

## Research Article

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# Sharp conditions on global existence and blow-up in a degenerate two-species and cross-attraction system

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**Abstract:** We consider a degenerate chemotaxis model with two-species and two-stimuli in dimension  $d \geq 3$  and find two critical curves intersecting at one point which separate the global existence and blow up of weak solutions to the problem. More precisely, above these curves (i.e. subcritical case), the problem admits a global weak solution obtained by the limits of strong solutions to an approximated system. Based on the second moment of solutions, initial data are constructed to make sure blow up occurs in finite time on and below these curves (i.e. critical and supercritical cases). In addition, the existence or non-existence of minimizers of free energy functional is discussed on the critical curves and the solutions exist globally in time if the size of initial data is small. We also investigate the crossing point between the critical lines in which a refined criteria in terms of the masses is given again to distinguish the dichotomy between global existence and blow up. We also show that the blow ups is simultaneous for both species.

**Keywords:** Degenerate parabolic system, chemotaxis, variational methods, global existence, blow up

**MSC:** 35K65, 92C17, 35J20, 35A01, 35B44

## 1 Introduction

The interaction motion of two cell populations in breast cancer cell invasion models in  $\mathbb{R}^d$  ( $d \geq 3$ ) has been described by the following chemotaxis system with two chemicals and nonlinear diffusion (cf. [20, 30])

$$\begin{cases} u_t = \Delta u^{m_1} - \nabla \cdot (u \nabla v), & x \in \mathbb{R}^d, t > 0, \\ -\Delta v = w, & x \in \mathbb{R}^d, t > 0, \\ w_t = \Delta w^{m_2} - \nabla \cdot (w \nabla z), & x \in \mathbb{R}^d, t > 0, \\ -\Delta z = u, & x \in \mathbb{R}^d, t > 0, \\ u(x, 0) = u_0(x), \quad w(x, 0) = w_0(x), & x \in \mathbb{R}^d, \end{cases} \quad (1.1)$$

where  $m_1, m_2 > 1$  are constants. Here,  $u(x, t)$  and  $w(x, t)$  denote the density of the macrophages and the tumor cells,  $v(x, t)$  and  $z(x, t)$  denote the concentration of the chemicals produced by  $w(x, t)$  and  $u(x, t)$ ,

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respectively. For simplicity, the initial data are assumed to satisfy

$$\begin{aligned} u_0 &\in L^1(\mathbb{R}^d; (1 + |x|^2)dx) \cap L^\infty(\mathbb{R}^d), \quad \nabla u_0^{m_1} \in L^2(\mathbb{R}^d) \quad \text{and} \quad u_0 \geq 0, \\ w_0 &\in L^1(\mathbb{R}^d; (1 + |x|^2)dx) \cap L^\infty(\mathbb{R}^d), \quad \nabla w_0^{m_2} \in L^2(\mathbb{R}^d) \quad \text{and} \quad w_0 \geq 0. \end{aligned} \quad (1.2)$$

Since the solutions to the Poisson equations can be written by the Newtonian potential such as

$$v(x, t) = \mathcal{K} \star w = c_d \int_{\mathbb{R}^d} \frac{w(y, t)}{|x - y|^{d-2}} dy, \quad z(x, t) = \mathcal{K} \star u = c_d \int_{\mathbb{R}^d} \frac{u(y, t)}{|x - y|^{d-2}} dy$$

with  $\mathcal{K}(x) = \frac{c_d}{|x|^{d-2}}$  and  $c_d$  is the surface area of the sphere  $\mathbb{S}^{d-1}$  in  $\mathbb{R}^d$ , the original system (1.1) can be regarded as the interaction between two populations

$$\begin{cases} u_t = \Delta u^{m_1} - \nabla \cdot (u \nabla \mathcal{K} \star w), & x \in \mathbb{R}^d, t > 0, \\ w_t = \Delta w^{m_2} - \nabla \cdot (w \nabla \mathcal{K} \star u), & x \in \mathbb{R}^d, t > 0, \\ u(x, 0) = u_0(x), \quad w(x, 0) = w_0(x), & x \in \mathbb{R}^d, \end{cases} \quad (1.3)$$

where it follows that the solutions obey the mass conservation

$$M_1 := \int_{\mathbb{R}^d} u(x, t) dx = \int_{\mathbb{R}^d} u_0(x) dx \quad \text{and} \quad M_2 := \int_{\mathbb{R}^d} w(x, t) dx = \int_{\mathbb{R}^d} w_0(x) dx.$$

The associated free energy functional  $\mathcal{F}$  for (1.1) or (1.3) is given by

$$\mathcal{F}[u(t), w(t)] = \frac{1}{m_1 - 1} \int_{\mathbb{R}^d} u^{m_1} dx + \frac{1}{m_2 - 1} \int_{\mathbb{R}^d} w^{m_2} dx - c_d \mathcal{H}[u, w],$$

which is non-increasing with respect to time since for smooth cases it satisfies the following decreasing property

$$\begin{aligned} \frac{d}{dt} \mathcal{F}[u(t), w(t)] &= - \int_{\mathbb{R}^d} u \left| \frac{m_1}{m_1 - 1} \nabla u^{m_1 - 1} - \nabla v \right|^2 dx \\ &\quad - \int_{\mathbb{R}^d} w \left| \frac{m_2}{m_2 - 1} \nabla w^{m_2 - 1} - \nabla z \right|^2 dx, \end{aligned}$$

where

$$\mathcal{H}[u, w] = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{u(x, t) w(y, t)}{|x - y|^{d-2}} dx dy.$$

The chemotaxis system consisting of only one population and one chemical signal is the well-known Keller-Segel model by taking into account volume filling constraints (see [9, 28, 38]) reading as

$$\begin{cases} u_t = \Delta u^{m_1} - \nabla \cdot (u \nabla \mathcal{K} \star u), & x \in \mathbb{R}^d, t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \quad (1.4)$$

which has been immensely investigated over the last decades. See [3, 13, 23, 28, 39] for the biological motivations and a complete overview of mathematical results for related more general aggregation-diffusion models. Here the diffusion exponent  $m_1$  is taken to be supercritical  $0 < m_1 < m_c := 2 - 2/d$ , critical  $m_1 = m_c$  and subcritical  $m_1 > m_c$  if  $d \geq 3$ . The critical number  $m_c$  is chosen to produce a balance between diffusion and potential drift in mass invariant scaling. For the subcritical  $m_1 > m_c$  in the sense that diffusion dominates, the solutions are globally solvable without any restriction on the size of the initial data [29, 43, 45]. However, in the supercritical case, the attraction is stronger leading to a coexistence of global existence of solutions

and blow-up behavior. More precisely, finite-time blow up occurs for large initial data, see [11] for  $m_1 = 1$ , [17] for  $m_1 = 2d/(d+2)$ , [16] for  $2d/(d+2) < m_1 < m_c$ , and [43] for  $1 < m_1 < m_c$ . But there also exists a global weak solution with decay properties under some smallness condition on the initial mass [4, 17, 18, 45]. The critical case  $m_1 = m_c$  is investigated in [6, 44] showing the existence of a sharp mass constant  $M^*$  allowing for a dichotomy: if  $\|u\|_1 = M_1 < M^*$  the solutions exist for all time, whereas if  $M_1 \geq M^*$  there exists solution with non-positive free energy functional blowing up. In addition, such similar dichotomy was found in [8, 19, 24] earlier in dimension  $d = 2$  and linear diffusion  $m_1 = 1$  for (1.4) with  $\mathcal{K}(x) = -1/(2\pi) \log |x|$ , where  $M^*$  was replaced by  $8\pi$ . We also note that the results in [7] prove that solutions blow up as a delta Dirac at the center of mass as time increases in critical mass  $M_1 = 8\pi$ . Sufficient conditions for nonlinear diffusion  $m_1 > 1$  to prevent blow up are derived in [9].

The variational viewpoint to analyse problems of the type (1.4) has also been an active field of research. For instance, there exist a lot of results about the properties of global minimizers of the corresponding free energy functional, including the existence, radial symmetry and uniqueness and so on, since they not only correspond to steady states of (1.4) in some particular cases, but also are candidates for the large time asymptotics of solutions to (1.4). Lion's concentration-compactness principle [36] (see also [2]) can be directly applied to the subcritical  $m_1 > m_c$  if  $d \geq 3$  and allows the existence of minimizer which further satisfies some regularities properties (see [15]). The uniqueness of minimizer in this case is ensured in [33] and it is shown that such minimizer is also an exponential attractor of solutions of (1.4) when the initial data is radially symmetric and compactly supported by using the mass comparison principle (see [29]). In the critical case  $m_1 = m_c$ , the free energy functional does not admit global minimizers except for the critical mass case  $M_1 = M^*$  introduced above [10]. The minimizers were used in [6] to describe the infinite time blow-up profile. For the nonlinear-diffusion in two dimension, the long time asymptotics of solutions is fully characterized in [14] based on the unique existence of radial minimizer [12]. We refer to [5] for a discussion on the existence of many stationary states for  $m_1 = 1$  and  $d = 2$  in the critical case  $M_1 = 8\pi$  and their basins of attraction.

Back to linear two-species system (1.1) in  $d = 2$ , similar to the role of the critical mass  $8\pi$  in (1.4) ([8, 19]), the critical curve  $M_1 M_2 - 4\pi(M_1 + M_2) = 0$  for two species was discovered in [22]: solutions exist globally if  $M_1 M_2 - 4\pi(M_1 + M_2) < 0$  and blow up occurs if  $M_1 M_2 - 4\pi(M_1 + M_2) > 0$ . The key tool for the proof of the global existence part is using the Moser-Trudinger inequality [42]. One can use partial results in [42] to check that minimizers indeed exist in the case  $M_1 M_2 - 4\pi(M_1 + M_2) = 0$ . We also mention that such nonlinear system (1.1) and the one population system (1.4) can be formally regarded as gradient flows of the free energy functional in the probability measure space with the Euclidean Wasserstein metric [1, 25]. For general  $n$ -component multi-populations chemotaxis system, in [26, 27] the authors have made considerable progress on these aspects and obtain the global arguments in the subcritical and critical cases. The Neumann initial-boundary value problem is analysed in [34, 35, 47, 48].

The aim of this paper is to give a thorough understanding of the well-posedness and asymptotic behavior for (1.1) or (1.3) in  $d \geq 3$  and to show the existence or non-existence of global minimizers in critical cases. We make use of bold faces  $\mathbf{m}, \mathbf{A}, \mathbf{B}, \mathbf{I}, \mathbf{M}, \dots$  to denote two-dimensional vectors throughout the paper and assume that  $\mathbf{A} = (a_1, a_2) \leq (\geq) \mathbf{B} = (b_1, b_2)$  means that  $a_1 \leq (\geq) b_1$  and  $b_1 \leq (\geq) b_2$ , respectively. If  $(u, w)$  is a solution of (1.3), then for any  $\lambda > 0$  the following scaling

$$u_\lambda(x, t) = \lambda^{m_2} u(\lambda^{\frac{m_1+m_2-m_1 m_2}{2}} x, \lambda^{m_1} t), \quad w_\lambda(x, t) = \lambda^{m_1} w(\lambda^{\frac{m_1+m_2-m_1 m_2}{2}} x, \lambda^{m_2} t)$$

is also a solution. Obviously, each of  $(u, w)$  preserves the  $L^1$ -norm if  $\mathbf{m} := (m_1, m_2)$  satisfy

$$m_1 m_2 + 2m_1/d = m_1 + m_2, \quad (1.5)$$

and

$$m_1 m_2 + 2m_2/d = m_1 + m_2, \quad (1.6)$$

respectively, whereas the above scaling becomes mass invariant for both  $u$  and  $w$  if and only if  $\mathbf{m} = (m_c, m_c)$ . The curves (1.5) and (1.6) can be shown to be the sharp conditions separating the global existence and blow

up. Our main result in Theorem 1.3 shows the following dichotomy: above the two red curves in Figure 1, in the sense that  $m_1 m_2 + 2m_1/d > m_1 + m_2$  or  $m_1 m_2 + 2m_2/d > m_1 + m_2$ , weak solutions globally exist and blow up occurs below the red curves for certain initial data regardless of their initial masses (see Theorem 1.3). Several results are also obtained at the critical curves (see Theorem 1.4). In addition, the two lines will intersect at the point  $(m_c, m_c)$ . Therefore, we consider the  $(m_1, m_2) \in (1, \infty)^2$  parameter range divided by the following three critical cases (red curve in Figure 1):

Line  $L_1$  :  $m_1 m_2 + 2m_1/d = m_1 + m_2$  with  $m_1 \in (m_c, d/2)$ ,  $m_2 \in (1, m_c)$ ;

Line  $L_2$  :  $m_1 m_2 + 2m_2/d = m_1 + m_2$  with  $m_1 \in (1, m_c)$ ,  $m_2 \in (m_c, d/2)$ ;

The intersection point  $\mathbf{I} := (m_c, m_c)$ .

Based on the above discussion, we say that  $\mathbf{m} = (m_1, m_2)$  is subcritical if

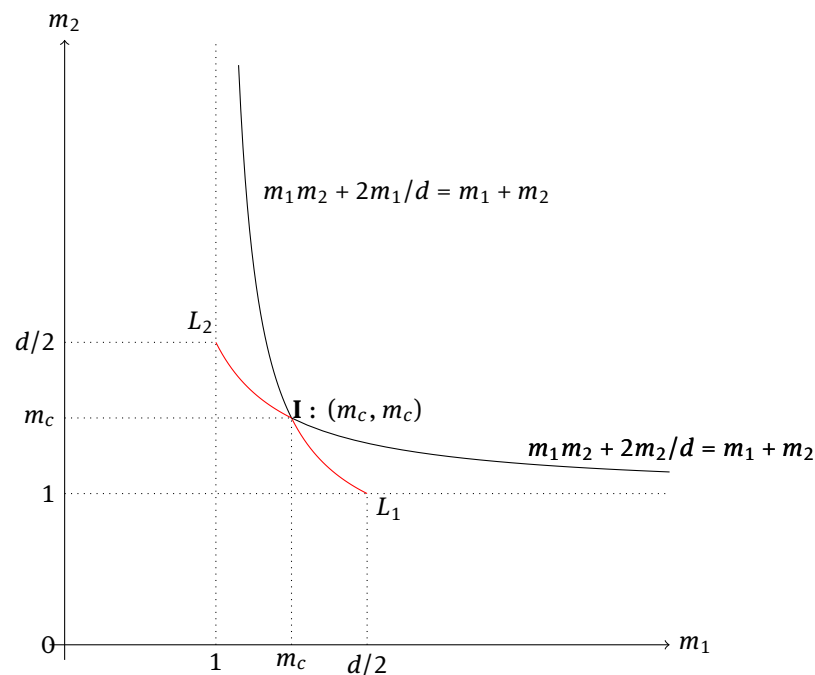


Fig. 1: Parameter lines determining the critical regimes.

$$m_1 m_2 + 2m_1/d > m_1 + m_2 \quad \text{or} \quad m_1 m_2 + 2m_2/d > m_1 + m_2,$$

and  $\mathbf{m} = (m_1, m_2)$  is supercritical if

$$m_1 m_2 + 2m_1/d < m_1 + m_2 \quad \text{and} \quad m_1 m_2 + 2m_2/d < m_1 + m_2.$$

Notice that this corresponds to being above (subcritical) or below (supercritical) the red curves in Figure 1. We also define subsets of  $L^1(\mathbb{R}^d)$  as

$$S_{M_1} := \{f \geq 0 : f \in L^1(\mathbb{R}^d) \cap L^{m_1}(\mathbb{R}^d) \text{ and } \|f\|_1 = M_1\}$$

and

$$S_{M_2} := \{g \geq 0 : g \in L^1(\mathbb{R}^d) \cap L^{m_2}(\mathbb{R}^d) \text{ and } \|g\|_1 = M_2\}.$$

Now the definition of weak solution for (1.1) or (1.3) is give as

**Definition 1.1.** Let  $m_1, m_2 > 1$ ,  $d \geq 3$  and  $T > 0$ . Suppose the initial data  $(u_0, w_0)$  satisfies some classical regularities (1.2). Then  $(u, w)$  of nonnegative functions defined in  $\mathbb{R}^d \times (0, T)$  is called a weak solution if

$$\begin{aligned} & i) (u, w) \in (C([0, T]; L^1(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d \times (0, T)))^2, \\ & (u^{m_1}, w^{m_2}) \in (L^2(0, T; H^1(\mathbb{R}^d)))^2; \\ & ii) (u, w) \text{ satisfies} \\ & \int_0^T \int_{\mathbb{R}^d} u \phi_{1t} dx dt + \int_{\mathbb{R}^d} u_0(x) \phi_1(x, 0) dx = \int_0^T \int_{\mathbb{R}^d} (\nabla u^{m_1} - u \nabla v) \cdot \nabla \phi_1 dx dt, \\ & \int_0^T \int_{\mathbb{R}^d} w \phi_{2t} dx dt + \int_{\mathbb{R}^d} w_0(x) \phi_2(x, 0) dx = \int_0^T \int_{\mathbb{R}^d} (\nabla w^{m_2} - w \nabla z) \cdot \nabla \phi_2 dx dt, \end{aligned}$$

for any test functions  $\phi_1 \in \mathcal{D}(\mathbb{R}^d \times [0, T))$  and  $\phi_2 \in \mathcal{D}(\mathbb{R}^d \times [0, T))$  with  $v = \mathcal{K} * w$  and  $z = \mathcal{K} * u$ .

For a given weak solution, we also define:

**Definition 1.2.** Let  $T > 0$ . Then  $(u, w)$  is called a free energy solution with some regular initial data  $(u_0, w_0)$  on  $(0, T)$  if  $(u, w)$  is a weak solution and moreover satisfies  $(u^{(2m_1-1)/2}, w^{(2m_2-1)/2}) \in (L^2(0, T; H^1(\mathbb{R}^d)))^2$  and

$$\begin{aligned} & \mathcal{F}[u(t), w(t)] + \int_0^t \int_{\mathbb{R}^d} u \left| \frac{m_1}{m_1 - 1} \nabla u^{m_1-1} - \nabla v \right|^2 dx ds \\ & + \int_0^t \int_{\mathbb{R}^d} w \left| \frac{m_2}{m_2 - 1} \nabla w^{m_2-1} - \nabla z \right|^2 dx ds \leq \mathcal{F}[u_0, w_0] \end{aligned} \quad (1.7)$$

for all  $t \in (0, T)$  with  $v = \mathcal{K} * w$  and  $z = \mathcal{K} * u$ .

Our first main result for (1.1) or (1.3) above or below lines  $L_1$  and  $L_2$  is:

**Theorem 1.3.** Let  $m_1, m_2 > 1$ . Suppose that the initial data  $(u_0, w_0)$  with  $\|u_0\|_1 = M_1$ ,  $\|w_0\|_1 = M_2$  fulfill (1.2). Then

- i) If  $\mathbf{m}$  is subcritical, there exists a global free energy solution.
- ii) If  $\mathbf{m}$  is supercritical, then one can construct large initial data ensuring blow up in finite time.

On the lines  $L_1, L_2$  and intersection point  $\mathbf{I}$ , our second main result is as follows.

