# Semigroup Representations: An Abstract Approach

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Trinity 1994

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## Abstract

**Chapter One** After the definitions and basic results required for the rest of the thesis, a notion of spectrum for semigroup representations is introduced and some relevant examples given.

**Chapter Two** Any semigroup representation by isometries on a Banach space may be dilated to a group representation on a larger Banach space. A new proof of this result is presented here, and a connection is shown to exist between the dilation and the trajectories of the dual representation. The problem of dilating various types of spaces, including partially ordered spaces, C*-algebras, and reflexive spaces, is discussed, and new dilation theorems are given for dual Banach spaces and von Neumann algebras.

**Chapter Three** In this chapter the spectrum of a representation is examined more closely with the aid of methods from Banach algebra theory. In the case where the representation is by isometries it is shown that the spectrum is non-empty, that it is compact if and only if the representation is norm-continuous, and that any isolated point in the unitary spectrum is an eigenvalue.

**Chapter Four** An analytic characterisation is given of the spectral conditions that imply a representation by isometries is invertible. For representations of $\mathbf{Z}_+^n$ this con-

dition is shown to be equivalent to polynomial convexity. Some topological conditions on the spectrum are also shown to imply invertibility.

**Chapter Five** The ideas of the previous chapters are applied to problems of asymptotic behaviour. Asymptotic stability is described in terms of the behaviour of the dual of a representation. Finally, the case when the unitary spectrum is countable is discussed in detail.

**To my parents,**

for never asking why

**and to Amy,**

for helping me answer this for myself.

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# Preface

Many results that originated in the study of the classical groups, $\mathbf{R}$ and $\mathbf{Z}$, were long ago generalised to the class of locally compact, abelian groups, and the resulting theory is in many ways clearer at this level of generality. The structure of semigroups is far less conducive to generalisation than that of groups however. There are results applicable to wide classes of semigroups, but the power of these is often little compared to even minor observations in specific cases. As a consequence, most research into semigroup representations has centred on the one-parameter semigroups $\mathbf{R}_+$ and $\mathbf{Z}_+$, with methods developed specially for those cases. The representations of $\mathbf{R}_+$, for example, each possess an *infinitesimal generator*, and the study of this generator can produce a great deal of information about the representation. Unfortunately however, this concept does not transfer even to $\mathbf{Z}_+$, let alone to more general semigroups.

This thesis attempts to develop methods which are transferable between different semigroups, but which are still of sufficient power to prove non-trivial results in the one-parameter cases.

The approach taken has been to link the theory of semigroups as closely as possible to that of groups, where generality has already been attained. Where representations are concerned, this has meant that most attention has had to be paid to representations by isometries. This is not so constricting as it may appear however, since problems in semigroup representation theory can sometimes be reduced to the isometric case. This is especially so for the study of asymptotics. Another aspect has been the development of a notion of spectrum for semigroup representations recently given in [10] which at once generalises the usual spectra of individual operators and of $C_0$-semigroups, and also links into the concept of the Arveson spectrum of a group representation.

The first half of chapter one contains most of the definitions and basic results needed later. In particular, the class of semigroups to be dealt with is defined. This includes the commonest examples in semigroup theory, $\mathbf{R}_+$ and $\mathbf{Z}_+$, as well as their multi-parameter equivalents, $\mathbf{R}_+^n$ and $\mathbf{Z}_+^n$. Representations of semigroups are then introduced along with a notion of spectrum. Theorems 1.4.3 and 1.5.2 demonstrate the existence of representations with arbitrary spectra; these examples are not only interesting in their own right, but they also show that the results presented later in the thesis are non-void.

Chapter two begins the investigation into the relation between semigroups and groups of isometries. In section 2.2 the trajectories of a representation are defined and an existence theorem is proved (2.2.2). This is used as a basis for proving a series of dilation theorems. The first, theorem 2.1.1, covers dilations of Banach spaces and is due to R. G. Douglas. Theorem 2.4.3 is a new result for dual Banach spaces, and this is specialised in theorem 2.5.1 to cover von Neumann algebras.

In chapter three the spectrum of a representation is examined more closely with the aid of methods from Banach algebra theory. Once again special attention is paid to representations by isometries, when it is shown that the spectrum is non-empty (3.1.1), that any isolated point in the unitary spectrum is an eigenvalue (3.4.5), and that it is compact if and only if the representation is norm-continuous (3.5.3). Some of the limitations of this approach are discussed.

The aim of the chapter four is to discover spectral conditions on a semigroup representation by isometries that will imply it is invertible. Several motivating examples are given and the problem is reduced to a question on ideals in $L^1(G)$ by theorem 4.2.2. However, since algebraic properties are not always practical, the majority of the chapter deals with sufficient topological conditions. Principal among these results are theorems 4.4.3 and 4.5.8.

In chapter five the asymptotic behaviour of representations is studied in light of the previous chapters. It includes an abstract characterisation of asymptotic stability in terms of the behaviour of the dual of the representation (theorem 5.2.1), and a detailed look into the asymptotic behaviour of semigroups with countable unitary spectra.

## Prerequisites

I have assumed that the reader has a general knowledge of functional analysis, so standard results are often used without comment. An acquaintance with abstract harmonic analysis is also desirable, but references are usually provided in these cases. (My standard text for this purpose is Rudin [35], although many of the results may also be found in Katznelson [20] or Hewitt and Ross [21, 22].) The theory of $C^*$ and von Neumann algebras is touched on in section 2.5, but little knowledge is assumed beyond that of elementary results and definitions. Chapter 3 relies fairly heavily on commutative Banach algebra theory, for which my standard reference is Stout [42].

In one sense however, this is a largely self-contained work. For although it deals exclusively with semigroups of operators, including the much studied $C_0$-semigroups, a reader who knows little about semigroup theory beyond the Hille-Yosida theorem would not be seriously incommoded.

Finally, a word on notation. Throughout this thesis $\mathbf{R}$, $\mathbf{Z}$, and $\mathbf{C}$ will have their usual meanings. In addition we define

$$\mathbf{R}_+ = \{x \in \mathbf{R} : x \geq 0\}; \qquad \mathbf{Z}_+ = \{n \in \mathbf{Z} : n \geq 0\}$$

$$\mathbf{C}_- = \{\lambda \in \mathbf{C} : \mathrm{Re}\, \lambda \leq 0\}; \qquad \mathbf{D} = \{\lambda \in \mathbf{C} : |\lambda| \leq 1\}.$$

# Acknowledgements

I am very grateful to my supervisor, Dr. C. J. K. Batty, who put up with me for three and a half years, and who always returned written work with comments far more quickly than I ever managed to give it to him in the first place. His mathematical advice was always excellent, although I frequently learned this by not heeding it. I am especially grateful for his generous behaviour in the area of attributing work.

The first three years of my research were funded by an SERC grant, while the last six months have been funded by my family, to whom I am deeply grateful both for their financial, practical, and moral support of me.

Of my friends I would like to thank Philippe Balland, who was often the only person willing to listen to me talk about my work, and Harry Quiney for showering me with (mostly) sound advice. Both Tchavdar Todorov and Amanda Michels did much to make my visits to Oxford during the last year easier and my thanks go also to them as they do to Kaye, the MCR steward at St. Catherine's, who has done so much to make it a friendly place.

Most of all though, I am grateful to Amy Butler, who has been a continual source of strength and encouragement throughout the last two and a half years of my research.

# Chapter 1

# Semigroup Representations

## 1.1 A Class of Semigroups

A *subsemigroup* of the group $G$ with addition as the group operation is any subset $S$ satisfying $S+S \subseteq S$. This thesis is concerned with the representations of a particular class of subsemigroups and with their relation to representations of the groups that contain them. The class consists of all subsemigroups $S$ of locally compact, abelian groups $G$ that satisfy four conditions, the first three of which may be stated at once.

(S1) $S$ is measurable with respect to the Haar measure of $G$;

(S2) the interior of $S$ with respect to the topology of $G$ is non-empty;

(S3) $G = S - S$.

The third is not too great a restriction since when it is not satisfied $S$ may always be considered as a subsemigroup of the LCA group $S - S$ instead. Equip $S$ with the topology and measure induced from $G$.

The dual group $\Gamma$ of $G$ is the set of all continuous characters on $G$. Similarly the dual of $S$, $S^*$, is defined to be the set of all non-zero, bounded, continuous complex homomorphisms of $S$. (These are called the *bounded semicharacters* of $S$ in some literature, and an alternative notation for $S^*$ is $\hat{S}$.) The topology and other aspects of $S^*$ will be discussed later in the chapter.

**Definition 1.1.1** *The* **Fourier transform** *of* $f \in L^1(S)$, $\hat{f} : S^* \to \mathbf{C}$, *is given by*

$$\hat{f} : \chi \mapsto \int_S f(s)\chi(s)\, ds \qquad (\chi \in S^*).$$

It should be noted that when $S = G$ this in fact corresponds to the standard definition of the inverse rather than the usual Fourier transform. For this reason the one defined here is sometimes called the *Fourier-Laplace transform* and I have tried to use that name where ambiguity could arise. The final restriction to be placed on $S$ may now be stated as

(S4) For any $\chi, \gamma \in S^*$ there exists an $f \in L^1(S)$ such that

$$0 \neq \hat{f}(\chi) \neq \hat{f}(\gamma).$$

Suppose $S$ is a subsemigroup of $G$ satisfying (S1)–(S3). If $\chi, \gamma \in S^*$ are distinct, then there must exist a (relatively) open subset $W$ of $S$ on which $\chi$ differs from $\gamma$. We may assume that $\overline{W}$ is compact and, by continuity, that $|\chi(s) - \gamma(s)| > \varepsilon$ for all $s \in \overline{W}$ and some $\varepsilon > 0$. If

$$f(s) = \begin{cases} \frac{1}{\chi(s)-\gamma(s)} & s \in \overline{W} \\ 0 & \text{otherwise} \end{cases},$$

then $f \in L^1(S)$ and $\hat{f}(\chi) - \hat{f}(\gamma) = |\overline{W}|$, the Haar measure of $\overline{W}$. Property (S4) would then follow if $|\overline{W}|$ were non-zero, but this is not generally the case as example 1.1.4 below shows. In particular (S4) does not follow from (S1)–(S3). However, if $S$ has a dense interior then $W$ may be assumed to be open in $G$ and since any non-empty open set in $G$ has non-zero Haar measure, (S4) holds in this case.

While more exotic examples of semigroups exist, the first two here are so important and will be returned to so many times that they are given explicitly.

**Example 1.1.2** *If* $S = \mathbf{Z}_+^n$, *then* $S^* = \mathbf{D}^n$.

Let $G = \mathbf{Z}^n$ and $S = \mathbf{Z}_+^n$; these clearly satisfy conditions (S1)–(S3) and, since the interior of $S$ is just $S$, they also satisfy (S4). It is well known that the dual group of $G$ may be regarded as the compact set $(\delta\mathbf{D})^n$, and in a similar manner the dual of $S$ may be identified with $\mathbf{D}^n$. So, for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{D}^n$, the semicharacter $\chi_\lambda$ of $S$ is given by

$$\chi_\lambda(m_1, m_2, \ldots, m_n) = \lambda_1^{m_1}\lambda_2^{m_2}\ldots\lambda_n^{m_n},$$

where we consider $\lambda_i^{m_i} = 1$ when $\lambda_i = m_i = 0$. That this does define a semicharacter of $S$ whenever $\lambda \in \mathbf{D}^n$ is obvious. In the other direction if $\chi \in S^*$, then $\chi = \chi_\lambda$, where $\lambda_i = \chi(e_i)$ and $e_i \in \mathbf{Z}_+^n$ is zero except in the i-th position where it is one.

**Example 1.1.3** *If* $S = \mathbf{R}_+^n$, *then* $S^* = \mathbf{C}_-^n$.

For $G = \mathbf{R}^n$, the dual group $\Gamma$ is equal to $i\mathbf{R}^n$ under the identification

$$\chi_\lambda(x_1, \ldots, x_n) = e^{\lambda_1 x_1}\cdots e^{\lambda_n x_n}.$$

Clearly $S = \mathbf{R}_+^n$ satisfies (S1)–(S4), and $S^*$ may be identified with the set $\mathbf{C}_-^n$ in a similar manner.

**Example 1.1.4** *There exists a subsemigroup of* $\mathbf{R}$ *satisfying (S1)–(S3) but not (S4).*

Let $G = \mathbf{R}$ and $S = \{0\} \cup [1, \infty)$; that $S$ satisfies (S1)–(S3) is trivial. It fails to fulfil condition (S4) however because if $\chi$ is the element of the dual given by

$$\chi(s) = \begin{cases} 1 & \text{when } s = 0 \\ 0 & \text{otherwise} \end{cases},$$

then $\hat{f}(\chi) = 0$ for any $f \in L^1(S)$.

From now on, unless explicitly stated to the contrary, $S$ will always stand for a subsemigroup of a locally compact, abelian group $G$ that satisfies (S1)–(S4). Note that it is neither required that $S$ contain 0, nor that it be closed. The interior of $S$ with respect to $G$ will be denoted $S^0$. Where any additional assumptions on $S$ are made they will be included in the relevant statements.

Before moving on to deal with the topology of $S^*$, a few elementary but useful properties of $S$ will be given.

**Lemma 1.1.5** *If $C$ is a compact subset of $G$, then there exists an $s \in S$ such that $s + C \subseteq S$.*

**Proof:** By condition (S2) there is an $s_0 \in S$ and a neighbourhood $W$ of 0 in $G$ such that $s_0 + W \subseteq S$. Using the compactness of $C$ we deduce the existence of a finite set of elements $t_1, \ldots, t_n$ in $C$ such that $\cup_{i=1}^n (t_i + W)$ contains $C$, and (S3) allows us to choose $u_1, \ldots, u_n, u'_1, \ldots, u'_n$ in $S$ such that $t_i = u'_i - u_i$ for each $i$. I claim that $s = s_0 + u_1 + \ldots + u_n$ is the required member of $S$.

If $w \in W$, then $t_i + w + s = (s_0 + w) + (u_1 + \ldots + u_{i-1} + u_{i+1} + \ldots + u_n) + u'_i$, which is certainly in $S$; hence $s + \cup_{i=1}^n (t_i + W) \subseteq S$, and since $s + C$ is contained in this set, the result follows.

**Lemma 1.1.6** *If $f \in L^1(G)$ and $\varepsilon > 0$, then there exists an $s \in S$ such that*

$$\int_{G\setminus(S-s)} |f(t)|\, dt < \varepsilon.$$

**Proof:** Let $C$ be some compact subset of $G$ such that $\int_{G\setminus C} |f| < \varepsilon$. By lemma 1.1.5 there is an $s$ for which $s + C \subseteq S$ and hence the result.

**Lemma 1.1.7** *For $\chi \in S^*$, $f, g \in L^1(S)$, and $s \in S$ the following two relations hold:*

$$\hat{f}(\chi)\hat{g}(\chi) = \widehat{f * g}(\chi),$$

*and*

$$\chi(s)\hat{f}(\chi) = \int_S f(t)\chi(s+t)\, dt = \hat{f}_s(\chi),$$

*where $f * g$ is the convolution of $f$ and $g$, and*

$$f_s(t) = \begin{cases} f(t-s) & \text{if } t-s \in S \\ 0 & \text{otherwise} \end{cases} .$$

**Proof:** Straightforward.

The dual of $S$ may be associated with the set of characters of $L^1(S)$ in exactly the same way that the dual of $G$ is with those of $L^1(G)$. The proof is almost identical to that for the group version [35, theorem 1.2.2].

**Lemma 1.1.8** *For each $\chi \in S^*$ the map $\phi_\chi : f \mapsto \hat{f}(\chi)$ is a non-zero, complex homomorphism of $L^1(S)$, and distinct elements of $S^*$ induce distinct homomorphisms of $L^1(S)$. Conversely, any non-zero, complex homomorphism of $L^1(S)$ may be obtained in this way.*

As a consequence of this, $S^*$ can be viewed as a locally compact Hausdorff space. The following analysis of the topology of $S^*$ is a natural analogue of that of $\Gamma$ [35, section 1.2.6].

**Lemma 1.1.9**

1. *$(s,\chi) \mapsto \chi(s)$ is a continuous function on $S \times S^*$;*
2. *For each compact $C \subseteq S$, $\delta > 0$, and $\chi \in S^*$ the set*

$$V(\chi, C, \delta) = \{\gamma \in S^* : |\gamma(s) - \chi(s)| < \delta \text{ for all } s \in C\}$$

   *is open in $S^*$;*

3. *The set of all such $V(\chi, C, \delta)$ is a base for the topology of $S^*$.*

A corollary is the observation that in examples 1.1.2 and 1.1.3 the topologies of $S^*$ are precisely the same as the topologies of $\mathbf{D}^n$ and $\mathbf{C}^n_-$ respectively. In future we will always regard the dual of $\mathbf{Z}^n_+$ to be identical to $\mathbf{D}^n$, and similarly for $\mathbf{R}^n_+$ and $\mathbf{C}^n_-$.

**Definition 1.1.10** *The* **unitary part** *of $S^*$ is defined to be*

$$S_u^* = \{\chi \in S^* : |\chi(s)| = 1 \text{ for all } s \in S\}.$$

The map $\Gamma \to S_u^*$ obtained by restriction is a bijection by (S3) and a homeomorphism by lemma 1.1.9. The two sets will often be identified in this thesis.

## 1.2 Representations

**Definitions 1.2.1** *For the purposes of this thesis, a* **representation** *of $S$ will be a strongly continuous homomorphism from $S$ into $\mathcal{B}(X)$ where $X$ is a Banach Space. In other words, a representation of $S$ will be a map $T : S \to \mathcal{B}(X)$ such that*

$$T(s)T(t) = T(s+t) \qquad (s, t \in S),$$

*and for each $x$ in $X$ the map*

$$s \mapsto T(s)x$$

*is norm-continuous. When 0 is in $S$ we will assume in addition that $T(0) = I$. A representation is* **bounded** *by $M$ if $\|T(s)\| \leq M$ for all $s \in S$.*

The *one-parameter* representations are both well known and much studied. When $S = \mathbf{Z}_+$ a representation of $S$ is given completely by its value at $s = 1$, $B$ say; for in this case, $T(n) = B^n$ for all $n \in \mathbf{Z}_+$. A representation of $S = \mathbf{R}_+$ is called a $C_0$-semigroup [15, 28, 23]. Each $C_0$-semigroup has a unique *generator*, and the study of these semigroups is closely related to the study of generators. For representations of general semigroups however, no such simplification is known. A representation of $\mathbf{R}_+^n$ or $\mathbf{Z}_+^n$ is said to be *multi-parameter*. In particular if $T$ is a representation of $\mathbf{R}_+^n$, then $T$ may be viewed as the product of $n$ commuting $C_0$-semigroups $T_i(s) = T(se_i)$, where $e_i$ is the n-tuple which is zero except in the i-th position where it is one.

A notion of Fourier transform exists for bounded representations $T$. For $f \in L^1(S)$, the bounded operator $\hat{f}(T)$ is defined by the equation

$$\hat{f}(T) : x \mapsto \int_S f(s)T(s)x\, ds \qquad (x \in X).$$

If $M$ is a bound for $T$, it is easy to show that $\|\hat{f}(T)\| \leq M\|f\|_1$ for all $f \in L^1(S)$.

**Definition 1.2.2** *A* **dual representation** *is a homomorphism $T$ from $S$ to $\mathcal{B}(X)$, where $X$ is a dual Banach Space, such that each $T(s)$ is continuous in the weak*-topology of $X$. For each $f \in X$, and $x \in X_*$ (the pre-dual of $X$), we require also that the map*

$$s \mapsto (T(s)f)x$$

*be continuous. As for strongly continuous representations, $T(0)$ is assumed to be $I$ when $0 \in S$, and $T$ is said to be bounded by $M$ if $\|T(s)\| \leq M$ for all $s \in S$.*

Assuming weak*-continuity of $T(s)$ is of course equivalent to assuming that it is the dual of some operator $T_*(s)$ acting on $X_*$. The map $s \mapsto T_*(s)$ will also be a homomorphism and is weakly continuous. It is a standard result that a semigroup of operators is strongly continuous if and only it is weakly continuous [48, p. 233], so $T_*$ satisfies the conditions of being a representation. A dual representation is thus precisely the dual of a representation. The reason it has its own definition is that occasionally it is the dual which is of most interest; this is the case when dealing with von Neumann algebras for example. In sections 1.4 and 1.5, examples of both representations and dual representations will be given.

## 1.3 The Spectrum

The following notion of spectrum was introduced in [10]. $T$ is assumed to be a bounded representation of $S$ on a Banach space $X$.

**Definitions 1.3.1** *The* **spectrum** *of $T$ with respect to $S$ is defined to be*

$$Sp(T,S) = \{\chi \in S^* : |\hat{f}(\chi)| \leq \|\hat{f}(T)\| \text{ for all } f \in L^1(S)\}.$$

*A semicharacter $\chi \in S^*$ is said to be an* **eigenvalue** *for $T$ if there exists a non-zero $x \in X$ such that $T(s)x = \chi(s)x$ for all $s \in S$. The set of all eigenvalues is called the*

**point spectrum** *and is denoted* $P\sigma(T, S)$. *If there exists a generalised sequence of elements* $x_\alpha$ *of norm one in* $X$ *such that*

$$\|T(s)x_\alpha - \chi(s)x_\alpha\| \longrightarrow 0$$

*uniformly for* $s$ *in compact subsets of* $S$ *as* $\alpha \longrightarrow \infty$, *then* $\chi$ *is said to be an approximate eigenvalue for* $T$. *The set of all such is called the* **approximate point spectrum** *and is denoted* $A\sigma(T, S)$.

*The* **unitary part** *of the spectrum,* $Sp_u(T, S)$, *is defined to be* $Sp(T, S) \cap S_u^*$, *and a similar definition holds for* $P\sigma_u(T, S)$ *and* $A\sigma_u(T, S)$.

Several facts about the spectrum are proved in [10]; the most important (for the purposes of this thesis) are restated here. Both the point and the approximate point spectrum of $T$ are contained in $Sp(T, S)$; moreover $Sp_u(T, S) = A\sigma_u(T)$. When $S = \mathbf{Z}_+$, $Sp_u(T, \mathbf{Z}_+) = \sigma(T(1)) \cap \delta\mathbf{D}$, and when $S = \mathbf{R}_+$, $Sp_u(T, \mathbf{R}_+) = \sigma(A) \cap i\mathbf{R}$ where $A$ is the generator of $T$. If $S = G$, then $T$ is a group representation and $Sp(T, G)$ is identical to the Arveson spectrum of $T$ [31, proposition 8.1.9]. The following lemma is also included in [10] with an outline of the method of proof, but it is so central to the work in this thesis that the proof is included in full here.

**Lemma 1.3.2** *If* $T$ *is a bounded representation of* $S$ *and if* $\mathcal{A}_T$ *is the closure in* $\mathcal{B}(X)$ *of the algebra* $\{\hat{f}(T) : f \in L^1(S)\}$, *then* $Sp(T, S)$ *may be identified with be the set of non-zero characters of* $\mathcal{A}_T$.

**Proof:** When $\chi \in Sp(T, S)$, the homomorphism

$$\phi_\chi : \hat{f}(T) \mapsto \hat{f}(\chi)$$

is continuous and may be extended to a character on $\mathcal{A}_T$. By assumption (S4) this is non-zero, and distinct points in the spectrum of $T$ induce distinct characters of $\mathcal{A}_T$.

On the other hand if $\phi$ is a character of $\mathcal{A}_T$, then

$$f \mapsto \phi(\hat{f}(T)) \qquad (f \in L^1(S))$$

is a character of $L^1(S)$. From lemma 1.1.8 it follows that $\phi(\hat{f}(T)) = \hat{f}(\chi)$ for some $\chi \in S^*$, which is to say $\phi = \phi_\chi$ for some $\chi \in Sp(T, S)$.

We now turn our attention to the spectrum of a dual representation. The definition given above will not do in this case because it assumes $T$ to be strongly continuous. The pre-dual of $T$ is strongly continuous however, and consequently $Sp(T_*, S)$ does exist. I therefore propose the following definition.

**Definition 1.3.3** *The spectrum of a bounded, dual representation is defined to be that of the pre-dual. It will also be denoted by $Sp(T, S)$ and context will decide on the meaning.*

This definition is not so arbitrary as it may at first appear as lemma 1.3.5 will make clear.

**Lemma 1.3.4** *Let $T$ be a bounded dual representation on $X$ and denote by $N$ the space of all $x \in X$ such that $s \mapsto T(s)x$ is norm continuous. For any $f \in L^1(S)$ and any $y \in X$, $y \circ \hat{f}(T_*) \in N$. If $\phi \in X_*$, $s_0 \in S_0$, and $\varepsilon > 0$, then there exists $x \in N$ with $\|x\| \leq 1$ such that $|x(T_*(s_0)\phi) - \|T_*(s_0)\phi\|\,| < \varepsilon$.*

Note that $N$ will be a closed, $T$-invariant subspace of $X$ and that $T|_N$ will be a representation in its own right. $T|_N$ is often called the *sun dual* of $T_*$.

**Proof:** For $y \in X$, $f \in L^1(S)$, and $\phi \in X_*$,

$$\begin{aligned} |T(s)y(\hat{f}(T_*)\phi) - T(t)y(\hat{f}(T_*)\phi)| &= \left| \int_S T(s)y(f(v)T_*(v)\phi\, dv) \right. \\ &\qquad \left. - \int_S T(t)y(f(v)T_*(v)\phi\, dv) \right| \\ &= \left| y\left( \int_S (f_s(v) - f_t(v))T_*(v)\phi\, dv \right) \right| \\ &\leq \|y\|\, \|f_s - f_t\|_1\, \|\phi\| \sup \|T(v)\|. \end{aligned}$$

But $\|f_s - f_t\|_1 \to 0$ as $s \to t$ so $\hat{f}(T_*)^* y \in N$. The final part of the lemma is easily deduced from the above and from strong continuity of $T_*$.

**Lemma 1.3.5** *Let $T$ be a bounded dual representation of $S$ on $X$. For any $\chi \in S_u^*$ the following are equivalent.*

1. $|\hat{f}(\chi)| \leq \|\hat{f}(T|_N)\|$ *for all* $f \in L^1(S)$*;*

2. $\chi \in A\sigma_u(T, S)$*;*

3. $\chi \in Sp_u(T_*, S)$.

**Proof:** The fact that 1 implies 2 follows from the work in [10] mentioned above. Suppose 2 is true and let $f_\alpha \in X$ be a generalised sequence of norm one elements such that $\|T(s)f_\alpha - \chi(s)f_\alpha\|$ tends to zero uniformly on compact subsets of $S$. Let $\eta > 0$ and choose, assuming the axiom of choice if necessary, an $x_\alpha$ for each $\alpha$ with the properties $1 \leq \|x_\alpha\| \leq (1+\eta)$ and $f_\alpha(x_\alpha) > 1/(1+\eta)$.

If $h$ is continuous and has compact support $C$, then for any $x \in X_*$.

$$\begin{aligned} |f_\alpha(\hat{h}(T_*)x_\alpha - \hat{h}(\chi))x_\alpha| &= \left|\int_C f_\alpha(h(s)T_*(s)x_\alpha)\,ds \right. \\ &\quad \left. - \int_C f_\alpha(h(s)\chi(s)x_\alpha)\,ds\right| \\ &\leq \int_C |h(s)f_\alpha(T_*(s)x_\alpha - \chi(s)x_\alpha)|\,ds \\ &\leq (1+\eta)\int_C |h(s)|\,\|T(s)f_\alpha - \chi(s)f_\alpha\|\,ds \\ &\rightarrow 0 \end{aligned}$$

as $\alpha \rightarrow \infty$. It follows that $|\hat{h}(\chi)| \leq (1+\eta)^2\|\hat{h}(T_*)\|$ and hence $\chi \in Sp(T_*, S)$.

It remains to prove that 3 implies 1. Given $s \in S^0$ there exists a sequence $g_\alpha \in L^1(S)$ such that $\|g_\alpha\|_1 = 1$ and $\|g_\alpha * h - h_s\|_1 \rightarrow 0$ for any $h \in L^1(S)$. So, for some $\alpha$,

$$\begin{aligned} |\hat{h}(\chi)| &= |\chi(s)\hat{h}(\chi)| \\ &\leq |\hat{g}_\alpha(\chi)\hat{h}(\chi)| + \varepsilon \\ &\leq \|\hat{g}_\alpha(T_*)\hat{h}(T_*)\| + \varepsilon. \end{aligned}$$

(We are here using the fact that $\chi \in S_u^*$.) Let $f \in X$ and $x \in X_*$ be of norm one such that

$$\begin{aligned} \|\hat{g}_\alpha(T_*)\hat{h}(T_*)\| &< |f(\hat{g}_\alpha(T_*)\hat{h}(T_*)x)| + \varepsilon \\ &= |(\hat{g}_\alpha(T_*)^* f)(\hat{h}(T_*)x)| + \varepsilon \\ &= |\hat{h}(T|_N)(\hat{g}_\alpha(T_*)^* f)x)| + \varepsilon \\ &\leq M\|\hat{h}(T|_N)\| + \varepsilon \end{aligned}$$

where $M$ is some bound for $T$. The step from $\hat{h}(T_*)^*$ to $\hat{h}(T|_N)$ is justified by lemma 1.3.4. Combining these two inequalities we may deduce that $|\hat{h}(\chi)| \leq M\|\hat{h}(T|_N)\|$ for any $h \in L^1(S)$ and $\chi \in Sp_u(T_*, S)$. Taking powers of $h$ we have

$$|\hat{h}(\chi)|^n = |\hat{h^n}(\chi)| \leq M\|\hat{h^n}(T|_N)\| \leq M\|\hat{h}(T|_N)\|^n.$$

Statement 1 of the lemma is now proved by taking n-th roots.

