Lipschitz Geometry and Rough Paths

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Abstract

Motivated by building a Lipschitz structure on the reachability set of a set of rough differential equations, we study first Lipschitz maps in full detail and in many aspects: we show embedding properties of Lipschitz spaces, study algebraic properties, local versus global behaviour of Lipschitz maps, give a version of the inverse function and the constant rank theorems and analyse flows of Lipschitz vector fields. Along the way, we give quantitative estimates, which is, after all, one of the main strengths of the Lipschitz geometry. Second, we combine our observations on the local properties of rough paths and Lipschitz maps to give a rather flexible structure on manifolds that allow the definition of rough paths on them and explain how the same process can be used to define a notion of coloured paths on manifolds, assuming that one can build a suitable functorial rule. Finally, we make use of those remarks to derive a quantitative condition that endow orbits of vector fields with the structure of a Lipschitz manifold and show, that under this condition, the space of solutions of rough differential equations driven by rough paths is the same as the one we obtain by driving the RDEs with smooth paths.

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Introduction

The problem in hand can be briefly described as studying the Lipschitz geometry and the geometric structure of the reachability set of a differential equation and whether points reached by driving the differential equation with a rough control signal can also be reached using only smooth signals. Given a family of vector fields $(f^1, \ldots, f^d)$ and an initial condition y_0, the reachability set associated with $(f^1, \ldots, f^d)$ and y_0 is the set of all terminal values of the solutions to the differential equation:

$$\begin{cases} \mathrm{d}y_t &= \displaystyle\sum_{i=1}^{d} f^i(y_t)\mathrm{d}x_t^i \quad ,\forall t \in [0,T] \\ y(0) &= y_0 \end{cases}$$

when the signal $(x^1, \ldots, x^d)$ takes all possible values. As mentioned above, rough paths theory is our main framework and we will then be considering all geometric rough paths as entry signals.

The approach taken here is inductive: we first look at the properties of signatures (which are the solutions to a particular R.D.E.) and the space they live in (the nilpotent free Lie group) then see whether these properties can be generalized to any R.D.E. The nilpotent free Lie group has been well studied in the past (and for which many equivalent definitions exist) and therefore, many of the properties we seek to prove come as trivial in the case of the signatures. However, these can be retrieved in the general case under suitable assumptions on the vector fields defining the R.D.E.

This is how this work is organized:

- In the first chapter, we introduce the signature, show how it can be seen as a solution to a differential equation then give a version of Chow's theorem that states that all truncated signatures live in the nilpotent free Lie group (of finite order) and that any element of this group is the signature of a smooth path. We also give a short review of the theory of rough paths.
- In Chapter 2, We formally write the problem we intend to solve and give a review of how far the study of this problem has gone.
- In Chapters 3 and 4, we improve our understanding of the Lipschitz geometry which is the right framework of working with rough paths on a manifold.
- Finally, Chapter 5 deals with the geometric structure of the reachability set of an R.D.E. and generalize Chow's theorem to a class of R.D.E.s where the vector fields involved satisfy some suitable quantitative assumptions.

Important Notations:

$\mathcal{S}_k$: The symmetric group of order k.

$\lfloor \gamma \rfloor$: The only integer such that $0 < \gamma - \lfloor \gamma \rfloor \leq 1$, γ being a real number.

$[\gamma]$: The integer part of a real number γ, i.e. the only integer such that $0 \leq \gamma - [\gamma] < 1$.

$\mathcal{D}_J$: The set of all finite subdivisions of a compact interval J.

$|D|$: The mesh of a subdivision D.

Δ_J : The simplex of all pairs $(s,t) \in J^2$ such that $s \leq t$, J being an interval.

$\mathcal{L}_c(E,F)$: The space of all continuous linear mappings from a normed vector space E to a normed vector space F.

I_k : The identity matrix of rank k.

Id_U : The identity map on the set U.

$\overline{A}$: The closure of a subset A of a topological space.

$B(x,\alpha)$: The ball centered at x of radius α.

T_pM : The tangent space to the manifold M at the point p.

Chapter 1

A review of the rough paths theory

The goal of this first introductory chapter is to introduce the concept of the signature of a path after providing the necessary tool to this definition, namely Stieltjes's integration theory. The differential equations theory associated with it is also presented as this will later be needed in providing an alternative definition of the signature and understanding the reachability problem in this case. We will give a version of Picard's theorem for solving a linear differential equation; a version that also deals with the continuity of the associated Itô map. We present tensors and by the same occasion formal tensor series before defining the signature of a path and studying the algebraic and analytic properties of signatures required for the rough paths theory. We finish by defining rough paths and giving a reminder of all the notions that will be necessary to us in the rest of this work, one of which will be the extension of the notion of Lipschitz (Hölder) maps. Most of the results in this chapter are admitted without proofs as they are very basic and can be found, for example, in [2] and [22]. We merely give some examples and prove some of the statements to familiarize the reader with any possibly new notions.

1.1 The concept of the p-variation

We introduce the notion of p-variation, crucial in both Stieltjes's integration theory and the rough paths theory. It is a way of measuring the amplitude of the oscillations of a path, independently of when said oscillations occur or the order in which they occur.

Definition 1.1. *Let $p \geq 1$, $(E, \|.\|)$ be a normed vector space and $T > 0$. Let $x : [0,T] \to E$ be a continuous path. For a finite subdivision $D = (t_i)_{0 \leq i \leq n}$ of $[0,T]$, we denote by $\|x\|_{p,D}$ the quantity:*

$$\|x\|_{p,D} := \left(\sum_{i=0}^{n-1} \|x_{t_{i+1}} - x_{t_i}\|^p \right)^{\frac{1}{p}} = \left(\sum_{D} \|x_{t_{i+1}} - x_{t_i}\|^p \right)^{\frac{1}{p}}$$

x is said to have a finite p-variation over $[0,T]$ if $\{\|x\|_{p,D} | D \in \mathcal{D}_{[0,T]}\}$ has a finite supremum. In this case, this supremum is called the p-variation of x over $[0,T]$ and is denoted by $\|x\|_{p,[0,T]}$. When $p = 1$, we say that the path x has bounded variation.

Remark 1.2. *The previous definition can easily be adapted for a metric space (E,d) but we will not use this generalisation in the rest of these notes.*

Examples: Let $T > 0$:

1. Let E be a normed vector space. A $\mathcal{C}^1$ E-valued path x over $[0,T]$ is of bounded variation over $[0,T]$ and $\|x\|_{1,[0,T]} \leq T\|x'\|_{\infty,[0,T]}$.
2. Let E be a normed vector space and $\alpha \leq 1$. An α-Hölder E-valued path over $[0,T]$ has finite $1/\alpha$-variation over $[0,T]$.
3. Let Ω be a probability space and B be a Brownian motion over $[0,T]$:
   - B has finite p-variation over $[0,T]$ almost surely, for any $p > 2$.
   - Seen as an $L^2(\Omega)$-valued path, B has finite 2-variation.

Proposition 1.3. *Let* $p \in (0,1)$, $(E, \|.\|)$ *be a normed vector space and* $T > 0$. *Let* $x : [0,T] \to E$ *be a continuous path. Then,* x *has a finite* p*-variation over* $[0,T]$ *if and only if* X *is constant over* $[0,T]$.

Proof. It is obvious that any constant path has a finite p-variation over the interval it is defined upon.

Suppose that x has a finite p-variation over $[0,T]$ and that it is not constant. Let $s, t \in [0,T]$ such that $s < t$ and $x_s \neq x_t$:

Let $n \in \mathbb{N}^*$. Then, by continuity of x, there exists a finite increasing sequence $(u_i)_{0 \leq i \leq n}$ such that $u_0 = s$ and $u_n (\leq t)$ and for every $i \in [\![0, n-1]\!]$ we have:

$$\|x_{u_{i+1}} - x_{u_i}\| = \frac{\|x_t - x_s\|}{n}.$$

By definition of the p-variation, we have then:

$$\sum_{i=0}^{n-1} \|x_{u_{i+1}} - x_{u_i}\|^p \leq \|x\|_{p,[0,T]}^p.$$

Hence:

$$\frac{\|x_t - x_s\|^p}{n^{p-1}} \leq \|x\|_{p,[0,T]}^p.$$

As the above inequality holds for every n, then the p-variation of x over $[0,T]$ should be infinite, which contradicts our initial assumption. Therefore, x is constant over $[0,T]$. □

Proposition 1.4. *Let* $p \geq 1$ *and* $(E, \|.\|)$ *be a Banach space.*

- *The set* $\mathcal{V}^p([0,T], E)$ *of all continuous paths from* $[0,T]$ *to* E *that have a finite* p*-variation over* $[0,T]$ *is a vector space when endowed with the natural operations of addition and multiplication by a scalar.*

- *The map:*

$$\begin{array}{rcl} \|.\|_{\mathcal{V}^p([0,T],E)} : \mathcal{V}^p([0,T],E) & \rightarrow & \mathbb{R}^+ \\ x & \mapsto & \|x\|_{p,[0,T]} + \sup_{t\in[0,T]} \|x_t\| \end{array}$$

  defines a norm on $\mathcal{V}^p([0,T],E)$ *called the p-variation norm.*

- $(\mathcal{V}^p([0,T],E), \|.\|_{\mathcal{V}^p([0,T],E)})$ *is a Banach space.*

- $\forall q \geq p : \mathcal{V}^p([0,T],E) \subseteq \mathcal{V}^q([0,T],E)$

The manipulation of p-variations is often made easier by the introduction of the notion of controls.

Definition 1.5. *A function* $\omega : \Delta_{[0,T]} \rightarrow \mathbb{R}_+$ *is said to be a control if it has the following properties:*

- ω *is continuous.*

- ω *is super-additive i.e.* $\omega(s,u) + \omega(u,t) \leq \omega(s,t) \quad , \forall\, 0 \leq s \leq u \leq t \leq T.$

- $\omega(t,t) = 0$, $\forall t \in [0,T]$.

Lemma 1.6 shows that to every path of finite p-variation, we can associate a "natural" control.

Lemma 1.6. *Let* $p \geq 1$*,* E *be a normed vector space and* $T > 0$*. Let* $x : [0,T] \rightarrow E$ *be a continuous path of finite p-variation over* $[0,T]$*. Then the function* ω *defined on* $\Delta_{[0,T]}$ *by* $\omega(s,t) = \|x\|^p_{p,[s,t]}$ *is a control.*

Conversely, if we can find a control that upper-bound the p^{th} power of the increments of a continuous path then this path is necessarily of finite p-variation. This is what theorem 1.7 states. Consequently, it also gives an easy way to prove the finiteness of the p-variation of a path without having to go back to the definition.

Theorem 1.7. *Let $p \geq 1$, E be a normed vector space and $T > 0$. Let $x : [0,T] \to E$ be a continuous path. There exists a control ω defined on $\Delta_{[0,T]}$ such that for every $(s,t) \in \Delta_{[0,T]}$ we have: $\|x_t - x_s\|^p \leq \omega(s,t)$ if and only if x has a finite p-variation over $[0,T]$ and is such that:*

$$\forall (s,t) \in \Delta_{[0,T]} : \|x\|_{p,[s,t]}^p \leq \omega(s,t)$$

We say in this case that the p-variation of x is controled by ω.

Proof. ($\Rightarrow$) Let $(s,t) \in \Delta_{[0,T]}$ and $D \in \mathcal{D}_{[s,t]}$. Using the super-additivity of the control, it is easy to show that: $\|x\|_{p,D}^p \leq \omega(s,t)$. By taking the supremum of the left-hand side term over all finite subdvisions of $[s,t]$, x has then a finite p-variation over $[0,T]$ controlled by ω.

($\Leftarrow$) The p-variation of x defines a natural control of its p-variation as stated in lemma 1.6. □

1.2 Stieltjes's integration theory and the related differential equations

Stieltjes's integration theory is almost a straightforward generalization of Riemann's integration theory. The construction of Stieltjes's integral and its basic properties are then given by the following theorem:

Theorem 1.8. *Let T be a positive real number. Let E and F be two Banach spaces. Let $x : [0,T] \to E$ be a path of bounded variation and $y : [0,T] \to \mathcal{L}_c(E,F)$ be a continuous path. For $t \in [0,T]$ and a subdivision $D = (t_i)_{0 \leq i \leq n}$ over $[0,t]$, we define the quantity*

$$\int_D y \mathrm{d}x := \sum_{i=0}^{n} y_{t_i}(x_{t_{i+1}} - x_{t_i}).$$

Then:

1. $\forall t \in [0,T]$, $\lim\limits_{|D|\to 0, D\in D_{[0,t]}} \int_D y\mathrm{d}x$ *exists and is denoted by* $\int_{[0,t]} y_u \mathrm{d}x_u$ *or* $\int_0^t y_u \mathrm{d}x_u$.

2. $\left(\int_{[0,t]} y_u \mathrm{d}x_u\right)_{0\le t\le T}$ *has bounded variation over* $[0,T]$.

3. *The* 1*-variation and the uniform norm of* $\int y_u \mathrm{d}x_u$ *can be controlled in the following way:*
$$\| \int_{[0,T]} y_u \mathrm{d}x_u\|_{\infty,[0,T]} \vee \| \int_{[0,T]} y_u \mathrm{d}x_u\|_{1,[0,T]} \le \|y\|_{\infty,[0,T]}\|x\|_{1,[0,T]},$$

Let E and F be two Banach spaces. Let T be a positive real number and consider a map $f : F \to \mathcal{L}(E,F)$ and a path $x : [0,T] \to E$. The definitions in the section above give a meaning to the following differential equation:

$$\begin{cases} \mathrm{d}y_t = f(y_t)\mathrm{d}x_t \quad ,\forall t \in [0,T] \\ y(0) = y_0 \end{cases} \tag{1.1}$$

Theorem 1.9 (Picard-Lindelöf). *Let T be a positive real number. Let E and F be two Banach spaces. Let $f : F \to \mathcal{L}(E,F)$ be a 1-Hölder map. For any E-valued path x of bounded variation and $y_0 \in F$, the differential equation:*

$$\begin{cases} \mathrm{d}y_t = f(y_t)\mathrm{d}x_t \quad ,\forall t \in [0,T] \\ y(0) = y_0 \end{cases} \tag{1.2}$$

admits a unique solution that we will denote by $I_f(x,y_0)$. This solution has bounded variation over $[0,T]$.
Moreover, the map (called the Itô map) $I_f : \mathcal{V}^1([0,T],E)\times F \to \mathcal{V}^1([0,T],F)$ is continuous.

1.3 The signature of a path

We start with defining the tensor algebra of a vector space.

Definition 1.10. *Let E be a vector space. For every $n \in \mathbb{N}^*$, let $E^{\otimes n}$ be the space of homogeneous tensors of E of degree n. We use the convention: $E^{\otimes 0} = \mathbb{R}$. The set of formal series of tensors of E, denoted by $T((E))$ is defined by the following:*

$$T((E)) := \{\mathbf{a} = (a_n)_{n\in\mathbb{N}} | \forall n \in \mathbb{N} : \quad a_n \in E^{\otimes n}\}$$

$T((E))$ has an algebra structure when endowed with the operations defined by the following: for $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$ and $\mathbf{b} = (b_n)_{n\in\mathbb{N}}$ in $T((E))$ and $\lambda \in \mathbb{R}$:

(Addition) $\mathbf{a} + \mathbf{b} = (a_n + b_n)_{n\in\mathbb{N}}$,

(Multiplication) $\mathbf{a} \otimes \mathbf{b} = (\sum_{k=0}^{n} a_k \otimes b_{n-k})_{n\in\mathbb{N}}$,

(Multiplication by a scalar) $\lambda.\mathbf{a} = (\lambda a_n)_{n\in\mathbb{N}}$.

Proposition 1.11. *Let E be a vector space. $(T((E)), +, ., \otimes)$ is a non-commutative (assuming dim(E)≥ 2) unital algebra with unit $\mathbf{1} = (1, 0, 0, \ldots)$. An element $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$ in $T((E))$ is invertible if and only if $a_0 \neq 0$. In this case, its inverse is given by the well-defined series:*

$$\mathbf{a}^{-1} = \frac{1}{a_0} \sum_{n\geq 0} \left(\mathbf{1} - \frac{\mathbf{a}}{a_0}\right)^n$$

.

Definition 1.12. *Let E be a Banach space and $T > 0$. Let $x : [0, T] \to E$ be a path of bounded variation. For $(s, t) \in \Delta_{[0,T]}$, we define the following sequence by induction* *(well-defined by theorem 1.8):*

$$\begin{cases} S^0(x)_{(s,t)} = 1 \\ S^n(x)_{(s,t)} = \int_{[s,t]} S^{n-1}(x)_{(s,u)} \otimes \mathrm{d}x_u \quad , \forall n \in \mathbb{N}^* \end{cases}$$

For every pair $(s,t) \in \Delta_{[0,T]}$*, the sequence* $(S^n(x)_{(s,t)})_{n\in\mathbb{N}}$*, simply denoted* $S(x)$*, is called the signature of* x *over* $[s,t]$*. For* $N \in \mathbb{N}^*$*,* $(S^n(x)_{(s,t)})_{n\le N}$*, simply denoted* $S_N(x)$*, is called the truncated signature of* x *over* $[s,t]$ *of degree* N*.*

The label "signature", implicitely implying the full characterization of a path, can be justified at many levels:

- Lyons and Hambly in [12] show that the signature of a path of bounded variation fully and uniquely characterizes the path in question up to a tree-like equivalence.
- In the context of differential equations, the signature of the control signal is the only needed information to get the solution. This can be shown for example easily and explicitely in the case where the map f in (1.2) is linear and continuous (details in [2] for example).

Lemma 1.13. *Let* E *be a vector space. Let* $m \in \mathbb{N}$ *be an integer and define*

$$B_m = \{\mathbf{a} = (a_n)_{n\in\mathbb{N}} | \forall i \in \{0,\dots,m\} \quad a_i = 0\}$$

Then is B_m *an ideal of* $T((E))$*.*

Definition 1.14. *Let* E *be a vector space. Let* $m \ge 0$ *be an integer. The truncated tensor algebra of order* m *of* E*, denoted by* $T^{(m)}(E)$*, is the quotient algebra* $T((E))/B_m$*. We will denote the canonical homomorphism* $T((E)) \to T((E))/B_m$ *by* π_m*. There is a natural identification between* $T^{(n)}(E)$ *and* $\bigoplus_{0\le i\le n} E^{\otimes i}$*.*

Definition 1.15 (Action of the Symmetric Group on Tensors)**.** *Let* $n \in \mathbb{N}^*$, $\sigma \in \mathcal{S}_n$ *and* E *be a vector space. We define the action of* σ *on the homogenous tensors of* E *of order* n *as a linear map by the following:*

$$\forall x_1, x_2, \dots, x_n \in E \quad \sigma(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(n)}$$

Definition 1.16. *Let* E *be a vector space. A norm* $\|.\|$ *on* $T((E))$ *is said to be admissible if the two following properties hold:*

1. $\forall n \in \mathbb{N}^*, \forall \sigma \in \mathcal{S}_n, \forall x \in E^{\otimes n} \quad \|\sigma x\| = \|x\|$.
2. $\forall n, m \in \mathbb{N}^*, \forall x \in E^{\otimes n}, \forall y \in E^{\otimes m} \quad \|x \otimes y\| \leq \|x\|\|y\|$.

We will assume in the rest of this chapter that $(E^{\otimes n})_{n \geq 1}$ are endowed with admissible norms. We will be discussing properties of norms on tensor product spaces more in detail in section 3.1.

Let $N \in \mathbb{N}^*$. The truncated signature of a path x of bounded variation considered as the path mapping every element t in $[0,T]$ to $S_N(x)_{(0,t)}$ can be seen as the solution to a particular differential equation:

Theorem 1.17. *Let* E *be a Banach space and* $N \in \mathbb{N}^*$*. Let* $x : [0,T] \to E$ *be a path of bounded variation and* $f_N : T^{(N)}(E) \to \mathcal{L}(E, T^{(N)}(E))$ *the map defined by:*

$$\forall x \in E, \forall (a_0, \dots, a_N) \in T^{(N)}(E): \quad f_N((a_0, \dots, a_N))(x) = (0, a_0 \otimes x, \dots, a_{N-1} \otimes x)$$

Then, the differential equation:

$$\begin{cases} \mathrm{d}S_t = f_N(S_t)\mathrm{d}x_t \quad, \forall t \in [0,T] \\ S_0 = (1, 0, \dots, 0) \end{cases} \tag{1.3}$$

admits a unique solution S *which is the path defined by:* $S_t = S_N(x)_{(0,t)}$*, for all* $t \in [0,T]$*.*

Proof. This is a simple application of Picard-Lindelöf's theorem (1.9). The differential equation (1.3) admits a unique solution. By definition of the signature, $t \mapsto S_N(x)_{(0,t)}$ is this unique solution. $\square$

1.4 Algebraic and analytic properties of the signature

We resume the notations of section 1.3. Let S (respectively S_n, for $n \in \mathbb{N}^*$) denote the map giving the signature (resp. the truncated signature of degree n) over $[0,T]$ of E-valued paths defined on $[0,T]$ that have bounded variation. We give below three basic properties of signatures that will be key in defining rough paths later.

We identify linear forms on $T((E))$ with the elements of the vector space $T(E^*)$ (the tensor algebra of E^*, E^* being the space of linear forms on E). We start by what we will be calling the shuffle product property:

Lemma 1.18. *For every two linear forms* e *and* f *defined on* $T((E))$*, there exists a linear form denoted by* $e \sqcup\!\sqcup f$ *such that:*

$$\forall x \in \mathcal{V}^1([0,T],E) \quad e(S(x)).f(S(x)) = (e \sqcup\!\sqcup f)(S(x))$$

$e \sqcup\!\sqcup f$ *is called the shuffle product of* e *and* f*.*

The following theorem is a straightforward consequence of lemma 1.18.

Theorem 1.19. *When endowed with the addition, the multiplication by scalars and the shuffle product operations, the set of the restrictions to the range of* S_n *of all linear forms* *on $T((E))$ is an algebra.*

The elements in the tensor algebra that satisfy the shuffle product property are of interest on their own:

Definition 1.20. *An element* $\mathbf{a}$ *of* $\tilde{T}((E))$ *that satisfies the shuffle product property, i.e.* $\forall e, f \in T(E^*) \quad e(\mathbf{a}).f(\mathbf{a}) = (e \sqcup\!\sqcup f)(\mathbf{a})$*, is called a group-like element. The set of all group-like elements is denoted by* $G^{(*)}$*.*

By lemma 1.18 then, the signatures of paths of bounded variation are group-like elements. It happens that $G^{(*)}$ has indeed a group structure:

Theorem 1.21. $(G^{(*)}, \otimes)$ *is a group.*

Remark 1.22. *Theorem 1.21 is crucial for the theory of rough paths (the extension theorem, see [23]) and the study of logarithms of signatures (see [2]).*

Let $n \in \mathbb{N}$ and let $G^{(n)} := \pi_n(G^{(*)})$. $G^{(n)}$ naturally contains the truncated signatures of degree n of all paths of bounded variation and has a smooth group structure:

Proposition 1.23. $G^{(n)}$ *is a closed Lie subgroup of* $\widetilde{T}^{(n)}(E)$ *called the free nilpotent group of step* n *over* E*.*

It appears that $G^{(*)}$ is much wider than the range of S. Let $n \in \mathbb{N}^*$. We have shown so far that $S_n(\mathcal{V}^1([0,T],E)) \subseteq G^{(n)}$ and that $S_n(\mathcal{V}^1([0,T],E))$ is exactly the set of terminal values to the equation (1.3). A geometric study of $G^{(n)}$ (see [10]), like the one we propose later in this work, shows that in fact $S_n(\mathcal{V}^1([0,T],E)) = G^{(n)}$, i.e.

$$\forall \mathbf{a} \in G^{(n)}, \exists x \in \mathcal{V}^1([0,T],E) \text{ such that: } \quad \pi_n(\mathbf{a}) = S_n(x)$$

Definition 1.24. *Let $t \in [0,T]$ and $x : [0,t] \to E$ and $y : [t,T] \to E$ be any two paths. We call the concatenation of x and y the path denoted by $x * y$ and defined on $[0,T]$ by:*

$$\begin{cases} (x*y)_u = x_u & , \text{if} \quad u \in [0,t] \\ (x*y)_u = y_u - y_t + x_t & , \text{if} \quad u \in [t,T] \end{cases}$$

We define in a similar way the concatenation of any two paths defined on two adjacent segments.

Remark 1.25. *The concatenation operation is associative.*

The second property that is of interest to us is what we call the multiplicativity property. The following theorem is due to Chen ([4]). For a proof, see also [2]

Theorem 1.26. *Let $t \in [0,T]$. Consider $x \in \mathcal{V}^1([0,t],E)$ and $y \in \mathcal{V}^1([t,T],E)$. Then $x * y \in \mathcal{V}^1([0,T],E)$ and:*

$$S(x*y)_{(0,T)} = S(x)_{(0,t)} \otimes S(y)_{(t,T)}$$

Remark 1.27. *Theorem 1.26 is usually used in the following way (which is equivalent to the previous formulation): for every path $x \in \mathcal{V}^1([0,T],E)$ and $s,u,t \in [0,T]$ such that $s \le u \le t$, we have:*

$$S(x)_{(s,t)} = S(x)_{(s,u)} \otimes S(x)_{(u,t)}$$

Remark 1.28. *For $x \in \mathcal{V}^1([0,T],E)$ and for $(s,t) \in \Delta_{[0,T]}$, $S(x)_{(s,t)} \in \widetilde{T}((E))$ and is therefore invertible. Consequently, using theorem 1.26:*

$$S(x)_{(s,t)} = (S(x)_{(0,s)})^{-1} \otimes S(x)_{(0,t)}.$$

It is then equivalent to study the path $u \mapsto S(x)_{(0,u)}$ on the interval $[0,T]$ or $(s,t) \mapsto S(x)_{(s,t)}$ on the simplex $\Delta_{[0,T]}$.

Finally, the norms of successive terms in a signature of a path of bounded variation decay in a factorial way in repect to the 1-variation:

Theorem 1.29. *Let x be in $\mathcal{V}^1([0,T],E)$, then:*

$$\forall n \in \mathbb{N}^* \quad \forall (s,t) \in \Delta_{[0,T]} \quad \|S^n(x)_{(s,t)}\| \leq \frac{\|x\|^n_{1,[s,t]}}{n!}.$$

1.5 Rough Paths

The theory of rough paths generalizes the concept of signatures to more irregular paths and provides the tools to solving differential equations driven by these without having to build a whole new theory of integration for each one of them (as in Itô's calculus). It is in this wider framework that we will introduce the reachability problem.

The concept of rough paths finds its source in the signature and the main analytic and algebraic properties that it satisfies. The space of geometric rough paths is indeed simply defined as the completion of the set of signatures of paths with bounded variation under a suitably chosen metric similar to the p-variation metric for paths introduced in section 1.1. After introducing the basic definitions and constructions, we state the fundamental theorem of rough differential equations needed in the subsequent chapters.

1.5.1 Multiplicative functionals

Definition 1.30. *Let E be a normed vector space and $T > 0$. Let X be a map on $\Delta_{[0,T]}$ with values in $T((E))$ (respectively in $T^{(n)}(E)$, with $n \in \mathbb{N}^*$). X is said to be a multiplicative functional (resp. a multiplicative functional of degree n) if the following holds:*

1. *X is continuous.*

2. $\forall t \in [0,T] \quad X_{(t,t)} = \mathbf{1}$.

3. *X is multiplicative:* $\forall\, 0 \le s \le u \le t \le T \quad X_{(s,t)} = X_{(s,u)} \otimes X_{(u,t)}$.

Remark 1.31. *When no confusion is possible, we may use the term multiplicative functional with no reference to its degree being finite or not.*

Remark 1.32. *We will use the notation X^i for the component of X of degree i.*

Is is clear, by Chen's theorem 1.26, that, for every path $x \in \mathcal{V}^1([0,T],E)$, $S(x)$ is a multiplicative functional and that for every $n \in \mathbb{N}^*$, $S_n(X)$ is a multiplicative functional of degree n. The concept of multiplicative functionals is a generalisation, then, of the Chen's multiplicativity property for signatures.

1.5.2 p-variation metric and controls

Definition 1.33. *Let $p \ge 1$. Let X be a map on $\Delta_{[0,T]}$ with values in $T((E))$ (respectively in $T^{(n)}(E)$, with $n \in \mathbb{N}^*$) and ω be a control over $[0,T]$. X is said to have a finite p-variation over $[0,T]$ controlled by ω if:*

$$\forall i \in \mathbb{N}^* (\text{resp.} \forall i \in [\![1,n]\!]) \quad \forall\, 0 \le s \le t \le T \quad \|X^i_{(s,t)}\| \le \frac{\omega(s,t)^{\frac{i}{p}}}{\beta_p \left(\frac{i}{p}\right)!}$$

where we write $x!$ for $\Gamma(x+1)$, with Γ being the usual extension of the factorial function and:

$$\beta_p = p\left(1 + \sum_{k=1}^{\infty} \left(\frac{2}{k}\right)^{\frac{[p]+1}{p}}\right)$$

If there exists a control such that the previous properties holds, we may say that X has a finite p-variation over $[0,T]$ without mentionning the control. We denote by $\mathcal{C}_{0,p}(\Delta_{[0,T]}, T^{[p]}(E))$ the set of continuous paths defined from $\Delta_{[0,T]}$ to $T^{[p]}(E)$ that have finite p-variation.

We see, that for $n = 1$, the concept of p-variation in definition 1.33 is the same as the one introduced in section 1.1.

Remark 1.34. *Notice, for the moment, that the value of β_p isn't of much importance unless for multiplicative functionals of infinite order. Indeed, let $p \geq 1$, $n \in \mathbb{N}^*$ and X a multiplicative functional of degree n. Assume there exists a control ω and n constants $C_1, \dots, C_n$ such that:*

$$\forall i \in [\![1, n]\!] \quad \forall\, 0 \leq s \leq t \leq T \quad \|X^i_{(s,t)}\| \leq \frac{\omega(s,t)^{\frac{i}{p}}}{C_i}$$

Then, by putting:

$$D = \max_{i \in [\![1,n]\!]} \left(\frac{\beta_p \left(\frac{i}{p}\right)!}{C_i} \right)^{\frac{p}{i}}$$

and defining the control $\tilde{\omega}$ on $\Delta_{[0,T]}$ by:

$$\forall (s,t) \in \Delta_{[0,T]} \quad \tilde{\omega}(s,t) = D\omega(s,t)$$

we retrieve that X is a multiplicative functional of degree n of finite p-variation according to definition 1.33.

By theorem 1.29, signatures have finite 1-variation. The p-variation control substitutes the factorial decay property of signatures.

Remark 1.35. *One can also easily notice that for $1 \leq q \leq p$, a multiplicative functional of finite q-variation is of finite p-variation.*

The next result (the neo-classical inequality) will be of a technical use to us in a subsequent chapter and it is crucial for the extension theorem for rough paths (see [13] for a complete proof or [22] for a loose version of the inequality).

Lemma 1.36 (Neo-classical inequality).

$$\forall p \geq 1, \, \forall n \in \mathbb{N}, \, \forall a, b \in \mathbb{R}^+ \quad \frac{1}{p} \sum_{k=0}^{n} \frac{a^{\frac{k}{p}} b^{\frac{n-k}{p}}}{(\frac{k}{p})!(\frac{n-k}{p})!} \leq \frac{(a+b)^{\frac{n}{p}}}{(\frac{n}{p})!}$$

The following lemma shows that a multiplicative functional of finite p-variation is determined by its terms of degree less than or equal to $[p]$ (with $p \in \mathbb{R}_+^*$). The extension theorem, which we state later, gives the reciprocal of this result: a multiplicative functional of degree $[p]$ and of finite p-variation can be extended to a multiplicative functional (of an arbitrary degree) of finite p-variation.

Lemma 1.37. *Let $p \geq 1$ and $n \in \mathbb{N}^*$ such that $n \geq [p]$ and X and Y be two multiplicative functionals (resp. multiplicative functionals of degree n) that have a finite p-variation over $[0, T]$. If $\pi_{[p]}(X) = \pi_{[p]}(Y)$, then $X = Y$.*

Proof. We are going to show that $X^{[p]+1} = Y^{[p]+1}$, terms of higher degrees are dealt with in the same way by strong induction.
Assume that $\pi_{[p]}(X) = \pi_{[p]}(Y)$. Let ω_1 (resp. ω_2) be a control of the p-variation of X (resp. of Y). Define ω on $\Delta_{[0,T]}$ by:

$$\omega(s,t) = \left(\frac{p}{\beta_p (\frac{[p]+1}{p})!} \right)^{\frac{p}{[p]+1}} (\omega_1(s,t) + \omega_2(s,t))$$

ω is a control function and for every $(s,t) \in \Delta_{[0,T]}$ (as a result of the Neo-classical inequality):

$$\frac{\omega_1(s,t)^{\frac{[p]+1}{p}}}{\beta_p \left(\frac{[p]+1}{p}\right)!} + \frac{\omega_2(s,t)^{\frac{[p]+1}{p}}}{\beta_p \left(\frac{[p]+1}{p}\right)!} \leq \frac{p}{\beta_p} \frac{(\omega_1(s,t) + \omega_2(s,t))^{\frac{[p]+1}{p}}}{\left(\frac{[p]+1}{p}\right)!} \leq \omega(s,t)^{\frac{[p]+1}{p}}$$

Let $(s,t) \in \Delta_{[0,T]}$. By the multicativity property and by the assumption made at the beginning of the proof, one has:

$$\begin{aligned} X^{[p]+1}_{(0,t)} - X^{[p]+1}_{(0,s)} &= \sum_{k=1}^{[p]} X^k_{(0,s)} \otimes X^{[p]+1-k}_{(s,t)} + X^{[p]+1}_{(s,t)} \\ &= \sum_{k=1}^{[p]} Y^k_{(0,s)} \otimes Y^{[p]+1-k}_{(s,t)} + X^{[p]+1}_{(s,t)} \\ &= Y^{[p]+1}_{(0,t)} - Y^{[p]+1}_{(0,s)} + X^{[p]+1}_{(s,t)} - Y^{[p]+1}_{(s,t)} \end{aligned}$$

Therefore, by the control inequality of $X^{[p]+1}_{(s,t)}$ and $Y^{[p]+1}_{(s,t)}$ and the remark earlier made about ω:

$$\|(X^{[p]+1}_{(0,t)} - Y^{[p]+1}_{(0,t)}) - (X^{[p]+1}_{(0,s)} - Y^{[p]+1}_{(0,s)})\| \leq \omega(s,t)^{\frac{[p]+1}{p}}$$

The path $u \to X^{[p]+1}_{(0,u)} - Y^{[p]+1}_{(0,u)}$ is continuous on $[0,T]$ and the inequality above shows, using proposition 1.7, that is has finite $\frac{p}{[p]+1}$-variation. Hence, by proposition 1.3, this path is constant. It is null as its initial value is null. □

Theorem 1.38 (Extension theorem). *Let $p \geq 1$ and $n \in \mathbb{N}^* \cup \{\infty\}$ such that $n \geq [p]$. Let E be a Banach space. Let X be a multiplicative functional of degree $[p]$ that has a finite p-variation over $[0,T]$ controlled by a control function ω. There exists a unique multiplicative functional $\tilde{X}$ of degree n that has a finite p-variation over $[0,T]$ and such that $\pi_{[p]}(X) = \pi_{[p]}(\tilde{X})$. Furthermore, the p-variation of $\tilde{X}$ is also controlled by ω.*

Remark 1.39. *Theorem 1.38 partially encompasses Young's integration theory (see [30]) in the sense that it allows the construction of the signature of a path of finite p-variation, when $p < 2$, using only the increments of the path.*

The signature of a path of bounded variation is, therefore, by theorem 1.38, the only multiplicative functional with finite 1-variation which component of degree 1 corresponds to the increments of said path.