**Theorem 1.4.** Let  $m_1, m_2 > 1$ . Suppose that the initial data  $(u_0, w_0)$  with  $\|u_0\|_1 = M_1$ ,  $\|w_0\|_1 = M_2$  fulfill (1.2). Then

- i) If  $\mathbf{m}$  is  $\mathbf{I}$ , then there exists a number  $M_c > 0$  such that if  $M_1 M_2 < M_c^2$ , solutions globally exist and if  $M_1 M_2 / (M_1^{m_c} + M_2^{m_c}) > M_c^{2/d} / 2$ , there exists a finite time blow-up solution. Moreover, non-zero global minimizers of  $\mathcal{F}$  exist in  $S_{M_1} \times S_{M_2}$  at the crossing point  $\mathbf{M} = (M_c, M_c)$ .

ii) If  $\mathbf{m}$  is on  $L_1$ , there exists a number  $M_{2c} > 0$  with the following properties: if  $M_2 < M_{2c}$ , solutions globally exist and  $\inf_{f \in S_{M_1}} \inf_{g \in S_{M_2}} \mathcal{F}[f, g] = 0$  if  $M_2 = M_{2c}$ , but there exist no non-zero global minimizers of  $\mathcal{F}$  in  $S_{M_1} \times S_{M_2}$ . In addition, blow-up solution exists if

$$\frac{\left( \int_{\mathbb{R}^d} u_0^{m_1/m_2} dx \right)^{m_2/m_1} \left( \int_{\mathbb{R}^d} w_0 dx \right)}{\left( \int_{\mathbb{R}^d} u_0^{m_1/m_2} dx \right)^{m_2} + \left( \int_{\mathbb{R}^d} w_0 dx \right)^{m_2}} > N_0 \text{ with some } N_0 > 0.$$

If  $\mathbf{m}$  is on  $L_2$ , there exists  $M_{1c} > 0$  with the similar properties for  $M_1$  and blow-up solution exists if

$$\frac{\left(\int_{\mathbb{R}^d} u_0 dx\right) \left(\int_{\mathbb{R}^d} w_0^{m_2/m_1} dx\right)^{m_1/m_2}}{\left(\int_{\mathbb{R}^d} u_0 dx\right)^{m_1} + \left(\int_{\mathbb{R}^d} w_0^{m_2/m_1} dx\right)^{m_1}} > N_0.$$

iii) A simultaneous blow-up phenomenon exists if  $\mathbf{m}$  is critical.

We summarize our second main result on the intersection point **I**, see Figure 2. The blue curve  $M_1 M_2 = M_c^2$  intersects with the green curve  $M_1 M_2 / (M_1^{m_c} + M_2^{m_c}) = M_c^{2/d} / 2$  at the point **J** =  $(M_c, M_c)$ . Theorem 1.4 implies that below the curve  $M_1 M_2 = M_c^2$  solutions globally exist and above the curve  $M_1 M_2 / (M_1^{m_c} + M_2^{m_c}) = M_c^{2/d} / 2$  blow up happens.

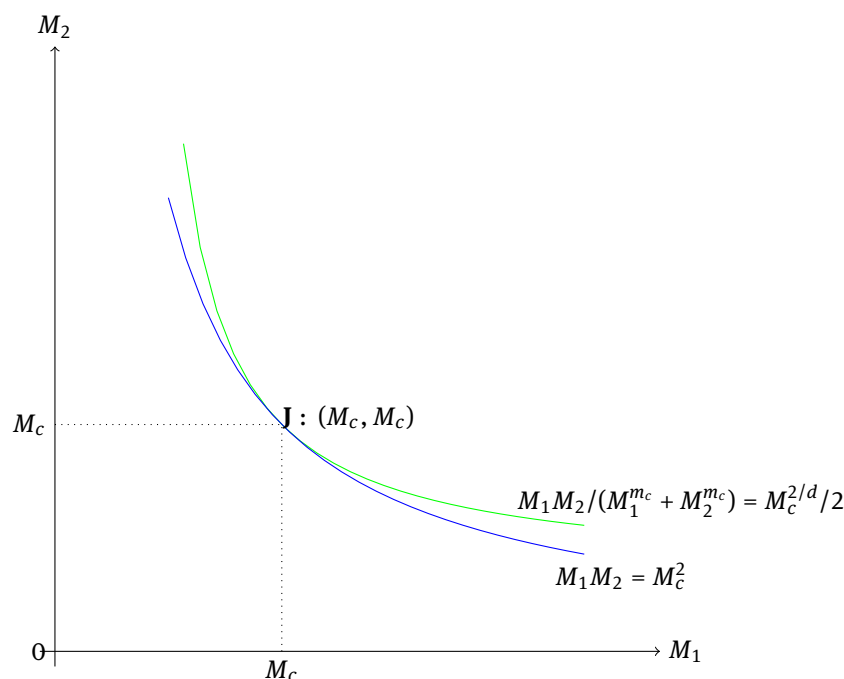


Fig. 2: Parameter lines on intersection point **I**.

It is an open problem to determine the sharp relation between the masses leading to dichotomy in the intersection point **I** and the long time asymptotics on the red curves  $L_1$  and  $L_2$  in Figure 1.

The organization of the paper is as follows: we first construct an approximated system for (1.1) in Section 2, and provide a sufficient condition for global existence of smooth solution and then obtain global weak solution or free energy solution of (1.1) by passing limits upon a prior estimate. Section 3 deals with properties of free energy functional, including the lower and upper bounds, and the existence or non-existence of non-zero minimizers if  $\mathbf{m}$  is critical. Finally, we prove that the solutions are global if  $\mathbf{m}$  is subcritical or critical with small initial data in Section 4 and construct blow-up solutions if  $\mathbf{m}$  is supercritical or critical with large masses in Section 5.

## 2 Approximated system

As mentioned in the introduction, we first consider an approximated system

$$\begin{cases} u_{\epsilon t}(x, t) = \Delta(u_{\epsilon} + \epsilon)^{m_1} - \nabla \cdot (u_{\epsilon} \nabla v_{\epsilon}), & x \in \mathbb{R}^d, t > 0, \\ v_{\epsilon} = \mathcal{K} * w_{\epsilon}, & x \in \mathbb{R}^d, t > 0, \\ w_{\epsilon t}(x, t) = \Delta(w_{\epsilon} + \epsilon)^{m_2} - \nabla \cdot (w_{\epsilon} \nabla z_{\epsilon}), & x \in \mathbb{R}^d, t > 0, \\ z_{\epsilon} = \mathcal{K} * u_{\epsilon}, & x \in \mathbb{R}^d, t > 0, \\ u_{\epsilon}(x, 0) = u_0^{\epsilon}(x) \geq 0, w_{\epsilon}(x, 0) = w_0^{\epsilon}(x) \geq 0, & x \in \mathbb{R}^d \end{cases} \quad (2.1)$$

with  $u_0^{\epsilon}$  and  $w_0^{\epsilon}$  being the convolution of  $u_0$  and  $w_0$  with a sequence of mollifiers and  $\|u_0^{\epsilon}\|_1 = \|u_0\|_1 = M_1$  and  $\|w_0^{\epsilon}\|_1 = \|w_0\|_1 = M_2$ . Then the uniform a priori estimate for solutions to (2.1) is given if  $m_1$  and  $m_2$  are suitably large, thus global weak solution or even free energy solution exists by letting  $\epsilon$  tends to 0.

By virtue of the local existence of strong solution for only one population chemotaxis system (see [43, Proposition 4.1]), one obtains:

**Lemma 2.1.** *Let  $m_1, m_2 > 1$ . Then there exists  $T_{\max}^{\epsilon} \in (0, \infty]$  denoting the maximal existence time such that (2.1) has a unique nonnegative strong solution  $(u_{\epsilon}, w_{\epsilon}) \in \left(\mathbb{W}_p^{2,1}(Q_T)\right)^2$  with some  $p > 1$ , where  $Q_T = \mathbb{R}^d \times (0, T)$  with  $T \in (0, T_{\max}^{\epsilon}]$  and*

$$\mathbb{W}_p^{2,1}(Q_T) := \{u \in L^p(0, T; W^{2,p}(\mathbb{R}^d)) \cap W^{1,p}(0, T; L^p(\mathbb{R}^d))\}.$$

Moreover, if  $T_{\max}^{\epsilon} < \infty$ , then

$$\lim_{t \rightarrow T_{\max}^{\epsilon}} [\|u_{\epsilon}(\cdot, t)\|_{\infty} + \|w_{\epsilon}(\cdot, t)\|_{\infty}] = \infty.$$

Now we recall the Hardy-Littlewood-Sobolev (HLS) inequality which we frequently use later (see [31] or [32, Chapter 4]).

**Lemma 2.2.** *Let  $0 < \lambda < d$ , and let the Riesz potential  $I_{\lambda}(h)$  of a function  $h$  be defined by*

$$I_{\lambda}(h)(x) = \frac{1}{|x|^{d-\lambda}} * h = \int_{\mathbb{R}^d} \frac{h(y)}{|x-y|^{d-\lambda}} dy, \quad x \in \mathbb{R}^d.$$

*Then for  $h \in L^{\kappa_1}(\mathbb{R}^d)$  and for  $\kappa_1, \kappa_2 > 1$  with  $\frac{1}{\kappa_2} = \frac{1}{\kappa_1} - \frac{\lambda}{d}$ , then there exists a sharp constant  $C_{HLS} = C_{HLS}(d, \lambda, \kappa_1) > 0$  such that*

$$\|I_{\lambda}(h)\|_{\kappa_2} \leq C_{HLS} \|h\|_{\kappa_1}.$$

An equivalent form of the HLS inequality can be stated that if

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{\lambda}{d},$$

and  $h_1 \in L^p(\mathbb{R}^d)$ ,  $h_2 \in L^q(\mathbb{R}^d)$  with  $p, q > 1$ , then there exists a  $C_{HLS} = C_{HLS}(d, \lambda, p) > 0$  such that

$$\left| \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{h_1(x)h_2(y)}{|x-y|^{d-\lambda}} dx dy \right| \leq C_{HLS} \|h_1\|_p \|h_2\|_q.$$

Inspired by [46], the global solvability of (2.1) can be achieved based on assumptions on the boundedness for  $\|u_{\epsilon}\|_{m_1}$  and  $\|w_{\epsilon}\|_{m_2}$  with some large  $m_1$  and  $m_2$ .

**Lemma 2.3.** Let  $T \in (0, T_{\max}^{\epsilon}]$ . Assume that  $\mathbf{m}$  satisfies

$$m_1 m_2 + 2m_1 m_2 / d > m_1 + m_2. \quad (2.2)$$

Suppose that there exists a constant  $C > 0$  such that  $(u_{\epsilon}, w_{\epsilon})$  of (2.1) with initial data  $(u_0^{\epsilon}, w_0^{\epsilon})$  being the convolution of  $(u_0, w_0)$  satisfies

$$\|u_{\epsilon}(t)\|_{m_1} \leq C \quad \text{and} \quad \|w_{\epsilon}(t)\|_{m_2} \leq C \quad \text{for } t \in (0, T). \quad (2.3)$$

Then there exists a constant  $C = C(d, m_1, m_2, u_0^{\epsilon}, w_0^{\epsilon}) > 0$  such that

$$\|(u_{\epsilon}(t), w_{\epsilon}(t))\|_r \leq C \quad \text{for } r \in [1, \infty) \quad \text{and } t \in (0, T) \quad (2.4)$$

and

$$\|(v_{\epsilon}(t), z_{\epsilon}(t))\|_r + \|(\nabla v_{\epsilon}(t), \nabla z_{\epsilon}(t))\|_r \leq C \quad \text{for } r \in (1, \infty] \quad \text{and } t \in (0, T). \quad (2.5)$$

*Proof.* We split the proof into three steps.

**Step 1.** The choices of  $p$  and  $q$ . We first show that there exist  $\bar{p} > 1$ ,  $\bar{q} > 1$ ,  $r_1 > 1$  and  $r_2 > 1$  such that for some  $p > \bar{p}$  and  $q > \bar{q}$  one has

$$p > \begin{cases} m_1 + 1, & \text{if } m_1 \geq \frac{d}{2}, m_2 \geq \frac{d}{2}, \\ \max \left\{ m_1 + 1, \frac{(m_1-1)(m_2-1)d}{d-2m_2}, \frac{m_1(d-2)}{2m_2} \right\}, & \text{if } m_1 \geq \frac{d}{2}, m_2 < \frac{d}{2}, \\ \max \left\{ m_1 + 1, \frac{dm_1^2+d-2m_1}{d-2m_1}, \frac{m_1(d-2)}{2m_2} \right\}, & \text{if } m_1 < \frac{d}{2}, m_2 \geq \frac{d}{2}, \\ \max \left\{ m_1 + 1, \frac{dm_1^2+d-2m_1}{d-2m_1}, \frac{(m_1-1)(m_2-1)d}{d-2m_2}, \frac{m_1(d-2)}{2m_2} \right\}, & \text{if } m_1 < \frac{d}{2}, m_2 < \frac{d}{2}, \end{cases} \quad (2.6)$$

$$\frac{1}{r_1} < 1 - \frac{d-2}{(q+m_2-1)d}, \quad (2.7)$$

$$\frac{1}{r_1} > \max \left\{ 1 - \frac{1}{m_2}, \frac{d-2}{d} \cdot \frac{p}{p+m_1-1} \right\}, \quad (2.8)$$

$$\frac{1}{r_2} > \frac{d-2}{d} \cdot \frac{1}{p+m_1-1}, \quad (2.9)$$

$$\frac{1}{r_2} < \min \left\{ \frac{1}{m_1}, 1 - \frac{d-2}{d} \cdot \frac{q}{q+m_2-1} \right\} \quad (2.10)$$

and

$$\frac{\frac{p}{m_1} - \frac{1}{r_1}}{1 - \frac{d}{2} + \frac{(p+m_1-1)d}{2m_1}} + \frac{\frac{1}{m_2} - 1 + \frac{1}{r_1}}{1 - \frac{d}{2} + \frac{(q+m_2-1)d}{2m_2}} < \frac{2}{d} \quad (2.11)$$

as well as

$$\frac{\frac{1}{m_1} - \frac{1}{r_2}}{1 - \frac{d}{2} + \frac{(p+m_1-1)d}{2m_1}} + \frac{\frac{q}{m_2} - 1 + \frac{1}{r_2}}{1 - \frac{d}{2} + \frac{(q+m_2-1)d}{2m_2}} < \frac{2}{d}. \quad (2.12)$$

In order to prove this claim let us first pick  $r_1 > 1$  and  $r_2 > 1$  fulfilling

$$r_1 < \min \left\{ \frac{d}{d-2}, \frac{m_2}{m_2-1} \right\} \quad (2.13)$$



and

$$r_2 > m_1, \quad (2.14)$$

and let

$$q := \frac{m_2(p-1)}{m_1} + 1. \quad (2.15)$$

In (2.15),  $p > m_1 + 1$  implies  $q > m_2 + 1$ . The assertions in (2.6)-(2.7) and (2.9) hold by choosing sufficiently large  $p \geq \bar{p}$  with some  $\bar{p} > 1$  and  $q \geq \bar{q}$  with some  $\bar{q} > 1$ .

To see the possible choice of  $r_1$  satisfying (2.7)-(2.8), we first observe that  $1 - \frac{1}{m_2} \geq \frac{d-2}{d} \cdot \frac{p}{p+m_1-1}$  is true for any  $p > 1$  if  $m_2 \geq \frac{d}{2}$ , and  $\frac{1}{r_1} > 1 - \frac{1}{m_2}$  holds by (2.13) and  $1 - \frac{1}{m_2} < 1 - \frac{d-2}{(q+m_2-1)d}$  for any  $q > 1$ . Thus the asserted  $r_1$  can be actually found. When  $m_2 < \frac{d}{2}$ , one has  $\frac{1}{r_1} > \frac{d-2}{d} \cdot \frac{p}{p+m_1-1} > 1 - \frac{1}{m_2}$ . The first inequality is guaranteed by (2.13) and the second is due to

$$\begin{aligned} \frac{d-2}{d} \cdot \frac{p}{p+m_1-1} > 1 - \frac{1}{m_2} &\iff \left( \frac{1}{m_2} - \frac{2}{d} \right) p > \frac{(m_1-1)(m_2-1)}{m_2} \\ &\iff p > \frac{(m_1-1)(m_2-1)d}{d-2m_2} \end{aligned}$$

by (2.6) if  $m_2 < \frac{d}{2}$ . Moreover, from (2.15) and (2.6),  $\frac{d-2}{d} \cdot \frac{p}{p+m_1-1} < 1 - \frac{d-2}{(q+m_2-1)d}$ . Therefore, one can also choose  $r_1 > 1$  satisfying (2.7)-(2.8) in the case  $m_2 < \frac{d}{2}$ .

Similar to the choice of  $r_2$ , if  $m_1 \geq \frac{d}{2}$  then it follows from (2.14) that  $\frac{1}{r_2} < \frac{1}{m_1} \leq 1 - \frac{d-2}{d} \cdot \frac{q}{q+m_2-1}$ , in which (2.9)-(2.10) can be satisfied due to  $\frac{d-2}{d} \cdot \frac{1}{p+m_1-1} < \frac{1}{m_1}$ . If  $m_1 < \frac{d}{2}$ , (2.6) implies  $\frac{d-2}{d} \cdot \frac{1}{p+m_1-1} < 1 - \frac{d-2}{d} \cdot \frac{q}{q+m_2-1} < \frac{1}{m_1}$ , and the assertion is true.

Since (2.2) ensures that

$$m_1/m_2 - m_1 < 2m_1/d - 1,$$

then

$$\begin{aligned} &\frac{\frac{p}{m_1} - \frac{1}{r_1}}{1 - \frac{d}{2} + \frac{(p+m_1-1)d}{2m_1}} + \frac{\frac{1}{m_2} - 1 + \frac{1}{r_1}}{1 - \frac{d}{2} + \frac{(q+m_2-1)d}{2m_2}} \\ &= \frac{\frac{p}{m_1} - \frac{1}{r_1}}{1 + \frac{(p-1)d}{2m_1}} + \frac{\frac{1}{m_2} - 1 + \frac{1}{r_1}}{1 + \frac{(q-1)d}{2m_2}} \\ &= \frac{\frac{p}{m_1} - \frac{1}{r_1}}{1 + \frac{(p-1)d}{2m_1}} + \frac{\frac{1}{m_2} - 1 + \frac{1}{r_1}}{1 + \frac{(p-1)d}{2m_1}} \\ &= \frac{p + \frac{m_1}{m_2} - m_1}{p + \frac{2m_1}{d} - 1} \cdot \frac{2}{d} < \frac{2}{d}, \end{aligned}$$

and

$$\begin{aligned} &\frac{\frac{1}{m_1} - \frac{1}{r_2}}{1 - \frac{d}{2} + \frac{(p+m_1-1)d}{2m_1}} + \frac{\frac{q}{m_2} - 1 + \frac{1}{r_2}}{1 - \frac{d}{2} + \frac{(q+m_2-1)d}{2m_2}} \\ &= \frac{\frac{q}{m_2} + \frac{1}{m_1} - 1}{1 + \frac{(p-1)d}{2m_1}} \\ &= \frac{p + \frac{m_1}{m_2} - m_1}{p + \frac{2m_1}{d} - 1} \cdot \frac{2}{d} < \frac{2}{d}, \end{aligned}$$

which implies (2.11)-(2.12).