**Corollary 1.3.6** *For any bounded representation $T$*

$$P\sigma_u(T^*, S) \subseteq Sp_u(T, S).$$

**Lemma 1.3.7** *If $U$ is a representation of $G$ by isometries, then for any subsemigroup $S$ satisfying (S1)-(S4)*

$$Sp_u(U, S) = Sp(U, G).$$

This lemma uses the association between $\Gamma$ and $S_u^*$ mentioned above of course.

**Proof:** $Sp(U, G) \subseteq Sp_u(U, S)$ is trivial. Suppose $\chi \in Sp_u(U, S)$ and $f \in L^1(G)$; by lemma 1.1.6, given $\varepsilon > 0$ there exists an $s \in S$ such that if $g \in L^1(S)$ is defined by $g(t) = f(t - s)$ for $t \in S$, then $\|g - f_s\|_1 < \varepsilon$. It follows that

$$|\hat{f}(\chi)| < |\hat{g}(\chi)| + \varepsilon \leq \|\hat{g}(U)\| + \varepsilon < \|\hat{f}(U)\| + 2\varepsilon,$$

so we are done.

## 1.4 Examples of Representations

In this section we investigate whether examples exist of representations by isometries with arbitrary (closed) unitary spectrum $E \subseteq \Gamma$. This will involve the theory of closed ideals in $L^1(G)$ for which a standard text is Rudin [35]. The main elements needed are as follows.

A closed ideal in $L^1(G)$ is necessarily translation-invariant; specifically this means that if $f$ is in the ideal, then so is $f_t$ for all $t \in G$, where

$$f_t : G \longrightarrow \mathbf{C} \qquad f_t(s) = f(s-t).$$

The zero set of a closed ideal $J$, $Z(J)$, is defined to be the set of all $\chi$ in $\Gamma$ such that $\hat{f}(\chi) = 0$ for all $f \in J$. The zero set is clearly closed. Since $\Gamma$ is identified with the maximal modular ideal space of $L^1(G)$, the zero set may be seen as the set of all maximal modular ideals which contain $J$. The theorem of Wiener tells us that if $J$ is a proper closed ideal, then $Z(J)$ is non-empty.

For $E$ a closed subset of $\Gamma$, define

$$I_E = \{f \in L^1(G) : \hat{f}|_E = 0\},$$

and let $J_E$ be the closure in $L^1(G)$ of

$$\{f \in L^1(G) : \hat{f}|_V = 0 \text{ for some open subset } V \text{ containing } E\}.$$

Standard theory tells us that $Z(J_E) = Z(I_E) = E$. Moreover, $J_E$ is the smallest closed ideal with zero set $E$, while $I_E$ is the largest. We now know enough to construct our examples.

Let $J$ be a closed ideal in $L^1(G)$ and denote $L^1(G)/J$ by $Y_J$. For each $t \in G$ define the linear operator $U_J(t) : Y_J \to Y_J$ for $t$ in $G$ by

$$U_J(t) : f + J \mapsto f_t + J.$$

Since $J$ is translation invariant, each $U_J(t)$ is a well-defined isometry, and it is easy to verify that $U_J$ is a bounded group representation.

**Theorem 1.4.1** *If $J$ is a closed ideal of $L^1(G)$, and if $Y_J$ and $U_J$ are as above, then $Sp(U_J, G) = Z(J)$.*

Simply by setting $J = J_E$, we have the following.

**Corollary 1.4.2** *If $E$ is a closed subset of $\Gamma$, then there exists a bounded group representation with spectrum $E$.*

**Proof of Theorem 1.4.1:** Suppose first that $\chi$ is not in $Z(J)$. By definition there exists an $f \in J$ such that $\hat{f}(\chi) \neq 0$.

If $g$ and $h$ are in $L^1(G)$, then

$$\begin{aligned} \hat{g}(U_J)(h+J) &= \int_G g(s)U_J(s)(h+J)\,ds \\ &= \int_G g(s)h(\cdot - s)\,ds + J \\ &= g * h + J. \end{aligned}$$

Consequently, if $g \in J$, then $g * h + J = J$ for all $h$; in particular $\hat{g}(U_J) = 0$. Applying this to the $f$ chosen above, we have $\|\hat{f}(U_J)\| = 0$, but since $|\hat{f}(\chi)| > 0$, the definition of spectrum tells us that $\chi \notin Sp(U_J, G)$. Hence $Sp(U_J, G) \subseteq Z(J)$.

Now assume on the contrary that $\chi$ is in $Z(J)$. Considered as an element of $L^\infty(G)$, $\chi$ acts on $L^1(G)$ by $f \mapsto \hat{f}(\chi)$, so by the definition of $Z(J)$, $\chi \in J^\perp$. For any $h(\cdot) \in J^\perp$, $(U_J^*(t)h)(\cdot) = h(\cdot + t)$; in other words $U_J^*$ works by left translation. In particular therefore, $(U_J^*(t)\chi)(\cdot) = \chi(t)\chi(\cdot)$ and $\chi$ may be seen simultaneously as an eigenvector and an eigenvalue for $U_J^*$. Finally, corollary 1.3.6 implies that $\chi \in Sp(U_J, G)$.

We now turn back to the question of whether, given an arbitrary closed subset of $\Gamma$, we can find a representation of $S$ by isometries with that set as its unitary spectrum. The trivial answer now that we have corollary 1.4.2 is yes, since we may simply restrict any group representation with spectrum $E$ down to $S$ and lemma 1.3.7 then tells us that this has the right unitary spectrum. But can we find a semigroup representation that does not derive from a group in this way? The full answer to this turns out to be quite complex and is covered more fully in chapter 4.

For a closed subset $E$ with the property that $L^1(S) + J_E$ is not dense in $L^1(G)$ however, we can directly construct an example of an isometric semigroup representation with unitary spectrum $E$ which is not the restriction to $S$ of some group representation.

For $J$ a closed ideal of $L^1(G)$ define

$$X_J = \overline{L^1(S) + J}/J.$$

$X_J$ is a closed subspace of $Y_J$ as defined above and moreover it is also $U_J(s)$-invariant for all $s$ in $S$. Let $T_J(s)$ denote $U_J(s)|_{X_J}$ for $s$ in $S$.

**Theorem 1.4.3** *$T_J$ is a representation of $S$ by isometries on $X_J$ with unitary spectrum $Z(J)$.*

**Proof:** Let $\|\cdot\|_J$ denote the norm of $Y_J$. If $f \in L^1(S)$, then

$$\hat{f}(T_J) = \hat{f}(U_J)|_{X_J}$$

by direct verification; so it is a triviality that $\|\hat{f}(T_J)\| \leq \|\hat{f}(U_J)\|$ for all $f$ in $L^1(S)$. Thus

$$Sp_u(T_J, S) \subseteq Sp_u(U_J, S) = Sp(U_J, G) = Z(J)$$

by lemma 1.3.7 and theorem 1.4.1. It remains to prove the other inclusion.

For any $\chi \in Z(J)$ define $\phi_\chi : X_J \to X_J$ to be the completion of

$$f + J \mapsto \hat{f}(\chi) \qquad (f \in L^1(S)).$$

$\phi_\chi$ is well-defined and continuous since $\hat{f}(\chi) = 0$ for all $f \in J$. Moreover if $J \neq L^1(G)$, then $\phi_\chi$ is non-zero by assumption (S4). For any $s \in S$ and $f \in L^1(S)$

$$\begin{aligned} T_J(s)^* \phi_\chi(f + J) &= \phi_\chi(f_s + J) = \hat{f}_s(\chi) \\ &= \chi(s)\hat{f}(\chi) = \chi(s)\phi_\chi(f + J); \end{aligned}$$

so $\phi_\chi$ is an eigenvector of $T_J^*$. Corollary 1.3.6 then completes the proof.

When $L^1(S) + J_E$ is dense in $L^1(G)$, $X_{J_E}$ is by definition equal to $Y_{J_E}$, and hence $T_{J_E}$ is the restriction of the group representation $U_{J_E}$ to $S$. Moreover, since $J_E$ is the smallest closed ideal with zero set $E$, any other $J$ with zero set $E$ will also have the property that $T_J$ is $U_J|_S$. So in this case we have not constructed an example of a representation in the sense discussed above. That this does not happen when $L^1(S) + J_E$ is not dense in $L^1(G)$ is straightforward to prove. See theorem 4.2.2 for the details.

A more obvious approach to constructing semigroup representations out of $L^1$ spaces would have been to consider quotients of $L^1(S)$. For example, we might have considered factoring out the closed ideal $L^1(S) \cap J_E$. The problem with this is that the spectrum of such a representation would not necessarily be $E$.

**Example 1.4.4** *If $S = \mathbf{Z}_+$ and $E = \{\lambda \in \mathbf{C} : |\lambda| = 1$ and $Re\,\lambda \geq 0\}$, then $J \cap l^1(\mathbf{Z}_+) = \{0\}$ for any $J$ with zero set $E$. In particular the shift semigroup on $l^1(\mathbf{Z}_+)/(J \cap l^1(\mathbf{Z}_+))$ has unitary spectrum $\delta\mathbf{D}$.*

This is because if a non-zero function is holomorphic in the unit disc and continuous on the boundary, then the intersection of its zero set with $\delta\mathbf{D}$ is null with respect to one-dimensional Lebesgue measure [42, theorem 20.2 and corollary 20.6]. In [19], Esterle, Strouse, and Zouakia show that the shift semigroup on $L^1(S)/(J_E \cap L^1(S))$ consists of isometries when $S = \mathbf{R}_+$ and $E$ is (closed and) countable. They further prove that the quotient space obtained is isometrically isomorphic to $L^1(G)/J_E$. This method is covered in more detail in lemma 5.4.5. Another approach to constructing semigroups with arbitrary closed spectra is to use $C_0$ spaces; this is simpler but the resultant representations are not so useful (see example 3.4.7).

## 1.5 Examples of Dual Representations

In this section we will ask a similar question of dual representations: given a closed subset $E$ of $\Gamma$ can we find a dual representation by isometries with unitary spectrum $E$ that is not derived from a group? (If $T$ is a bounded representation with unitary spectrum $E$, then $T^*$ is a dual representation with $Sp_u(T^*, S) = E$; so if the isometric requirement is removed then the question has already been answered.)

Let $X = (L^1(S) + J_E)^\perp \subseteq L^\infty(G)$. This is non-trivial if and only if $L^1(S) + J_E$ is not dense in $L^1(G)$. Define $T(s) : X \to X$ for $s$ in $S$ by

$$T(s) : \phi(\cdot) \mapsto \phi(\cdot + s).$$

In other words $T(s)$ is a left-shift operator on $X$, which is well-defined since $L^1(S)+J_E$ is right-shift invariant. The map $s \mapsto T(s)$ is easily seen to be a homomorphism. The pre-dual of $X$, $X_*$, is given by

$$X_* = L^1(G)/\overline{(L^1(S) + J_E)}$$

and the pre-dual of $T(s)$ is given by

$$T_*(s) : f(\cdot) + \overline{(L^1(S) + J_E)} \mapsto f(\cdot - s) + \overline{(L^1(S) + J_E)}.$$

Since the right-shift group on $L^1(G)$ is strongly continuous, it follows that $T_*$ is also strongly continuous and hence $T$ is a dual representation of $S$ on $X$. Furthermore, each $T(s)$ is an isometry.

By looking at the definition of $X$, and considering $L^1(G)^*$ to be $L^\infty(G)$ as usual, we may see that an element $\phi$ of $X$ is an element of ${J_E}^\perp$ that is equal to zero almost everywhere on $S$. If $X$ contains a non-zero element $\phi(\cdot)$, which it will if $L^1(S) + J_E$ is not dense in $L^1(G)$, then for some $s$ in $S$, $\phi(\cdot - s)$ is not contained in $X$. This is because if $W \subseteq G$ is some set of positive measure on which $\phi$ is non-zero, then by regularity of the Haar measure, we may find a compact subset $C$ of $W$, also with non-zero Haar measure. By lemma 1.1.5, there exists an $s$ in $S$ such that $s + C$

is contained in $S$. Hence $\phi(\cdot - s)$ is nowhere zero on $s + C$ and in particular it is not in $X$. This property of $X$ and $T$ (whereby for every $x$ in $X$ there exists an $s$ with $x \notin T(s)[X]$) is closely related to the fact that $T_*$ is an asymptotically stable representation. (See theorem 5.2.1.)

So far we have proved that $T$ is a dual representation by isometries, and that it is not the restriction to $S$ of some group. We now turn our attention to the spectrum of $T$. With the methods and results used so far, we are unable even to prove that the spectrum is non-empty, but for the present purposes it will be enough to show that it is contained in $E$.

Suppose $\chi$ is not in $E$ and choose $f \in J_E$ such that $\hat{f}(\chi) = 1$. Then as in the proof of theorem 1.4.3 it may be shown that

$$\int_G h(t)T_*(t)(g + \overline{(L^1(S) + J_E)})\, dt = h * g + \overline{(L^1(S) + J_E)}$$

for all $g, h \in L^1(G)$. So in particular, $f * g \in J_E$ for all $g$, and $\hat{f}(T_*)$ must be zero. We thus have $\hat{f}(\chi) > \|\hat{f}(T_*)\|$ and hence $\chi$ is not in the spectrum of $T_*$. In the lemma that follows *non-invertible* simply means the semigroup representation is not the restriction to $S$ of some group representation.

**Lemma 1.5.1** *If $E$ is a closed subset of $\Gamma$ for which there exists a non-invertible, dual representation of $S$ by isometries with unitary spectrum contained in $E$, then there exists a non-invertible, dual representation of $S$ by isometries with unitary spectrum exactly $E$.*

**Proof:** Given a non-invertible, dual representation $T$ on $X$, we define the Banach space $\tilde{X}$ to be

$$\tilde{X} = X \times J_E{}^{\perp}$$

–the norm being given by $\|(x,\phi)\|_{\tilde{X}} = \max(\|x\|, \|\phi\|_\infty)$. $\tilde{X}$ has a pre-dual $\tilde{X}_*$ given by $\tilde{X}_* = X_* \times Y_{J_E}$, where $Y_{J_E}$ is as defined above, and the norm of $\tilde{X}_*$ is given by the $l^1$-sum. We further define $\tilde{T}$ on $\tilde{X}$ in the obvious way:

$$\tilde{T}(s) : (x, \phi) \mapsto (T(s)x, U_{J_E}{}^*\phi).$$

$\tilde{T}$ is a dual representation of $S$ by isometries, and since

$$\|\hat{f}(\tilde{T})\| = \max(\|\hat{f}(T)\|, \|\hat{f}(U_{J_E})\|),$$

it now follows easily that $Sp_u(\tilde{T}, S) = E$. The non-invertibility of $\tilde{T}$ follows from that of $T$.

Putting all this together we have

**Theorem 1.5.2** *If $E$ is a closed subset of $\Gamma$ such that $L^1(S) + J_E$ is not dense in $L^1(G)$, then there exists a non-invertible dual representation by isometries with unitary spectrum $E$.*

Both this theorem and corollary 1.4.2 are optimal: no non-invertible, isometric representations or dual representations with unitary spectrum $E$ exist when $L^1(S) + J_E$ is dense in $L^1(G)$. This will be proved in theorems 4.2.2 and 4.6.1. Another way to construct non-invertible, dual representations will be given in example 3.4.9.

# Notes

**Section 1.1** The conditions placed on $S$ are the minimum required for most of the results in this thesis to hold true. The examples $S = \mathbf{Z}_+^n$ and $S = \mathbf{R}_+^n$ satisfy the stronger condition that $S$ is the closure in $G$ of its interior and it will occasionally be necessary to assume this to be true. Subsemigroups satisfying this extra condition were studied by Arens and Singer [5] (in relation to the algebra $L^1(S)$). A feature of such semigroups is that $S_u^*$ is always a proper subset of $S^*$ (this is due to Rieffel [34]). The class of subsemigroups satisfying (S1)–(S4) is larger than this set however. For example, let $G = \mathbf{R}^2$ and consider $S$ to be the union of the sets $(1,1) + \mathbf{R}_+^2$ and $\{(s,s) : s \geq 0\}$; the dual semigroup $S^*$ is $\mathbf{C}_-^2$ and in particular $S$ satisfies (S1)-(S4), but $S$ does not have dense interior.

It is possible to prove that $S^*$ is strictly larger than $\Gamma$ in more general subsemigroups however.

**Theorem** *If $S$ is a closed subsemigroup of an LCA group $G$ such that $S$ is not contained in any proper closed subgroup of $G$, then 1 is a limit point of $\{\chi \in S^* : 0 < \chi < 1\}$.*

The proof of this may be found in [46] along with a multitude of other results concerning topological semigroups.

**Section 1.2** There are many books which cover $C_0$-semigroup theory including Hille and Phillips [23] and, more recently, Davies [15] and Nagel (Ed.) [28]. The basic theory, such as the definition of generator and the Hille-Yosida theorem, is included in the general texts on functional analysis Dunford and Schwartz [17] and Rudin [37].

**Section 1.3**

As so much of this thesis revolves around the study of $\mathcal{A}_T$, it is worth looking at it more closely. When $S = \mathbf{Z}_+$, $\mathcal{A}_T$ is simply the closed subalgebra of $\mathcal{B}(X)$ generated by $T(1)$. For $S = \mathbf{R}_+$, $\mathcal{A}_T$ contains all of the resolvent operators $R_\lambda = (\lambda - A)^{-1}$ (where $A$ generates $T$) because of the resolvent equation

$$R_\lambda x = \int_0^\infty e^{-\lambda t} T(t)x \, dt \qquad (x \in X)$$

which holds for all $\lambda$ in the left half plane. It is not surprising therefore that study of $\mathcal{A}_T$ will yield spectral-like results.

**Section 1.4** Theorem 1.4.3 was suggested by C. J. K. Batty, who first proved the second inclusion. His argument, based on the fact that $S$ satisfies the Følner condition, demonstrates that $\chi$ must be in the approximate point spectrum.

**Section 1.5** When $L^1(S) + J_E$ is dense in $L^1(G)$, the dual representation defined on $X = (L^1(S) + J_E)^\perp$ is trivial; in particular its spectrum is empty. If $L^1(S) + J_E$ is not dense, then theorem 3.1.1 implies that the unitary spectrum is non-empty, but in general it will not be equal to $E$. For example it will contain no relatively isolated points and, in the case $S = \mathbf{R}_+^n$, it will have no relatively open, compact subsets.

# Chapter 2

# Dilation Theory

## 2.1 The Standard Dilation Theorem

The following theorem is valid for any subsemigroup of $G$ satisfying (S1)–(S4) of chapter one.

**Theorem 2.1.1** *Let $T$ be a representation of $S$ by isometries on a Banach space $X$. There exists a Banach space $Y$, an isometric embedding $\pi : X \to Y$, and a representation $U : G \to \mathcal{B}(Y)$ by isometries with the following properties:*

1. $U(s)\pi x = \pi T(s)x \qquad (x \in X,\ s \in S)$;
2. $\{U(-s)\pi x : x \in X,\ s \in S\}$ *is dense in* $Y$;
3. $Sp(U,G) = Sp_u(T,S)$.

For a representation by isometries $T$, the $U$ so obtained will be referred to as *the group dilation* or *the standard group dilation* of $T$. $Y$ will be called the *dilated Banach space.*

This theorem is essentially due to R. G. Douglas [18]. His version does not include any continuity properties on $T$; he proves a dilation theorem for more general semigroups of isometries and then shows that various continuity properties are preserved.

He does not include property 3 of the theorem, but this is a consequence of the first two properties as will be shown below.

It is easy to see that the pair $(U, Y)$ is essentially unique, and, partly as a consequence of this, there are a number of ways to construct it. My own version, which is given below, is longer than many, but has the advantage that the dilated space is fairly easy to visualise. Before proceeding to this construction, we will prove that property 3 is deducible from 1 and 2.

**Proof of property 3:** If $f \in L^1(S)$ and $x \in X$, then by property 1,

$$\hat{f}(U)\pi x = \int_S f(s)U(s)\pi x\, ds = \int_S \pi(f(s)T(s)x)\, ds = \pi \hat{f}(T)x.$$

Hence

$$\begin{aligned} \|\hat{f}(U)\| &= \sup_{\|y\|\leq 1} \|\hat{f}(U)y\| \\ &= \sup_{s\in S} \sup_{\|x\|\leq 1} \|\hat{f}(U)U(-s)\pi x\| \qquad \text{by property 2} \\ &= \sup_{\|x\|\leq 1} \|\pi(\hat{f}(T)x)\| \\ &= \|\hat{f}(T)\|. \end{aligned}$$

It follows that $Sp(T, S) = Sp(U, S)$, and property 3 is then a consequence of lemma 1.3.7.

It is clear that if $T$ is not a representation by isometries, then the theorem could not be true as stated since this would contradict property 1, $\pi$ and $U$ being isometries. If the requirement that $\pi$ be an isometry is dropped, then problems arise with property 3. For example, let $A$ be some bounded operator on a non-trivial Hilbert space $H$ such that the $C_0$-semigroup generated by $A$ is contractive (so $A$ is dissipative). Consider $T$, the $C_0$-semigroup generated by $A - \lambda I$ for some $\lambda > 0$. If $U$ were a non-trivial dilation of $T$ by isometries satisfying properties 1 and 2 but with $\pi$ not isometric, then property 3 is not satisfied. The reason is simply that $Sp_u(T, \mathbf{R}_+) = \emptyset$ but $Sp(U, G)$ must be non-empty (see theorem 3.1.1). Since for the purposes of this thesis the spectral property is very important, we will digress no further here trying to cover non-isometric cases. (See the notes at the end of this chapter for further references however.)

## 2.2 Trajectories

**Definitions 2.2.1** *Let $V : S \to \mathcal{B}(Z)$ be a homomorphism where $Z$ is some Banach space; if $0 \in S$, then we require $V(0) = I$. We will say that $z : G \to Z$ is a* **trajectory** *for $V$ if it satisfies*

$$V(s)(z(t)) = z(s+t) \qquad (s \in S, t \in G),$$

*and if in addition there exists an $M$ such that $\|z(t)\| \leq M$ for all $t \in G$. When the second condition is not satisfied $z$ will be called an* **unbounded trajectory**. *For any $x \in Z$ a* **trajectory for $V$ through $x$** *is a trajectory $z$ such that $z(0) = x$.*

When $G = \mathbf{R}$ and $V$ is a $C_0$-semigroup a trajectory may be interpreted as being a 'complete' solution of the associated differential equation $\frac{df}{ds} = Af$; that is a solution over the whole of $\mathbf{R}$. Of course there may be many trajectories through a particular $x$, or there may be none.

No assumptions regarding continuity were placed either on $V$ or on $z$ by the definition, and this allows us to consider trajectories of both representations and dual representations. The latter are of considerable theoretical interest. Phóng has proved that when $T$ is a $C_0$-semigroup of isometries on a non-trivial Banach space with generator $A$, then $T^*$ has non-trivial trajectories whenever $i\mathbf{R} \not\subseteq \sigma(A)$ [33]. The following theorem is an obvious improvement on this result.

**Theorem 2.2.2** *Let $T$ be a representation of $S$ by isometries on $X$. There exists a trajectory through every $\phi$ in $X^*$, which is bounded by $\|\phi\|$.*

**Proof (given theorem 2.1.1):** Let $U$ acting on $Y$ be the group dilation as given by 2.1.1. For $\phi \in X^*$, let $\psi$ be a norm-preserving extension of $\pi x \mapsto \phi(x)$ to the whole of $Y$. The required trajectory for $T^*$ through $\phi$ may then be defined as

$$\zeta(t)x = (U(t)^*\psi)(\pi x) \qquad (x \in X,\, t \in G).$$

Theorem 2.2.2 is no mere corollary of theorem 2.1.1 however, as they are in fact equivalent; this will be demonstrated below. First we will show that 2.2.2 may be proved without resort to 2.1.1. This proof requires Tychonoff's theorem [37, p. 368]. It is possible to avoid it, but only by restricting the class of semigroups we can use to those which satisfy property (S5) below; the method by which this may be achieved is used in the proof of theorem 2.4.3.

**Proof (without theorem 2.1.1):** Let $\phi \in X^*$; assume without loss of generality that $\|\phi\| = 1$. For each $s$ in $S$ we construct a function as follows: let $\psi_s$ be a norm-preserving extension of $T(s)x \mapsto \phi(x)$ to the whole of $X$ and define $f_s : G \to X^*$ by

$$f_s(t) = \begin{cases} T(s+t)^*\psi_s & \text{if } s+t \in S \\ \phi & \text{otherwise} \end{cases} .$$

One must assume the axiom of choice to do this for general $S$ of course.

The unit ball of $X^*$, $B$ say, is weak*-compact, so $B^S$ is compact in the appropriate product topology by Tychonoff's theorem. Order $S$ by defining

$$s \le t \qquad \text{if and only if} \qquad t - s \in S \cup \{0\}.$$

The set $\{f_s : s \in S\}$ forms a net in $B^S$ and so by compactness we may choose a convergent subnet $(f_{s_\alpha})$ with limit point $f(\cdot)$ say.

Suppose $s_1, s_2, t_1, t_2 \in S$ with $s_1 - s_2 = t_1 - t_2$. Since $s_\alpha \to \infty$ through the semigroup, we may pick a $\beta$ such that $s_\alpha > t_2$ and $s_\alpha > s_2$ for all $\alpha > \beta$. Then

$$\begin{aligned} T(t_1)^* f(-t_2) &= T(t_1)^* \lim_{\alpha} f_{s_\alpha}(-t_2) \\ &= T(t_1)^* \lim_{\alpha > \beta} f_{s_\alpha}(-t_2) \end{aligned}$$

$$\begin{aligned}
&= T(t_1)^* \lim_{\alpha>\beta} T(s_\alpha - t_2)^* \psi_{s_\alpha} \\
&= \lim_{\alpha>\beta} T(s_\alpha + t_1 - t_2)^* \psi_{s_\alpha} \\
&= \lim_{\alpha>\beta} T(s_\alpha + s_1 - s_2)^* \psi_{s_\alpha} \\
&= T(s_1)^* f(-s_2).
\end{aligned}$$

It is valid therefore to define

$$\zeta(t) = T(t_1)^* f(-t_2) \qquad (t = t_1 - t_2,\ t_1, t_2 \in S).$$

This is a trajectory for $T^*$; for if $t = t_1 - t_2$ and $s \in S$, then

$$\begin{aligned}
T(s)^* \zeta(t_1 - t_2) &= T(s)^* T(t_1)^* f(-t_2) \\
&= T(s + t_1)^* f(-t_2) \\
&= \zeta(s + t_1 - t_2).
\end{aligned}$$

Choose any $s \in S$ and let $\alpha$ be such that $s_\beta > s$ for all $\beta > \alpha$. Then

$$\begin{aligned}
\zeta(0) &= T(s)^* f(-s) \\
&= T(s)^* \lim_{\beta>\alpha} f_{s_\beta}(-s) \\
&= \lim_{\beta>\alpha} T(s)^* T(s_\beta - s)^* \psi_{s_\beta} \\
&= \lim_{\beta>\alpha} T(s_\beta)^* \psi_{s_\beta} = \phi.
\end{aligned}$$

Finally, for any $t = t_1 - t_2$,

$$\|\zeta(t)\| = \|T(t_1)^* f(-t_2)\| \leq \|f(-t_2)\| \leq 1,$$

since $f(\cdot) \in B^S$.

**Definition 2.2.3** *Let $\mathcal{C}_T$ denote the set of all bounded trajectories for $T^*$. Define addition and scalar multiplication pointwise: so for $f(\cdot), g(\cdot) \in \mathcal{C}_T$ and $\lambda \in \mathbf{C}$, $(f + \lambda g)(\cdot)$ is the trajectory $(f + \lambda g)(t) = f(t) + \lambda g(t)$. The norm on $\mathcal{C}_T$ is given by*

$$\|\phi(\cdot)\|_\infty = \sup_{t \in G} \|\phi(t)\|.$$

This is clearly a normed vector space. It is also complete, for suppose $(\phi_n)$ is a Cauchy sequence in $\mathcal{C}_T$; it is elementary to see that there exists a bounded function $\phi(\cdot)$ that is the uniform limit of $(\phi_n)$. For $s \in S$ and $t \in G$,

$$\begin{aligned} T(s)^*\phi(t) &= T(s)^* \lim_n \phi_n(t) = \lim_n T(s)^*\phi_n(t) \\ &= \lim_n \phi_n(s+t) = \phi(s+t). \end{aligned}$$

So $\phi(\cdot) \in \mathcal{C}_T$ and $\mathcal{C}_T$ must be a Banach space.

Now define a homomorphism $L : G \longrightarrow \mathcal{B}(\mathcal{C}_T)$ by

$$(L(t)\phi)(\cdot) = \phi(t+\cdot) \qquad (t \in G,\ \phi \in \mathcal{C}_T).$$

$L$ may be seen as a group of left-shift operators on $\mathcal{C}_T$, which, although not strongly continuous and therefore not a group representation, turns out to be important to the theory of dilations.