We are now ready to give a definition for rough paths:

Definition 1.40. *Let $p \geq 1$ and E be a Banach space. A p-rough path is a multiplicative functional of degree $[p]$ that has finite p-variation. The space of p-rough paths over $[0,T]$ is denoted by $\Omega_p^{[0,T]}(E)$.*

Definition 1.41. *We define on $\mathcal{C}_{0,p}(\Delta_{[0,T]}, T^{[p]}(E))$ the p-variation metric, denoted by $\tilde{d}_p$, by the following:*

$$\tilde{d}_p(X,Y) = \max_{0 \leq i \leq [p]} \sup_{D \in \mathcal{D}_{[0,T]}} \left(\sum_D \| X^i_{(t_j,t_{j+1})} - Y^i_{(t_j,t_{j+1})} \|^{\frac{p}{i}} \right)^{\frac{1}{p}}$$

for all $X, Y \in \mathcal{C}_{0,p}(\Delta_{[0,T]}, T^{[p]}(E))$.

(For a subdivision $D = (t_j)_{0 \leq j \leq n}$:

$$\sum_D \| X^i_{(t_j,t_{j+1})} - Y^i_{(t_j,t_{j+1})} \|^{\frac{p}{i}} := \sum_{j=0}^{n-1} \| X^i_{(t_j,t_{j+1})} - Y^i_{(t_j,t_{j+1})} \|^{\frac{p}{i}})$$

Definition 1.42. *Let $p \geq 1$ and E be a Banach space. Let $X \in \mathcal{C}_{0,p}(\Delta_{[0,T]}, T^{[p]}(E))$ and $(X(n))_{n \in \mathbb{N}}$ be a sequence of elements of $\mathcal{C}_{0,p}(\Delta_{[0,T]}, T^{[p]}(E))$. We say that $(X(n))_{n \in \mathbb{N}}$ converges to X in the p-variation topology, if there exists a control function ω and a sequence $(a(n))_{n \in \mathbb{N}}$ converging to 0 such that:*

1. *ω controls the p-variation of X and of $X(n)$ for each $n \in \mathbb{N}$.*
2. *$\forall n \in \mathbb{N}, \forall (s,t) \in \Delta_{[0,T]}, \forall i \in [\![1,[p]]\!] : \| X^i_{s,t} - X^i_{s,t}(n) \| \leq a(n) \omega(s,t)^{i/p}$.*

Convergence in the p-variation topology and the p-variation metric are "almost" equivalent in the following way:

Proposition 1.43. *Let $p \geq 1$ and E be a Banach space. Let $X \in \mathcal{C}_{0,p}(\Delta_{[0,T]}, T^{[p]}(E))$ and $(X(n))_{n \in \mathbb{N}}$ be a sequence of elements of $\mathcal{C}_{0,p}(\Delta_{[0,T]}, T^{[p]}(E))$.*

- *if $(X(n))_{n \in \mathbb{N}}$ converges to X in the p-variation metric, then the convergence also occurs in the p-variation topology.*

- *if* $(X(n))_{n\in\mathbb{N}}$ *converges to* X *in the* p*-variation topology, then there exists a subsequence of* $(X(n))_{n\in\mathbb{N}}$ *converging to* X *in the* p*-variation metric.*

The completeness of the space of rough paths is an important property. See [22] for a complete proof:

Theorem 1.44. $(\Omega_p^{[0,T]}(E), \tilde{d}_p)$ *is a complete metric space.*

1.5.3 Geometric p-rough paths

Definition 1.45. *We define the set of geometric* p*-rough paths to be the closure of the set* $\{S_{[p]}(x) | x \in \mathcal{V}^1([0,T], E)\}$ *in the* p*-variation metric. It is denoted by* $G\Omega_p^{[0,T]}(E)$*.*

Rough paths do not take into account the shuffle product property satisfied by signatures. We show that geometric rough paths do not have this "setback":

Proposition 1.46. *Let* X *be a geometric* p*-rough path over* $[0,T]$*. Then* X *is an element of* $\Omega_p^{[0,T]}(E)$ *(in particular,* X *is multiplicative and has finite* p*-variation) and for every* $(s,t) \in \Delta_{[0,T]}$*,* $X_{(s,t)} \in G^{([p])}$*.*

Proof. As the signatures of paths of bounded variation satisfy these algebraic and analytic properties, this proposition is a direct consequence of theorem 1.44 and the continuity of linear forms of $T^{[p]}(E)$ in the p-variation metric. □

Remark 1.47. *It is interesting to see the analogy between the restriction of* $\tilde{d}_p$ *on the space of geometric rough paths and the* p*-variation norm for paths taking their values in the Carnot group* $(G^{[p]}, \otimes)$ *when endowed with the homogeneous norm:* $\|.\| : (1, g_1, \ldots, g_{[p]}) \mapsto \max_{1\leq i\leq [p]} \|i! g_i\|^{1/i}$ *(see [23]).*

Like for simple paths, we can define the concatenation of paths taking their values in the truncated tensor algebra:

Definition 1.48. *Let $n \in \mathbb{N}^*$. Let E be a vector space. Let $s, u, t \in \mathbb{R}$ such that $s \leq u \leq t$. Let X (resp. Y) be a functional defined on $\Delta_{[s,u]}$ (resp. on $\Delta_{[u,t]}$) with values in $T^n(E)$. We define the concatenation of X and Y, denoted $X * Y$, to be the functional over $\Delta_{[s,t]}$ defined as follows: for $(a, b) \in \Delta_{[s,t]}$*

$$(X * Y)_{(a,b)} = \begin{cases} X_{(a,b)} & , \quad \textit{if } b \leq u \\ X_{(a,u)} \otimes Y_{(a,u)} & , \quad \textit{if } a \leq u \leq b \\ Y_{(a,b)} & , \quad \textit{if } u \leq a \end{cases}$$

The following theorem is straight-forward:

Theorem 1.49. *Let $p \geq 1$. Let E be a Banach space. Let $s, u, t \in \mathbb{R}$ such that $s \leq u \leq t$. Let X (resp. Y) be a functional defined on $\Delta_{[s,u]}$ (resp. on $\Delta_{[u,t]}$) with values in $T^{[p]}(E)$. Then:*

- *If X and Y are multiplicative functionals, then $X * Y$ is a multiplicative functional;*
- *If X and Y have finite p-variation, then $X * Y$ has finite p-variation;*
- *If X and Y are geometric p-rough paths, then $X * Y$ is a geometric p-rough path.*

1.5.4 The integral of Lipschitz one-forms along geometric p-rough paths

When trying to make sense of integrals of one-forms along rough paths, it is very important to be able to control the smoothness (in terms of variation) of the image of the path under the one-form. It appears that Lipschitz maps, in the sense of Stein [25], are the appropriate type of maps to use in this frame:

Definition 1.50. *Let $n \in \mathbb{N}$ and $0 < \varepsilon \leq 1$. Let E and F be two normed vector spaces and U be a subset of E. We will use, when there is no ambiguity, the same notation $\|.\|$*

to designate norms on $E^{\otimes k}$, for $k \in [\![1, n]\!]$, and the norm on F. For every $k \in [\![0, n]\!]$, let $f^k : U \to \mathcal{L}(E^{\otimes k}, F)$ be a map with values in the space of the symmetric k-linear mappings from E to F. The collection $f = (f^0, f^1, \dots, f^n)$ is said to be Lipschitz of degree $n + \varepsilon$ on U (or in short a Lip $-(n+\varepsilon)$ map) if there exist a constant M and $n+1$ maps $R_k : E \times E \to \mathcal{L}(E^{\otimes k}, F)$, $k \in [\![0, n]\!]$ (called the associated remainders) such that for all $k \in [\![0, n]\!]$:

$$\sup_{x \in U} \|f^k(x)\| \leq M$$

$$\forall x, y \in U, \forall v \in E^{\otimes k} : f^k(x)(v) = \sum_{j=k}^{n} f^j(y)\left(\frac{v \otimes (x-y)^{\otimes (j-k)}}{(j-k)!}\right) + R_k(x, y)(v),$$

$$\forall x, y \in U : \|R_k(x, y)\| \leq M \|x - y\|^{n+\varepsilon-k}.$$

The smallest constant M for which the properties above hold is called the Lipschitz-$(n+\varepsilon)$ norm of f and is denoted by $\|f\|_{Lip-(n+\varepsilon)}$.

Remark 1.51. *On any open subset of U (and in particular on the interior of U), $f^1, \dots, f^n$ are the successive derivatives of f^0. However, these maps are not necessarily uniquely determined by f^0 on an arbitrary subset of U.*

If $f^0 : U \to F$ is a map such that there exist $f^1, \dots, f^n$ such that $(f^0, f^1, \dots, f^n)$ is Lip $-$ $(n + \varepsilon)$, we will often say that f^0 is Lip $-$ $(n + \varepsilon)$ with no mention of $f^1, \dots, f^n$. We will call the functions $R_0, R_1, \dots, R_n$ its associated remainders.

It will be useful to us, and contrary to custom in most studies of rough paths on Banach spaces, to attach a starting point to our geometric rough paths as we will be mostly dealing with integrals of rough paths, rough differential equations and rough paths on manifolds; all of which require assigning a starting point to a rough path. A geometric rough path will be a pair (x, X), where x is called the starting point and X is a geometric rough path in the sense of definition 1.45. We will denote the "new space" of geometric rough paths with starting points in the same way as above, and when there is ambiguity, we will introduce an equivalence relation $\sim$ on $G\Omega_p^{[0,T]}(E)$ that makes two rough paths with the same increments equivalent, i.e.:

$$(x, X) \sim (y, Y) \Leftrightarrow X = Y$$

On $G\Omega_p^{[0,T]}(E)$ we define a metric d_p as the product metric of $\tilde{d}_p$ and the norm on E:

$$d_p((x, X), (y, Y)) = \max(\|x - y\|, \tilde{d}_p(X, Y))$$

Theorem 1.52. *Let $\gamma, p \in \mathbb{R}$ such that $\gamma > p \geq 1$. Let E and F be two Banach spaces. Let $\alpha : E \to \mathcal{L}_c(E, F)$ be a Lip $-$ $(\gamma - 1)$ one-form. There exists a unique continuous map:*

$$I_\alpha : (G\Omega_p^{[0,T]}(E), d_p) \longrightarrow (G\Omega_p^{[0,T]}(F)/\sim, \tilde{d}_p)$$

such that, for all $x \in \mathcal{V}^1([0, T], E)$, $I_\alpha(x_0, S_{[p]}(x)) = S_{[p]}(\int \alpha(x)\mathrm{d}x)$. For a geometric p-rough path with a starting point (x_0, X), we denote:

$$I_\alpha(x_0, X) = \int \alpha(x_0, X)\mathrm{d}X$$

Definition 1.53. *Let $\gamma, p \in \mathbb{R}$ such that $\gamma > p \geq 1$. Let E and F be two Banach spaces. Let $f : E \to F$ be a Lip$-\gamma$ map. We define the image of a geometric p-rough path (x, X) in E by f to be the geometric p-rough path in F given by $(f(x), \int \mathrm{d}f(x, X)\mathrm{d}X)$.*

Remark 1.54. *It is clear by theorem 1.52 that the map giving the images of geometric p-rough paths by a Lip$-\gamma$ map is continuous in the p-variation metric (assuming $\gamma > p$).*

1.6 Rough Differential Equations

We can now give a meaning to differential equations driven by rough paths in a way that is generalizes solving differential equations with signals with bounded variation:

Definition 1.55. *Let $\gamma, p \in \mathbb{R}$ such that $\gamma > p \geq 1$ and $T \geq 0$. Let E and F be two Banach spaces, with $T(E)$ endowed with admissible norms. Let $f : F \to \mathcal{L}_c(E, F)$ be a Lip-$(\gamma - 1)$ map. For any pair $(X, y_0) \in G\Omega_p(E) \times F$, a solution to the rough differential equation:*

$$\begin{cases} \mathrm{d}Y_t = f(Y_t)\mathrm{d}X_t \quad , \forall t \in [0, T] \\ Y_0 = y_0 \end{cases}$$

is a geometric p-rough path Z in $E \times F$ with starting point $(0, y_0)$ satisfying the fixed point property

$$Z = \int h(Z)\mathrm{d}Z$$

where h is the map:

$$\begin{array}{rccc} h : & E \times F & \longrightarrow & \mathcal{L}(E, F) \\ & (a, b) & \longmapsto & ((u, v) \mapsto (u, f(b)u)) \end{array}$$

and is such that $\pi(Z) = X$, where π is the natural projection from $E \times F$ to E.

Theorem 1.56. *Let $\gamma, p \in \mathbb{R}$ such that $\gamma > p \geq 1$ and $T \geq 0$. Let E and F be two Banach spaces, with $T(E)$ endowed with admissible norms. Let $f : F \to \mathcal{L}_c(E, F)$ be a Lip $- \gamma$ map. For any pair $(X, y_0) \in G\Omega_p(E) \times F$, the rough differential equation:*

$$\begin{cases} \mathrm{d}Y_t = f(Y_t)\mathrm{d}X_t \quad , \forall t \in [0, T] \\ Y_0 = y_0 \end{cases} \tag{1.4}$$

admits a unique solution. Moreover, the Itô map:

$$I_f : (G\Omega_p(E)/\sim \times F, d_p) \to (G\Omega_p(E \times F), d_p)$$

mapping every pair (X, y_0) *to the solution to (1.4) is continuous.*

Another strong argument if favour of Lipschitz maps is that they allow us to get quantitative estimates of the convergence rate of the Picard iterations used to construct a solution to (1.4) in contrast to asymptotic estimates that one usually obtains when working with “traditional” smooth maps and this makes an important difference when computing solutions numerically.

Chapter 2

The reachability problem

We now have all the tools necessary to introduce the reachability problem. In this short chapter, we will highlight the main achievements made so far in this direction by other authors and explain how we aim to solve it in the case of rough differential equations.

2.1 Formulation of the problem

Let $d \in \mathbb{N}$, $\gamma, p \in \mathbb{R}$ such that $\gamma > p \geq 1$ and $T > 0$. Let E be a finite-dimensional vector space. Let $(f^1, \ldots, f^d)$ be a family of $\mathrm{Lip} - \gamma$ vector fields on E and define the $\mathrm{Lip} - \gamma$ map $f := (f^1, \ldots, f^d)$ in the following way:

$$\begin{array}{rccl} f: & E & \longrightarrow & \mathcal{L}(\mathbb{R}^d, E) \\ & y & \longmapsto & ((x^1, \ldots, x^d) \mapsto \sum_{i=1}^{d} f^i(y) x^i) \end{array}$$

Given a geometric p-rough path X in $\mathbb{R}^d$ ($X \in G\Omega_p(\mathbb{R}^d)$) and $y_0 \in E$, we consider the (rough) differential equation associated with f and the initial condition y_0 and driven by

X:

$$\begin{cases} \mathrm{d}Y_t &= f(Y_t)\mathrm{d}X_t \quad ,\forall t \in [0,T] \\ y(0) &= y_0 \end{cases} \tag{2.1}$$

If $x = (x^1, \ldots, x^d)$ is a path in $\mathbb{R}^d$ such that it is the trace of the geometric p-rough path X, i.e. $\forall (s,t) \in \Delta_{[0,T]} : x_t - x_s = X^1_{s,t}$ and when no confusion is possible, we rewrite the equation (2.1) in its classical form (i.e. exactly as in the case where both x and X have finite 1-variation):

$$\begin{cases} \mathrm{d}Y_t &= \sum_{i=1}^{d} f^i(Y_t)\mathrm{d}x^i_t, \forall t \in [0,T] \\ y(0) &= y_0 \end{cases} \tag{2.2}$$

By theorem 1.56, the solution (X, Y) to such equation exists and is unique. Let $y(X)$ be the trace of its projection into E, i.e. $y(X) = y_0 + Y^1_{0,\cdot}$, and denote its terminal value by $y_T(X)$.

Definition 2.1 (Reachability set)**.** *Let G be a family of geometric p-rough paths in $\mathbb{R}^d$. We call the reachability set associated with the family of rough differential equations (2.1) defined by the vector fields $(f^1, \ldots, f^d)$, the initial condition y_0 and the set of controls G the set:*

$$\mathcal{R}(y_0, f, G) = \{y_T(X) | X \in G\}$$

In the following chapters, given an initial condition y_0 and a family $f := (f^1, \ldots, f^d)$ of $\mathrm{Lip} - \gamma$ vector fields on E, we set out to answer the following questions:

1. Does $\mathcal{R}(y_0, f, G\Omega_p(\mathbb{R}^d))$ have a nicely identifiable geometric structure?
2. Can we identify a simple family G of geometric p-rough paths in $\mathbb{R}^d$, for example the signatures of a certain class of smooth paths, such that:

$$\mathcal{R}(y_0, f, G\Omega_p(\mathbb{R}^d)) = \mathcal{R}(y_0, f, G)?$$

Let us make the remark that by the continuity of the Itô map associated to (2.1) and the very definition of a geometric rough path, given a geometric p-rough path X in $\mathbb{R}^d$, one can approximate the solution $Y(X)$ in the d_p norm by the solution to the R.D.E. when it is driven by a path of bounded variation. Therefore, using the notations above, one can approximate $y(X)$ by driving the R.D.E. (2.1) by a path of bounded variation. The question we ask here is slightly different: we are only interested in the terminal value $y_T(X)$ of the output of the system and, as a price for this weaker condition, we want to attain the exact value $y_T(X)$ by driving the R.D.E. (2.1) by a *simpler path*, the nature of which we aim to define in the rest of this work.

2.2 Chow's theorem

In the case of the equation (1.3), we know that $G^{(n)}$ contains all the terminal values of the solutions whenever the control signal is a path of bounded variation. The reachability problem (known, in this particular case, as the connectivity problem) asks the reciprocal question: is every point in $G^{(n)}$ the terminal value of a solution to (1.3)? And, in this case, is it true that we can reach every point in $G^{(n)}$ as the terminal value of a solution to (1.3) when the class of control signals is restricted, for example, to paths of bounded variation? If we assume that E is finite-dimensional, then the answer is affirmative and is given by Chow's theorem. This is achieved by closely studying the geometric structure of the reachability set (here $G^{(n)}$, a Lie group) as we will set out to do later.

2.2.1 Piece-wise integral curves

Let us give first a definition of the flow of a vector field and state the fundamental theorem of ordinary differential equations (O.D.E.s) which ensures the existence and uniqueness of flows:

Definition 2.2. *Let I be an open interval. Let M be a $\mathcal{C}^1$-manifold and A be a vector field on M. A $\mathcal{C}^1$-path $\gamma : I \to M$ is said to be an integral curve of A if:*

$$\forall t \in I : \quad \gamma'(t) = A(\gamma(t))$$

If $0 \in I$ and $x = \gamma(0)$, we say that x is the starting point of γ. If, furthermore, U is a subset of M and $\tilde{A} : I \times U \to M$ is such that, for every $x \in U$, $t \mapsto \tilde{A}(t, x)$ is an integral curve of A starting at x, we say then that $\tilde{A}$ is a local flow (or global flow if $I \times U = \mathbb{R} \times M$) of A on $I \times U$.

Remark 2.3. *Note that, in the general case, the existence of a global flow does not necessarily hold, even when the vector field is smooth. One can only get the existence of local flows.*

We can however state the following theorem when one is working in a Banach space and the vector field is Lipschitz continuous:

Theorem 2.4. *Let A be a $Lip-1$ vector field on a Banach space E. Then there exists a unique global flow of A on $\mathbb{R} \times E$.*

Remark 2.5. *Note that theorem 2.4 is a special case of theorem 1.9.*

Definition 2.6. *Let M be a $\mathcal{C}^1$-manifold and $p \in M$. Let $\mathcal{D}$ be a family of vector fields on M. A piece-wise integral curve of $\mathcal{D}$ starting at p is a path starting at p and that is a concatenation of finitely many integral curves of vector fields contained in $\mathcal{D}$ or $-\mathcal{D}$. The set of all points of M that can be reached by piece-wise integral curves of $\mathcal{D}$ starting at p is called the $\mathcal{D}$-orbit at p and will be denoted by $\mathbb{L}_p(\mathcal{D})$. If $q \in \mathbb{L}_p(\mathcal{D})$, we say that p and q can be joined by a piece-wise integral curve of $\mathcal{D}$.*

Example 2.7. *Consider the differential equation (2.2). If* $\mathcal{D} = span\{f^1, \dots, f^d\}$*, then* $\mathbb{L}_{y_0}(\mathcal{D})$ *is the reachability set associated to the set of controls consisting of all piecewise linear paths in* $\mathbb{R}^d$ *and the starting point* $y_0 \in E$*.*

Proof. Let $a = (a_i)_{1 \leq i \leq d}$ be a family of real numbers and $(\vec{e}_i)_{1 \leq i \leq d}$ be the canonical basis of $\mathbb{R}^d$. Consider the signal x_a defined over $[0, T]$ by:

$$x_a(t) = t \sum_{i=1}^{d} a_i \vec{e}_i$$

Then the trace of the solution to the equation (2.2) when driven by x_a is given by solving the O.D.E.:

$$\begin{cases} \mathrm{d}y_{y_0,a}(t) & = & \sum_{i=1}^{d} a_i f^i(y(t))\mathrm{d}t, \forall t \in [0, T] \\ y_{y_0,a}(0) & = & y_0 \end{cases}$$

If we denote G_a to be the vector field $\sum_{i=1}^{d} a_i f^i$, then $y_{y_0,a}$ is simply the flow of G_a starting at y_0, i.e.:

$$\forall t : y_{y_0,a}(t) = \tilde{G}_a(t, y_0)$$

Let $a^1, \dots, a^r$ be elements of $\mathbb{R}^d$ and $(t_0 = 0, t_1, \dots, t_r = T)$ be a subdivision of $[0, T]$. For every $i \in [\![1, r]\!]$, let x^i be an affine path whose direction is a^i. Let x be the concatenation of the affine paths $x^1, \dots, x^r$. Define the sequence of paths $(y^i)_{1 \leq i \leq r}$ in the following recursive way:

1. y^1 is defined over $[0, t_1]$ by: $y^1(t) = \tilde{G}_{a^1}(t, y_0)$.
2. For $i \in [\![2, r]\!]$, y^i is defined over $[t_{i-1}, t_i]$ by $y^i(t) = \tilde{G}_{a^i}(t, y^{i-1}(t_{i-1}))$:

Then y, the trace of the solution to the equation (2.2) when driven by x, is given by the concatenation $y^1 * y^2 * \cdots * y^r$. As y^i is a flow of a vector field in $\mathcal{D}$, for every $i \in [\![1, r]\!]$, and y is starting at y_0, then $y_T(x) \in \mathbb{L}_{y_0}(\mathcal{D})$. The reciprocal is obtained immediately by a reverse argumentation of the above. $\square$

2.2.2 Chow's theorem and consequences

Notation. *For a family of vector fields* $\mathcal{D}$ *on a manifold* M*, we denote by* $\mathcal{D}^*$ *the closed algebra of* $\mathcal{D}$ *under Lie brackets.*

Chow's connectivity theorem can now be stated in the following way (see [5, 10]):

Theorem 2.8 (Chow)**.** *Let* M *be a connected smooth manifold and let* $\mathcal{D}$ *be a finite family of smooth vector fields in* M *such that, for every* $p \in M$, T_pM *is the span of the value at* p *of the vector fields in* $\mathcal{D}^*$*. Then every two points in* M *can be joined by a piece-wise integral curve of* $\mathcal{D}$*, i.e.* $M = \mathbb{L}_p(\mathcal{D})$*, where* p *is an arbitrary point in* M*.*

As a straightforward corollary of the theorem 2.8, we have the following:

Corollary 2.9. *Let* $d, n \in \mathbb{N}^*$*. Let* $G^{(n)}(\mathbb{R}^d)$ *be the free nilpotent group of step* n *over* $\mathbb{R}^d$*. Then:*

$$G^{(n)}(\mathbb{R}^d) = \mathbb{L}_{(1,0,\dots,0)}(\{f_1, \dots, f_n\})$$

where, for every $i \in [\![1, n]\!]$, f^i *is the left-invariant vector field in* $G^{(n)}(\mathbb{R}^d)$*, whose value at* $(1, 0, \dots, 0)$ *is* $\frac{\partial}{\partial x_i}|_{(1,0,\dots,0)}$*.*

Proof. Follows directly from the existence of a global exponential map between $G^{(n)}(\mathbb{R}^d)$ and its Lie algebra (see for example [2]) and theorem 2.8. $\square$

We introduce now axis paths in order to write $\mathbb{L}_p(\mathcal{D})$ (and $G^{(n)}(\mathbb{R}^d)$) as the reachability set of a reduced set of geometric rough paths:

Definition 2.10. *Let* $d \in \mathbb{N}^*$*. Denote by* $(\vec{e}_1, \dots, \vec{e}_d)$ *the canonical basis of* $\mathbb{R}^d$*. We call an axis path in* $\mathbb{R}^d$ *(with unit speed) a continuous path* x *defined over a compact interval* J

such that there exists a finite subdivision $(t_i)_{0\leq i\leq n}$ *of* J *such that for every* $i \in [\![1,n]\!]$*, there exists* $j \in [\![1,d]\!]$ *and* $\varepsilon \in \{-1;1\}$ *such that* $x_t - x_{t_{i-1}} = \varepsilon(t-t_{i-1})\vec{e}_j$*, for all* $t_{i-1} \leq t \leq t_i$*.*

Lemma 2.11. *Axis paths have finite variation.*

Example 2.12. *Consider the differential equation (2.2). If* $\mathcal{D} = \{\pm f^1, \ldots, \pm f^d\}$*, then* $\mathbb{L}_{y_0}(\mathcal{D})$ *is the reachability set associated to the set of controls consisting of all axis paths with unit speed in* $\mathbb{R}^d$ *and the starting point* $y_0 \in E$*.*

Proof. This proof is very similar to the proof to 2.7. Let $(\vec{e_i})$ be the canonical basis of $\mathbb{R}^d$. Let $j \in [\![1,d]\!]$ and $\varepsilon \in \{-1;1\}$. Consider the signal x_j defined over $[0,T]$ by:

$$x_j(t) = \varepsilon t \vec{e}_j$$

Then the trace of the solution to the equation (2.2) when driven by x_j is given by solving the O.D.E.:

$$\begin{cases} \mathrm{d}y_{y_0,\varepsilon,j}(t) &= \varepsilon f^j(y(t))\mathrm{d}t, \forall t \in [0,T] \\ y_{y_0,\varepsilon,j}(0) &= y_0 \end{cases}$$

If we denote $G_{\varepsilon,j}$ to be the vector field εf^j, then $y_{y_0,\varepsilon,j}$ is simply the flow of $G_{\varepsilon,j}$ starting at y_0, i.e.:

$$\forall t : y_{y_0,\varepsilon,j}(t) = \tilde{G}_{\varepsilon,j}(t, y_0)$$

Let $\varepsilon_1, \ldots, \varepsilon_r$ be elements of $\{-1;1\}$, $j_1, \ldots, j_r$ be elements of $[\![1,d]\!]$ and $(t_0 = 0, t_1, \ldots, t_r = T)$ be a subdivision of $[0,T]$. For every $i \in [\![1,r]\!]$, let x^i be an affine path defined over $[t_{i-1}, t_i]$ whose direction is $\varepsilon_i \vec{e}_{j_i}$. Let x be the concatenation of the affine paths $x^1, \ldots, x^r$. Define the sequence of paths $(y^i)_{1\leq i\leq r}$ in the following recursive way:

1. y^1 is defined over $[0,t_1]$ by: $y^1(t) = \tilde{G}_{\varepsilon_1,j_1}(t, y_0)$.
2. For $i \in [\![2,r]\!]$, y^i is defined over $[t_{i-1}, t_i]$ by $y^i(t) = \tilde{G}_{\varepsilon_i,j_i}(t, y^{i-1}(t_{i-1}))$:

Then y, the trace of the solution to the equation (2.2) when driven by x, is given by the concatenation $y^1 * y^2 * \cdots * y^r$. As y^i is a flow of a vector field in $\mathcal{D}$, for every $i \in [\![1, r]\!]$, and y is starting at y_0, then $y_T(x) \in \mathbb{L}_{y_0}(\mathcal{D})$. The reciprocal is again obtained immediately by a reverse argumentation of the above. □

As a main result of this section, we prove that, in the finite-dimensional case, $G^{(n)}$ corresponds to the reachability set associated with the set of axis paths as controls:

Theorem 2.13. *Let* $d, n \in \mathbb{N}^*$, $p \geq 1$ *and* $T > 0$. *Let* $G^{(n)}(\mathbb{R}^d)$ *be the free nilpotent group of step* n *over* $\mathbb{R}^d$. *Let* f_n *be defined as in the equation (1.3) and* $\mathcal{AP}_{[0,T]}$ *denote the set of the truncated signatures of order* n *of all axis paths in* $\mathbb{R}^d$ *defined over* $[0, T]$. *Then:*

$$G^{(n)}(\mathbb{R}^d) = \mathcal{R}((1, 0, \ldots, 0), f_n, G\Omega_p(\mathbb{R}^d)) = \mathcal{R}((1, 0, \ldots, 0), f_n, \mathcal{AP}_{[0,T]})$$

Proof. By proposition 1.46, $\mathcal{R}((1, 0, \ldots, 0), f_n, G\Omega_p(\mathbb{R}^d)) \subseteq G^{(n)}(\mathbb{R}^d)$, and we naturally have

$$\mathcal{R}((1, 0, \ldots, 0), f_n, \mathcal{AP}_{[0,T]}) \subseteq \mathcal{R}((1, 0, \ldots, 0), f_n, G\Omega_p(\mathbb{R}^d))$$

By corollary 2.9, $G^{(n)}(\mathbb{R}^d) = \mathbb{L}_{(1,0,\ldots,0)}(\{f_1, \ldots, f_n\})$, and by 2.12, $\mathbb{L}_{(1,0,\ldots,0)}(\{f_1, \ldots, f_n\}) = \mathcal{R}((1, 0, \ldots, 0), f_n, \mathcal{AP}_{[0,T]})$. Therefore:

$$G^{(n)}(\mathbb{R}^d) = \mathcal{R}((1, 0, \ldots, 0), f_n, \mathcal{AP}_{[0,T]})$$

which concludes this proof. □

2.3 Review of the literature and strategy

A variety of facets and versions of this problem have been studied for more than five decades now. In pure differential geometry, the reachability problem -as we aim to solve it- is divided into two parts:

- The integrability problem tries to answer the question whether one can build a submanifold whose tangent space is given. Frobenius' theorem can be seen as a primary attempt to answer this question. However, it has the main drawback of assuming that the dimension of the span of the values of the vector fields at any point is the same (see [18] for example).

- The connectivity problem is very well-known but only deals with half the issue: knowing that a set has a smooth manifold structure, can any two points of the said set be joined by a horizontal path? It is worth mentionning at this point that when the answer is affirmative (like in Chow's theorem), one can introduce a Carnot-Carathodory metric on the manifold as the length of the shortest horizontal path joining two points.

In order to focus our attention on the main features appearing in this context, we introduce the notion of distributions:

Definition 2.14. *Let M be a $\mathcal{C}^1$-manifold. A distribution Δ on M is a mapping that associates to every point $p \in M$ a linear subspace of T_pM which will be denoted by $\Delta(p)$.*

Definition 2.15. *Let M be a $\mathcal{C}^1$-manifold. Let $\mathcal{D}$ be a set of vector fields on M and Δ and Γ two distributions on M.*

- *We say that $\mathcal{D}$ spans Δ if, for every $p \in M$, we have:*

$$\Delta(p) = span\{A(p) | A \in \mathcal{D}\}$$

- *We say that Δ contains Γ and write $\Gamma \subseteq \Delta$ if, for every $p \in M$, $\Gamma(p) \subseteq \Delta(p)$.*

Remark 2.16. *Given a set of vector fields* $\mathcal{D}$ *on a* $\mathcal{C}^1$*-manifold* M*, there exists a natural distribution spanned by* $\mathcal{D}$ *and which is the smallest one as such. This distribution will be denoted by* $\mathcal{L}(\mathcal{D})$ $(\mathcal{L}(\mathcal{D}(p)) = span\{A(p)|A \in \mathcal{D}\}, \forall p \in M)$*.*

Definition 2.17. *Let* M *be a* $\mathcal{C}^1$*-manifold. Let* Δ *be a distribution on* M*. We say that* Δ *is involutive if, for every two vector fields* X, Y *on* M *such that* $\mathcal{L}(\{X, Y\}) \subseteq \Delta$ *and every* $p \in M$*, we have* $[X, Y](p) \in \Delta(p)$*.*

Definition 2.18. *Let* M *be a* $\mathcal{C}^1$*-manifold and* Δ *a distribution on* M*. Let* S *be an immersed* $\mathcal{C}^1$*-submanifold of* M*. We say that* S *is an integral submanifold of* Δ *if the tangent space at any point* $s \in S$ *is equal to* $\Delta(s)$*.*

Theorem 2.19 (Frobenius [9])**.** *Let* M *be a* $\mathcal{C}^1$*-manifold and* Δ *be an involutive distribution on* M *such that* $\{dim(\Delta(p))|p \in M\}$ *is a singleton. Then for every point* $p \in M$ *there exists a maximal integral submanifold of* Δ *(in the sense of inclusion) containing* p*.*

One of the first complete answers to the reachability problem as presented here and which combined both the study of the geometric structure of the reachability set and the question of the replacement of a control by another is to be found in control theory:

- In a series of papers ([14], [15] and [16]), R. Hermann proved that under certain circumstances, the orbit of a family of smooth vector fields has a structure of a connected smooth manifold. In [19], C. Lobry weakens Hermann's assumption and replaces it with the "locally of finite type" hypothesis. The idea behind these approaches is to build an integral manifold of a nice distribution that satisfies Chow's hypothesis then deduce that it is actually the reachability set associated to axis paths.