**Step 2. Inequalities for both  $u$  and  $w$ .** For  $p > 1$  and  $q > 1$ , we test (2.1)<sub>1</sub> by  $u_\epsilon^{p-1}$  and integrate to find that

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^d} u_\epsilon^p dx &= -(p-1) \int_{\mathbb{R}^d} u_\epsilon^{p-2} \nabla u_\epsilon \cdot (\nabla(u_\epsilon + \epsilon))^{m_1} - u_\epsilon \nabla v_\epsilon dx \\ &\leq -\frac{4m_1(p-1)}{(p+m_1-1)^2} \int_{\mathbb{R}^d} |\nabla u_\epsilon^{\frac{p+m_1-1}{2}}|^2 dx - \frac{p-1}{p} \int_{\mathbb{R}^d} u_\epsilon^p \Delta v_\epsilon dx \\ &= -\frac{4m_1(p-1)}{(p+m_1-1)^2} \int_{\mathbb{R}^d} |\nabla u_\epsilon^{\frac{p+m_1-1}{2}}|^2 dx + \frac{p-1}{p} \int_{\mathbb{R}^d} u_\epsilon^p w_\epsilon dx \end{aligned}$$

with  $-\Delta v_\epsilon = w_\epsilon$ , and similarly,

$$\frac{1}{q} \frac{d}{dt} \int_{\mathbb{R}^d} w_\epsilon^q dx \leq -\frac{4m_2(q-1)}{(q+m_2-1)^2} \int_{\mathbb{R}^d} |\nabla w_\epsilon^{\frac{q+m_2-1}{2}}|^2 dx + \frac{q-1}{q} \int_{\mathbb{R}^d} u_\epsilon w_\epsilon^q dx$$

holds by multiplying (2.1)<sub>3</sub> by  $w_\epsilon^{q-1}$  and  $-\Delta z_\epsilon = u_\epsilon$ . Then

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^d} u_\epsilon^p dx + \frac{1}{q} \frac{d}{dt} \int_{\mathbb{R}^d} w_\epsilon^q dx &+ \frac{4m_1(p-1)}{(p+m_1-1)^2} \int_{\mathbb{R}^d} |\nabla u_\epsilon^{\frac{p+m_1-1}{2}}|^2 dx \\ &+ \frac{4m_2(q-1)}{(q+m_2-1)^2} \int_{\mathbb{R}^d} |\nabla w_\epsilon^{\frac{q+m_2-1}{2}}|^2 dx \\ &\leq \frac{p-1}{p} \int_{\mathbb{R}^d} u_\epsilon^p w_\epsilon dx + \frac{q-1}{q} \int_{\mathbb{R}^d} u_\epsilon w_\epsilon^q dx, \end{aligned} \quad (2.16)$$

where

$$\int_{\mathbb{R}^d} u_\epsilon^p w_\epsilon dx \leq \left( \int_{\mathbb{R}^d} u_\epsilon^{pr_1} dx \right)^{\frac{1}{r_1}} \left( \int_{\mathbb{R}^d} w_\epsilon^{r'_1} dx \right)^{\frac{1}{r'_1}} \quad (2.17)$$

and

$$\int_{\mathbb{R}^d} u_\epsilon w_\epsilon^q dx \leq \left( \int_{\mathbb{R}^d} u_\epsilon^{r_2} dx \right)^{\frac{1}{r_2}} \left( \int_{\mathbb{R}^d} w_\epsilon^{qr'_2} dx \right)^{\frac{1}{r'_2}} \quad (2.18)$$

by Hölder's inequality with  $r_1, r_2 > 1$ ,  $r'_1 = \frac{r_1}{r_1-1}$  and  $r'_2 = \frac{r_2}{r_2-1}$ . We begin with estimating the right sides of (2.17)-(2.18) based on the choices of  $p, q, r_1$  and  $r_2$  in **Step 1**. The assumption (2.6) ensures

$$pr_1 > m_1, \quad (2.19)$$

and

$$pr_1 < \frac{(p+m_1-1)d}{d-2} \quad (2.20)$$

by (2.8). Then by a variant of the Gagliardo-Nirenberg inequality (see [45, Lemma 6]),

$$\|\varphi\|_{k_2} \leq C^{\frac{2}{r+m-1}} \|\varphi\|_{k_1}^{1-\sigma} \|\nabla \varphi^{\frac{r+m-1}{2}}\|_2^{\frac{2\sigma}{r+m-1}} \quad (2.21)$$

with  $m \geq 1$ ,  $k_1 \in [1, r+m-1]$  and  $1 \leq k_1 \leq k_2 \leq \frac{(r+m-1)d}{d-2}$  with  $d \geq 3$ ,  $\sigma = \frac{r+m-1}{2} \left( \frac{1}{k_1} - \frac{1}{k_2} \right) \left( \frac{1}{d} - \frac{1}{2} + \frac{r+m-1}{2k_1} \right)^{-1}$ , we pick  $r = p$ ,  $m = m_1$ ,  $k_1 = m_1$ ,  $k_2 = pr_1$  in (2.21) and use (2.19)-(2.20) to find

$$\left( \int_{\mathbb{R}^d} u_\epsilon^{pr_1} dx \right)^{\frac{1}{r_1}} = \|u_\epsilon\|_{pr_1}^p \leq C \|u_\epsilon\|_{m_1}^{p(1-\sigma)} \|\nabla u_\epsilon^{\frac{p+m_1-1}{2}}\|_2^{p \frac{2\sigma}{p+m_1-1}}$$

with

$$\sigma = \frac{p + m_1 - 1}{2} \frac{\frac{1}{m_1} - \frac{1}{pr_1}}{\frac{1}{d} - \frac{1}{2} + \frac{p+m_1-1}{2m_1}} \in (0, 1),$$

where invoking (2.3) we further obtain

$$\left( \int_{\mathbb{R}^d} u_\epsilon^{pr_1} dx \right)^{\frac{1}{r_1}} \leq C \|\nabla u_\epsilon^{\frac{p+m_1-1}{2}}\|_2^{\frac{\frac{p}{m_1} - \frac{1}{r_1}}{\frac{1}{d} - \frac{1}{2} + \frac{p+m_1-1}{2m_1}}}.$$

Likewise, (2.7)-(2.8) warrants that

$$m_2 < r'_1 < \frac{(q + m_2 - 1)d}{d - 2},$$

which allows one to make use of the Gagliardo-Nirenberg inequality and the upper bound for  $\|w\|_{m_2}$  in (2.3) to estimate

$$\left( \int_{\mathbb{R}^d} w_\epsilon^{r'_1} dx \right)^{\frac{1}{r'_1}} = \|w_\epsilon\|_{r'_1} \leq C \|\nabla w_\epsilon^{\frac{q+m_2-1}{2}}\|_2^{\frac{\frac{1}{m_2} - \frac{1}{r'_1}}{\frac{1}{d} - \frac{1}{2} + \frac{q+m_2-1}{2m_2}}}.$$

Then

$$\begin{aligned} & \left( \int_{\mathbb{R}^d} u_\epsilon^{pr_1} dx \right)^{\frac{1}{r_1}} \left( \int_{\mathbb{R}^d} w_\epsilon^{r'_1} dx \right)^{\frac{1}{r'_1}} \\ & \leq C \|\nabla u_\epsilon^{\frac{p+m_1-1}{2}}\|_2^{\frac{\frac{p}{m_1} - \frac{1}{r_1}}{\frac{1}{d} - \frac{1}{2} + \frac{p+m_1-1}{2m_1}}} \cdot \|\nabla w_\epsilon^{\frac{q+m_2-1}{2}}\|_2^{\frac{\frac{1}{m_2} - \frac{1}{r'_1}}{\frac{1}{d} - \frac{1}{2} + \frac{q+m_2-1}{2m_2}}} \\ & = C \|\nabla u_\epsilon^{\frac{p+m_1-1}{2}}\|_2^{\frac{\frac{p}{m_1} - \frac{1}{r_1}}{\frac{1}{d} - \frac{1}{2} + \frac{p+m_1-1}{2m_1}}} \cdot \|\nabla w_\epsilon^{\frac{q+m_2-1}{2}}\|_2^{\frac{\frac{1}{m_2} - 1 + \frac{1}{r'_1}}{\frac{1}{d} - \frac{1}{2} + \frac{q+m_2-1}{2m_2}}}. \end{aligned} \quad (2.22)$$

To estimate the right side of (2.18), we use (2.10) and (2.9) to obtain

$$m_1 < r_2 < \frac{(p + m_1 - 1)d}{d - 2}.$$

Then the Gagliardo-Nirenberg inequality implies

$$\left( \int_{\mathbb{R}^d} u_\epsilon^{r_2} dx \right)^{\frac{1}{r_2}} \leq C \|\nabla u_\epsilon^{\frac{p+m_1-1}{2}}\|_2^{\frac{\frac{1}{m_1} - \frac{1}{r_2}}{\frac{1}{d} - \frac{1}{2} + \frac{p+m_1-1}{2m_1}}}$$

by (2.3). We also obtain

$$m_2 < qr'_2 < \frac{(q + m_2 - 1)d}{d - 2}$$

by (2.10) and (2.15), and choose  $r = q$ ,  $m = m_2$ ,  $k_1 = m_2$ ,  $k_2 = qr'_2$  in (2.21) to see that

$$\begin{aligned} & \left( \int_{\mathbb{R}^d} w_\epsilon^{qr'_2} dx \right)^{\frac{1}{r'_2}} = \|w_\epsilon\|_{qr'_2}^q \leq C \|w_\epsilon\|_{m_2}^{q(1-\sigma)} \|\nabla w_\epsilon^{\frac{q+m_2-1}{2}}\|_2^{q\frac{2\sigma}{q+m_2-1}} \\ & \leq C \|\nabla w_\epsilon^{\frac{q+m_2-1}{2}}\|_2^{\frac{\frac{q}{m_2} - \frac{1}{r'_2}}{\frac{1}{d} - \frac{1}{2} + \frac{q+m_2-1}{2m_2}}} \\ & = C \|\nabla w_\epsilon^{\frac{q+m_2-1}{2}}\|_2^{\frac{\frac{q}{m_2} - 1 + \frac{1}{r'_2}}{\frac{1}{d} - \frac{1}{2} + \frac{q+m_2-1}{2m_2}}} \end{aligned}$$

with

$$\sigma = \frac{q + m_2 - 1}{2} \frac{\frac{1}{m_2} - \frac{1}{qr_2'}}{\frac{1}{d} - \frac{1}{2} + \frac{q+m_2-1}{2m_2}}.$$

Then

$$\begin{aligned} & \left( \int_{\mathbb{R}^d} u_\epsilon^{r_2} dx \right)^{\frac{1}{r_2}} \left( \int_{\mathbb{R}^d} w_\epsilon^{qr_2'} dx \right)^{\frac{1}{r_2}} \\ & \leq C \|\nabla u_\epsilon^{\frac{p+m_1-1}{2}}\|_2^{\frac{\frac{1}{m_1} - \frac{1}{r_2}}{\frac{1}{d} - \frac{1}{2} + \frac{p+m_1-1}{2m_1}}} \|\nabla w_\epsilon^{\frac{q+m_2-1}{2}}\|_2^{\frac{\frac{q}{m_2} - 1 + \frac{1}{r_2}}{\frac{1}{d} - \frac{1}{2} + \frac{q+m_2-1}{2m_2}}}, \end{aligned}$$

which combines with (2.16) and (2.22) ensures that

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^d} u_\epsilon^p dx + \frac{1}{q} \frac{d}{dt} \int_{\mathbb{R}^d} w_\epsilon^q dx + \frac{4m_1(p-1)}{(p+m_1-1)^2} \int_{\mathbb{R}^d} |\nabla u_\epsilon^{\frac{p+m_1-1}{2}}|^2 dx \\ & + \frac{4m_2(q-1)}{(q+m_2-1)^2} \int_{\mathbb{R}^d} |\nabla w_\epsilon^{\frac{q+m_2-1}{2}}|^2 dx \\ & \leq \frac{p-1}{p} \left( \int_{\mathbb{R}^d} u_\epsilon^{pr_1} dx \right)^{\frac{1}{r_1}} \left( \int_{\mathbb{R}^d} w_\epsilon^{r_1'} dx \right)^{\frac{1}{r_1}} \\ & + \frac{q-1}{q} \left( \int_{\mathbb{R}^d} u_\epsilon^{r_2} dx \right)^{\frac{1}{r_2}} \left( \int_{\mathbb{R}^d} w_\epsilon^{qr_2'} dx \right)^{\frac{1}{r_2}} \\ & \leq C \|\nabla u_\epsilon^{\frac{p+m_1-1}{2}}\|_2^{\frac{\frac{p}{m_1} - \frac{1}{r_1}}{\frac{1}{d} - \frac{1}{2} + \frac{p+m_1-1}{2m_1}}} \cdot \|\nabla w_\epsilon^{\frac{q+m_2-1}{2}}\|_2^{\frac{\frac{1}{m_2} - 1 + \frac{1}{r_1}}{\frac{1}{d} - \frac{1}{2} + \frac{q+m_2-1}{2m_2}}} \\ & + C \|\nabla u_\epsilon^{\frac{p+m_1-1}{2}}\|_2^{\frac{\frac{1}{m_1} - \frac{1}{r_2}}{\frac{1}{d} - \frac{1}{2} + \frac{p+m_1-1}{2m_1}}} \cdot \|\nabla w_\epsilon^{\frac{q+m_2-1}{2}}\|_2^{\frac{\frac{q}{m_2} - 1 + \frac{1}{r_2}}{\frac{1}{d} - \frac{1}{2} + \frac{q+m_2-1}{2m_2}}}. \end{aligned} \quad (2.23)$$

**Step 3. Boundedness for  $u_\epsilon$  and  $w_\epsilon$  in  $L^p$ - and  $L^q$ -spaces.** Let  $\gamma_1 > 0, \gamma_2 > 0$  be such that  $\gamma_1 + \gamma_2 < 2$ . For  $\epsilon > 0$ , a direct application of Young's inequality implies that

$$\alpha^{\gamma_1} \beta^{\gamma_2} \leq \epsilon(\alpha^2 + \beta^2) + C. \quad (2.24)$$

From **Step 1**, there exist some  $p > \bar{p}$  and  $q > \bar{q}$  with some  $\bar{p} > 1$  and  $\bar{q} > 1$  such that

$$\frac{\frac{p}{m_1} - \frac{1}{r_1}}{\frac{1}{d} - \frac{1}{2} + \frac{p+m_1-1}{2m_1}} + \frac{\frac{1}{m_2} - 1 + \frac{1}{r_1}}{\frac{1}{d} - \frac{1}{2} + \frac{q+m_2-1}{2m_2}} < 2$$

and

$$\frac{\frac{1}{m_1} - \frac{1}{r_2}}{\frac{1}{d} - \frac{1}{2} + \frac{p+m_1-1}{2m_1}} + \frac{\frac{q}{m_2} - 1 + \frac{1}{r_2}}{\frac{1}{d} - \frac{1}{2} + \frac{q+m_2-1}{2m_2}} < 2,$$

where

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^d} u_\epsilon^p dx + \frac{1}{q} \frac{d}{dt} \int_{\mathbb{R}^d} w_\epsilon^q dx + \frac{2m_1(p-1)}{(p+m_1-1)^2} \int_{\mathbb{R}^d} |\nabla u_\epsilon^{\frac{p+m_1-1}{2}}|^2 dx \\ & + \frac{2m_2(q-1)}{(q+m_2-1)^2} \int_{\mathbb{R}^d} |\nabla w_\epsilon^{\frac{q+m_2-1}{2}}|^2 dx \leq C \end{aligned} \quad (2.25)$$

by (2.23)-(2.24). One may invoke the Gagliardo-Nirenberg inequality with  $\|u\|_1 = M_1$  and  $\|w\|_1 = M_2$  and Young's inequality to obtain

$$\begin{aligned} \frac{1}{p} \int_{\mathbb{R}^d} u_\epsilon^p dx &= \frac{1}{p} \|u_\epsilon\|_p^p \leq C \|\nabla u_\epsilon^{\frac{p+m_1-1}{2}}\|_2^{\frac{p-1}{\frac{1}{d}-\frac{1}{2}+\frac{p+m_1-1}{2}}} \\ &\leq \frac{2m_1(p-1)}{(p+m_1-1)^2} \int_{\mathbb{R}^d} |\nabla u_\epsilon^{\frac{p+m_1-1}{2}}|^2 dx + C \end{aligned}$$

and

$$\frac{1}{q} \int_{\mathbb{R}^d} w_\epsilon^q dx \leq \frac{2m_2(q-1)}{(q+m_2-1)^2} \int_{\mathbb{R}^d} |\nabla w_\epsilon^{\frac{q+m_2-1}{2}}|^2 dx + C$$

by the fact that

$$\frac{p-1}{\frac{1}{d}-\frac{1}{2}+\frac{p+m_1-1}{2}} < 2 \quad \text{and} \quad \frac{q-1}{\frac{1}{d}-\frac{1}{2}+\frac{q+m_2-1}{2}} < 2.$$

Writing  $y(t) = \frac{1}{p} \int_{\mathbb{R}^d} u_\epsilon^p dx + \frac{1}{q} \int_{\mathbb{R}^d} w_\epsilon^q dx$ , we obtain from (2.25) that

$$y'(t) + y(t) \leq C \quad \text{for } t \in (0, T).$$

Then

$$\|u_\epsilon(t)\|_p \leq C \quad \text{and} \quad \|w_\epsilon(t)\|_q \leq C \quad \text{for } t \in (0, T),$$

which implies that (2.4) holds for any  $r > \max\{\bar{p}, \bar{q}\}$ . Together with an interpolation with  $L^1$  any intermediate space can be obtained.

**Step 4.** The regularities of  $v$  and  $z$ . As

$$v_\epsilon = \mathcal{K} * w_\epsilon = c_d \int_{\mathbb{R}^d} \frac{w_\epsilon(y)}{|x-y|^{d-2}} dy, \quad z_\epsilon = c_d \int_{\mathbb{R}^d} \frac{u_\epsilon(y)}{|x-y|^{d-2}} dy,$$

an application of the HLS inequality ensures that for  $r \in (d/(d-1), \infty)$ , we have

$$\begin{aligned} \|\nabla v_\epsilon\|_r &\leq c_d(d-2) \|I_1(w_\epsilon)\|_r \leq C \|w_\epsilon\|_{dr/(d+r)}, \\ \|\nabla z_\epsilon\|_r &\leq C \|u_\epsilon\|_{dr/(d+r)}. \end{aligned} \tag{2.26}$$

Furthermore, observing that the Calderon-Zygmund inequality yields the existence of a constant  $C = C(r) > 0$  with  $r \in (1, \infty)$  such that

$$\begin{aligned} \|\partial_{x_i} \partial_{x_j} v_\epsilon\|_r &\leq C \|w_\epsilon\|_r, \\ \|\partial_{x_i} \partial_{x_j} z_\epsilon\|_r &\leq C \|u_\epsilon\|_r, \quad (1 \leq i, j \leq d), \end{aligned}$$

we combine this with (2.4), (2.26) and the Morrey's inequality to see that

$$\|(v_\epsilon(t), z_\epsilon(t))\|_r + \|(\nabla v_\epsilon(t), \nabla z_\epsilon(t))\|_r \leq C \quad \text{for } r \in (1, \infty] \quad \text{and } t \in (0, T).$$

Thus we finish our proof.  $\square$

Upon the boundedness arguments in Lemma 2.3, we obtain a global weak solution by letting a subsequence of  $\epsilon$  approaches to 0.

**Lemma 2.4.** Under the same assumption in Lemma 2.3, there exists  $C > 0$  independent of  $\epsilon$  such that strong solution  $(u_\epsilon, w_\epsilon)$  of (2.1) satisfies

$$\|(u_\epsilon(t), w_\epsilon(t))\|_\infty \leq C \quad \text{for all } t \in (0, T). \tag{2.27}$$

Moreover, there exists a global weak solution  $(u, w)$  of (1.1) which also satisfies a uniform estimate.

