In the case of two products, a tax on product 1 will reduce the firm’s supply of  $x_1$ , and similarly to (11) the net impact on consumer surplus is positive when

$$s_2 \frac{\pi_{12}}{\pi_{22}} - s_1 > 0 . \quad (15)$$

Here, if  $s$  decreases with  $x_1$  then (15) is very likely to hold. This is because negative  $s_1$  implies that products are complements, and this is likely to mean that products are also complements in the profit function in the sense that  $\pi_{12} > 0$ . Since  $s_1 < 0$  implies  $s_2 > 0$ , it then follows that (15) holds.

More generally, a similar argument to that in Proposition 4 shows that any differentiable demand system (regardless of whether products are substitutes or complements) can lead to the surplus paradox with a suitable choice of cost function. For let  $u(x)$  be some utility function, leading to revenue  $r(x)$  and consumer surplus  $s(x)$ . Pick a quantity vector  $x^*$  such that  $r$  is increasing in  $x_1$  and  $x_2$  at  $x^*$ . The function  $s$  cannot have both  $s_1 = 0$  and  $s_2 = 0$  at  $x^*$ , and so label products so that  $s_2(x^*) \neq 0$ . We seek a concave profit function  $\pi$  such that (15) holds when evaluated at  $x = x^*$ . For any constants  $\pi_{11}$ ,  $\pi_{22}$  and  $\pi_{12}$  such that  $\pi_{11} < 0$  and  $\pi_{11}\pi_{22} > \pi_{12}^2$ , the quadratic function (14) is concave, is maximized at  $x = x^*$  (with positive profits), and has second derivatives equal to the constant terms  $\pi_{11}$ ,  $\pi_{22}$  and  $\pi_{12}$ . Since these constant terms can be freely chosen (subject to the requirements  $\pi_{11} < 0$  and  $\pi_{11}\pi_{22} > \pi_{12}^2$ ), we can always find such a profit function where  $\pi_{12}/\pi_{22}$  satisfies (15).

---

<sup>14</sup>Chen and Schwartz (2015) analyze the effect on consumer surplus (and welfare) of mean-preserving spreads of unit cost in a setting with separate single-product markets that have the same demand conditions. As noted earlier, the surplus paradox cannot arise in the single-product case.

<sup>15</sup>This is because

$$-\sum x_i \frac{\partial}{\partial x_i} s(x) = -\sum x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} u(x) > 0 ,$$

where the inequality is due to the matrix of second derivatives of  $u$  being negative definite, and so at  $\partial s / \partial x_k > 0$  for at least one product  $k$ .



(We choose  $\pi_{12}$  to be positive or negative according to whether  $s_2$  is negative or positive.) The final step is then to choose the cost function  $c(x) = r(x) - \pi(x)$ , which implements the profit function. Since  $r(x)$  is increasing at  $x = x^*$ , so is  $c(x)$ . With such a cost function, a small tax  $t_1 > 0$  on product 1 will induce the firm to deliver higher surplus  $s$  to consumers. Thus we have:

**Proposition 5** *For any differentiable two-product demand system there exists a cost function such that the surplus paradox arises.*

## 4.2 An Edgeworth-Ramsey connection

The surplus paradox has a simple connection with Ramsey pricing. Recall that the monopolist in our model is assumed to maximize  $\phi(x) = \pi(x) + \alpha s(x)$ , where  $\alpha$  is the weight on consumer surplus relative to profit. A natural question is how the optimal quantity  $x_k$  of product  $k$  varies with  $\alpha$ . As the next proposition records, the answer is that the rate that consumer surplus varies with the cost of product  $k$  is equal to minus the rate that  $x_k$  varies with  $\alpha$ . Therefore, if  $x_k$  decreases with  $\alpha$  then the surplus paradox occurs and consumer surplus increases with the tax  $t_k$ . Thus there is a close Edgeworth-Ramsey connection.