**Proof of Theorem 2.1.1:** Define $\pi : X \longrightarrow \mathcal{C}_T^*$ by

$$\pi x : \phi(\cdot) \mapsto \phi(0)x \qquad (x \in X,\ \phi \in \mathcal{C}_T).$$

It is clear that $\pi x$ is well-defined, linear, and also that its norm is not greater than that of $x$. It follows that $\pi$ is also well-defined and that it is bounded by one.

Given $x \in X$, there exists $\phi$ of norm one in $X^*$ such that $\phi(x) = \|x\|$. By theorem 2.2.2 there exists a $\psi(\cdot)$ in $\mathcal{C}_T$, also of norm one, satisfying $\psi(0) = \phi$. In particular,

$$\|\pi x\| \geq |\pi x(\psi(\cdot))| = |\psi(0)x| = |\phi(x)| = \|x\|,$$

and $\pi$ is seen to be an isometry. The subset $\{L(-s)^*\pi x : s \in S,\ x \in X\}$ of $\mathcal{C}_T^*$ may be shown to be a linear subspace; let $Y$ denote its closure and let $U(t)$ denote $L(t)^*|_Y$. I claim that these together with $\pi$ satisfy the requirements of the theorem.

Firstly, for $s \in S$, $x \in X$, and $\phi(\cdot) \in \mathcal{C}_T$,

$$\begin{aligned} (U(s)\pi x)(\phi(\cdot)) &= (L(s)^*\pi x)(\phi(\cdot)) = \pi x(\phi(s+\cdot)) \\ &= \phi(s)x = T(s)^*\phi(0)x \\ &= \phi(0)(T(s)x) = \pi(T(s)x)\phi(\cdot). \end{aligned}$$

In other words, $U(s)\pi(x) = \pi T(s)x$; this is property 1 of the theorem. It also follows from this that $Y$ is $L^*$-, and hence $U$-, invariant. Property 2 of the theorem follows directly from the definition of $Y$. Since property 3 was shown above to follow from the first two, all that remains to prove is the strong continuity of $U$.

Let $t = s_1 - s_2$ with $s_1, s_2 \in S$, and choose $s_0$ in the interior of $S$. If $t_\alpha$ is a net tending to $t$ in $G$, then $t_\alpha + s_2 + s_0$ is a net tending to $s_1 + s_0$. Since $s_0$ is an interior point, so is $s_1 + s_0$ and eventually all $t_\alpha + s_2 + s_0$ are going to be in $S$. From this and strong continuity of $T$, we may deduce that

$$U(t_\alpha + s_2 + s_0)\pi x \to U(t + s_2 + s_0)\pi x \qquad \text{as } \alpha \to \infty.$$

Isometricity of $U(-s_2 - s_0)$ is now enough to ensure the result.

The relationship between the trajectory space $\mathcal{C}_T$ and the dilated Banach space becomes clearer in the following lemma.

**Lemma 2.2.4** *Let $T$ be a representation of $S$ by isometries, and let $Y$ be the dilated Banach space given by theorem 2.1.1. The trajectory space $\mathcal{C}_T$ is isometrically isomorphic to $Y^*$.*

**Proof:** Suppose first that $f \in Y^*$ and define $F(\cdot) \in \mathcal{C}_T$ by

$$F(t) : x \mapsto f(U(t)\pi x) \qquad (t \in G,\, x \in X).$$

A moment's verification shows that this does indeed define a trajectory for $T^*$ and that it is bounded by $\|f\|$. By property 2 of theorem 2.1.1, for each $\varepsilon > 0$ there exists $s \in S$ and $x \in X$ with $\|x\| = 1$ such that $|f(U(-s)\pi x)| > \|f\| - \varepsilon$; in other words $|F(-s)x| > \|f\| - \varepsilon$ and hence $\|F(\cdot)\|_\infty = \|f\|$.

In the other direction, suppose $F(\cdot) \in \mathcal{C}_T$ and define $f$ in $Y^*$ to be the unique extension of

$$f : U(-s)\pi x \mapsto F(-s)x \qquad (s \in S,\, x \in X).$$

This is well-defined, for if $U(-s)\pi x = U(-t)\pi y$, then $T(t)x = T(s)y$ by isometricity of $\pi$ and hence

$$\begin{aligned} F(-s)x &= T(t)^*F(-s-t)x = F(-s-t)T(t)x \\ &= F(-s-t)T(s)y = F(-t)y. \end{aligned}$$

Clearly the two processes above are the inverses of each above, and hence define an isometric isomorphism between $\mathcal{C}_T$ and $Y^*$. Furthermore, under this mapping the actions of $L$ and $U^*$ correspond exactly.

**Corollary 2.2.5** *If $(U,Y)$ is the dilation of $T$, then $\mathcal{C}_U \simeq Y^* \simeq \mathcal{C}_T$.*

**Example 2.2.6** *Let $X = L^1(S)$ and $T$ be the right-shift representation. The dilated Banach space of $X$ is $L^1(G)$, and the dilated group representation consists of the right-shift operators. $\pi$ is the natural embedding of these spaces. The trajectory space is $L^\infty(G)$.*

The calculation of the dilations of $X$ and $T$ is trivial. Lemma 2.2.4 tells us that $\mathcal{C}_T = L^1(G)^*$, but we will demonstrate this independently (at least for reasonable $S$).

Let $F(\cdot) \in \mathcal{C}_T$. This means that for each $s$ in $S$, $F(s) \in X^*$, and so we may consider $F(s)$ to a function in $L^\infty(S)$. Now consider the relationship between $F(s)$ and $F(0)$ where $s$ is in $S$: we know that $F(s) = T(s)^*F(0)$, and that $T$ is a right-shift. Hence $F(s)$ is equal (almost everywhere) to $F(0)$ shifted along by $s$ to the left in $L^\infty(S)$. Consequently, if we consider $L^\infty(S)$ as a subset of $L^\infty(G)$, and shift $F(s)$ by $s$ to the left in $L^\infty(G)$, then it will agree on $S$ with $F(0)$. By shifting each $F(s)$ to the left by $s$ in this way, we can build up a function defined on all $G$. (In fact there needs to be an additional assumption on $S$ to do this; for example (S5) below will suffice.) So if we start with a trajectory in $\mathcal{C}_T$, we end up with a function in $L^\infty(G)$. In the other direction, a trajectory may be obtained from an $L^\infty(G)$ function $f$ by restricting $f_{-t}$ to $S$ for each $t \in G$ and considering it as an element of $L^1(S)^*$.

## 2.3 Structural Stability under Dilations

Once theorem 2.1.1 is known for Banach spaces the question naturally arises as to which properties of the original space and representation are preserved under the dilation. This section deals with some of the easier cases. Throughout $T$ will denote a representation of $S$ by isometries on $X$, and $(U, Y, \pi)$ will be the dilation given by theorem 2.1.1. $Y^0$ will denote the dense subspace of $Y$ given by

$$Y^0 = \{U(-s)\pi x : x \in X, s \in S\}.$$

### 2.3.1 Hilbert Spaces

As mentioned in the notes below, there is a Hilbert space version of theorem 2.1.1 originating in the work of Itô [24]: if $X$ is a Hilbert space, then so is $Y$. It is straightforward to deduce this from the Banach space statement by defining an inner-product on $Y^0$ by

$$\langle U(-s)\pi x, U(-t)\pi y\rangle = \langle T(t)x, T(s)y\rangle$$

for all $s, t \in S$ and $x, y \in X$. This may then be extended to the whole of $Y$.

### 2.3.2 Banach Algebras

When $X$ is a Banach algebra we will assume in addition that each $T(s)$ is an algebra homomorphism. We may then define a multiplication naturally on $Y^0$ by the following formula.

$$U(-s)\pi x \cdot U(-t)\pi y = U(-s-t)\pi(T(t)x \cdot T(s)y) \qquad (s, t \in S,\ x, y \in X).$$

This may be extended to the whole of $Y$ so that $Y$ is also a Banach algebra. In this case $U$ will be a group of isometric algebra automorphisms.

### 2.3.3 $C^*$-algebras

When $X$ is a $C^*$-algebra we assume that $T$ is a semigroup of $*$-homomorphisms. Multiplication is then defined on $Y$ as above, and the star operation in $Y^0$ is given by

$$(U(-s)\pi x)^* = U(-s)\pi(x^*),$$

which we again must complete to the whole of $Y$. So

$$\begin{aligned} \|(U(-s)\pi x)^*(U(-s)\pi x)\| &= \|U(-s)\pi x^* x\| \\ &= \|x^* x\| = \|x\|^2 \\ &= \|U(-s)\pi x\|^2 \end{aligned}$$

and hence $Y$ is also a $C^*$-algebra. $U$ is a group of $*$-automorphisms. This leads easily to a theorem.

**Theorem 2.3.4** *Suppose $T$ is a representation of $S$ by isometric $*$-homomorphisms on a $C^*$-algebra. For any $\chi \in \Gamma$, if $\chi \in Sp_u(T,S)$, then $-\chi \in Sp_u(T,S)$, where $(-\chi)(t) = \chi(t)^{-1}$.*

**Proof:** This follows from the above comments, from theorem 2.1.1 part 3, and from [31, corollary 8.3.4], where it is proved that the conclusion required is true for any representation of $G$ by $*$-automorphisms on a $C^*$-algebra.

### 2.3.5 Partially Ordered Spaces

Suppose $X$ has a partial order $\leq$ with the property that there exists some $\alpha$ such that

$$0 \leq x \leq y \Rightarrow \|x\| \leq \alpha\|y\|;$$

the norm is then called $\alpha$-dominating. In particular if $x \leq y$ and $y \leq x$, then $0 \leq x - y \leq 0$ and hence $x = y$.

If $T$ is a representation of $S$ by positive isometries, that is if $x \geq 0$ implies $T(s)x \geq 0$ for all $x \in X$ and $s \in S$, then the dilated Banach space $Y$ may be equipped with a partial order with respect to which $\pi$ and $U$ are positive.

To see this begin by defining a partial order on $Y^0$ by declaring $U(-s)\pi x \leq U(-t)\pi y$ if and only if there exists a $v \in S$ such that $T(t+v)x \leq T(s+v)y$.

This is a well-defined relation, for if $U(-s)\pi x \leq U(-t)\pi y = U(-t')\pi y'$, then there exists $v \in S$ such that $T(t+v)x \leq T(s+v)y$. Since $T(t')y = T(t)y'$, it follows that

$$T(t'+v+t)x \leq T(s+v+t')y = T(s+v+t)y',$$

and hence $U(-s)\pi x \leq U(-t')\pi y'$.

Suppose $U(-s)\pi x \leq U(-t)\pi y$ and $U(-s')\pi x' \leq U(-t')\pi y'$. There exists $v, v' \in S$ such that $T(t+v)x \leq T(s+v)y$ with an equivalent relation holding for $v'$. By positivity of $T$,

$$T(t+v+(t'+s'+v'))x \leq T(s+v+(t'+s'+v'))y$$

and

$$T(t'+v'+(t+s+v))x' \leq T(s'+v'+(t+s+v))y'.$$

So, denoting $v+v'$ by $w$,

$$T(w+(t'+t))(T(s')x+T(s)x') \leq T(w+(s'+s))(T(t')y+T(t)y'),$$

which, by the definition, implies that

$$U(-s-s')(T(s')x+T(s)x') \leq U(-t-t')(T(t')y+T(t)y').$$

But this just says $U(-s)\pi x + U(-s')\pi x' \leq U(-t)\pi y + U(-t')\pi y'$. In a similar manner the other properties of a partial order may be verified on $Y^0$. It may also be shown that $U(t)$ is a positive operator for each $t \in G$.

If $0 \leq U(-s)\pi x \leq U(-t)\pi y$, then for some $v$, $0 \leq T(t+v)x \leq T(s+v)y$. By the assumption on the norm of $X$, $\|x\| \leq \alpha\|y\|$ from which we may deduce that the norm on $Y^0$ is also $\alpha$-dominating. The partial order may be completed to the whole of $Y$ by setting $y \geq 0$ if it is the limit of positive elements of $Y^0$; with respect to this ordering the norm of $Y$ is $\alpha$-dominating and each operator $U(t)$ is positive.

Note that while $\pi$ is a positive isometry, the implication $\pi x \geq 0 \Rightarrow x \geq 0$ is true for all $x \in X$ only when $T$ is such that $T(s)x \geq 0 \Rightarrow x \geq 0$ for all $s \in S$ and $x \in X$.

### 2.3.6 Banach Lattices

This time we assume that $T$ consists of lattice homomorphisms. We prove that $Y^0$ can be made naturally into a Riesz space. The completion of this is then a Banach lattice.

Since $T(s)x \geq 0$ if and only if $x \geq 0$ for isometric lattice homomorphisms, the partial order on $Y^0$ defined in the above section can be given here as

$$U(-s)\pi x \geq U(-t)\pi y \qquad \text{if and only if} \qquad T(t)x \geq T(s)y.$$

I claim that

$$U(-s-t)\pi(T(t)x \vee T(s)y) = \sup\{U(-s)\pi x, U(-t)\pi y\}$$

for each $x, y \in X$, and $s, t \in S$. First note that it is clearly an upper bound for $U(-s)\pi x$ and $U(-t)\pi y$. Suppose that $U(-p)\pi z$ is another upper bound. (We may assume that an upper bound is of this form because we are working in $Y^0$, not its closure.) Then

$$T(s)z \geq T(p)x \qquad \text{and} \qquad T(t)z \geq T(p)y$$

and hence

$$T(s+t)z \geq T(p+t)x \qquad \text{and} \qquad T(s+t)z \geq T(s+p)y.$$

So

$$\begin{aligned} T(s+t)z &\geq T(p+t)x \vee T(s+p)y \\ &= T(p)(T(t)x \vee T(s)y), \end{aligned}$$

which, stated otherwise, says that

$$U(-p)\pi z \geq U(-s-t)\pi(T(t)x \vee T(s)y).$$

$U|_{Y^0}$ will now be a group representation by lattice automorphisms. It follows that $Y$ is a Banach lattice and $U$ is a group of lattice automorphisms.

### 2.3.7 Reflexive Spaces

The property of reflexivity is rather a fragile one, and when we dilate a reflexive space, the result is not in general another reflexive space. Of course if $X$ is actually a Hilbert space, then reflexivity is preserved since $Y$ will be a Hilbert space, and there exist examples of reflexive spaces which are not Hilbert spaces for which reflexivity is preserved.

**Example 2.3.8** *If $X = L^p(S)$ for some $1 < p < \infty$ and $T$ is the right-shift representation, then the dilated Banach space is reflexive.*

The dilated Banach space is $L^p(G)$ of course.

**Example 2.3.9** *There exists a representation by isometries on a reflexive space such that the dilated Banach space is not reflexive.*

Let $l_n^\infty$ denote the space consisting of n-tuples equipped with the sup-norm, and let $X$ be the infinite $l^2$-sum

$$X = \bigoplus_{n=1}^{\infty} l_n^\infty$$

The dual of $X$ is the infinite $l^2$-sum of $l_n^1$ spaces, and $X$ is reflexive. We define a representation of $\mathbf{Z}_+$ by isometries in the following fashion.

Let $R_n : l_n^\infty \longrightarrow l_{n+1}^\infty$ by

$$R_n : (\zeta_1, \zeta_2, \ldots, \zeta_n) \mapsto (0, \zeta_1, \zeta_2, \ldots, \zeta_n).$$

Each $R_n$ is an isometry. Now define $T : X \longrightarrow X$ by the relation

$$T : (x_n)_{n=1}^{\infty} \mapsto (0, R_1 x_1, R_2 x_2, \ldots, R_n x_n, \ldots),$$

where $x_n \in l_n^{\infty}$. The representation is simply $\{T^n : n \in \mathbf{Z}_+\}$.

The dilated space $Y$ is the infinite $l^2$ sum $\oplus_{-\infty}^{\infty} c_0$, where the $c_0$-sequences are assumed to run from $-\infty$ to $\infty$. The dilated group representation is then given by

$$U : (y_n)_{-\infty}^{\infty} \mapsto (z_n)_{-\infty}^{\infty},$$

where each $y_n = (\ldots, y_n(-1), y_n(0), y_n(1), \ldots) \in c_0$ and $z_n \in c_0$ is given by $z_n(k) = y_{n-1}(k-1)$ $(k \in \mathbf{Z})$. $Y$ is not a reflexive space.

All of the above have been essentially trivial – either the structure goes across easily or it does not go across at all. Harder cases will be covered in the next two sections.

## 2.4 The Dual Dilation Theorem

Let $T : S \to \mathcal{B}(X)$ be a dual representation by isometries.

**Definitions 2.4.1** *A dual group representation $U : G \to \mathcal{B}(Y)$ is a* **dual dilation** *of $T$ if there exists a weak*-continuous isometry $\pi : X \to Y$ such that*

$$\pi(T(s)x) = U(s)\pi x \qquad (s \in S,\, x \in X).$$

*$U$ will be said to be* **minimal** *if it satisfies the condition*

$$\{U(-s)\pi x : s \in S, \|x\| \leq 1\} \textit{ is weak*-dense in the unit ball of } Y.$$

The more natural definition of minimality would have been that $\{U(-s)\pi x : s \in S, x \in X\}$ must be weak*-dense in $Y$. The (apparently) stronger definition given here is needed to prove theorem 2.4.3 and lemma 2.4.7. The standard dilation theorem 2.1.1 was valid for any semigroup satisfying conditions (S1)–(S4) of chapter one. My proof of the dual version unfortunately requires $S$ to satisfy the following additional property.

(S5) There exists a sequence $(s_n)$ in $S$ satisfying

(i) For each $n$ in $\mathbf{Z}_+$, $s_{n+1} - s_n \in S$;

(ii) For all $t$ in $S$, there exists an $m$ such that $s_m - t \in S$.

Both $\mathbf{Z}_+^k$ and $\mathbf{R}_+^k$ satisfy (S5) (with, for example, the k-tuples $s_n = (n, n, \ldots, n)$), so this extra restriction is not too great. In fact the following is true:

**Proposition 2.4.2** *If $S$ is a subsemigroup of the compactly generated LCA group $G$, then $S$ satisfies (S5) whenever it satisfies conditions (S1)-(S4).*

**Proof:** To say that $G$ is compactly generated means that there exists a compact neighbourhood $C$ of 0 in $G$ such that $G = \cup_{n\geq 1} C_n$, where $C_1 = C$ and $C_{n+1} = C_n + C$ for each $n$. (See for example [35, p. 41].) By lemma 1.1.5 applied to the compact set $-C$, there is an $s \in S$ such that $s - t \in S$ for all $t \in C$. If $s_n := ns$, then for each $n \in \mathbf{Z}_+$, $s_n - t \in S$ for all $t \in C_n$. It now follows easily that $(s_n)$ is a sequence satisfying the demands of condition (S5).

**Theorem 2.4.3** *Suppose $S$ is a subsemigroup of $G$ satisfying conditions (S1)-(S5). If $T$ is a dual representation of $S$ by isometries on a dual Banach space $X$, then there exists a minimal dual dilation $(U, Y, \pi)$ of $T$, which is also maximal in the sense that for any other minimal dual dilation $(W, Z, \theta)$, there exists an isometry $\alpha_* : Z_* \longrightarrow Y_*$ that intertwines $W_*$ with $U_*$, and satisfies ${\alpha_*}^* \pi = \theta$.*

This maximal-minimal property of $(U, Y, \pi)$ characterises it uniquely up to isometric isomorphism. At first sight it may seem odd that the isometry should be between the pre-duals as one would think that the natural mapping would be between $Z$ and $Y$. For example, one could try to map $W(-s)\theta x$ to $U(-s)\pi x$; both of these have norm $\|x\|$ and the mapping as defined so far is linear. But as it happens, this will not work. To prove the theorem we will require some preliminary results.

**Lemma 2.4.4** *Suppose $S$ is a subsemigroup of $G$ satisfying conditions (S1)–(S5) and let $T$ be a dual representation of $S$ by isometries on $X$. For every $\phi \in X_*$ and $\varepsilon > 0$ there is a trajectory $\psi(\cdot)$ of $T_*$ through $\phi$ such that $\|\psi(\cdot)\|_\infty < \|\phi\| + \varepsilon$. In particular $\psi(s) \in X_*$ for each $s$ in $G$.*

Considered as an element of $X^*$, the existence of a trajectory for $T^*$ through $\phi$ is given by theorem 2.2.2; but one cannot guarantee that $\psi(s)$ is in $X_*$ for all $s$. So we must construct a trajectory that meets our requirements from scratch. The extra condition imposed on $S$ is necessary to prove this lemma. If it could be proved without this extra condition, then the dual dilation theorem would not require it either.

**Proof:** Let $(s_n)$ be the sequence in $S$ given by (S5) and let $s_0 = 0$. We shall create a sequence $(\phi_n)$ by induction satisfying these three conditions:

1. $\phi_n \in X_* \qquad (n = 0, 1, 2, \ldots);$
2. $T_*(s_{n+1} - s_n)\phi_{n+1} = \phi_n \qquad (n = 0, 1, 2, \ldots);$
3. $\|\phi_n\| < \|\phi\| + \varepsilon \qquad (n = 0, 1, 2, \ldots).$

This is an easy consequence of a standard result in functional analysis: if $B$ is some linear operator on $Z$ for which $B^*$ is an isometry, then given any $z \in Z$, there exists a $v \in Z$ such that $Bv = z$ and $\|v\| < \|z\| + \varepsilon$. (See for example [37, lemma 4.13b].) Suppose we have $\phi_0, \ldots, \phi_k$ satisfying the conditions, then since $T(s_{k+1} - s_k)$ is an isometry, this result allows us to pick a suitable $\phi_{k+1} \in X_*$.

For $t \in G$ let $t_1, t_2 \in S$ be such that $t = t_1 - t_2$. By assumption, there exists a $k$ with $s_k - t_2 \in S$ and consequently $t + s_k \in S$. Suppose $n$ is also such that $t + s_n \in S$; then assuming without loss of generality that $k \geq n$,

$$T_*(s_k + t)\phi_k = T_*(s_n + (s_{n+1} - s_n) + \ldots + (s_k - s_{k-1}) + t)\phi_k$$

$$
\begin{aligned}
&= T_*(s_n + t)T_*(s_{n+1} - s_n) \cdots T_*(s_k - s_{k-1})\phi_k \\
&= T_*(s_n + t)T_*(s_{n+1} - s_n) \cdots T_*(s_{k-1} - s_{k-2})\phi_{k-1} \\
&= T_*(s_n + t)\phi_n.
\end{aligned}
$$

The following function $\psi : G \to X_*$ may therefore be defined without ambiguity. For $t \in G$ let

$$\psi(t) = T_*(s_k + t)\phi_k$$

where $k$ is such that $s_k + t \in S$. Trivially, $T_*(s)\psi(t) = \psi(s+t)$ and $\|\psi(\cdot)\|_\infty < \|\phi\| + \varepsilon$. Finally note that $\psi(0) = T_*(s_1)\phi_1 = \phi$.

**Lemma 2.4.5** *For any $x \in X$ and $\varepsilon > 0$ there exists a trajectory $\zeta(\cdot)$ for $T_*$ with the following properties:*

1. *the function $s \mapsto \zeta(s)$ is uniformly norm-continuous;*
2. *$|\zeta(0)x - \|x\|| < \varepsilon$;*
3. *$\|\zeta(\cdot)\|_\infty \leq 1$.*

**Proof:** If $\psi(\cdot)$ is a trajectory for $T_*$, then it follows from lemma 1.1.5 and the strong continuity of $T_*$ that $t \mapsto \psi(t)$ is norm-continuous. For $\varepsilon > 0$ let $V$ be an open neighbourhood of 0 in $G$ with compact closure $C$ such that $\|\psi(s) - \psi(0)\| < \varepsilon/2$ for all $s \in C$. Assume also that $|C|$ is finite and define

$$\zeta(t) = \frac{1}{C}\int_C \psi(t+s)\,ds \qquad (t \in G).$$

It is easily verified that $\zeta$ is another trajectory for $T_*$, that it is bounded by $\|\psi(\cdot)\|_\infty$, that it is uniformly continuous, and that $\|\zeta(0) - \psi(0)\| < \varepsilon/2$.

If $x \in X$ is zero, then the lemma is trivial, so assume without loss of generality that $\|x\| = 1$. There exists $\phi \in X_*$ with $\|\phi\| < 1$ such that $|\phi(x) - \|x\|| < \varepsilon/2$. By lemma 2.4.4, there is a trajectory $\psi(\cdot)$ for $T_*$ through $\phi$ with $\|\psi(\cdot)\|_\infty \leq 1$. Construct $\zeta$ from this $\psi$ as described above. The only property left to verify is number 2:

$$\begin{aligned} |\zeta(0)x - \|x\|| &\leq |\zeta(0)x - \psi(0)x\| + |\psi(0)x - \|x\|| \\ &< \varepsilon. \end{aligned}$$

**Proposition 2.4.6** *A dual dilation $(W, Z, \theta)$ of $T$ is minimal if and only if*

$$\|\psi\| = \sup_{t \in G} \|\theta_* W_*(t)\psi\| \tag{2.1}$$

*for all $\psi$ in $Z_*$.*

**Proof:** Suppose the dilation is minimal and that $\psi$ is an element of $Z_*$. If $\psi = 0$ there is nothing to prove, so let us assume that $\|\psi\| = 1$. Let $z \in Z$ be of norm one such that $z(\psi) = 1$. By minimality, $z$ is in the weak*-closure of the set $\{W(-s)\theta x : s \in S, \|x\| = 1\}$ and we may hence choose an $s \in S$ and an $x \in X$ of norm one such that $|(W(-s)\theta x)(\psi) - z(\psi)|$ is arbitrarily small. Since

$$\sup_{t \in G} \|\theta_* W_*(t)\psi\| \geq \|(W(-s)\theta x)\| = 1,$$

the non-trivial half of equation 2.1 is proved.

Suppose instead that $(W, Z, \theta)$ is a dual dilation for which the equality 2.1 holds. Let $M$ denote $\{W(-s)\theta x : s \in S, \|x\| \leq 1\}$. If $z \in Z$ is not in the weak*-closure of $M$, then there exists $\psi \in Z_*$ such that $\sup_{m \in M} \operatorname{Re} m(\psi) < \operatorname{Re} z(\psi)$ [37, theorem 3.4(b)]. It is straightforward to show that

$$\sup_{t \in G} \|\theta_* W_*(t)\psi\| = \sup_{m \in M} |m(\psi)| = \sup_{m \in M} \operatorname{Re} m(\psi),$$

and hence equation 2.1 implies that $\|\psi\| < |z(\psi)|$ from which we deduce $\|z\| > 1$.

**Proof of Theorem 2.4.3:** Let $\mathcal{C}_T$ be the space of all trajectories of $T^*$, and let $Y_*$ be that subspace of $\mathcal{C}_T$ consisting of those $\phi(\cdot)$ for which $\phi(t) \in X_*$ for all $t \in G$ and such that $t \mapsto \phi(t)$ is uniformly norm-continuous. Lemma 2.4.5 implies that $Y_*$ is non-trivial if $X$ is, and it is clearly closed and $L$-invariant (where $L$ is the left-shift group on $\mathcal{C}_T$). Denote $L|_{Y^*}$ by $U_*$ and the linear map $\phi(\cdot) \mapsto \phi(0)$ from $Y_*$ to $X_*$ by $\pi_*$. I claim that these form the dual dilation required.

The strong continuity of $U$ follows from the construction of $Y_*$, for if $\phi(\cdot) \in Y_*$, then

$$\|U_*(s)\phi(\cdot) - U_*(t)\phi(\cdot)\|_\infty = \sup_{u \in G} \|\phi(s+u) - \phi(t+u)\|$$

which tends to zero as $s$ tends to $t$ by uniform norm-continuity of $\phi(\cdot)$. Hence $U = (U_*)^*$ is a dual representation of $G$ on $Y = (Y_*)^*$.

The map $\pi = (\pi_*)^*$ is clearly bounded by one. Suppose $x \in X$ and $\varepsilon > 0$. Then by lemma 2.4.5 there exists a $\zeta(\cdot) \in Y_*$ of norm one such that $|\zeta(0)x - \|x\|| < \varepsilon$; in particular $\|\pi x\| \geq |\zeta(0)x| > \|x\| - \varepsilon$ and we have shown that $\pi$ is an isometry. For $x \in X$, $s \in S$, and $\phi(\cdot) \in Y_*$,

$$\begin{aligned} U(s)\pi x(\phi(\cdot)) &= \pi x(\phi(\cdot + s)) = x(\phi(s)) \\ &= x(T_*(s)\phi(0)) = T(s)x(\phi(0)) \\ &= \pi(T(s)x)(\phi(\cdot)); \end{aligned}$$

so $U(s)\pi = \pi T(s)$ and the proof of the existence of a dual dilation is complete. Minimality is trivial given proposition 2.4.6.

For $(W, Z, \theta)$ another minimal dual dilation, define $\alpha_* : Z_* \longrightarrow Y_*$ by

$$\alpha_* : z \mapsto \phi(\cdot) \qquad \text{where} \qquad \phi(t) = \theta_* W_*(t) z.$$

It is easily verified that $\phi(\cdot)$ is a trajectory for $T_*$. It is uniformly norm-continuous because $W_*(\cdot)$ is a strongly continuous group representation. Proposition 2.4.6 implies that $\|\phi(\cdot)\|_\infty = \|z\|$ and hence $\alpha_*$ is an isometry. Finally, the intertwining properties are easily verified.