*Proof.* Relying on Lemma 2.3, we apply the Moser's iteration technique to obtain a priori estimate for the solution in  $L^\infty$ . Then this local solution can be extended globally in time from the extensibility criterion in Lemma 2.1, which indeed establishes (2.27), see [45, Proposition 10]. Moreover, testing the first equation in (2.1) by  $\partial_t(u_\epsilon + \epsilon)^{m_1}$  and integrating over  $\mathbb{R}^d$ , we make use of Young's inequality to see that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |\nabla(u_\epsilon + \epsilon)^{m_1}|^2 dx + \frac{4m_1}{(m_1 + 1)^2} \int_{\mathbb{R}^d} \left| \partial_t(u_\epsilon + \epsilon)^{\frac{m_1+1}{2}} \right|^2 dx \\ &= - \frac{2m_1}{m_1 + 1} \int_{\mathbb{R}^d} (u_\epsilon + \epsilon)^{\frac{m_1-1}{2}} \nabla \cdot (u_\epsilon \nabla v_\epsilon) \cdot \partial_t(u_\epsilon + \epsilon)^{\frac{m_1+1}{2}} dx \\ &\leq \frac{2m_1}{(m_1 + 1)^2} \int_{\mathbb{R}^d} \left| \partial_t(u_\epsilon + \epsilon)^{\frac{m_1+1}{2}} \right|^2 dx \\ &\quad + C(m_1) \|\nabla v_\epsilon\|_{L^\infty(Q_T)}^2 \int_{\mathbb{R}^d} |\nabla(u_\epsilon + \epsilon)^{\frac{m_1+1}{2}}|^2 dx + \\ &\quad + C(m_1) (\|u_\epsilon\|_{L^\infty(Q_T)} + \epsilon)^{m_1+1} \int_{\mathbb{R}^d} |\Delta v_\epsilon|^2 dx. \end{aligned} \quad (2.28)$$

We then test the first equation in (2.1) by  $u_\epsilon$  to find that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} u_\epsilon^2 dx + \frac{4m_1}{(m_1 + 1)^2} \int_{\mathbb{R}^d} |\nabla(u_\epsilon + \epsilon)^{\frac{m_1+1}{2}}|^2 dx \\ &\leq C(m_1) \|u_\epsilon\|_{L^\infty(Q_T)}^2 \int_{\mathbb{R}^d} |\Delta v_\epsilon|^2 dx. \end{aligned} \quad (2.29)$$

Here, we obtain from (2.28), (2.29) and the regularities of initial data and  $(u, v, w, z)$  obtained in Lemma 2.3 and (2.27) that

$$\int_{\mathbb{R}^d} |\nabla(u_\epsilon + \epsilon)^{m_1}|^2 \leq C,$$

where  $C$  does not depend on  $\epsilon$ . Then  $\nabla u_\epsilon^{m_1} \in L^\infty((0, T); L^2(\mathbb{R}^d))$ , and  $\nabla w_\epsilon^{m_2} \in L^\infty((0, T); L^2(\mathbb{R}^d))$  is true by similar procedure. All in all, there exists  $(u, v, w, z)$  with the regularities given in Definition 1.1 such that, up to a subsequence,  $\epsilon_n \rightarrow 0$ ,

$$\begin{aligned} & u_{\epsilon_n} \rightarrow u \text{ strongly in } C([0, T]; L_{loc}^p(\mathbb{R}^d)) \text{ and a.e. in } \mathbb{R}^d \times (0, T), \\ & \nabla u_{\epsilon_n}^{m_1} \rightharpoonup \nabla u^{m_1} \text{ weakly-}^* \text{ in } L^\infty((0, T); L^2(\mathbb{R}^d)), \\ & w_{\epsilon_n} \rightarrow w \text{ strongly in } C([0, T]; L_{loc}^p(\mathbb{R}^d)) \text{ and a.e. in } \mathbb{R}^d \times (0, T), \\ & \nabla w_{\epsilon_n}^{m_2} \rightharpoonup \nabla w^{m_2} \text{ weakly-}^* \text{ in } L^\infty((0, T); L^2(\mathbb{R}^d)), \\ & v_{\epsilon_n}(t) \rightarrow v(t) \text{ strongly in } L_{loc}^r(\mathbb{R}^d) \text{ and a.e. in } (0, T), \\ & \nabla v_{\epsilon_n}(t) \rightarrow \nabla v(t) \text{ strongly in } L_{loc}^r(\mathbb{R}^d) \text{ and a.e. in } (0, T), \\ & \Delta v_{\epsilon_n}(t) \rightharpoonup \Delta v(t) \text{ weakly in } L_{loc}^r(\mathbb{R}^d) \text{ and a.e. in } (0, T), \\ & z_{\epsilon_n}(t) \rightarrow z(t) \text{ strongly in } L_{loc}^r(\mathbb{R}^d) \text{ and a.e. in } (0, T), \\ & \nabla z_{\epsilon_n}(t) \rightarrow \nabla z(t) \text{ strongly in } L_{loc}^r(\mathbb{R}^d) \text{ and a.e. in } (0, T), \\ & \Delta z_{\epsilon_n}(t) \rightharpoonup \Delta z(t) \text{ weakly in } L_{loc}^r(\mathbb{R}^d) \text{ and a.e. in } (0, T), \end{aligned}$$

where  $p \in (1, \infty)$ ,  $r \in (1, \infty]$  and  $T \in (0, \infty)$ . Since the above convergence can be found in [45, Section 4], we omit the main proof here. Therefore, we have a global weak solution  $(u, v, w, z)$  over  $\mathbb{R}^d \times (0, T)$  with  $T > 0$ .  $\square$

The weak solution obtained in Lemma 2.4 is also a free energy solution given in Definition 1.2. The proof comes from [43].

**Lemma 2.5.** *Consider a global weak solution in Lemma 2.4, then it is also a global free energy solution  $(u, w)$  of (1.1) given in Definition 1.2.*

*Proof.* Define a weight function

$$\psi(|x|) = \begin{cases} 1, & \text{for } 0 \leq |x| \leq 1, \\ 1 - 2(|x| - 1)^2, & \text{for } 1 < |x| \leq \frac{3}{2}, \\ 2(2 - |x|)^2, & \text{for } \frac{3}{2} < |x| < 2, \\ 0, & \text{for } |x| \geq 2, \end{cases}$$

and define  $\psi_l$  by  $\psi_l(x) := \psi\left(\frac{|x|}{l}\right)$  for any  $x \in \mathbb{R}^d$  and  $l = 1, 2, 3, \dots$ . Evidently,

$$|\nabla \psi_l(x)| \leq \frac{C}{l} (\psi_l(x))^{\frac{1}{2}} \quad \text{and} \quad |\Delta \psi_l(x)| \leq \frac{C}{l^2}$$

is valid with some  $C > 0$ . Denote

$$\begin{aligned} \mathcal{F}[u_\epsilon(t), w_\epsilon(t)] &:= \frac{1}{m_1 - 1} \int_{\mathbb{R}^d} (u_\epsilon + \epsilon)^{m_1} \psi_l(x) dx + \frac{1}{m_2 - 1} \int_{\mathbb{R}^d} (w_\epsilon + \epsilon)^{m_2} \psi_l(x) dx \\ &\quad - \int_{\mathbb{R}^d} u_\epsilon v_\epsilon dx \\ &= \frac{1}{m_1 - 1} \int_{\mathbb{R}^d} (u_\epsilon + \epsilon)^{m_1} \psi_l(x) dx - \int_{\mathbb{R}^d} u_\epsilon v_\epsilon dx \\ &\quad + \frac{1}{m_2 - 1} \int_{\mathbb{R}^d} (w_\epsilon + \epsilon)^{m_2} \psi_l(x) dx - \int_{\mathbb{R}^d} w_\epsilon z_\epsilon dx \\ &\quad + \int_{\mathbb{R}^d} \nabla v_\epsilon \cdot \nabla z_\epsilon dx. \end{aligned}$$

Since

$$\begin{aligned} &\frac{1}{m_1 - 1} \frac{d}{dt} (u_\epsilon + \epsilon)^{m_1} \psi_l - \frac{d}{dt} (u_\epsilon v_\epsilon) + u_\epsilon v_{\epsilon t} \\ &\quad = \nabla \cdot (\nabla (u_\epsilon + \epsilon)^{m_1} - u_\epsilon \nabla v_\epsilon) \cdot \left( \frac{m_1 (u_\epsilon + \epsilon)^{m_1 - 1}}{m_1 - 1} \psi_l - v_\epsilon \right), \\ &\frac{1}{m_2 - 1} \frac{d}{dt} (w_\epsilon + \epsilon)^{m_2} \psi_l - \frac{d}{dt} (w_\epsilon z_\epsilon) + w_\epsilon z_{\epsilon t} \\ &\quad = \nabla \cdot (\nabla (w_\epsilon + \epsilon)^{m_2} - w_\epsilon \nabla z_\epsilon) \cdot \left( \frac{m_2 (w_\epsilon + \epsilon)^{m_2 - 1}}{m_2 - 1} \psi_l - z_\epsilon \right) \end{aligned}$$

by testing (2.1)<sub>1</sub> by  $\frac{m_1(u_\epsilon + \epsilon)^{m_1-1}}{m_1-1} \psi_l - v_\epsilon$  and (2.1)<sub>3</sub> by  $\frac{m_2(w_\epsilon + \epsilon)^{m_2-1}}{m_2-1} \psi_l - z_\epsilon$ , then the derivative of  $\mathcal{F}[u_\epsilon(t), w_\epsilon(t)]$  with respect to time is

$$\begin{aligned} \frac{d}{dt} \mathcal{F}[u_\epsilon(t), w_\epsilon(t)] &= \frac{1}{m_1-1} \frac{d}{dt} \int_{\mathbb{R}^d} (u_\epsilon + \epsilon)^{m_1} \psi_l(x) dx - \frac{d}{dt} \int_{\mathbb{R}^d} u_\epsilon v_\epsilon dx \\ &\quad + \int_{\mathbb{R}^d} u_\epsilon v_{\epsilon t} dx + \frac{1}{m_2-1} \frac{d}{dt} \int_{\mathbb{R}^d} (w_\epsilon + \epsilon)^{m_2} \psi_l(x) dx \\ &\quad - \frac{d}{dt} \int_{\mathbb{R}^d} w_\epsilon z_\epsilon dx + \int_{\mathbb{R}^d} w_\epsilon z_{\epsilon t} dx \\ &= - \int_{\mathbb{R}^d} (\nabla(u_\epsilon + \epsilon)^{m_1} - u_\epsilon \nabla v_\epsilon) \cdot \nabla \left( \frac{m_1(u_\epsilon + \epsilon)^{m_1-1}}{m_1-1} \psi_l - v_\epsilon \right) dx \\ &\quad - \int_{\mathbb{R}^d} (\nabla(w_\epsilon + \epsilon)^{m_2} - w_\epsilon \nabla z_\epsilon) \cdot \nabla \left( \frac{m_2(w_\epsilon + \epsilon)^{m_2-1}}{m_2-1} \psi_l - z_\epsilon \right) dx, \end{aligned}$$

which can be written as

$$\begin{aligned} \frac{d}{dt} \mathcal{F}[u_\epsilon(t), w_\epsilon(t)] &= - \int_{\mathbb{R}^d} \left[ (u_\epsilon + \epsilon) \nabla \left( \frac{m_1}{m_1-1} (u_\epsilon + \epsilon)^{m_1-1} - v_\epsilon \right) + \epsilon \nabla v_\epsilon \right] \\ &\quad \cdot \left[ \nabla \left( \frac{m_1}{m_1-1} (u_\epsilon + \epsilon)^{m_1-1} - v_\epsilon \right) \psi_l \right. \\ &\quad \left. + \left( \frac{m_1}{m_1-1} (u_\epsilon + \epsilon)^{m_1-1} - v_\epsilon \right) \nabla \psi_l + \nabla v_\epsilon (\psi_l - 1) + v_\epsilon \nabla \psi_l \right] dx \\ &\quad - \int_{\mathbb{R}^d} \left[ (w_\epsilon + \epsilon) \nabla \left( \frac{m_2}{m_2-1} (w_\epsilon + \epsilon)^{m_2-1} - z_\epsilon \right) + \epsilon \nabla z_\epsilon \right] \\ &\quad \cdot \left[ \nabla \left( \frac{m_2}{m_2-1} (w_\epsilon + \epsilon)^{m_2-1} - z_\epsilon \right) \psi_l \right. \\ &\quad \left. + \left( \frac{m_2}{m_2-1} (w_\epsilon + \epsilon)^{m_2-1} - z_\epsilon \right) \nabla \psi_l + \nabla z_\epsilon (\psi_l - 1) + z_\epsilon \nabla \psi_l \right] dx \\ &= - \int_{\mathbb{R}^d} I_1 \times J_1 dx - \int_{\mathbb{R}^d} I_2 \times J_2 dx. \end{aligned} \tag{2.30}$$

With  $U_\epsilon := \frac{m_1}{m_1-1} (u_\epsilon + \epsilon)^{m_1-1} - v_\epsilon$ , we expand the term  $-\int_{\mathbb{R}^d} I_1 \times J_1 dx$  to find that

$$\begin{aligned} - \int_{\mathbb{R}^d} I_1 \times J_1 dx &= - \int_{\mathbb{R}^d} (u_\epsilon + \epsilon) \psi_l |\nabla U_\epsilon|^2 dx - \int_{\mathbb{R}^d} (u_\epsilon + \epsilon) (U_\epsilon + v_\epsilon) \nabla U_\epsilon \cdot \nabla \psi_l dx \\ &\quad - \int_{\mathbb{R}^d} (u_\epsilon + \epsilon) (\psi_l - 1) \nabla U_\epsilon \cdot \nabla v_\epsilon dx - \epsilon \int_{\mathbb{R}^d} \nabla(\psi_l U_\epsilon) \cdot \nabla v_\epsilon dx \\ &\quad - \epsilon \int_{\mathbb{R}^d} \nabla(v_\epsilon (\psi_l - 1)) \cdot \nabla v_\epsilon dx, \end{aligned}$$

where by defining  $\Omega_l = \{x \in \mathbb{R}^d : l < |x| < 2l\}$ , upon using Young's inequality, Hölder's inequality and  $(a+b)^m \leq 2^m(a^m + b^m)$  with  $a, b > 0$  and  $m > 1$ , with any  $\eta \in (0, 1)$  we deduce from  $|\nabla \psi_l| \leq \frac{C}{l} (\psi_l)^{\frac{1}{2}}$  and



$\text{supp}|\nabla\psi_l| = \overline{\Omega_l}$  that

$$\begin{aligned}
 - \int_{\mathbb{R}^d} (u_\epsilon + \epsilon)(U_\epsilon + v_\epsilon) \nabla U_\epsilon \cdot \nabla \psi_l dx &\leq \eta \int_{\mathbb{R}^d} (u_\epsilon + \epsilon) \psi_l |\nabla U_\epsilon|^2 dx \\
 &\quad + \frac{C}{\eta l^2} \left( \|u_\epsilon\|_{2m_1-1}^{2m_1-1} + \epsilon^{2m_1-1} |\Omega_l| \right), \\
 - \int_{\mathbb{R}^d} (u_\epsilon + \epsilon)(\psi_l - 1) \nabla U_\epsilon \cdot \nabla v_\epsilon dx &= \int_{\mathbb{R}^d} (1 - \psi_l) \nabla (u_\epsilon + \epsilon)^{m_1} \cdot \nabla v_\epsilon dx \\
 &\quad + \int_{\mathbb{R}^d} (u_\epsilon + \epsilon)(\psi_l - 1) |\nabla v_\epsilon|^2 dx \\
 &\leq \int_{\mathbb{R}^d} (u_\epsilon + \epsilon)^{m_1} \nabla \psi_l \cdot \nabla v_\epsilon dx \\
 &\quad - \int_{\mathbb{R}^d} (1 - \psi_l)(u_\epsilon + \epsilon)^{m_1} \Delta v_\epsilon dx \\
 &\leq \int_{\mathbb{R}^d} (u_\epsilon + \epsilon)^{m_1} w_\epsilon (1 - \psi_l) dx \\
 &\quad + \frac{C}{l} \int_{\mathbb{R}^d} (u_\epsilon^{m_1} + \epsilon^{m_1}) |\nabla v_\epsilon| dx, \\
 -\epsilon \int_{\mathbb{R}^d} \nabla(\psi_l U_\epsilon) \cdot \nabla v_\epsilon dx &= -\epsilon \int_{\mathbb{R}^d} \psi_l U_\epsilon w_\epsilon dx \leq \epsilon \|w_\epsilon\|_{L^1} \|U_\epsilon\|_{L^\infty}, \\
 -\epsilon \int_{\mathbb{R}^d} \nabla(v_\epsilon(\psi_l - 1)) \cdot \nabla v_\epsilon dx &\leq \epsilon \int_{\mathbb{R}^d} w_\epsilon v_\epsilon (1 - \psi_l) dx.
 \end{aligned}$$

The regularities of  $(u_\epsilon, v_\epsilon, w_\epsilon)$  from Lemmas 2.3-2.4 assert that

$$\begin{aligned}
 - \int_{\mathbb{R}^d} I_1 \times J_1 dx &\leq - (1 - \eta) \int_{\mathbb{R}^d} (u_\epsilon + \epsilon) \psi_l |\nabla U_\epsilon|^2 dx \\
 &\quad + \frac{C}{\eta l^2} \left( \|u_\epsilon(t)\|_{2m_1-1}^{2m_1-1} + \epsilon^{2m_1-1} |\Omega_l| \right) \\
 &\quad + \int_{\mathbb{R}^d} (u_\epsilon + \epsilon)^{m_1} w_\epsilon (1 - \psi_l) dx + \frac{C}{l} \int_{\mathbb{R}^d} (u_\epsilon^{m_1} + \epsilon^{m_1}) |\nabla v_\epsilon| dx \\
 &\quad + \epsilon \|w_\epsilon\|_{L^1} \|U_\epsilon\|_{L^\infty} + \epsilon \int_{\mathbb{R}^d} w_\epsilon v_\epsilon (1 - \psi_l) dx \\
 &\leq - (1 - \eta) \int_{\mathbb{R}^d} (u_\epsilon + \epsilon) \psi_l |\nabla U_\epsilon|^2 dx + \frac{C}{\eta l^2} \left( 1 + \epsilon^{2m_1-1} |\Omega_l| \right) \\
 &\quad + C \int_{\mathbb{R}^d} w_\epsilon (1 - \psi_l) dx + \frac{C}{l} + \epsilon C.
 \end{aligned}$$

Doing a similar argument for  $-\int_{\mathbb{R}^d} I_2 \times J_2 dx$ , and integrating (2.30) over time shows that

$$\begin{aligned} \mathcal{F}[u_\epsilon(t), w_\epsilon(t)] &\leq \mathcal{F}[u_{0\epsilon}, w_{0\epsilon}] \\ &\quad - (1-\eta) \int_0^t \int_{\mathbb{R}^d} (u_\epsilon + \epsilon) \psi_l \left| \frac{m_1}{m_1-1} \nabla(u_\epsilon + \epsilon)^{m_1-1} - \nabla v_\epsilon \right|^2 \\ &\quad - (1-\eta) \int_0^t \int_{\mathbb{R}^d} (w_\epsilon + \epsilon) \psi_l \left| \frac{m_2}{m_2-1} \nabla(w_\epsilon + \epsilon)^{m_2-1} - \nabla z_\epsilon \right|^2 \\ &\quad + \frac{CT}{\eta l^2} \left( 1 + \epsilon^{2m_1-1} |\Omega_l| + \epsilon^{2m_2-1} |\Omega_l| \right) \\ &\quad + C \int_0^t \int_{\mathbb{R}^d} (u_\epsilon + w_\epsilon)(1 - \psi_l) + \frac{CT}{l} + \epsilon CT \text{ for } t \in (0, T), \end{aligned}$$

where as  $\epsilon$  tends to 0,

$$\begin{aligned} \mathcal{F}[u(t), w(t)] &\leq \mathcal{F}[u_0, w_0] - (1-\eta) \int_0^t \int_{\mathbb{R}^d} u \psi_l \left| \frac{m_1}{m_1-1} \nabla u^{m_1-1} - \nabla v \right|^2 \\ &\quad - (1-\eta) \int_0^t \int_{\mathbb{R}^d} w \psi_l \left| \frac{m_2}{m_2-1} \nabla w^{m_2-1} - \nabla z \right|^2 \\ &\quad + C \int_0^t \int_{\mathbb{R}^d} (u_0 + w_0)(1 - \psi_l) + \frac{CT}{\eta l^2} + \frac{CT}{l} \text{ for } t \in (0, T) \end{aligned}$$

by the claimed convergence in Lemma 2.4 and a lower semi-continuity of the free energy dissipation. Finally as  $l \rightarrow +\infty$  and  $\eta \rightarrow 0$ ,

$$\begin{aligned} \mathcal{F}[u(t), w(t)] &\leq \mathcal{F}[u_0, w_0] - \int_0^t \int_{\mathbb{R}^d} u \left| \frac{m_1}{m_1-1} \nabla u^{m_1-1} - \nabla v \right|^2 \\ &\quad - \int_0^t \int_{\mathbb{R}^d} w \left| \frac{m_2}{m_2-1} \nabla w^{m_2-1} - \nabla z \right|^2 \text{ for } t \in (0, T). \end{aligned}$$

Therefore,  $(u, w)$  is a free energy solution by the definition. □

### 3 The free energy functional

Now we concentrate on a deeper analysis of the energy functional  $\mathcal{F}$  given by

$$\mathcal{F}[u(t), w(t)] = \frac{1}{m_1-1} \int_{\mathbb{R}^d} u^{m_1} dx + \frac{1}{m_2-1} \int_{\mathbb{R}^d} w^{m_2} dx - c_d \mathcal{H}[u, w]$$

with decay property  $\mathcal{F}[u(t), w(t)] \leq \mathcal{F}[u_0, w_0]$  for  $t \geq 0$ , where

$$\mathcal{H}[u, w] = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{u(x)w(y)}{|x-y|^{d-2}} dx dy = \int_{\mathbb{R}^d} u(x) I_2(w)(x) dx = \int_{\mathbb{R}^d} w(y) I_2(u)(y) dy.$$

The estimate for  $\mathcal{H}$  can be given as follows.