**Proposition 6** *Suppose that per-unit tax  $t_k$  is imposed on product  $k$ . Then*

$$\frac{\partial s}{\partial t_k} = -\frac{\partial x_k}{\partial \alpha} . \quad (16)$$

**Proof.** Define

$$\hat{\phi}(t_k, \alpha) \equiv \max_x : \pi(x) - t_k x_k + \alpha s(x)$$

as maximum weighted welfare with Ramsey parameter  $\alpha$  and tax  $t_k$ . By the envelope theorem

$$\frac{\partial \hat{\phi}}{\partial t_k} = -x_k \quad ; \quad \frac{\partial \hat{\phi}}{\partial \alpha} = s$$

and the symmetry of the cross derivatives of  $\hat{\phi}$  entails (16). ■

A simple revealed preference argument shows that  $s$  necessarily increases with  $\alpha$ , and so Proposition 6 can be interpreted as saying that the surplus paradox arises for a cost increase for product  $k$  if the most profitable quantity of product  $k$  needed to achieve a target consumer surplus  $s$  decreases with  $s$ . In addition, since  $s$  increases with  $\alpha$  and  $s(x)$

cannot be a decreasing function of all quantities  $x_k$ , it cannot be that the surplus paradox occurs for cost increases in *all* products. In particular, with two products the surplus paradox can occur for a cost increase for at most one of the products.

Using Proposition 6 we can apply Ramsey pricing results to understand when the surplus paradox might arise. For instance, AV connects Ramsey pricing to Cournot competition. In particular, Section IIC in AV implies that Cournot competition between  $m$  symmetric multiproduct firms with cost function satisfying

$$c(x) \text{ is convex and homogeneous degree 1} \tag{17}$$

has the same outcome as the monopoly Ramsey problem with weight  $\alpha = (m - 1)/m$ . Proposition 6 then implies that, if entry into the Cournot market would have caused equilibrium supply of product  $k$  to fall, then an industry-wide tax on that product will cause consumer surplus to rise.

A theme of AV is that under certain conditions optimal quantities move *equiproportionately* as the Ramsey weight  $\alpha$  on consumer surplus varies, in which case the surplus paradox cannot occur. It is well known that the equiproportional property holds near the first-best (when  $\alpha$  is close to 1) when (17) holds (see section IIB in AV). Thus when  $\alpha \approx 1$  the only way to obtain either paradox is to have cost functions outside the class (17). For instance, as already noted, Hotelling (1932, section 7) gives an example with linear demand and a *quadratic* cost function in which Edgeworth's paradox arises with  $\alpha = 1$ . Section III in AV considers the situation where the cost function satisfies (17) and consumer surplus  $s$  is *homothetic* in  $x$ , i.e.,  $s$  is an increase function of the scalar "composite quantity"  $q(x)$  where  $q(\cdot)$  is homogenous degree 1 in outputs  $x$ . (Consumer surplus is homothetic in  $x$  when utility  $u$  is homothetic, and also when demands are linear or take a Logit form. More generally, Proposition 2 in AV shows that  $s$  is homothetic in  $x$  if the utility function  $u$  takes the form  $u(x) = h(x) + g(q(x))$ , where  $h$  and  $q$  are homogeneous degree 1 functions.) Proposition 3 in AV shows in this case that Ramsey quantities increase equiproportionately as  $\alpha$  increases, so that neither paradox can occur. The reason is that the cost of producing composite quantity  $q$  increases if the cost of any component product rises, and this induces the firm to reduce  $q$  and so reduce consumer surplus.

An implication of Proposition 6 is that the surplus paradox can be found wherever the profit-maximizing quantity of some product exceeds its efficient level, for in that case there must be a range of  $\alpha$  over which  $x_k$  falls with  $\alpha$ . This phenomenon of excessive monopoly

supply of one product can be viewed as the quantity analogue to a monopoly price being below marginal cost. We now give two examples of this phenomenon. They share the feature that total quantity  $X$  is unchanged as  $\alpha$  varies, so (unless the profit-maximizing and efficient allocation of that quantity happened to coincide) one product must be in greater supply with profit-maximization than with marginal cost pricing.