- H.J. Sussmann studied the problem more deeply and showed that the reachability set always has a smooth structure and gave a necessary and sufficient condition for

> the tangent space of the reachability set to be only generated by the span of the vector fields we start with (see [26] for example.)

While a structure of a smooth manifold on the reachability set answers the connectivity problem, it still has the drawback of not being suited to R.D.E.s, and, in general, to solving ordinary differential equations globally. In [3], Cass, Litterer and Lyons introduced the concept of Lipschitz manifolds and rough paths on them. We will base our work on this new structure to get our global quantitative answers. Our approach will be based on mimicking Sussman's strategy, but this time with the aim of endowing the reachability set (or the set of orbits, more precisely) with a Lipschitz structure so that we can solve rough differential equations on it. We identify then the solutions to these "manifold" R.D.E.s and the classical ones for (2.1) when solved in the ambient Euclidean space.

Chapter 3

Lipschitz maps and their flows

L.C. Young, in [30], uses the concept of p-variation ($p \geq 1$) to generalize Stieltjes' integration theory to paths of finite p-variation. In building a theory of differential equations using the aformentioned work, one needs to be able to control the smoothness (in terms of variation) of the image of a path of finite p-variation under the involved vector fields. It appears that Lipschitz maps, introduced by Stein in [25], are the appropriate type of maps to use in this frame and the wider one of rough paths, introduced by Lyons in [21]. Lipschitz maps correspond to the Hölder functions in the classical polynomial regularity structure (see Hairer [11]) which one can use to build solutions to certain types of SPDEs. This chapter regroups some of the basic properties of Lipschitz maps and their flows. While some of the results presented here have already appeared in other papers, most of them are either only and classically known in the case of smooth maps and need to be proved in the Lipschitz case for a better understanding of the Lipschitz geometry or are new and necessary for the development of numerical methods for rough paths. Below, we set out to answer basic and natural questions about this class of maps: do they have a nice embedding structure? How can they be linked to the more familiar class of $\mathcal{C}^n$ maps? Are they stable under compostion? etc.

3.1 Lipschitz maps

3.1.1 Norms on tensor product of spaces

One important feature one has to pay attention to when deriving properties of Lipschitz maps is the nature of the norms involved. We study here three types of norms which will enable us to prove the most needed results for the exposition of our work on rough paths.

Definition 3.1 (Projective property)**.** *Let E be a normed vector space. Let $n \in \mathbb{N}^*$. We say that $(E^{\otimes k})_{1\leq k\leq n}$ (respectively $(E^{\otimes k})_{k\geq 1}$) are endowed with norms satisfying the projective property if, for every $k \in [\![1, n]\!]$ (resp. $k \geq 1$) and $p, q \in \mathbb{N}$ such that $p + q = k$ and every $a \in E^{\otimes p}, b \in E^{\otimes q}$, we have $\|a \otimes b\| \leq \|a\|\|b\|$.*

When at least such one exists, norms satisfying the projective property are abundant in the following sense:

Proposition 3.2. *Let E be a normed vector space and $n \in \mathbb{N}^*$. Suppose $(\|.\|_k)_{1\leq k\leq n}$ are norms on $(E^{\otimes k})_{1\leq k\leq n}$ satisfying the projective property, then the norms $(\alpha^k\|.\|_k)_{1\leq k\leq n}$ and $(\beta\|.\|_k)_{1\leq k\leq n}$, where $\alpha > 0$ and $\beta \geq 1$, also satisfy the projective property.*

Example 3.3. *Let E be a finite dimensional vector space and let $(\vec{e}_1, \dots, \vec{e}_p)$ be a basis for E. Let $n \in \mathbb{N}^*$. Let $k \in [\![1, n]\!]$ and $p \geq 1$ and define the norms $\|.\|_{p,k}$ and $\|.\|_{\infty,k}$ on $E^{\otimes k}$ by the following: for $x \in E^{\otimes k}$, if $(x_{i_1,\dots,i_k})_{1\leq i_1,\dots,i_k\leq p}$ are the coordinates of x in the basis $(\vec{e}_{i_1} \otimes \cdots \otimes \vec{e}_{i_k})_{1\leq i_1,\dots,i_k\leq p}$ of $E^{\otimes k}$, i.e.:*

$$x = \sum_{1\leq i_1,\dots,i_k\leq p} x_{i_1,\dots,i_k}\vec{e}_{i_1} \otimes \cdots \otimes \vec{e}_{i_k}$$

then:

$$\|x\|_{p,k} = \left(\sum_{1\leq i_1,\dots,i_k\leq p} |x_{i_1,\dots,i_k}|^p\right)^{1/p} \quad \text{and} \quad \|x\|_{\infty,k} = \max_{1\leq i_1,\dots,i_k\leq p} |x_{i_1,\dots,i_k}|$$

Then $(\|.\|_{p,k})_{1\leq k\leq n}$ *and* $(\|.\|_{\infty,k})_{1\leq k\leq n}$ *are norms on* $(E^{\otimes k})_{1\leq k\leq n}$ *satisfying the projective property.*

Definition 3.4. *Let* E *and* F *be two normed vector spaces and* $u : E \to F$ *be a linear map. Let* $n \in \mathbb{N}^*$. *We define the map* $u^{\otimes n} : E^{\otimes n} \to F^{\otimes n}$ *as the unique linear map satisfying:*

$$\forall v_1, \dots, v_n \in E : \quad u^{\otimes n}(v_1 \otimes \cdots \otimes v_n) = u(v_1) \otimes \cdots \otimes u(v_1)$$

Definition 3.5 (Compatible norms)**.** *Let* E *and* F *be two normed vector spaces. Let* $n \in \mathbb{N}^*$ *and* $C \geq 0$. *We say that* $(E^{\otimes k})_{1\leq k\leq n}$ *and* $(F^{\otimes k})_{1\leq k\leq n}$ *are endowed with* C*-compatible norms if, for every bounded linear map* $u : E \to F$ *and every* $k \in [\![1, n]\!]$, *we have* $\|u^{\otimes k}\| \leq C\|u\|^k$. *When the value of* C *is irrelevant, we may simply say that the norms are compatible.*

Examples 3.6. *Let* E *be a finite dimensional vector space and let* $(\vec{e}_1, \dots, \vec{e}_p)$ *be a basis for* E. *Let* $n \in \mathbb{N}^*$. *Let* F *be a normed vector space. We assume that we have norms on* $(F^{\otimes k})_{1\leq k\leq n}$ *satisfying the projective property. Then:*

- *The norms* $(\|.\|_{1,k})_{1\leq k\leq n}$ *on* $(E^{\otimes k})_{1\leq k\leq n}$ *are* 1*-compatible with the norms on* $(F^{\otimes k})_{1\leq k\leq n}$.
- *The norms* $(\|.\|_{\infty,k})_{1\leq k\leq n}$ *(resp.* $(p^k\|.\|_{\infty,k})_{1\leq k\leq n}$*) on* $(E^{\otimes k})_{1\leq k\leq n}$ *are* p^n*-compatible (resp.* 1*-compatible) with the norms on* $(F^{\otimes k})_{1\leq k\leq n}$.
- *Let* $q > 1$. *The norms* $(\|.\|_{q,k})_{1\leq k\leq n}$ *(resp.* $(p^{k(1-1/q)}\|.\|_{q,k})_{1\leq k\leq n}$*) on* $(E^{\otimes k})_{1\leq k\leq n}$ *are* $p^{n(1-1/q)}$*-compatible (resp.* 1*-compatible) with the norms on* $(F^{\otimes k})_{1\leq k\leq n}$.

Remark 3.7. *As shown in the case of the norms given in the previous examples, given* C*-compatible norms, it is always possible to define new norms on* $(E^{\otimes k})_{1\leq k\leq n}$ *so that the new norms are* 1*-compatible and that the new norms on* $(E^{\otimes k})_{1\leq k\leq n}$ *satisfy the projective property if the original ones do.*

Definition 3.8 (Action of the Symmetric Group on Tensors)**.** *Let $n \in \mathbb{N}^*$, $\sigma \in \mathcal{S}_n$ and E be a vector space. We define the action of σ on the homogenous tensors of E of order n as a linear map by the following:*

$$\forall x_1, x_2, \dots, x_n \in E \quad \sigma(x_1 \otimes x_2 \otimes \dots \otimes x_n) = x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \dots \otimes x_{\sigma(n)}$$

Definition 3.9 (Symmetric norms)**.** *Let E be a vector space and $n \in \mathbb{N}^*$. The norm on $E^{\otimes n}$ is said to be symmetric if:*

$$\forall n \in \mathbb{N}^*, \forall \sigma \in \mathcal{S}_n, \forall x \in E^{\otimes n} \quad \|\sigma(x)\| = \|x\|$$

We show now how to control the Lipschitz norm of the Cartesian product of two Lipschitz maps.

Proposition 3.10. *Let $\gamma > 0$. Let E, F and G be normed vector spaces. Let U be a subset of E and let f (resp. g) be a map defined on U with values in F (resp. G). Let h be the map defined on U by $h = (f, g)$. Then:*

- *If f and g are Lip$-\gamma$ and $F \times G$ is endowed with the l^p norm ($p \in [1, \infty]$), then h is also Lip$-\gamma$ and $\|h\|_{Lip-\gamma}$ is less than or equal to the l^p norm of $(\|f\|_{Lip-\gamma}, \|g\|_{Lip-\gamma})$.*

- *If the norm $\|.\|_F$ on F and the norm $\|.\|_{F\times G}$ on $F \times G$ are such that there exists $C > 0$ satisfying:*

  $$\forall (x, y) \in F \times G : \quad \|x\|_F \leq C\|(x, y)\|_{F\times G}$$

  (note that the l^p norms on $F \times G$ satisfy this property, for $p \in [1, \infty]$), and if h is Lip $-\gamma$ then f is Lip $-\gamma$ and $\|f\|_{Lip-\gamma} \leq C\|h\|_{Lip-\gamma}$.

3.1.2 Local characterization and embeddings

Once the concept of Lipschitzness understood, one of the first and the most natural questions one may ask is whether Lip $-\gamma$ maps are Lip $-\gamma'$, for $\gamma \geq \gamma' > 0$. We deal first with the trivial case where the domain of definition of the map is bounded:

Lemma 3.11. *Let $\gamma, \gamma' > 0$ such that $\gamma' < \gamma$. Let E and F be two normed vector spaces and U be a bounded subset of E. Let $f : U \to F$ be a Lip $-\gamma$ map. We assume that $(E^{\otimes k})_{1 \leq k \leq \lfloor \gamma \rfloor}$ are endowed with norms satisfying the projective property. Then f is Lip $-\gamma'$ and if $L \geq 0$ is larger than or equal to the diameter of U then:*

$$\|f\|_{Lip-\gamma'} \leq \|f\|_{Lip-\gamma} \max\left(1, \sum_{j=\lfloor \gamma' \rfloor+1}^{\lfloor \gamma \rfloor} \frac{L^{j-\gamma'}}{(j-\lfloor \gamma' \rfloor)!} + L^{\gamma-\gamma'}\right)$$

Proof. Let $n, n' \in \mathbb{N}$, $(\varepsilon, \varepsilon') \in (0,1]^2$ such that $\gamma = n + \varepsilon$ and $\gamma' = n' + \varepsilon'$. Let $f^1, \ldots, f^n$ be maps on U such that $(f, f^1, \ldots, f^n)$ is Lip $-\gamma$ and let $R_0, \ldots, R_n$ be the associated remainders. For $k \in [\![0, n']\!]$, define $S_k : U \times U \to \mathcal{L}(E^{\otimes k}, F)$ as follows:

$$\forall x, y \in U, \forall v \in E^{\otimes k} : S_k(x,y)(v) = \sum_{j=n'+1}^{n} f^j(y)\left(\frac{v \otimes (x-y)^{\otimes(j-k)}}{(j-k)!}\right) + R_k(x,y)(v)$$

By a straightforward computation, one gets that, for all $x, y \in U$:

$$\|S_k(x,y)\| \leq \|f\|_{\mathrm{Lip}-\gamma}\left(\sum_{j=n'+1}^{n} \frac{L^{j-\gamma'}}{(j-k)!} + L^{\gamma-\gamma'}\right)\|x-y\|^{\gamma'-k}$$

By recognising the S_i's in the expansion formulas of the f_i's, we see therefore that $(f, f^1, \ldots, f^{n'})$ is Lip $-\gamma'$ with $S_0, \ldots, S_{n'}$ as remainders and:

$$\|f\|_{\mathrm{Lip}-\gamma'} \leq \|f\|_{\mathrm{Lip}-\gamma} \max\left(1, \sum_{j=n'+1}^{n} \frac{L^{j-\gamma'}}{(j-n')!} + L^{\gamma-\gamma'}\right)$$

□

Remark 3.12. *With the notations of the previous lemma, we have the following simple control:*

$$\sum_{j=\lfloor\gamma'\rfloor+1}^{\lfloor\gamma\rfloor} \frac{L^{j-\gamma'}}{(j-\lfloor\gamma'\rfloor)!} + L^{\gamma-\gamma'} \leq eL^{\gamma-\gamma'}$$

The aim now is to be able to go from the case where the domain of definition of the map is bounded to a more general one. This gives us an important local characterization of Lipschitz maps:

Lemma 3.13. *Let $\gamma > 0$. Let E and F be two normed vector spaces and U be a subset of E. For every $k \in [\![0, \lfloor\gamma\rfloor]\!]$, let $f^k : U \to \mathcal{L}(E^{\otimes k}, F)$ be a map with values in the space of the symmetric k-linear mappings from E to F. We assume that $(E^{\otimes k})_{1\leq k\leq\lfloor\gamma\rfloor}$ are endowed with norms satisfying the projective property and that there exists $\delta > 0$ and $C \geq 0$ such that, for every $x \in U$, $f_{|B(x,\delta)\cap U}$ is Lip $-\gamma$ with a norm less than or equal to C (where $f = (f^0, \ldots, f^{\lfloor\gamma\rfloor})$). Then f is Lip $-\gamma$ and:*

$$\|f\|_{Lip-\gamma} \leq C \max\left(1, \max_{0\leq k\leq\lfloor\gamma\rfloor} \frac{1}{\delta^{\gamma-k}}(1+\sum_{j=0}^{\lfloor\gamma\rfloor-k}\frac{\delta^j}{j!})\right)$$

Proof. Let $k \in [\![0, \lfloor\gamma\rfloor]\!]$. We already know that $\sup_{x\in U}\|f^k(x)\| \leq C$. Define $R_k : U\times U \to \mathcal{L}(E^{\otimes k}, F)$ as follows:

$$\forall x, y \in E, \forall v \in E^{\otimes k}: \quad R_k(x,y)(v) = f^k(x)(v) - \sum_{j=k}^{\lfloor\gamma\rfloor} f^j(y)(\frac{v\otimes(x-y)^{\otimes(j-k)}}{(j-k)!})$$

Let $x, y \in U$. If $\|x-y\| < \delta$, then, as $f_{|B(x,\delta)\cap U}$ is Lip $-\gamma$, we have:

$$\|R_k(x,y)\| \leq C\|x-y\|^{\gamma-k}$$

Assume that $\|x-y\| \geq \delta$, then, as the $(E^{\otimes k})_{1\leq k\leq n}$ are endowed with norms satisfying the projective property, we obtain:

$$\begin{aligned}
\frac{\|R_k(x,y)\|}{\|x-y\|^{\gamma-k}} &\leq \frac{\|f^k(x)\|}{\|x-y\|^{\gamma-k}} + \sum_{j=k}^{\lfloor\gamma\rfloor} \frac{\|f^j(y)\|}{\|x-y\|^{\gamma-j}(j-k)!} \\
&\leq C\left(\frac{1}{\delta^{\gamma-k}} + \sum_{j=k}^{\lfloor\gamma\rfloor} \frac{1}{\delta^{\gamma-j}(j-k)!}\right) \\
&\leq \frac{C}{\delta^{\gamma-k}}\left(1 + \sum_{j=0}^{\lfloor\gamma\rfloor-k} \frac{\delta^j}{j!}\right)
\end{aligned}$$

We deduce then that f is $\mathrm{Lip}-\gamma$ on U with the suggested upper-bound of $\|f\|_{\mathrm{Lip}-\gamma}$. $\square$

Remark 3.14. *With the notations of the previous lemma, we have:*

$$\max_{0\leq k\leq n} \frac{1}{\delta^{\gamma-k}}(1 + \sum_{j=0}^{n-k} \frac{\delta^j}{j!}) \leq (\frac{1+e}{\delta^\gamma}) \max(1, \delta^{\lfloor\gamma\rfloor})$$

We can now state the following natural embedding theorem:

Theorem 3.15. *Let $\gamma, \gamma' > 0$ such that $\gamma' < \gamma$. Let E and F be two normed vector spaces and U be a subset of E. We assume that $(E^{\otimes k})_{1\leq k\leq \lfloor\gamma\rfloor}$ are endowed with norms satisfying the projective property. Let $f : U \to F$ be a $Lip-\gamma$ map. Then f is $Lip-\gamma'$ and there exists a constant $M_{\gamma,\gamma'}$ (depending only on γ and γ') such that $\|f\|_{Lip-\gamma'} \leq M_{\gamma,\gamma'}\|f\|_{Lip-\gamma}$*

Proof. Let $\delta > 0$. Let $x \in U$, f is $\mathrm{Lip}-\gamma$ on $B(x,\delta)\cap U$ with a $\mathrm{Lip}-\gamma$ norm less than or equal to $\|f\|_{\mathrm{Lip}-\gamma}$. Then, by lemma 3.11, f is $\mathrm{Lip}-\gamma'$ on $B(x,\delta)\cap U$ and:

$$\|f\|_{\mathrm{Lip}-\gamma',B(x,\delta)\cap U} \leq \|f\|_{\mathrm{Lip}-\gamma} \max\left(1, \sum_{j=\lfloor\gamma'\rfloor+1}^{\lfloor\gamma\rfloor} \frac{(2\delta)^{j-\gamma'}}{(j-\lfloor\gamma'\rfloor)!} + (2\delta)^{\gamma-\gamma'}\right)$$

Using now lemma 3.13, we deduce that f is $\mathrm{Lip}-\gamma'$ on U with a $\mathrm{Lip}-\gamma'$ controlled as follows:

$$\begin{aligned}\|f\|_{\mathrm{Lip}-\gamma'} \leq \quad & \|f\|_{\mathrm{Lip}-\gamma} \max\left(1, \sum_{j=\lfloor\gamma'\rfloor+1}^{\lfloor\gamma\rfloor} \frac{(2\delta)^{j-\gamma'}}{(j-\lfloor\gamma'\rfloor)!} + (2\delta)^{\gamma-\gamma'}\right). \\ & \max\left(1, \max_{0\leq k\leq\lfloor\gamma'\rfloor} \frac{1}{\delta^{\gamma'-k}}(1+\sum_{j=0}^{\lfloor\gamma'\rfloor-k}\frac{\delta^j}{j!})\right)\end{aligned}$$

The above inequality holding for every $\delta > 0$, we can make it sharper by taking the infimum of the right-hand side over all possible positive values of δ. This ends the proof. □

Remark 3.16.

$$\begin{aligned}M_{\gamma,\gamma'} \quad = \quad & \inf_{\delta>0}\{\max\left(1, \sum_{\lfloor\gamma'\rfloor+1}^{\lfloor\gamma\rfloor} \frac{(2\delta)^{j-\gamma'}}{(j-\lfloor\gamma'\rfloor)!} + (2\delta)^{\gamma-\gamma'}\right). \\ & \max\left(1, \max_{0\leq k\leq\lfloor\gamma'\rfloor} \frac{1}{\delta^{\gamma'-k}}(1+\sum_{0}^{\lfloor\gamma'\rfloor-k}\frac{\delta^j}{j!})\right)\}\end{aligned}$$

By considering the value $\delta = 1/2$ for the prove above, we get the following estimate:

$$M_{\gamma,\gamma'} \leq 2^{\gamma'}e(1+e^{1/2}) \leq 2^{\gamma}e(1+e^{1/2})$$

which has the additional advantage of being dependent on only one of the variables γ and γ'.

As highlighted in [3], a simpler proof of theorem 3.15 can be given when the domain of definition of the map is open and convex. We first give a characterization of Lipschitz maps in this case, which also gives a very useful recursive definition of Lipschitzness. The proof of the following is trivial and can be found if needed in [3] for example:

Lemma 3.17. *Let $n \in \mathbb{N}$, $0 < \varepsilon \leq 1$ and $C \geq 0$. Let E and F be two normed vector spaces and U be a subset of E. Let $f : U \to F$ be a map and for every $k \in [\![1, n]\!]$,* *let* $f^k : U \to \mathcal{L}(E^{\otimes k}, F)$ *be a map with values in the space of the symmetric* k*-linear mappings from* E *to* F*. We consider the two following assertions:*

(A1) $(f, f^1, \ldots, f^n)$ *is Lip* $-$ $(n+\varepsilon)$ *and* $\|f\|_{Lip-(n+\varepsilon)} \leq C$.

(A2) f *is* n *times differentiable, with* $f^1, \ldots, f^n$ *being its successive derivatives.* $\|f\|_\infty$, $\|f^1\|_\infty$, $\ldots$, $\|f^n\|_\infty$ *are upper-bounded by* C *and for all* $x, y \in U$: $\|f^n(x) - f^n(y)\| \leq C\|x-y\|^\varepsilon$.

If U *is open then* $(\mathbf{A1}) \Rightarrow (\mathbf{A2})$*. If, furthermore,* U *is convex then* $(\mathbf{A1}) \Leftrightarrow (\mathbf{A2})$*.*

When the domain of a Lipschitz map is open, convex and bounded, we get a sharper estimate than the one obtained in lemma 3.11:

Lemma 3.18. *Let* $\gamma, \gamma' > 0$ *such that* $\gamma' < \gamma$*. Let* E *and* F *be two normed vector spaces and* U *be an open convex bounded subset of* E*. Let* $f : U \to F$ *be a Lip* $-\gamma$ *function. We assume that* $(E^{\otimes k})_{1 \leq k \leq \lfloor \gamma \rfloor}$ *are endowed with norms satisfying the projective property. Then* f *is Lip* $-\gamma'$ *and if* $L \geq 0$ *is larger than or equal to the diameter of* U *then:*

$$\|f\|_{Lip-\gamma'} \leq \|f\|_{Lip-\gamma} \max\left(1, L^{\min(\lfloor \gamma' \rfloor + 1, \gamma) - \gamma'}\right)$$

Proof. Uses the characterization in lemma 3.17 and, if $\lfloor \gamma' \rfloor < \lfloor \gamma \rfloor$, the fundamental theorem of calculus. □

Always in the case of an open convex domain, we also get a sharper control of the Lipschitz norm from the uniform local behaviour of the map:

Lemma 3.19. *Let* $\gamma > 0$*. Let* E *and* F *be two normed vector spaces and* U *be an open convex subset of* E*. Let* $f : U \to F$ *be a map such that there exists* $\delta > 0$ *and* $C \geq 0$ *such that, for every* $x \in U$*,* $f_{|B(x,\delta) \cap U}$ *is Lip* $-\gamma$ *with a norm less than or equal to* C*. Then* f *is Lip* $-\gamma$ *and:*

$$\|f\|_{Lip-\gamma} \leq C \max\left(1, \frac{2}{\delta^{\gamma - \lfloor \gamma \rfloor}}\right)$$

Proof. Uses the characterization in lemma 3.17 and the same technique as in the proof of lemma 3.13. If necessary, a complete proof can be found for example in [3]. □

Theorem 3.15 now becomes:

Theorem 3.20. *Let* $\gamma, \gamma' > 0$ *such that* $\gamma' < \gamma$. *Let* E *and* F *be two normed vector spaces and* U *be an open convex subset of* E. *We assume that* $(E^{\otimes k})_{1 \leq k \leq \lfloor \gamma \rfloor}$ *are endowed with norms satisfying the projective property. Let* $f : U \to F$ *be a* $Lip - \gamma$ *map. Then* f *is* $Lip - \gamma'$ *and there exists a positive constant* $m_{\gamma,\gamma'}$ *such that:*

$$\|f\|_{Lip-\gamma'} \leq m_{\gamma,\gamma'} \|f\|_{Lip-\gamma}$$

Remark 3.21. *We can choose the constant in the previous theorem such that:*

$$m_{\gamma,\gamma'} = \inf_{\delta > 0} \max\left(1, (2\delta)^{\min(\lfloor \gamma' \rfloor + 1, \gamma) - \gamma'}\right) \max\left(1, \frac{2}{\delta^{\gamma' - \lfloor \gamma' \rfloor}}\right)$$

And we have in this case $m_{\gamma,\gamma'} \leq 4$.

The following theorem about smooth maps is very useful and comes as an easy consequence of all the above:

Theorem 3.22. *Let* $\gamma > 0$. *A map that is* $(\lfloor \gamma \rfloor + 1)$ *times continuously differentiable and is such that its derivatives are bounded on a given convex set is Lipschitz-*γ *on that set. Its Lipschitz-*γ *norm can be upper-bounded by the following constant:*

$$L_\gamma = m_{\lfloor \gamma \rfloor + 1, \gamma} \max\{\|f\|_\infty, \|f^1\|_\infty, \ldots, \|f^{\lfloor \gamma \rfloor}\|_\infty, \|f^{\lfloor \gamma \rfloor + 1}\|_\infty\}$$

Remark 3.23. *Following remark 3.21:*

$$L_\gamma \leq 4 \max\{\|f\|_\infty, \|f^1\|_\infty, \ldots, \|f^{\lfloor \gamma \rfloor}\|_\infty, \|f^{\lfloor \gamma \rfloor + 1}\|_\infty\}$$

3.1.3 Composition of Lipschitz functions

Composition with linear maps

As one would expect, a well-defined composition of two Lipschitz maps is also Lipschitz. We start first with the simple case where one of the maps is linear as the derivatives are easier to extract, though, technically, a continuous linear map defined on the whole space is not necessarily Lipschitz (as its values are not necessary uniformly bounded).

Proposition 3.24. *Let E, F and G be three normed vector spaces and U be a subset of E. Let $\gamma > 0$ and let $f : U \to F$ be a $Lip - \gamma$ map. Let $u : F \to G$ a bounded linear map. Then $u \circ f$ is $Lip - \gamma$ and $\|u \circ f\|_{Lip-\gamma} \leq \|u\|\|f\|_{Lip-\gamma}$.*

Proof. Let $n \in \mathbb{N}$ such that $\gamma \in (n, n+1]$. Let $f^1, \dots, f^n$ be maps on U such that $(f, f^1, \dots, f^n)$ is $\mathrm{Lip} - \gamma$ and let $R_0, \dots, R_n$ be the associated remainders. Let $g = u \circ f$ and for every $k \in [\![1, n]\!]$, let g^k and S_k be defined as follows:

$$\forall x, y \in E, \forall v \in E^{\otimes k} : g^k(x)(v) = u(f^k(x)(v)), \quad S_k(x,y)(v) = u(R_k(x,y)(v))$$

Then it is easy to check that $(g, g^1, \dots, g^n)$ is $\mathrm{Lip} - \gamma$ with $S_0, \dots, S_n$ as remainders and with a $\mathrm{Lip} - \gamma$ norm upper-bounded by $\|u\|\|f\|_{\mathrm{Lip}-\gamma}$. □

Remark 3.25. *Although a linear map in general is not Lipschitz, we can restrict ourselves, in the previous proposition, to a bounded domain of F so that the restriction of u on that domain is Lipschitz. We will be then in the case of a composition of two Lipschitz maps but we don't get a control of the Lipschitz norm as sharp as the one in proposition 3.24.*

Proposition 3.26. *Let $\gamma > 0$ and E, F and G be three normed vector spaces. We assume that $(E^{\otimes k})_{1 \leq k \leq \lfloor \gamma \rfloor}$ and $(F^{\otimes k})_{1 \leq k \leq \lfloor \gamma \rfloor}$ are endowed with compatible norms. Let $f : F \to G$*

be a Lip $-\gamma$ *map and* $u : E \to F$ *a bounded linear map. Then* $f \circ u$ *is Lip* $-\gamma$ *and* $\|f \circ u\|_{Lip-\gamma} \leq \|f\|_{Lip-\gamma} \max(1, \|u\|^\gamma)$.

Proof. Let $f^1, \dots, f^n$ be maps defined on F such that $(f, f^1, \dots, f^n)$ is Lip $-\gamma$ and let $R_0, \dots, R_n$ be the associated remainders. Let $g = f \circ u$ and, for $k \in [\![1, n]\!]$, let $g^k : E \to \mathcal{L}(E^{\otimes k}, G)$ and $S_k : E \times E \to \mathcal{L}(E^{\otimes k}, G)$ be the maps defined by:

$$\forall x, y \in E, \forall v \in E^{\otimes k} : g^k(x)(v) = f^k(u(x))(u^{\otimes k}(v)), S_k(x,y)(v) = R_k(u(x), u(y))(u^{\otimes k}(v))$$

Let $k \in [\![0, n]\!]$, $x, y \in E$ and $v \in E^{\otimes k}$. Then we have, using the previous definitions:

$$\begin{aligned}
g^k(x)(v) &= f^k(u(x))(u^{\otimes k}(v)) \\
&= \sum_{j=k}^{n} f^j(u(y)) \left(\frac{u^{\otimes k}(v) \otimes (u(x-y))^{\otimes j-k}}{(j-k)!} \right) + R_k(u(x), u(y))(u^{\otimes k}(v)) \\
&= \sum_{j=k}^{n} f^j(u(y)) \left(\frac{u^{\otimes k}(v) \otimes u^{\otimes (j-k)}(x-y)^{\otimes (j-k)}}{(j-k)!} \right) + S_k(x,y)(v) \\
&= \sum_{j=k}^{n} f^j(u(y)) \left(u^{\otimes j}\left(\frac{v \otimes (x-y)^{\otimes (j-k)}}{(j-k)!}\right) \right) + S_k(x,y)(v) \\
&= \sum_{j=k}^{n} g^j(y) \left(\frac{v \otimes (x-y)^{\otimes (j-k)}}{(j-k)!} \right) + S_k(x,y)(v)
\end{aligned}$$

And $\|g^k(x)\| \leq \|f\|_{\mathrm{Lip}-\gamma} \|u\|^k$ and $\|S_k(x,y)\| \leq \|f\|_{\mathrm{Lip}-\gamma} \|u\|^\gamma \|x-y\|^{\gamma-k}$. Hence, $(g, g^1, \dots, g^n)$ is Lip $-\gamma$ (with $(S_0, \dots, S_n)$ as remainders) and

$$\begin{aligned}
\|g\|_{\mathrm{Lip}-\gamma} &\leq \max(\|f\|_{\mathrm{Lip}-\gamma}, \|f\|_{\mathrm{Lip}-\gamma}\|u\|, \dots, \|f\|_{\mathrm{Lip}-\gamma}\|u\|^n, \|f\|_{\mathrm{Lip}-\gamma}\|u\|^\gamma) \\
&\leq \|f\|_{\mathrm{Lip}-\gamma} \max(1, \|u\|^\gamma)
\end{aligned}$$

□

Remark 3.27. *If* $(E^{\otimes k})_{1 \leq k \leq \lfloor \gamma \rfloor}$ *and* $(F^{\otimes k})_{1 \leq k \leq \lfloor \gamma \rfloor}$ *are not necessarily endowed with com-*

patible norms, then $f \circ u$ is still $Lip - \gamma$ and:

$$\|f \circ u\|_{Lip-\gamma} \leq \|f\|_{Lip-\gamma} \max_{0 \leq k \leq \lfloor \gamma \rfloor} \|u^{\otimes k}\|(1 \vee \|u\|^{\gamma-k})$$

General cases

Definition 3.28 (Equivalence property)**.** *Let E be a normed vector space. Let $n \in \mathbb{N}^*$. We say that $(E^{\otimes k})_{1 \leq k \leq n}$ (respectively $(E^{\otimes k})_{k \geq 1}$) are endowed with norms satisfying the equivalence property if there exists a constant $c \in \mathbb{R}$ such that, for every $k \in [\![1, n]\!]$ (resp. $k \geq 1$) and $p, q \in \mathbb{N}$ such that $p + q = k$ and every $a \in E^{\otimes p}, b \in E^{\otimes q}$, we have $\|a\|\|b\| \leq c\|a \otimes b\|$.*

The image of a product of Lipschitz maps by a bilinear map is also Lipschitz:

Proposition 3.29. *Let E, F, G and H be normed vector spaces and U be a subset of E. Let $\gamma > 0$ and let $f : U \to F$ and $g : U \to G$ be two $Lip - \gamma$ maps. Let $B : F \times G \to H$ a continuous bilinear map. We assume that $(E^{\otimes k})_{k \geq 1}$ are endowed with norms satisfying the properties of projection, symmetry and equivalence. For $k \in [\![1, n]\!]$, define the map B^k on U by:*

$$\forall x \in U, \forall v \in E^{\otimes k} : \quad B^k(x)(v) = \sum_{\substack{i \in [\![0,k]\!] \\ \sigma \in \mathcal{S}_k}} \frac{B(f^i(x), g^{k-i}(x))}{i!(k-i)!} \sigma(v)$$

Then $B(f, g) : U \to H$, when endowed with $B^1, \ldots, B^n$ as successive derivatives, is $Lip-\gamma$ and there exists a constant C (depending only on γ) such that:

$$\|B(f, g)\|_{Lip-\gamma} \leq C\|B\|\|f\|_{Lip-\gamma}\|g\|_{Lip-\gamma}$$

Remark 3.30. *For $i, j \in [\![0, n]\!]$, $x, y \in U$, $B(f^i(x), g^j(y))$ is the unique map defined of*

$E^{\otimes(i+j)}$ *by the following:*

$$\forall v_1,\dots,v_{i+j}\in E : B(f^i(x),g^j(y))(v_1\otimes\cdots\otimes v_{i+j}) = B(f^i(x)(v_1\otimes\cdots\otimes v_i), g^j(y)(v_{i+1}\otimes\cdots\otimes v_{i+j}))$$

The idea behind the proof is rather simple but contains notions and ideas that will be very important to the proof of the main theorem of this section.