**Lemma 3.1.** Let  $\eta > 0$ , and let  $m_1, m_2, m > 1$ . If

$$m < d/2 \quad \text{and} \quad mm_2 + 2mm_2/d \geq m + m_2, \quad (3.1)$$

then for any  $f \in L^m(\mathbb{R}^d)$  and  $g \in L^1(\mathbb{R}^d) \cap L^{m_2}(\mathbb{R}^d)$ , there holds

$$|\mathcal{H}[f, g]| \leq \eta \|f\|_m^m + C\eta^{-\frac{1}{m-1}} \|g\|_1^{\frac{mm_2+2mm_2/d-m-m_2}{(m-1)(m_2-1)}} \|g\|_{m_2}^{\frac{m_2-2mm_2/d}{(m-1)(m_2-1)}}. \quad (3.2)$$

Moreover, if

$$m < d/2 \quad \text{and} \quad mm_1 + 2mm_1/d \geq m + m_1, \quad (3.3)$$

then for any  $f \in L^1(\mathbb{R}^d) \cap L^{m_1}(\mathbb{R}^d)$  and  $g \in L^m(\mathbb{R}^d)$ , there holds

$$|\mathcal{H}[f, g]| \leq C\eta^{-\frac{1}{m-1}} \|f\|_1^{\frac{m_1m+2m_1m/d-m_1-m}{(m_1-1)(m-1)}} \|f\|_{m_1}^{\frac{m_1-2m_1m/d}{(m_1-1)(m-1)}} + \eta \|g\|_m^m. \quad (3.4)$$

*Proof.* Fixing  $m \in (1, d/2)$ , using Hölder's inequality with  $\frac{1}{m} + \frac{m-1}{m} = 1$  and the HLS inequality with  $\lambda = 2$  in Lemma 2.2, we find that

$$\mathcal{H}[f, g] = \int_{\mathbb{R}^d} f(x) I_2(g)(x) dx \leq \|f\|_m \|I_2(g)\|_{\frac{m}{m-1}} \leq C_{\text{HLS}} \|f\|_m \|g\|_{\frac{md}{(d+2)m-d}}. \quad (3.5)$$

Since the assumption  $m + m_2 \leq mm_2 + 2mm_2/d$  ensures that

$$1 < \frac{md}{(d+2)m-d} \leq m_2,$$

then if  $g \in L^1(\mathbb{R}^d) \cap L^{m_2}(\mathbb{R}^d)$  with  $m_2 > 1$ , the following interpolation inequality holds:

$$\|g\|_{\frac{md}{(d+2)m-d}} \leq \|g\|_1^{\theta_1} \|g\|_{m_2}^{1-\theta_1}$$

with  $\frac{(d+2)m-d}{md} = \theta_1 + \frac{1-\theta_1}{m_2}$ ,  $\theta_1 \in (0, 1)$ . Hence

$$\begin{aligned} |\mathcal{H}[f, g]| &\leq C_{\text{HLS}} \|f\|_m \|g\|_1^{\theta_1} \|g\|_{m_2}^{1-\theta_1} \\ &\leq \eta \|f\|_m^m + C\eta^{-\frac{1}{m-1}} \|g\|_1^{\frac{mm_2+2mm_2/d-m-m_2}{(m-1)(m_2-1)}} \|g\|_{m_2}^{\frac{m_2-2mm_2/d}{(m-1)(m_2-1)}}, \end{aligned}$$

by Young's inequality, which implies (3.2). (3.4) can be also proved if (3.3) holds.  $\square$

We establish several variants to the HLS inequality on the lines  $L_1, L_2$  and the intersection point  $\mathbf{I}$ .

**Lemma 3.2.** Let  $\mathbf{m}$  be on  $L_1$ , and let  $f \in L^{m_1}(\mathbb{R}^d)$  and  $g \in L^1(\mathbb{R}^d) \cap L^{m_2}(\mathbb{R}^d)$ . Then

$$C_\star := \sup_{f \neq 0, g \neq 0} \left\{ \frac{|\mathcal{H}[f, g]|}{\|f\|_{m_1} \|g\|_1^{2/d} \|g\|_{m_2}^{1-2/d}} \right\} < \infty.$$

If  $\mathbf{m}$  is on  $L_2$ , and  $f \in L^1(\mathbb{R}^d) \cap L^{m_1}(\mathbb{R}^d)$  and  $g \in L^{m_2}(\mathbb{R}^d)$ , then

$$C_\star := \sup_{f \neq 0, g \neq 0} \left\{ \frac{|\mathcal{H}[f, g]|}{\|f\|_1^{2/d} \|f\|_{m_1}^{1-2/d} \|g\|_{m_2}} \right\} < \infty.$$

In addition, assume that  $\mathbf{m}$  is  $\mathbf{I}$  and  $(f, g) \in (L^1(\mathbb{R}^d) \cap L^{m_c}(\mathbb{R}^d))^2$ . Then

$$C_c := \sup_{f \neq 0, g \neq 0} \left\{ \frac{\mathcal{H}[f, g]}{\|f\|_1^{1/d} \|f\|_{m_c}^{m_c/2} \|g\|_1^{1/d} \|g\|_{m_c}^{m_c/2}} \right\} < \infty. \quad (3.6)$$

*Proof.* If  $\mathbf{m}$  is on  $L_1$ , then  $m_1 \in (m_c, d/2)$  and using (3.5) with  $m = m_1$  we have

$$|\mathcal{I}[f, g]| \leq C_{\text{HLS}} \|f\|_{m_1} \|g\|_{\frac{m_1 d}{(d+2)m_1 - d}} \leq C_{\text{HLS}} \|f\|_{m_1} \|g\|_1^{2/d} \|g\|_{m_2}^{1-2/d}.$$

Therefore,  $C_*$  is finite and bounded above by  $C_{\text{HLS}}$ . It is also easy to see that  $C_*$  is controlled by  $C_{\text{HLS}}$  if  $\mathbf{m}$  is on  $L_2$ . Finally, with the help of the HLS inequality and Hölder's inequality, we find that

$$|\mathcal{I}[f, g]| \leq C_{\text{HLS}} \|f\|_{\frac{2d}{d+2}} \|g\|_{\frac{2d}{d+2}} \leq C_{\text{HLS}} \|f\|_1^{1/d} \|f\|_{m_c}^{m_c/2} \|g\|_1^{1/d} \|g\|_{m_c}^{m_c/2}$$

if  $\mathbf{m}$  is **I**. Then the definition of  $C_c$  is valid.  $\square$

Define

$$M_{1c} = (c_d C_*)^{-d/2} (m_2/(m_2 - 1))^{d/2} (m_1 - 1)^{-d(m_2-1)/(2m_2)},$$

$$M_{2c} = (c_d C_*)^{-d/2} (m_1/(m_1 - 1))^{d/2} (m_2 - 1)^{-d(m_1-1)/(2m_1)},$$

and

$$M_c = (2/[c_d C_c(m_c - 1)])^{d/2}.$$

The lower and upper bounds for  $\mathcal{F}$  in the sets  $S_{M_1} \times S_{M_2}$  are given next.

**Lemma 3.3.** *Let  $(f, g)$  satisfy  $f \in S_{M_1}$  and  $g \in S_{M_2}$ . If  $\mathbf{m}$  is on  $L_1$ , then*

$$\begin{aligned} & (c_d C_*)^{\frac{m_1}{m_1-1}} (m_1 - 1)^{\frac{m_1}{m_1-1}} m_1^{-\frac{m_1}{m_1-1}} \left( M_{2c}^{\frac{2m_1}{d(m_1-1)}} - M_2^{\frac{2m_1}{d(m_1-1)}} \right) \|g\|_{m_2}^{m_2} \\ & \leq \mathcal{F}[f, g] \leq \frac{2}{m_1 - 1} \|f\|_{m_1}^{m_1} \\ & + (c_d C_*)^{\frac{m_1}{m_1-1}} (m_1 - 1)^{\frac{m_1}{m_1-1}} m_1^{-\frac{m_1}{m_1-1}} \left( M_{2c}^{\frac{2m_1}{d(m_1-1)}} + M_2^{\frac{2m_1}{d(m_1-1)}} \right) \|g\|_{m_2}^{m_2} \end{aligned} \quad (3.7)$$

and

$$\inf_{f \in S_{M_1}} \inf_{g \in S_{M_2}} \mathcal{F}[f, g] = 0, \text{ if } M_2 \in (0, M_{2c}].$$

If  $\mathbf{m}$  is on  $L_2$ , then

$$\mathcal{F}[f, g] \geq (c_d C_*)^{\frac{m_2}{m_2-1}} (m_2 - 1)^{\frac{m_2}{m_2-1}} m_2^{-\frac{m_2}{m_2-1}} \left( M_{1c}^{\frac{2m_2}{d(m_2-1)}} - M_1^{\frac{2m_2}{d(m_2-1)}} \right) \|f\|_{m_1}^{m_1} \quad (3.8)$$

and

$$\inf_{f \in S_{M_1}} \inf_{g \in S_{M_2}} \mathcal{F}[f, g] = 0, \text{ if } M_1 \in (0, M_{1c}]. \quad (3.9)$$

If  $\mathbf{m}$  is **I**, then

$$\mathcal{F}[f, g] \geq \frac{(c_d C_c)^2 (m_c - 1)}{4} \left( M_c^{\frac{4}{d}} - M_1^{\frac{2}{d}} M_2^{\frac{2}{d}} \right) \|g\|_{m_c}^{m_c}$$

or

$$\mathcal{F}[f, g] \geq \frac{(c_d C_c)^2 (m_c - 1)}{4} \left( M_c^{\frac{4}{d}} - M_1^{\frac{2}{d}} M_2^{\frac{2}{d}} \right) \|f\|_{m_c}^{m_c}.$$

Furthermore,

$$\inf_{f \in S_{M_1}} \inf_{g \in S_{M_2}} \mathcal{F}[f, g] = 0, \text{ if } M_1 M_2 \in (0, M_c^2]. \quad (3.10)$$

*Proof.* By Lemma 3.2,  $\mathcal{H}$  satisfies

$$\begin{aligned} |\mathcal{H}[f, g]| &\leq C_* \|f\|_{m_1} \|g\|_1^{\frac{2}{d}} \|g\|_{m_2}^{1-\frac{2}{d}} \\ &\leq \frac{1}{c_d(m_1-1)} \|f\|_{m_1}^{m_1} \\ &\quad + C_* (c_d C_*)^{\frac{1}{m_1-1}} \left( \frac{m_1-1}{m_1} \right)^{\frac{m_1}{m_1-1}} \|g\|_1^{\frac{2m_1}{d(m_1-1)}} \|g\|_{m_2}^{(1-\frac{2}{d})\frac{m_1}{m_1-1}} \\ &= \frac{1}{c_d(m_1-1)} \|f\|_{m_1}^{m_1} \\ &\quad + C_* (c_d C_*)^{\frac{1}{m_1-1}} \left( \frac{m_1-1}{m_1} \right)^{\frac{m_1}{m_1-1}} \|g\|_1^{\frac{2m_1}{d(m_1-1)}} \|g\|_{m_2}^{m_2}. \end{aligned}$$

Then  $\mathcal{F}$  can be estimated by

$$\begin{aligned} \mathcal{F}[f, g] &= \frac{1}{m_1-1} \|f\|_{m_1}^{m_1} + \frac{1}{m_2-1} \|g\|_{m_2}^{m_2} - c_d \mathcal{H}[f, g] \\ &\geq \frac{1}{m_2-1} \|g\|_{m_2}^{m_2} \\ &\quad - (c_d C_*)^{\frac{m_1}{m_1-1}} \left( \frac{m_1-1}{m_1} \right)^{\frac{m_1}{m_1-1}} \|g\|_1^{\frac{2m_1}{d(m_1-1)}} \|g\|_{m_2}^{m_2} \\ &= (c_d C_*)^{\frac{m_1}{m_1-1}} \left( \frac{m_1-1}{m_1} \right)^{\frac{m_1}{m_1-1}} \left( M_{2c}^{\frac{2m_1}{d(m_1-1)}} - M_2^{\frac{2m_1}{d(m_1-1)}} \right) \|g\|_{m_2}^{m_2} \end{aligned}$$

and

$$\begin{aligned} \mathcal{F}[f, g] &\leq \frac{2}{m_1-1} \|f\|_{m_1}^{m_1} \\ &\quad + (c_d C_*)^{\frac{m_1}{m_1-1}} \left( \frac{m_1-1}{m_1} \right)^{\frac{m_1}{m_1-1}} \left( M_{2c}^{\frac{2m_1}{d(m_1-1)}} + M_2^{\frac{2m_1}{d(m_1-1)}} \right) \|g\|_{m_2}^{m_2}. \end{aligned}$$

In the case  $M_2 \leq M_{2c}$ , since  $\mathcal{F} \geq 0$ , then the infimum is nonnegative. Taking

$$h_1(x, t) = \frac{M_1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}} \quad \text{and} \quad h_2(x, t) = \frac{M_2}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}},$$

it is obvious that  $h_i \in L^1(\mathbb{R}^d)$  with  $\|h_i\|_1 = M_i$ ,  $i = 1, 2$ , satisfy

$$\|h_i\|_{m_i}^{m_i} = O(t^{-\frac{d(m_i-1)}{2}}),$$

which implies that  $h_i \in S_{M_i}$  and that  $\mathcal{F}[h_1, h_2]$  tends to 0 as  $t \rightarrow \infty$ . Therefore,

$$\inf_{f \in S_{M_1}} \inf_{g \in S_{M_2}} \mathcal{F}[f, g] = 0.$$

If  $\mathbf{m}$  is on  $L_2$ , we have (3.8) by the HLS inequality and Hölder's inequality, and take  $h_i$  above to see (3.9).

If  $\mathbf{m}$  is  $\mathbf{I}$ , since

$$|\mathcal{H}[f, g]| \leq C_c M_1^{\frac{1}{d}} M_2^{\frac{1}{d}} \|f\|_{m_c}^{\frac{m_c}{2}} \|g\|_{m_c}^{\frac{m_c}{2}} \leq \frac{1}{c_d(m_c-1)} \|f\|_{m_c}^{m_c} + \frac{M_1^{\frac{2}{d}} M_2^{\frac{2}{d}}}{c_d(m_c-1) M_c^{\frac{4}{d}}} \|g\|_{m_c}^{m_c}$$

or

$$|\mathcal{H}[f, g]| \leq \frac{M_1^{\frac{2}{d}} M_2^{\frac{2}{d}}}{c_d(m_c-1) M_c^{\frac{4}{d}}} \|f\|_{m_c}^{m_c} + \frac{1}{c_d(m_c-1)} \|g\|_{m_c}^{m_c}$$

by Young's inequality, then  $\mathcal{F}$  satisfies

$$\begin{aligned} \mathcal{F}[f, g] &\geq \frac{1}{m_c-1} \|f\|_{m_c}^{m_c} + \frac{1}{m_c-1} \|g\|_{m_c}^{m_c} - c_d C_c M_1^{\frac{1}{d}} M_2^{\frac{1}{d}} \|f\|_{m_c}^{\frac{m_c}{2}} \|g\|_{m_c}^{\frac{m_c}{2}} \\ &\geq \frac{(c_d C_c)^2 (m_c-1)}{4} \left( \frac{4}{(c_d C_c (m_c-1))^2} - M_1^{\frac{2}{d}} M_2^{\frac{2}{d}} \right) \|g\|_{m_c}^{m_c} \end{aligned}$$

or

$$\mathcal{F}[f, g] \geq \frac{(c_d C_c)^2 (m_c - 1)}{4} \left( \frac{4}{(c_d C_c (m_c - 1))^2} - M_1^{\frac{2}{d}} M_2^{\frac{2}{d}} \right) \|f\|_{m_c}^{m_c}.$$

One finally obtains from

$$\mathcal{F}[f, g] \leq \frac{2}{m_c - 1} \|f\|_{m_1}^{m_1} + \frac{(c_d C_c)^2 (m_c - 1)}{4} \left( \frac{4}{(c_d C_c (m_c - 1))^2} + M_1^{\frac{2}{d}} M_2^{\frac{2}{d}} \right) \|g\|_{m_c}^{m_c}$$

that (3.10) is true by taking  $f = h_1$  and  $g = h_2$ .  $\square$

The characterization of non-zero minimizers of  $\mathcal{F}$  in  $S_{M_1} \times S_{M_2}$  on critical lines and point is the goal in this subsection. If  $\mathbf{m}$  is  $\mathbf{I}$ , the existence of global minimizers is guaranteed in the follow. The proof is inspired by [6, Proposition 3.5].

**Theorem 3.4.** *Let  $\mathbf{m}$  be  $\mathbf{I}$ . Then there exist a pair of nonnegative, radially symmetric and non-increasing functions  $(f^*, g^*) \in \left(L^1(\mathbb{R}^d) \cap L^{m_c}(\mathbb{R}^d)\right)^2$  such that*

$$\mathcal{H}[f^*, g^*] = C_c.$$

*In addition, there exists a minimizer  $(f, g) \in S_{M_1} \times S_{M_2}$  of  $\mathcal{F}$  if  $M_1 = M_2 = M_c$ , which satisfies*

$$f(x) = g(x) = \begin{cases} \frac{1}{R_0^d} \left[ \zeta \left( \frac{x-x_0}{R_0} \right) \right]^{d/(d-2)}, & \text{if } x \in B(x_0, R_0), \\ 0, & \text{if } x \in \mathbb{R}^d \setminus B(x_0, R_0) \end{cases}$$

*with some  $R_0 > 0$  and  $x_0 \in \mathbb{R}^d$ , where  $\zeta$  is the unique positive radial classical solution to the Lane-Emden equation*

$$\begin{cases} -\Delta \zeta = \frac{m_c-1}{m_c} \zeta^{1/(m_c-1)}, & x \in B(0, 1), \\ \zeta = 0, & x \in \partial B(0, 1). \end{cases}$$

*Proof.* We claim that if  $C_c$  in (3.6) is obtained by some non-zero  $f$  and  $g$ , then  $g = c_0 f$  with some  $c_0$ . This is easily verified by the positive definiteness of  $|x - y|^{-(d-2)}$ , see [32, Theorem 9.8]. In fact, suppose that there exist a pair of maximizing nonnegative functions  $(f, g) \in \left(L^1(\mathbb{R}^d) \cap L^{m_c}(\mathbb{R}^d)\right)^2$  such that

$$\mathcal{H}[f, g] = C_c \|f\|_1^{\frac{1}{d}} \|f\|_{m_c}^{\frac{m_c}{2}} \|g\|_1^{\frac{1}{d}} \|g\|_{m_c}^{\frac{m_c}{2}}.$$

Then by [32, Theorem 9.8] and the HLS inequality,

$$\begin{aligned} \mathcal{H}[f, g] &\leq \sqrt{\mathcal{H}[f, f]} \cdot \sqrt{\mathcal{H}[g, g]} \\ &\leq C_c \|f\|_1^{\frac{1}{d}} \|f\|_{m_c}^{\frac{m_c}{2}} \|g\|_1^{\frac{1}{d}} \|g\|_{m_c}^{\frac{m_c}{2}}. \end{aligned} \quad (3.11)$$

However, (3.11) is an equality if and only if  $g = c_0 f$  with some constant  $c_0$ .

Note that

$$C_c = \sup_{f \neq 0} \left\{ \frac{\mathcal{H}[f, f]}{\|f\|_1^{\frac{2}{d}} \|f\|_{m_c}^{m_c}}, f \in L^1(\mathbb{R}^d) \cap L^{m_c}(\mathbb{R}^d) \right\}. \quad (3.12)$$

The existence of a maximizing nonnegative, radially symmetric and non-increasing  $f^*$  with  $\|f^*\|_1 = \|f^*\|_{m_c} = 1$  for (3.12) has been given in [6, Proposition 3.3]. So choosing  $g^* = f^*$ , then  $\mathcal{H}[f^*, g^*] = C_c$  and the first conclusion has been proved.

To derive minimizers for  $\mathcal{F}$  in the situation  $M_1 = M_2 = M_c$ , with  $f := M_c f^*$  and  $g := M_c f^*$  we have  $(f, g) \in S_{M_1} \times S_{M_2}$  with  $\|f\|_1 = \|f\|_{m_c} = M_c$ ,  $\|g\|_1 = \|g\|_{m_c} = M_c$ . After a careful computation we infer that

$$\mathcal{F}[f, g] = 0$$

by the definition of  $M_c$  and thus  $(f, g)$  is a non-zero global minimizer of  $\mathcal{F}$  in  $S_{M_1} \times S_{M_2}$ . The precise description of the set of minimizers of  $\mathcal{F}$  was derived in [6, Proposition 3.5], we omit it here and have proved the second conclusion.  $\square$

On  $L_1$ , we assert that there is no non-zero minimizer of  $\mathcal{F}$  in  $S_{M_1} \times S_{M_2}$  if  $M_2 = M_{2c}$ . The proof includes two steps: the first one is to derive the nonexistence of non-trivial classical solution to a Lane-Emden system (see Lemma 3.5), and the second is to make a contradiction by the achievement of Euler-Lagrange equalities which consist of the Lane-Emden system on the assumption that minimizers of its free energy exist (see Theorem 3.6).