In fact the pre-dual of any dual dilation may be mapped to $Y_*$ in this way, but the map $\alpha_*$ will not in general be an isometry. (Proposition 2.4.6 could be restated as saying that the dilation is minimal if and only if this map is an isometry.) One useful feature of minimal dual dilations is shown by the following lemma.

**Lemma 2.4.7** *If $(U, Y, \pi)$ is a minimal dual dilation of a dual representation $T$ of $S$ by isometries, then $Sp(U, G) = Sp_u(T, S)$*

**Proof:** For each $y \in Y_*$ and $f \in L^1(S)$,

$$\begin{aligned}
\hat{f}(T_*)(\pi_* y) &= \int_S f(s) T_*(s)(\pi_* y)\, ds \\
&= \int_S \pi_*(f(s) U_*(s) y)\, ds \\
&= \pi_*(\hat{f}(U_*) y).
\end{aligned}$$

So

$$\|\hat{f}(T_*)\| = \sup_{\|y\| \leq 1} \|\hat{f}(T_*)\pi_* y\| \leq \|\hat{f}(U_*)\|$$

and hence $Sp(T_*, S) \subseteq Sp(U_*, S)$ for any dual dilation.

Using the particular property of a minimal dilation, if $y \in Y_*$, then

$$\begin{aligned}
\|\hat{f}(U_*) y\| &= \sup_{\|\phi\| \leq 1,\, \phi \in Y} |\phi(\hat{f}(U_*) y)| \\
&= \sup_{\|\psi\| \leq 1,\, \psi \in X} \sup_{t \in S} |U(-t) \pi \psi(\hat{f}(U_*) y)| \\
&= \sup |\psi(\pi_* U_*(-t) \hat{f}(U_*) y)| \\
&= \sup |\psi(\hat{f}(T_*)(\pi_* U_*(-t) y)| \\
&\leq \|\hat{f}(T_*)\| \sup_{t \in S} \|\pi_* U_*(-t) y\| \\
&\leq \|\hat{f}(T_*)\|\, \|y\|.
\end{aligned}$$

Thus $\|\hat{f}(U_*)\| \leq \|\hat{f}(T_*)\|$, which means that $Sp(U_*, S) \subseteq Sp(T_*, S)$.

## 2.5 Dilating von Neumann Algebras

We will adopt Sakai's notion of von Neumann algebras as being $C^*$-algebras that are also dual Banach spaces. A representation of $S$ by $*$-endomorphisms on $X$ is a dual representation where each $T(s)$ is an isometric $*$-homomorphism of $X$. We further assume that $T(s)1 = 1$ for all $s \in S$.

**Theorem 2.5.1** *Let $X$ be a von Neumann algebra and $T$ a dual representation by $*$-endomorphisms. The Banach space $Y$ given by theorem 2.4.3 is a von Neumann algebra, $U$ is a group of $*$-automorphisms, and $\pi$ is a normal $*$-homomorphism.*

The proof relies on the Arens construction for multiplication in the second dual of a Banach algebra which we recall here.

Let $A$ be a Banach algebra. For each $a \in A$ and $f \in A^*$, define $f_a \in A^*$ by

$$f_a : b \mapsto f(ab) \qquad (b \in A).$$

For $f \in A^*$ and $F \in A^{**}$, define $Ff \in A^*$ by

$$Ff : a \mapsto F(f_a) \qquad (a \in A).$$

Finally, for $F, G \in A^{**}$ the element $GF \in A^{**}$ is defined by

$$GF : f \mapsto G(Ff).$$

In the case when $A$ is a $C^*$-algebra this multiplication coincides with the natural multiplication of the universal enveloping von Neumann algebra.

**Proof of Theorem 2.5.1:** Let $N$ be the space of all $x \in X$ such that $s \mapsto T(s)x$ is norm-continuous. As in lemma 1.3.4, this is a closed subspace of $X$. Furthermore

$$\begin{aligned}
\|T(s)(xy) - T(t)(xy)\| &= \|T(s)x \cdot T(s)y - T(t)x \cdot T(t)y\| \\
&\leq \|(T(s)x - T(t)x)T(s)y\| \\
&\qquad + \|T(t)x(T(s)y - T(t)y)\| \\
&\leq \|T(s)x - T(t)x\| \, \|y\| + \|T(s)y - T(t)y\| \, \|x\|,
\end{aligned}$$

which tends to zero as $s \to t$. Similarly

$$\begin{aligned}
\|T(s)(x^*) - T(t)(x^*)\| &= \|(T(s)x - T(t)x)^*\| \\
&= \|T(s)x - T(t)x\|
\end{aligned}$$

also tends to zero.

This demonstrates that $N$ is a $C^*$-subalgebra of $X$. It is trivially $T$-invariant, and so we may consider $T|_N$ as a (strongly continuous) representation on $N$. By section 2.3.3, the dilation of $(T|_N, N)$ consists of another $C^*$-algebra, call it $M$, and a group of $*$-automorphisms, $V$ say. Denote the isometric $*$-homomorphism of $N$ into $M$ by $\theta$.

$M^{**}$ is a von Neumann algebra, and its multiplication is given by the Arens product. We will now construct an isometry from $Y_*$ to $M^*$, where $Y_*$ is the space of all uniformly norm-continuous trajectories of $T_*$ (as in theorem 2.4.3). Let $\phi(\cdot) \in Y_*$ and define $\beta\phi(\cdot) \in M^*$ to be the extension of

$$\beta\phi(\cdot) : V(-s)\theta x \mapsto x(\phi(-s)).$$

By inspection of the definition it may be seen that $\beta\phi$ is of norm less than or equal to $\|\phi(\cdot)\|$.

Given $\varepsilon > 0$, there exists $s \in S$ with the property

$$\|\phi(-s)\| > \|\phi(\cdot)\|_\infty - \varepsilon$$

and, by lemma 1.3.4, we may choose an $x \in N$ of norm one such that

$$|x(\phi(-s)) - \|\phi(-s)\|\,| < \varepsilon.$$

Combining these we have

$$|\beta\phi(\cdot)(V(-s)\theta x)| = |x(\phi(-s))| > \|\phi(\cdot)\|_\infty - 2\varepsilon$$

and $\beta$ is shown to be an isometry. Let $E$ denote $\beta[Y_*]$. The aim of the rest of the proof is to show that $E^\perp$ is a two-sided ideal of $M^{**}$ and hence that $Y \simeq E^* = M^{**}/E^\perp$ is a von Neumann algebra.

If $s \in S$, $x \in X$, and $\phi(\cdot) \in Y_*$, then $a := V(-s)\theta x \in M$ and $f = \beta\phi(\cdot) \in E$. I claim that $f_a$ is also in $E$. This will be demonstrated by direct construction of an appropriate trajectory in $Y_*$.

For $t_1, t_2 \in S$, define $\psi_{t_1,t_2} : X \to \mathbf{C}$ by

$$\psi_{t_1,t_2} : y \mapsto (T(t_2)x \cdot T(s+t_1)y)\phi(-t_2 - s).$$

A moment's inspection shows this to be linear. It is also bounded since

$$\begin{aligned} |\psi_{t_1,t_2}(y)| &\leq \|T(t_2)x \cdot T(s+t_1)y\| \, \|\phi(-t_2 - s)\| \\ &\leq \|x\| \, \|y\| \, \|\phi(\cdot)\|_\infty. \end{aligned}$$

If $s_1, s_2 \in S$ are such that $s_1 - s_2 = t_1 - t_2$, then

$$\begin{aligned} \psi_{s_1,s_2}(y) &= (T(s_2)x \cdot T(s+s_1)y)\phi(-s_2 - s) \\ &= (T(s_2)x \cdot T(s+s_1)y)\phi(t_1 - t_2 - s_1 - s) \\ &= T(t_1)(T(s_1 + t_2 - t_1)x \cdot T(s+s_1)y)\phi(-t_2 - s_1 - s) \\ &= (T(s_1 + t_2)x \cdot T(s + t_1 + s_1)y)\phi(-t_2 - s_1 - s) \\ &= (T(t_2)x \cdot T(s+t_1)y)\phi(-t_2 - s) \\ &= \psi_{t_1,t_2}(y). \end{aligned}$$

We may therefore define $\psi(t) = \psi_{t_1,t_2}$ for $t = t_1 - t_2$ without ambiguity.

If $u \in S$, and $t = t_1 - t_2$, then

$$\begin{aligned} T(u)^*\psi(t)y &= \psi(t)(T(u)y) \\ &= (T(t_2)x \cdot T(s+t_1)T(u)y)\phi(-t_2 - s) \\ &= (T(t_2)x \cdot T(s+t_1+u)y)\phi(-t_2 - s) \\ &= \psi_{t_1+u,t_2}(y) = \psi(u+t)(y). \end{aligned}$$

Hence $\psi(\cdot)$ is a trajectory for $T^*$ which is bounded by $\|x\| \, \|\phi(\cdot)\|_\infty$.

Standard von Neumann algebra theory tells us that if $y_\alpha \to y$ in the weak*-topology of $X$, then $zy_\alpha \to zy$ in the weak*-topology for all $z \in X$ (see for example, [39, p. 11]). This, together with the weak-continuity of $T$, implies that

$$(T(t_2)x \cdot T(s+t_1)y_\alpha) \longrightarrow T(t_2)x \cdot T(s+t_1)y$$

in the weak*-topology. In particular, since each $\phi(t) \in X_*$,

$$\psi_{t_1,t_2}(y_\alpha) \longrightarrow \psi_{t_1,t_2}(y);$$

hence $\psi(t) \in X_*$ for each $t \in G$, and so $\psi$ is a trajectory for $T_*$.

If either $x$ or $\phi(\cdot)$ is zero, then $\psi = 0$ which is uniformly norm-continuous; so assume this is not the case. Using uniform norm-continuity of $s \mapsto T(s)x$ and norm-continuity of $t \mapsto \phi(\cdot + t)$ we may, given $\varepsilon > 0$, choose an open neighbourhood $V_\varepsilon$ of 0 in $G$ such that

$$\|T(u)x - T(u')x\| < \frac{\varepsilon}{2\|\phi(\cdot)\|_\infty}$$

whenever $u, u' \in S$ satisfy $u' - u \in V_\varepsilon$, and also

$$\|\phi(\cdot) - \phi(\cdot + v)\|_\infty < \frac{\varepsilon}{2\|x\|}$$

for all $v \in V_\varepsilon$. For $t, t' \in G$ with $t - t' \in V_\varepsilon$, it is possible to express $t = t_1 - t_2$ and $t' = t_1 - t_3$ for some $t_1, t_2, t_3$ in $S$. Note that $t_3 - t_2 = t - t' \in V_\varepsilon$.

For any $y \in X$,

$$\begin{aligned}
|\psi(t)y - \psi(t')y| &= |(T(t_2)x \cdot T(t_1 + s)y)\phi(-t_2 - s) \\
&\qquad -(T(t_3)x \cdot T(t_1 + s)y)\phi(-t_3 - s)| \\
&\leq |(T(t_2)x \cdot T(t_1 + s)y)\phi(-t_2 - s) \\
&\qquad -(T(t_3)x \cdot T(t_1 + s)y)\phi(-t_2 - s)| \\
&\quad +|(T(t_3)x \cdot T(t_1 + s)y)\phi(-t_2 - s) \\
&\qquad -(T(t_3)x \cdot T(t_1 + s)y)\phi(-t_3 - s)| \\
&\leq \|T(t_2)x - T(t_3)x\| \, \|y\| \, \|\phi(\cdot)\|_\infty \\
&\qquad +\|x\| \, \|y\| \, \|\phi(-t_2 - s) - \phi(-t_3 - s)\| \\
&\leq \frac{\varepsilon}{2\|\phi(\cdot)\|_\infty}\|y\| \, \|\phi(\cdot)\|_\infty + \|x\| \, \|y\| \frac{\varepsilon}{2\|x\|} \\
&= \varepsilon\|y\|.
\end{aligned}$$

Hence $\psi(\cdot)$ is uniformly norm-continuous and as such is in $Y_*$. Finally we show that $\beta\psi = f_a$. For $t \in S$ and $y \in N$,

$$\begin{aligned} \beta\psi(V(-t)\theta y) &= y(\psi(-t)) \\ &= (T(t)x \cdot T(s)y)\phi(-t-s) \\ &= \beta\phi(V(-s-t)\theta(T(t)x \cdot T(s)y)) \\ &= \beta\phi(a \cdot V(-t)\theta y) \\ &= f_a(V(-t)\theta y). \end{aligned}$$

For $a \in M$ not of the form $V(-s)\theta x$ the same result can be shown using an approximation argument and the above work. So, for all $a \in M$ and $f \in E$, we have proved that $f_a$ is also in $E$.

If $F \perp E$ and $f \in E$, then for all $a \in M$,

$$Ff(a) = F(f_a) = 0.$$

For any other $G \in M^{**}$, $GF(f) = G(Ff) = 0$ for all $f \in E$; thus $GF \perp E$ and $E^\perp$ is shown to be a left ideal. One may similarly demonstrate that $E^\perp$ is a right ideal.

Now that we know $E^\perp$ is an ideal, and since we also know it is $V^{**}$-invariant, we may deduce that $Y \simeq M^{**}/E^\perp$ is a von Neumann algebra equipped with a weakly-continuous group of $*$-automorphisms. By construction $\pi|_N : N \to Y$ is a $*$-homomorphism; it remains to prove that $\pi : X \to Y$ is also a $*$-homomorphism. For any $x, y \in X$ and $s_0 \in S_0$, $T_*(s_0)x$ is in the weak$^*$-closure of $N$ by lemma refL7, from which fact it may be shown that $U(s_0)\pi(xy^*) = U(s_0)(\pi(x)^* \cdot \pi(y))$. The result now follows from injectivity of $U(s_0)$.

# Notes

**Section 2.1** When $X$ is a Hilbert space and $S = \mathbf{Z}_+$, theorem 2.1.1 is due to Itô [24] and the equivalent $C_0$-semigroup version was proved by J. L. B. Cooper [13].

Douglas proved the Banach space version while attempting to simplify Itô's proof, but it seems that his result it not so well known as the Hilbert space versions. Arveson and Kishimoto give their own proof in their 1992 paper [8] saying that they did not know a reference for the result. I also proved the result independently.

A dilation theorem for contractions on a Hilbert space does exist, but it does not have the same form. In this version the isometry $B \in \mathcal{B}(H)$ is said to be a dilation of the contractive operator $A \in \mathcal{B}(K)$ if $K \leq H$ and $\langle A^n x, y \rangle = \langle B^n x, y \rangle$ for all $n \in \mathbf{Z}_+$ and $x, y \in K$. In [43] it was proved that any contraction on a Hilbert space has a dilation in this sense. A similar result for $C_0$-semigroups appeared in [14]. All of this material is well covered in [44].

**Section 2.2**

The definition of trajectories given differs slightly from that in some other literature. For example, in [33] a trajectory as defined here is referred to as a *bounded, complete trajectory*, but for the purposes of this thesis the briefer notation will cause no ambiguities. In [33], Phóng proves the existence of non-trivial trajectories for $C_0$-semigroups satisfying certain conditions and goes on to use them to prove results about the asymptotic behaviour of the semigroup. His existence results are all superseded by theorem 2.2.2.

**Section 2.4** The main idea behind the proof of the dual dilation theorem is due to C. J. K. Batty, who constructed it by considering a certain subspace of the dual of the standard dilated Banach space: dilate $T$ to a group of isometries $V$ on $Z$. Now define $Y_*$ to be the subspace of $Z^*$ that consists of $f$ for which $V(s)^* f$ is norm-continuous and $\theta^* V(s)^* f$ is weak*-continuous for each $s$ in $S$; this will be the pre-dual of the required minimal dual dilation. The treatment here in terms of trajectories for $T_*$ is my own version of this proof. The details of the two proofs are in essence identical. In particular both proofs require the additional assumption (S5), which my proof makes clear is required in lieu of any compactness of the unit ball in $X_*$. (Weak*-

compactness made construction of trajectories possible in $X^*$ for any semigroup by using Tychonoff's theorem.)

**Section 2.5**

The level of knowledge of $C^*$ and von Neumann algebra theory required for this section is minimal, but for a good general theory see for example [45], [31], or [39].

The minimality condition of dual dilations may be weakened in the von Neumann algebra case to

$$\{U(-s)\pi x : s \in S, x \in X\} \text{ is weak}^*\text{-dense in } Y.$$

This is because of Kaplansky's density theorem ([39, p. 11] or [31, theorem 2.3.3]).

The problem of dilating von Neumann algebras has been widely studied recently, but the collection of results obtained so far is rather motley. In [8] it is proved that it is always possible to dilate an $E_0$-semigroup on an $I_\infty$ factor (that is a representation of $\mathbf{R}_+$ by $*$-endomorphisms on some $\mathcal{B}(H)$) to a weakly continuous group of $*$-automorphisms on a larger $I_\infty$ factor. The method relies on a theorem specific to $\mathbf{R}_+$ however, so generalisation along those lines is difficult. The proof of theorem 2.5.1 was partially inspired by it however. A more serious problem is that the dilation is not minimal in the sense of this thesis and is not in any way unique among possible dilations.

Another recent paper is that of Dinh [16] who proved a dilation theorem for when $G$ is a countable, dense subgroup of the real line and $S$ is the subsemigroup of positive elements of $G$. Once again he proves that a representation of $S$ by $*$-endomorphisms on a type $I_\infty$ factor may be dilated to a group on a larger $I_\infty$ factor, and moreover his construction has a minimality property. He does have to assume that $T$ has an *intertwining semigroup* however, and his methods are again very specific to the particular example.

Theorem 2.5.1 is very frugal in the amount of information it gives concerning the structure of the dilated space. Indeed it will in general be so massive that any structure possessed by the original space will probably be entirely lost. Kishimoto has found an example of a representation of $\mathbf{Z}_+$ on an $I_\infty$ factor with several dilations, each on a space of different type. In general the identity element of $Y$ may be decomposed uniquely into the sum of $U$-invariant, central projections $p_I, p_{II}, p_{III}$, and $p_{IV}$, where each space $p_i Y$ is either trivial or is of type $i$. Furthermore, it would be possible to characterise all minimal von Neumann algebra dilations of $T$ in this way in terms of $U$-invariant, central projections in $Y$. I do not think much is to be gained by this approach however.

The research into this problem at the moment appears to be part of a larger thrust towards a classification theory which I do not pretend to understand. See for example [6, 7].

# Chapter 3

# Further Analysis of the Spectrum

## 3.1 A Fundamental Theorem

This chapter lays the foundations for the work in the final two chapters of the thesis by making a more systematic study of the spectrum of a semigroup representation. The methods involved are closely related to commutative Banach algebra theory, a general knowledge of which is assumed. (All the necessary background is contained in Stout [42] or Wermer [47] for example.)

The first theorem appeared in [10] although the proof given here is my own.

**Theorem 3.1.1** *If $T$ is a representation of $S$ by isometries on a non-trivial Banach space, then $Sp_u(T,S)$ is non-empty.*

**Proof:** Let $U : G \to \mathcal{B}(Y)$ denote the dilated group representation as given by theorem 2.1.1. As $X$ is non-trivial and $X \subseteq Y$ it follows that $Y$ is non-trivial and hence the Arveson spectrum of $U$, $Sp(U,G)$, is non-empty. By theorem 2.1.1, this is equal to $Sp_u(T,S)$ and hence that too must be non-empty.

The theorem fails to be true if the isometric condition is dropped, but it follows easily from the above theorem that the unitary spectrum of any bounded, non asymptotically stable representation is also non-empty ($T$ is non asymptotically stable if there exists an $x$ for which $T(s)x$ does not tend to zero as $s$ tends to infinity.) This will be covered by theorem 5.3.1.

## 3.2 S-Convexity and Šilov Boundaries

**Definition 3.2.1** *Let $S$ be a subsemigroup of $G$ satisfying conditions (S1) to (S4) given in chapter one. For any closed $E \subseteq S^*$, the $S$-**hull** (or $S$-**convex hull**) of $E$ is defined as follows:*

$$S\text{-}hull\ E = \{\chi \in S^* : |\hat{f}(\chi)| \leq \sup_{\gamma \in E} |\hat{f}(\gamma)| \ for\ all\ f \in L^1(S)\}.$$

*When $E = S$-hull $E$, $E$ will be said to be $S$-**convex**. Where ambiguity will not arise, the notation $\widehat{E}$ will sometimes be used instead of $S$-hull $E$.*

The best known use of this concept occurs when $S = \mathbf{Z}_+^n$, where it is known as polynomial convexity. Since $|\hat{f}(\gamma)| \leq \|f\|_1$ for all $\gamma \in S^*$, it is clear that for any dense subspace $Y$ of $L^1(S)$

$$S\text{-hull } E = \{\chi \in S^* : |\hat{f}(\chi)| \leq \sup_{\gamma \in E} |\hat{f}(\gamma)| \text{ for all } f \in Y\}.$$

Suppose $S = \mathbf{Z}_+^n$ and that $f$ is in the dense subspace of $l^1(\mathbf{Z}_+^n)$ consisting of elements with only finitely many non-zero values. Then, remembering that $S^*$ is the unit polydisc in $\mathbf{C}^n$,

$$\hat{f}(z_1, \ldots, z_n) = \Sigma_{i \in \mathbf{Z}_+^n} f(i_1, \ldots, i_n) {z_1}^{i_1} \ldots {z_n}^{i_n}$$

and in particular $\hat{f}$ is a polynomial in $n$ variables. Since all polynomials may be obtained from transforms of such $f$, we have

$$\mathbf{Z}_+^n\text{-hull } E = \{z \in \mathbf{C}^n : |p(z)| \leq \sup_{w \in E} |p(w)| \text{ for all polynomials } p\}$$

which is the definition of the polynomially convex hull. (Strictly speaking, the definition of $\mathbf{Z}_+^n$-hull would also require that $|z| \leq 1$, but it is not hard to see that this is automatically satisfied when the other conditions are.)

**Lemma 3.2.2** *Let $T$ be a bounded representation of $S$. For any closed subset $E$ of $Sp(T,S)$, the $S$-hull of $E$ is contained in $Sp(T,S)$. In particular $Sp(T,S)$ is $S$-convex.*

**Proof:** Each $\gamma \in E$ is also in $Sp(T,S)$, so $|\hat{f}(\gamma)| \leq \|\hat{f}(T)\|$ for all $f \in L^1(S)$; in particular $\sup_{\gamma \in E} |\hat{f}(\gamma)| \leq \|\hat{f}(T)\|$. It follows from this and from the definition of $S$-hull $E$ that if $\chi \in S$-hull $E$, then $|\hat{f}(\chi)| \leq \|\hat{f}(T)\|$.

Let $A$ be a commutative Banach algebra with Gelfand spectrum $M$. A closed subset $E$ of $M$ is said to be a boundary for $A$ if

$$|\hat{a}(\phi)| \leq \sup_{\psi \in E} |\hat{a}(\psi)|$$

for all $a \in A$ and $\phi \in M$. Clearly $M$ is itself a boundary for $A$. In fact there is always a minimal boundary for $M$ (in the sense that any other (closed) boundary for $A$ contains it); it is called the *Šilov boundary* of $A$ and is denoted $\check{S}(A)$. For any bounded representation $T$ of $S$, $\mathcal{A}_T$ is itself a commutative Banach algebra. The Šilov boundary of $\mathcal{A}_T$ may be seen as a closed subset of the spectrum of $T$. (See notes at the end of the chapter for references.)

**Lemma 3.2.3** *Let $T$ be a bounded representation of $S$. The spectrum of $T$ is the $S$-hull of the Šilov boundary of $\mathcal{A}_T$.*

**Proof:** If $\chi$ is in $Sp(T,S)$ but not in $S$-hull $\check{S}(\mathcal{A}_T)$, then there must exist an $f$ in $L^1(S)$ such that

$$|\hat{f}(\chi)| > \sup_{\gamma \in \check{S}(\mathcal{A}_T)} |\hat{f}(\gamma)|;$$

but this contradicts the fact that $\check{S}(\mathcal{A}_T)$ is a boundary for $A$. The other inclusion follows from lemma 3.2.2.

**Lemma 3.2.4** *If $T$ is a bounded representation of $S$, then $Sp_u(T,S) \subseteq \check{S}(\mathcal{A}_T)$.*

**Proof:** Let $\chi$ be in $Sp_u(T, S)$. Regarding $\chi$ as an element of $\Gamma$, and choosing any open neighbourhood $V$ of zero in $\Gamma$, we may find a function $f$ in $L^1(G)$ such that $\hat{f}(\chi) = 1$ and $\hat{f} = 0$ outside $\chi + V$ [35, theorem 2.6.1]. By lemma 1.1.6 there exists an $s$ in $S$ such that

$$\int_{G\setminus(S-s)} |f(t)|\, dt < \frac{1}{4}.$$

Setting $g_s(t) = f(t - s)$ for $t \in S$ we have $g_s \in L^1(S)$. For $\gamma$ in $\Gamma$

$$\begin{aligned} \left||\hat{g}_s(\gamma)| - |\hat{f}(\gamma)|\right| &\leq |\hat{g}_s(\gamma) - \gamma(s)\hat{f}(\gamma)| \\ &= \left|\int_S f(t-s)\gamma(t)\, dt - \int_G f(t)\gamma(t+s)\, dt\right| \\ &= \left|\int_{G\setminus(S-s)} f(t)\gamma(t)\, dt\right| \\ &\leq \int_{G\setminus(S-s)} |f(t)|\, dt \\ &< \frac{1}{4}; \end{aligned}$$

in particular $|\hat{g}_s(\gamma)| < \frac{1}{4}$ if $\gamma \notin V$, but $|\hat{g}_s(\chi)| > \frac{3}{4}$. So there exist elements of $\mathcal{A}_T$, $\hat{g}_s(T)$ for example, that peak on $\chi + V$, and since this is true for any neighbourhood $V$, $\chi$ must be in the Šilov boundary.

The next two results are of considerable value in the study of representations by isometries. They tells us that all the information obtainable from the spectrum of $T$ can in theory be deduced from the unitary part.

**Lemma 3.2.5** *If $T$ is a representation of $S$ by isometries, then*

$$\check{S}(\mathcal{A}_T) = Sp_u(T, S) = A\sigma_u(T, S).$$

**Proof:** The fact that the Šilov boundary of $\mathcal{A}_T$ is always contained in the approximate point spectrum is proved in [10, proposition 2.5], and the other implication comes from 3.2.4.

**Corollary 3.2.6** *For any representation $T$ of $S$ by isometries,*

$$Sp(T, S) = S\text{-}hull\ Sp_u(T, S).$$

*In particular, $Sp(T, S) = Sp_u(T, S)$ if and only if $Sp_u(T, S)$ is $S$-convex.*

## 3.3 Limitations of this Approach

Before applying corollary 3.2.6 and the other results above to the study of isometries, we digress to show some of the limitations of the Banach algebra approach in the non-isometric case. It would not be possible to expect that equality of the Šilov boundary of $\mathcal{A}_T$ and the unitary spectrum of $T$ should hold for general bounded representations: consider the case where $T$ is the representation of $\mathbf{Z}_+$ consisting of powers of $\frac{1}{2}I$; the spectrum of $T$ is $\{\frac{1}{2}\}$, but the unitary spectrum is empty. Could it be the case, however, that the Šilov boundary and the approximate point spectrum always coincide?

**Example 3.3.1** *There exists a bounded representation of $S$ for which the Šilov boundary of $\mathcal{A}_T$ differs from the approximate point spectrum of $T$.*

Let $E$ be a closed, non $S$-convex subset of $\Gamma$. The representation by isometries $T_{J_E}$ (as defined in chapter 1) has unitary spectrum $E$, and hence has spectrum $\widehat{E}$ by corollary 3.2.6. For ease of notation we shall call $T_{J_E}$ simply $T$, and similarly let $X$ denote $X_{J_E}$. Pick any $\chi$ in $\widehat{E} \setminus E$ (which is non-empty by assumption), and define $Y$ and $V$ as follows: let $Y$ be the Banach space $X \times \mathbf{C}$ with $\|(x, \lambda)\| = \max(\|x\|, |\lambda|)$, and let $V(s) : Y \to Y$ by

$$V(s) : (x, \lambda) \mapsto (T(s)x, \chi(s)\lambda).$$

$V$ is a bounded representation of $S$ on $Y$. It is easy to show that $\|\hat{f}(V)\| = \|\hat{f}(T)\|$ for any $f \in L^1(S)$, so $\mathcal{A}_V$ and $\mathcal{A}_T$ are isomorphic. In particular, the spectrum of $T$ is equal to that of $V$, and similarly $\check{S}(\mathcal{A}_T) = \check{S}(\mathcal{A}_V)$, both being $E$. The point spectra of $T$ and $V$ differ however. As the point spectrum is trivially contained in the approximate point spectrum, the example is found.

This example points out an important limitation of the notion of spectrum considered here, for while it does give a surprising amount of information concerning the representation, it is somewhat clumsy at distinguishing between non-isometric representations. Even for isometric $C_0$-semigroups it is less precise than simply considering the spectrum of the generator.