*Hotelling preferences:* In the spirit of Hotelling (1929), consider a firm with two products located at each end of the unit interval  $[0, 1]$  with unit costs  $c_1$  and  $c_2$  respectively. Consumers of mass 1 are uniformly distributed along the line and wish to buy or other product (or neither). Their willingness to pay for a product is  $1 - \tau z$ , where  $z$  is their distance travelled and  $\tau$  is the transport cost. Assume  $0 \leq c_2 - c_1 < \tau$ , which ensures an interior solution with both profit maximization and with marginal-cost pricing. Assume also that  $\frac{1}{2}(c_1 + c_2) < 1 - \tau$ , which ensures that the firm will optimally choose to serve all consumers, and so total output does not depend on costs over this range. With marginal cost pricing the quantity of the high-cost product is

$$\tilde{x}_1 = \frac{1}{2} \left( 1 - \frac{c_1 - c_2}{\tau} \right) ,$$

whereas with profit-maximization it is

$$\hat{x}_1 = \frac{1}{2} \left( 1 - \frac{c_1 - c_2}{2\tau} \right) > \tilde{x}_1 .$$

So there is more asymmetry between  $x_1$  and  $x_2$  with marginal cost pricing than with profit-maximizing monopoly. In the latter case, increasing  $c_1$  has the effect of increasing asymmetry, which is good for consumers and so we have the surplus paradox. The reason is that the consumer indifferent between products gets zero surplus, and the surplus of others is  $\tau$  times their distance from the indifferent consumer. The average distance increases with asymmetry. In this example total quantity is at the efficient level with profit-maximization but is inefficiently allocated between products whenever cost levels differ. Thus increasing the cost of the more costly product 1 will benefit consumers in aggregate, and the surplus paradox always exists.

*Discrete choice where one valuation is known:* Again there are two products and a unit mass of consumers. Here, the valuation for product 1 is known to be  $v_1 \equiv 1$ , while the valuation  $v_2$  for product 2 is uniformly distributed on the interval  $[0, 1]$ . Suppose initially

that  $c_1 = c_2 = 0$  in which case the firm can obtain first-best profit by choosing  $p_1 = 1$  and fully extracting consumer surplus (and setting  $p_2 \geq 1$  so that consumers do not wish to buy product 2).

If cost  $c_1$  is now increased to  $c_1 > 0$ , the firm will wish instead to supply some consumers with the costless product 2. The most profitable strategy is to choose  $p_1 = 1$  and to choose  $0 < p_2 < 1$ , in which case all consumers with  $v_2 > p_2$  will buy product 2 and all other consumers will buy product 1, yielding profit

$$p_2(1 - p_2) + (1 - c_1)p_2 . \quad (18)$$

This profit is maximized at  $p_2 = 1 - \frac{1}{2}c_1$  which falls with  $c_1$ . Since consumers gain no surplus from product 1, it follows that consumer surplus rises with  $c_1$ . The most profitable supply for product 1 is then  $\hat{x}_1 = 1 - \frac{1}{2}c_1$ , while the supply of this product with marginal-cost pricing is  $\tilde{x}_1 = 1 - c_1$ , which is smaller.<sup>16</sup> The same result would occur if the known-value product 1 were eliminated altogether. Indeed product elimination may be seen as the ultimate cost increase. For product elimination to increase consumer surplus is not uncommon. For example, in the  $n$ -product version of the standard discrete choice model with independent uniformly-distributed values, profit-maximization leaves consumers with little surplus when  $n$  is large, as a consumer's maximum value is likely to be close to 1, unlike when  $n = 1$  when the firm cannot fully extract consumer surplus.