Proof. Let $\varepsilon\in(0,1]$. We prove our statement by induction on n. For $n=0$, the proof of the statement is trivial and is left as an exercise. Let $n\in\mathbb{N}$. We assume the statement true for n and let us prove it for $n+1$. Let E, F, G and H be normed vector spaces and U be a subset of E, and let $f : U\to F$ and $g : U\to G$ be two $\mathrm{Lip}-(n+1+\varepsilon)$ maps and $B : F\times G\to H$ be a continuous bilinear map. We will show that $(Z, Z^1,\dots,Z^{n+1})$ is $\mathrm{Lip}-(n+1+\varepsilon)$ where, $Z := B(f,g)$ and for $k\in[\![1,n+1]\!]$, $x\in U$ and $v\in E^{\otimes k}$:

$$Z^k(x)(v) = \sum_{\substack{i\in[\![0,k]\!]\\ \sigma\in\mathcal{S}_k}} \frac{B(f^i(x),g^{k-i}(x))}{i!(k-i)!}\sigma(v)$$

For $k\in[\![0,n+1]\!]$, let R_k (resp. S_k, T_k) be the remainder of order k associated to f (resp. g, Z). Let $x,y\in U$. Writing the Taylor expansion of f and g and using the bilinearity of B, we get:

$$Z(x) = \sum_{i=0}^{n+1} Z^i(y)\left(\frac{(x-y)^{\otimes i}}{i!}\right) + T_0(x,y)$$

where:

$$\begin{aligned} T_0(x,y) \;=\; & B(R_0(x,y),g(x)) + B(f(x),S_0(x,y)) + B(R_0(x,y),S_0(x,y)) + \\ & \sum_{\substack{i,j\in[\![0,n+1]\!]\\ i+j>n+1}} B(f(y),g(y))\left(\tfrac{(x-y)^{\otimes(i+j)}}{i!j!}\right) \end{aligned}$$

It is obvious then that we can bound the $Z^k(x)$'s and $T_0(x,y)$ adequately (appropriate exponents can be obtained by reasoning over balls of the same size and then using lemma

3.13).

We prove now that Z^1 is $\mathrm{Lip}-(n+\varepsilon)$ with a Taylor expansion that we can identify with that of Z and with a well bounded $\mathrm{Lip}-(n+\varepsilon)$ norm:
f^1 and g (resp. f and g^1) are both $\mathrm{Lip}-(n+\varepsilon)$. Therefore, by the induction hypothesis, $B(f^1,g)$ (resp. $B(f,g^1)$) is $\mathrm{Lip}-(n+\varepsilon)$. Hence $(Z^1,(Z^1)^1\dots,(Z^1)^n)$ is $\mathrm{Lip}-(n+\varepsilon)$, where, for $k\in[\![1,n]\!]$, $x\in U$ and $v\in E^{\otimes k}$:

$$(Z^1)^k(x)(v)=\sum_{\substack{i\in[\![0,k]\!]\\ \sigma\in\mathcal{S}_k}}\frac{B((f^1)^i(x),g^{k-i}(x))+B(f^i(x),(g^1)^{k-i}(x))}{i!(k-i)!}\sigma(v)$$

and there exists a constant $c_{n,\varepsilon}$ such that:

$$\|Z^1\|_{\mathrm{Lip}-(n+\varepsilon)}\le c_{n,\varepsilon}\|B\|\|f\|_{\mathrm{Lip}-(n+1+\varepsilon)}\|g\|_{\mathrm{Lip}-(n+1+\varepsilon)}$$

To end the proof, we only have to make the identification between $(Z^1)^k$ and Z^{k+1}, for all $k\in[\![1,n]\!]$. Let then $k\in[\![1,n]\!]$, $x\in U$ and $v_1,\dots,v_k,v_{k+1}\in E$ and define: $v=v_1\otimes\cdots\otimes v_k$. For $\sigma\in\mathcal{S}_{k+1}$ and $i\in[\![0,k+1]\!]$, let $\sigma^{1,i}(v\otimes v_{k+1})$ and $\sigma^{2,i}(v\otimes v_{k+1})$ be the only elements of $E^{\otimes i}$ and $E^{\otimes(k+1-i)}$ respectively such that:

$$\sigma(v\otimes v_{k+1})=\sigma^{1,i}(v\otimes v_{k+1})\otimes\sigma^{2,i}(v\otimes v_{k+1})$$

Studying the position of v_{k+1} in $Z^{k+1}(x)(v\otimes v_{k+1})$, we are naturally led into dividing the sum into the two following parts:

$$\begin{aligned} Z^{k+1}(x)(v\otimes v_{k+1}) &= \sum_{\substack{i\in[\![0,k+1]\!]\\ \sigma\in\mathcal{S}_{k+1}}} \tfrac{B(f^i(x)(\sigma^{1,i}(v\otimes v_{k+1})),g^{k+1-i}(x)(\sigma^{2,i}(v\otimes v_{k+1})))}{i!(k+1-i)!} \\ &= \sum_{i=1}^{k+1}\left(\sum_{\substack{\sigma\in\mathcal{S}_{k+1}\\ \sigma^{-1}(k+1)\le i}} \tfrac{B(f^i(x)(\sigma^{1,i}(v\otimes v_{k+1})),g^{k+1-i}(x)(\sigma^{2,i}(v\otimes v_{k+1})))}{i!(k+1-i)!}\right)+ \\ &\quad \sum_{i=0}^{k}\left(\sum_{\substack{\sigma\in\mathcal{S}_{k+1}\\ \sigma^{-1}(k+1)> i}} \tfrac{B(f^i(x)(\sigma^{1,i}(v\otimes v_{k+1})),g^{k+1-i}(x)(\sigma^{2,i}(v\otimes v_{k+1})))}{i!(k+1-i)!}\right) \end{aligned}$$

For every $\sigma\in\mathcal{S}_{k+1}$ let $\tau_\sigma\in\mathcal{S}_k$ be defined as follows:

$$\tau_\sigma(j)=\begin{cases}\sigma(j) & \text{, if } j<\sigma^{-1}(k+1)\\ \sigma(j+1) & \text{, if } j\ge\sigma^{-1}(k+1)\end{cases}$$

The map $\sigma\mapsto\tau_\sigma$ is surjective and for each $\tau\in\mathcal{S}_k$, there exists exactly $(k+1)$ elements $\sigma\in\mathcal{S}_{k+1}$ such that $\tau=\tau_\sigma$. More precisely, for $i\in[\![0,k+1]\!]$:

$$\mathcal{S}_k=\{\tau_\sigma:\sigma\in\mathcal{S}_{k+1},\sigma^{-1}(k+1)\le i\} \text{ and } \mathrm{card}\{\sigma\in\mathcal{S}_{k+1}:\tau=\tau_\sigma,\sigma^{-1}(k+1)\le i\}=i$$

Similarly

$$\mathcal{S}_k=\{\tau_\sigma:\sigma\in\mathcal{S}_{k+1},\sigma^{-1}(k+1)> i\} \text{ and } \mathrm{card}\{\sigma\in\mathcal{S}_{k+1}:\tau=\tau_\sigma,\sigma^{-1}(k+1)> i\}=k+1-i$$

Let $i\in[\![1,k+1]\!]$. Since $f^i(x)$ is symmetric then, for every $\sigma\in\mathcal{S}_{k+1}$ such that $\sigma^{-1}(k+1)\le i$, we have:

$$f^i(x)(\sigma^{1,i}(v\otimes v_{k+1}))=f^i(x)(\tau_\sigma^{1,i-1}(v)\otimes v_{k+1})$$

which gives:

$$\sum_{\substack{\sigma\in\mathcal{S}_{k+1}\\ \sigma^{-1}(k+1)\leq i}} \frac{B(f^i(x)(\sigma^{1,i}(v\otimes v_{k+1})), g^{k+1-i}(x)(\sigma^{2,i}(v\otimes v_{k+1})))}{i!(k+1-i)!} =$$
$$\sum_{\tau\in\mathcal{S}_k} \frac{B((f^1)^{i-1}(x)(\tau_\sigma^{1,i-1}(v))(v_{k+1}), g^{k+1-i}(x)(\tau_\sigma^{2,i-1}(v)))}{(i-1)!(k+1-i)!}$$

Summing over all $i \in [\![1, k+1]\!]$:

$$\sum_{i=1}^{k+1} \sum_{\substack{\sigma\in\mathcal{S}_{k+1}\\ \sigma^{-1}(k+1)\leq i}} \frac{B(f^i(x)(\sigma^{1,i}(v\otimes v_{k+1})), g^{k+1-i}(x)(\sigma^{2,i}(v\otimes v_{k+1})))}{i!(k+1-i)!} =$$
$$\sum_{j=0}^{k} \sum_{\tau\in\mathcal{S}_k} \frac{B((f^1)^{j}(x)(\tau^{1,j}(v))(v_{k+1}), g^{k-j}(x)(\tau^{2,j}(v)))}{j!(k-j)!} =$$
$$\sum_{j=0}^{k} \sum_{\tau\in\mathcal{S}_k} \frac{B((f^1)^{j}(x)(.)(v_{k+1}), g^{k-j}(x))}{j!(k-j)!}(\tau(v))$$

We deal with the other term by using a similar idea. We finally get $Z^{k+1}(x)(v\otimes v_{k+1})$ is equal to:

$$\sum_{j=0}^{k} \sum_{\tau\in\mathcal{S}_k} \frac{B((f^1)^{j}(x)(.)(v_{k+1}), g^{k-j}(x)) + B(f^j(x), (g^1)^{k-j}(x)(.)(v_{k+1}))}{j!(k-j)!}(\tau(v))$$

which is exactly $(Z^1)^k(x)(v)(v_{k+1})$. This ends this proof. □

Remark 3.31. *By proposition 3.29, the real-valued Lip-γ functions form an algebra under point-wise multiplication.*

Remark 3.32. *If $E\otimes F$ is endowed with a norm satisfying the projective property, then the tensor product of an E-valued Lipschitz map by an F-valued Lipschitz map is also Lipschitz as a direct consequence of proposition 3.29.*

We are ready now to show that a well-defined composition of Lipschitz maps is itself Lipschitz. We start first with the following simple case:

Lemma 3.33. *Let E, F and G be three normed vector spaces. Let U be a subset of E and V be a subset of F. Let $\varepsilon \in (0,1]$. We assume that $(E^{\otimes k})_{k\geq 1}$ and $(F^{\otimes k})_{k\geq 1}$ are endowed with norms satisfying the projective property. Let $f : U \to F$ and $g : V \to G$ be two $Lip-(1+\varepsilon)$ maps such that $f(U) \subseteq V$. Then $g \circ f$ is $Lip-(1+\varepsilon)$ and, there exists a constant C_ε (depending only on ε) such that:*

$$\|g \circ f\|_{Lip-(1+\varepsilon)} \leq C_\varepsilon \|g\|_{Lip-(1+\varepsilon)} \max(\|f\|^{(1+\varepsilon)}_{Lip-(1+\varepsilon)}, 1)$$

Proof. Let $f = (f^0, f^1) : U \to F$ and $g = (g^0, g^1) : V \to G$ be two $\mathrm{Lip}-(1+\varepsilon)$ maps such that $f(U) \subseteq V$, and with remainders denoted by (R_0, R_1) and (S_0, S_1) respectively. In this case, $(g \circ f)^1$ is simply the map given by the chain rule and which maps every element x in U to $g^1(f^0(x)) \circ f^1(x)$. Let $x, y \in U$. Then the Taylor expansions of g and f enable us to write $g^0(f^0(x))$ in function of $g^0(f^0(y))$ and $(g \circ f)^1(y)$ as follows:

$$g^0(f^0(x)) = g^0(f^0(y)) + g^1(f^0(y))(f^1(y)(x-y)) + g^1(f^0(y))(R_0(x,y)) + S_0(f^0(x), f^0(y))$$

Let $T_0 : U \times U \to G$ and $T_1 : U \times U \to \mathcal{L}(E, G)$ be defined as follows:

$$\begin{aligned} \forall x, y \in U, \forall v \in E: \quad T_0(x,y) &= g^1(f^0(y))(R_0(x,y)) + S_0(f^0(x), f^0(y)) \\ T_1(x,y) &= (g \circ f)^1(x) - (g \circ f)^1(y) \end{aligned}$$

Let $x, y \in U$. It is easy to see that:

$$\|(g^0 \circ f^0)(x)\| \leq \|g\|_{\mathrm{Lip}-(1+\varepsilon)}$$

and that:

$$\|(g \circ f)^1(x)\| \leq \|g\|_{\mathrm{Lip}-(1+\varepsilon)} \|f\|_{\mathrm{Lip}-(1+\varepsilon)}$$

Furthermore, we have:

$$\|g^1(f^0(y))(R_0(x,y))\| \leq \|g\|_{\mathrm{Lip}-(1+\varepsilon)}\|f\|_{\mathrm{Lip}-(1+\varepsilon)}\|x-y\|^{1+\varepsilon}$$

By theorem 3.15, f is also $\mathrm{Lip}-1$ and there exists a constant $M_{1,\varepsilon}$ depending only on 1 and ε such that $\|f\|_{\mathrm{Lip}-1} \leq M_{1,\varepsilon}\|f\|_{\mathrm{Lip}-(1+\varepsilon)}$. Consequently, we can write:

$$\|S_0(f^0(x), f^0(y))\| \leq \|g\|_{\mathrm{Lip}-(1+\varepsilon)} M_{1,\varepsilon}^{1+\varepsilon}\|f\|_{\mathrm{Lip}-(1+\varepsilon)}^{1+\varepsilon}\|x-y\|^{1+\varepsilon}$$

Hence:

$$\|T_0(x,y)\| \leq \|g\|_{\mathrm{Lip}-(1+\varepsilon)}(\|f\|_{\mathrm{Lip}-(1+\varepsilon)} + M_{1,\varepsilon}^{1+\varepsilon}\|f\|_{\mathrm{Lip}-(1+\varepsilon)}^{1+\varepsilon})\|x-y\|^{1+\varepsilon}$$

By writing

$$T_1(x,y) = g^1(f^0(x))(f^1(x) - f^1(y)) + (g^1(f^0(x)) - g^1(f^0(y)))(f^1(y))$$

and by using similar techniques as above, we get the inequality:

$$\|T_1(x,y)\| \leq \|g\|_{\mathrm{Lip}-(1+\varepsilon)}(\|f\|_{\mathrm{Lip}-(1+\varepsilon)} + M_{1,\varepsilon}^{\varepsilon}\|f\|_{\mathrm{Lip}-(1+\varepsilon)}^{1+\varepsilon})\|x-y\|^{\varepsilon}$$

Therefore, $g \circ f$ is $\mathrm{Lip}-(1+\varepsilon)$ (with the suggested Taylor expansion) and by defining, for example, $C_{1,\varepsilon} = 2\max(1, M_{1,\varepsilon}^{1+\varepsilon})$, we obtain:

$$\|g \circ f\|_{\mathrm{Lip}-(1+\varepsilon)} \leq C_{1,\varepsilon}\|g\|_{\mathrm{Lip}-(1+\varepsilon)} \max(\|f\|_{\mathrm{Lip}-(1+\varepsilon)}^{1+\varepsilon}, 1)$$

□

Notation. *For any finite set* $\{\alpha_1, \ldots, \alpha_r\}$, $\mathcal{S}_{\alpha_1,\ldots,\alpha_r}$ *denotes the set of all bijections from*

$\{\alpha_1, \ldots, \alpha_r\}$ *onto itself.*

We will need the combinatorial result stated in the next lemma before we can proceed:

Lemma 3.34. *Let V be a non-empty set. For $j \in \mathbb{N}^*$ and $w \in V^{\otimes j}$, we will denote by Sym(w) the symmetric part of w. Let $N \in \mathbb{N}^*$ and $v_1, \ldots, v_N \in V$, then:*

$$\sum_{\sigma \in \mathcal{S}_N} \sum_{\tau \in \mathcal{S}_{\sigma(1),\ldots,\sigma(i)}} v_{\tau\circ\sigma(1)} \otimes \cdots \otimes v_{\tau\circ\sigma(i)} \otimes \mathit{Sym}(v_{\sigma(i+1)} \otimes \cdots \otimes v_{\sigma(N)})$$

equals:

$$i!(N-i)! \sum_{1 \le r_1 < \cdots < r_i \le k} \sum_{\tau \in \mathcal{S}_{r_1,\ldots,r_i}} v_{\tau(r_1)} \otimes \cdots \otimes v_{\tau(r_i)} \otimes v_{r_{i+1}} \otimes \cdots \otimes v_{r_N}$$

where, in order to have a compact expression, for every $1 \le r_1 < \cdots < r_i \le N$, we mean by $r_{i+1}, \ldots, r_N$ the integers that are such that $r_{i+1} < \cdots < r_N$ and $\{r_{i+1}, \ldots, r_N\} = [\![1, N]\!] - \{r_1, \ldots, r_i\}$.

Proof. For $\sigma, \tilde{\sigma} \in \mathcal{S}_k$ such that $\sigma([\![1, i]\!]) = \tilde{\sigma}([\![1, i]\!])$, we have:

$$\sum_{\tau \in \mathcal{S}_{\sigma(1),\ldots,\sigma(i)}} v_{\tau\circ\sigma(1)} \otimes \cdots \otimes v_{\tau\circ\sigma(i)} = \sum_{\tau \in \mathcal{S}_{\tilde{\sigma}(1),\ldots,\tilde{\sigma}(i)}} v_{\tau\circ\tilde{\sigma}(1)} \otimes \cdots \otimes v_{\tau\circ\tilde{\sigma}(i)}$$

and:

$$\mathrm{Sym}(v_{\sigma(i+1)} \otimes \cdots \otimes v_{\sigma(N)}) = \mathrm{Sym}(v_{\tilde{\sigma}(i+1)} \otimes \cdots \otimes v_{\tilde{\sigma}(N)})$$

In the light of the previous identities, we rewrite $\mathcal{S}_N$ as the following disjoint union:

$$\mathcal{S}_N = \bigcup_{1 \le r_1 < \cdots < r_i \le N} \{\sigma \in \mathcal{S}_N |\, \sigma([\![1, i]\!]) = \{r_1, \ldots, r_i\}\}$$

Notice that, for $1 \le r_1 < \cdots < r_i \le N$, the set $\{\sigma \in \mathcal{S}_N | \sigma([\![1, i]\!]) = \{r_1, \ldots, r_i\}\}$ has

$i!(N-i)!$ elements. Therefore:

$$\sum_{\sigma\in\mathcal{S}_k} v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(i)}\otimes \mathrm{Sym}(v_{\sigma(i+1)}\otimes\cdots\otimes v_{\sigma(N)})$$

is equal to:

$$\sum_{1\leq r_1<\cdots<r_i\leq N}\ \sum_{\tau\in\mathcal{S}_{r_1,\ldots,r_i}} v_{\tau(r_1)}\otimes\cdots\otimes v_{\tau(r_i)}\otimes \sum_{\sigma\in\mathcal{S}_N,\sigma(\llbracket 1,i\rrbracket)=\{r_1,\ldots,r_i\}} \mathrm{Sym}(v_{\sigma(i+1)}\otimes\cdots\otimes v_{\sigma(N)})$$

which itself is equal to:

$$i!(N-i)!\sum_{1\leq r_1<\cdots<r_i\leq N}\ \sum_{\tau\in\mathcal{S}_{r_1,\ldots,r_i}} v_{\tau(r_1)}\otimes\cdots\otimes v_{\tau(r_i)}\otimes v_{r_{i+1}}\otimes\cdots\otimes v_{r_N}$$

□

The following composition result in the general case has already appeared in [3] but with a slight mistake in the control of the Lipschitz norm of the composition map that we correct here along with giving a full and detailed proof that differs from the one suggested in the aforementioned paper:

Theorem 3.35. *Let E, F and G be three normed vector spaces. Let U be a subset of E and V be a subset of F. Let $\gamma>0$. We assume that $(E^{\otimes k})_{k\geq 1}$ and $(F^{\otimes k})_{k\geq 1}$ are endowed with norms satisfying the projective property. Let $f: U\to F$ and $g: V\to G$ be two $Lip-\gamma$ maps such that $f(U)\subseteq V$. Then $g\circ f$ is $Lip-\gamma$ and, if $\gamma\geq 1$, there exists a constant C_γ (depending only on γ) such that:*

$$\|g\circ f\|_{Lip-\gamma}\leq C_\gamma\|g\|_{Lip-\gamma}\max(\|f\|_{Lip-\gamma}^{\gamma},1)$$

Proof. The idea of the proof is very simple but rather technical and long. We leave the case $\gamma\leq 1$ as an easy and straightforward exercise. Let $\varepsilon\in(0,1]$. We will prove the following by induction:

Claim. *For all $n \in \mathbb{N}^*$, for any normed vector spaces E, F, G and H, such that $(E^{\otimes k})_{k\geq 1}$ and $(F^{\otimes k})_{k\geq 1}$ are endowed with norms satisfying the projective property, and any subsets U of E and V of F, there exists a real constant $C_{n,\varepsilon}$ (depending only on n and ε) such that if $f = (f^0, \dots, f^n) : U \to F$ and $g = (g^0, \dots, g^n) : V \to G$ are two $Lip - (n+\varepsilon)$ maps such that $f(U) \subseteq V$, then $g \circ f = (g^0 \circ f^0, (g\circ f)^1, \dots, (g \circ f)^n)$ is $Lip - (n+\varepsilon)$ and:*

$$\|g \circ f\|_{Lip-(n+\varepsilon)} \leq C_{n,\varepsilon} \|g\|_{Lip-(n+\varepsilon)} \max(\|f\|^{n+\varepsilon}_{Lip-(n+\varepsilon)}, 1)$$

where, for every $k \in [\![1, n]\!]$, $y \in U$, and $v \in E^{\otimes k}$, $(g\circ f)^k(y)(v)$ is given by the following formula:

$$(g \circ f)^k(y)(v) = \sum_{j=1}^{k} \frac{g^j(f(y))}{j!} \sum_{\substack{1\leq i_1,\dots,i_j\leq n \\ i_1+\dots+i_j=k}} \frac{f^{i_1}(y)\otimes\cdots\otimes f^{i_j}(y)}{i_1!\cdots i_j!}\Big(\sum_{\sigma\in\mathcal{S}_k}(\sigma(v))\Big)$$

The case $n = 1$ has been proved in lemma 3.33.

Let now $n \in \mathbb{N}^*$. We assume that the assertion is true for n and let us prove it for $n+1$. Let $f = (f^0, \dots, f^{n+1}) : U \to F$ and $g = (g^0, \dots, g^{n+1}) : V \to G$ be two $\mathrm{Lip} - (n + 1 + \varepsilon)$ functions such that $f(U) \subseteq V$, and with remainders denoted by $R_0, \dots, R_{n+1}$ and $S_0, \dots, S_{n+1}$ respectively. Let $x, y \in U$. Using the Taylor expansion of g, we have:

$$g^0(f^0(x)) = g^0(f^0(y)) + \sum_{j=1}^{n+1} g^j(f^0(y))\Big(\frac{(f^0(x) - f^0(y))^{\otimes j}}{j!}\Big) + S_0(f^0(x), f^0(y))$$

Define $P_0(x, y) = R_0(x, y)$, and for every $k \in [\![1, n+1]\!]$: $P_k(x, y) = f^k(y)\frac{(x-y)^{\otimes k}}{k!}$. Having in mind these notations and the Taylor expansion of f and the one we want to get for

$g \circ f$, define:

$$\begin{aligned} T_0(x,y) &= S_0(f^0(x), f^0(y)) + \sum_{j=1}^{n+1} \tfrac{g^j(f^0(y))}{j!} \left(\sum_{\substack{0 \le i_1,\ldots,i_j \le n+1 \\ i_1 \cdots i_j = 0}} P_{i_1}(x,y) \otimes \cdots \otimes P_{i_j}(x,y) + \right. \\ & \left. \sum_{\substack{1 \le i_1,\ldots,i_j \le n+1 \\ i_1 + \cdots + i_j > n+1}} f^{i_1}(y) \otimes \cdots \otimes f^{i_j}(y) \tfrac{(x-y)^{\otimes(i_1+\cdots+i_j)}}{i_1! \cdots i_j!} \right) \end{aligned}$$

Then, we can simply write:

$$\begin{aligned} g^0(f^0(x)) &= g^0(f^0(y)) + T_0(x,y) + \\ & \sum_{k=1}^{n+1} \sum_{j=1}^{k} \frac{g^j(f^0(y))}{j!} \left(\sum_{\substack{1 \le i_1,\ldots,i_j \le n+1 \\ i_1 + \cdots + i_j = k}} f^{i_1}(y) \otimes \cdots \otimes f^{i_j}(y) \frac{(x-y)^{\otimes k}}{i_1! \cdots i_j!} \right) \\ &= g^0(f^0(y)) + \sum_{k=1}^{n+1} (g \circ f)^k(y) \frac{(x-y)^{\otimes k}}{k!} + T_0(x,y) \end{aligned}$$

Assume that $\|x-y\| < 2$. It is an easy exercise then to show that there exists a constant $M_{n,\varepsilon}$ (depending only on n and ε) such that:

$$\|T_0(x,y)\| \le M_{n,\varepsilon} \|g\|_{\mathrm{Lip}-(n+1+\varepsilon)} \max(\|f\|^{(n+1+\varepsilon)}_{\mathrm{Lip}-(n+1+\varepsilon)}, 1) \|x-y\|^{n+1+\varepsilon}$$

We have also that $\|g^0 \circ f^0\|_\infty \le \|g\|_{\mathrm{Lip}-(n+1+\varepsilon)}$. All that remains to do then to end the proof is to show that $((g \circ f)^1, \ldots, (g \circ f)^{n+1})$ and the associated remainders satisfy the appropriate Taylor expansion with well controlled uniform convergence norms. This will show that $g \circ f$ is $\mathrm{Lip} - (n+1+\varepsilon)$ on every intersection of a ball of radius 1 with U with a uniformly controlled $\mathrm{Lip} - (n+1+\varepsilon)$ norm and we can then conclude using lemma 3.13.

Define the following maps:

$$\begin{array}{rccc} \varphi : & U & \to & \mathcal{L}_c(E,F) \times \mathcal{L}_c(F,G) \\ & x & \mapsto & (f^1(x), g^1(f^0(x))) \end{array}$$

and:

$$\begin{array}{rccc} \psi : & \mathcal{L}_c(E,F) \times \mathcal{L}_c(F,G) & \to & \mathcal{L}_c(E,G) \\ & (u,v) & \mapsto & v \circ u \end{array}$$

As g^1, f^0 and f^1 are all $\mathrm{Lip}-(n+\varepsilon)$, then φ is also $\mathrm{Lip}-(n+\varepsilon)$. ψ is a continuous bilinear map with norm 1 (it is also smooth and is therefore $\mathrm{Lip}-(n+\varepsilon)$ on any bounded set). As $(g \circ f)^1 = \psi \circ \varphi$ then $((g \circ f)^1, \dots, ((g \circ f)^1)^n)$ is $\mathrm{Lip}-(n+\varepsilon)$ and there exists a constant $C_{n,\varepsilon}$ depending only on n and ε (using proposition 3.29) such that:

$$\|(g \circ f)^1\|_{\mathrm{Lip}-(n+\varepsilon)} \leq C_{n,\varepsilon} \|g\|_{\mathrm{Lip}-(n+\varepsilon+1)} \max(\|f\|^{n+\varepsilon+1}_{\mathrm{Lip}-(n+\varepsilon+1)}, 1)$$

Now, we only have to identify, for every $k \in [\![1,n]\!]$, $((g \circ f)^1)^k$ with $(g \circ f)^{k+1}$.
Let $k \in [\![1,n]\!]$. Let $x \in U$ and $v \in E^{\otimes k}$. By the induction hypothesis, we have:

$$((g \circ f)^1)^k(x)(v) = \sum_{j=1}^{k} \frac{\psi^j(\varphi(x))}{j!} \sum_{\substack{1 \leq i_1, \dots, i_j \leq n \\ i_1 + \cdots + i_j = k}} \frac{\varphi^{i_1}(x) \otimes \cdots \otimes \varphi^{i_j}(x)}{i_1! \cdots i_j!} (\sum_{\sigma \in \mathcal{S}_k} \sigma(v))$$

which, by studying the successive derivatives of ψ (which is straight-forward since it is a bilinear map) and φ, gives two simple formulas depending on whether $k = 1$ or $k > 1$. For $k = 1$, it is an easy exercise to see that, for every $v_1, v_2 \in E$, we have:

$$(((g \circ f)^1)^1(x)(v_1))(v_2) = (g \circ f)^2(x)(v_1 \otimes v_2)$$

Assume then that $k > 1$ and for $v = v_1 \otimes \cdots \otimes v_k$, $v_1, \dots, v_k \in E$, $((g \circ f)^1)^k(x)(v)$ is the sum of four terms:

$$
\begin{aligned}
&(g^1 \circ f)(x) \circ (f^1)^k(x)(v) + (g^1 \circ f)^k(x)(v) \circ f^1(x) + \\
&\frac{1}{2} \sum_{\sigma \in \mathcal{S}_k} \sum_{i=1}^{k-1} \frac{1}{i!(k-i)!} \\
&((g^1 \circ f)^i(x)(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(i)}) \circ (f^1)^{k-i}(x)(v_{\sigma(i+1)} \otimes \cdots \otimes v_{\sigma(k)}) + \\
&(g^1 \circ f)^{k-i}(x)(v_{\sigma(i+1)} \otimes \cdots \otimes v_{\sigma(k)}) \circ (f^1)^i(x)(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(i)}))
\end{aligned}
$$

Let $v_1, \ldots, v_{k+1} \in E$. We use the formula for the successive derivatives given by the induction hypothesis to simplify the four terms in the sum above:

First term:

$$(g^1 \circ f)(x) \circ (f^1)^k(x)(v_1 \otimes \cdots \otimes v_k)(v_{k+1}) = g^1(f(x))(f^{k+1}(x)(v_1 \otimes \cdots \otimes v_{k+1}))$$

Second term:

$$
\begin{aligned}
&((g^1 \circ f)^k(x)(v_1 \otimes \cdots \otimes v_k) \circ f^1(x))(v_{k+1}) \\
&= \sum_{j=1}^{k} \frac{g^{j+1}(f(x))}{j!} \left(\sum_{\substack{1 \le i_1, \ldots, i_j \le n \\ i_1 + \cdots + i_j = k}} \frac{f^{i_1}(x) \otimes \cdots \otimes f^{i_j}(x) \otimes f^1(x)}{i_1! \cdots i_j!} (\sum_{\sigma \in \mathcal{S}_k} \sigma(v_1 \otimes \cdots \otimes v_k) \otimes v_{k+1}) \right)
\end{aligned}
$$

Third and fourth term: They are dealt with in the same manner and are actually equal. Consequently, we will admit below that they are indeed equal and show how to deal with the third term only. Let $i \in [\![1, k-1]\!]$ and $\sigma \in \mathcal{S}_k$. Using the induction hypothesis for $g^1 \circ f$, we get that $(g^1 \circ f)^i(x)(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(i)})$ is equal to:

$$\sum_{p=1}^{i} \frac{(g^1)^p(f(x))}{p!} \left(\sum_{\substack{1 \le m_1, \ldots, m_p \le n \\ m_1 + \cdots + m_p = i}} \frac{f^{m_1}(x) \otimes \cdots \otimes f^{m_p}(x)}{m_1! \cdots m_p!} (\sum_{\tau \in \mathcal{S}_{\sigma(1), \ldots, \sigma(i)}} v_{\tau \circ \sigma(1)} \otimes \cdots \otimes v_{\tau \circ \sigma(i)}) \right)$$

As $g^{p+1}(f(x))$ is symmetric, this equals, when composed with $(f^1)^{k-i}(x)(v_{\sigma(i+1)} \otimes \cdots \otimes v_{\sigma(k)})(v_{k+1})$:

$$\sum_{p=1}^{i} \frac{g^{p+1}(f(x))}{p!} \Big(\sum_{\substack{1 \leq m_1, \ldots, m_p \leq n \\ m_1 + \cdots + m_p = i}} \frac{f^{m_1}(x) \otimes \cdots \otimes f^{m_p}(x) \otimes f^{k-i+1}(x)}{m_1! \cdots m_p!}$$

$$\Big(\sum_{\tau \in \mathcal{S}_{\sigma(1), \ldots, \sigma(i)}} v_{\tau \circ \sigma(1)} \otimes \cdots \otimes v_{\tau \circ \sigma(i)} \otimes v_{\sigma(i+1)} \otimes \cdots \otimes v_{\sigma(k)} \otimes v_{k+1} \Big) \Big)$$

As $f^{k-i+1}(x)$ is symmetric, by using lemma 3.34, we get that the third term:

$$\sum_{\substack{1 \leq i \leq k-1 \\ \sigma \in \mathcal{S}_k}} \frac{(g^1 \circ f)^i(x)}{i!(k-i)!} (v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(i)}) \circ (f^1)^{k-i}(x)(v_{\sigma(i+1)} \otimes \cdots \otimes v_{\sigma(k)})(v_{k+1})$$

is equal to:

$$\sum_{p=1}^{k-1} \frac{g^{p+1}(f(x))}{p!} \sum_{i=p}^{k-1} \Big(\sum_{\substack{1 \leq m_1, \ldots, m_p \leq n \\ m_1 + \cdots + m_p = i}} \frac{f^{m_1}(x) \otimes \cdots \otimes f^{m_p}(x) \otimes f^{k-i+1}(x)}{m_1! \cdots m_p!}$$

$$\Big(\sum_{1 \leq r_1 < \cdots < r_i \leq k} \sum_{\tau \in \mathcal{S}_{r_1, \ldots, r_i}} v_{\tau(r_1)} \otimes \cdots \otimes v_{\tau(r_i)} \otimes v_{r_{i+1}} \otimes \cdots \otimes v_{r_k} \otimes v_{k+1} \Big) \Big)$$

Now, the strategy is to compare the terms of the same degree in $(g \circ f)^{k+1}(x)(v)$ and $((g \circ f)^1)^k(x)(v)$ (to simplify, we call degree the integer i in the expression $g^i(f(x))$). The term of degree 1 in $(g \circ f)^{k+1}(x)(v_1 \otimes \cdots \otimes v_{k+1})$ is:

$$g^1(f(x)) \Big(\frac{f^{k+1}(x)}{(k+1)!} \Big(\sum_{\sigma \in \mathcal{S}_{k+1}} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k+1)} \Big) \Big)$$

and as $f^{k+1}(x)$ is symmetric and $\mathrm{card}(\mathcal{S}_{k+1}) = (k+1)!$, this is equal to the term of degree 1 in $((g \circ f)^1)^k(x)(v_1 \otimes \cdots \otimes v_k)(v_{k+1})$ (which corresponds to the first term here).

The term of degree $k+1$ in $(g\circ f)^{k+1}(x)(v_1\otimes\cdots\otimes v_{k+1})$ is:

$$\frac{g^{k+1}(f(x))}{(k+1)!}(f^1(x)\otimes\cdots\otimes f^1(x))(\sum_{\sigma\in\mathcal{S}_{k+1}} v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(k+1)})$$

As $g^{k+1}(f(x))$ is symmetric, then this term is equal to

$$g^{k+1}(f(x))(f^1(x)\otimes\cdots\otimes f^1(x))(v_1\otimes\cdots\otimes v_{k+1})$$

and the same argument proves that is exactly the term of degree $k+1$ in

$$((g\circ f)^1)^k(x)(v_1\otimes\cdots\otimes v_k)(v_{k+1})$$

(which only appears in the second term here).