For example, if we consider the right-shift operators on $L^1(\mathbf{R})$, $T$ say, then $T$ has unitary spectrum $i\mathbf{R}$ and corollary 3.2.6 implies $Sp(T, S) = \mathbf{C}_-$. But if we consider the generator $A$ of $T$, then $\sigma(A)$ is only $i\mathbf{R}$. Put loosely, one might say that if the unitary spectrum of $T$ is large, then it 'covers up' the details that would otherwise be given by the non-unitary part. This is a direct consequence of corollary 3.2.6. The best results obtained so far from this notion of spectrum arise when the unitary spectrum is small; in particular when it is $S$-convex, and it to these cases which we will shortly turn our attention. First however, we will show that, even when the unitary spectrum of the representation is $S$-convex, the above example can still occur.

**Example 3.3.2** *Assume that $S^0$ is dense in $S$. For any closed subset $E$ of $\Gamma$ there exists a representation of $S$ with unitary spectrum $E$ for which the approximate point spectrum and Šilov boundary differ.*

To prove this we require the following two lemmas, which are of independent interest.

**Lemma 3.3.3** *Assume $S^0$ to be dense in $S$ and let $T$ be any bounded representation of $S$.*

*If $\chi \in S^*$ is such that there exists $M < \infty$ and a compact $C \subseteq S$ for which*

$$\|T(s)\| \leq M|\chi(s)|$$

*holds for all* $s \in S \setminus C$*, then* $Sp(T,S) \subseteq \chi S^* = \{\chi\gamma : \gamma \in S^*\}$*, where* $\chi\gamma$ *is the element of* $S^*$ *defined by* $\chi\gamma(s) = \chi(s)\gamma(s)$*.*

**Proof:** Suppose $\lambda \in Sp(T,S) \setminus \chi S^*$. An element $\lambda$ of $S^*$ is in $\chi S^*$ if and only if $|\lambda(s)| \leq |\chi(s)|$ for all $s \in S$, so here we are assuming that there exists a $t \in S^0$ such that $|\lambda(t)| > |\chi(t)|$. If $K$ denotes the closure in $S$ of the set $\{nt : n \in \mathbf{Z}_+\}$, then $K$ is a subsemigroup of $S$ not wholly contained in $C$. For if it were, then $K$ would itself be a compact subsemigroup, which is only possible when it is also a group. So in particular, $-t \in K$ and hence $|\lambda(t)| = |\chi(t)| = 1$, contradicting the assumption on $t$. Let $n$ be large enough such that $nt \notin C$ and also

$$|\lambda(nt)| > 2M|\chi(nt)|.$$

Since $nt \in S^0$ there exists an open subset $V$ of $S \setminus C$ for which $|\lambda(v)| > 2M|\chi(v)|$ holds whenever $v \in V$. We may assume further that $V$ has compact closure which does not intersect with $C$. Define $f \in L^1(S)$ by

$$f(s) = \begin{cases} \frac{1}{\lambda(s)} & s \in \overline{V} \\ 0 & \text{otherwise} \end{cases}.$$

But for any $x \in X$,

$$\begin{aligned} \|\hat{f}(T)x\| &\leq \int_{\overline{V}} \frac{\|T(s)x\|}{|\lambda(s)|}\, ds \\ &\leq \int_{\overline{V}} \frac{M|\chi(s)|}{2M|\chi(s)|} \|x\|\, ds = \frac{|\overline{V}|}{2}\|x\|. \end{aligned}$$

This leads to a contradiction since $|V| \neq 0$ and $\lambda \in Sp(T,S)$ implies that $|V| = \hat{f}(\lambda) \leq \|\hat{f}(T)\|$.

When $S = \mathbf{Z}_+$ this lemma is equivalent to the fact that the spectral radius of $T(1)$ is less than or equal to $\lim_{n\to\infty} \|T(1)^n\|^{\frac{1}{n}}$. When $S = \mathbf{R}_+$ it is the well known fact that if $\|T(s)\| \leq Me^{-|\lambda|s}$ for all $s$, then for all $\mu > -|\lambda|$, $\mu \notin \sigma(A)$.

**Lemma 3.3.4** *Assume that* $S^0$ *is dense in* $S$ *and let* $T$ *be a bounded representation of* $S$ *on* $X$*. For any* $\chi$ *in* $S^*$ *such that* $\chi(s)$ *is never zero, the set of operators* $V$

*defined by* $V(s) = \chi(s)T(s)$ *for* $s$ *in* $S$ *form a bounded representation of* $S$ *on* $X$*; furthermore*

$$Sp(V, S) = \{\chi\gamma : \gamma \in Sp(T, S)\},$$

*and similar equations relate the point and approximate point spectra of* $T$ *and* $V$.

**Proof:** The fact that $V$ forms a bounded representation is trivial. By lemma 3.3.3, any character in the spectrum of $V$ may be written in the form $\chi\gamma$ for some $\gamma$ in $S^*$. Now $\chi\gamma \in Sp(V, S)$ if and only if

$$|\hat{f}(\chi\gamma)| \leq \|\hat{f}(V)\|$$

for all $f$ in $C(S)$ with compact support (that is, all $f$ in $C_c(S)$). If we let $f\chi$ denote the function $f(\cdot)\chi(\cdot)$, then this may be restated as $\chi\gamma \in Sp(V, S)$ if and only if

$$|\widehat{f\chi}(\gamma)| \leq \|\widehat{f\chi}(T)\|$$

for all $f \in C_c(S)$. The function $f = g/\chi$ is in $C_c(S)$ for any $g \in C_c(S)$; so

$$|\hat{g}(\gamma)| = |\widehat{f\chi}(\gamma)| \leq \|\widehat{f\chi}(T)\| = \|\hat{g}(T)\|.$$

We have shown that $\gamma$ is in $Sp(T, S)$. The other inclusion is even more straightforward, as are the results for the point and approximate point spectra.

When this lemma is restated in terms of $C_0$-semigroups it is similar to the elementary fact that if $A$ is the generator of $T$ and if Re $\lambda \leq 0$, then $A - \lambda I$ is the generator of the $C_0$-semigroup $e^{-\lambda s}T(s)$ and, of course, $\sigma(A - \lambda I) = \{\mu - \lambda : \mu \in \sigma(A)\}$. (In fact it is not exactly this result for while $\sigma(A) \cap i\mathbf{R} = Sp_u(T, \mathbf{R}_+)$ is always true, $\sigma(A) \neq Sp(T, \mathbf{R}_+)$ in general.)

Returning to example 3.3.2, the proof is now straightforward. Let $T$ be some example of a representation for which the approximate point spectrum differs from the Šilov boundary, and let $\chi$ be in $S^* \backslash S_u^*$; $\chi T$ is then another example of a representation for which the approximate point spectrum differs from the Šilov boundary. For any closed subset $E$ of $\Gamma$ we know that there exist examples of representations with Šilov boundary $E$. In a manner similar to the proof of example 3.3.1 above, we may form a new Banach space by adding together these two spaces and similarly construct a new representation. It is elementary to check that this new representation has the desired properties.

## 3.4 Advantages of this Approach

Having shown some of the limitations, we now turn to some benefits of studying the Banach algebra $\mathcal{A}_T$.

**Lemma 3.4.1** *Let $T$ be a representation of $S$ by isometries. If $Sp_u(T, S)$ is countable, then $Sp(T, S) = Sp_u(T, S)$.*

**Proof:** The result follows from lemma 3.2.5 and a standard result of commutative Banach algebra theory which states that if the Gelfand spectrum is not equal to the Šilov boundary, then the Šilov boundary is uncountable. (See for example [42, corollary 11.4].)

**Lemma 3.4.2** *Let $T$ be a representation of $S$ by isometries on $X$. If $\chi$ is isolated in $Sp_u(T, S)$ (regarded as a subset of $\Gamma$), then $\chi$ is isolated in $Sp(T, S)$.*

**Proof:** If a point is isolated in the relative topology of the Šilov boundary of a commutative Banach algebra, then it is isolated in the whole Gelfand spectrum. This is apparently a standard result from the theory of Banach algebras, but the only reference I know of it comes as an exercise in [47, p. 55]. So I shall give my own proof here.

Let $A$ be a commutative Banach algebra with Gelfand spectrum $M$. Let $E$ denote the Šilov boundary of $A$ minus the (relatively) isolated point $\chi$. $E$ is obviously closed and moreover it is cannot be a boundary for $A$. In particular $\chi$ is not in the $A$-hull of $E$ and hence there exists an $a \in A$ such that $\hat{a}(\chi) > 1$ and $\sup_{\lambda\in E} |\hat{a}(\lambda)| < 1$.

Assume for contradiction that $\chi$ is not isolated in $M$. Then by continuity of $\hat{a}$ there exists a $\gamma$ in $M$, not equal to $\chi$, such that $|\hat{a}(\gamma)| > 1$. Let $b \in A$ be such that $\hat{b}(\chi) = 0$ and $\|b\| \leq 1$. If for all such $b$ it followed that $\hat{b}(\gamma) = 0$, then the ideal associated with $\gamma$ would contain that of $\chi$ which contradicts maximality; let us assume further therefore that $\hat{b}(\gamma) \neq 0$. Taking powers of $a$ if necessary, we could also assume that $\sup_{\lambda\in E} |\hat{a}(\lambda)| < |\hat{b}(\gamma)|$; but we would then be in the impossible situation in which $\widehat{ab}(\chi) = 0$, and

$$\sup_{\lambda\in E} |\widehat{ab}(\lambda)| < \|b\|\, |\hat{b}(\gamma)| < |\widehat{ab}(\gamma)|.$$

The proof of the lemma is now easy: $Sp_u(T,S)$ is equal to $\check{S}(\mathcal{A}_T)$ by lemma 3.2.5, so any $\chi$ isolated in $Sp_u(T,S)$ is isolated in $Sp(T,S)$ by the above.

In the following lemma we use the usual notations: $T$ is a bounded representation of $S$ on a Banach space $X$, $\mathcal{A}_T = \{\hat{f}(T) : f \in L^1(S)\}^-$, and for each $\chi \in Sp(T,S)$, $\phi_\chi$ is the extension to the whole of $\mathcal{A}_T$ of $\hat{f}(T) \mapsto \hat{f}(\chi)$.

**Lemma 3.4.3** *If $P$ is an idempotent element of $\mathcal{A}_T$, and if $V$ denotes the bounded representation of $S$ obtained by restricting $T$ to $\{Px : x \in X\}$, then*

$$Sp(V,S) = \{\chi \in Sp(T,S) : \phi_\chi(P) = 1\}.$$

A few things should be noted before we prove this lemma. Firstly $P$ commutes with each $T(s)$, so the subspace $P[X]$ is $T$-invariant. It follows immediately that $V$ is a well-defined, bounded representation. Secondly, since each $\phi_\chi$ is a character of $\mathcal{A}_T$, it must happen that either $\phi_\chi(P) = 1$ or $\phi_\chi(P) = 0$.

**Proof:** Let $W$ denote $P[X]$. If $f \in L^1(S)$, then

$$P\hat{f}(T) = 0 \iff \hat{f}(T)Px = 0 \text{ for all } x \in X$$

$$\iff \hat{f}(T)y = 0 \text{ for all } y \in W$$
$$\iff \hat{f}(V) = 0.$$

We may thus see that $\alpha : P\hat{f}(T) \mapsto \hat{f}(V)$ is a well-defined one-to-one map from $\{P\hat{f}(T) : f \in L^1(S)\}$ to $\mathcal{A}_V$. Furthermore, since

$$\begin{aligned}
\|\hat{f}(V)\| &= \sup_{\|y\|\leq 1} \|\hat{f}(V)y\| = \sup_{\|y\|\leq 1} \|\hat{f}(T)y\| \\
&= \sup_{\|y\|\leq 1} \|\hat{f}(T)Py\| \leq \|\hat{f}(T)P\|,
\end{aligned}$$

we see that $\alpha$ is continuous and may therefore extend it to a map from $P\mathcal{A}_T = \{PB : B \in \mathcal{A}_T\}$ to $\mathcal{A}_V$. $\alpha$ is easily seen to be a homomorphism. For $f \in L^1(S)$,

$$\begin{aligned}
\|\hat{f}(V)\| &= \sup_{\|y\|\leq 1} \|\hat{f}(V)y\| \\
&= \frac{1}{\|P\|} \sup_{\|y\|\leq\|P\|} \|\hat{f}(T)y\| \\
&\geq \frac{1}{\|P\|} \sup_{\|x\|\leq 1} \|\hat{f}(T)Px\| \\
&= \frac{1}{\|P\|}\|\hat{f}(T)P\|.
\end{aligned}$$

We have shown that $\alpha$ is a homeomorphism from $P\mathcal{A}_T$ to $\mathcal{A}_V$.

It is a triviality that $Sp(V,S) \subseteq Sp(T,S)$. In the other direction, suppose $\chi \in Sp(V,S)$ and consider $\alpha^*(\phi_\chi)$. This is easily seen to be a multiplicative functional on $\mathcal{A}_T$, but it may be zero. To see when this occurs, note that for any $f \in L^1(S)$,

$$\alpha^*(\phi_\chi)(P\hat{f}(T)) = \phi_\chi(\hat{f}(V)) = \hat{f}(\chi);$$

in other words $\alpha^*(\phi_\chi)$ is $\phi_\chi$ (thought of as a character on $\mathcal{A}_T$) restricted to $P\mathcal{A}_T$. But $P\hat{f}(T) = PP\hat{f}(T)$ for all $f$, so $\alpha^*(\phi_\chi)$ will be non-zero if and only if $\phi_\chi(P) = 1$.

In order to make use of the above lemma we will need to find suitable idempotents. But this is easy as the following theorem shows.

**Theorem 3.4.4** *Let $T$ be a bounded representation of $S$ on $X$. If $E$ is a compact, open subset of $Sp(T,S)$, then there exists a closed, $T$-invariant subspace $W$ of $X$ for which $Sp(T|_W, S) = E$.*

**Proof:** The Šilov idempotent theorem gives us the $P$ required to apply the preceding lemma.

Note that the condition that $E$ is compact may be removed if more care is taken over the condition of openness: if we adjoin the identity to $\mathcal{A}_T$, then the Gelfand spectrum of this will be $Sp(T, S)$ together with a possible point at infinity. If the closure of $E$ in this larger set is open, then we may conclude the same result. For the purposes of this thesis however, we require only the compact case. The converse of the theorem is clearly false.

**Theorem 3.4.5** *If $\chi$ is isolated in the unitary spectrum of $T$, a representation of $S$ by isometries, then $\chi$ is in the point spectrum of $T$.*

**Proof:** By lemma 3.4.2 and theorem 3.4.4 there exists a (necessarily non-trivial) closed subspace $W$ of $X$ for which the restriction $V$ of $T$ to isometries on $W$ has spectrum $\{\chi\}$. Let $(U, Y, \pi)$ denote the dilation of $(V, W)$ given by theorem 2.1.1. For any $f \in L^1(S)$, $\pi\hat{f}(V) = \hat{f}(U)\pi$, so if in particular $\hat{f}(\chi) = 0$, then $\hat{f}(U) = 0$ and hence $\hat{f}(V) = 0$. (The fact that $\hat{f}(U) = 0$ if $\hat{f}(\chi) = 0$ follows almost immediately from the definition of the Arveson spectrum [31, 8.1.6].)

So, for all $f \in L^1(S)$ and $w \in W$,

$$\hat{f}(V)(V(s)w - \chi(s)w) = (f_s - \chi(s)f)\hat{\ }(V)w = 0,$$

which implies that $T(s)w = V(s)w = \chi(s)w$ for all $w \in W$.

The converse to the theorem is obviously false. Consider for example the closed linear span of $\Gamma$ in $L^\infty(G)$ with the shift operators forming a group representation; in this case the point spectrum equals the unitary spectrum, both being $\Gamma$. The theorem is false for general bounded representations as the next example shows.

**Example 3.4.6** *There exists a bounded representation of $\mathbf{Z}_+$ with unitary spectrum $\{1\}$, for which $1 \notin P\sigma(T)$.*

Let $X = l^1$ and define $T : X \to X$ by

$$T : (x_n) \mapsto ((1 - \frac{1}{n})x_n) \qquad (x_n) \in l^1.$$

The spectrum is $\{1 - \frac{1}{n} : n \geq 1\} \cup \{1\}$ but 1 is not in the point spectrum. One could construct similar examples for other semigroups $S$. The following example is elementary given the results in this chapter.

**Example 3.4.7** *Let $E$ be a closed subset of $\Gamma$ and let $\hat{E}$ denote its $S$-convex hull. If $X$ is the closure of $\{\hat{f}|_{\hat{E}} : f \in L^1(S)\}$ in $C_0(\hat{E})$, and if $T(s) : X \to X$ by*

$$(T(s)f)(\chi) = \chi(s)f(\chi) \qquad (f \in X, \chi \in \hat{E}),$$

*then $T$ is a representation of $S$ by isometries with unitary spectrum $E$. If $E$ is not $S$-convex, then $T$ is not the restriction to $S$ of a group representation.*

To see the last statement, suppose that $\chi \in \hat{E} \setminus E$ and choose $f \in X$ with $f(\chi) = 1$. Given $\varepsilon > 0$, there exists $s \in S$ such that $|\chi(s)| < \varepsilon$. So if $T$ were invertible, then there would exist $g \in X$ satisfying $T(s)g = f$ and $\|g\| = \|f\|$. In particular $\chi(s)g(\chi) = 1$, from which it would follow that $\|g\| > 1/\varepsilon$; this is a contradiction. It is unclear whether $T$ is invertible when $E$ is $S$-convex.

**Lemma 3.4.8** *Let $E$ be a closed subset of $\Gamma$. An element $\chi$ of $S^*$ is in the $S$-hull of $E$ if and only if there exists a $\phi \in {J_E}^\perp$ such that $\phi|_S = \chi$.*

**Proof:** We know from theorem 1.4.3 that if $T$ is the representation by isometries on $X = \overline{L^1(S) + J_E}/J_E$ induced by the right-shift operators on $L^1(S)$, then $Sp_u(T, S) = E$. For $\chi \in S$-hull $E$ the linear functional

$$\phi_\chi : f + J_E \mapsto \hat{f}(\chi)$$

is bounded, and so it may be extended to a $\phi_\chi \in X^*$. From the definition of $X$ it follows that

$$X^* = {J_E}^\perp / (L^1(S) + J_E)^\perp.$$

In particular there must exist $\phi$ in $J_E{}^\perp$ such that $\hat{f}(\chi) = \int_S f(s)\phi(s)\,ds$ for all $f \in L^1(S)$, so it must be true that $\phi|_S = \chi$ almost everywhere.

In the other direction, if $\phi \in J_E{}^\perp$ is equal to $\chi$ when restricted to $S$, then by a similar line of argument $\chi \in Sp(T, S)$. From corollary 3.2.6 it follows that $\chi \in S$-hull $E$.

**Example 3.4.9** *If $E$ is not $S$-convex, then there exists an example of a dual representation of $S$ by isometries with unitary spectrum contained in $E$ that is not the restriction to $S$ of some dual representation of $G$.*

If $\chi$ is in $\widehat{E} \setminus E$, then by the above lemma there exists a $\phi$ in $J_E{}^\perp$ such that $\phi|_S = \chi$. Let $X$ be the weak*-closed subspace of $J_E{}^\perp$ spanned by all left-shifts of $\phi$; that is the set of all $\phi(\cdot + s)$ for $s \in S$. It is easy to show that for any $\psi \in X$ and $s, t \in S$, $\psi(s+t) = \chi(s)\psi(t)$. In particular $X$ is not right-shift invariant.

If $\gamma \in \Gamma \setminus E$, then there exists $f \in J_E$ such that $\hat{f}(\gamma) = 1$. Because $J_E{}^\perp$ is weak*-closed, it follows that $X \subseteq J_E^\perp$ and hence $J_E \subseteq X_\perp$. It is easy to deduce from this that $\hat{f}(T_*) = 0$, and thence $\gamma \notin Sp(T_*, S) = Sp(T, S)$.

**Theorem 3.4.10** *Let $T$ be a representation of $S$ by isometries. The following are equivalent.*

1. *$T$ is norm-continuous;*
2. *The unitary spectrum of $T$ is compact;*
3. *The spectrum of $T$ is compact.*

This result is a semigroup version of a group representation theorem: namely, a group representation $U$ is norm-continuous if and only if $Sp(U, G)$ is compact [31, 8.1.12].

**Proof:** 1 $\iff$ 2. Let $U : G \to \mathcal{B}(Y)$ denote the standard group dilation as given by theorem 2.1.1. Since $Sp_u(T, S) = Sp(U, G)$ the equivalence of 1 and 2 will follow directly from the group version once it is established that $T$ is norm-continuous if and only if $U$ is. One direction is trivial: if $U$ is norm-continuous, then that of $T$ follows immediately because $T(s) = U(s)|_X$ for all $s \in S$.

Assume instead that $T$ is norm-continuous and let $t \in G$; we will show that $U$ is norm-continuous at $t$. Let $V$ be some open neighbourhood of $t$ with compact closure, and let $\varepsilon > 0$. By lemma 1.1.5 there exists $s \in S$ such that $s + V \subseteq S$ and in particular $s + t$ is in the interior of $S$. Norm-continuity of $T$ implies the existence an open neighbourhood $W$ of 0 such that $\|T(s + t) - T(s + t + w)\| < \varepsilon$ when $w \in W$. It now follows that

$$\begin{aligned} \|U(t) - U(t + w)\| &= \|U(t + s) - U(t + w + s)\| \\ &= \sup_{\|y\| \leq 1} \|U(t + s)(I - U(w))y\| \\ &= \sup_{s' \in S} \sup_{\|x\| \leq 1} \|U(t + s - s')(I - U(w))\pi x\| \\ &= \sup_{\|x\| \leq 1} \|T(t + s)x - T(w + s + t)x\| \\ &< \varepsilon. \end{aligned}$$

Hence $U$ is also norm-continuous.

2 $\iff$ 3. Denote $Sp_u(T, S)$ by $E$. We know from corollary 3.2.6 that $Sp(T, S)$ is the $S$-hull of $E$, which we will denote by $\widehat{E}$. Assume for contradiction that $\widehat{E}$ is not compact. Let $\tilde{\mathcal{A}}_T$ denote the algebra $\mathcal{A}_T$ with the identity appended. The Gelfand spectrum of $\tilde{\mathcal{A}}_T$ is $\widehat{E}$ with a point added at infinity, $\phi_\infty : \hat{f}(T) + \lambda I \mapsto \lambda$. By [42, lemma 28.6], if $E$ is a compact subset of the Gelfand spectrum of a unital commutative Banach algebra, then $\widehat{E}$ is compact; in this context this means that $Sp(T, S)$ is compact. The other implication is trivial.

## 3.5 Dual Results

Some of the above results concerning representation by isometries have their equivalents for dual representation by isometries. A few of these will be given explicitly here. They may all be proved using lemmas 1.3.4 and 1.3.5 together with the appropriate strongly continuous results.

**Theorem 3.5.1** *If $T$ is a dual representation of $S$ by isometries on a non-trivial Banach space, then $Sp_u(T, S)$ is non-empty.*

**Theorem 3.5.2** *If $T$ is a dual representation by isometries then any isolated point in the unitary spectrum of $T$ is an eigenvalue for $T$.*

**Theorem 3.5.3** *If $T$ is a dual representation by isometries, then the following are equivalent.*

1. *$T_*$ is norm-continuous;*
2. *$T$ is norm-continuous;*
3. *The unitary spectrum of $T$ is compact;*

## Notes

**Section 3.1** The proof of theorem 3.1.1 that appeared in [10] is similar to the one presented here in that it reduces the problem to one of group representations. Their proof relies on a result of Arens [4, theorem 3.93] which in effect allows $T$ acting on the Banach algebra $\mathcal{A}_T$ to be dilated to a group representation on a larger algebra.

The non-emptiness of the Arveson spectrum is proved in [28, p. 91], but it quite easy to see anyway. Let $J = \{f \in L^1(G) : \hat{f}(U) = 0\}$; $J$ is clearly a closed ideal in $L^1(G)$. $U$ is non-trivial if and only if $J \neq L^1(G)$, so in this case the zero set of $J$

must be non-empty; but the zero set of $J$ is precisely the Arveson spectrum of $U$ by [31, proposition 8.1.9].

**Section 3.2** The notion of hulls is well established and all that is new here is the notation, which is specific to this situation. If $A$ is a commutative Banach algebra and $C$ is a closed subset of the Gelfand spectrum $M$ of $A$, then the $\hat{A}$-hull of $C$ is defined to be the set

$$\{\phi \in M : |\hat{a}(\phi)| \leq \sup_{\psi \in C} |\hat{a}(\psi)| \text{ for all } a \in A\},$$

and $C$ is $\hat{A}$-convex whenever $\hat{A}$-hull $C = C$. (See for example [42, section 28]). In our situation then, we could have used the notations $\widehat{L^1(S)}$-hull and $\widehat{L^1(S)}$-convex, but these seem unnecessarily long for the purposes of the thesis.

The Šilov boundary is often introduced in the context of uniform algebras (which are closed subalgebras of $C(K)$ containing the constants, where $K$ is a compact Hausdorff space). It is given in this form in [47] for example. If $X$ is a locally compact space, then a function algebra on $X$ is a subalgebra on $C_0(X)$ that separates points of $X$ from each other and from zero. In [42, theorem 7.4] it is proved that any function algebra possesses a minimal boundary. This still does not cover the case of general commutative Banach algebras such as $\mathcal{A}_T$, since in general they will not be semisimple, so in this case the Šilov boundary of the algebra $A$ always refers to the minimal boundary of the function algebra $\{\hat{a} : a \in A\}$, $\hat{a}$ being the Gelfand transform of $a$. In particular, $\check{S}(\mathcal{A}_T)$ is the minimal boundary of $\{\hat{f}|_X : f \in L^1(S)\}$, where $X = Sp(T, S)$. See for example [42, p. 43].

**Section 3.4** The results from Lemma 3.4.2 to theorem 3.4.5 are inspired by [10, proposition 4.1] where it is proved that if the unitary spectrum of a representation by isometries is countable, then the point spectrum is non-empty.

# Chapter 4

# Invertibility

## 4.1 Introduction and Examples

**Definition 4.1.1** *A representation $T$ of $S$ by isometries is* **invertible** *if for each $s \in S$, the operator $T(s)$ is invertible.*

If $T$ is invertible we may define $U$ as follows: for each $t$ in $G$, let

$$U(t) = T(s_1)T(s_2)^{-1}$$

for any $s_1, s_2 \in S$ such that $s_1 - s_2 = t$. The existence of such $s_1, s_2$ is guaranteed by condition (S3) of chapter one, and it is straightforward to show independence of choice. Suppose $t \in G$ and that $V$ is an open neighbourhood of $t$ with compact closure; then by lemma 1.1.5 there exists $s \in S$ such that $s + V \subseteq S$. In particular $s + t \in S^0$ and norm-continuity of $s + v \mapsto T(s + v)x$ implies that $v \mapsto U(v)x$ is also norm-continuous. Hence $T$ is the restriction to $S$ of the group representation $U$. An equivalent definition of invertibility would be that $T$ is invertible if and only the dilated Banach space given by theorem 2.1.1 is equal to the (undilated) original.

Consider the following examples.

**Example 4.1.2** *If $T$ is a representation by isometries with unitary spectrum $\{\chi\}$ for some $\chi \in \Gamma$, then $T$ is invertible.*

Indeed, in this case $T(s)x = \chi(s)x$ for all $s$ in $S$ and $x$ in $X$. (See the proof of theorem 3.4.5.)

**Example 4.1.3** *If $T$ is a representation of $\mathbf{Z}_+$ by isometries and if*

$$Sp_u(T, \mathbf{Z}_+) \neq \delta\mathbf{D},$$

*then $T$ is invertible.*

The unitary spectrum of the semigroup $\{T(n) : n \in \mathbf{Z}_+\}$ is equal to $\sigma(T(1)) \cap \delta\mathbf{D}$. It is a standard result in operator theory that the spectrum of an isometry is either the whole of $\mathbf{D}$ or is contained entirely in $\delta\mathbf{D}$. (This is because if the spectrum of $T(1)$ is not the whole of the unit disc, then there must be a boundary point of the spectrum in the interior of the disc. This is impossible however, since the boundary of the spectrum is contained in the approximate point spectrum, which in turn is contained in $\delta\mathbf{D}$ since $T(1)$ is an isometry.)

**Example 4.1.4** *If $T$ is a $C_0$-semigroup of isometries and if $Sp_u(T, \mathbf{R}_+) \neq i\mathbf{R}$, then $T$ is invertible.*

The unitary part of the spectrum of $T$ is equal to $\sigma(A) \cap i\mathbf{R}$ where $A$ is the generator. In [26], Lyubich and Phóng use an argument involving the Hille-Yosida Theorem to show that $T$ is invertible whenever $\sigma(A) \cap i\mathbf{R} \neq i\mathbf{R}$.

Motivated by these examples this chapter attempts to answer the following question: what conditions on a closed subset $E$ of $\Gamma$ are sufficient to ensure that any representation with unitary spectrum $E$ is invertible? Any set with this property will be called a set of automatic invertibility for $S$. So far the examples have indicated that the condition is merely $E \neq \Gamma$. The next example shows that this is not the case for general $S$.