**Lemma 3.5.** *Let  $M_1, M_2, \rho > 0$ , and let  $m_1 > 1$  and  $m_2 > 1$ . Consider a Lane-Emden system*

$$\begin{cases} -\Delta \vartheta(x) = \frac{m_1-1}{m_1} \zeta^{\frac{1}{m_2-1}}(x), & x \in \Omega_1 = \mathbb{R}^d, \\ -\Delta \zeta(x) = \frac{m_2-1}{m_2} \vartheta^{\frac{1}{m_1-1}}(x), & x \in \Omega_2 = B(0, \rho), \\ \zeta(x) = 0, & x \in \mathbb{R}^d \setminus \Omega_2. \end{cases} \quad (3.13)$$

*Then (3.13) does not admit any nonnegative and non-trivial classical solution  $(\vartheta, \zeta) \in \left(L^{1/(m_1-1)}(\mathbb{R}^d) \cap L^{m_1/(m_1-1)}(\mathbb{R}^d)\right) \times \left(L^{1/(m_2-1)}(\mathbb{R}^d) \cap L^{m_2/(m_2-1)}(\mathbb{R}^d)\right)$  with  $\|\vartheta^{1/(m_1-1)}\|_1 = M_1$  and  $\|\zeta^{1/(m_2-1)}\|_1 = M_2$ , provided that  $\mathbf{m}$  is on  $L_1$ .*

*Proof.* Let

$$q := \frac{1}{m_1-1} \in \left(\frac{2}{d-2}, \frac{d}{d-2}\right).$$

The existence/nonexistence of solutions to the general form of Lane-Emden system has been investigated in [37, 40, 41], for example. However, the solvability of (3.13) involving both whole space and bounded domains is not yet known as far as we know. Here, we assert that there exists no non-trivial classical solution for (3.13) if  $\mathbf{m}$  is on  $L_1$ .

Consider the following properties: Suppose that  $\omega \in C^2(\mathbb{R}^d)$  is non-trivial and satisfies  $\Delta \omega \leq 0$ ,  $x \in \mathbb{R}^d$ . Then

$$\omega(x) \geq C|x|^{2-d}, \quad |x| \geq 1 \quad (3.14)$$

by the strong maximum principle (see [40, Proposition 3.4]). Relying on the finiteness of  $\|\vartheta\|_q$ , we have the following contradiction: For  $R > 1$ ,

$$M_1 \geq \int_{B(0,R)} \vartheta^q = c_d \int_0^R \int_{\mathbb{S}^{d-1}} \vartheta^q(r, \theta) r^{d-1} dS(\theta) dr,$$

where one combines with the fact that  $\Delta \vartheta \leq 0$  for  $x \in \Omega_1 = \mathbb{R}^d$  and (3.14) to see that

$$\begin{aligned} M_1 &\geq C \int_1^R r^{d-1+q(2-d)} dr = C \int_1^R r^{\frac{dm_1+2-2d}{m_1-1}-1} dr \\ &= \frac{C(m_1-1)}{dm_1+2-2d} \left( R^{\frac{dm_1+2-2d}{m_1-1}} - 1 \right) \rightarrow \infty \text{ as } R \rightarrow \infty \end{aligned}$$

due to  $m_1 > m_c = 2 - 2/d$ . So (3.13) has no non-trivial and nonnegative classical solution.  $\square$

**Theorem 3.6.** *Let  $\mathbf{m}$  be on  $L_1$ . For all  $M_2 \leq M_{2c}$ , then  $\mathcal{F}$  does not admit any non-zero minimizer in  $S_{M_1} \times S_{M_2}$ .*

*Proof.* The left inequality in (3.7) in Lemma 3.3 makes sure that there exists no minimizer if  $M_2 < M_{2c}$ . Thus we only consider  $M_2 = M_{2c}$  and prove it by contradiction.

**Step 1.** *Necessary conditions for global minimizers of  $\mathcal{F}$ .* We assume that minimizers exist and try to present some basic properties of them. Suppose that  $(f^*, g^*) \in S_{M_1} \times S_{M_2}$  is a minimizer of  $\mathcal{F}$  in the sense that  $\mathcal{F}[f^*, g^*] = 0$ . Then

$$\begin{aligned} & \frac{1}{m_1 - 1} \|f^*\|_{m_1}^{m_1} + \frac{1}{m_2 - 1} \|g^*\|_{m_2}^{m_2} = c_d \mathcal{H}[f^*, g^*] \\ & \leq c_d C_* \|f^*\|_{m_1} \|g^*\|_1^{2/d} \|g^*\|_{m_2}^{1-2/d} \\ & \leq \frac{1}{m_1 - 1} \|f^*\|_{m_1}^{m_1} + (c_d C_*)^{\frac{m_1}{m_1-1}} \left( \frac{m_1 - 1}{m_1} \right)^{\frac{m_1}{m_1-1}} \|g^*\|_1^{\frac{2m_1}{d(m_1-1)}} \|g^*\|_{m_2}^{\left(1-\frac{2}{d}\right)\frac{m_1}{m_1-1}} \\ & = \frac{1}{m_1 - 1} \|f^*\|_{m_1}^{m_1} + \frac{1}{m_2 - 1} M_{2c}^{-\frac{2m_1}{d(m_1-1)}} \|g^*\|_1^{\frac{2m_1}{d(m_1-1)}} \|g^*\|_{m_2}^{m_2} \\ & = \frac{1}{m_1 - 1} \|f^*\|_{m_1}^{m_1} + \frac{1}{m_2 - 1} M_{2c}^{-\frac{2m_1}{d(m_1-1)}} M_2^{\frac{2m_1}{d(m_1-1)}} \|g^*\|_{m_2}^{m_2} \\ & = \frac{1}{m_1 - 1} \|f^*\|_{m_1}^{m_1} + \frac{1}{m_2 - 1} \|g^*\|_{m_2}^{m_2} \end{aligned} \quad (3.15)$$

by the HLS inequality, Young's inequality, the definition of  $M_{2c}$  and  $M_2 = M_{2c}$ . As a consequence of (3.15), we obtain that

$$\begin{aligned} \|f^*\|_{m_1}^{m_1} &= \frac{1}{m_2 - 1} M_{2c}^{-\frac{2m_1}{d(m_1-1)}} \|g^*\|_1^{\frac{2m_1}{d(m_1-1)}} \|g^*\|_{m_2}^{m_2} \\ &= \frac{1}{m_2 - 1} M_{2c}^{-\frac{2m_1}{d(m_1-1)}} M_2^{\frac{2m_1}{d(m_1-1)}} \|g^*\|_{m_2}^{m_2} \\ &= \frac{1}{m_2 - 1} \|g^*\|_{m_2}^{m_2} \end{aligned} \quad (3.16)$$

and

$$\mathcal{H}[f^*, g^*] = C_* \|f^*\|_{m_1} \|g^*\|_1^{2/d} \|g^*\|_{m_2}^{1-2/d} = \frac{m_1}{c_d(m_1 - 1)(m_2 - 1)} \|g^*\|_{m_2}^{m_2}.$$

**Step 2.** *The Euler-Lagrange equalities.* Let  $f$  and  $g$  be symmetric rearrangement of  $f^*$  and  $g^*$ . Then  $(f, g) \in S_{M_1} \times S_{M_2}$  satisfies

$$\|f\|_{m_1}^{m_1} = \|f^*\|_{m_1}^{m_1} = \frac{1}{m_2 - 1} \|g^*\|_{m_2}^{m_2} = \frac{1}{m_2 - 1} \|g\|_{m_2}^{m_2} \quad (3.17)$$

and

$$\mathcal{H}[f, g] \geq \mathcal{H}[f^*, g^*]$$

by (3.16) and the Riesz rearrangement properties [31, Lemma 2.1]. Obviously,  $\mathcal{F}[f, g] = 0$  and  $(f, g)$  is also a minimizer of  $\mathcal{F}$ . Note that

$$c_d \mathcal{H}[f, g] = \frac{m_1}{m_1 - 1} \|f\|_{m_1}^{m_1} = \frac{m_1}{(m_1 - 1)(m_2 - 1)} \|g\|_{m_2}^{m_2}. \quad (3.18)$$

Given  $\Omega_{10} = \{x \in \mathbb{R}^d : f(x) = 0\}$  and  $\Omega_{1+} = \{x \in \mathbb{R}^d : f(x) > 0\}$  and introduce  $\phi_1 \in C_0^\infty(\mathbb{R}^d)$  with  $\phi_1(x) = \phi_1(-x)$  and

$$\psi_1(x) = \frac{f(x)}{M_1} \left( \phi_1(x) - \frac{1}{M_1} \int_{\mathbb{R}^d} f(x) \phi_1(x) dx \right).$$

Then for  $f \in S_{M_1}$  and fix  $\epsilon \in (0, \epsilon_0 := M_1(2\|\phi_1\|_\infty)^{-1})$ , there holds

$$\|f + \epsilon \psi_1\|_1 = M_1$$



and

$$\begin{aligned} f + \epsilon \psi_1 &= f \left( 1 + \frac{\epsilon}{M_1} \left( \phi_1(x) - \frac{1}{M_1} \int_{\mathbb{R}^d} f(x) \phi_1(x) dx \right) \right) \\ &\geq f \left( 1 - \frac{2 \|\phi_1\|_\infty \epsilon}{M_1} \right) \geq 0, \end{aligned}$$

which implies that  $f + \epsilon \psi_1 \in S_{M_1}$ . Moreover,  $\text{supp}(\psi_1) \subset \overline{\Omega}_{1+}$ . Then

$$\frac{\mathcal{F}[f + \epsilon \psi_1, g] - \mathcal{F}[f, g]}{\epsilon} = \frac{1}{m_1 - 1} \int_{\Omega_{1+}} \frac{(f + \epsilon \psi_1)^{m_1} - f^{m_1}}{\epsilon} - \int_{\mathbb{R}^d} \mathcal{K} \star g(x) \psi_1(x) dx.$$

According to  $\mathcal{F}[f + \epsilon \psi_1, g] \geq \mathcal{F}[f, g]$ , as  $\epsilon \rightarrow 0$ , Lebesgue's dominated convergence theorem shows that

$$\int_{\mathbb{R}^d} \left( \frac{m_1}{m_1 - 1} f^{m_1-1}(x) - \mathcal{K} \star g(x) \right) \psi_1(x) dx \geq 0.$$

By replacing  $-\psi_1$  by  $\psi_1$ , one also obtains from above to see that

$$\int_{\mathbb{R}^d} \left( \frac{m_1}{m_1 - 1} f^{m_1-1}(x) - \mathcal{K} \star g(x) \right) \psi_1(x) dx = 0,$$

where

$$\begin{aligned} 0 &= \frac{1}{M_1} \int_{\mathbb{R}^d} \left( \frac{m_1}{m_1 - 1} f^{m_1-1}(x) - \mathcal{K} \star g(x) \right) f(x) \phi_1(x) dx \\ &\quad - \frac{1}{M_1^2} \int_{\mathbb{R}^d} f(x) \phi_1(x) dx \cdot \int_{\mathbb{R}^d} \left( \frac{m_1}{m_1 - 1} f^{m_1}(x) - \mathcal{K} \star f(x) g(x) \right) dx \\ &= \frac{1}{M_1} \int_{\mathbb{R}^d} \left( \frac{m_1}{m_1 - 1} f^{m_1-1}(x) - \mathcal{K} \star g(x) \right) f(x) \phi_1(x) dx \end{aligned}$$

by (3.18). For any choice of symmetric test function  $\phi_1 \in C_0^\infty(\mathbb{R}^d)$ , we also obtain

$$\frac{m_1}{m_1 - 1} f^{m_1-1}(x) - \mathcal{K} \star g(x) = 0 \text{ a.e. in } \overline{\Omega}_{1+}. \quad (3.19)$$

Now we intend to extend above equality to the whole space. Denote  $\phi_1 \in C_0^\infty(\mathbb{R}^d)$  with  $\phi_1(x) = \phi_1(-x)$  and  $\phi_1 \geq 0$ . Define

$$\psi_1(x) = \phi_1 - \frac{f(x)}{M_1} \int_{\mathbb{R}^d} \phi_1(x) dx. \quad (3.20)$$

Then for  $f \in S_{M_1}$  and fix  $\epsilon \in (0, M_1 (\|\phi_1\|_\infty |\text{supp}(\phi_1)|)^{-1})$ , we have  $f + \epsilon \psi_1 \in S_{M_1}$  due to  $\|f + \epsilon \psi_1\|_1 = M_1$  and

$$f + \epsilon \psi_1 \geq f \left( 1 - \frac{\epsilon}{M_1} \int_{\mathbb{R}^d} \phi_1(x) dx \right) \geq 0 \text{ in } \overline{\Omega}_{1+}$$

and outside  $\overline{\Omega}_{1+}$  since  $\phi_1 \geq 0$ . Following above similar arguments, one has

$$\int_{\mathbb{R}^d} \left( \frac{m_1}{m_1 - 1} f^{m_1-1}(x) - \mathcal{K} \star g(x) \right) \psi_1(x) dx \geq 0,$$

in which we make use of the definition of  $\psi_1$  in (3.20) to see that

$$\int_{\mathbb{R}^d} \left( \frac{m_1}{m_1 - 1} f^{m_1-1}(x) - \mathcal{K} * g(x) \right) \phi_1(x) dx \geq 0.$$

Then

$$\frac{m_1}{m_1 - 1} f^{m_1-1}(y) - \mathcal{K} * g(x) \geq 0 \text{ a.e. in } \mathbb{R}^d.$$

Hence for almost every  $x \in \Omega_{10}$ ,

$$\frac{m_1}{m_1 - 1} f^{m_1-1}(y) = 0 = \mathcal{K} * g(x),$$

which together with (3.19) implies that

$$\frac{m_1}{m_1 - 1} f^{m_1-1} = \mathcal{K} * g(x) \text{ a.e. in } \mathbb{R}^d. \quad (3.21)$$

For  $g$ , arguing similarly as above and we define  $\Omega_{20} = \{x \in \mathbb{R}^d : g(x) = 0\}$  and  $\Omega_{2+} = \{x \in \mathbb{R}^d : g(x) > 0\}$  and introduce  $\phi_2 \in C_0^\infty(\mathbb{R}^d)$  with  $\phi_2(x) = \phi_2(-x)$  and

$$\psi_2(x) = \frac{g(x)}{M_2} \left( \phi_2(x) - \frac{1}{M_2} \int_{\mathbb{R}^d} g(x) \phi_2(x) dx \right).$$

Then for  $g \in S_{M_2}$  and fix  $\epsilon \in (0, M_2(2\|\phi_2\|_\infty)^{-1})$ , there holds  $g + \epsilon\psi_2 \in S_{M_2}$ . Then

$$\begin{aligned} \frac{\mathcal{F}[f, g + \epsilon\psi_2] - \mathcal{F}[f, g]}{\epsilon} &= \frac{1}{m_2 - 1} \int_{\Omega_{2+}} \frac{(g + \epsilon\psi_2)^{m_2} - g^{m_2}}{\epsilon} dy \\ &\quad - \int_{\mathbb{R}^d} \mathcal{K} * f(y) \psi_2(y) dy, \end{aligned}$$

where by Lebesgue's dominated convergence theorem again,

$$\int_{\mathbb{R}^d} \left( \frac{m_2}{m_2 - 1} g^{m_2-1}(y) - \mathcal{K} * f(y) \right) \psi_2(y) dy \geq 0,$$

and replacing  $-\psi_2$  by  $\psi_2$ , it follows that

$$\int_{\mathbb{R}^d} \left( \frac{m_2}{m_2 - 1} g^{m_2-1}(y) - \mathcal{K} * f(y) \right) \psi_2(y) dy = 0.$$

Then (3.17) and (3.18) imply that

$$\begin{aligned} 0 &= \frac{1}{M_2} \int_{\mathbb{R}^d} \left( \frac{m_2}{m_2 - 1} g^{m_2-1}(y) - \mathcal{K} * f(y) \right) g(y) \phi_2(y) dy \\ &\quad - \frac{1}{M_2^2} \int_{\mathbb{R}^d} g(y) \phi_2(y) dy \cdot \int_{\mathbb{R}^d} \left( \frac{m_2}{m_2 - 1} g^{m_2}(y) - \mathcal{K} * f(y) g(y) \right) dy \\ &= \frac{1}{M_2} \int_{\mathbb{R}^d} \left( \frac{m_2}{m_2 - 1} g^{m_2-1}(y) - \mathcal{K} * f(y) \right) g(y) \phi_2(y) dy \\ &\quad + \frac{2m_1}{M_2^2(d - 2m_1)} \|g\|_{m_2}^{m_2} \int_{\mathbb{R}^d} g(y) \phi_2(y) dy \\ &= \frac{1}{M_2} \int_{\mathbb{R}^d} \left( \frac{m_2}{m_2 - 1} g^{m_2-1}(y) - \mathcal{K} * f(y) + \frac{2m_1 \|g\|_{m_2}^{m_2}}{M_2(d - 2m_1)} \right) g(y) \phi_2(y) dy \end{aligned}$$

on  $L_1$ . Therefore,

$$\frac{m_2}{m_2-1} g^{m_2-1} - \mathcal{K} \star f + \frac{2m_1}{M_2(d-2m_1)} \|g\|_{m_2}^{m_2} = 0 \quad a.e. \text{ in } \overline{\Omega}_{2+}. \quad (3.22)$$

To extend the whole space, we repeat the previous argument for  $f$ . Denote  $\phi_2 \in C_0^\infty(\mathbb{R}^d)$  with  $\phi_2(x) = \phi_2(-x)$  and  $\phi_2 \geq 0$ . Define

$$\psi_2(x) = \phi_2 - \frac{g(x)}{M_2} \int_{\mathbb{R}^d} \phi_2(x) dx.$$

Then for  $g \in S_{M_2}$  and fix  $\epsilon \in (0, M_2 (\|\phi_2\|_\infty \text{supp}(\phi_2))^{-1})$ , we have  $g + \epsilon \psi_2 \in S_{M_2}$  due to  $\|g + \epsilon \psi_2\|_1 = M_2$  and  $g + \epsilon \psi_2 \geq 0$ , as well as

$$\int_{\mathbb{R}^d} \left( \frac{m_2}{m_2-1} g^{m_2-1}(y) - \mathcal{K} \star f(y) \right) \psi_2(y) dy \geq 0.$$

Then taking account of the definition of  $\psi_2$ , we see that

$$\frac{m_2}{m_2-1} g^{m_2-1}(y) - \mathcal{K} \star f(y) + \frac{2m_1}{M_2(d-2m_1)} \|g\|_{m_2}^{m_2} \geq 0 \quad a.e. \text{ in } \mathbb{R}^d,$$

which together with (3.22) implies that

$$\frac{m_2}{m_2-1} g^{m_2-1} = \left( \mathcal{K} \star f - \frac{2m_1}{M_2(d-2m_1)} \|g\|_{m_2}^{m_2} \right)_+ \quad a.e. \text{ in } \mathbb{R}^d. \quad (3.23)$$

Since  $g$  is radially symmetric and non-increasing, there exists  $\rho \in (0, \infty]$  such that

$$\Omega_{2+} \subset B(0, \rho) \text{ and } \Omega_{20} \subset \mathbb{R}^d \setminus B(0, \rho),$$

and from (3.23) we obtain

$$\frac{m_2}{m_2-1} g^{m_2-1} = \mathcal{K} \star f - \frac{2m_1}{M_2(d-2m_1)} \|g\|_{m_2}^{m_2} \quad a.e. \text{ in } B(0, \rho).$$