Finally, let $j\in[\![1,k-1]\!]$. The term of degree $j+1$ in the second and third term of $(g\circ f)^{k+1}(x)(v_1\otimes\cdots\otimes v_{k+1})$ is the image under $\frac{g^{j+1}(f(x))}{j!}$ of the compactly written sum:

$$\sum_{i=j}^{k}(\sum_{\substack{1\le m_1,\dots,m_j\le n\\ m_1+\cdots+m_j=i}} \frac{f^{m_1}(x)\otimes\cdots\otimes f^{m_j}(x)\otimes f^{k-i+1}(x)}{m_1!\cdots m_j!}$$
$$(\sum_{1\le r_1<\cdots<r_i\le k}\ \sum_{\tau\in\mathcal{S}_{r_1,\dots,r_i}} v_{\tau(r_1)}\otimes\cdots\otimes v_{\tau(r_i)}\otimes v_{r_{i+1}}\otimes\cdots\otimes v_{r_k}\otimes v_{k+1}))$$

Using the symmetries of both $g^{j+1}(f(x))$ and the $f^i(x)$'s, this can be shown to have the same image under $g^{j+1}(f(x))$ as:

$$\frac{1}{j+1}\sum_{\substack{1\le m_1,\dots,m_{j+1}\le n\\ m_1+\cdots+m_{j+1}=k+1}} \frac{f^{m_1}(x)\otimes\cdots\otimes f^{m_{j+1}}(x)}{m_1!\cdots m_{j+1}!}(\sum_{\sigma\in\mathcal{S}_{k+1}}(\sigma(v_1\otimes\cdots\otimes v_{k+1})))$$

which ends the proof. □

Remark 3.36. *We can obtain an easier proof for theorem 3.35 by using the extension* *theorems that will be introduced in subsection 3.1.5. The inequality will still prove hard to get and will involve a constant depending on the dimension of the spaces, an inconvenient that we don't have in the proof presented above.*

3.1.4 A quantitative estimate

In this section, we give some more precise local quantitative estimates (in the Lipschitz norm) if the value of a Lipschitz map at a point is known.

Theorem 3.37. *Let* $\gamma, \gamma' > 0$ *such that* $\gamma' < \gamma$. *Let* E *and* F *be two normed vector spaces,* U *be a subset of* E *and* $x_0 \in U$. *Let* $f = (f^0, \dots, f^{\lfloor \gamma \rfloor})$ *be a* $Lip - \gamma$ *map on* U *with values in* F. *Assume that for all* $k \in [\![0, \lfloor \gamma \rfloor]\!] : f^k(x_0) = 0$. *We also assume that* $(E^{\otimes k})_{1 \leq k \leq \lfloor \gamma \rfloor}$ *are endowed with norms satisfying the projective property. Then for all* $\delta > 0$, *one has:*

$$\|f\|_{Lip-\gamma', B(x_0,\delta)\cap U} \leq \|f\|_{Lip-\gamma} \max\left(\delta^{\gamma}, \delta^{\gamma-\lfloor\gamma'\rfloor}, \delta^{\gamma-\gamma'}\left(\sum_{j=\lfloor\gamma'\rfloor+1}^{\lfloor\gamma\rfloor} \frac{2^{j-\gamma'}}{(j-\lfloor\gamma'\rfloor)!} + 2^{\gamma-\gamma'}\right)\right)$$

Proof. Let $n, n' \in \mathbb{N}$ and $\varepsilon, \varepsilon' \in (0,1]$ such that $\gamma = n + \varepsilon$ and $\gamma' = n' + \varepsilon'$. Denote by $(R_0, \dots, R_n)$ the remainders associated to f. Let $\delta > 0$ and let $k \in [\![0, n']\!]$. Let $x \in B(x_0, \delta) \cap U$ and $v \in E^{\otimes k}$. Then, as f is $\mathrm{Lip} - \gamma$ and that $f^k(x_0) = 0$ for all $k \in [\![0, n]\!]$, we get:

$$\begin{aligned}
\|f^k(x)(v)\| &= \|\textstyle\sum_k^n f^j(x_0)(\frac{v\otimes(x-x_0)^{\otimes(j-k)}}{(j-k)!}) + R_k(x,x_0)(v)\| \\
&= \|R_k(x,x_0)(v)\| \\
&\leq \|f\|_{\mathrm{Lip}-\gamma}\|x-x_0\|^{\gamma-k}\|v\| \\
&\leq \|f\|_{\mathrm{Lip}-\gamma}\delta^{\gamma-k}\|v\|
\end{aligned}$$

Therefore, $\sup\limits_{x\in B(x_0,\delta)\cap U} \|f^k(x)\| \leq \|f\|_{\mathrm{Lip}-\gamma}\delta^{\gamma-k}$, for all $k \in [\![0, n']\!]$.

Let $k \in [\![0, n']\!]$. We define $S_k : U \times U \to \mathcal{L}(E^{\otimes k}, F)$ (the new remainder) by:

$$S_k(x,y)(v) = f^k(x)(v) - \sum_{j=k}^{n'} f^j(y)(\frac{v \otimes (x-y)^{\otimes(j-k)}}{(j-k)!})$$

Let $x, y \in B(x_0, \delta) \cap U$ and $v \in E^{\otimes k}$. Writing the Taylor expansion of f as a $\mathrm{Lip}-\gamma$ map, we get the following identity:

$$S_k(x,y)(v) = \sum_{j=n'+1}^{n} f^j(y)(\frac{v \otimes (x-y)^{\otimes(j-k)}}{(j-k)!}) + R_k(x,y)(v)$$

which, using our new upper-bound for $\|f^k\|_{\infty, B(x_0,\delta)\cap U}$, leads to the inequality:

$$\begin{aligned} \|S_k(x,y)(v)\| &\leq \|f\|_{\mathrm{Lip}-\gamma} \left(\sum_{j=n'+1}^{n} \delta^{\gamma-j} \frac{\|x-y\|^{j-k}}{(j-k)!} + \|x-y\|^{\gamma-k} \right) \|v\| \\ &\leq \|f\|_{\mathrm{Lip}-\gamma} \|x-y\|^{\gamma'-k} \left(\sum_{j=n'+1}^{n} \delta^{\gamma-j} \frac{\|x-y\|^{j-\gamma'}}{(j-k)!} + \|x-y\|^{\gamma-\gamma'} \right) \|v\| \end{aligned}$$

Therefore:

$$\sup_{x,y \in B(x_0,\delta)\cap U} \|S_k(x,y)\| \leq \|f\|_{\mathrm{Lip}-\gamma} \|x-y\|^{\gamma'-k} \delta^{\gamma-\gamma'} \left(\sum_{j=\lfloor\gamma'\rfloor+1}^{\lfloor\gamma\rfloor} \frac{2^{j-\gamma'}}{(j-\lfloor\gamma'\rfloor)!} + 2^{\gamma-\gamma'} \right)$$

which ends the proof. □

Remark 3.38.

$$\max\left(\delta^{\gamma}, \delta^{\gamma-\lfloor\gamma'\rfloor}, \delta^{\gamma-\gamma'} \left(\sum_{j=\lfloor\gamma'\rfloor+1}^{\lfloor\gamma\rfloor} \frac{2^{j-\gamma'}}{(j-\lfloor\gamma'\rfloor)!} + 2^{\gamma-\gamma'} \right) \right) \leq (\delta^{\gamma} \vee \delta^{\gamma-\gamma'})(e^2 - 1 + 2^{\gamma-\gamma'})$$

Remark 3.39. *With the notations of the previous theorem, if we only have $f^k(x_0) = 0$ for $k \in [\![0, n']\!]$, the result remains essentially true but with a slightly different upper-bound (that still converges to 0 as δ goes to 0). However, in the cases where $\gamma' = \gamma$*

or if there exists $k \in [\![0, n']\!]$ *such that* $f^k(x_0) \neq 0$ *then we cannot get a better control of* $\|f\|_{Lip-\gamma', B(x_0,\delta)\cap U}$ *than* $\|f\|_{Lip-\gamma}$ *as the example below shows (we can, nevertheless, improve the control of* $\|f^k\|_{\infty, B(x_0,\delta)\cap U}$ *for all* $k \in [\![0, n]\!]$ *in the first case),*

Example 3.40. *Consider the function* $f : x \mapsto x$ *defined on* $(-1, 1)$*. As* f *is smooth,* f *is Lipschitz of any degree.*

- *On one hand,* f *is* $Lip - 1$, $\|f\|_{Lip-1} = 1$ *and* $f(0) = 0$*. On the other hand, for any* $\delta \in (0, 1]$*, there does not exist a constant* λ *strictly less than* 1 *such that:*

  $$\forall x, y \in (\delta, -\delta) : \quad |f(x) - f(y)| \leq \lambda |x - y|$$

  Therefore :

  $$\forall \delta \in (0, 1] : \quad \|f\|_{Lip-1,(\delta,-\delta)} = 1$$

  However, we still have $\|f\|_{\infty,(\delta,-\delta)} \underset{\delta \to 0}{\longrightarrow} 0$

- f *is* $Lip - 2$, $\|f\|_{Lip-2} = 1$ *and* $f(0) = 0$*. However,* $f'(0) \neq 0$ *and:*

  $$\forall \delta \in (0, 1] : \quad \|f\|_{Lip-3/2,(\delta,-\delta)} = 1$$

Remark 3.41. *Using theorem 3.37, one can easily then compare in the Lip-*γ' *norm two Lip-*γ *maps, when* $\gamma' < \gamma$*, which values and "successive derivatives"' values (in the sense of a Lipschitz map) agree at one point.*

3.1.5 Extension theorems

One of the most interesting and still open problems in Lipschitz geometry, and classical analysis in general, is about the existence of extensions of Lipschitz (or smooth) maps to the whole space and the control of the Lipschitz norm of the extension. This is known as Whitney's extension problem and canbe informally stated in the following way:

> Given an arbitrary set A and a map $f : A(\subseteq E) \to F$, where E and F are vector spaces:
>
> 1. In which ways can one define f to be a *smooth* map on A so that, if $\mathring{A}$ (the interior of A) is not empty, $f_{|\mathring{A}}$ is smooth in the classical sense?
> 2. Given such a definition, can we extend f to the whole space E so that this extension is smooth in the classical sense?

Since Whitney introduced it in a series of three seminal papers [27, 28, 29], several mathematicians have been working on this problem, mostly in the case where both E and F are finite dimensional. Whitney himself was the first one to suggest a solution in the case where A is the closure of a region. The answers to this question are of crucial importance. For instance, the theory of rough paths requires that the vector fields appearing in a rough differential equation be Stein-Lipschitz. As linear and polynomial functions in particular are not in this class of functions, using Whitney's theorem allows to extend the restriction of polynomials to compact sets (which are Stein-Lipschitz) to the whole space in a way that they stay Stein-Lipschitz. Another illustration would be the construction of a suitable function from sampled data so that one can work in the appropriate class of functions associated to the experiment's physical model.

We will state below two other examples of such results: one in which one can extend Lipschitz maps of any degree to the whole space, but at the cost of amplifying the Lipschitz norm; and another one where the extension has the same Lipschitz norm as the map we start with but which is currently only obtained for Lipschitz-1 maps in the framework of Hilbert spaces.

Stein introduced the class of Lipschitz functions in [25] and gave a Whitney extension theorem in this case. Lipschitz maps have the advantage (among others) of making sense even on discrete sets:

Theorem 3.42 (Stein [25]). *Let $\gamma \geq 1$. Let E and F be two finite dimensional vector spaces and K be a closed subset of E. There exists continuous a linear map sending every F-valued Lip-γ map f defined on K to an F-valued Lip-γ map $\tilde{f}$ defined on E such that $\tilde{f}_{|K} = f$. Moreover, the norm of the linear extension map depends only on γ and the dimensions of E and F.*

Theorem 3.43 (Kirszbraun [17]). *Let H_1 and H_2 be two Hilbert spaces. Let A be a subset of H_1, $K \geq 0$ and $f : A \to H_2$ be a map such that:*

$$\forall x, y \in A : \quad \|f(x) - f(y)\| \leq K\|x - y\|$$

(i.e. f is 1-Hölder). Then there exists a map $\tilde{f} : H_1 \to H_2$ such that $\tilde{f}_{|A} = f$ and:

$$\forall x, y \in H_1 : \quad \|\tilde{f}(x) - \tilde{f}(y)\| \leq K\|x - y\|$$

Moreover, if f is bounded (i.e. f is Lip-1) then $\tilde{f}$ can be chosen to be bounded and such that $\sup_A \|f\| = \sup_{H_1} \|\tilde{f}\|$.

Fields medallist C. Fefferman has been working on a variety of versions of this problem, including one that deals with appropriately approximating f with a *smooth* map, which can be of enormous use in practice when one is collecting a finite sample of data (see [6, 7, 8]).

3.2 Flows of Lipschitz vector fields

Notation. *Under the assumption of existence, we will be denoting the flow of a vector field* A *by* $\tilde{A}$*; by* $\tilde{A}_t$*, for* $t \in \mathbb{R}$*, the map* $x \mapsto \tilde{A}(t,x)$*; and by* $\tilde{A}_x$*, for* $x \in M$*, the map* $t \mapsto \tilde{A}(t,x)$*.*

In the following, we will need the comparison lemma (which is also used to prove theorem 2.4).

Lemma 3.44 (Comparison lemma)**.** *Let* I *be an open interval. Let* $d \in \mathbb{N}^*$ *and* $u : I \to \mathbb{R}^d$ *be a differentiable map such that there exists* $a > 0$ *and* $b \geq 0$ *such that:*

$$\forall t \in I : \|u'(t)\| \leq a\|u(t)\| + b$$

Then, if $t_0 \in I$*, we have:*

$$\forall t \in I : \|u(t)\| \leq e^{a|t-t_0|}\|u(t_0)\| + \frac{b}{a}(e^{a|t-t_0|} - 1)$$

We start by proving some basic properties of flows:

Lemma 3.45. *Let* A *be a* $Lip-1$ *vector field on a Banach space* E *and* $\tilde{A}$ *its global flow. Then:*

- $\forall t \in \mathbb{R}, \forall y, \tilde{y} \in E : \|\tilde{A}(t,y) - \tilde{A}(t,\tilde{y})\| \leq e^{|t|\|A\|_{Lip-1}}\|y - \tilde{y}\|$.
- $\tilde{A}$ *is locally* 1*-Hölder: for all* $t, \tilde{t} \in \mathbb{R}$ *and* $y, \tilde{y} \in E$*:*

$$\|\tilde{A}(t,y) - \tilde{A}(\tilde{t},\tilde{y})\| \leq e^{(|t|\wedge|\tilde{t}|)\|A\|_{Lip-1}}\|y - \tilde{y}\| + \|A\|_\infty|t - \tilde{t}|$$

- $\forall T, r \in \mathbb{R}_+^*, \forall x_0 \in E : \quad \tilde{A}((-T,T) \times B(x_0,r)) \subseteq B(x_0, r + T\|A\|_\infty)$.

Proof. The lemma is trivial in the case where $A = 0$. Assume then that $A \neq 0$. Let $y, \tilde{y} \in E$. Define u on $\mathbb{R}$ by the identity: $u(t) = \tilde{A}(t, y) - \tilde{A}(t, \tilde{y})$. Note that $u(0) = y - \tilde{y}$. u is differentiable and:

$$\forall t \in \mathbb{R} : u'(t) = A(\tilde{A}(t, y)) - A(\tilde{A}(t, \tilde{y}))$$

Therefore, for all $t \in \mathbb{R}$, $\|u'(t)\| \leq \|A\|_{\mathrm{Lip}-1} \|u(t)\|$. Hence, by the comparison lemma 3.44 (note that $\|A\|_{\mathrm{Lip}-1} > 0$):

$$\forall t \in \mathbb{R} : \qquad \|\tilde{A}(t, y) - \tilde{A}(t, \tilde{y})\| = \|u(t)\| \leq e^{|t| \|A\|_{\mathrm{Lip}-1}} \|y - \tilde{y}\|$$

Let $t, \tilde{t} \in \mathbb{R}$ and assume that $|t| \leq |\tilde{t}|$:

$$\begin{aligned} \|\tilde{A}(t, y) - \tilde{A}(\tilde{t}, \tilde{y})\| &\leq \|\tilde{A}(t, y) - \tilde{A}(t, \tilde{y})\| + \|\tilde{A}(t, \tilde{y}) - \tilde{A}(\tilde{t}, \tilde{y})\| \\ &\leq e^{|t| \|A\|_{\mathrm{Lip}-1}} \|y - \tilde{y}\| + \| \textstyle\int_{\tilde{t}}^{t} A(\tilde{A}(u, \tilde{y})) \mathrm{d}u \| \\ &\leq e^{|t| \|A\|_{\mathrm{Lip}-1}} \|y - \tilde{y}\| + \|A\|_{\infty} |t - \tilde{t}| \end{aligned}$$

Therefore, $\tilde{A}$ is locally 1-Hölder continuous and, by taking $(t, y) = (0, x_0)$ in the previous inequality, we see that $\tilde{A}((-T, T) \times B(x_0, r)) \subseteq B(x_0, r + T\|A\|_{\infty})$. □

We show now that the flows of differentiable vector fields are differentiable too:

Lemma 3.46. *Let $0 < \varepsilon \leq 1$ and $d \in \mathbb{N}^*$. Let A be a Lip$-(1+\varepsilon)$ vector field on $\mathbb{R}^d$ and $\tilde{A}$ be its global flow. Then $\tilde{A}$ is continuously differentiable and, if $(\vec{e}_1, \ldots, \vec{e}_d)$ is a basis for $\mathbb{R}^d$, then for all $(t, y) \in \mathbb{R} \times \mathbb{R}^d$:*

$$\|\partial_t \tilde{A}(t, y)\| \leq \|A\|_{Lip-(1+\varepsilon)}$$

and

$$\|\partial_{x_i}\tilde{A}(t,y)\| \leq e^{|t|\|A\|_{Lip-1}}\|\vec{e}_i\|$$

Proof. Let $y \in \mathbb{R}^d$. By definition of the flow:

$$\forall t \in \mathbb{R} : \quad \tilde{A}(t,y) = y + \int_0^t A(\tilde{A}(u,y))\mathrm{d}u$$

As both A and $\tilde{A}(.,y)$ are continuous, $t \mapsto \tilde{A}(t,y)$ is then continuously differentiable and, for all $t \in \mathbb{R} : \partial_t\tilde{A}(t,y) = A(\tilde{A}(t,y))$. Moreover:

$$\forall (t,y) \in \mathbb{R} \times \mathbb{R}^d : \|\partial_t\tilde{A}(t,y)\| \leq \|A\|_\infty \tag{3.1}$$

We will prove now that $\tilde{A}$ is continously differentiable in space. Let $(\vec{e}_1, \ldots, \vec{e}_d)$ be a basis for $\mathbb{R}^d$ and $T > 0$. Let $i \in [\![1, d]\!]$. For $h \in \mathbb{R}^*$, we define the map Δ_h^i on $(-T,T) \times \mathbb{R}^d$ by the relation:

$$\Delta_h^i(t,y) = \frac{\tilde{A}(t, y + h\vec{e}_i) - \tilde{A}(t,y)}{h}$$

We are going to show that the sequence $(\Delta_h^i)_{|h|>0}$ converges uniformly on $(-T,T)\times\mathbb{R}^d$ (as h goes to zero). Let $h \in \mathbb{R}^*$. For $(t,y) \in (-T,T) \times \mathbb{R}^d$, lemma 3.45 gives the inequality:

$$\|\Delta_h^i(t,y)\| \leq e^{T\|A\|_{\mathrm{Lip}-1}}\|\vec{e}_i\| \tag{3.2}$$

A being $\mathrm{Lip} - (1+\varepsilon)$, let R be a map defined on $\mathbb{R}^d \times \mathbb{R}^d$ with values in $\mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$ such that, for all $a, b \in \mathbb{R}^d$:

$$A(a) = A(b) + \mathrm{d}A(b)(a-b) + R(a,b)$$

and

$$\|R(a,b)\| \leq \|A\|_{\mathrm{Lip}-(1+\varepsilon)}\|a-b\|^{1+\varepsilon}$$

Δ_h^i is obviously continuously differentiable in time. Let $(t,y) \in (-T,T) \times \mathbb{R}^d$:

$$\begin{aligned} \partial_t \Delta_h^i(t,y) &= \tfrac{1}{h}(A(\tilde{A}(t,y+h\vec{e}_i)) - A(\tilde{A}(t,y))) \\ &= \tfrac{1}{h}(\mathrm{d}A(\tilde{A}(t,y))(\tilde{A}(t,y+h\vec{e}_i) - \tilde{A}(t,y)) + \\ &\quad R(\tilde{A}(t,y+h\vec{e}_i), \tilde{A}(t,y))) \\ &= \mathrm{d}A(\tilde{A}(t,y))(\Delta_h^i(t,y)) + \tfrac{1}{h}R(\tilde{A}(t,y+h\vec{e}_i), \tilde{A}(t,y)) \end{aligned}$$

Let $\tilde{h} \in \mathbb{R}^*$. From the calculation above and the inequality (3.2), we get that:

$$\begin{aligned} \|\partial_t \Delta_h^i(t,y) - \partial_t \Delta_{\tilde{h}}^i(t,y)\| &\leq \|\mathrm{d}A(\tilde{A}(t,y))(\Delta_h^i(t,y) - \Delta_{\tilde{h}}^i(t,y))\| + \\ &\quad \|\tfrac{1}{h}R(\tilde{A}(t,y+h\vec{e}_i), \tilde{A}(t,y))\| + \\ &\quad \|\tfrac{1}{\tilde{h}}R(\tilde{A}(t,y+\tilde{h}\vec{e}_i), \tilde{A}(t,y))\| \\ &\leq \|A\|_{\mathrm{Lip}-(1+\varepsilon)}(\|\Delta_h^i(t,y) - \Delta_{\tilde{h}}^i(t,y)\| + \\ &\quad |h|^\varepsilon \|\Delta_h^i(t,y)\|^{1+\varepsilon} + |\tilde{h}|^\varepsilon \|\Delta_{\tilde{h}}^i(t,y)\|^{1+\varepsilon}) \\ &\leq \|A\|_{\mathrm{Lip}-(1+\varepsilon)}(\|\Delta_h^i(t,y) - \Delta_{\tilde{h}}^i(t,y)\| + \\ &\quad 2(|h| \vee |\tilde{h}|)^\varepsilon (e^{T\|A\|_{\mathrm{Lip}-1}} \|\vec{e}_i\|)^{1+\varepsilon}) \end{aligned}$$

Therefore, using the comparison lemma and the fact that $\Delta_h^i(0,y) = \Delta_{\tilde{h}}^i(0,y) = \vec{e}_i$, we get the following inequality:

$$\|\Delta_h^i - \Delta_{\tilde{h}}^i\|_{\infty,(-T,T)\times\mathbb{R}^d} \leq 2(|h| \vee |\tilde{h}|)^\varepsilon (e^{T\|A\|_{\mathrm{Lip}-1}} \|\vec{e}_i\|)^{1+\varepsilon} (e^{T\|A\|_{\mathrm{Lip}-(1+\varepsilon)}} - 1)$$

We therefore see that $(\Delta_h^i)_{|h|>0}$ converges uniformly on $(-T,T) \times \mathbb{R}^d$ and that $\partial_{x_i}\tilde{A}$ exists (as its limit). As for every h, Δ_h^i is continuous, $\partial_{x_i}\tilde{A}$ is also continous. We also get the following inequality by passing to the limit in the inequality (3.2):

$$\forall (t,y) \in (-T,T) \times \mathbb{R}^d : \quad \|\partial_{x_i}\tilde{A}(t,y)\| \leq e^{T\|A\|_{\mathrm{Lip}-1}} \|\vec{e}_i\|$$

□

Lemma 3.47. *Let $0 < \varepsilon \leq 1$. Let $d \in \mathbb{N}^*$, $T > 0$, $r > 0$ and $x_0 \in \mathbb{R}^d$. Let and A be a $Lip - (1+\varepsilon)$ vector field on $\mathbb{R}^d$ and $\tilde{A}$ its global flow. Then $\mathrm{d}\tilde{A}$ is ε-Hölder on $(-T,T) \times B(x_0,r)$ and there exists a constant C depending on $\|A\|_{Lip-(1+\varepsilon)}$, ε, T and r such that $\|\mathrm{d}\tilde{A}\|_{Lip-\varepsilon} \leq C$.*

Proof. The lemma is trivial in the case where $A = 0$. Assume then that $A \neq 0$. In the following, we endow $\mathbb{R} \times \mathbb{R}^d$ with the l^∞ norm that we will denote by N.

Let $(t,y), (\tilde{t},\tilde{y}) \in (-T,T) \times B(x_0,r)$. Using the definition of a $\mathrm{Lip} - 1$ map and our preliminary study of $\tilde{A}$, we get the following inequality:

$$\begin{aligned}
\|\partial_t \tilde{A}(t,y) - \partial_t \tilde{A}(\tilde{t},\tilde{y})\| &= \|A(\tilde{A}(t,y)) - A(\tilde{A}(\tilde{t},\tilde{y}))\| \\
&\leq \|A\|_{\mathrm{Lip}-1} \|\tilde{A}(t,y) - \tilde{A}(\tilde{t},\tilde{y})\| \\
&\leq \|A\|_{\mathrm{Lip}-1} (e^{T\|A\|_{\mathrm{Lip}-1}} \|y-\tilde{y}\| + \|A\|_\infty |t-\tilde{t}|) \\
&\leq \|A\|_{\mathrm{Lip}-1} (e^{T\|A\|_{\mathrm{Lip}-1}} (2r)^{1-\varepsilon} + \|A\|_\infty (2T)^{1-\varepsilon}). \\
&\quad N((t,y) - (\tilde{t},\tilde{y}))^\varepsilon
\end{aligned}$$

Hence, $\partial_t \tilde{A}$ is ε-Hölder on $(-T,T) \times B(x_0,r)$.

Let $(\vec{e}_1, \ldots, \vec{e}_d)$ is a basis for $\mathbb{R}^d$. Let $i \in [\![1,d]\!]$ and $(y,\tilde{y}) \in B(x_0,r)^2$. Define the map v on $(-T,T)$ by the identity $v(t) = \partial_{x_i}\tilde{A}(t,y) - \partial_{x_i}\tilde{A}(t,\tilde{y})$. Since $\partial_{x_i}\tilde{A}$ satisfies the following differential equation:

$$\forall t \in (-T,T): \quad \partial_{x_i}\tilde{A}(t,y) = \vec{e}_i + \int_0^t \mathrm{d}A(\tilde{A}(u,y)) \partial_{x_i}\tilde{A}(u,y) \mathrm{d}u$$

v is then continuously differentiable. Using the fact that A is $\mathrm{Lip} - (1+\varepsilon)$, the controls obtained in lemmas 3.45 and 3.46, we get the following inequality, for all $t \in (-T,T)$:

$$\|v'(t)\| \leq \|A\|_{\mathrm{Lip}-(1+\varepsilon)} e^{(1+\varepsilon)T\|A\|_{\mathrm{Lip}-1}} \|\vec{e}_i\| \|y-\tilde{y}\|^\varepsilon + \|A\|_{\mathrm{Lip}-(1+\varepsilon)} \|v(t)\|$$

Noting that $\|A\|_{\text{Lip}-(1+\varepsilon)} > 0$ and using the comparison lemma, we then get, for all $t \in (-T, T)$:

$$\|\partial_{x_i}\tilde{A}(t,y) - \partial_{x_i}\tilde{A}(t,\tilde{y})\| \leq e^{(1+\varepsilon)T\|A\|_{\text{Lip}-1}}\|\vec{e}_i\|\|y-\tilde{y}\|^{\varepsilon}(e^{T\|A\|_{\text{Lip}-(1+\varepsilon)}} - 1)$$

Let $(t,\tilde{t}) \in (-T,T)^2$. Using, successively, the differential equation satisfied by $\partial_{x_i}\tilde{A}(.,\tilde{y})$, the fact that A is $\text{Lip}-(1+\varepsilon)$ and finally the lemma 3.46, one gets:

$$\begin{array}{rcl} \|\partial_{x_i}\tilde{A}(t,\tilde{y}) - \partial_{x_i}\tilde{A}(\tilde{t},\tilde{y})\|\|\| & = & \| \int_{\tilde{t}}^{t}(\mathrm{d}A(\tilde{A}(u,\tilde{y}))\partial_{x_i}\tilde{A}(u,\tilde{y})\mathrm{d}u\| \\ & \leq & |t-\tilde{t}|\|A\|_{\text{Lip}-(1+\varepsilon)}\|\partial_{x_i}\tilde{A}(.,\tilde{y})\|_{\infty,[\tilde{t},t]} \\ & \leq & |t-\tilde{t}|\|A\|_{\text{Lip}-(1+\varepsilon)}e^{T\|A\|_{\text{Lip}-1}}\|\vec{e}_i\| \end{array}$$

Finally, we get the inequality:

$$\begin{array}{rcl} \|\partial_{x_i}\tilde{A}(t,y) - \partial_{x_i}\tilde{A}(\tilde{t},\tilde{y})\| & \leq & \|\partial_{x_i}\tilde{A}(t,y) - \partial_{x_i}\tilde{A}(t,\tilde{y})\| + \|\partial_{x_i}\tilde{A}(t,\tilde{y}) - \partial_{x_i}\tilde{A}(\tilde{t},\tilde{y})\| \\ & \leq & e^{T\|A\|_{\text{Lip}-1}}\|\vec{e}_i\|(e^{\varepsilon T\|A\|_{\text{Lip}-1}}(e^{T\|A\|_{\text{Lip}-(1+\varepsilon)}} - 1) + \\ & & (2T)^{1-\varepsilon}\|A\|_{\text{Lip}-(1+\varepsilon)})N((t,y)-(\tilde{t},\tilde{y}))^{\varepsilon} \end{array}$$

Therefore, $\partial_{x_i}\tilde{A}$ is ε-Hölder, which ends this part of the proof. Define the following constants:

$$\left\{\begin{array}{rcl} m_1 & = & \|A\|_{\text{Lip}-1}(e^{T\|A\|_{\text{Lip}-1}}(2r)^{1-\varepsilon} + \|A\|_{\text{Lip}-(1+\varepsilon)}(2T)^{1-\varepsilon}) \\ m_2 & = & (e^{\varepsilon T\|A\|_{\text{Lip}-1}}(e^{T\|A\|_{\text{Lip}-(1+\varepsilon)}} - 1) + (2T)^{1-\varepsilon}\|A\|_{\text{Lip}-(1+\varepsilon)})e^{T\|A\|_{\text{Lip}-1}}\max\limits_{1\leq i\leq d}\|\vec{e}_i\| \\ m_3 & = & \|A\|_{\text{Lip}-(1+\varepsilon)} \\ m_4 & = & e^{T\|A\|_{\text{Lip}-1}}\max\limits_{1\leq i\leq d}\|\vec{e}_i\| \end{array}\right.$$

Then $\|\mathrm{d}\tilde{A}\|_{\text{Lip}-\varepsilon} \leq \max\limits_{1\leq i\leq 4} m_i$. □

Finally, we show that flows of Lipschitz vector fields are also Lipschitz on bounded sets and have a well-controlled Lipschitz norm:

Theorem 3.48. *Let* $n, d \in \mathbb{N}^*$ *and* $0 < \varepsilon \leq 1$. *Let* A *be a* $Lip-(n+\varepsilon)$ *vector field on* $\mathbb{R}^d$. *Let* $x_0 \in \mathbb{R}^d$, $T > 0$ *and* $r > 0$ *and* $\tilde{A}$ *the local flow of* A *defined on* $(-T, T) \times B(x_0, r)$. *Then* $\tilde{A}$ *is* $Lip-(n+\varepsilon)$ *on* $(-T, T) \times B(x_0, r)$ *and there exists a constant* C *depending only on* T, r, $\|x_0\|$, n, ε *and* $\|A\|_{Lip-(n+\varepsilon)}$ *such that* $\|\tilde{A}\|_{Lip-(n+\varepsilon)} \leq C$.

Proof. We will prove the theorem by induction. The previous lemma states that if that A is $\mathrm{Lip}-(1+\varepsilon)$ then $\tilde{A}$ is $\mathrm{Lip}-(1+\varepsilon)$ on $(-T, T) \times B(x_0, r)$ and that the $\mathrm{Lip}-\varepsilon$ norm of its derivative can be controlled by a constant depending only on T, r, ε and $\|A\|_{\mathrm{Lip}-(1+\varepsilon)}$. $\|\tilde{A}\|_\infty$ is itself controlled by a constant depending on the same aforementioned variables and $\|x_0\|$.

Let $n \in \mathbb{N}^*$. Assume that the assertion is true for $\mathrm{Lip}-(n+\varepsilon)$ vector fields and let us prove it when A is $\mathrm{Lip}-(n+1+\varepsilon)$. By the induction hypothesis, we know that $\tilde{A}$ is $\mathrm{Lip}-(n+\varepsilon)$. For $(t, y) \in (-T, T) \times B(x_0, r)$, we know that:

$$\partial_t \tilde{A}(t, y) = A(\tilde{A}(t, y))$$

Hence, $\partial_t \tilde{A}$ is $\mathrm{Lip}-(n+\varepsilon)$ and $\|\partial_t \tilde{A}\|_{\mathrm{Lip}-(n+\varepsilon)}$ can be upper-bounded by a constant depending only on n, ε, $\|A\|_{\mathrm{Lip}-(n+1+\varepsilon)}$ and $\|\tilde{A}\|_{\mathrm{Lip}-(n+\varepsilon)}$ (the latter being less than $\|\tilde{A}\|_\infty \vee \|\mathrm{d}\tilde{A}\|_{\mathrm{Lip}-(n+\varepsilon-1)}$).