**Example 4.1.5** *There exists a non-invertible representation of $\mathbf{Z}_+^2$ by isometries with unitary spectrum not equal to $(\delta\mathbf{D})^2$.*

Let $T$ be an isometry on some Banach space $X$ and let $E$ denote $\sigma(T) \cap \delta\mathbf{D}$. Define $U : \mathbf{Z}_+^2 \to \mathcal{B}(X)$ by

$$U(n,m) = T^{n+m} \qquad (n, m \in \mathbf{Z}_+).$$

Clearly $U$ is a representation by isometries which is invertible if and only if $T$ is invertible. Suppose $(\lambda, \mu)$ is in $Sp_u(U, \mathbf{Z}_+^2)$ so that $|\lambda| = |\mu| = 1$. We know that $(\lambda, \mu)$ must be in the approximate point spectrum of $U$ and hence there exists a sequence $(x_k)$ of norm one elements in $X$ such that

$$\|U(n,m)x_k - \lambda^n \mu^m x_k\| \longrightarrow 0$$

as $k \to 0$ for all $n, m \in \mathbf{Z}_+$. In particular

$$\|T^n x_k - \lambda^n x_k\| \longrightarrow 0$$

for each $n$ and hence $\lambda \in E$; similarly, $\mu \in E$. Furthermore, since $U(1,0) = U(0,1) = T$,

$$\begin{aligned} |\lambda - \mu| &= \|(\lambda - \mu)x_k\| \\ &\leq \|U(1,0)x_k - \lambda x_k\| + \|U(0,1)x_k - \mu x_k\| \\ &\longrightarrow 0 \end{aligned}$$

as $k \to \infty$. We must conclude that $\lambda = \mu$ and

$$Sp_u(U, \mathbf{Z}_+^2) = \{(\lambda, \lambda) : \lambda \in E\}.$$

If we choose a $T$ which is non-invertible, then by example 4.1.3, $E = \delta\mathbf{D}$. So in this case $Sp_u(U, \mathbf{Z}_+^2) = \{(\lambda, \lambda) : |\lambda| = 1\}$, which is not the whole of $\delta\mathbf{D}^2$ even though $U$ is non-invertible.

This final example is a companion to example 4.1.5.

**Example 4.1.6** *Let $T$ be an invertible isometry with $\sigma(T) = \delta\mathbf{D}$ and define*

$$U(n,m) = T^{n-m} \qquad n,m \in \mathbf{Z}_+.$$

*The unitary spectrum of $U$ is given by*

$$Sp_u(U, \mathbf{Z}_+^2) = \{(\lambda, \overline{\lambda}) : |\lambda| = 1\} := E.$$

The proof is similar to that of example 4.1.5. This time $U$ is invertible and while this does not, of course, constitute a proof that $E$ is a set of automatic invertibility for $\mathbf{Z}_+^2$, we will see by corollary 4.3.5 that this is in fact the case.

Topologically, the sets in examples 4.1.5 and 4.1.6 are identical even though one is a set of automatic invertibility and the other is not. The difference is an analytic one.

## 4.2 An Analytic Characterisation of Invertibility

For $E \subseteq \Gamma$ closed, $J_E$ is defined as in section 1.4. Let

$$X_E = \overline{L^1(S) + J_E/J_E},$$

$T_E$ be the right-shift semigroup representation on $X_E$, and let $\|\cdot\|_E$ denote the norm of $L^1(G)/J_E$. (In 1.4 these are called $X_{J_E}$, $T_{J_E}$, and $\|\cdot\|_{J_E}$ respectively; the reason for the change is merely to simplify notation.) By theorem 1.4.3, $Sp_u(T_E, S) = E$, so $Sp(T_E, S) = S$-hull $E$ by corollary 3.2.6.

**Lemma 4.2.1** *If $T$ is a representation of $S$ by isometries with unitary spectrum $E$ on some Banach space $X$, then for any $f$ in $L^1(S)$*

$$\|\hat{f}(T)\| \leq \|f + J_E\|_E.$$

**Proof:** Let $f$ be in $L^1(S)$ and $g$ be in $J_E$. If $U$ is the group dilation of $T$ as given by theorem 2.1.1, then $Sp(U, G) = E$. By the theory of isometric group representations [31, section 8.1], if $g \in J_E$, then $\hat{g}(U) = 0$. It follows that

$$\begin{aligned} \|\hat{f}(T)\| &= \sup_{\|x\|\leq 1} \|\hat{f}(T)x\| \\ &= \sup_{\|x\|\leq 1} \|\hat{f}(U)\pi x\| \\ &\leq \|\hat{f}(U)\| = \|\hat{f}(U) - \hat{g}(U)\| \\ &\leq \|f - g\|_1. \end{aligned}$$

Taking the infimum over all $g$ in $J_E$ gives the required inequality.

While this lemma looks rather straightforward, it in fact forms a basis for many of the results which follow. It is not true for bounded representations in general – consider for example any $C_0$-semigroup $T$ satisfying $\|T(t)\| \leq e^{-t}$ for all $t$. The first use of the lemma is the following theorem which is the principal analytic characterisation of the sets we seek.

**Theorem 4.2.2** *The following three conditions are equivalent:*

1. *$T_E$ is invertible;*

2. *$\overline{(L^1(S) + J_E)} = L^1(G)$;*

3. *Every representation of $S$ by isometries on a Banach space with unitary spectrum $E$ is invertible.*

**Proof:** Suppose 1 is true. Let $f$ be in $L^1(G)$ and $\varepsilon > 0$. By lemma 1.1.6 there exists an $s \in S$ satisfying

$$\int_{G\setminus(S-s)} |f(t)|\, dt < \frac{\varepsilon}{2};$$

if $g$ denotes the function $f \cdot 1_{S-s}$, this may be written as $\|g - f\|_1 < \frac{\varepsilon}{2}$. Because $g_s \in L^1(S)$, there exists by assumption an $h$ in $L^1(S)$ such that

$$\|T_E(s)(h + J_E) - (g_s + J_E)\|_E < \frac{\varepsilon}{2},$$

and hence there is a $k$ in $J_E$ satisfying

$$\|h + k - g\|_1 = \|h_s + k_s - g_s\|_1 < \frac{\varepsilon}{2}.$$

It now follows trivially that $\|f - (h + k)\|_1 < \varepsilon$ and thus $f \in \overline{(L^1(S) + J_E)}$.

Suppose 2 is true and let $T$ be any representation of $S$ by isometries with unitary spectrum $E$ on some Banach space $X$ . Choose $s_0$ in the interior of $S$ and suppose $s \in S$, $x \in X$ with $\|x\| = 1$, and $\varepsilon > 0$. By strong continuity of $T$, there exists a neighbourhood $V$ of $s_0$, contained in $S$, with the property

$$\|T(t)x - T(s_0)x\| < \frac{\varepsilon}{2}$$

holding for all $t$ in $V$. Without loss of generality assume that $|V| < \infty$.

Setting $f = \frac{1}{|V|}1_V$ we then have

$$\begin{aligned}
\|\hat{f}(T)x - T(s_0)x\| &= \left\| \int_V \frac{1}{|V|} T(t)x\,dt - T(s_0)x \right\| \\
&= \frac{1}{|V|} \left\| \int_V T(t)x - T(s_0)x\,dt \right\| \\
&< \frac{\varepsilon}{2}.
\end{aligned}$$

Regard $f$ as an $L^1(G)$ function and consider $f(\cdot + s + s_0)$. By assumption, there exists a $g$ in $L^1(S)$ with

$$\|g(\cdot) - f(\cdot + s + s_0) + J_E\|_E < \frac{\varepsilon}{2}.$$

If $y = \hat{g}(T)x$, then

$$\begin{aligned}
\|T(s)y - x\| &= \|T(s + s_0)\hat{g}(T)x - T(s_0)x\| \\
&\leq \|T(s + s_0)\hat{g}(T)x - \hat{f}(T)x\| + \|\hat{f}(T)x - T(s_0)x\| \\
&< \|\hat{g}_{s+s_0}(T) - \hat{f}(T)\| + \frac{\varepsilon}{2}.
\end{aligned}$$

By lemma 4.2.1 therefore,

$$\begin{aligned}
\|T(s)y - x\| &< \|g(\cdot - s - s_0) - f(\cdot) + J_E\|_E + \frac{\varepsilon}{2} \\
&< \varepsilon.
\end{aligned}$$

Since $T(s)$ is an isometry it follows that $T(s)[X]$ is closed and so the above shows $x$ to be in $T(s)[X]$; hence $T(s)[X] = X$ and $T(s)$ is invertible.

The final equivalence, 3 implies 1, is trivial given theorem 1.4.1.

The characterisation given by theorem 4.2.2, while interesting theoretically, is not very easy to use on a practical level. After all, the set of functions whose Fourier transforms annihilate a given set $E$ is difficult sometimes to find, and then there is the rather tricky problem of spectral synthesis to handle. The next three sections continue the investigation into sets of automatic invertibility in the hope of finding other, more practical characterisations. The first of these continues the analytic approach and deals especially with norm-continuous representations.

## 4.3 $S$-Convexity and Invertibility

**Theorem 4.3.1** *Let $E$ be a closed subset of $\Gamma$ (or equivalently of $S_u^*$). Each of the following statements implies those that follow it.*

1. *$L^1(S) + J_E$ is dense in $L^1(G)$;*
2. *$L^1(S) + I_E$ is dense in $L^1(G)$, where $I_E = \{f \in L^1(G) : \hat{f}|_E = 0\}$;*
3. *$\{\hat{f}|_E : f \in L^1(S)\}$ is dense in $C_0(E)$;*
4. *$E$ is $S$-convex.*

**Proof:** $1 \Longrightarrow 2$. This is trivial since $J_E \subseteq I_E$.

$2 \Longrightarrow 3$. We know from the Stone-Weierstrass approximation theorem that the set of $\hat{f}$ for $f$ in $L^1(G)$ is a dense subset of $C_0(\Gamma)$. So if $\varepsilon > 0$ and $h \in C_0(E)$, then there exists $f \in L^1(G)$ such that $\sup_{\chi \in E} |\hat{f}(\chi) - h(\chi)| < \frac{\varepsilon}{2}$. By assumption there exists a $g \in L^1(S)$ and a $k \in I_E$ satisfying $\|g + k - f\|_1 < \frac{\varepsilon}{2}$ and it is then trivial to show that $\sup_{\chi \in E} |\hat{g}(\chi) - h(\chi)| < \varepsilon$.

$3 \Longrightarrow 4$. This is really a special case of a standard result from Banach algebra theory [47, p. 55]. Suppose $\chi \in S$-hull $E$; then by definition, the map from $\{\hat{f}|_E : f \in L^1(S)\}$ to $\mathbf{C}$ via $\hat{f}|_E \mapsto \hat{f}(\chi)$ is continuous. Since the domain of this map is dense in $C_0(E)$, it may be uniquely extended to a multiplicative functional on $C_0(E)$. This must correspond to one of the characters of $C_0(E)$; in other words $\chi \in E$.

In fact it is possible to skip very quickly from 1 to 4 by noting that the Gelfand spectrum of $\overline{(L^1(S)+J_E)}/J_E$ is the $S$-hull of $E$ while that of $L^1(G)/J_E$ is simply $E$. Since these algebras are identical it must be the case that $E$ is $S$-convex. In the following lemma $S^0$ denotes the interior of $S$, $\mathcal{A}_T$ is the closure in $\mathcal{B}(X)$ of $\{\hat{f}(T) : f \in L^1(S)\}$, and $\tilde{\mathcal{A}}_T$ is $\mathcal{A}_T$ augmented by the identity.

**Lemma 4.3.2** *If $T$ is norm-continuous, then for each $s$ in the closure of $S^0$, the interior of $S$, the following holds:*

$$\sigma(T(s)) \subseteq \{\chi(s) : \chi \in Sp(T,S)\} \cup \{0\}.$$

**Proof:** For each $s \in S^0$ and $\varepsilon > 0$ there exists a neighbourhood $V$ of $S$ such that $|V| < \infty$ and $\|T(s) - T(v)\| < \varepsilon$ for all $v \in V$. So if $f = 1/|V|\,1_V$, then $\|\hat{f}(T) - T(s_0)\| < \varepsilon$ and hence $T(s) \in \mathcal{A}_T$ for all $s \in S^0$.

The Gelfand spectrum of $\tilde{\mathcal{A}}_T$ is $Sp(T,S) \cup \{\phi_\infty\}$, where the characters are given by

$$\phi_\chi : \hat{g}(T) \mapsto \hat{g}(\chi),$$

for $\chi$ in the spectrum of $T$. (See lemma 1.3.2. For $f$ defined as above, it follows that

$$|\phi_\chi(\hat{f}(T)) - \chi(s)| = |\hat{f}(\chi) - \chi(s)| < \varepsilon.$$

Choosing a sequence of such $\hat{f}(T)$ approaching $T(s)$ it is clear that

$$\sigma_{\tilde{\mathcal{A}}_T}(T(s)) = \{\chi(s) : \chi \in Sp(T,S)\} \cup \{0\}.$$

The lemma now follows for $s \in S^0$ since clearly $\sigma(T(s)) = \sigma_{\mathcal{B}(X)}(T(s)) \subseteq \sigma_{\tilde{\mathcal{A}}_T}(T(s))$.

When $s \in S \cap \overline{S^0}$, $T(s)$ may be approximated in norm by elements of the form $T(s_0)$ where $s_0 \in S^0$. The general result then follows from the above.

**Theorem 4.3.3** *Let $E$ be a closed subset of $\Gamma$. If $E$ is compact and $S$-convex, then $L^1(S) + J_E$ is dense in $L^1(G)$.*

**Proof:** Let $s$ be in the interior of $S^0$ and let $T$ be any representation by isometries with unitary spectrum $E$. Since $E$ is $S$-convex it is immediate that $|\chi(s)| = 1$ for all $\chi \in Sp(T, S)$. By theorem 3.4.10 $T$ is norm-continuous, so lemma 4.3.2 implies that $\sigma(T(s)) \subseteq \{\lambda : |\lambda| = 1\} \cup \{0\}$. But this is only possible when $T(s)$ is invertible. For any $t \in S$, whether or not in the interior, $s + t \in S^0$ and hence $T(s + t)$ is invertible by the above; it follows trivially that $T(t)$ is also invertible. Theorem 4.2.2 now gives the result.

The exact converse of this theorem is untrue. To see this consider $S = \mathbf{R}_+$ and $E \neq i\mathbf{R}$ which, as mentioned in example 4.1.4 above, does imply automatic invertibility.

**Corollary 4.3.4** *Let $E$ be a compact subset of $\Gamma$. The following statements are equivalent.*

1. *Every representation of $S$ by isometries on a Banach space with unitary spectrum $E$ is invertible.*
2. *$L^1(S) + J_E$ is dense in $L^1(G)$;*
3. *$L^1(S) + I_E$ is dense in $L^1(G)$;*
4. *$\{\hat{f}|_E : f \in L^1(S)\}$ is dense in $C(E)$;*
5. *$E$ is $S$-convex.*

Note that this result covers all the norm-continuous cases by theorem 3.4.10.

**Corollary 4.3.5** *Let $E$ be a closed subset of the unit polydisc $(\delta\mathbf{D})^n$. The following statements are equivalent.*

1. *$l^1(\mathbf{Z}_+^n) + J_E$ is dense in $l^1(\mathbf{Z}^n)$;*

2. *$l^1(\mathbf{Z}_+^n) + I_E$ is dense in $l^1(\mathbf{Z}^n)$;*

3. *$\{\hat{f}|_E : f \in l^1(\mathbf{Z}_+^n)\}$ is dense in $C(E)$;*

4. *$E$ is polynomially convex.*

When $S = \mathbf{Z}_+^n$ it is also the case that a totally disconnected unitary spectrum implies invertibility though this is by no means a trivial result (except when $n = 1$). The work of H. Alexander and G. Stolzenburg has involved establishing topological conditions which imply polynomial convexity and from [41] and [1] the following corollary may be obtained.

**Corollary 4.3.6** *Let $T$ be a representation of $\mathbf{Z}_+^n$ by isometries on a Banach space. If $Sp_u(T, S)$ is contained in a Jordan arc, or if it is totally disconnected, then $T$ is invertible.*

## 4.4 Countable Unitary Spectra and Invertibility

**Proposition 4.4.1** *Let $T$ be a representation of $S$ by isometries and let $M$ denote the intersection $\cap_{s \in S} T(s)[X]$. $T$ induces a (unique) representation by isometries $U$ on the quotient space $X/M$ with the property that the unitary spectrum of $U$ contains no (relatively) isolated point.*

**Proof:** First it should be noted that $M$ is trivially a $T$-invariant vector subspace of $X$. Moreover each $T(s)[X]$ is closed since $T(s)$ is an isometry, and therefore $M$ is also closed. If $x \in M$ and $s \in S$, then $T(s)y = x$ for some $y \in X$; we will show that $y$ is also in $M$. For any $t \in S$, there exists by assumption a $z \in X$ such that $T(s+t)z = x$; so $T(s)(T(t)z - y) = 0$. But $T(s)$ is an isometry, so $T(t)z = y$; in particular, $y \in M$. It may now be seen that $T|_M$ is an invertible isometry, although of course $M$ could be trivial.

For $s \in S$ define $U(s) : X/M \to X/M$ by

$$U(s) : x + M \mapsto T(s)x + M \qquad (x \in X).$$

Since each $T(s)|_M$ is invertible, $U$ is a representation by isometries.

Suppose for contradiction that $\chi$ is isolated in $Sp_u(U, S)$. By theorem 3.4.5 $\chi$ must be an eigenvalue of $U$, so there exists $x \in X \setminus M$ such that $x + M$ is an associated eigenvector. Define $N$ to be the (necessarily closed) subspace spanned by $M$ and $x$. For any $s \in S$ we know that $U(s)(x+M) = \chi(s)(x+M)$, so there must exist a $y \in M$ such that $T(s)x = \chi(s)x + y$. In turn there must exist a $z \in M$ satisfying $y = T(s)z$ and we may rearrange the above to obtain

$$x = T(s)(\chi(-s)(x - z));$$

in particular $x$ is in $T(s)[X]$. This is true for all $s$, so we are forced to conclude that $x \in M$, contradicting the first assumptions on $x$.

Before the next theorem recall the following standard fact.

**Lemma 4.4.2** *If $E$ is a closed, countable subset of a locally compact Hausdorff space, then the set of isolated points of $E$ is dense in $E$. In particular if $E$ is non-empty, then it must have isolated points.*

**Proof:** $E$ is a locally compact Hausdorff space in its own right. For each point $x$ in $E$ let $V_x = E \setminus \{x\}$; $V_x$ is an open subset which is dense in $E$ if $x$ is not isolated. The intersection of all of the $V_x$ for non-isolated $x$ is dense in $E$ by Baire's Theorem; but this is precisely the set of isolated points.

**Theorem 4.4.3** *Let $T$ be a representation of $S$ by isometries. If the unitary spectrum of $T$ is countable, then $T$ is invertible.*

I have several different proofs of this theorem. The first one presented here is the most elegant and is also in keeping with other methods used in the chapter. The second (which in fact predates the first) is longer, but is in some ways more elementary and is interesting enough to be included as well. A third proof follows almost directly from a result due to Loomis [25] and from theorem 4.2.2, but the details are not given here.

**Proof of Theorem 4.4.3 (a):** Let $U$ be the induced representation on $X/M$ as given by proposition 4.4.1. For any $f \in L^1(S)$ and $x \in X$, $\hat{f}(U)(x+M) = \hat{f}(T)x + M$, so

$$\|\hat{f}(U)(x+M)\| = \|\hat{f}(T)x + M\| \leq \|\hat{f}(T)\| \, \|x\|.$$

Taking the infimum over all $x+m$ for $m \in M$ shows that $\|\hat{f}(U)\| \leq \|\hat{f}(T)\|$ and hence $Sp(U,S) \subseteq Sp(T,S)$.

If $Sp_u(U,S)$ is non-empty, then it must be countable and have an isolated point by lemma 4.4.2. But this contradicts proposition 4.4.1, so we are forced to conclude that $Sp_u(U,S)$ is empty. By theorem 3.1.1, this is impossible unless $X/M$ is the trivial space $\{0\}$; in other words $X = M$ and $T$ is an invertible representation.

**Corollary 4.4.4** *If $E \subseteq \Gamma$ is closed and countable, then $L^1(S)+J_E$ is dense in $L^1(G)$ and the set of (restricted) Fourier transforms of $L^1(S)$ functions is dense in $C_0(E)$.*

**Proof:** This follows from theorems 4.4.3, 4.2.2, and 4.3.1.

The second proof of theorem 4.4.3 requires a number of preliminary results. These are related to the work in chapter two and show the close link that exists between isometric semigroup representations and group representations even when the semigroup is not invertible. The following proposition is so elementary that I have not included the proof.

**Proposition 4.4.5** *Let $T$ be a bounded representation of $S$. If $g(\cdot)$ is a trajectory for $T^*$, then for any $f \in L^1(S)$, $h(\cdot)$ defined by*

$$h(s) : x \mapsto g(s)(\hat{f}(T)x)$$

*is also a trajectory for $T^*$. Moreover $h$ is uniformly norm-continuous in the sense that for any $\varepsilon > 0$ there exists an open neighbourhood of $0$, $V \subseteq G$, such that*

$$\|h(s) - h(t)\| < \varepsilon$$

*whenever $s - t \in V$.*

For brevity, here and subsequently uniform norm-continuity of a trajectory will simply be referred to as uniform continuity.

**Proposition 4.4.6** *Let $T$ be a bounded representation of $S$. If $g(\cdot)$ is a trajectory for $T^*$, $\varepsilon > 0$, and if $x$ is such that $g(0)x = 1$, then there exists a uniformly continuous trajectory $h(\cdot)$ for $T^*$ satisfying $|h(0)x - 1| < \varepsilon$. Furthermore, if $g(s) = 0$ for some $s \in S$, then there exists a $t \in S$ such that $h(t) = 0$.*

This is similar to lemma 2.4.5 with $T^*$ instead of $T_*$, and the proof is almost identical. The only real difference is the last statement, but this is straightforward.

**Corollary 4.4.7** *If $T$ is a bounded representation of $S$, then $T^*$ has a non-zero trajectory if and only if $T^*$ has a non-zero, uniformly continuous trajectory.*

The advantage of considering uniformly continuous trajectories is that the set of all such forms a closed subspace of $\mathcal{C}_T$, the space of all trajectories for $T^*$ (see section 2.2), on which the group of shifts $g(\cdot) \mapsto g(\cdot + t)$ is strongly continuous. On the whole of $\mathcal{C}_T$ this may not be the case. The following proposition sums up the important properties.

**Proposition 4.4.8** *Let $T$ be a bounded representation of $S$ and let $Z$ denote the space of all uniformly continuous trajectories equipped with the norm*

$$\|g(\cdot)\|_\infty = \sup_{t\in G} \|g(t)\|$$

*and with pointwise addition and scalar multiplication. For each $t \in G$, let $U(t) : Z \to Z$ be the isometry $U(t) : g(\cdot) \mapsto g(\cdot + t)$. $Z$ is a Banach space and $U$ is a group representation on $Z$ which satisfies $Sp(U,G) \subseteq Sp_u(T,S)$. Furthermore $P\sigma(U) = P\sigma_u(T^*)$.*

**Proof:** We know from lemma 2.2.4 that $\mathcal{C}_T$ is a Banach space and $Z$ is clearly a subspace. Also, since the uniform limit of uniformly continuous functions is continuous, $Z$ must be closed. The definition of uniformly continuous with respect to trajectories implies immediately that $U$ is strongly continuous and is hence a group representation.

If $f \in L^1(S)$, $h(\cdot) \in Z$, $t \in G$, and $x \in X$, then

$$\begin{aligned}(\hat{f}(U)h)(t)x &= \int_S f(s)U(s)h(t)x\,ds \\ &= \int_S f(s)h(s+t)x\,ds \\ &= \int_S h(t)(f(s)T(s)x)\,ds \\ &= h(t)(\hat{f}(T)x).\end{aligned}$$

It follows that $\|\hat{f}(U)\| \le \|\hat{f}(T)\|$ and hence $Sp(U,S) \subseteq Sp(T,S)$. Lemma 1.3.7 completes the proof of the first part.

If $\phi(\cdot)$ is such that $U(t)\phi(\cdot) = \chi(t)\phi(\cdot)$ for all $t \in G$, then in particular

$$T(t)^*\phi(0) = \phi(t) = \chi(t)\phi(0) \qquad (t \in S).$$

So if $\phi(\cdot) \neq 0$, then $\phi(0) \neq 0$ and hence $\chi$ is also an eigenvalue for $T^*$. On the other hand, if $\psi \neq 0$ is such that $T(t)^*\psi = \chi(t)\psi$ for all $t \in S$ and some $\chi \in \Gamma$, then $\phi(\cdot) = \chi(\cdot)\psi$ is an eigenvector for $U$.

We are now ready for the second proof of theorem 4.4.3.

**Proof of Theorem 4.4.3 (b):** Let $T$ be a representation of $S$ by isometries with countable unitary spectrum. Suppose for contradiction that $T$ is not invertible; then there must exist $s \in S$ with $T(s)$ not invertible. Since $T(s)$ is an isometry this must mean that $T(s)$ is not surjective and so there will exist a $g \in X^*$ such that $T(s)^*g = 0$ but $g(x) = 1$ for some $x$. By theorem 2.2.2 there exists a trajectory for $T^*$ through $g$, and by proposition 4.4.6 we deduce the existence of a non-zero, uniformly continuous trajectory $h$ satisfying $h(t) = 0$ for some $t \in G$. Let $Y^0$ be the set of all such trajectories.

If $h_1, h_2 \in Y^0$, then there exist $t_1, t_2 \in G$ such that $h_1(t_1) = h_2(t_2) = 0$. For any $\lambda \in \mathbf{C}$,

$$(h_1 + \lambda h_2)(t_1 + t_2) = T(t_2)^*h_1(t_1) + \lambda T(t_1)^*h_2(t_2) = 0;$$

thus $Y^0$ is a linear subspace of $Z$ (as defined in proposition 4.4.8). Let $Y$ denote the closure of $Y^0$ in $Z$. If $h \in Y^0$, then it is trivial to show that $U(t)h$ is also in $Y^0$ and it follows that $Y$ is also $U$-invariant. Since $Y$ is a subspace of $Z$, $Sp(U|_Y, G) \subseteq Sp(U, G) \subseteq Sp_u(T, S)$ which is countable. By theorem 3.1.1 $Sp_u(T, S)$ is non-empty (since $Y$ is non-trivial) and so by lemma 4.4.2 it contains an isolated point. By the standard theory of group representations any isolated point must be an eigenvalue. (Note that this follows from theorem 3.4.5 but that the group version is easier to prove.) So there must exist a $\chi \in \Gamma$ and an $h(\cdot) \in Y$ with $h(\cdot) \neq 0$ such that $U(t)h(\cdot) = \chi(t)h(\cdot)$ for all $t \in G$. I claim that this is a contradiction.

For any $g \in Y$ there exists $k \in Y^0$ such that $\|g - k\|_\infty \leq \|g\|_\infty/2$. By the definition of $Y^0$ there exists $t \in G$ with $k(t) = 0$, so it follows that $\|g(t)\| < \|g\|_\infty/2$. The contradiction comes from the fact that if $h$ is as above, then $h(s) = \chi(s)h(0)$ for all $s \in G$; in particular $\|h(t)\| = \|h\|_\infty$.

## 4.5 The case $S = \mathbf{R}_+^n$

**Lemma 4.5.1** *Let $T$ be a representation of $\mathbf{R}_+^n$ by isometries. If $Sp_u(T,S)$ is compact, then $T$ is invertible. In particular, any compact subset of $i\mathbf{R}^n$ is $\mathbf{R}_+^n$-convex.*

**Proof:** $T$ is norm-continuous by theorem 3.4.10, so lemma 4.3.2 implies that for each $s \in \mathbf{R}_+^n$,

$$\sigma(T(s)) \subseteq \{\chi(s) : \chi \in Sp(T, \mathbf{R}_+^n)\} \cup \{0\}.$$

Also by theorem 3.5.3, the whole spectrum must be compact, so it follows that for any $s \in \mathbf{R}_+^n$ the set $\{\chi(s) : \chi \in Sp(T, \mathbf{R}_+^n)\}$ is not the whole of $\mathbf{D}$; in fact it does not contain some open neighbourhood of 0. Since $T(s)$ is an isometry, either its spectrum is contained in $\delta\mathbf{D}$ or else it is the whole of $\mathbf{D}$. Since the latter is not true, we deduce that each $T(s)$ must be invertible.

The method of proof used will generalise slightly to other semigroups: for if $S$ is any semigroup with the property that for all $\chi \in S^*$, $\chi(s)$ is never zero, then the same result holds. This is also true for propositions 4.5.2 and 4.5.3 below.

**Proposition 4.5.2** *Let $T$ be a representation of $\mathbf{R}_+^n$ by isometries and let $M$ denote the intersection $\cap_{s \in \mathbf{R}_+^n} T(s)[X]$. $T$ induces a (unique) representation by isometries $U$ on the quotient space $X/M$ which has the property that the spectrum of $U$ has no non-empty, open, compact subsets.*

The proof of this is almost identical to that of proposition 4.4.1, with the role of theorem 3.4.5 being taken by lemma 4.5.1

**Proposition 4.5.3** *If $T$ is a representation of $\mathbf{R}_+^n$ by isometries, and if $Sp(T, \mathbf{R}_+^n)$ is the union of its open, compact sets, then $T$ is invertible.*

**Proof:** In the same manner as in the first proof of theorem 4.4.3, $Sp(U, \mathbf{R}^n_+) \subseteq Sp(T, \mathbf{R}^n_+)$. This is then the union of open, compact subsets which will contradict proposition 4.5.2 unless $M = X$ and $T$ is invertible.