Hence such symmetric non-increasing minimizer  $(f, g) \in S_{M_1} \times S_{M_2}$  of  $\mathcal{F}$  satisfies the following Euler-Lagrange equalities

$$\begin{cases} \frac{m_1}{m_1-1} f^{m_1-1}(x) = \mathcal{K} \star g(x) \quad a.e. \text{ in } \mathbb{R}^d, \\ \frac{m_2}{m_2-1} g^{m_2-1}(x) = \mathcal{K} \star f(x) - \frac{2m_1}{M_2(d-2m_1)} \|g\|_{m_2}^{m_2} \quad a.e. \text{ in } B(0, \rho). \end{cases} \quad (3.24)$$

**Step 3. The regularities of minimizer.** From (3.24)<sub>1</sub>, one invokes the HLS inequality in Lemma 2.2 to see for  $g \in L^1(\mathbb{R}^d) \cap L^{m_2}(\mathbb{R}^d)$  that

$$f \in L^p(\mathbb{R}^d) \text{ with } p \in \left[ \frac{d(m_1-1)}{d-2}, \frac{d(m_1-1)m_2}{d-2m_2} \right],$$

where once more using the HLS inequality again, one concludes that

$$\mathcal{K} \star f \in L^q(\mathbb{R}^d) \text{ with } q \in \begin{cases} \left[ \frac{d(m_1-1)}{d-2m_1}, \frac{d(m_1-1)m_2}{d-2m_1m_2} \right], & \text{if } d > 2m_1m_2, \\ \left[ \frac{d(m_1-1)}{d-2m_1}, \infty \right), & \text{if } d \leq 2m_1m_2. \end{cases}$$

In particular,  $\mathcal{K} \star f \in L^{\frac{m_2}{m_2-1}}(\mathbb{R}^d)$  since  $m_1 + m_2 = 2m_1/d + m_1m_2 \leq 2m_1m_2/d + m_1m_2$  and

$$\frac{m_2}{m_2-1} \in \left[ \frac{d(m_1-1)}{d-2m_1}, \frac{d(m_1-1)m_2}{(d-2m_1m_2)_+} \right).$$

Consequently,  $g^{m_2-1} \in L^{\frac{m_2}{m_2-1}}(\mathbb{R}^d)$ , which excludes  $\rho = \infty$  in (3.24)<sub>2</sub>. Hence  $\rho < \infty$  and

$$\frac{m_2}{m_2-1} g^{m_2-1}(x) = \begin{cases} \mathcal{K} \star f(x) - \frac{2m_1}{M_2(d-2m_1)} \|g\|_{m_2}^{m_2}, & \text{if } |x| < \rho, \\ 0, & \text{if } |x| > \rho \end{cases}$$

by the monotonicity of  $g$ . Moreover, a bootstrap argument ensures that

$$(f, g) \in (L^\infty(\mathbb{R}^d))^2.$$

Letting  $\vartheta := f^{m_1-1}$  and  $\varsigma := g^{m_2-1}$ , we readily infer from (3.24)<sub>1</sub> that

$$\vartheta(x) = \frac{m_1-1}{m_1} \mathcal{K} \star \varsigma^{\frac{1}{m_2-1}}(x) \text{ a.e. in } \mathbb{R}^d,$$

and invoke [21, Theorem 9.9] to have  $\vartheta \in W^{2,r}(B(0, \rho))$  with  $r \in (m_1, \infty)$  and  $-\Delta \vartheta = \frac{m_1-1}{m_1} \varsigma^{\frac{1}{m_2-1}}$  a.e.  $x \in \mathbb{R}^d$ . Furthermore, from the expression for  $\varsigma$  such as

$$\varsigma(x) = \frac{m_2-1}{m_2} \mathcal{K} \star \vartheta^{\frac{1}{m_1-1}}(x) - \frac{2m_1(m_2-1)}{m_2 M_2(d-2m_1)} \|\varsigma\|_{m_2/(m_2-1)}^{m_2/(m_2-1)}, \quad x \in B(0, \rho),$$

by means of the regularity of  $\vartheta$  and [21, Lemma 4.2], we obtain  $\varsigma \in C^2(B(0, \rho))$  with  $-\Delta \varsigma = \frac{m_2-1}{m_2} \vartheta^{\frac{1}{m_1-1}}$  in  $B(0, \rho)$  and [21, Lemma 4.1] ensures that  $\varsigma \in C^1(\mathbb{R}^d)$ . Then  $\varsigma(x) = 0$  if  $|x| = \rho$  and  $\varsigma$  is a classical solution to

$$\begin{cases} -\Delta \varsigma(x) = \frac{m_2-1}{m_2} \vartheta^{\frac{1}{m_1-1}}(x), & x \in B(0, \rho), \\ \varsigma(x) = 0, & x \in \partial B(0, \rho). \end{cases} \quad (3.25)$$

With the smoothness of  $\varsigma$ , [21, Lemma 4.2] applies so as to assert that  $\vartheta \in C^2(\mathbb{R}^d)$  and

$$-\Delta \vartheta(x) = \frac{m_1-1}{m_1} \varsigma^{\frac{1}{m_2-1}}(x), \quad x \in \mathbb{R}^d. \quad (3.26)$$

**Step 4. Contradiction.** (3.25)-(3.26) consist of the Lane-Emden system (3.13). However, it has been proved that there exists no non-trivial classical solution of (3.13) if  $\mathbf{m}$  is on  $L_1$ , which makes a contradiction.  $\square$

**Remark 3.7.** Let  $\mathbf{m}$  be on  $L_2$ , there exists no non-zero minimizer for  $\mathcal{F}$  in  $S_{M_1} \times S_{M_2}$  with  $M_1 \leq M_{1c}$ .

## 4 The global existence

This section deals with the global solvability of (1.1) in the subcritical case. We first present a local existence and extensibility criterion of free energy solutions to (1.1). Note that this theorem also provides the simultaneous blow-up argument in Section 5.

**Theorem 4.1.** Let  $m_1, m_2 > 1$ . Under assumption (1.2) on the initial data  $(u_0, w_0)$  with  $\|u_0\|_1 = M_1$ ,  $\|w_0\|_1 = M_2$ , then there exists  $T_{\max} \in (0, \infty]$  and a free energy solution  $(u, w)$  over  $\mathbb{R}^d \times (0, T_{\max})$  of (1.1) such that either  $T_{\max} = \infty$  or  $T_{\max} < \infty$  and

$$\lim_{t \rightarrow T_{\max}} (\|u(\cdot, t)\|_\infty + \|w(\cdot, t)\|_\infty) = \infty. \quad (4.1)$$

Moreover, let  $\mathbf{m}$  be subcritical or critical. Then if  $T_{\max} < \infty$ ,

$$\lim_{t \rightarrow T_{\max}} \|u(\cdot, t)\|_{m_1} = \lim_{t \rightarrow T_{\max}} \|w(\cdot, t)\|_{m_2} = \infty. \quad (4.2)$$

*Proof.* For  $(u_0, w_0)$  satisfying (1.2), local existence and (4.1) can be proved by approximation arguments (similar to those in the proof of Theorem 1.1 in [43] for instance). To see (4.2), since the solution is globally solved

if both  $\|u\|_{m_1}$  and  $\|w\|_{m_2}$  are uniformly bound in the subcritical or critical case due to Lemmas 2.3-2.5, then it is sufficient to show that the two terms  $\|u\|_{m_1}$  and  $\|w\|_{m_2}$  are governed by one other with some constants.

Since

$$\frac{1}{m_1 - 1} \int_{\mathbb{R}^d} u^{m_1} + \frac{1}{m_2 - 1} \int_{\mathbb{R}^d} w^{m_2} \leq c_d \mathcal{H}[u, w] + \mathcal{F}[u_0, w_0], \quad (4.3)$$

then it needs to control the term  $\mathcal{H}$  at the right side of (4.3). For  $m \in (1, d/2)$  satisfying (3.1), Lemma 3.1 yields that

$$|\mathcal{H}[f, g]| \leq \eta \|f\|_m^m + C \eta^{-\frac{1}{m-1}} \|g\|_1^{\frac{mm_2+2mm_2/d-m-m_2}{(m-1)(m_2-1)}} \|g\|_{m_2}^{\frac{m_2-2mm_2/d}{(m-1)(m_2-1)}} \quad (4.4)$$

for some  $f \in L^m(\mathbb{R}^d)$  and  $g \in L^1(\mathbb{R}^d) \cap L^{m_2}(\mathbb{R}^d)$  with  $\eta > 0$ . If  $m_1 < d/2$ , choosing  $m = m_1$  in (4.4), then

$$\begin{aligned} & \frac{1}{m_1 - 1} \int_{\mathbb{R}^d} u^{m_1} + \frac{1}{m_2 - 1} \int_{\mathbb{R}^d} w^{m_2} \\ & \leq c_d \eta \|u\|_{m_1}^{m_1} + c_d C \eta^{-\frac{1}{m_1-1}} M_2^{\frac{m_1 m_2 + 2m_1 m_2/d - m_1 - m_2}{(m_1-1)(m_2-1)}} \|w\|_{m_2}^{\frac{m_2 - 2m_1 m_2/d}{(m_1-1)(m_2-1)}} + \mathcal{F}[u_0, w_0] \\ & \leq c_d \eta \|u\|_{m_1}^{m_1} + c_d C \eta^{-\frac{1}{m_1-1}} \|w\|_{m_2}^{m_2} + C \end{aligned}$$

by Young's inequality, since

$$\frac{m_2 - 2m_1 m_2/d}{(m_1 - 1)(m_2 - 1)} \leq m_2$$

if  $m_1 m_2 + 2m_1/d \geq m_1 + m_2$  holds. Taking  $\eta$  small enough, we have

$$\|u(t)\|_{m_1}^{m_1} \leq C \|w(t)\|_{m_2}^{m_2} + C \quad \text{for } t \in (0, T_{\max}) \quad (4.5)$$

and if  $\eta$  is sufficiently large, we see that

$$\|w(t)\|_{m_2}^{m_2} \leq C' \|u(t)\|_{m_1}^{m_1} + C' \quad \text{for } t \in (0, T_{\max}). \quad (4.6)$$

Therefore, (4.2) holds by (4.1), (4.5)-(4.6). In fact, suppose that  $T_{\max} < \infty$  and (4.2) does not hold. Then the finiteness of  $\|u(t)\|_{m_1}$  or  $\|w(t)\|_{m_2}$  ensures both norms are finite by means of (4.5)-(4.6), which actually implies that  $T_{\max} = \infty$  due to Lemma 2.4. This is a contradiction.

However, if  $m_1 \geq d/2$ , we pick  $m \in (1, d/2)$  such that

$$\frac{m_2}{m_2 + 2/d - 1} < m < d/2,$$

and next take interpolation inequality to find that

$$\|u\|_m^m \leq \|u\|_1^{\frac{m_1-m}{m_1-1}} \|u\|_{m_1}^{\frac{m_1(m-1)}{m_1-1}}.$$

Upon

$$\frac{m_2 - 2mm_2/d}{(m-1)(m_2-1)} < m_2,$$

then (4.4) implies that

$$\begin{aligned} |\mathcal{H}[u, w]| & \leq \eta \|u\|_1^{\frac{m_1-m}{m_1-1}} \|u\|_{m_1}^{\frac{m_1(m-1)}{m_1-1}} + C \eta^{-\frac{1}{m-1}} \|w\|_1^{\frac{mm_2+2mm_2/d-m-m_2}{(m-1)(m_2-1)}} \|w\|_{m_2}^{\frac{m_2-2mm_2/d}{(m-1)(m_2-1)}} \\ & = \eta M_1^{\frac{m_1-m}{m_1-1}} \|u\|_{m_1}^{\frac{m_1(m-1)}{m_1-1}} + C \eta^{-\frac{1}{m-1}} M_2^{\frac{mm_2+2mm_2/d-m-m_2}{(m-1)(m_2-1)}} \|w\|_{m_2}^{\frac{m_2-2mm_2/d}{(m-1)(m_2-1)}} \\ & \leq \eta \|u\|_{m_1}^{m_1} + \eta^{-\frac{1}{m-1}} \|w\|_{m_2}^{m_2} + C \end{aligned} \quad (4.7)$$

with  $\|u\|_1 = M_1$  and  $\|w\|_1 = M_2$ . Hence (4.5)-(4.6) are valid by picking suitable  $\eta > 0$ . By the same token, the case  $m_1 m_2 + 2m_2/d \geq m_1 + m_2$  is also true for both  $m_2 < d/2$  and  $m_2 \geq d/2$ . The proof is finished.  $\square$

The global existence result in the subcritical case is the subject of our next theorem.

**Theorem 4.2.** *Let  $m_1, m_2 > 1$ . Suppose that the initial data  $(u_0, w_0)$  with  $\|u_0\|_1 = M_1$ ,  $\|w_0\|_1 = M_2$  fulfills (1.2). Then if  $\mathbf{m}$  is subcritical, (1.1) has a global free energy solution given in Definition 1.2.*

**Remark 4.3.** *If  $m_1 \geq d/2$  or  $m_2 \geq d/2$ , the conclusion in Theorem 4.2 holds for all  $m_2 > 1$  or  $m_1 > 1$ .*

*Proof.* In the case  $m_1 m_2 + 2m_1/d > m_1 + m_2$  and  $m_1 < d/2$ , since  $\frac{m_2 - 2m_1 m_2/d}{(m_1 - 1)(m_2 - 1)} < m_2$ , then Lemma 3.1 warrants that

$$\begin{aligned} |\mathcal{H}[u, w]| &\leq \frac{1}{2c_d(m_1 - 1)} \|u\|_{m_1}^{m_1} + C \|w\|_1^{\frac{m_1 m_2 + 2m_1 m_2/d - m_1 - m_2}{(m_1 - 1)(m_2 - 1)}} \|w\|_{m_2}^{\frac{m_2 - 2m_1 m_2/d}{(m_1 - 1)(m_2 - 1)}} \\ &\leq \frac{1}{2c_d(m_1 - 1)} \|u\|_{m_1}^{m_1} + \frac{1}{2c_d(m_2 - 1)} \|w\|_{m_2}^{m_2} + C \end{aligned}$$

by Young's inequality. Then substituting (4.3) into above, we have

$$\begin{aligned} \frac{1}{m_1 - 1} \int_{\mathbb{R}^d} u^{m_1} dx + \frac{1}{m_2 - 1} \int_{\mathbb{R}^d} w^{m_2} dx \\ \leq \frac{1}{2(m_1 - 1)} \int_{\mathbb{R}^d} u^{m_1} dx + \frac{1}{2(m_2 - 1)} \int_{\mathbb{R}^d} w^{m_2} dx + C. \end{aligned}$$

As a corollary,

$$\|u\|_{m_1} \leq C \quad \text{and} \quad \|w\|_{m_2} \leq C. \quad (4.8)$$

If  $m_1 \geq \frac{d}{2}$ , we recalculate (4.7) carefully and also have (4.8), in which the global existence of free energy solution is immediate from Theorem 4.1. The other case  $m_1 m_2 + 2m_2/d > m_1 + m_2$  is similar.  $\square$

Also on the critical lines, we obtain global existence results reading as

**Theorem 4.4.** *Let  $\mathbf{m}$  be on  $L_1$ , and let  $(u, w)$  be a free energy solution of (1.1) with  $(u_0, w_0)$  satisfying (1.2) on  $[0, T_{\max})$  with  $T_{\max}$  given in Theorem 4.1. If*

$$M_2 < M_{2c}, \quad (4.9)$$

*then  $T_{\max} = \infty$ . The subcritical condition (4.9) will be replaced by  $M_1 < M_{1c}$  on  $L_2$ . Moreover, if  $\mathbf{m}$  is  $\mathbf{I}$ , one has  $T_{\max} = \infty$  if  $M_1 M_2 < M_c^2$ .*

*Proof.* We just infer from (1.7) and Lemma 3.3 that

$$\begin{aligned} (c_d C_*)^{\frac{m_1}{m_1 - 1}} (m_1 - 1)^{\frac{m_1}{m_1 - 1}} m_1^{-\frac{m_1}{m_1 - 1}} \left( M_{2c}^{\frac{2m_1}{d(m_1 - 1)}} - M_2^{\frac{2m_1}{d(m_1 - 1)}} \right) \|w\|_{m_2}^{m_2} \\ \leq \mathcal{F}[u, w] \leq \mathcal{F}[u_0, w_0]. \end{aligned}$$

Due to (4.9), there exists  $C > 0$  such that for all  $t \in [0, T_{\max})$  we have  $\|w\|_{m_2} \leq C$ . Then the extensibility criterion in Theorem 4.1 makes sure that  $T_{\max} = \infty$ . The other cases can be similarly obtained.  $\square$

## 5 Blow up

Our last section concerns finite-time blow-up phenomenon when  $\mathbf{m}$  is critical or super-critical. These results actually show that lines  $L_i$ ,  $i = 1, 2$  are optimal in view of the global existence for sub-critical case. The following second moment estimate of solutions can be achieved in a straightforward computation.

**Lemma 5.1.** Let  $(u_0, w_0)$  satisfy (1.2), and let  $(u, w)$  be a free energy solution of (1.1) on  $[0, T_{\max})$  with  $T_{\max} \in (0, \infty]$ . Then

$$\frac{d}{dt}I(t) = G(t) \text{ for all } t \in (0, T_{\max}),$$

where

$$I(t) := \int_{\mathbb{R}^d} |x|^2 (u(x, t) + w(x, t)) dx$$

and

$$\begin{aligned} G(t) := & 2d \int_{\mathbb{R}^d} u^{m_1}(x, t) dx + 2d \int_{\mathbb{R}^d} w^{m_2}(x, t) dx \\ & - 2c_d(d-2) \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{u(x, t)w(y, t)}{|x-y|^{d-2}} dx dy. \end{aligned}$$

*Proof.* We present a formal computation for the proof of this lemma. Otherwise, one can easily invoke some localisation arguments in [8, Lemma 2.1] or [43, Lemma 6.2] to give a complete rigorous proof. We differentiate the second moment to see that

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^d} |x|^2 (u(x, t) + w(x, t)) dx \\ &= \int_{\mathbb{R}^d} |x|^2 (\Delta u^{m_1} - \nabla \cdot (u \nabla v)) dx + \int_{\mathbb{R}^d} |x|^2 (\Delta w^{m_2} - \nabla \cdot (w \nabla z)) dx \\ &= 2d \int_{\mathbb{R}^d} u^{m_1}(x, t) dx + 2d \int_{\mathbb{R}^d} w^{m_2}(x, t) dx \\ &+ 2 \iint_{\mathbb{R}^d \times \mathbb{R}^d} [x \cdot \nabla \mathcal{K}(x-y)] u(x, t) w(y, t) dx dy \\ &+ 2 \iint_{\mathbb{R}^d \times \mathbb{R}^d} [x \cdot \nabla \mathcal{K}(x-y)] u(y, t) w(x, t) dx dy. \end{aligned}$$

With  $\mathcal{K}(x) = c_d \frac{1}{|x|^{d-2}}$ , we have

$$\begin{aligned} & 2 \iint_{\mathbb{R}^d \times \mathbb{R}^d} [x \cdot \nabla \mathcal{K}(x-y)] u(x, t) w(y, t) dx dy \\ &= -2c_d(d-2) \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(x-y) \cdot x}{|x-y|^d} u(x, t) w(y, t) dx dy \\ &= -2c_d(d-2) \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|x|^2}{|x-y|^d} u(x, t) w(y, t) dx dy \\ &+ 2c_d(d-2) \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{x \cdot y}{|x-y|^d} u(x, t) w(y, t) dx dy \\ &= -c_d(d-2) \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|x|^2}{|x-y|^d} u(x, t) w(y, t) dx dy \\ &- c_d(d-2) \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|y|^2}{|x-y|^d} u(y, t) w(x, t) dx dy \\ &+ 2c_d(d-2) \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{x \cdot y}{|x-y|^d} u(x, t) w(y, t) dx dy \end{aligned}$$

and

$$\begin{aligned}
 & 2 \iint_{\mathbb{R}^d \times \mathbb{R}^d} [x \cdot \nabla \mathcal{K}(x-y)] u(y, t) w(x, t) dx dy \\
 &= -c_d(d-2) \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|x|^2}{|x-y|^d} u(y, t) w(x, t) dx dy \\
 &\quad - c_d(d-2) \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|y|^2}{|x-y|^d} u(x, t) w(y, t) dx dy \\
 &\quad + 2c_d(d-2) \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{x \cdot y}{|x-y|^d} u(x, t) w(y, t) dx dy.
 \end{aligned}$$

Combining above equations, it follows that

$$\begin{aligned}
 & \frac{d}{dt} \int_{\mathbb{R}^d} |x|^2 (u(x, t) + w(x, t)) dx = 2d \int_{\mathbb{R}^d} u^{m_1}(x, t) dx + 2d \int_{\mathbb{R}^d} w^{m_2}(x, t) dx \\
 &\quad - c_d(d-2) \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|x|^2 + |y|^2}{|x-y|^d} u(x, t) w(y, t) dx dy \\
 &\quad - c_d(d-2) \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|x|^2 + |y|^2}{|x-y|^d} u(y, t) w(x, t) dx dy \\
 &\quad + 4c_d(d-2) \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{x \cdot y}{|x-y|^d} u(x, t) w(y, t) dx dy \\
 &= 2d \int_{\mathbb{R}^d} u^{m_1}(x, t) dx + 2d \int_{\mathbb{R}^d} w^{m_2}(x, t) dx \\
 &\quad - 2c_d(d-2) \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{u(x, t) w(y, t)}{|x-y|^{d-2}} dx dy,
 \end{aligned}$$

which readily implies the lemma.  $\square$

We construct initial data which ensures the non-positivity of  $G(0)$ .