These examples shared the feature that the firm's choice of total output  $X$  was unaffected by cost changes over the relevant range of costs. In effect, the firm's choice of quantities was taken from the constrained set  $\mathfrak{X} = \{x \mid X = \sum_k x_k = 1\}$ . Likewise, in the situation with uniform pricing in section 3.2, we saw that the uniform price decreases with the cost of a product when the constrained set  $\mathfrak{X}$  took the form of a downward-sloping curve in  $\mathbb{R}_+^2$ . This result holds quite generally. If the firm chooses its quantities from a constrained set  $\mathfrak{X}$  that has the property that an increase in one quantity necessarily causes another quantity to fall, then the surplus paradox will hold. Proposition 6 continues to hold when quantities are chosen from a (suitably smooth) constrained set  $\mathfrak{X}$  rather than  $\mathbb{R}_+^n$ , and as long as increasing the Ramsey weight  $\alpha$  has *any* impact on its choice of  $x$  it

---

<sup>16</sup>The same effect can be seen less transparently in the more familiar situation where the valuation of each product is independently and uniformly distributed on  $[0, 1]$ . Then one can show that when  $c_2 = 0$  and when  $c_1$  is sufficiently close to one, consumer surplus rises when  $c_1$  is increased still further.

must then cause one quantity to fall. For instance, if the firm operates under some form of average price regulatory constraint then this will usually entail this form of quantity constraint  $\mathfrak{X}$ , and so an increase in some product’s cost will cause consumer surplus to rise.

## 5 Conclusion

Edgeworth’s paradox highlights that comparative statics in multi-product settings can be very different from what happens in the familiar single-product case. We have provided various simple examples in which all prices fell as a cost level increased, and have shown that this possibility always exists for some cost function in the two-product case with substitutes. For the case of linear demands and marginal costs we established that a matrix is a possible cost-passthrough matrix if and only if it is similar to a positive definite matrix, and that its eigenvectors provide a way to express profit and revenue in terms of composite quantities with no cross-effects. We then explored the milder consumer surplus paradox, and related it to Ramsey pricing. A common theme was that the paradox in either form involves the most profitable output of product  $k$  decreasing with consumer surplus. This is akin to product  $k$  being an inferior good in consumer theory—i.e., one for which demand decreases as income rises. Although Edgeworth’s pricing paradox is rarer than the surplus paradox, examples of either kind are not hard to find once one considers situations outside the most familiar specifications for multiproduct cost and demand systems.

## REFERENCES

- Armstrong, M. and J. Vickers (2018), ‘Multiproduct pricing made simple’, *Journal of Political Economy*, 126, 1444-1471.
- Ballantine, C.S. (1968), ‘Products of positive definite matrices, III’, *Journal of Algebra*, 10, 174-182.
- Chen, Y. and M. Schwartz (2015), ‘Differential pricing when costs differ: a welfare analysis’, *RAND Journal of Economics*, 46, 442-460.
- Edgeworth, F. (1925), ‘The pure theory of monopoly’, in *Papers Relating to Political Economy*, Macmillan, London, for the Royal Economic Society.
- Edlin, A. and C. Shannon (1998), ‘Strict monotonicity in comparative statics’, *Journal of Economic Theory*, 81, 201-219.

Hotelling, H. (1929), ‘Stability in competition’, *Economic Journal*, 39, 41-57.

Hotelling, H. (1932), ‘Edgeworth’s taxation paradox and the nature of demand and supply functions’, *Journal of Political Economy*, 40, 577-616.

Miklos-Thal, J. and G. Shaffer (2021), ‘Pass-through as an economic tool: on exogenous competition, social incidence, and price discrimination’, *Journal of Political Economy*, 129, 323-335.

Strang, G. (2006), *Linear Analysis and its Application*, 4th edition, Belmont CA, Thomson, Brooks/Cole.

Weyl, G., and M. Fabinger (2013), ‘Pass-through as an economic tool: principles of incidence under imperfect competition’, *Journal of Political Economy*, 121, 528–583.