Let Y be the vector field on $\mathbb{R}^d \times \mathbb{R}^d$ defined by:

$$\forall a, b \in \mathbb{R}^d : \quad Y(a, b) = \begin{pmatrix} A(a) \\ \mathrm{d}A(a)(b) \end{pmatrix}$$

It is then a simple exercise to show that, for all $M > 0$, Y is a $\mathrm{Lip}-(n+\varepsilon)$ vector field on $\mathbb{R}^d \times B(0, M)$ and to bound $\|Y\|_{\mathrm{Lip}-(n+\varepsilon)}$ with a constant depending only on n, ε and

$\|A\|_{\mathrm{Lip}-(n+1+\varepsilon)}$ and M. Moreover:

$$\|Y\|_{\infty,\mathbb{R}^d\times B(0,M)} \leq \|A\|_{\mathrm{Lip}-(n+1+\varepsilon)}(1\vee M)$$

Define $m = \max\limits_{1\leq i\leq d}\|\vec{e}_i\|$ and take:

$$M = (1\vee r\vee m\vee \|x_0\|)(1+T\|A\|_{\mathrm{Lip}-(n+1+\varepsilon)})$$

Let $(\vec{e}_1,\dots,\vec{e}_d)$ be a basis for $\mathbb{R}^d$. Let $i\in [\![1,d]\!]$ and let α be the local flow of Y on $(-T,T)\times B((x_0,\vec{e}_i),r\vee m)$. Then (lemma 3.45):

$$\begin{aligned}\alpha((-T,T)\times B((x_0,\vec{e}_i),r\vee m)) &\subseteq B((x_0,\vec{e}_i),r\vee m+T\|Y\|_{\infty,B((x_0,e_i),r\vee m)})\\ &\subseteq B(0,M)\end{aligned}$$

Using the induction hypothesis and proposition 3.10, α is $\mathrm{Lip}-(n+\varepsilon)$ and $\|\alpha\|_{\mathrm{Lip}-(n+\varepsilon)}$ can be controlled with a constant depending only on $\|A\|_{\mathrm{Lip}-(n+\varepsilon+1)}$, T, r, $\|x_0\|$, n and ε. Using the uniqueness of solutions to differential equations with Lipshitz vector fields, we get, for every $(t,y)\in(-T,T)\times B(x_0,r)$:

$$\alpha(t,(y,\vec{e}_i)) = \begin{pmatrix}\tilde{A}(t,y)\\ \partial_{x_i}\tilde{A}(t,y)\end{pmatrix}$$

Therefore, $\partial_{x_i}\tilde{A}$ is $\mathrm{Lip}-(n+\varepsilon)$ and that $\|\partial_{x_i}\tilde{A}\|_{\mathrm{Lip}-(n+\varepsilon)}$ can be upper-bounded by a constant depending only on $\|A\|_{\mathrm{Lip}-(n+\varepsilon+1)}$, n, ε, T, r and $\|x_0\|$, which ends the proof. $\square$

3.3 Constant Rank theorems for Lipschitz maps

The two versions of the constant rank theorem in this section and the related techniques are classical in the case of smooth maps and the literature is abundant in this matter (see for example [18]). As the reader may notice, we will only be assuming that the derivatives are Lipschitz (instead of the maps themselves) as this is a less demanding requirement to get our quantitative estimates.

We will be working in the finite-dimensional case and will assume that the norms on tensor spaces satisfy all the norm properties presented in section 3.1. Norms of continuous linear maps are computed as subordinate norms.

3.3.1 The inverse function theorem

We will need the following well-known lemma:

Lemma 3.49. *Let E and F be two normed vector spaces. Let U and V be two open subsets of E and F respectively and let $\varphi : U \to V$ be a homeomorphism. Let $x \in U$ and assume that φ is differentiable at x and that $\mathrm{d}\varphi(x)$ is invertible and continuous. Then φ^{-1} is differentiable at $\varphi(x)$ and $\mathrm{d}\varphi^{-1}(\varphi(x)) = (\mathrm{d}\varphi(x))^{-1}$.*

When working with a Lipschitz map and one knows that it is of maximal rank at a point, one can quantify the size of the domain on which said map stays of maximal rank:

Lemma 3.50. *Let $\gamma \geq 1$. Let E, F and G be normed vector spaces. Let U be a subset of E and let $f : U \to \mathcal{L}(F, G)$ be a Lip-γ. Let $x_0 \in U$ and $M_1, M_2 \in (0, \infty)$ and assume that $\|f\|_{Lip\text{-}\gamma} \leq M_1$:*

1. *There exists $\delta > 0$ depending only on γ and $M_1 M_2$ such that:*

$$\forall x \in \overline{B(x_0, \delta)} \cap U : \quad \|f(x) - f(x_0)\| \leq \frac{1}{2M_2}$$

In particular, if $f(x_0)$ is invertible and $\|f(x_0)^{-1}\| \leq M_2$, then f is invertible on $\overline{B(x_0,\delta)} \cap U$.

2. *Assume that F and G are finite dimensional of dimension p and m respectively, that f is of rank less than or equal to $k \in \mathbb{N}$ and that $f(x_0)$ is of maximal rank k. Identify f with a matrix of functions $(f_{i,j})_{(i,j)\in[\![1,m]\!]\times[\![1,p]\!]}$. Let $(i_1,\dots,i_k)$ and $(j_1,\dots,j_k)$ be, respectively, strictly ordered subsets of $[\![1,m]\!]$ and $[\![1,p]\!]$ such that $M = (f_{i_r,j_l}(x_0))_{1\leq r,l\leq k}$ is invertible. We assume that $\|M^{-1}\| \leq M_2$. Then there exists $\delta > 0$ that depends only on γ and M_1M_2 such that, for all $x \in \overline{B(x_0,\delta)} \cap U$, $f(x)$ is of rank k.*

Proof. 1. Let $n \in \mathbb{N}$ and $\varepsilon \in (0,1]$ such that $\gamma = n+\varepsilon$. Let $R : U \times U \to \mathcal{L}(F,G)$ be such that for all $x, y \in U$:

$$f(x) = f(y) + \sum_{k=1}^{n} f^k(y)\left(\frac{(x-y)^{\otimes k}}{k!}\right) + R(x,y)$$

and

$$\|R(x,y)\| \leq M_1\|x-y\|^{n+\varepsilon}$$

We get from the above that, for $x \in U$:

$$\|f(x) - f(x_0)\| \leq M_1\left(\sum_{k=1}^{n}\frac{\|x-x_0\|^k}{k!} + \|x-x_0\|^{n+\varepsilon}\right)$$

It suffices to choose δ such that:

$$\forall 0 \leq t \leq \delta : \quad \sum_{k=1}^{n}\frac{t^k}{k!} + t^{n+\varepsilon} \leq \frac{1}{2M_1M_2}$$

which proves the claim.

2. The previous result insures that we can find $\delta > 0$ that depends only on γ and M_1M_2 such that the square matrix $(f_{i_r,j_l}(x))_{1\leq r,l\leq k}$ is invertible for all $x \in \overline{B(x_0,\delta)} \cap U$. Therefore the rank of $(f_{i,j}(x))_{(i,j)\in[\![1,m]\!]\times[\![1,p]\!]}$ is larger than or equal to k on $\overline{B(x_0,\delta)} \cap U$. Since the rank of f is always less than or equal to k, then $f(x)$ is necessarily of rank k on $\overline{B(x_0,\delta)} \cap U$.

□

Definition 3.51. *Let $\gamma > 0$. Let E and F be two normed vector spaces, U be a subset of E and V a subset of F. A map $f : U \to V$ is said to be a Lipschitz diffeomorphism of degree γ (a $Lip-\gamma$ diffeomorphism in short) if f is $Lip-\gamma$ and bijective and f^{-1} is also $Lip-\gamma$.*

Theorem 3.52 (Inverse Function)**.** *Let $n \in \mathbb{N}^*$ and $\varepsilon \in (0,1]$. Let U be an open subset of a Banach space E and let $\varphi : U \to E$ be a differentiable map such that $\mathrm{d}\varphi$ is $Lip-(n+\varepsilon-1)$. Let $x_0 \in U$ and assume that $\mathrm{d}\varphi(x_0)$ is invertible. Then, for every $M_1 > 0$ and $M_2 > 0$ such that $\|\mathrm{d}\varphi\|_{Lip-(n+\varepsilon-1)} \leq M_1$ and $\|\mathrm{d}\varphi(x_0)^{-1}\| \leq M_2$, there exists a constant δ, depending only on n, ε and M_1M_2, such that for every $\alpha \in (0,\delta]$ satisfying $\overline{B(x_0,\alpha)} \subseteq U$ there exists a constant c depending only on n, ε, $\|x_0\|$, M_1 and M_2 such that:*

- *$\varphi : B(x_0,\alpha) \cap \varphi^{-1}(V_0) \to V_0$ is a $Lip-(n+\varepsilon)$-diffeomorphism, where $V_0 = \varphi(x_0) + \mathrm{d}\varphi(x_0)(B(0,\alpha/2))$.*
- *$B(x_0,\alpha/3) \subseteq B(x_0,\alpha) \cap \varphi^{-1}(V_0)$.*
- *$\|\varphi^{-1}_{|V_0}\|_{Lip-(n+\varepsilon)} \leq c$.*

Proof. Let $M_1, M_2 > 0$ and assume that $\|\mathrm{d}\varphi\|_{\mathrm{Lip}-(n+\varepsilon-1)} \leq M_1$ and $\|\mathrm{d}\varphi(x_0)^{-1}\| \leq M_2$. We start by proving that, on a well-chosen bounded subset of U, φ is injective and satisfies some key inequalities. As $\mathrm{d}\varphi$ is Lipschitz, then by lemma 3.50, we can find $\delta > 0$

depending only on $n+\varepsilon$ and M_1M_2 such that:

$$\forall x \in \overline{B(x_0,\delta)} \cap U : \quad \|\mathrm{d}\varphi(x) - \mathrm{d}\varphi(x_0)\| \leq \frac{1}{2M_2} \left(\leq \frac{1}{2\|\mathrm{d}\varphi(x_0)^{-1}\|} \right)$$

Let $0 < \alpha \leq \delta$ be such that $\overline{B(x_0,\alpha)} \subseteq U$. For all $x, \tilde{x} \in \overline{B(x_0,\alpha)}$, we have then

$$\|\mathrm{d}\varphi(x_0)^{-1} \circ \mathrm{d}\varphi(x) - \mathrm{Id}\| \leq \frac{1}{2} \tag{3.3}$$

and consequently

$$\|\mathrm{d}\varphi(x_0)^{-1}(\varphi(x) - \varphi(\tilde{x})) - (x - \tilde{x})\| \leq \frac{1}{2}\|x - \tilde{x}\| \tag{3.4}$$

$$\frac{1}{2}\|x - \tilde{x}\| \leq \|\mathrm{d}\varphi(x_0)^{-1}(\varphi(x) - \varphi(\tilde{x}))\| \leq \frac{3}{2}\|x - \tilde{x}\| \tag{3.5}$$

Thus, φ is injective on $\overline{B(x_0,\alpha)}$.

Denote $V_0 = \varphi(x_0) + \mathrm{d}\varphi(x_0)B(0,\alpha/2)$ and let $y \in V_0$. We prove now that there exists a unique point $x \in B(x_0,\alpha)$ such that $\varphi(x) = y$:

Let G be the map:

$$\begin{array}{rccl} G: & \overline{B(x_0,\alpha)} & \longrightarrow & E \\ & x & \longmapsto & x + \mathrm{d}\varphi(x_0)^{-1}(y - \varphi(x)) \end{array}$$

Note that $x \in \overline{B(x_0,\alpha)}$ is a fixed point of G if and only if $\varphi(x) = y$. Let $x \in \overline{B(x_0,\alpha)}$, then by (3.4):

$$\|G(x) - x_0\| \leq \|\mathrm{d}\varphi(x_0)^{-1}(y - \varphi(x_0))\| + \|x - x_0 + \mathrm{d}\varphi(x_0)^{-1}(\varphi(x) - \varphi(x_0))\| < \alpha$$

Therefore $G(\overline{B(x_0,\alpha)}) \subseteq \overline{B(x_0,\alpha)}$ and by (3.4), we conclude that G is a contraction. It has therefore a unique fixed point, denote it by $\tilde{x}$. Then we have:

$$\|\tilde{x} - x_0\| = \|G(\tilde{x}) - x_0\| < \alpha$$

Hence $\tilde{x} \in B(x_0, \alpha)$, which proves the claim. Note that the inequality (3.5) shows that $B(x_0, \alpha/3) \subseteq B(x_0, \alpha) \cap \varphi^{-1}(V_0)$.

We show now that φ is a homeomorphism when restricted to a specific domain. $\varphi : B(x_0, \alpha) \cap \varphi^{-1}(V_0) \to V_0$ is continuous and bijective. Therefore, $\varphi^{-1} : V_0 \to B(x_0, \alpha) \cap \varphi^{-1}(V_0)$ exists and is 1-Hölder by (3.5). Hence φ is a homeomorphism (from $B(x_0, \alpha) \cap \varphi^{-1}(V_0)$ onto V_0).

Now lemma 3.49 (together with the inequality (3.3)) shows that φ^{-1} is differentiable at every point of V_0 and that for every $y \in V_0$, $\mathrm{d}\varphi^{-1}(y) = (\mathrm{d}\varphi(\varphi^{-1}(y)))^{-1}$. We will show by induction that φ^{-1} is $\mathrm{Lip} - (n + \varepsilon)$. More precisely, we will prove that for every $k \in [\![1, n]\!]$, φ^{-1} is $\mathrm{Lip} - (k + \varepsilon)$ and that there exists a constant H_k depending only on n, ε, M_1 and M_2 such that $\|\mathrm{d}\varphi^{-1}\|_{\mathrm{Lip}-(k+\varepsilon-1)} \leq H_k$. But let us first make some remarks:

- V_0 being open and convex, we can then use the criteria in lemma 3.17 to show that φ^{-1} is $\mathrm{Lip} - (n + \varepsilon)$.
- $\|\varphi^{-1}\|_{\infty, V_0} \leq \alpha + \|x_0\| \leq \delta + \|x_0\|$.
- If we denote by i the inversion map on $\overline{B_{\mathcal{L}(E)}(\mathrm{d}\varphi(x_0), \frac{1}{2M_2})}$ (which is a smooth map and thus Lipschitz), $\mathrm{d}\varphi^{-1}$ can then be seen as the composition map of φ^{-1}, $\mathrm{d}\varphi$ and i:

  $$\mathrm{d}\varphi^{-1} : V_0 \xrightarrow{\varphi^{-1}} B(x_0, \alpha) \xrightarrow{\mathrm{d}\varphi} \overline{B_{\mathcal{L}(E)}(\mathrm{d}\varphi(x_0), \frac{1}{2M_2})} \xrightarrow{i} \mathcal{L}(E)$$

  For $\gamma > 0$, let C_γ denote the $\mathrm{Lip} - (\gamma)$ norm of i.

We start now our induction. For $k = 1$, we know that φ^{-1} is bounded. Let $y, \tilde{y} \in V_0$, then we have, using that $\mathrm{d}\varphi$ is ε-Hölder and that φ^{-1} is 1-Hölder:

$$\begin{aligned} \|\mathrm{d}\varphi^{-1}(y) - \mathrm{d}\varphi^{-1}(\tilde{y})\| &\leq \|i(\mathrm{d}\varphi(\varphi^{-1}(y))) - i(\mathrm{d}\varphi(\varphi^{-1}(\tilde{y})))\| \\ &\leq C_1\|\mathrm{d}\varphi(\varphi^{-1}(y)) - \mathrm{d}\varphi(\varphi^{-1}(\tilde{y}))\| \\ &\leq C_1\|\mathrm{d}\varphi\|_{\mathrm{Lip}-\varepsilon}\|\varphi^{-1}(y) - \varphi^{-1}(\tilde{y})\|^{\varepsilon} \\ &\leq (2M_2)^{\varepsilon}C_1\|\mathrm{d}\varphi\|_{\mathrm{Lip}-\varepsilon}\|y - \tilde{y}\|^{\varepsilon} \end{aligned}$$

Hence, $\mathrm{d}\varphi^{-1}$ is ε-Hölder. Written as a composition map, we see that $\mathrm{d}\varphi^{-1}$ is bounded (by C_1). Consequently, φ^{-1} is $\mathrm{Lip}-(1+\varepsilon)$. Following theorem 3.15, let $m > 0$ be constant dependent only on n and ε, such that $\|\mathrm{d}\varphi\|_{\mathrm{Lip}-\varepsilon} \leq m\|\mathrm{d}\varphi\|_{\mathrm{Lip}-(n+\varepsilon-1)}$. Then:

$$\|\mathrm{d}\varphi^{-1}\|_{\mathrm{Lip}-\varepsilon} \leq H_1 \text{ , where } H_1 = C_1 \max(1, (2M_2)^{\varepsilon} m M_1)$$

Let $k \in [\![1, n-1]\!]$. We assume that φ^{-1} is $\mathrm{Lip}-(k+\varepsilon)$ and that there exists a constant H_k depending only on n, ε, M_1 and M_2 such that:

$$\|\mathrm{d}\varphi^{-1}\|_{\mathrm{Lip}-(k+\varepsilon-1)} \leq H_k$$

As $\mathrm{d}\varphi^{-1} = i \circ \mathrm{d}\varphi \circ \varphi^{-1}$ and i, $\mathrm{d}\varphi$ and φ^{-1} are all $\mathrm{Lip}-(k+\varepsilon)$, then, by lemma 3.35, $\mathrm{d}\varphi^{-1}$ is $\mathrm{Lip}-(k+\varepsilon)$ with a Lipschitz norm less than a constant H_{k+1} depending only on $k+\varepsilon$, $C_{k+\varepsilon}$ (which depends only on k, ε, M_1 and M_2), $\|\mathrm{d}\varphi\|_{\mathrm{Lip}-(k+\varepsilon)}$ (which can be controlled using only M_1, k, n and ε by corollary 3.15) and $\|\varphi^{-1}\|_{\mathrm{Lip}-(k+\varepsilon)}$ (which is less than $H_k \vee (\delta + \|x_0\|)$; δ, we recall, depends only on n, ε and M_1M_2), which ends the induction. Consequently, $\varphi : B(x_0, \alpha) \cap \varphi^{-1}(V_0) \to V_0$ is a $\mathrm{Lip}-(n+\varepsilon)$ diffeomorphism. $\square$

3.3.2 The constant rank theorem

Definition 3.53 (Local Inverse)**.** *Let E and F be two topological spaces. Let U be a subset of E and $\varphi : U \to F$ be a map. We say that an E-valued map $\hat{\varphi}$ defined on a subset of F containing $\varphi(U)$ is a local inverse of φ on U if $\hat{\varphi} \circ \varphi_{|U} = \mathrm{Id}_U$.*

Definition 3.54 (Immersions)**.** *Let E and F be two topological vector spaces. Let U be a subset of E and $\varphi : U \to F$ be a differentiable map. We say that φ is an immersion if, for every $x \in U$, $\mathrm{d}\varphi(x)$ is injective.*

In the following theorem, we will use the l^∞ norms. The statement of the theorem and the subsequent proof adapt easily in the case of other norms.

Theorem 3.55 (Constant Rank)**.** *Let $(n, p, q, k) \in (\mathbb{N}^*)^3 \times \mathbb{N}$, $\varepsilon \in (0, 1]$ and M_1 and M_2 be two positive real numbers. Let U be an open subset of $\mathbb{R}^p$. Let $\varphi = (\varphi_1, \dots, \varphi_q) : U \to \mathbb{R}^q$ be a differentiable map of constant rank k such that $\mathrm{d}\varphi$ is $Lip - (n + \varepsilon - 1)$. Let $x_0 \in U$, $(i_1, \dots, i_k)$ and $(j_1, \dots, j_k)$ be, respectively, strictly ordered subsets of $[\![1, q]\!]$ and $[\![1, p]\!]$ such that $M = (\frac{\partial \varphi_{i_r}}{\partial x_{j_l}}(x_0))_{1 \le r,l \le k}$ is invertible. We assume that $\|\mathrm{d}\varphi\|_{Lip-(n+\varepsilon-1)} \le M_1$ and $\|M^{-1}\| \le M_2$. Then, there exist two constants δ and c, depending only on n, ε, M_1 and M_2 such that for every $\alpha \in (0, \delta]$ such that $\overline{B(x_0, \alpha)} \subseteq U$, we have:*

- *A $Lip - (n + \varepsilon)$ diffeomorphism $f : U_0 \to H$ centered at x_0, where U_0 and H are two open subsets of $\mathbb{R}^p$ and $B(x_0, \alpha/3) \subseteq U_0 \subseteq U$.*
- *A diffeomorphism g defined on an open subset W of $\mathbb{R}^q$ centered at $\varphi(x_0)$ and containing $\varphi(U_0)$ such that $\mathrm{d}g$ is $Lip - (n + \varepsilon - 1)$ and the restriction of g to $\varphi(U_0)$ or to any bounded set of W is $Lip - (n + \varepsilon)$.*

such that, for all $(x_1, \dots, x_p) \in H$:

$$g \circ \varphi \circ f^{-1}(x_1, \dots, x_p) = (x_1, \dots, x_k, 0, \dots, 0)$$

Morever, if $k = p$, then $\varphi_{|U_0}$ is an injective immersion and admits a local inverse $\hat{\varphi}$ on U_0 that is Lip $-$ $(n+\varepsilon)$ and such that

$$\|\mathrm{d}\hat{\varphi}\|_{Lip-(n+\varepsilon-1)} \leq C_{n+\varepsilon} c \|\mathrm{d}\varphi\|_{Lip-(n+\varepsilon-1)}$$

where $C_{n+\varepsilon}$ is the constant specified in theorem (3.35).

$$\begin{tikzcd}
(B(x_0, \alpha/3) \subseteq) U_0 (\subseteq \mathbb{R}^p) \arrow[r, "\varphi"] \arrow[d, "f"'] & (\varphi(U_0) \subseteq) W (\subseteq \mathbb{R}^q) \arrow[d, "g"] \\
H \arrow[r, "\pi"] & g(W)
\end{tikzcd}$$

Proof. We start first by two changes of variables that will enable us later to see φ almost as a projection of the first k variables. We will indentify $\mathbb{R}^p$ (resp. $\mathbb{R}^q$) with $\mathbb{R}^k \oplus \mathbb{R}^{p-k}$ (resp. $\mathbb{R}^k \oplus \mathbb{R}^{q-k}$). For $x \in \mathbb{R}^p$, we denote by $\overline{(x_{j_l})_{1\leq l \leq k}}$ the image of x by the projection onto $\mathbb{R}^{p-k}$ which kernel is the span of $((e_{j_l})_{1 \leq l \leq k})$, where $(e_i)_{1\leq i \leq p}$ is the canonical basis of $\mathbb{R}^p$. We define in a similar way the vector $\overline{(z_{i_r})_{1 \leq r \leq k}}$ for $z \in \mathbb{R}^q$. Now, let f_1 and g_1 be the two following diffeomorphisms:

$$\begin{array}{rccl}
f_1: & U & \to & \tilde{U}(= f_1(U)) \\
 & x & \mapsto & ((x_{j_l})_{1\leq l\leq k}, \overline{(x_{j_l})_{1\leq l \leq k}})
\end{array}$$

and:

$$\begin{array}{rccl}
g_1: & \mathbb{R}^q & \to & \mathbb{R}^q \\
 & z & \mapsto & ((z_{i_r})_{1\leq r\leq k}, \overline{(z_{i_r})_{1\leq r \leq k}})
\end{array}$$

We also define $\tilde{\varphi} = (\tilde{\varphi}_1, \ldots, \tilde{\varphi}_q) := g_1 \circ \varphi \circ {f_1}^{-1}$, $A = (\tilde{\varphi}_1, \ldots, \tilde{\varphi}_k)$ and $B = (\tilde{\varphi}_{k+1}, \ldots, \tilde{\varphi}_q)$. Then, $\mathrm{d}\tilde{\varphi}$, $\mathrm{d}A$ and $\mathrm{d}B$ are Lip $-$ $(n + \varepsilon - 1)$ and we have:

$$\max(\|\mathrm{d}A\|_{\mathrm{Lip}-(n+\varepsilon-1)}, \|\mathrm{d}B\|_{\mathrm{Lip}-(n+\varepsilon-1)}) = \|\mathrm{d}\tilde{\varphi}\|_{\mathrm{Lip}-(n+\varepsilon-1)} = \|\mathrm{d}\varphi\|_{\mathrm{Lip}-(n+\varepsilon-1)}$$

Let f_2 be the map defined on $\tilde{U}(\subseteq \mathbb{R}^k \oplus \mathbb{R}^{p-k})$ by:

$$f_2(a,b) = (A(a,b) - A(f_1(x_0)), b - ((f_1(x_0))_j)_{k-p\leq j\leq p})$$

Then f_2 is differentiable at every point of $\tilde{U}$ and $\mathrm{d}f_2$ is $\mathrm{Lip}-(n+\varepsilon-1)$ with:

$$\|\mathrm{d}f_2\|_{\mathrm{Lip}-(n+\varepsilon-1)} \leq \max(1, \|\mathrm{d}\varphi\|_{\mathrm{Lip}-(n+\varepsilon-1)})$$

The representation matrix of $\mathrm{d}f_2(f_1(x_0))$ in the canonical basis of $\mathbb{R}^p$ is under the form:

$$\begin{pmatrix} M & \tilde{M} \\ (0) & \mathrm{I}_{p-k} \end{pmatrix}$$

where $\tilde{M}$ is some matrix in $\mathcal{M}_{k,p-k}$. Hence $\mathrm{d}f_2(f_1(x_0))$ is invertible and

$$\begin{aligned} \|\mathrm{d}f_2(f_1(x_0))^{-1}\| &\leq \max(1, \|M^{-1}\|, \|M^{-1}\tilde{M}\|) \\ &\leq \max(1, \|M^{-1}\| \max(1, C_{p,q}\|\mathrm{d}\varphi\|_{\mathrm{Lip}-(n+\varepsilon-1)})) \end{aligned}$$

where $C_{p,q}$ is an integer depending only on p and q. Using theorem 3.52, let δ (resp. c) be a constant depending on n, ε, M_1 and M_2 (resp. depending on n, ε, M_1, M_2 and $\|f_1(x_0)\|$) such that, for every $\alpha \in (0,\delta]$ such that $\overline{B(f_1(x_0),\alpha)} \subseteq \tilde{U}$, the map:

$$f_2 : B(f_1(x_0),\alpha) \cap f_2^{-1}(H) \to H$$

is a $\mathrm{Lip}-(n+\varepsilon)$ diffeomorphism, where:

$$H = \mathrm{d}f_2(f_1(x_0))(B(0,\alpha/2))$$

(we drop again the restriction signs $._{|}$) and:

$$\|f_2^{-1}\|_{\text{Lip}-(n+\varepsilon)} \leq c$$

We also have $B(f_1(x_0), \alpha/3) \subseteq B(f_1(x_0), \alpha) \cap f_2^{-1}(H)$. We can define $U_0 = f_1^{-1}(B(f_1(x_0), \alpha) \cap f_2^{-1}(H))$.

We prove now that $\tilde{\varphi} \circ {f_2}^{-1}$ is independent of the second variable (when identifying $\mathbb{R}^p$ with $\mathbb{R}^k \oplus \mathbb{R}^{p-k}$). Write ${f_2}^{-1}$ under the form:

$${f_2}^{-1}(y_1, y_2) = (C(y_1, y_2), D(y_1, y_2))$$

Then the identity $f_2 \circ {f_2}^{-1} = \text{Id}$ shows that for every $(y_1, y_2) \in H$:

$$A(C(y_1, y_2), D(y_1, y_2)) - A(f_1(x_0)) = y_1$$

which allows writing $\tilde{\varphi} \circ {f_2}^{-1}$ under the form:

$$\tilde{\varphi} \circ {f_2}^{-1}(y_1, y_2) = (y_1 + A(f_1(x_0)), B({f_2}^{-1}(y_1, y_2)))$$

For lighter expressions, we define $\tilde{B}$ on H by $\tilde{B} = B \circ {f_2}^{-1}$. As $\mathrm{d}\tilde{\varphi}$ is of rank k, $\mathrm{d}(\tilde{\varphi} \circ {f_2}^{-1})$ is of rank at most k on H. For $(y_1, y_2) \in H$, the representation matrix of $\mathrm{d}(\tilde{\varphi} \circ {f_2}^{-1})(y_1, y_2)$ in the canonical bases of $\mathbb{R}^p$ and $\mathbb{R}^q$ is under the form:

$$\begin{pmatrix} \mathrm{I}_k & (0) \\ \frac{\partial \tilde{B}}{\partial y_1}(y_1, y_2) & \frac{\partial \tilde{B}}{\partial y_2}(y_1, y_2) \end{pmatrix}$$

As this matrix is of order k, then, $\frac{\partial \tilde{B}}{\partial y_2}(y_1, y_2) = 0$. Therefore, we see that $\tilde{\varphi} \circ {f_2}^{-1}$ is independent of the second variable. Define then F on $\pi_{\mathbb{R}^k}(H)$ by $F(a) = \tilde{B}(a, b_a)$, where for $a \in \pi_{\mathbb{R}^k}(H)$, b_a is any element such that $(a, b_a) \in H$, and $\pi_{\mathbb{R}^k} : \mathbb{R}^k \oplus \mathbb{R}^{p-k} \to \mathbb{R}^k$ is the projection in the first variable. We have then, for all $(y_1, y_2) \in H$:

$$\tilde{\varphi} \circ {f_2}^{-1}(y_1, y_2) = (y_1 + A(f_1(x_0)), F(y_1))$$

We end this proof by defining a final diffeomorphism. Define the open set:

$$\tilde{W} = \{(z_1, z_2) \in \mathbb{R}^q | z_1 - A(f_1(x_0)) \in \pi_{\mathbb{R}^k}(H)\}$$

Let g_2 be the map defined on $\tilde{W}$ by:

$$g_2(z_1, z_2) = (z_1 - A(f_1(x_0)), z_2 - F(z_1 - A(f_1(x_0))))$$

$\mathrm{d}g_2$ is clearly $\mathrm{Lip} - (n + \varepsilon - 1)$ and g_2 is a $\mathrm{Lip} - (n + \varepsilon)$ diffeomorphism when restricted to a bounded set of $\tilde{W}$ or to $f_1(U_0)$ and for $(y_1, y_2) \in H$:

$$g_2 \circ \tilde{\varphi} \circ {f_2}^{-1}(y_1, y_2) = (y_1, 0)$$

By defining $f = f_2 \circ f_1$, $g = g_2 \circ g_1$ and $W = g_1^{-1}(\tilde{W})$, we get the claimed result.

Assume now that $k = p$. Then $g \circ \varphi \circ f^{-1} = i_p$, where

$$\begin{array}{rccc} i_p : & \mathbb{R}^p & \rightarrow & \mathbb{R}^p \oplus \mathbb{R}^{q-p} \\ & x & \mapsto & (x, 0) \end{array}$$

Let $\hat{\varphi}$ be the map $f^{-1} \circ \pi_p \circ g$ defined on W, where $\pi_p : \mathbb{R}^p \oplus \mathbb{R}^{q-p} \to \mathbb{R}^p$ is the projection on the first p variables. Then $\hat{\varphi}$ is a local inverse of φ on U_0 and a $\mathrm{Lip} - (n + \varepsilon)$ map on

$\varphi(U_0)$. Writing $\hat{\varphi}$ as the composition of the two maps f^{-1} and $\pi_p \circ g$, one gets the control

$$\|\mathrm{d}\hat{\varphi}\|_{\mathrm{Lip}-(n+\varepsilon-1)} \leq C_{n+\varepsilon}\|\mathrm{d}f^{-1}\|_{\mathrm{Lip}-(n+\varepsilon-1)}\|\mathrm{d}f(x_0)\|$$

where $C_{n+\varepsilon}$ is the constant specified in theorem (3.35). □

Chapter 4

Rough paths on manifolds

As it has been been proved in many other articles (see for example [19] and [26]), under suitable circumstances, the reachability set can be endowed with a structure of a smooth manifold. This is not, however, the suitable context for trying to generalize the concept of rough paths on a manifold. Indeed, when one is working with smooth maps instead of Lipschitz maps and trying to solve a rough differential equation, one loses the quantitative estimates controlling the convergence of the Picard sequence. Moreover, even with a definition of rough paths in smooth manifolds, ordinary and rough differential equations can only be solved locally in such case. In this chapter, we present the foundations of the Lipschitz geometry, as introduced in [3], along with the main findings that generalise the classical theory of rough paths in Banach spaces. Then we give what we believe to be a minimalistic frame for defining rough paths on a manifold that is both less rigid than the classical one and emphasized on the local behaviour of rough paths. We end by explaining how this same idea can be used to define any notion of coloured paths on a manifold.

4.1 The Lipschitz geometry

We first introduce, following [3][1], the general definitions of Lipschitz manifolds and Lipschitz maps and one-forms on them.

Definition 4.1. *Let $\gamma \geq 1$. Let $n \in \mathbb{N}^*$ and let M be an n-topological manifold. Let I be a countable set and, for every $i \in I$, U_i be an open subset of M and $\phi_i : M \to \mathbb{R}^n$ be a map such that its restriction on U_i defines a homeomorphism. We say that $((\phi_i, U_i))_{i\in I}$ is a Lipschitz-γ atlas if the following properties are satisfied:*

- *$(U_i)_{i\in I}$ is a pre-compact locally finite cover of M;*
- *There exists $R > 0$ such that, for every $i \in I$: $\phi_i(U_i) = B(0,1)$ and $\phi_i(M)$ is a compact subset contained in $B(0,R)$;*
- *There exists $\delta \in (0,1)$, such that $(U_i^\delta)_{i\in I}$ covers M, where, for every $i \in I$: $U_i^\delta = \phi_i^{-1}{}_{|U_i}(B(0,1-\delta))$;*
- *There exists $L > 0$, such that, for every $i, j \in I$, $\phi_j \circ (\phi_{i|U_i})^{-1} : B(0,1) \to \mathbb{R}^n$ is Lipschitz-γ and $\|\phi_j \circ (\phi_{i|U_i})^{-1}\|_{Lip-\gamma} \leq L$.*

With the constants above, we will say that M is a Lipschitz-γ manifold with constants (R, δ, L). For $i \in I$, (ϕ_i, U_i) is said to be a Lipschitz-γ chart on M.

Notation. *For a Lip-γ chart (ϕ, U) on a Lip-γ manifold M and for $\alpha \in [0,1]$, we will denote by U^α the set $(\phi_{|U})^{-1}(B(0,1-\alpha))$.*

Example 4.2. *As one would expect, finite-dimensional vector spaces can be endowed with a natural Lipschitz-γ manifold structure of any degree $\gamma \geq 1$.*

Proof. Indeed, let $\gamma \geq 1$. Let V be a finite dimensional space and let $(e_1, \ldots, e_n)$ be a basis for V. Let φ be a Lip-γ extension on V of $Id_{B(0,1)}$ with support in $B(0,2)$. For

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[1] Up to some corrections.