**Lemma 5.2.** *Let  $\mathbf{m}$  be critical or super-critical. There exists initial data  $(u_0, w_0)$  satisfying (1.2), and fulfilling*

$$\begin{aligned}
 & \frac{\left( \int_{\mathbb{R}^d} u_0^{1/t_1} dx \right)^{t_1} \left( \int_{\mathbb{R}^d} w_0^{1/t_2} dx \right)^{t_2}}{\left( \int_{\mathbb{R}^d} u_0^{1/t_1} dx \right)^{t_1 m_1} + \left( \int_{\mathbb{R}^d} w_0^{1/t_2} dx \right)^{t_2 m_2}} \\
 & > \begin{cases} N_0, & \text{if } m_1 m_2 + 2 \max\{m_1, m_2\}/d \leq m_1 + m_2 < m_1 m_2 + 2/dm_1 m_2, \\ 2N_0, & \text{if } m_1 + m_2 \geq m_1 m_2 + 2/dm_1 m_2, \end{cases}
 \end{aligned} \tag{5.1}$$

and

$$G(0) < 0, \tag{5.2}$$

where

$$\iota_1 := \frac{2m_2}{(m_1 + m_2 - m_1 m_2)d} \quad \text{and} \quad \iota_2 := \frac{2m_1}{(m_1 + m_2 - m_1 m_2)d}, \tag{5.3}$$

$$N_0 = \frac{(d/c_d)^{2-2/d}}{2^{1+2/d}(d-2)} (1 + \iota_1)(1 + \iota_2)$$

and  $G$  is given in Lemma 5.1.



*Proof.* Consider the following functions having the same compact support as initial data of form

$$\begin{aligned} u_0(x) &= A \left( 1 - \frac{|x|^d}{a^d} \right)_+^{t_1}, \quad x \in \mathbb{R}^d, \\ w_0(x) &= B \left( 1 - \frac{|x|^d}{a^d} \right)_+^{t_2}, \quad x \in \mathbb{R}^d, \end{aligned} \quad (5.4)$$

where  $A, B > 0$  denote the maximum of initial data and  $a > 0$  denotes the size of the corresponding supports. Such constructions in (5.4) are inspired by [44, Section 6] which deals with one population Keller-Segel system.

In the **Case 1:**  $m_1 m_2 + 2 \max\{m_1, m_2\}/d \leq m_1 + m_2 < m_1 m_2 + 2m_1 m_2/d$ , one has

$$\begin{aligned} \int_{\mathbb{R}^d} u_0^{m_1} dx &= A^{m_1} \int_{\mathbb{R}^d} \left( 1 - \frac{|x|^d}{a^d} \right)_+^{\frac{2m_1 m_2}{(m_1+m_2-m_1 m_2)d}} dx \\ &= A^{m_1} \int_{\mathbb{R}^d} \left( 1 - \frac{|x|^d}{a^d} \right)_+^{\frac{2m_1 m_2}{(m_1+m_2-m_1 m_2)d}-1} \left( 1 - \frac{|x|^d}{a^d} \right)_+ dx \\ &\leq A^{m_1} \int_{\mathbb{R}^d} \left( 1 - \frac{|x|^d}{a^d} \right)_+ dx \\ &= c_d a^d A^{m_1} / (2d) \end{aligned} \quad (5.5)$$

and

$$\int_{\mathbb{R}^d} w_0^{m_2} dx \leq c_d a^d B^{m_2} / (2d).$$

For the **Case 2:**  $m_1 + m_2 > m_1 m_2 + 2m_1 m_2/d$ ,

$$\begin{aligned} \int_{\mathbb{R}^d} u_0^{m_1} dx &\leq A^{m_1} \int_{|x|<a} 1 dx = c_d a^d A^{m_1} / d, \\ \int_{\mathbb{R}^d} w_0^{m_2} dx &\leq c_d a^d B^{m_2} / d. \end{aligned}$$

The coupled term can be estimated as

$$\begin{aligned} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{u_0(x) w_0(y)}{|x-y|^{d-2}} dx dy &\geq \min_{|x|, |y| \leq a} |x-y|^{-(d-2)} \int_{\mathbb{R}^d} u_0(x) dx \cdot \int_{\mathbb{R}^d} w_0(y) dy \\ &\geq a^{-(d-2)} \int_{\mathbb{R}^d} A \left( 1 - \frac{|x|^d}{a^d} \right)_+^{t_1} dx \cdot \int_{\mathbb{R}^d} B \left( 1 - \frac{|x|^d}{a^d} \right)_+^{t_2} dx \\ &= \frac{c_d^2 a^{d+2}}{d^2(1+t_1)(1+t_2)} AB. \end{aligned} \quad (5.6)$$

Since

$$G(0) \leq c_d a^d A^{m_1} + c_d a^d B^{m_2} - \frac{2c_d^3 a^{d+2}(d-2)}{d^2(1+t_1)(1+t_2)} AB \quad (5.7)$$

by (5.5)-(5.6), to show (5.2), it only needs to show the right side of (5.7) is negative in the sense that

$$\frac{AB}{A^{m_1} + B^{m_2}} a^2 > N_1 \quad (5.8)$$

with

$$N_1 = \frac{d^2}{2c_d^2(d-2)} (1+t_1)(1+t_2)$$

in the **Case 1**, whereas the right side will be replaced by  $2N_1$  in the **Case 2**.

Since

$$\begin{aligned}\int_{\mathbb{R}^d} u_0^{1/t_1} dx &= A^{1/t_1} \int_{\mathbb{R}^d} \left(1 - \frac{|x|^d}{a^d}\right)_+ dx = c_d a^d A^{1/t_1} / (2d), \\ \int_{\mathbb{R}^d} w_0^{1/t_2} dx &= B^{1/t_2} \int_{\mathbb{R}^d} \left(1 - \frac{|x|^d}{a^d}\right)_+ dx = c_d a^d B^{1/t_2} / (2d)\end{aligned}$$

imply that

$$A = \left(2d \int_{\mathbb{R}^d} u_0^{1/t_1} dx / c_d\right)^{t_1} a^{-t_1 d}, \quad B = \left(2d \int_{\mathbb{R}^d} w_0^{1/t_2} dx / c_d\right)^{t_2} a^{-t_2 d},$$

then (5.8) can be rewritten as

$$\begin{aligned}\frac{AB}{A^{m_1} + B^{m_2}} a^2 &= \frac{\left(2d \int_{\mathbb{R}^d} u_0^{1/t_1} dx / c_d\right)^{t_1} \left(2d \int_{\mathbb{R}^d} w_0^{1/t_2} dx / c_d\right)^{t_2}}{\left(2d \int_{\mathbb{R}^d} u_0^{1/t_1} dx / c_d\right)^{t_1 m_1} + \left(2d \int_{\mathbb{R}^d} w_0^{1/t_2} dx / c_d\right)^{t_2 m_2}} \\ &= (2d/c_d)^{2/d} \frac{\left(\int_{\mathbb{R}^d} u_0^{1/t_1} dx\right)^{t_1} \left(\int_{\mathbb{R}^d} w_0^{1/t_2} dx\right)^{t_2}}{\left(\int_{\mathbb{R}^d} u_0^{1/t_1} dx\right)^{t_1 m_1} + \left(\int_{\mathbb{R}^d} w_0^{1/t_2} dx\right)^{t_2 m_2}} \\ &> N_1 \quad (\text{or } 2N_1 \text{ for the Case 2}).\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\frac{\left(\int_{\mathbb{R}^d} u_0^{1/t_1} dx\right)^{t_1} \left(\int_{\mathbb{R}^d} w_0^{1/t_2} dx\right)^{t_2}}{\left(\int_{\mathbb{R}^d} u_0^{1/t_1} dx\right)^{t_1 m_1} + \left(\int_{\mathbb{R}^d} w_0^{1/t_2} dx\right)^{t_2 m_2}} \\ &> \begin{cases} (c_d/2d)^{2/d} N_1, & \text{if } m_1 m_2 + 2 \max\{m_1, m_2\}/d \leq m_1 + m_2 \\ < m_1 m_2 + 2m_1 m_2/d, \\ 2(c_d/2d)^{2/d} N_1, & \text{if } m_1 + m_2 \geq m_1 m_2 + 2m_1 m_2/d, \end{cases}\end{aligned}$$

which yields  $G(0) < 0$ . □

The blow-up results state that

**Theorem 5.3.** *Let  $\mathbf{m}$  be critical or super-critical. Then one can find some initial data  $(u_0, w_0)$  satisfying (1.2) such that free energy solution  $(u, w)$  of (1.1) with  $(u, w)|_{t=0} = (u_0, w_0)$  blows up in finite time.*

*Proof.* For a given initial data  $(u_0, w_0)$  in (5.4) satisfying (5.1), then  $G(0) < 0$  from Lemma 5.2. By the continuity argument, there exists  $T^* > 0$  such that

$$G(t) < G(0)/2 \quad \text{for all } t \in [0, T^*],$$

where from Lemma 5.1, one obtains  $\frac{d}{dt} I(t) < G(0)/2$  for all  $t \in [0, T^*]$ . Integrating by parts, it follows that

$$I(T^*) < I(0) + G(0)T^*/2. \quad (5.9)$$

As

$$\begin{aligned}
 I(0) &= \int_{\mathbb{R}^d} |x|^2 \left( A \left( 1 - \frac{|x|^d}{a^d} \right)_+^{t_1} + B \left( 1 - \frac{|x|^d}{a^d} \right)_+^{t_2} \right) dx \\
 &= A \int_{|x| \leq a} |x|^2 \left( 1 - \frac{|x|^d}{a^d} \right)_+^{t_1} dx + B \int_{|x| \leq a} |x|^2 \left( 1 - \frac{|x|^d}{a^d} \right)_+^{t_2} dx \\
 &= c_d A \int_0^a \left( 1 - \frac{r^d}{a^d} \right)_+^{t_1} r^{d+1} dr + c_d B \int_0^a \left( 1 - \frac{r^d}{a^d} \right)_+^{t_2} r^{d+1} dr \\
 &= (c_d a^{d+2} A N_2) / d + (c_d a^{d+2} B N_3) / d
 \end{aligned} \tag{5.10}$$

with  $t_1, t_2$  given in (5.3) and

$$N_2 := \int_0^1 (1-r)^{t_1} r^{2/d} dr < \infty \quad \text{and} \quad N_3 := \int_0^1 (1-r)^{t_2} r^{2/d} dr < \infty,$$

then inserting (5.7) and (5.10) into (5.9), the right side of (5.9) should be negative if we may fix small  $a > 0$  such that

$$\begin{aligned}
 &\frac{T^*}{2} \cdot \left[ \frac{2c_d^3 a^{d+2} (d-2)}{d^2 (1+t_1)(1+t_2)} AB - c_d a^d A^{m_1} - c_d a^d B^{m_2} \right] \\
 &\geq (c_d a^{d+2} A N_2) / d + (c_d a^{d+2} B N_3) / d.
 \end{aligned}$$

More precisely, if

$$\begin{aligned}
 &\frac{dT^*}{2} \cdot \left[ \frac{2^{1+2/d} (d-2)}{(1+t_1)(1+t_2)} \left( \frac{c_d}{d} \right)^{2-2/d} \cdot \left( \int_{\mathbb{R}^d} u_0^{1/t_1} dx \right)^{t_1} \left( \int_{\mathbb{R}^d} w_0^{1/t_2} dx \right)^{t_2} \right. \\
 &\quad \left. - \left( \int_{\mathbb{R}^d} u_0^{1/t_1} dx \right)^{m_1 t_1} - \left( \int_{\mathbb{R}^d} w_0^{1/t_2} dx \right)^{m_2 t_2} \right] \\
 &\geq (2d/c_d)^{(1-m_1)t_1} \left( \int_{\mathbb{R}^d} u_0^{1/t_1} dx \right)^{t_1} a^{d t_2} N_2 \\
 &\quad + (2d/c_d)^{(1-m_2)t_2} \left( \int_{\mathbb{R}^d} w_0^{1/t_2} dx \right)^{t_2} a^{d t_1} N_3,
 \end{aligned}$$

this leads to a contradiction after time  $T^*$  since  $I(t)$  is always nonnegative for all  $t > 0$ . Hence the solutions blow up in finite time.  $\square$

If  $\mathbf{m}$  is **I**, Theorem 5.3 shows that the blow up condition (5.1) can be written as

$$\frac{M_1 M_2}{M_1^{m_c} + M_2^{m_c}} > \frac{1}{2(d-2)} \cdot \left( \frac{2d}{c_d} \right)^{m_c}, \tag{5.11}$$

since

$$\frac{d}{dt} I(t) = G(t) = 2(d-2) \mathcal{F}[u(t), w(t)] \leq 2(d-2) \mathcal{F}[u_0, w_0] = G(0) < 0$$

if (5.11) holds, then the second moment will be negative after some time and it contradicts the non-negativity of  $u$  and  $w$ .

We improve blow-up arguments if  $\mathbf{m}$  is **I** by using a different method and summarize the blows up results on the lines  $L_1, L_2$  and intersection point **I** as

**Theorem 5.4.** Let  $\mathbf{m}$  be critical. Suppose that  $(u, w)$  is a free energy solution of (1.1) with  $\|u_0\|_1 = M_1$ ,  $\|w_0\|_1 = M_2$  fulfilling (1.2).

If  $\mathbf{m}$  is on  $L_1$ , for sufficiently small size of the supports of  $(u_0, w_0)$  one asserts that blow up happens if

$$\frac{\left(\int_{\mathbb{R}^d} u_0^{m_1/m_2} dx\right)^{m_2/m_1} \left(\int_{\mathbb{R}^d} w_0 dx\right)}{\left(\int_{\mathbb{R}^d} u_0^{m_1/m_2} dx\right)^{m_2} + \left(\int_{\mathbb{R}^d} w_0 dx\right)^{m_2}} > N_0$$

with  $N_0$  given in Lemma 5.2.

If  $\mathbf{m}$  is on  $L_2$ , for sufficiently small size of the supports of  $(u_0, w_0)$  blow-up solution can be constructed if

$$\frac{\left(\int_{\mathbb{R}^d} u_0 dx\right) \left(\int_{\mathbb{R}^d} w_0^{m_2/m_1} dx\right)^{m_1/m_2}}{\left(\int_{\mathbb{R}^d} u_0 dx\right)^{m_1} + \left(\int_{\mathbb{R}^d} w_0^{m_2/m_1} dx\right)^{m_1}} > N_0.$$

If  $\mathbf{m}$  is  $\mathbf{I}$ , blow up occurs if

$$M_1 M_2 / (M_1^{m_c} + M_2^{m_c}) > M_c^{2/d} / 2.$$

Finally, let  $(u, w)$  blow up in finite time  $T_{\max}$ . Then  $T_{\max} < \infty$  implies that

$$\lim_{t \rightarrow T_{\max}} \|u\|_{m_1} = \lim_{t \rightarrow T_{\max}} \|w\|_{m_2} = \infty.$$

*Proof.* The asserted blow-up conditions on the lines  $L_1$  and  $L_2$  just follow from Lemma 5.2 and Theorem 5.3. If  $\mathbf{m}$  is  $\mathbf{I}$ , note that for any  $M_1^* > 0$  and  $M_2^* > 0$  such that

$$M_1^* M_2^* / (M_1^{*m_c} + M_2^{*m_c}) = M_c^{2/d} / 2, \quad (5.12)$$

there exists nonnegative function  $(u^*, w^*)$  with  $\|u^*\|_1 = M_1^*$ ,  $\|w^*\|_1 = M_2^*$  fulfilling  $\mathcal{F}[u^*, w^*] = 0$ .

This can be seen by the fact that  $C_c$  in (3.6) is

$$C_c = \sup_{f \neq 0} \left\{ \frac{\mathcal{H}[f, f]}{\|f\|_1^{2/d} \|f\|_{m_c}^{m_c}}, f \in L^1(\mathbb{R}^d) \cap L^{m_c}(\mathbb{R}^d) \right\}$$

from Theorem 3.4. From [6, Proposition 3.3], for any  $M_1^* > 0$  there exists nonnegative, radially symmetric and non-increasing function  $u^* \in L^1(\mathbb{R}^d) \cap L^{m_c}(\mathbb{R}^d)$  with  $\|u^*\|_1 = M_1^*$  such that

$$\|u^*\|_{m_c}^{m_c} = C_c^{-1} \|u^*\|_1^{-2/d} \mathcal{H}[u^*, u^*]. \quad (5.13)$$

Define  $w^* = M_2^* / M_1^* u^*$ . Then  $w^* \in L^1(\mathbb{R}^d) \cap L^{m_c}(\mathbb{R}^d)$  with  $\|w^*\|_1 = M_2^*$  and

$$\mathcal{F}[u^*, w^*] = 0$$

by (5.12) and the definition of  $M_c$ . Then

$$c_d \mathcal{H}[u^*, w^*] = c_d M_2^* / M_1^* \mathcal{H}[u^*, u^*] = \frac{1}{m_c - 1} \left( 1 + \left( \frac{M_2^*}{M_1^*} \right)^{m_c} \right) \|u^*\|_{m_c}^{m_c}.$$

Given  $u_0 = M_1 / M_1^* u^*$  and  $w_0 = M_2 / M_2^* w^*$  with  $\|u_0\|_1 = M_1$  and  $\|w_0\|_1 = M_2$ , then

$$\begin{aligned} \mathcal{F}[u_0, w_0] &= \frac{1}{m_c - 1} \|u_0\|_{m_c}^{m_c} + \frac{1}{m_c - 1} \|w_0\|_{m_c}^{m_c} - c_d \mathcal{H}[u_0, w_0] \\ &= \frac{1}{m_c - 1} \left[ \left( \frac{M_1}{M_1^*} \right)^{m_c} + \left( \frac{M_2}{M_1^*} \right)^{m_c} - \frac{M_1 M_2}{M_1^* M_2^*} \left( 1 + \left( \frac{M_2^*}{M_1^*} \right)^{m_c} \right) \right] \|u^*\|_{m_c}^{m_c} \\ &< 0, \end{aligned}$$

since

$$M_1 M_2 / (M_1^{m_c} + M_2^{m_c}) > M_1^* M_2^* / (M_1^{*m_c} + M_2^{*m_c}) = M_c^{2/d} / 2.$$

If  $(u, w)$  is the corresponding free energy solution with initial data  $(u_0, w_0)$ , then

$$\mathcal{F}[u(t), w(t)] \leq \mathcal{F}[u_0, w_0] < 0, \quad t > 0$$

by the decreasing property of  $\mathcal{F}$ . From Lemma 5.1, it follows that blow up occurs.

To see the simultaneous blow-up phenomenon, from extensibility criterion in Theorem 4.1 we have

$$C \|w(t)\|_{m_2}^{m_2} + C \leq \|u(t)\|_{m_1}^{m_1} \leq C' \|w(t)\|_{m_2}^{m_2} + C' \quad \text{for } t \in (0, T_{\max})$$

with some  $C > 0$  and  $C' > 0$  if  $m$  is critical. Then all assertions have been proved.  $\square$

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