As it stands this result is not really in the spirit of the rest of this chapter because the condition is placed on the whole of the spectrum rather than on the unitary part alone. That such a spectrum must in fact be entirely unitary follows from 3.2.6, 4.2.2, and 4.3.1, but that only helps after the event. The rest of this section is dedicated to fitting proposition 4.5.3 in with the other results of the chapter; this will culminate in theorem 4.5.8.

Define $m_n : \tilde{\mathbf{C}}^n \to \tilde{\mathbf{C}}^n$, where $\tilde{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, by

$$m_n : z_i \mapsto \frac{z_i + 1}{z_i - 1} \qquad i = 1, \ldots, n.$$

In particular $m_1(1) = \infty$, $m_1(\infty) = 1$, and $m_n$ is a self-inverting transform for any $n$.

**Proposition 4.5.4** *For $E$ a closed subset of $i\mathbf{R}^n$, the $\mathbf{R}^n_+$-convex hull of $E$ is given by*

$$\mathbf{R}^n_+\text{-}hull\ E = \{z \in \mathbf{C}^n_- : m_n(z) \in \mathbf{Z}^n_+\text{-}hull\ (\overline{m_n(E)})\}.$$

**Proof:** Assume first that $z \in \mathbf{C}^n_-$ is such that $m_n(z) \in \mathbf{Z}^n_+$-hull $(\overline{m_n(E)})$; we will show that $z$ is in the $\mathbf{R}^n_+$-hull of $E$. For any $f \in L^1(\mathbf{R}^n_+)$ define $g : \mathbf{D}^n \mapsto \mathbf{C}$ by

$$g(w) = \begin{cases} \hat{f}(m_n(w)) & \text{if } w_j \neq 1 \quad (j = 1, 2, \ldots, n) \\ 0 & \text{otherwise} \end{cases} .$$

The transform $m_n$ is analytic in the interior of $\mathbf{D}^n$ and $\hat{f}$ is analytic on $\mathbf{C}^n_- \setminus i\mathbf{R}^n$; so $g$ must also be analytic in the interior of $\mathbf{D}^n$. On $(\delta\mathbf{D})^n$, excluding those points where one of the co-ordinates is zero, it is easy to show that $g$ is continuous. For the rest of $(\delta\mathbf{D})^n$, if $w_1 = 1$ (for example), and $z_k \to w_1$, then $m_1(z_k) \to \infty$. Consequently $g(z_k) \to 0$ by the Riemann-Lebesgue Lemma. Summing up, $g$ is analytic on the interior and continuous on the whole of $\mathbf{D}^n$. Any such function may be approximated uniformly by polynomials on $\mathbf{D}^n$. Given $\varepsilon > 0$, let $p$ be a polynomial such that $|g(z) - p(z)| < \varepsilon$ for all $z \in \mathbf{D}^n$. Then

$$\begin{aligned} |\hat{f}(z)| &= |g(m_n(z))| \\ &\leq |p(m_n(z))| + \varepsilon \\ &\leq \sup_{v \in \overline{m_n(E)}} |p(v)| + \varepsilon \\ &= \sup_{x \in E} |p(m_n(x))| + \varepsilon \\ &\leq \sup_{x \in E} |g(m_n(x))| + 2\varepsilon \\ &= \sup_{x \in E} |\hat{f}(x)| + 2\varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$ it follows that $z \in \mathbf{R}^n_+$-hull $E$ and we have proved one inclusion. The second inclusion is more tricky.

Suppose $z \in \mathbf{R}^n_+$-hull $E$: so $|\hat{f}(z)| \leq \sup_{x \in E} |\hat{f}(x)|$ for all $f \in L^1(\mathbf{R}^n_+)$. We want to show that $m_n(z) \in \mathbf{Z}^n_+$-hull $\overline{m_n(E)}$, which is to say that we want $|p(m_n(z))| \leq \sup_{x \in E} |p(m_n(x))|$ for all polynomials $p$. The obvious approach would be to try to show that $p \circ m_n$ could be approximated uniformly by Fourier transforms of $L^1(\mathbf{R}^n_+)$ functions for all polynomials $p$ since the result would then be immediate. Unfortunately this is not the case. (Consider for example the polynomial that is constantly 1: then $p \circ m_n(z) = 1$ which cannot be approximated uniformly by transforms by the Riemann-Lebesgue lemma.)

Let $q$ be the polynomial $q(z) = (z_1 - 1)(z_2 - 1)\ldots(z_n - 1)$, and define $\widehat{A}$ to be the closure of $\{\hat{f} : f \in L^1(\mathbf{R}^n_+)\}$ in the topology of uniform convergence on $\mathbf{C}^n_-$. What we will prove is that

(1) For any polynomial $p$ the function $q(m_n(\cdot))p(m_n(\cdot)) \in \widehat{A}$, and

(2) that this is sufficient to imply the result.

(1) For $r = 0, 1, \ldots, n$ define

$$\mathcal{E}_r = \text{span}\,\{q(z)(1 - \lambda_1 z_1)^{-1} \ldots (1 - \lambda_r z_r)^{-1} : |\lambda_i| < 1\}^{-},$$

where the closure is taken in the topology of uniform convergence on $\mathbf{D}^n$. For $\lambda$ in the interior of $\mathbf{D}^n$, each $\mu_i = m_1(\lambda_i)$ for $i = 1, 2, \ldots n$ satisfies $\operatorname{Re}\mu_i < 0$. The function $f : \mathbf{R}_+^n \to \mathbf{C}$ defined by

$$f(t_1, t_2, \ldots, t_n) = \exp(\mu_1 t_1 + \ldots + \mu_u t_n)$$

is therefore in $L^1(\mathbf{R}_+^n)$. Furthermore

$$\hat{f}(m_n(z)) = (-\frac{1}{2})^n q(\lambda) q(z)(1 - \lambda_1 z_1)^{-1} \ldots (1 - \lambda_n z_n)^{-1},$$

so $\hat{f} \circ m_n \in \mathcal{E}_n$. The point is that if $g \in \mathcal{E}_n$, then it may be approximated uniformly on $\mathbf{D}^n$ by Fourier-Laplace transforms of linear combinations of such $f$. In other words if $g \in \mathcal{E}_n$, then $g \circ m_n \in \hat{A}$. We must therefore prove that $q(z)p(z) \in \mathcal{E}_n$ for all polynomials $p$.

**Inductive Hypothesis (I)** *Suppose* $1 \leq r \leq n$, $1 \leq k < \infty$, *and that* $p$ *is any polynomial depending only on the variables* $z_1, z_2, \ldots, z_{r-1}$. *If* $q(z)p(z) \in \mathcal{E}_{r-1}$ *and* $q(z)p(z){z_r}^i \in \mathcal{E}_r$ *for* $0 \leq i < k$, *then* $q(z)p(z){z_r}^k \in \mathcal{E}_r$.

**Proof:** If $g \in \mathcal{E}_{r-1}$, then it may be approximated by functions of the form $k_i(z) = q(z)(1 - \lambda_1 z_1)^{-1} \ldots (1 - \lambda_{r-1} z_{r-1})^{-1}$. So it is easy to see that for any $|\alpha| < 1$, $g(z)(1 - \alpha z_r)^{-1}$ is approximated by the functions $k_i(z)(1 - \alpha z_r)^{-1}$. In particular, since $q(z)p(z) \in \mathcal{E}_{r-1}$ by assumption, it follows that $q(z)p(z)(1 - \alpha z_r)^{-1} \in \mathcal{E}_r$. Also by assumption, $q(z)p(z){z_r}^i \in \mathcal{E}_r$ for $0 \leq i < k$, and thus

$$\sum_{i=k}^{\infty} q(z)p(z)\alpha^{i-k}{z_r}^i = \alpha^{-k} q(z)p(z) \left((1 - \alpha z_r)^{-1} - \Sigma_{i=0}^{k-1} \alpha^i {z_r}^i\right) \in \mathcal{E}_r$$

whenever $0 < |\alpha| < 1$. As $\alpha \to 0$ this function converges uniformly on $\mathbf{D}^n$ to $q(z)p(z){z_r}^k$, which therefore belongs to $\mathcal{E}_r$ as required.

We use the inductive hypothesis as follows: by definition, $q(z) \in \mathcal{E}_0$, so use of (I) tells us that $q(z){z_1}^k \in \mathcal{E}_1$ for any integer $k$. Hence any polynomial $p$ in one variable $q(z)p(z_1) \in \mathcal{E}_1$. Using this as a base case one may use (I) again to show that

$q(z)p(z_1){z_2}^k \in \mathcal{E}_2$ for any $k$ and deduce that $q(z)p(z) \in \mathcal{E}_2$ for any polynomial in two variables. Repeating this as often as necessary we have the required objective: for any polynomial $p$, the function $q(\cdot)p(\cdot) \in \mathcal{E}_n$.

(2) By straightforward approximation arguments it can be seen that for any $h \in \hat{A}$ and $z \in \mathbf{R}^n_+$-hull $E$, $|h(z)| \leq \sup_{x \in E} |h(x)|$. From part (1) we can now say that

$$\begin{aligned} |q(m_n(z))p(m_n(z))| &\leq \sup_{w \in m_n(E)} |q(w)p(w)| \\ &\leq 2^n \sup_{w \in m_n(E)} |p(w)| \end{aligned}$$

for all polynomials $p$. Replacing $p(w)$ by $p(w)^k$ and taking $k$-th roots, it follows that

$$|p(m_n(z))| \leq \left( \frac{2^n}{|q(m_n(z))|} \right)^{\frac{1}{k}} \sup_{w \in m_n(E)} |p(w)|.$$

Letting $k$ tend to infinity we deduce that $|p(m_n(z))| \leq \sup_{w \in m_n(E)} |p(w)|$, and hence $m_n(z) \in \mathbf{Z}^n_+$-hull $(\overline{m_n(E)})$.

At first sight this proposition may appear to say that $E$ is $\mathbf{R}^n_+$-convex if and only if $\overline{m_n(E)}$ is polynomially convex. But this is not the case, for while it does say that $E$ is $\mathbf{R}^n_+$-convex whenever $\overline{m_n(E)}$ is polynomially convex, it does not imply the reverse. If $w \in \mathbf{Z}^n_+$-hull $m_n(E)$ is such that $w_i = 1$ for some $i$, then it can never be of the form $m_n(z)$ for some $z$ in $\mathbf{C}^n_-$. In particular it may be possible that some point is in the hull of $\overline{m_n(E)}$, not be in $\overline{m_n(E)}$ itself, and yet $E$ could be still be $\mathbf{R}^n_+$-convex.

**Example 4.5.5** *There exists an $\mathbf{R}^2_+$-convex set $E \subseteq i\mathbf{R}^2$ such that $\overline{m_2(E)}$ is not polynomially convex.*

Define $E \subseteq i\mathbf{R}^2$ by

$$E = \{i(n, q_n) : n \geq q\},$$

where $q_n$ is an enumeration of the rationals. $E$ is countable and hence $\mathbf{R}^2_+$-convex by lemma 3.4.1. But

$$\overline{m_2(E)} = m_2(E) \cup \{(1, \lambda) : |\lambda| = 1\};$$

this is not polynomially convex since its hull contains $(z, \lambda)$ for all $|z| \leq 1$, $|\lambda| = 1$.

The following proposition is due to H. Alexander (private communication).

**Proposition 4.5.6** *For $n = 1, 2, \ldots$ define*

$$Z_n = \{z \in \mathbf{D}^n : z_j = 1 \text{ for some } 1 \leq j \leq n\}.$$

*If $E$ is a compact set such that $Z_n \subseteq E \subseteq Z_n \cup \delta\mathbf{D}^n$ and $E \setminus Z_n$ is the union of its compact, relatively open subsets, then $E$ is polynomially convex.*

**Proof:** The proof is by induction on $n$.

Case $n = 1$: $Z_1 = \{1\}$, so if $E \setminus Z_1$ contains a relatively open, compact subset, then in particular $E$ cannot be the whole of $\delta\mathbf{D}$ and hence $E$ is polynomially convex.

Suppose now that the result holds for $n - 1$ dimensions and that $E$ is as in the statement. We wish to show that if $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in \mathbf{Z}_+^n$-hull $E$, then $\zeta \in E$. There are three cases.

Case 1: $\zeta_j = 1$ for some $j$. This is simple since $\zeta \in Z_n \subseteq E$.

Case 2: $|\zeta_1| = 1$, $\zeta_1 \neq 1$. Let $E_0 = \{w \in \mathbf{C}^{n-1} : (\zeta_1, w) \in E\}$. Since $\{z \in \mathbf{Z}_+^n$-hull $E : z_1 = \zeta_1\}$ is a peak set in $\mathbf{Z}_+^{n-1}$-hull $E$ we have $(\zeta_2, \ldots, \zeta_n) \in \mathbf{Z}_+^n$-hull $E_0$. However, $Z_{n-1} \subseteq E_0 \subseteq Z_{n-1} \cup \delta\mathbf{D}^{n-1}$ and $E_0 \setminus Z_{n-1}$ is the union of its compact, relatively open subsets, so the inductive hypothesis implies that $\mathbf{Z}_+^{n-1}$-hull $E_0 = E_0$. By definition this means $\zeta \in E$.

Case 3: $|\zeta_1| < 1$, $\zeta_j \neq 1 (j = 2, 3, \ldots, n)$. It will be shown that this case contradicts the assumption that $\zeta \in \mathbf{Z}_+^n$-hull $E$.

Let $q$ be the polynomial

$$q(z) = (z_1 - 1)(z_2 - 1) \ldots (z_n - 1),$$

so that $q(z) = 0$ if $z \in Z_n$ and $q(\zeta) \neq 0$. If F is defined to be the set $\{z \in E : |q(z)| \geq \frac{1}{2}|q(\zeta)|\}$, then $F$ is a compact subset of $E \setminus Z_n$. Now since $E \setminus Z_n$ is the union of its compact, relatively open subsets, we can choose a compact, relatively open subset $E_1$

of $E \setminus Z_n$ such that $F \subseteq E_1$. Setting $E_2 = E \setminus E_1$ we note that $E_2$ is also compact and open in $E$. Moreover since $|q(z)| < \frac{1}{2}|q(\zeta)|$ for all $z \in E_2$, we have in particular $\zeta \notin \mathbf{Z}^n_+\text{-hull } E_2$.

Let $V_1$ and $V_2$ be disjoint , open subsets of $\mathbf{C}^n$ such that

$$E_1 \subseteq V_1 \subseteq \{z \in \mathbf{C}^n : z_1 = re^{i\theta},\, r > 0,\, 0 < \theta < 2\pi\} \qquad \text{and} \qquad E_2 \subseteq V_2.$$

Now, using cases 1 and 2 above,

$$\begin{aligned} \bigcap_{0<r<1} \{z \in \mathbf{Z}^n_+\text{-hull } E : r \le |z_1| \le 1\} &= \{z \in \mathbf{Z}^n_+\text{-hull } E : |z_1| = 1\} \\ &= \{z \in E : |z_1| = 1\} \subseteq V_1 \cup V_2. \end{aligned}$$

The set $(\delta\mathbf{D})^n \setminus (V_1 \cup V_2)$ is compact and we may thus choose an $r > 0$ such that

$$\{z \in \mathbf{Z}^n_+\text{-hull } E : r \le |z_1| \le 1\} \subseteq V_1 \cup V_2.$$

Let

$$Q = \{z \in \mathbf{Z}^n_+\text{-hull } E : r \le |z_1| \le 1\} \cap V_1;$$

$Q$ is compact and relatively open in $\{z \in \mathbf{Z}^n_+\text{-hull } E : r \le |z_1| \le 1\}$.

The argument is now completed by means of techniques originating in the work of Stolzenburg [41]. Since $z \mapsto \log z$ is holomorphic in the neighbourhood $V_1$ of $Q$, it follows from the Local Maximum Modulus Principle [41, (1.6)] that

$$\delta_{\mathbf{C}}\{\log z_1 : z \in Q\} \subseteq \{\log z_1 : z \in (\delta_{\mathbf{Z}^n_+\text{-hull } E} Q) \cup (Q \cap E)\}.$$

Now we may deduce from the relations

$$\delta_{\mathbf{Z}^n_+\text{-hull } E} Q = \{z \in Q : |z_1| = 1 \text{ or } r\}$$

and

$$Q \cap E \subseteq E_1 \subseteq \{z : |z_1| = 1\}$$

that

$$\delta_{\mathbf{C}}\{\log z_1 : z \in Q\} \subseteq \{w \in \mathbf{C} : \operatorname{Re} w = 0 \text{ or } \log r\}.$$

Since $\log z_1$ takes its values in a horizontal strip it follows that

$$\{\log z_1 : z \in Q\} \subseteq \{w \in \mathbf{C} : \text{Re } w = 0 \text{ or } \log r\},$$

and hence

$$Q \subseteq \{z \in \mathbf{D}^n : |z_1| = 1 \text{ or } \log r\}.$$

Thus the compact set $\{z \in Q : |z_1| = 1\}$ is relatively open in $\mathbf{Z}_+^n$-hull $E$, and since $E \setminus \{z \in Q : |z_1| = 1\} = E_2$, the Šilov idempotent theorem tells us that

$$\mathbf{Z}_+^n\text{-hull } E = \{z \in Q : |z_1| = 1\} \cup \mathbf{Z}_+^n\text{-hull } E_2.$$

But $\zeta \notin \mathbf{Z}_+^n$-hull $E_2$ and $|\zeta_1| < 1$, so this is a contradiction.

**Corollary 4.5.7** *Let $E$ be a closed subset of $i\mathbf{R}^n$. If $E$ is the union of its compact, relatively open subsets, then $\mathbf{R}_+^n$-hull $E = E$.*

**Proof:** If $m_n$ is as in proposition 4.5.4, then $m_n(E)$ is also the union of its compact, relatively open subsets and $\overline{m_n(E)}$ is contained in $m_n(E) \cup Z_n$. The set $Z_n \cup m_n(E)$ satisfies the conditions required by proposition 4.5.6 and must therefore be polynomially convex. The result follows from 4.5.4.

We are now prepared to state an improved version of proposition 4.5.3.

**Theorem 4.5.8** *Let $T$ be a representation of $\mathbf{R}_+^n$ by isometries on a Banach space. If $Sp_u(T, \mathbf{R}_+^n)$ is the union of its compact, relatively open subsets, then $T$ is invertible.*

**Proof:** By corollary 4.5.7 $Sp_u(T, S)$ is $\mathbf{R}_+^n$-convex and corollary 3.2.6 together with proposition 4.5.3 then give the result.

This theorem clearly covers the case when the unitary spectrum is totally disconnected and $S = \mathbf{R}_+^n$. In fact the condition that $E$ is the union of its relatively open, compact subsets is equivalent to saying that the connected components of $E$ are bounded. (I am grateful to Robin Knight for showing me a proof of this.)

## 4.6 Dual Invertibility Theorems

We will now ask whether a similar set of theorems hold for dual representations. Specifically, which closed subsets $E$ of $\Gamma$ have the property that any dual representation by isometries with unitary spectrum $E$ are necessarily invertible? The answer turns out to be exactly the same as for the strongly continuous cases.

**Theorem 4.6.1** *Let $E$ be a closed subset of $\Gamma$. The following are equivalent.*

1. $\overline{(L^1(S) + J_E)} = L^1(G)$;

2. *Every dual representation of $S$ by isometries with unitary spectrum $E$ is invertible.*

**Proof:** Suppose 1 is true and that $T$ has unitary spectrum $E$. Assume for simplcity that 0 is the the closure of $S^0$; the proof is similar for the more general case. By lemma 1.3.4 we know that $X$ has a weak*-dense subspace $Y$ on which $T$ is strongly continuous, and moreover the unitary spectrum of this reduced representation is contained in $E$ by lemma 1.3.5. Theorem 4.2.2 implies that this reduced representation is invertible.

For any $x \in X_*$,

$$\|x\| = \sup_{\|y\|\leq 1,\, y\in Y} |y(x)| = \sup_{\|T(-s)y\|\leq 1,\, y\in Y} |T(s)y(x)| = \sup_{\|y\|\leq 1,\, y\in Y} |y(T(s))x|.$$

In other words $T_*$ is itself isometric, and must therefore also be invertible.

In the other direction we have already shown in theorem 1.5.2 that if 1 fails to be true then there exists a non-invertible dual representation by isometries with unitary spectrum $E$.

## Notes

The literature on the work in this chapter is to my knowledge very small. Indeed I know of no-one else who has considered the questions found here other than in the one-parameter cases. The $\mathbf{R}_+$ case, covered by example 4.1.4, was proved in [26] as part of an asymptotic stability theorem. (See also theorem 5.3.1 of this thesis.) When this stability theorem was generalised to cover other semigroups in [10], the question of invertibility for general $S$ and countable unitary spectrum arose, but was avoided. My own work in this area began when I tried to answer this question and obtained theorem 4.4.3. Example 4.5.5 is due to C. J. K. Batty as is the idea of using the transformations $m_n$ in proposition 4.5.4.

The problem of characterising polynomially convex subsets of $\mathbf{C}^n$ is very difficult and has been a subject of many years research. See [42, pp. 368-402] for a glimpse into this area. The case when $E \subseteq \mathbf{D}^n$ is thankfully easier than for more general $E$, but even here there are still difficulties and I am very grateful to Professor Alexander for showing me a proof of proposition 4.5.6. Further analysis of the unit polydiscs, $\mathbf{D}^n$, may be found in [38].

Another very difficult area touched on in this chapter is that of characterising the semisimple Banach algebras $C_0(X)$ where $X$ is a locally compact or even compact Hausdorff space. This is perhaps the most important question tackled in Stout [42]. It arises when asking whether statement 4 of theorem 4.3.1 implies any of the others, particularly 3. We know by corollary 4.3.4 that the answer is yes when $E \subseteq \Gamma$ is compact, but it seems a hard question for non-compact $E$. Specifically, if $E$ is $S$-convex, then the closed subalgebra of $C_0(E)$ spanned by $\{\hat{f}|_E : f \in L^1(S)\}$ has the same Gelfand spectrum as $C_0(E)$. Does it follow that it equals $C_0(E)$? It is not true in general that a uniformly closed function algebra on $X$ with spectrum $X$ must be $C(X)$ and pathological examples exist which contradict many hypothesis that have been made. So if $4 \Rightarrow 3$ is true for theorem 4.3.1, then it will be by the good graces of the $L^1$ spaces and not of function algebras.

# Chapter 5

# Applications to Asymptotic Stability

## 5.1 Introduction

Let $S$ be a subsemigroup of a locally compact, abelian group $G$ and assume that $S$ satisfies conditions (S1)–(S4) of chapter one. An ordering of $S$ may be defined as follows: for $s, t \in S$

$$t \leq s \qquad \text{if and only if} \qquad s - t \in S \cup \{0\}.$$

The statement $s \to \infty$ will always mean convergence through this ordering. So $s_\alpha \to \infty$ as $\alpha \to \infty$ means that for all $t \in S$ there exists $\beta$ such that $s_\alpha - t \in S$ for all $\alpha > \beta$.

**Definition 5.1.1** *A bounded representation $T$ of $S$ is said to be asymptotically stable if for all $x$ in $X$,*

$$\lim_{s\to\infty} \|T(s)x\| = 0.$$

The final chapter of this thesis will apply ideas developed in the previous chapters to problems of asymptotic stability. It will be divided into three main sections, loosely corresponding to the methods and results of chapters 2, 3, and 4 respectively.

## 5.2 Asymptotic Stability and Trajectories

This section is the most abstract of the three and makes no demands on the spectrum of a representation. The notion of trajectory was defined in 2.2.1.

**Theorem 5.2.1** *Let $T$ be a bounded representation of $S$ on $X$. $T$ is asymptotically stable if and only if $T^*$ has no non-zero trajectories.*

**Proof:** We employ a construction due to Lyubich and Phóng which has since been called the *limit isometric semigroup*. For any $x$ in $X$ let

$$l(x) = \limsup_{s \to \infty} \|T(s)x\|,$$

and let $L = \{x : l(x) = 0\}$. Denote by $Y$ the Banach space obtained by completing the vector space $X/L$ with respect to the norm $\|x + L\|_0 = l(x)$. The operators $\tilde{T}(s) : Y \to Y$ induced by $T$ form a representation of $S$. They are isometries by construction.

There is a one-to-one correspondence between $Y^*$ and the subspace of $X^*$ consisting of those $g$ that satisfy

$$|g(x)| \leq Ml(x) \qquad \text{for all } x \in X \tag{5.1}$$

for some $M < \infty$. This correspondence is given by $g(x) = f(x + L)$ for $x \in X$.

Suppose $g$ satisfies 5.1 and let $f$ be the associated member of $Y^*$. By theorem 2.2.2, there is a trajectory for $\tilde{T}^*$ running through $f$; this trajectory corresponds to a mapping $G \to X^*$ given by the above mentioned correspondence which it is not difficult to show is itself a trajectory for $T^*$ running through $g$.

On the other hand, let $g(\cdot)$ be any trajectory for $T^*$. For all $x \in X$ and $s \in S$,

$$\begin{aligned} |g(0)x| &= |T(s)^* g(-s)x| \\ &= |g(-s)T(s)x| \\ &\leq \|g(\cdot)\|_\infty \|T(s)x\|, \end{aligned}$$

and hence $|g(0)x| \leq M \limsup_{s\to\infty} \|T(s)x\| = Ml(x)$, where $M = \|g(\cdot)\|_\infty$. In particular, there is a trajectory through $g \in X^*$ if and only if $g$ satisfies 5.1. The proof is completed by mentioning that such non-zero $g$ exist if and only if $Y \neq \{0\}$, which in turn is true only when $T$ is not asymptotically stable.

The first corollary to this theorem was part of its proof, but it is of sufficient interest to deserve its own statement.

**Corollary 5.2.2** *If $T$ is a bounded representation of $S$, then for each $g$ in $X^*$ there exists a trajectory for $T^*$ through $g$ if and only if $g$ satisfies*

$$|g(x)| \leq M \limsup_{s\to\infty} \|T(s)x\|$$

*for all $x \in X$ and some $M$.*

**Corollary 5.2.3** *If $T$ is a bounded representation of $S$, then for each $x \in X$*

$$\limsup_{s\to\infty} \|T(s)x\| \neq 0$$

*if and only if there exists a trajectory through some $g \in X^*$ satisfying $g(x) = 1$.*

Using proposition 4.4.6 and corollary 4.4.7, the above results can be sharpened as follows.

**Theorem 5.2.4** *Let $T$ be a bounded representation of $S$. For each $x \in X$, $T(s)x$ fails to tend to zero if and only if there exists a uniformly norm-continuous trajectory for $T^*$ through some $g \in X^*$ such that $g(x) = 1$. In particular $T$ is asymptotically stable if and only if there exist no non-zero, uniformly norm-continuous trajectories for $T^*$.*

When $T$ is a representation by isometries theorem 2.2.2 assures us of a plentiful supply of trajectories and is consistent with all the results above. So we will turn to various non-isometric cases.

**Example 5.2.5** *Let $X = L^1(S)$, where $S$ is such that $S \cap -S \subseteq \{0\}$, and let $T$ be the semigroup representation consisting of left-shift operators on $X$. $T$ is asymptotically stable and has no non-zero trajectories.*

For any non-zero $\phi$ in $L^\infty(S)$ there exists a set of positive Haar measure in $S$ such that $\phi \neq 0$ anywhere on that set. By regularity of the measure, there exists a compact subset $K$ of $S$, also with non-zero measure, on which $\phi$ is nowhere zero. Applying lemma 1.1.5 to the semigroup $-S$ there exists $s_0 \in S$ such that $(K - s_0) \cap S = \emptyset$. So for any $\psi \in L^\infty(S)$, $T^*(s_0)\psi$ will be zero on $K$ and in particular cannot equal $\phi$. As a consequence no complete trajectory can pass through $\phi$. Similarly it can be demonstrated that $T$ is asymptotically stable.

**Example 5.2.6** *Suppose $K$ is a compact subset of $\mathbf{C}_-$ and let $K_u$ denote $K \cap i\mathbf{R}$. Consider the multiplier $C_0$-semigroup on $X = C(K)$ given by*

$$(T(s)f)(z) = e^{sz}f(z) \qquad (s \in \mathbf{R}_+, z \in K).$$

*Corollaries 5.2.2 and 5.2.3 are verifiable in this case.*

The dual of $X$ is the space $M(K)$ of complex, regular measures on $K$. For $s \in S$, $T(s)^*\mu$ is given by $d(T(s)^*\mu)(\cdot) = e^{s\cdot}d\mu(\cdot)$. It follows that there exists a (bounded) trajectory through some $\mu \in M(K)$ if and only if $\operatorname{supp}\mu \in K_u$. On the other hand, for $f \in C(K)$, $T(s)f$ will converge to zero exactly in those cases when $f|_{K_u} = \emptyset$; in fact $l(f) = \sup_{\lambda \in K_u} |f(\lambda)|$. It is now easy to verify the corollaries.