$x \in V$, let φ_x be the map $(y \mapsto \varphi(y-x))$. Then $(\varphi_x, B(x,1))_{x\in I}$ is a Lip-γ atlas on V, where:

$$I = \left\{ \sum_{i=1}^{n} \frac{k_i}{2} e_i |\, k_1, \ldots, k_n \in \mathbb{Z} \right\}$$

□

Example 4.3. *For $\gamma \geq 1$, compact $\mathcal{C}^{[\gamma]+1}$-manifolds are Lip-γ manifolds (see [3]).*

The product of two Lipschitz manifolds comes with a natural Lipschitz manifold structure:

Proposition 4.4. *Let $\gamma \geq 1$ and $m, n \in \mathbb{N}$. Let M (respectively N) be a Lip-γ m-manifold (resp. n-manifold) with atlas $(\phi_i, U_i)_{i\in I}$ (resp. $(\psi_j, V_j)_{j\in J}$). When endowed with the product topology and the charts $(\phi_i \times \psi_j, U_i \times V_j)_{(i,j)\in I\times J}$, $M \times N$ has a natural Lip-γ structure; where, for $(i,j) \in I \times J$, $\phi_i \times \psi_j$ is defined as follows:*

$$\begin{array}{rccl} \phi_i \times \psi_j : & M \times N & \longrightarrow & (\mathbb{R}^m \times \mathbb{R}^n, \|.\|_\infty) \\ & (p,q) & \longmapsto & (\phi_i(p), \psi_j(q)) \end{array}$$

Definition 4.5. *Let $\gamma_0, \gamma \in \mathbb{R}$ such that $\gamma_0 \geq \gamma$. Let M be a $Lip-\gamma_0$ manifold with an atlas $\{(\phi_i, U_i), i \in I\}$ and E be a normed vector space. A map $f : M \to E$ is said to be $Lip-\gamma$ if there exists a constant C such that, for every $i \in I$, $f \circ \phi_{i|U_i}^{-1} : B(0,1) \to E$ is $Lip-\gamma$ with a Lipschitz norm at most C. The smallest constant C for which this property holds is called the $Lip-\gamma$ norm of f and is denoted by $\|f\|_{Lip-\gamma}$.*

Definition 4.6. *Let $\gamma_0, \gamma \in \mathbb{R}$ such that $\gamma_0 \geq \gamma$ and $d \in \mathbb{N}$. Let M be a $Lip-\gamma_0$ manifold with an atlas $\{(\phi_i, U_i), i \in I\}$ and E be a normed vector space. An E-valued one-form α on M is said to be $Lip-\gamma$ if there exists a constant C such that, for every $i \in I$:*

$$(\phi_{i|U_i}^{-1})^* \alpha : B(0,1) \to \mathcal{L}(\mathbb{R}^d, E)$$

is Lip $-\gamma$ with a Lipschitz norm at most C. The smallest constant C for which this property holds is called the Lip $-\gamma$ norm of α and is denoted by $\|\alpha\|_{Lip-\gamma}$.

An important property that is absent in [3] and that we underline here is that, on a Lip $-\gamma$ manifold, it does not make sense to talk about Lip $-\gamma_1$ functions or Lip $-\gamma_2$ one-forms if $\gamma_1 > \gamma$ or $\gamma_2 > (\gamma - 1)$. Indeed, with the notations of definition 4.5, if for $i \in I$, $f \circ \phi_{i|U_i}^{-1} : B(0,1) \to E$ is Lip $-\gamma_1$ (where $\gamma_1 > \gamma$), then there is no canonical way to show, based only on the definition of a Lip $-\gamma$ atlas, that, for $j \in I$, the restriction of $f \circ \phi_{j|U_j \cap U_i}^{-1} : B(0,1) \to E$ on any non-trivial subset of $U_j \cap U_i$ is more than Lip $-\gamma$. The same principle applies to defining Lipschitz one-forms. A clearer and more rigorous way to state the above is obtained by translating it in the language of equivalent Lipschitz structures, which are not discussed in this work.

An analogous result to the next lemma was stated in [3]. This is, however, a more general statement and includes a correction on the quantitative result that has been communicated to the authors.

Lemma 4.7. *Let $p, n \in \mathbb{N}$. Let M be a topological space. Let U and V be intersecting open subsets of M and ϕ (respectively ψ) be a continuous $\mathbb{R}^p$-valued (resp. $\mathbb{R}^n$-valued) map on M such that $\phi_{|U}$ (resp. $\psi_{|V}$) is a homeomorphism onto $B_{\mathbb{R}^p}(0,1)$ (resp. bijection into $B_{\mathbb{R}^n}(0,1)$). Assume that $\psi \circ (\phi_{|U})^{-1}$ is Lip-γ and let $m \in U \cap (\psi_{|V})^{-1}(B_{\mathbb{R}^n}(0, 1-\delta))$, for a given $\delta \in (0,1)$. Then, for any $r \in [0, \frac{\delta}{\|\psi \circ (\phi_{|U})^{-1}\|_{Lip\text{-}1}})$, we have:*

$$(\phi_{|U})^{-1}(B_{\mathbb{R}^p}(\phi(m), r) \cap B_{\mathbb{R}^p}(0,1)) \subseteq (\psi_{|V})^{-1}(B_{\mathbb{R}^n}(\psi(m), \delta))$$

Proof. Let $r \in [0, \frac{\delta}{\|\psi \circ (\phi_{|U})^{-1}\|_{\text{Lip-1}}})$. First notice that:

$$m \in (\phi_{|U})^{-1}(B_{\mathbb{R}^p}(\phi(m), r) \cap B_{\mathbb{R}^p}(0,1)) \cap (\psi_{|V})^{-1}(B_{\mathbb{R}^n}(\psi(m), \delta))$$

Suppose there exists $\tilde{m}$ such that:

$$\tilde{m} \in (\phi_{|U})^{-1}(B_{\mathbb{R}^p}(\phi(m), r) \cap B_{\mathbb{R}^p}(0,1)) - (\psi_{|V})^{-1}(B_{\mathbb{R}^n}(\psi(m), \delta))$$

Consider the continuous map g defined on $[0,1]$ by:

$$t \mapsto (\phi_{|U})^{-1}((1-t)\phi(m) + t\phi(\tilde{m}))$$

Then $g(0) \in (\psi_{|V})^{-1}(\overline{B_{\mathbb{R}^n}(\psi(m), r\|\psi \circ (\phi_{|U})^{-1}\|_{\text{Lip-1}})})$ and $g(1) \notin (\psi_{|V})^{-1}(B_{\mathbb{R}^n}(\psi(m), \delta))$, with the observation that:

$$(\psi_{|V})^{-1}(\overline{B_{\mathbb{R}^n}(\psi(m), r\|\psi \circ (\phi_{|U})^{-1}\|_{\text{Lip-1}})}) \subseteq (\psi_{|V})^{-1}(B_{\mathbb{R}^n}(\psi(m), \delta))$$

Therefore, there exists $t_0 \in (0,1)$ such that:

$$g(t_0) \in (\psi_{|V})^{-1}(B_{\mathbb{R}^n}(\psi(m), \delta)) - (\psi_{|V})^{-1}(\overline{B_{\mathbb{R}^n}(\psi(m), r\|\psi \circ (\phi_{|U})^{-1}\|_{\text{Lip-1}})})$$

Meaning in particular that

$$r\|\psi \circ (\phi_{|U})^{-1}\|_{\text{Lip-1}} < \|\psi(g(t_0)) - \psi(m)\| \tag{4.1}$$

By construction $g(t_0) \in (\phi_{|U})^{-1}(B_{\mathbb{R}^p}(\phi(m), r) \cap B_{\mathbb{R}^p}(0,1))$. Since $\psi \circ (\phi_{|U})^{-1}$ is Lip-1, then:

$$\|\psi \circ (\phi_{|U})^{-1}(\phi(g(t_0))) - \psi \circ (\phi_{|U})^{-1}(\phi(m))\| \leq \|\psi \circ (\phi_{|U})^{-1}\|_{\text{Lip-1}}\|\phi(g(t_0)) - \phi(m)\|$$

Implying that:

$$\|\psi(g(t_0)) - \psi(m)\| \leq r\|\psi \circ (\phi_{|U})^{-1}\|_{\text{Lip-1}}$$

which is a contradiction of (4.1). ☐

The following constructions in this section are natural and classical in the literature of smooth manifolds and have been stated or alluded to in [3] in the context of Lipschitz structures. However, we give full proofs with our statements in order to correct some errors in the arguments of the aforementioned paper.

Lemma 4.8. *Let $\gamma \geq \gamma' > 0$ such that $\gamma \geq 1$ and $n \in \mathbb{N}^*$. Let M be a Lip-γ n-manifold and (ϕ, U) be a chart on the Lip-γ atlas on M. Let f be a W-valued map on M, where W is a vector space. Assume that $f \circ (\phi_{|U})^{-1}$ is Lip-γ' and that there exists $\alpha > 0$ such that supp$(f) \subseteq (\phi_{|U})^{-1}(B(0, 1-\alpha))$. Then f is Lip-γ' on M.*

Proof. Let (V, ψ) be a Lip-γ chart on M. Let $r \in (0, \frac{\alpha}{2M_{1,\gamma}L})$ (with the notations of theorem 3.15). Let $x \in B(0,1)$. Suppose that $(\psi_{|V})^{-1}(x) \in U^{\alpha/2}$, then by lemma 4.7, we have:

$$(\psi_{|V})^{-1}(B(x,r) \cap B(0,1)) \subseteq (\phi_{|U})^{-1}(B(\phi((\psi_{|V})^{-1}(x)), \alpha/2)) \subseteq U$$

Therefore, when restricted to $B(x,r) \cap B(0,1)$, the map $f \circ (\psi_{|V})^{-1}$ is equal to

$$f \circ (\phi_{|U})^{-1} \circ \phi \circ (\psi_{|V})^{-1}$$

which is a Lip-γ' map with norm less than $C_{\gamma'} \|f\|_{\text{Lip}-\gamma'} \max(1, (M_{\gamma',\gamma} L)^{\gamma'})$ (by theorems 3.15 and 3.35). Assume now that $(\psi_{|V})^{-1}(x) \notin U^{\alpha/2}$. Then

$$(\psi_{|V})^{-1}(B(x,r) \cap B(0,1)) \cap U^{\alpha} = \varnothing$$

Indeed, if we assume that it is not the case and if $y \in (\psi_{|V})^{-1}(B(x,r) \cap B(0,1)) \cap U^{\alpha}$, then again by lemma 4.7:

$$(\psi_{|V})^{-1}(B(\psi(y), r) \cap B(0,1)) \subseteq (\phi_{|U})^{-1}(B(\phi(y), \alpha/2)) \subseteq U^{\alpha/2}$$

which implies that $(\psi_{|V})^{-1}(x) \in U^{\alpha/2}$, which will constitute a contradiction. Therefore, when restricted to $B(x,r) \cap B(0,1)$, the map $f \circ (\psi_{|V})^{-1}$ is identically 0, hence Lip-γ'. We deduce then, by lemma 3.19 , that $f \circ (\psi_{|V})^{-1}$ is Lip-γ' on $B(0,1)$ with norm less than:

$$C_{\gamma'} \|f\|_{\mathrm{Lip}-\gamma'} \max(1, (M_{\gamma',\gamma} L)^{\gamma'}) \max\left(1, \frac{2}{r^{\gamma' - \lfloor \gamma' \rfloor}}\right)$$

Hence, f is Lip-γ' on M. □

Using the above lemma, we can show that we can easily extend Lipschitz maps defined on a domain inside a chart to the whole (Lipschitz) manifold, more specifically:

Lemma 4.9. *Let $\gamma \geq \gamma' > 0$ such that $\gamma \geq 1$, $\alpha \in (0,1)$ and $n \in \mathbb{N}^*$. Let M be a Lip-γ n-manifold with constants (R, δ, L) and (ϕ, U) be a chart on the Lip-γ atlas on M. Let f be a V-valued Lip-γ' map defined on a subset of U^α, where V is a finite dimensional space. Then, we can extend f to a Lip-γ' map on M with support in $U^{\alpha/2}$ and a Lip-γ' norm depending only on $\|f\|_{Lip-\gamma'}$, α, L, γ and γ'.*

Proof. Consider the map $\hat{f}$ defined as $f \circ (\phi_{|U})^{-1}$ on the image of the domain of f by ϕ. Then $\hat{f}$ is Lip-γ' of norm $\|f\|_{\mathrm{Lip}-\gamma'}$. By Whitney's extension theorem, let F be a Lip-γ' extension of $\hat{f}$ to $\mathbb{R}^n$ with support in $B(0, 1-\alpha/2)$ with a Lip-γ' norm depending only on α, γ' and $\|f\|_{\mathrm{Lip}-\gamma'}$. Consider g defined on M as follows:

$$\begin{array}{rccl} g: & M & \longrightarrow & V \\ & p & \longmapsto & \begin{cases} F \circ \phi(p) & \text{, if } p \in U; \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

Notice that g is identically 0 on $M - U^{\alpha/2}$. Then by lemma 4.8, g is Lip-γ' on M. □

Lemma 4.10. *Let $\gamma \geq 1$ and $n \in \mathbb{N}^*$. Let M be a Lip-γ n-manifold with constants (R, δ, L) and (ϕ, U) be a chart on the Lip-γ atlas on M. There exists a Lip-γ function*

$c : M \to \mathbb{R}$ *such that:*

1. *supp*$(c) \subseteq U^{\delta/2}$;
2. $c_{|U^\delta}$ *is identically* 1;
3. $\forall p \in M : 0 \leq c(p) \leq 1$;

Proof. Mimic the proof of lemma 4.9 to extend the map that is identically 1 on U^δ. □

Like in the framework of smooth maps and manifolds, one can extend Lipschitz maps on a Lipschitz manifold by constructing a suitable partition of unity.

Theorem 4.11 (Partition of unity). *Let* $\gamma \geq 1$, M *be a Lip-*γ *manifold with constants* (R, δ, L) *and* $(\phi_i, U_i)_{i\in I}$ *be a Lip-*γ *atlas on* M. *Then there exists a family of real-valued functions* $(f_i)_{i\in I}$, *called a partition of unity adapted to the atlas* $(\phi_i, U_i)_{i\in I}$, *satisfying the following conditions:*

- $\forall i \in I : \quad$ *supp*$(f_i) \subseteq U_i^{\delta/2}$;
- $\forall i \in I : \quad f_i$ *is Lip-*γ;
- $\forall i \in I, \forall x \in M : \quad f_i(x) \geq 0$;
- $\forall x \in M : \quad \sum\limits_{i\in I} f_i(x) = 1.$

Proof. For every $i \in I$, let $c_i : M \to \mathbb{R}$ be a function satisfying the conditions of lemma 4.10 on the chart (ϕ_i, U_i). Define the map $f_i : M \to \mathbb{R}$, for every $i \in I$, by the following:

$$p \longmapsto \frac{c_i(p)}{\sum\limits_{j\in I} c_j(p)}$$

First notice that as $\text{supp}(c_j) \subseteq U_j^{\delta/2}$, for every $j \in I$, and that every point of M belongs to finitely many $U_j^{\delta/2}$'s, then the sum $\sum_{j\in I} c_j(p)$ is finite, for every $p \in M$. Moreover, as every point of M belongs to at least one of the U_j^{δ}'s, on which the c_js are identically 1, then the sum $\sum_{j\in I} c_j(p)$ is non-zero, for every $p \in M$. Therefore, the maps f_i's are well defined. By lemma 4.8, all the f_i's are Lip-γ. By construction, they obviously satisfy all the other conditions of the theorem. □

As a combination of Whitney's extension theorem and the existence of a suitable Lipschitz partition of unity on Lipschitz manifolds, we can extend Lipschitz maps to the whole manifold with a well-controlled Lipschitz norm:

Theorem 4.12. *Let $\gamma \geq \gamma' > 0$ such that $\gamma \geq 1$ and $n \in \mathbb{N}^*$. Let M be a Lip-γ n-manifold with constants (R, δ, L), $(\phi_i, U_i)_{i\in I}$ be a Lip-γ atlas on M and K be a precompact subset of M. Define:*

$$J = \{i \in U_i |\, U_i^{\delta/2} \cap K \neq \varnothing\}$$

Let f be a V-valued Lip-γ' map defined on K, where V is a finite dimensional space. Then, we can extend f to a Lip-γ' map on M with support in $\cup_J U_i^{\delta/2}$ and with a Lip-γ' norm depending only on $\|f\|_{Lip-\gamma'}$, K, δ, L, γ and γ'.

Proof. Let $(f_i)_{i\in I}$ be a partition of unity adapted to the atlas $(\phi_i, U_i)_{i\in I}$. For every $i \in J$, let (by lemma 4.9) g_i be an extension of $f_{|U_i^{\delta/2}\cap K}$ to a Lip-γ' map on M with support in $U_i^{\delta/4}$. Define:

$$\begin{array}{rccl} g: & M & \longrightarrow & V \\ & p & \longmapsto & \displaystyle\sum_{i\in J} f_i(p) g_i(p) \end{array}$$

Recall that J is finite. g is cleary Lip-γ' as a sum of Lip-γ' maps. Let $p \in K$, then $\sum_J f_i(p) = \sum_{J_p} f_i(p) = 1$ where $J_p = \{i \in I |\, p \in U_i^{\delta/2}\}$ (since $\mathrm{supp}(f_i) \subseteq U_i^{\delta/2}$, for all $i \in I$) and for $i \in J_p$, $g_i(p) = f_{|U_i^{\delta/2}\cap K} = g(p)$. Therefore $g(p) = f(p)$. □

4.2 Rough paths on a manifold

In this section, we generalize the notion of rough paths to manifolds. As we don't have a natural notion of linearity and iterated integrals on a manifold, we have to consider a different approach than the one arising from the p-variational properties of paths and signatures. As we will hint to later, integrals of Lipschitz one-forms along rough paths do characterize the path; this is the first direction that we will take to define our rough paths. Additionally, in the absence of a natural translation, a rough path on a manifolds comes attached with a starting point. All the following results appeared in [3]. We only add some minor corrections to some of the statements in light of the amendments of the previous section.

Definition 4.13. *Let $\gamma_0, p \in \mathbb{R}$ such that $\gamma_0 > p \geq 1$ and $T \geq 0$. Let M be a $Lip-\gamma_0$ manifold and $x \in M$. X is a geometric p-rough path over $[0,T]$ on M starting at x if, for every $\gamma \in \mathbb{R}$ such that $\gamma_0 \geq \gamma > p$ and every Banach space E, the following conditions are satisfied:*

1. *X maps $Lip-(\gamma-1)$ E-valued one-forms on M to E-valued geometric p-rough paths (in the classical sense).*

2. *For every Banach space F and every compactly supported $Lip-\gamma$ map $\psi : M \to F$ and every E-valued $Lip-(\gamma-1)$ one-form α on F we have:*

$$X(\psi^*\alpha) = \int \alpha(\psi(x), X(\psi_*))\mathrm{d}X(\psi_*)$$

3. *There exists a control ω such that for every E-valued $Lip-(\gamma-1)$ one-form α on M, $X(\alpha)$ is controlled by $\|\alpha\|_{Lip-(\gamma-1)}\omega$, i.e.:*

$$\forall (s,t) \in \Delta_{[0,T]}, \forall i \in [\![1,[p]]\!] : \quad \|X(\alpha)^i_{(s,t)}\| \leq \frac{(\|\alpha\|_{Lip-(\gamma-1)}\omega(s,t))^{i/p}}{\beta_p(i/p)!}$$

Contrary to the classical framework, in the context of manifolds, we do not need to make a difference between rough paths and geometric rough paths (as only the latter are defined). Consequently, we drop the word "geometric" when talking about geometric rough paths on manifolds. Moreover, in the classical sense, geometric rough paths are determined by the values of the integral of compactly supported one-forms along them. In order to make the correspondance one-to-one between the concept of a classic geometric rough path on a finite-dimensional space and a rough path on the same space when endowed with its canonical Lipschitz structure, we define the following equivalence relation:

Definition 4.14. *Let $\gamma_0, p \in \mathbb{R}$ such that $\gamma_0 > p \geq 1$. Let M be a Lip $-\gamma_0$ manifold. We say that two p-rough paths X and $\tilde{X}$ on M are equivalent, and we write $X \sim \tilde{X}$, if they have the same starting point and if, for every $\gamma \in \mathbb{R}$ such that $\gamma_0 \geq \gamma > p$ and for every Banach space valued one-form α on M that is compactly supported and Lip $-(\gamma - 1)$, we have $X(\alpha) = \tilde{X}(\alpha)$.*

We give now a hint on how to build a one-to-one correspondance between rough paths in the classical sense and in a Lip-γ manifold, when said manifold is a finite-dimensional vector space V:

- Given a geometric p-rough path (x, X) on V, we define the rough path Z in the manifold by the starting point x and the functional sending every Banach space-valued and compactly supported Lip-$(\gamma - 1)$ one-form α to the classical rough path:

$$Z(\alpha) = \int \alpha(x, X) \mathrm{d}X$$

- Conversely, given a rough path Z on V in the manifold sense, we are tempted to retrieve a rough path in the classical sense by integrating Z against $\mathrm{d}Id_V$. Since dId_V is not compactly supported (this is important for the definition to be consistent across the equivalence classes of the equivalence relation introduced in definition 4.14) and that Id_V is not Lip-γ (which is key in the consistency condition), we can intuitively replace Id_V with a compactly supported Lip-γ extension of its restriction on a set containing the "support" of Z. The notion of support is not yet defined at this point but we can still have a good guess at a set containing it by using the control of the p-variation of Z.

When working on a manifold, one has to ensure that certain properties are invariant by the change of charts (or, in other words, by the change of local parametrisation). For this reason, we define the pushforward of rough paths by conveniently chosen Lipschitz maps:

Lemma 4.15. *Let $\gamma_0, p \in \mathbb{R}$ such that $\gamma_0 > p \geq 1$. Let M and N be $Lip - \gamma_0$ manifolds and $f : M \to N$ be a map such that there exists a constant C_f such that, for all $\gamma \in (p, \gamma_0]$ and every $Lip - (\gamma - 1)$ Banach space valued one-form α on M, we have:*

$$\|f^*\alpha\|_{Lip-(\gamma-1)} \leq C_f \|\alpha\|_{Lip-(\gamma-1)}$$

Then f induces a pushforward map f_* from p-rough paths on M to p rough-paths on N defined as follows: for every p-rough path X over $[0,T]$ on M starting at x, f_*X starts at $f(x)$ and for every $Lip-(\gamma-1)$ Banach space valued one form α on M, where $\gamma \in (p, \gamma_0]$, $f_*X(\alpha)$ is given by:

$$f_*X(\alpha) = X(f^*\alpha)$$

The next proposition shows that there exists a particular class of Lipschitz maps that induce pushforwards of rough paths. In particular, we can ascertain that the pushforwards of rough paths by coordinate maps or transition maps are also rough paths:

Proposition 4.16. [2] *Let $\gamma, p \in \mathbb{R}$ such that $\gamma > p \geq 1$. Let M be a $Lip-\gamma$ manifold and W be a Banach space. Let $f : M \to W$ be a $Lip-\gamma$ map and α be a $Lip-(\gamma-1)$ Banach space valued one-form on W. Then $f^*\alpha$ is Lip-$(\gamma-1)$ and there exists a constant C_γ depending only on γ such that:*

$$\|f^*\alpha\|_{Lip-(\gamma-1)} \leq C_\gamma \|\alpha\|_{Lip-(\gamma-1)} \|f\|_{Lip-\gamma} \max(\|f\|^{\gamma}_{Lip-\gamma}, 1)$$

Like in the classical case, we can define the concatenation of two rough paths. In the context of manifolds, this is important on its own since one usually works locally on coordinate domains to solve ordinary or rough differential equations before attempting to make sense of a global solution:

Definition 4.17. *Let $\gamma, p \in \mathbb{R}$ such that $\gamma > p \geq 1$. Let M be a $Lip-\gamma$ manifold. Let $s \leq u \leq t$. Let X (respectively Y) be a p-rough path over $[s,u]$ (resp. $[u,t]$) on M with starting point x (resp. y). We define the concatenation of X and Y, denoted by $X*Y$, to be the functional Z over $[s,t]$ mapping every Banach space-valued Lip-$(\gamma-1)$ compactly supported one-form α to the classical rough path $X(\alpha) * Y(\alpha)$.*

Unlike the classical case, the concatenation of two rough paths on a manifold is not necessarily a rough path. This is due to the fact that rough paths on a manifold come attached with a starting point and that we have no natural notion of translation. Therefore, for this concatenation to be a rough path, we have to make sure that the two rough paths in question have starting and "ending" points that agree, in the following sense:

Definition 4.18. *Let $\gamma, p, s, u, t \in \mathbb{R}$ such that $\gamma > p \geq 1$ and $s \leq u \leq t$. Let M be a $Lip-\gamma$ manifold. Let X (respectively Y) be a p-rough path on M over $[s,u]$ (resp. $[u,t]$) with starting point x (resp. y). We say that X has an end point consistent with the starting point y of Y if for every Banach valued compactly supported Lip-γ map f on*

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[2] we correct again the exponent in the inequality compared to the result that appeared in [3].

M, we have:

$$f(x) + X(f_*)^1_{s,u} = f(y)$$

In this case, we can check the consistency condition for the concatenation of two rough paths and prove that it is also a rough path:

Proposition 4.19. *Let $\gamma, p \in \mathbb{R}$ such that $\gamma > p \geq 1$. Let M be a $Lip - \gamma$ manifold. Let X (respectively Y) be a p-rough path on M with starting point x (resp. y). We assume that X has an end point consistent with the starting point of Y. Then $X * Y$ is a p-rough path.*

Well-defined concatenations of rough paths constitute an associative operation:

Lemma 4.20. *Let $\gamma, p \in \mathbb{R}$ such that $\gamma > p \geq 1$. Let M be a $Lip - \gamma$ manifold. Let X, Y and Z be p-rough paths on M such that:*

- *X has an end point consistent with the starting point of Y;*
- *Y has an end point consistent with the starting point of Z.*

Then:

- *X has an end point consistent with the starting point of $Y * Z$;*
- *$X * Y$ has an end point consistent with the starting point of Z;*
- *$X * (Y * Z) = (X * Y) * Z$*

In this case, we simple denote $X * Y * Z := X * (Y * Z)$.

Since this notion of rough paths does not attach, for the moment, an underlying path on the manifold to a rough path, we define a notion of support based on the images by rough paths of one-forms supported in different sets:

Definition 4.21. *Let $\gamma, p \in \mathbb{R}$ such that $\gamma > p \geq 1$. Let M be a Lip $-\gamma$ manifold and U be an open subset of M. Let X be a p-rough path on M. We say that X misses U if for every Banach space-valued, compactly supported Lip $-(\gamma-1)$ one-form α on M, we have:*

$$supp(\alpha) \subseteq U \Rightarrow X(\alpha) = 0$$

Definition 4.22. *Let $\gamma, p \in \mathbb{R}$ such that $\gamma > p \geq 1$. Let M be a Lip $-\gamma$ manifold and X be a p-rough path on M with starting point x. We call the support of X the closed subset of M defined by:*

$$supp(X) = \{x\} \cup \left(M - \bigcup_{X \text{ misses } U} U \right)$$

Naturally, this notion of support is consistent with the definition of support for a classical rough path:

Proposition 4.23. *Let $\gamma, p \in \mathbb{R}$ such that $\gamma > p \geq 1$ and $T > 0$. Let V be a finite dimensional vector space endowed with its canonical structure of a Lip-γ manifold. Let (x, X) be a geometric p-rough path on V over $[0, T]$ (in the classical sense), and let Z be the p-rough path associated to it in the manifold sense, i.e. for every Banach space-valued Lip $-(\gamma-1)$ compactly one-form α on V: $Z(\alpha) = \int \alpha(x, X)\mathrm{d}X$. Then:*

$$supp(Z) = \left\{ x + X^1_{0,t} | \, t \in [0, T] \right\}$$

Theorem 4.24. *Let $\gamma, p \in \mathbb{R}$ such that $\gamma > p \geq 1$. Let M be a Lip $-\gamma$ manifold and X be a p-rough path on M. Then the support of X is compact.*

A rough path on a manifold can be “localised” into rough paths that have their support contained in the domain of only one chart at a time:

Theorem 4.25. *Let $\gamma, p \in \mathbb{R}$ such that $\gamma > p \geq 1$ and $T > 0$. Let M be a Lip $-\gamma$ manifold and X be a p-rough path over $[0, T]$ on M with starting point x. Let ω be a* *control of the p-variation of X. There exists $N \in \mathbb{N}^*$ depending only on $\omega(0,T)$ and a collection $(x_i, X^i, s_i, (\phi_i, U_i))_{1\leq i\leq N}$ such that:*

1. $0 \leq s_1 \leq \cdots \leq s_N = T$*;*
2. *For all $i \in [\![1, N]\!]$, (ϕ_i, U_i) is a Lip-γ chart on M;*
3. *For all $i \in [\![1, N]\!]$, X^i is a p-rough path over $[s_{i-1}, s_i]$ on M (with $s_0 = 0$) with starting point x_i;*
4. *For all $i \in [\![1, N-1]\!]$, X^i has an end point consistent with the starting point of X^{i+1};*
5. *For all $i \in [\![1, N]\!]$, supp$(X^i) \subseteq U_i$;*
6. $X = X_1 * \cdots * X_N$*.*

A sequence with such properties is called a localising sequence for X.

4.3 Rough differential equations on a manifold

Still following the presentation of [3], we now introduce rough differential equations on a manifold. In the following, let $\gamma_0, p \in \mathbb{R}$ such that $\gamma_0 > p \geq 1$. Let M and N be $\mathrm{Lip} - \gamma_0$ manifolds and denote by $\pi : M \times N \to M$ the natural projection onto M. Let g be a map on $M \times N$ such that for every $x \in M$, $g(x, .)$ is a linear map from T_xM to $\tau(N)$, the space of all vector fields on N. For all $(x, y) \in M \times N$, let $\Gamma_{(x,y)} : T_xM \to T_{(x,y)}(M \times N)$ be the linear map defined by:

$$\forall v \in T_xM : \quad \Gamma_{(x,y)}(v) = (v, g(x, y)(v))$$

We will call Γ the connection associated to g.

Let now $T \geq 0$, $(x_0, y_0) \in M \times N$ and X be a p-rough path over $[0, T]$ on M starting at

x_0. We will be considering the following rough differential equation:

$$\begin{cases} \mathrm{d}Y_t = g(X_t, Y_t)\mathrm{d}X_t \quad , \forall t \in [0, T] \\ Y_0 = y_0 \end{cases} \tag{4.2}$$

A solution to (4.2) can be defined in the following way:

Definition 4.26. *We say a geometric p-rough path Z on $M \times N$ is a solution to (4.2) if it satisfies the following conditions:*

1. *Its starting point is (x_0, y_0).*

2. *$\pi_* Z \sim X$.*

3. *For every $\gamma \in (p, \gamma_0]$ and every compactly supported $Lip - (\gamma - 1)$ one-form α on $M \times N$ taking values in a Banach space, α^Γ is $Lip - (\gamma - 1)$ and:*

$$Z(\alpha) = Z(\alpha^\Gamma)$$

where α^Γ is the one-form defined on $M \times N$ defined by:

$$\alpha_z^\Gamma(v) = \alpha_z \circ \Gamma_z \circ (\pi_*)(v)$$

for all $z \in M \times N$ and $v \in T_z(M \times N)$.

Definition 4.26 is consistent with the classical solution to an R.D.E.:

Theorem 4.27. *Let $\gamma > p \geq 1$. Let V and W be two finite dimensional vector spaces that we endow with their natural Lip-γ structures. Let $f : V \times W \to \mathcal{L}_c(V, W)$ be a Lip-γ map. Let (x, X) be a p-rough path on V and $y_0 \in W$. The solutions to the rough* *differential equation:*

$$\begin{cases} \mathrm{d}Y_t = f(X_t, Y_t)\mathrm{d}X_t & ,\forall t \in [0,T] \\ Y_0 = y_0 \end{cases}$$

both in the classical and the manifold sense exist and are the same.

Definition 4.28. *let $\gamma \geq 1$. Let M and N be two Lip-γ manifolds with dimensions m and n respectively. Let g be a map defined on M, such that for all $x \in M$, $g(x,.)$ is a linear map from T_xM to $\tau(N)$, the space of vector fields on N. For every (ϕ, U) and (ψ, V) Lip-γ charts on M and N respectively we define:*

$$\begin{array}{rlcl} g_{(\phi,\psi)}: & B_{\mathbb{R}^m}(0,1) \times B_{\mathbb{R}^n}(0,1) & \to & \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) \\ & (x,y) & \mapsto & \left(v \mapsto \psi_* \left(g((\phi_{|U})^{-1}(x), (\psi_{|V})^{-1}(y))(\phi_{|U})_*^{-1}(v)\right)\right) \end{array}$$

We say that g is Lip-γ, if there exists a real constant C such that, for every (ϕ, U) and (ψ, V) Lip-γ charts on M and N respectively, $g_{(\phi,\psi)}$ is Lip-γ with a Lip-γ norm less than or equal to C.

Theorem 4.29. *Let $\gamma \in \mathbb{R}$ such that $\gamma_0 \geq \gamma > p$. Suppose that g is $Lip - \gamma$. Then the equation (4.2) has a solution.*

4.4 Topology on intervals

We present here some basic topological properties of covers of intervals which will be of use when studying rough paths locally.

Definition 4.30. *A collection $(K_i)_{i \in I}$ of subsets of a topological space J is said to cover J (or is a cover for J) if $J = \cup_{i \in I} K_i$.*

Definition 4.31. *A collection $(K_i)_{i \in I}$ of compact subsets of a topological space J that is locally finite and covers J is called a compact cover for J.*

The proof of the following lemma is straightforward:

Lemma 4.32. *Each interval* J *of* $\mathbb{R}$ *admits a compact cover.*

Lemma 4.33. *Let* J *be an interval and* $(K_i)_{i\in I}$ *be a compact cover for* J*. Then* I *is countable. Moreover, if* J *is compact, then* I *is finite.*

Proof.

- Let $a, b \in \mathbb{R}$ such that $a \leq b$ and let $(K_i)_{i\in I}$ be a locally finite collection of subsets of $[a, b]$ covering $[a, b]$ (note that this is a weaker assumption than $(K_i)_{i\in I}$ being a compact cover for $[a, b]$). Assume that I is infinite. Then, it is an easy exercise to construct two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ such that:

$$\begin{cases} a_0 = a, b_0 = b \text{ and } \forall n \in \mathbb{N} : a_n \leq b_n; \\ (a_n)_{n\in\mathbb{N}} \text{ is non-decreasing and } (b_n)_{n\in\mathbb{N}} \text{ is non-increasing;} \\ \forall n \in \mathbb{N} : \quad b_n - a_n = \frac{(b-a)}{2^n}; \\ \forall n \in \mathbb{N} : \quad [a_n, b_n] \text{ intersects infinitely many } K_i\text{'s} \end{cases}$$

The two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are adjacent, converge and have a common limit $c \in [a, b]$. Then any neighborhood of c intersects infinitely many K_i's, which contradicts the fact that $(K_i)_{i\in I}$ is locally finite. Therefore, I is finite. $(K_i)_{i\in I}$ being a compact cover of $[a, b]$ is then just a special case of the one discussed above.