## 5.3 Countable Unitary Spectra and Asymptotic Stability

When $Sp_u(T, S)$ is countable, the use of trajectories in the study of asymptotics may be entirely replaced by the use of unitary eigenvectors. The rest of the chapter will be devoted to studying this case and justifying the claim.

The first example is the A-B-L-P theorem (named after its discoverers: W. Arendt, C. J. K. Batty, I. Lyubich, and V. Q. Phóng), which is stated here in a way to show its connection to theorem 5.2.1.

**Theorem 5.3.1** *If $T$ is a bounded representation of $S$ and if $Sp_u(T,S)$ is countable, then $T$ is asymptotically stable if and only if $P\sigma_u(T^*) = \emptyset$.*

**Proof:** Let $Z$ be the space of all uniformly norm-continuous trajectories of $T^*$. If $T$ is not asymptotically stable, then $Z \neq \{0\}$ by theorem 5.2.4. The spectrum of $U$ is non-empty (theorem 3.1.1) and hence it must contain an isolated point (lemma 4.4.2); this point must be an eigenvalue of $U$ by theorem 3.4.5. All of the above steps are reversible.

The trajectory space may be avoided by considering the limit isometric semigroup $\tilde{T}$ on $Y$; theorems 3.1.1 and 3.4.5 then imply the result in a similar manner. This is the approach taken in [10], the difference lying in the proof of theorem 3.1.1. The choice is really a matter of preference, but I feel the proof I have given is more elementary since it avoids having to know that an isolated point in the unitary spectrum of $T$ is isolated in the whole spectrum (and hence avoids resort to Šilov boundaries). The following lemma will be used on several occasions.

**Lemma 5.3.2** *If $E$ is closed and countable, then $f \in J_E$ for all $f \in L^1(G)$ such that $\hat{f}|_E = 0$.*

**Proof:** This is a consequence of the general Tauberian theorem [35, theorem 7.2.4].

An improvement on the basic A-B-L-P theorem was given in [27] for the $C_0$-semigroup case. It was later proved by a very different method in [9, theorem 8], and was extended to the general case in [10]. It runs as follows: if $Sp_u(T,S)$ is countable, $T$ is bounded, and if $E\sigma_u(T) = \emptyset$, then $X$ may be decomposed into the direct sum of two closed, $T$-invariant subspaces $L = \{x \in X : T(s)x \to 0\}$ and $M$, the closed subspace spanned by the unitary eigenvectors for $T$. ($E\sigma_u(T)$ is the set of all $\chi \in P\sigma_u(T^*)$ for which there exists an associated eigenvector $f$, but no $x \in X$ satisfies both $f(x) = 1$ and $T(s)x = \chi(s)x$ for all $s$. It is called the *ergodic spectrum.*) The proof is very similar to the proof of the standard theorem.

Since the ergodic spectrum is contained in the unitary point spectrum of $T^*$, theorem 5.3.1 is an obvious corollary of this result. When $X$ is a reflexive space, the ergodic spectrum is always empty. For non reflexive spaces it can fail to be empty even for norm-continuous group representation by isometries with countable spectrum as the following example shows.

**Example 5.3.3** *Let $E$ be a countable, closed subset of $\Gamma$ and let $T = T_{J_E}$ and $X = X_{J_E}$ be as in chapter 1. The point spectrum of $T$ is the set of all isolated points in $E$, while the ergodic spectrum is the set of all limit points.*

The eigenvectors for $T^*$ are precisely the scalar multiples of the $L^\infty(G)$ functions $\chi \in E$.

If $\chi \in P\sigma_u(T)$, then there exists $f \in L^1(G) \setminus J_E$ such that $f_t - \chi(t)f \in J_E$ for all $t \in G$. In particular, for all $\gamma \in E$,

$$\gamma(s)\hat{f}(\gamma) = \hat{f}_s(\gamma) = \chi(s)\hat{f}(\gamma),$$

so $\hat{f}(\gamma) = 0$ for all $\gamma \in E \setminus \{\chi\}$. By lemma 5.3.2 $\hat{f}(\chi) \neq 0$, so continuity of $\hat{f}$ implies that $\chi$ is isolated in $E$. On the other hand, if $\chi$ is an isolated point in $E$, then we can find an $f$ such that $\hat{f}(\chi) = 1$ but $\hat{f}$ is zero elsewhere on $E$. It follows that $f + J_E$ is an eigenvector for $T$ with eigenvalue $\chi$.

Any point in $P\sigma_u(T^*)$ is in $E\sigma_u(T^*)$ unless there exists an $f \in L^1(G)$ such that $f_t - \chi(t) \in J_E$ for all $t \in G$ and $\hat{f}(\chi) = 1$. By the above this happens whenever $\chi$ is a limit point of $E$.

## 5.4 Extending the A-B-L-P Theorem

As mentioned in [3], it is not possible to weaken either the countability or the boundedness assumptions of theorem 5.2.1 even in the $C_0$-semigroup case. The goal of this section is to produce a version of the theorem with no restriction on the point spectrum of $T^*$. In all the results that follow we will assume that $T$ is contractive rather than simply bounded; the results may be reformulated for the bounded case, but they are less clear. And since any bounded representation may be renormed to become contractive, the complication seems unnecessary.

**Lemma 5.4.1** *Let $T$ be a contractive representation of $S$ on $X$ with unitary spectrum $E$. For any $x \in X$ and $f \in L^1(S)$,*

$$\lim_{s\to\infty} \|T(s)\hat{f}(T)x\| \leq \|f + J_E\| \lim_{s\to\infty} \|T(s)x\|.$$

*In particular,* $\lim_{s\to\infty} \|T(s)\hat{f}(T)\| \leq \|f + J_E\|$.

**Proof:** Consider the limit isometric representation $\tilde{T}$ on $Y$ (as described in the proof of theorem 5.2.1 above). The unitary spectrum of $\tilde{T}$ is contained in $E$, so lemma 4.2.1 implies that $\|\hat{f}(\tilde{T})\| \leq \|f + J_E\|$ for any $f$ in $L^1(S)$. Since $\hat{f}(T)x + L = \hat{f}(\tilde{T})(x + L)$, the result follows from the definition of the norm on $Y$.

The most important part of the lemma, the last statement, may be proved by another method. It is longer, but as it is interesting in its own right, I will briefly describe it here. Define a seminorm on $L^1(S)$ by setting

$$p(f) = \lim_{s\to\infty} \|\hat{f}(T)T(s)\|.$$

Factor out the null space and complete with respect to $p$ to obtain a new Banach space $Z$. The semigroup of right-shift operators induces an isometric representation $V$ on $Z$ whose unitary spectrum is contained in $E$. After some easy technical work, an application of lemma 4.2.1 to $V$ yields the result.

**Corollary 5.4.2** *Let $T$ be a bounded representation of $S$. If $f \in L^1(S)$ is of spectral synthesis with respect to $Sp_u(T,S)$, then*

$$\lim_s \|\hat{f}(T)T(s)\| = 0.$$

**Definition 5.4.3** *A bounded representation $T$ will be called* **trivially asymptotically stable (t.a.s.)** *if the only $x \in X$ satisfying $\lim_{s\to\infty} \|T(s)x\| = 0$ is zero.*

**Lemma 5.4.4** *If $T$ is a t.a.s., contractive representation with unitary spectrum $E$, then*

$$\|\hat{f}(T)\| \leq \|f + J_E^+\|,$$

*where $J_E^+$ is the ideal in $L^1(S)$ given by $J_E^+ = L^1(S) \cap J_E$.*

**Proof:** By corollary 5.4.2, $\lim_{s\to\infty} \|\hat{g}(T)T(s)\| = 0$ whenever $g \in J_E^+$, so in particular $\|T(s)\hat{g}(T)x\| \to 0$ for any $x \in X$; but $T$ is trivially asymptotically stable, so $\hat{g}(T)x = 0$ for all $x \in X$ and hence $\hat{g}(T) = 0$. The result now follows immediately because $\|\hat{f}(T)\| \leq \|f\|_1$ for all $f$ in $L^1(S)$.

By itself this lemma would probably not be of much interest; but it comes into its own here because of the following lemma. In its $C_0$-semigroup form it is due to Esterle, Strouse, and Zouakia, but in fact their method of proof generalises without difficulty.

**Lemma 5.4.5** *If $E$ is a countable, closed subset of $\Gamma$, then the map $f + J_E^+ \mapsto f + J_E$ of $L^1(S)/J_E^+$ into $L^1(G)/J_E$ is an isometric isomorphism.*

**Proof:** Let $s_0 \in S^0$ and let $(f_\alpha)$ be a net of norm one elements in $L^1(S)$ such that

$$\|g * f_\alpha - g_{s_0}\|_1 \longrightarrow 0$$

as $\alpha \to \infty$ for all $g \in L^1(G)$. If $f'_\alpha(\cdot)$ denotes $f_\alpha(-\cdot)$, then for any $\psi \in L^\infty(G)$,

$$\psi * f'_\alpha(g) = \psi(f_\alpha * g) \longrightarrow \psi(g_{s_0})$$

for all $g \in L^1(G)$, so

$$\|\psi\| = \|\psi_{-s_0}\| \leq \liminf_{\alpha\to\infty} \|\psi * f'_\alpha\|.$$

On the other hand, for each $\alpha$, $\|\psi * f'_\alpha\| \leq \|\psi\|$ and it follows that

$$\|\psi\| = \lim_{\alpha\to\infty} \|\psi * f'_\alpha\|$$

for all $\psi \in L^\infty(G)$.

Now suppose $\phi \in J_E^\perp$ and write $\phi = \phi_1 + \phi_2$, where $\phi_1|_{G\setminus S} = \phi_2|_S = 0$. For $g \in L^1(S)$, $\phi_2 * f'_\alpha(g) = \phi_2(f_\alpha * g) = 0$; hence

$$\|(\phi * f'_\alpha)|_S\| = \|(\phi_1 * f'_\alpha)|_S\| \leq \|\phi_1 * f'_\alpha\|.$$

Each $\phi * f'_\alpha$ is bounded and uniformly continuous, so applying a result of Loomis [25], we deduce that $\phi * f'_\alpha$ is almost periodic; in particular $\|(\phi * f'_\alpha)|_S\| = \|\phi * f'_\alpha\|$. From the above,

$$\|\phi\| = \lim_{\alpha\to\infty} \|\phi * f'_\alpha\| = \lim_{\alpha\to\infty} \|(\phi * f'_\alpha)|_S\| \leq \lim_{\alpha\to\infty} \|\phi_1 * f'_\alpha\| = \|\phi_1\|.$$

But $\|\phi_1\| \leq \|\phi\|$ trivially, so $\|\phi\| = \|\phi|_S\|$ for all $\phi \in J_E^\perp$.

The map $f + J_E^+ \mapsto f + J_E$ is obviously injective, and since its dual is $\phi \mapsto \phi|_S$ for $\phi \in J_E^\perp$, the lemma now follows.

**Corollary 5.4.6** *If $T$ is a t.a.s. contractive representation with countable unitary spectrum $E$, then*

$$\|\hat{f}(T)\| \leq \|f + J_E\|$$

*for all $f \in L^1(S)$.*

**Proposition 5.4.7** *If $T$ is a t.a.s., contractive representation of $S$ with countable unitary spectrum $E$, then there exists a continuous homomorphism*

$$\pi : L^1(G)/J_E \to \mathcal{A}_T$$

*such that for all $s \in S$ and $f \in L^1(S)$,*

$$\pi(f_s + J_E) = T(s)\pi(f + J_E) = \pi(f + J_E)T(s).$$

**Proof:** By lemma 5.4.5 it is sufficient to define the action of $\pi$ only on $L^1(S)/J_E$. Let $\pi : L^1(S)/J_E \to \mathcal{A}_T$ by

$$\pi : f + J_E \mapsto \hat{f}(T).$$

By corollary 5.4.6 this map is well defined and bounded by one. The rest is easy verification.

**Proposition 5.4.8** *If $T$ is a t.a.s., contractive representation of $S$ with compact, countable unitary spectrum $E$, then $\hat{f}(T) = I$ for any $f \in L^1(S)$ such that $\hat{f}|_E = 1$.*

**Proof:** Suppose $f \in L^1(S)$ satisfies $\hat{f}|_E = 1$. By lemma 5.3.2, $f * f - f \in J_E$ and hence $\hat{f}(T)^2 = \hat{f}(T)$ by corollary 5.4.6. Let $M$ be the closed linear subspace $\{x : \hat{f}(T)x = 0\}$; $M$ is $T$-invariant and $\hat{f}(T|_M) = 0$.

Since $M \leq X$ it is clear that $Sp_u(T|_M, S) \subseteq Sp_u(T, S)$. On the other hand, if $\chi \in Sp_u(T|_M, S)$, then by definition

$$|\hat{f}(\chi)| \leq \|\hat{f}(T|_M)\| = 0;$$

hence $Sp_u(T|_M, S) = Sp_u(T|_M, S) \cap E = \emptyset$. By theorem 5.3.1, $T|_M$ must be asymptotically stable, but $T$ is t.a.s., so $M = \{0\}$. It now follows that $\hat{f}(T)x = x$ for all $x \in X$.

**Theorem 5.4.9** *If $T$ is a trivially asymptotically stable, contractive representation with countable unitary spectrum, then for each $s \in S$, $T(s)$ is an invertible isometry.*

**Proof:** We will first deal with the case when $E = Sp_u(T,S)$ is compact.

By [35, theorem 2.6.8] there exists a $k \in L^1(G)$ such that $\hat{k}|_E = 1$ and $\|k\|_1 < 1+\varepsilon$. Lemma 5.4.5 implies that for some $h \in L^1(S)$, $h + J_E = k + J_E$; so in particular $\hat{h}|_E = 1$. Let

$$U(t) = \pi(h_t + J_E) \qquad (t \in G).$$

(Here $\pi$ is as in proposition 5.4.7.) If $h' \in L^1(G)$ also satisfies $\hat{h'}|_E = 1$, then $h - h' \in J_E$ by lemma 5.3.2, so this definition is independent of the choice of $h$ (and hence of $\varepsilon$). For any $s, t \in G$,

$$\begin{aligned} U(t+s) &= \pi(h_{t+s} + J_E) \\ &= \pi((h_t + J_E) * (h_s + J_E)) \\ &= \pi(h_t + J_E)\pi(h_s + J_E) \\ &= U(t)U(s) \end{aligned}$$

since $\pi$ is a homomorphism. Furthermore, proposition 5.4.8 tells us that $\hat{h}(T) = I$; so $U(0) = I$ and

$$U(s) = \pi(h_s + J_E) = T(s)\pi(h + J_E) = T(s)\hat{h}(T) = T(s)$$

for all $s \in S$. Strong continuity of $U$ follows from continuity of $\pi$ and the strong continuity of the shift operators on $L^1(G)$. Finally, for all $t \in G$,

$$\begin{aligned} \|U(t)\| &= \|\pi(h_t + J_E)\| \leq \|h_t + J_E\| \\ &= \|h + J_E\| = \|k + J_E\| \\ &\leq \|k\|_1 \\ &< 1 + \varepsilon. \end{aligned}$$

Since $U$ was independent of the choice of $h$ it follows that $\|U(t)\| \leq 1$ for all $t \in G$. In particular each $T(s)$ is an invertible isometry.

Assume now that $E$ is not compact. Suppose $g \in L^1(G)$ has compactly supported Fourier transform and let $V$ be an open subset of $\Gamma$ containing $\operatorname{supp} \hat{g}$ such that

$V \cap E$ is compact. Choose $f \in L^1(G)$ with $\hat{f}|_{E\cap V} = 1$ and $\hat{f}|_{E\setminus V} = 0$; by lemma 5.3.2 $f * f - f \in J_E$ and hence $P := \pi(f + J_E)$ is an idempotent in $\mathcal{A}_T$. If $Y$ denotes $P[X]$, then lemma 3.4.3 implies that

$$Sp_u(T(\cdot)|_Y, S) = \{\chi \in Sp_u(T, S) : \hat{f}(\chi) = 1\} = E \cap V.$$

Since $T(\cdot)|_Y$ is also contractive and t.a.s. it follows from the above that $T(s)|_Y$ is an invertible isometry for each $s \in S$. Now let $B$ denote $\pi(g + J_E)$ and note that because $\hat{f}\hat{g} = \hat{g}$ on $E$, lemma 5.3.2 implies that $PB = B$; in particular $Bx \in Y$ and so $\|T(s)Bx\| = \|Bx\|$ for all $x \in X$, $s \in S$.

Let $s_0$ be in the interior of $S$ and $x \in X$ be non-zero. For any $\varepsilon > 0$ there exists $k \in L^1(S)$ such that

$$\|\hat{k}(T)x - T(s_0)x\| < \frac{\varepsilon}{4}.$$

By [35, theorem 2.6.6] there is a $g \in L^1(G)$ such that $\hat{g}$ has compact support and $\|k - k * g\|_1 < \varepsilon/4\|x\|$. If $B$ denotes $\pi(g + J_E)$ and $y = \pi(k + J_E)x = \hat{k}(T)x$, then

$$\begin{aligned}
\|By - T(s_0)x\| &\leq \|\pi(g + J_E)\pi(k + J_E)x - \pi(k + J_E)x\| + \|\pi(k + J_E)x - T(s_0)x\| \\
&= \|\pi((g * k - k) + J_E)x\| + \|\hat{k}(T)x - T(s_0)x\| \\
&< \frac{\varepsilon}{4\|x\|}\|x\| + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.
\end{aligned}$$

Since $\operatorname{supp}\hat{g}$ is compact, $\|T(s)By\| = \|By\|$ for all $s \in S$ by the work above. Hence

$$\begin{aligned}
|\|T(s + s_0)x\| - \|T(s_0)x\|| &\leq \|T(s + s_0)x - T(s)By\| \\
&\quad + |\|T(s)By\| - \|By\|| + \|By - T(s_0)x\| \\
&\leq \|T(s)\|\,\|T(s_0)x - By\| + 0 + \|By - T(s_0)x\| \\
&< \varepsilon.
\end{aligned}$$

It now follows that $\|T(s + s_0)x\| = \|T(s_0)x\|$ for all $x \in X$ and $s \in S$. Let $N$ denote the closure of $T(s_0)[X]$; $N$ is a $T$-invariant subspace of $X$ and, moreover, by the above we know that $T(s)|_M$ is an isometry for each $s \in S$. It is clear that the unitary spectrum of this reduced representation is contained in $E$, so by theorem 4.4.3 each $T(s)|_M$ is invertible. The result now follows easily from the fact that each $T(s)$ is injective.

Perhaps the most striking part of this theorem is not that $T$ is invertible, but that it is isometric. The following example gives an insight into why this is true.

**Example 5.4.10** *Consider the $C_0$-semigroup $T$ consisting of the right-shift operators on $L^1(\mathbf{R})$ normed with*

$$\|f\| = \int_{-\infty}^{0} |f(s)|\, ds + \frac{1}{2}\int_{0}^{\infty} |f(s)|\, ds.$$

*$T$ is invertible and contractive, but not isometric.*

*However, if $E$ is a closed, countable set, and if the ideal $J_E$ of $L^1(\mathbf{R})$ is factored out, then the induced $C_0$-semigroup is in fact isometric.*

Let $\|\cdot + J_E\|_1$ denote the usual norm on $L^1(\mathbf{R})/J_E$ and $\|\cdot + J_E\|$ the quotient norm induced from the norm defined above. It is immediate that for all $f \in L^1(\mathbf{R})$,

$$\frac{1}{2}\|f + J_E\|_1 \leq \|f + J_E\| \leq \|f + J_E\|_1.$$

If $f \in L^1(\mathbf{R}_+)$, then we know from lemma 5.4.5 that $\|f + J_E\|_1 = \|f + J_E^+\|_1$. So given $\varepsilon > 0$ there exists $h \in J_E \cap L^1(\mathbf{R}_+)$ such that $\|f - h\|_1 < \|f + J_E\|_1 + \varepsilon$. But restricted to $L^1(\mathbf{R}_+)$, the norms $\|\cdot\|$ and $\frac{1}{2}\|\cdot\|_1$ agree, and hence

$$\|f + J_E\| \leq \|f - h\| \leq \frac{1}{2}(\|f + J_E\| + \varepsilon).$$

The above shows that $\|f + J_E\| = \frac{1}{2}\|f + J_E\|_1$ for all $f \in L^1(\mathbf{R}_+)$, and from this we can deduce the equality for all $f \in L^1(\mathbf{R})$ because $L^1(S)/J_E$ is isomorphic to $L^1(G)/J_E$.

**Lemma 5.4.11** *If $T$ is a representation of $S$ on $X$ by isometries, and if $Sp_u(T, S)$ is countable, then $X^*$ is the weak*-closure of the linear span of the unitary eigenvectors of $T^*$.*

**Proof:** Let $M^0$ denote the linear span of the unitary eigenvectors of $T^*$. If the lemma is false, then ${M^0}_\perp$ must be non-trivial so assume for contradiction that this is the case. From theorem 4.4.3, $T$ is invertible, so we will consider it as a group representation.

Since $M^0$ is clearly a $T^*$-invariant subspace of $X^*$, ${M^0}_\perp$ is a closed, $T$-invariant subspace of $X$. The restricted group representation $T|_{{M^0}_\perp}$ is thus defined on a non-trivial Banach space and consequently $Sp(T|_{{M^0}_\perp}, G)$ is non-empty. As it is contained in a countable set it must have an isolated point $\chi$ which must be an eigenvalue for $T$; let $x$ be an associated eigenvector.

We may assume without loss of generality that $\chi = 1$, so $T(t)x = x$ for all $t$. Since $G$ is amenable there exists a translation invariant functional $\phi$ on $L^\infty(G)$ such that $\phi(1) = 1$. Choose any $g \in X^*$ satisfying $g(x) = 1$, and define $\psi_y \in L^\infty(G)$ for each $y \in X$ as follows: $\psi_y(t) = g(T(t)y)$ for $t \in G$. If $f : X \to \mathbf{C}$ is defined by

$$f : y \mapsto \phi(\psi_y),$$

then it may be verified that $f \in X^*$, that $f(x) = 1$, and that $T(t)^* f = f$ for all $t \in G$. Such an $f$ contradicts the fact that $x \in {M^0}_\perp$.

**Theorem 5.4.12** *Let $T$ be contractive representation of $S$ on $X$ and let $M$ denote the weak*-closed linear span of the unitary eigenvectors of $T^*$. If $Sp_u(T, S)$ is countable, then*

1. *$T^*(s)|_M$ is an invertible isometry for each $s \in S$;*

2. *$M$ is the largest subspace of $X^*$ for which this is true;*

3. *For all $x \in X$*
$$\lim_S \|T(s)x\| = \|x + M_\perp\| = \inf_{y \in M_\perp} \|x - y\|.$$

**Proof:** Let $L = \{x \in X : \|T(s)x\| \to 0\}$. We will consider the quotient space $X/L$, but, unlike in the construction of the limit isometric semigroup, we will re-norm it. Instead we note that the induced representation $\tilde{T}$ on $X/L$ is contractive in the quotient norm and that it has countable unitary spectrum. $\tilde{T}$ is also trivially asymptotically stable. To see this let $x \in X$ and set $l = \lim_{s\to\infty} \|T(s)x + L\|$. For $\varepsilon > 0$ there exists $y \in L$ and $t \in S$ such that $\|T(t)x - y\| < l + \varepsilon/2$. So, for $s \in S$,

$$\begin{aligned} \|T(s+t)x\| &\leq \|T(s+t)x - T(s)y\| + \|T(s)y\| \\ &\leq \|T(t)x - y\| + \|T(s)y\| \\ &< l + \frac{\varepsilon}{2} + \|T(s)y\|. \end{aligned}$$

Since $y \in L$ it follows that for some $s \in S$ $\|T(s+t)x\| < l + \varepsilon$, and hence $\lim_{s\to\infty} \|T(s)x\| \leq l$. On the other hand, $\|T(s)x + L\| \leq \|T(s)x\|$ for all $x \in X$, $s \in S$, and hence

$$\lim_{s\to\infty} \|T(s)x + L\| = \lim_{s\to\infty} \|T(s)x\| \qquad (x \in X).$$

In particular $\tilde{T}$ is t.a.s.. Theorem 5.4.9 implies that $\tilde{T}$ is invertible and isometric.

If $N$ is a closed subspace of $X^*$ such that $T^*|_N$ is invertible and isometric, then any functional $f \in N$ may be used to form a trajectory for $T^*$ simply by the action of $T^*|_N$ on $f$. Corollary 5.2.2 then implies that each $f \in N$ must annihilate $L$. Since this holds in particular for the closed linear span of the unitary eigenvectors of $T^*$, $M$ must be contained in $L^\perp$. By lemma 5.4.11 the span of the unitary eigenvectors of $\tilde{T}^*$ is weak* dense in $L^\perp$, but these are precisely the unitary eigenvector of $T^*$. Hence $M = L^\perp$, and all the claims of the theorem now follow.

**Corollary 5.4.13** *Let $T$ be a representation of $S$ with countable unitary spectrum. For any $x \in X$, $\lim_s \|T(s)x\| = 0$ if and only if $f(x) = 0$ for all unitary eigenvectors $f$ of $T^*$.*

Naturally if $P\sigma_u(T^*) = \emptyset$ as in the statement of theorem 5.3.1, this corollary implies the asymptotic stability of $T$. Just as 5.3.1 can be viewed as a version of 5.2.1 when the unitary spectrum of $T$ is countable with unitary eigenvectors replacing the role of trajectories, so the above corollary can be viewed as a refinement of corollary

5.2.3 in the same case. The following example shows that so far as information from the eigenvectors themselves goes (as opposed to their span), corollary 5.4.13 is about as much as we can expect.

**Example 5.4.14** *Let $K$ denote the set of all unitary eigenvectors of $T^*$. The above corollary cannot be strengthened to say that*

$$\lim_s \|T(s)x\| = \sup_{k\in K,\,\|k\|\le 1} |k(x)|.$$

Consider $L^1(G)/J_E$ for some compact, countable set $E$ with the associated group representation. In this case the above formula would be true if and only if $\|f+J_E\|_1 = \sup_{\chi\in E}|\hat{f}(\chi)|$ for all $f \in L^1(G)$. In particular this would imply that $E$ was a Helson set [35, p.114]. It is known that there exist compact, countable sets which are not Helson sets.

**Corollary 5.4.15** *If $T$ is contractive and $Sp_u(T,S)$ is countable, then $T|_{N_\perp}$ is an invertible isometric representation, where*

$$N = \{\phi \in X^* : \|T(s)^*\phi\| \to 0\}.$$

*In particular, if $P\sigma_u(T)$ is empty, then $N$ is weak*-dense in $X^*$.*

This is a sort of dual result to theorem 5.4.12 though it is not as powerful since it offers little information when $N$ is weak*-dense in $X^*$.

**Proof:** We will deal first with the case when $N = \{0\}$. Let $K$ be the space of $\phi \in X^*$ such that $s \mapsto T(s)^*\phi$ is norm-continuous. By theorem 5.4.9 $T(s)^*|_K$ is an invertible isometry. For all $x \in X$ and $s \in S_0$

$$\|T(s)x\| = \sup\{|T(s)^*f(x)| : f \in K,\ \|f\| \le 1\}$$

by lemma 1.3.4. Since $T^*$ is invertible on $K$ it follows that $\|T(s_0)x\| = \|x\|$ and $T$ is a representation by isometries, which must be invertible. For the general case consider $T$ restricted to $N_\perp$.

If $N_\perp$ is non-trivial, then $T$ must have a unitary eigenvector by theorem 3.4.5 and the remainder of the corollary is a result of this fact.

## Notes

**Section 5.2** The link between trajectories and asymptotic stability was demonstrated by Phóng in [33]. He proves that if $T$ is a bounded $C_0$-semigroup which is not asymptotically stable and if one of the following conditions is satisfied:

1. There is an $s_0 > 0$ such that $T(s_0)$ has dense range;

2. $\sigma(A) \not\supset i\mathbf{R}$, where $A$ generates $T$;

then there is a non-trivial (bounded) trajectory for the sun-dual semigroup $T^{\odot}$.

**Section 5.3** There are now a number of very different proofs for theorem 5.3.1, at least in its $C_0$-semigroup form. The first version of the theorem appeared in [40] and covered norm-continuous $C_0$-semigroups. It was generalised to strongly continuous $C_0$-semigroups in [26] and [3], the latter paper also presenting a $\mathbf{Z}_+$ version. For $\mathbf{R}_+$, the result was later proved using arguments from harmonic analysis in [19]. In [32] it was extended to norm-continuous representations of the general class of semigroups covered in this thesis. The generalisation to the form stated in 5.3.1 came in [10]. Yet another proof came for the $C_0$-semigroup case was given in [33].

The name 'ergodic spectrum' given to $E\sigma(T)$ arises from a mean-ergodic theorem for $C_0$-semigroups which states that $\mu \notin E\sigma(T)$ if and only if

$$\lim_{t\to\infty} \frac{1}{t} \int_0^\infty e^{-\mu s} T(s)\, ds$$

exists in the strong operator topology. See for example [15, chapter 5].

**Section 5.4** For $S = \mathbf{R}_+$ corollary 5.4.2 was shown first by Esterle, Strouse, and Zouakia in [19] using methods of harmonic analysis. They went on to use it as a vital step towards proving theorem 5.2.1. For norm-continuous $T$ and general $S$ it was proved by Phóng [32], while the first proof for strongly continuous $T$ appeared in [10]. The proof given in this thesis is considerably more concise than those mentioned.

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