- Let J be an arbitrary interval and assume that $(K_i)_{i\in I}$ is a compact cover for J. Then we can construct two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ such that $(a_n)_{n\in\mathbb{N}}$ is non-increasing and $(b_n)_{n\in\mathbb{N}}$ is non-decreasing, $a_n \leq b_n$ for all $n \in \mathbb{N}$ and $J = \cup_{n\in\mathbb{N}}[a_n, b_n]$. For every $n \in \mathbb{N}$, define:

$$I_n = \{i \in I | K_i \cap [a_n, b_n] \neq \varnothing\}$$

By the previous discussion, I_n is finite for every $n \in \mathbb{N}$. Moreover $I = \cup_{n\in\mathbb{N}} I_n$.

Therefore I is countable.

□

Lemma 4.34. *If $(K_i)_{i\in I}$ is a compact cover for an interval J, and if for each $i \in I$, $(K_{i_j})_{j\in I_i}$ is a compact cover for K_i then $\{K_{i_j} | j \in I_i, i \in I\}$ is also a compact cover for J.*

Proof. First note that:

$$\begin{aligned} \cup_{i\in I, j\in I_j} K_{i_j} &= \cup_{i\in I}(\cup_{j\in I_j} K_{i_j}) \\ &= \cup_{i\in I} K_i \\ &= J \end{aligned}$$

The K_{i_j}'s are all compact subsets of J.

Let $x \in J$. Let $\mathcal{V}_{x,J}$ be a neighborhood of x in J that intersects finitely many K_i's. As I_i is finite for every $i \in I$ (lemma 4.33), then $\mathcal{V}_{x,J}$ intersects finitely many K_{i_j}'s. □

Definition 4.35. *Let $(U_i)_{i\in I}$ and $(V_j)_{j\in J}$ be two collections of sets. We say that $(V_j)_{j\in J}$ is a refinement of $(U_i)_{i\in I}$ if, for every $j \in J$, there exists $i \in I$ such that $V_j \subseteq U_i$.*

Lemma 4.36. *Given any cover of an interval J by open sets $(O_i)_{i\in I}$ there exists a compact cover $\mathcal{K}$ for J that is a refinement of $(O_i)_{i\in I}$.*

Proof. Let $(K_h)_{h\in H}$ be a compact cover of J. Let $h \in H$ and $x \in K_h$. As $x \in J$, then there exists $i \in I$ and $\alpha_x > 0$ such that $(x - \alpha_x, x + \alpha_x) \subseteq O_i$. Denote $I_x = [x - \alpha_x/2, x + \alpha_x/2] \cap K_h$. Then I_x is a compact subset of K_h that is contained in O_i. As K_h is compact and $((x - \alpha_x/2, x + \alpha_x/2))_{x\in K_h}$ covers K_h, then there exists a finite subset $P \subseteq K_h$ such that $((x - \alpha_x/2, x + \alpha_x/2))_{x\in P}$ covers K_h. And therefore $(I_x)_{x\in P}$ is a compact cover of K_h. We conclude using lemma 4.34. □

Lemma 4.37. *Let $(K_i)_{i\in I}$ be a compact cover for a compact interval $[a, b]$. Then there exists a (finite) subdivision $(a_j)_{0\leq j\leq n}$ of $[a, b]$ such that:*

$$\forall j \in [\![0, n-1]\!], \exists i \in I \textit{ such that: } [a_j, a_{j+1}] \subseteq K_i$$

Proof. It can be done by induction on the number of elements in I for example. $\square$

4.5 Locally Lipschitz maps

Definition 4.38. *Let* $\gamma \in (0,+\infty[$. *Let* E *and* F *be two normed vector spaces,* U *be a subset of* E *and* $f : U \to F$ *be a map. We say that* f *is locally Lipschitz-*γ *if, for every* $x \in U$*, there exists a neighborhood* $\mathcal{V}_{x,U}$ *of* x *in* U *such that* $f_{|\mathcal{V}_{x,U}}$ *is Lipschitz-*γ*.*

Example 4.39. *Let* $\gamma \in (0,+\infty[$. *Smooth maps of class* $\mathcal{C}^{\lfloor\gamma\rfloor+1}$ *(in particular linear and polynomial maps) and Lipschitz-*γ *maps are obviously locally Lipschitz-*γ*.*

Unlike the notion of Lipschitzness, local Lipschitzness can easily be defined recursively on open sets:

Lemma 4.40. *Let* $\gamma \in (1,+\infty[$. *Let* E *and* F *be two normed vector spaces,* U *be an open subset of* E *and* $f : U \to F$ *be a differentiable map. Then its derivative* $\mathrm{d}f$ *is locally Lipschitz-*$(\gamma-1)$ *if and only if* f *is locally Lipschitz-*γ*.*

Proof. Notice that, on every ball $B(x,\alpha) \subseteq U$ on which $\mathrm{d}f$ is locally Lipschitz-$(\gamma-1)$, one can use the fundamental theorem of calculus to bound f and we deduce that the restriction of f on this set is Lipschitz-γ using lemma 3.17. The converse is obvious. $\square$

Local Lipschitzness is naturally conserved under composition:

Lemma 4.41. *Let* $\gamma \in [1,+\infty[$. *Let* E*,* F *and* G *be normed vector spaces,* U *be a subset of* E *and* V *be a subset of* F*. Let* $f : U \to F$ *and* $g : V \to G$ *be two locally Lipschitz-*γ *maps such that* $f(U) \subseteq V$*. Then* $g \circ f$ *is locally Lipschitz-*γ*.*

Proof. Direct consequence of theorem 3.35. $\square$

Local Lipschitz paths conserve the smoothness, in the sense of variation, of paths:

Theorem 4.42. *Let $p, \gamma \in [1, +\infty[$. Let E and F be two normed vector spaces, U be a subset of E and J a compact interval. Let $f : U \to F$ be a locally Lip-γ map over U and $x : J \to U$ a path with finite p-variation. Then $f \circ x : J \to F$ is of finite p-variation.*

Proof. Let ω be a control of the p-variation of x over J. Let $t \in J$. Let L_t be an open neighborhood of x_t in U such that $f_{|L_t}$ is Lipschitz-γ; denote its Lip-1 norm by M_t. Let $\mathcal{V}_{t,J}$ an open neighborhood of t in J such that $x_u \in L_t$, for all $u \in \mathcal{V}_{t,J}$. Let $(K_j)_{j \in I}$ be a compact cover of J that is a refinement of $(\mathcal{V}_{t,J})_{t \in J}$ (lemma 4.36). For every $j \in I$, let $t_j \in J$ such that $K_j \subseteq \mathcal{V}_{t_j,J}$. Finally, let $(a_i)_{0 \leq i \leq n}$ be a subdivision of J such that for all $i \in [\![0, n]\!]$, there exists $j_i \in I$ such that $[a_i, a_{i+1}] \subseteq K_{j_i}$. Denote $M = \sup_{i \in [\![0,n-1]\!]} M_{t_{j_i}}$. Let $s, u \in J$. Let $q, r \in [\![0, n]\!]$ such that

$$a_q \leq s \leq a_{q+1} \leq \cdots \leq a_{r-1} \leq u \leq a_r$$

Then we have, from the fact that f is Lip-1 on each of the $L_{t_{j_i}}$ with a norm less than $M_{t_{j_i}}$, for $i \in [\![0, n-1]\!]$:

$$\begin{aligned} \|f(x_s) - f(x_u)\| &\leq \|f(x_s) - f(x_{a_{q+1}})\| + \sum_{k=q+1}^{r-2} \|f(x_{a_k}) - f(x_{a_{k+1}})\| + \\ &\quad \|f(x_{a_{r-1}}) - f(x_u)\| \\ &\leq M_{t_{j_q}} \|x_s - x_{a_{q+1}}\| + \sum_{k=q+1}^{r-2} M_{t_{j_k}} \|x_{a_k} - x_{a_{k+1}}\| + \\ &\quad M_{t_{j_{r-1}}} \|x_{a_{r-1}} - x_u\| \\ &\leq M(\omega(s, a_{q+1})^{1/p} + \sum_{k=q+1}^{r-2} \omega(a_k, a_{k+1})^{1/p} + \omega(a_{r-1}, u)^{1/p}) \end{aligned}$$

which, using the super-additivity of ω, gives the control:

$$\|f(x_s) - f(x_u)\|^p \leq M^p n^{p-1} \omega(s, u)$$

Therefore, $f \circ x$ is of finite p-variation (note that M and n depend only on f and x). $\square$

4.6 Local properties of rough paths and their integrals against locally Lipschitz one-forms

4.6.1 Local properties of rough paths

For the sake of the developments we want to expose in the remaining of this chapter, we take a final and a slightly different approach to rough paths that will highlight their local properties:

Definition 4.43. *Let E be a Banach space and $p \geq 1$. For an open subset U of E, a local geometric p-rough path in U is a triple (x, X, J) such that:*

- *J is an interval.*
- *x is a U-valued path over J.*
- *X is the limit in the p-variation metric of a sequence of truncated signatures of degree $[p]$ of E-valued paths of bounded variation over J (i.e. a geometric p-rough path in the sense of definition 1.45: $X \in G\Omega_p^J(E)/\sim$) which trace is x, i.e. for all $(s,t) \in \Delta_J$, $X_{s,t}^1 = x_t - x_s$.*

The set of local geometric p-rough paths in U will be denoted $G\Omega_p(U)$.

Remark 4.44. *The trace of a geometric p-rough path is of finite p-variation.*

Lemma 4.45. *Let E be a normed space, $p \in [1, +\infty[$ and J be an interval. Let X and Y be two multiplicative functionals on J. Let $(K_i)_{i \in I}$ be a compact cover for J such that, for all $i \in I$, $X_{|K_i}$ and $Y_{|K_i}$ are equal. Then X and Y are equal on J.*

Proof. Let $(a,b) \in \Delta_J$. Then $(K_i \cap [a,b])_{i\in S}$, where $S = \{i \in I | K_i \cap [a,b] \neq \varnothing\}$, is a compact cover for $[a,b]$. Let $(a_j)_{0\leq j\leq n}$ be a subdivision of $[a,b]$ such that for all $j \in [\![0, n-1]\!]$, there exists $i \in S$ such that $[a_j, a_{j+1}] \subseteq K_i$. Then, by assumption:

$$\forall j \in [\![0, n-1]\!] : \quad X_{a_j,a_{j+1}} = Y_{a_j,a_{j+1}}$$

Therefore:

$$X_{a_0,a_1} \otimes \cdots \otimes X_{a_{n-1},a_n} = Y_{a_0,a_1} \otimes \cdots \otimes Y_{a_{n-1},a_n}$$

which, by using the multiplicativity of X and Y, gives: $X_{a,b} = Y_{a,b}$. Since this holds for all $(a,b) \in \Delta_J$, then $X = Y$. □

The following is a basic inequality showing how to control the distance between two geometric rough paths on a compact interval based on their distances on a compact cover of said interval:

Lemma 4.46. *Let E be a Banach space, $p \in [1, +\infty[$. Let $T > 0$. Let $Y, Z \in G\Omega_p^{[0,T]}(E)/\sim$. Let ω be a control function that controls the p-variation of Y and Z and let $a \in \mathbb{R}$ such that:*

$$\forall (s,t) \in \Delta_{[0,T]}, \forall k \in [\![0, [p]]\!] : \quad \|Y^k_{s,t} - Z^k_{s,t}\| \leq a\frac{\omega(s,t)^{k/p}}{\beta_p(k/p)!}$$

Let $\mathcal{D} = (s_i)_{0\leq i\leq r}$ be a subdivision of $[0,T]$. Then:

$$\tilde{d}_p(Y,Z) \leq (r+1)^{1-1/p} \max_{1\leq j\leq [p]} \left((\frac{2a(r-1)}{\beta_p(j/p)!})^{p/j}\omega(0,T) + \sum_{i=0}^{r-1} \tilde{d}_p^{[s_i,s_{i+1}]}(Y,Z)^p \right)^{1/p}$$

Proof. Let $j \in [\![1, [p]]\!]$. Let $s, t, u \in [0,T]$ such that $s \leq t \leq u$. Since Y and Z are multiplicative functionals, we have the following:

$$\begin{aligned}
Y^j_{s,u} - Z^j_{s,u} &= \sum_{r=0}^{j} \left(Y^r_{s,t} \otimes Y^{j-r}_{t,u} - Z^r_{s,t} \otimes Z^{j-r}_{t,u}\right) \\
&= \sum_{r=0}^{j} \left((Y^r_{s,t} - Z^r_{s,t}) \otimes Y^{j-r}_{t,u} + Z^r_{s,t} \otimes (Y^{j-r}_{t,u} - Z^{j-r}_{t,u})\right) \\
&= (Y^j_{s,t} - Z^j_{s,t}) + (Y^j_{t,u} - Z^j_{t,u}) + \sum_{r=1}^{j-1} \left((Y^r_{s,t} - Z^r_{s,t}) \otimes Y^{j-r}_{t,u} + \right. \\
&\quad \left. Z^r_{s,t} \otimes (Y^{j-r}_{t,u} - Z^{j-r}_{t,u})\right)
\end{aligned}$$

For $(s,u) \in \Delta_{[0,T]}$, define $V_{s,u} = Y^j_{s,u} - Z^j_{s,u}$. For a subdivision $\mathcal{D} = (s_i)_{0 \le i \le q}$ of a compact sub-interval of $[0,T]$, define $V^{\mathcal{D}}_{s_0,s_r} = \sum_0^{q-1} V_{s_i,s_{i+1}}$.

Let $(s,u) \in \Delta_{[0,T]}$ and let $\mathcal{D} = (s_i)_{0 \le i \le q}$ be a subdivision of $[s,u]$. Define $\mathcal{D}' = (s_0, s_2, \ldots, s_q)$. Then the previous identity along with the neo-classical inequality (lemma 1.36) shows that:

$$\begin{aligned}
\|V^{\mathcal{D}}_{s,u} - V^{\mathcal{D}'}_{s,u}\| &\le \tfrac{2a}{\beta_p (j/p)!} \omega(s_0, s_2)^{j/p} \\
&\le \tfrac{2a}{\beta_p (j/p)!} \omega(s,u)^{j/p}
\end{aligned}$$

By repeating the process finitely many times, we get the control:

$$\|V^{\mathcal{D}}_{s,u} - V_{s,u}\| \le \frac{2a(q-1)}{\beta_p (j/p)!} \omega(s,u)^{j/p}$$

Now, take $\Delta = (t_i)_{0 \le i \le q}$ to be a subdivision of $[0,T]$. Let $i \in [\![0, q-1]\!]$. Define the subdivision $\mathcal{D} \cap [t_i, t_{i+1}]$ of $[t_i, t_{i+1}]$ to be $(m_l) := (t_i, s_{\tilde{r}}, \ldots, s_{\tilde{r}+n}, t_{i+1})$, where $\tilde{r}$ and n are such that $s_{\tilde{r}-1} < t_i \leq s_{\tilde{r}}$ and $s_{\tilde{r}+n} \leq t_{i+1} < s_{\tilde{r}+n+1}$. Then we have:

$$\begin{aligned}
\|V_{t_i,t_{i+1}}\|^{p/j} &\leq \left(\|V_{t_i,t_{i+1}}^{\mathcal{D}\cap[t_i,t_{i+1}]} - V_{t_i,t_{i+1}}\| + \|V_{t_i,t_{i+1}}^{\mathcal{D}\cap[t_i,t_{i+1}]}\|\right)^{p/j} \\
&\leq \left(\|V_{t_i,t_{i+1}}^{\mathcal{D}\cap[t_i,t_{i+1}]} - V_{t_i,t_{i+1}}\| + \sum_{\mathcal{D}\cap[t_i,t_{i+1}]} \|Y^j_{m_l,m_{l+1}} - Z^j_{m_l,m_{l+1}}\|\right)^{p/j} \\
&\leq (r+1)^{p/j-1}\Bigg(\|V_{t_i,t_{i+1}}^{\mathcal{D}\cap[t_i,t_{i+1}]} - V_{t_i,t_{i+1}}\|^{p/j} + \\
&\qquad \sum_{\mathcal{D}\cap[t_i,t_{i+1}]} \|Y^j_{m_l,m_{l+1}} - Z^j_{m_l,m_{l+1}}\|^{p/j}\Bigg) \\
&\leq (r+1)^{p/j-1}\Bigg(\left(\tfrac{2a(r-1)}{\beta_p(j/p)!}\right)^{p/j}\omega(t_i,t_{i+1}) + \\
&\qquad \sum_{\mathcal{D}\cap[t_i,t_{i+1}]} \|Y^j_{m_l,m_{l+1}} - Z^j_{m_l,m_{l+1}}\|^{p/j}\Bigg)
\end{aligned}$$

Finally:

$$\sum_{\Delta} \|V_{t_i,t_{i+1}}\|^{p/j} \leq (r+1)^{p/j-1}\left(\left(\frac{2a(r-1)}{\beta_p(j/p)!}\right)^{p/j}\omega(0,T) + \sum_{i=0}^{r-1} \tilde{d}_p^{[s_i,s_{i+1}]}(Y,Z)^p\right)$$

and therefore:

$$\tilde{d}_p(Y,Z) \leq (r+1)^{1-1/p} \max_{1\leq j\leq [p]} \left(\left(\frac{2a(r-1)}{\beta_p(j/p)!}\right)^{p/j}\omega(0,T) + \sum_{i=0}^{r-1} \tilde{d}_p^{[s_i,s_{i+1}]}(Y,Z)^p\right)^{1/p}$$

□

Lemma 4.47. *Let E be a Banach space, $p \in [1,+\infty[$ and J be a compact interval. Let $X \in G\Omega_p^{[0,T]}(E)/\sim$ and let $X(n)_{n\in\mathbb{N}}$ be a sequence of elements of $G\Omega_p^{[0,T]}(E)/\sim$. Suppose there exists a compact cover $(K_i)_{i\in I}$ for J such that, for all $i \in I$, $(X(n)_{|K_i})_{n\in\mathbb{N}}$ converges to $X_{|K_i}$ for $\tilde{d}_p^{K_i}$. Then $(X(n))_{n\in\mathbb{N}}$ converges to X for $\tilde{d}_p$.*

Proof. Let $\mathcal{D} = (s_i)_{0\leq i\leq r}$ be a subdivision of $[0,T]$ such that for each $i \in [\![0, r-1]\!]$, there exists $j \in I$ such that $[s_i, s_{i+1}] \subseteq K_j$. Let ω be a control function over $[0,T]$ and $(a_n)_{n\in\mathbb{N}}$ a sequence of non-negative numbers converging to 0 such that:

- X and $X(n)$, for every $n \in \mathbb{N}$, are controlled in p-variation by ω;

- $\forall i \in [\![1, r-1]\!], \forall n \in \mathbb{N}, \forall (s,t) \in \Delta_{[s_i, s_{i+1}]}, \forall j \in [\![1, [p]]\!]$:

$$\|X(n)^j_{s,t} - X^j_{s,t}\| \leq a(n) \frac{\omega(s,t)^{j/p}}{\beta_p (j/p)!}$$

Note that such a control and sequence exist by proposition 1.43 and the fact that the convergences occur on finitely many intervals. Then by the previous lemma:

$$\tilde{d}_p(X, X(n)) \leq (r+1)^{1-1/p} \max_{1 \leq j \leq [p]} \left(\sum_{i=0}^{r-1} \tilde{d}_p^{[s_i, s_{i+1}]}(X, X(n))^p + (\frac{2a(n)(r-1)}{\beta_p (j/p)!})^{p/j} \omega(0,T) \right)^{1/p}$$

which trivially converges to 0. □

4.6.2 Local properties of integrals of rough paths

As studying rough paths locally (on compact covers) is enough to characterize the whole path, we can see how to define the integral of a locally Lipschitz one-form α along a geometric rough path (x, X): we simply construct the integral locally on regions where α is Lipschitz and ensure that the integral is the same on overlapping regions; which we can do easily by noting that this is indeed the fact for signatures of paths of bounded variation then taking the limit in the appropriate variation metric.

Next, we show that the integral of a locally Lipschitz one-form along geometric rough paths is, as expected, continuous when varying the path:

Theorem 4.48. *Let $p, \gamma \in [1, +\infty[$ such that $\gamma > p$. Let E and F be two Banach spaces and U be an open subset of E. Let $\alpha : U \to \mathcal{L}(E, F)$ be a locally Lip-$(\gamma - 1)$ one-form.*

Then, the map:

$$\begin{array}{rccl} I_\alpha: & (G\Omega_p^{[0,T]}(U), d_p) & \longrightarrow & (G\Omega_p^{[0,T]}(F)/\sim, \tilde{d}_p) \\ & (x, X) & \mapsto & \int \alpha(x(0), X)dX \end{array}$$

is continuous.

Proof. Let $(x, X) \in G\Omega_p^{[0,T]}(U)$ and let $(x(n), X(n))_{n\in\mathbb{N}}$ be a sequence of elements of $G\Omega_p^{[0,T]}(U)$ converging to (x, X) in d_p.

Let $t \in [0, T]$. Let $r_t > 0$ such that $B(x_t, r_t) \subseteq U$ and α is Lip-$(\gamma - 1)$ on $B(x_t, r_t)$. Let $\eta_t > 0$ such that, for all $s \in [0, T]$, if $|s - t| < \eta_t$ then $x_s \in B(x_t, r_t/3)$. Finally, denote by O_t the (relatively) open interval $(t - \eta_t, t + \eta_t) \cap [0, T]$. Then $(O_t)_{t\in[0,T]}$ defines a subdivision $\sigma = (s_i)_{0\leq i\leq q}$ on $[0, T]$ such that, for all $i \in [\![0, q-1]\!]$, there exists $t_i \in [0, T]$ such that $[s_i, s_{i+1}] \subseteq O_{t_i}$ (combination of the lemmas 4.36 and 4.37).

Let now $i \in [\![0, q-1]\!]$ and $\varepsilon > 0$. The map:

$$\begin{array}{rccl} I_\alpha^i: & (G\Omega_p^{[s_i,s_{i+1}]}(B(x_{t_i}, r_{t_i})), d_p^i) & \longrightarrow & (G\Omega_p^{[s_i,s_{i+1}]}(F)/\sim, \tilde{d}_p^i) \\ & (y, Y) & \mapsto & \int \alpha(y(s_i), Y)\mathrm{d}Y \end{array}$$

is continuous. Let $\gamma_i > 0$ such that, for $(y, Y) \in G\Omega_p^{[s_i,s_{i+1}]}(B(x_{t_i}, r_{t_i}))$:

$$\|y(s_i) - x(s_i)\| \vee d_p^i(Y, X_{|[s_i,s_{i+1}]}) \leq \gamma_i \Rightarrow \tilde{d}_p^i(I_\alpha^i(y(s_i), Y), I_\alpha^i(x(s_i), X_{|[s_i,s_{i+1}]})) \leq \varepsilon$$

Let $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$:

$$n \geq N \Rightarrow \|x(n)(0) - x(0)\| \vee d_p(X(n), X) \leq \gamma_i/2 \wedge (r_{t_i}/3)$$

Let $n \in \mathbb{N}$ such that $n \geq N$, then we have:

$$\begin{aligned} \|x(n)(s_i) - x(s_i)\| &\leq \|x(n)(0) - x(0)\| + \|X(n)^1_{0,s_i} - X^1_{0,s_i}\| \\ &\leq \|x(n)(0) - x(0)\| + d_p(X(n), X) \\ &\leq \gamma_i \end{aligned}$$

Furthermore, for all $u \in [s_i, s_{i+1}]$:

$$\begin{aligned} \|x(n)(u) - x(t_i)\| &\leq \|x(n)(0) - x(0)\| + \|X(n)^1_{0,u} - X^1_{0,u}\| + \|x(u) - x(t_i)\| \\ &< \|x(n)(0) - x(0)\| + d_p(X(n), X) + r_{t_i}/3 \\ &< r_{t_i} \end{aligned}$$

Meaning, in particular, that $x(n)([s_i, s_{i+1}]) \subseteq B(x_{t_i}, r_{t_i})$. Therefore:

$$\tilde{d}^i_p \left(\int \alpha(x(n)(s_i), X(n)) \mathrm{d}X(n), \int \alpha(x(s_i), X) \mathrm{d}X \right) \leq \varepsilon$$

which implies that:

$$\int \alpha(x(n)(s_i), X(n)) \mathrm{d}X(n)_{|[s_i, s_{i+1}]} \xrightarrow[n \to \infty]{\tilde{d}^i_p} \int \alpha(x(s_i), X) \mathrm{d}X_{|[s_i, s_{i+1}]}$$

We conclude using lemma 4.47. □

Lemma 4.49. *Let $p, \gamma \in [1, +\infty[$ such that $\gamma > p$. Let E and F be two Banach spaces and U be an open subset of E and J an interval. Let $f : U \to F$ be a locally Lip-γ map over U and $x : J \to U$ a path with bounded variation. Then:*

$$(f(x), \int df(x, S_{[p]}(x)) dS_{[p]}(x), J) = (f(x), S_{[p]}(f(x)), J)$$

Proof. First notice that $S_{[p]}(f(x))$ is well-defined since $f(x)$ has bounded variation by theorem 4.42. Let $s, t \in J$ such that $s \leq t$:

1. As $\mathrm{d}f$ is continuous and x is of bounded variation, then $\int \mathrm{d}f(x, S_{[p]}(x))\mathrm{d}S_{[p]}(x)$ has finite 1-variation and is equal to (the signature of) the Stieltjes integral $\int \mathrm{d}f(x)\mathrm{d}x$. Therefore:
$$\left(\int \mathrm{d}f(x, S_{[p]}(x))\mathrm{d}S_{[p]}(x)\right)^1_{u,v} = f(x_v) - f(x_u)$$
for all $(u, v) \in \Delta_{[s,t]}$.

2. $S_{[p]}(f(x))$ has finite 1-variation and $S_{[p]}(f(x))^1_{u,v} = f(x_v) - f(x_u)$ for all $(u, v) \in \Delta_{[s,t]}$.

Two multiplicative functionals that have finite 1-variation and which terms of the 1^{st} degree agree are equal by lemma 1.37. Therefore, the sought identity stands. □

4.7 Rough paths and categories

4.7.1 Categories and functors

To highlight the minimalist framework on which we can define the notion of rough paths, we will be using the language of categories.

Definition 4.50 (Category). *Let* $(U_i)_{i\in I}$ *be a family of sets. For every* $i, j \in I$*, let* $\hom(U_i, U_j)$ *be a set of mappings from* U_i *to* U_j*. Assume that there exists a family of binary operations* $(\psi_{i,j,k})_{i,j,k\in I}$ *satisfying the following properties:*

1. *For every* $i, j, k \in I$*,* $\psi_{i,j,k}$ *is a mapping from* $\hom(U_i, U_j) \times \hom(U_j, U_k)$ *into* $\hom(U_i, U_k)$*.*

2. ***(Associativity)*** *For every* $i, j, k, l \in I$*,* $f \in \hom(U_i, U_j)$*,* $g \in \hom(U_j, U_k)$ *and* $h \in \hom(U_k, U_l)$*:*
$$\psi_{i,k,l}(\psi_{i,j,k}(f, g), h) = \psi_{i,j,l}(f, \psi_{j,k,l}(g, h))$$

3. ***(Existence of identities)*** *For every* $i \in I$*, there exists* $id_{U_i} \in \mathrm{hom}(U_i, U_i)$ *such that, for all* $j \in I$*,* $g \in \mathrm{hom}(U_i, U_j)$ *and* $h \in \mathrm{hom}(U_j, U_i)$*, we have:*

$$\psi_{i,i,j}(id_{U_i}, g) = g \quad \textit{and} \quad \psi_{j,i,i}(h, id_{U_i}) = h$$

The triple:

$$\mathcal{C} = ((U_i)_{i\in I}, (\mathrm{hom}(U_i, U_j))_{i,j\in I}, (\psi_{i,j,k})_{i,j,k\in I})$$

is then called a category. For all $i, j, k \in I$*,* U_i *is called an object of* $\mathcal{C}$*, an element in* $\mathrm{hom}(U_i, U_j)$ *(also denoted by* $\mathrm{hom}_{\mathcal{C}}(U_i, U_j)$*) is called either an arrow, a morphism or a homomorphism in* $\mathcal{C}$*. The collection* $(\mathrm{hom}_{\mathcal{C}}(U_i, U_j))_{i,j\in I}$ *is simply denoted* $\mathrm{hom}(\mathcal{C})$*. The binary operation* $\psi_{i,j,k}$ *is called a composition of morphisms and is simply denoted by* $\circ$*. For* $i, j, k \in I$*,* $f \in \mathrm{hom}(U_i, U_j)$ *and* $g \in \mathrm{hom}(U_j, U_k)$*,* $\psi_{i,j,k}(f, g)$ *is denoted* $g \circ f$*.*

Remark 4.51. *If* $\mathcal{C} = ((U_i)_{i\in I}, (\mathrm{hom}_{\mathcal{C}}(U_i, U_j))_{i,j\in I}, \circ)$ *is a category, and with the notations of the previous definition, the associativity and the existence of identities axioms can be rewritten in the folloowing way:*

$$h \circ (g \circ f) = (h \circ g) \circ f$$

and

$$g \circ id_{U_i} = g \quad \textit{and} \quad id_{U_i} \circ h = h$$

Example 4.52. *By taking a family of sets considered as objects and all maps between these sets considered as arrows and the composition of maps as binary operations we obtain a category, usually called the category of sets.*

Example 4.53. *Let* $(G_i)_{i\in I}$ *be a family of groups. For every* $i, j \in I$*, let* $\mathrm{hom}(G_i, G_j)$ *be the set of all group homomorphisms from* G_i *to* G_j*. Then* $\mathcal{C} = ((G_i)_{i\in I}, (\mathrm{hom}(G_i, G_j))_{i,j\in I}, \circ)$

is a category (called category of groups).

Example 4.54. *Let $(M_i)_{i\in I}$ be a family of topological spaces (respectively smooth manifolds). For every $i,j \in I$, let $\hom(M_i,M_j)$ be the set of all continuous maps (resp. smooth maps) from M_i to M_j. Then $\mathcal{C} = ((M_i)_{i\in I}, (\hom(M_i,M_j))_{i,j\in I}, \circ)$ is a category.*

Definition 4.55. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two categories. Let F be a mapping such that:*

1. *For every object U in $\mathcal{C}_1$, $F(U)$ is an object in $\mathcal{C}_2$;*
2. *For every two objects U and V in $\mathcal{C}_1$ and an arrow $f \in \hom_{\mathcal{C}_1}(U,V)$, $F(f)$ is in $\hom_{\mathcal{C}_2}(F(U),F(V))$;*
3. *For all objects U, V and W in $\mathcal{C}_1$, and arrows $f \in \hom_{\mathcal{C}_1}(U,V)$ and $g \in \hom_{\mathcal{C}_1}(V,W)$ $F(g\circ f) = F(g)\circ F(f)$ and $F(id_U) = id_{F(U)}$.*

Then F is called a functor (or a functorial rule) from $\mathcal{C}_1$ to $\mathcal{C}_2$.

Example 4.56. *Let $\mathcal{C}_1 = ((G_i)_{i\in I}, (\hom(G_i,G_j))_{i,j\in I}, \circ)$ be a category of groups. For every $i,j \in I$, let $\hom_2(G_i,G_j)$ be the collection of all maps from G_i to G_j. Let F be the identity rule associating, for every $i,j\in I$, G_i to itself and every element of $\hom(G_i,G_j)$ to itself. Then $\mathcal{C}_2 = ((G_i)_{i\in I}, (\hom_2(G_i,G_j))_{i,j\in I}, \circ)$ is a category of sets and F defines a functorial rule from $\mathcal{C}_1$ to $\mathcal{C}_2$.*

Example 4.57. *Let $\mathcal{C}_1 = ((G_i)_{i\in I}, (\hom(G_i,G_j))_{i,j\in I}, \circ)$ be a category such that, for every $i,j\in I$, G_i is a Lie group and $\hom(G_i,G_j)$ is a collection of Lie group homomorphisms from G_i to G_j. For every $i,j\in I$, Denote by Lie(G_i) the Lie algebra of G_i and let $\hom_2(G_i,G_j)$ be the collection of all linear maps from Lie(G_i) to Lie(G_j). Then $\mathcal{C}_2 = ((G_i)_{i\in I}, (\hom_2(G_i,G_j))_{i,j\in I}, \circ)$ is a category. Let F be the rule mapping, for every $i,j\in I$, G_i to Lie(G_i) and every element of $\varphi \in \hom(G_i,G_j)$ to φ_*. Then F defines a functorial rule from $\mathcal{C}_1$ to $\mathcal{C}_2$.*

4.7.2 A functorial rule

Let E be a Banach space and $p, \gamma \in [1, +\infty[$ such that $\gamma > p$. Let $\mathcal{C}$ be a category whose objects are open subsets of E. The arrows of $\mathcal{C}$ between two objects U and V are the locally Lipschitz-γ maps between U and V whose set will be denoted $\hom_{\mathcal{C}}(U, V)$.

For U and V objects of $\mathcal{C}$ and $f \in \hom_{\mathcal{C}}(U, V)$, we denote by f_* the following map:

$$\begin{array}{rccc} f_* : & G\Omega_p(U) & \longrightarrow & G\Omega_p(V) \\ & (x, X, J) & \longmapsto & (f(x), \int \mathrm{d}f(x, X)\mathrm{d}X, J) \end{array}$$

Theorem 4.58. *The rule that assigns to every object U in $\mathcal{C}$ the object $G\Omega_p(U)$ and to every morphism $f \in \hom_{\mathcal{C}}(U, V)$, where U and V are objects in $\mathcal{C}$, the map f_*, is functorial.*

Proof. For every open subsets U, V and W of E that are objects in $\mathcal{C}$, we need to prove the following:

1. $(Id_U)* = Id_{G\Omega_p(U)}$;

2. $\forall f \in \hom_{\mathcal{C}}(U, V), \forall g \in \hom_{\mathcal{C}}(V, W) : (g \circ f)_* = g_* \circ f_*$.

Let U, V and W be open subsets of E that are objects in $\mathcal{C}$. Let $f \in \hom_{\mathcal{C}}(U, V)$ and $g \in \hom_{\mathcal{C}}(V, W)$. Let x be a U-valued path over J with bounded variation. By lemma 4.49

$$\begin{aligned} f_*(x, S_{[p]}(x), J) &= (f(x), \textstyle\int \mathrm{d}f(x, S_{[p]}(x))\mathrm{d}S_{[p]}(x), J) \\ &= (f(x), S_{[p]}(f(x)), J) \end{aligned}$$

as $f(x)$ has bounded variation (lemma 4.42), we similarly have:

$$g_*(f(x), S_{[p]}(f(x)), J) = (g \circ f(x), S_{[p]}(g \circ f(x)), J)$$

and as $g \circ f$ is locally Lip-γ:

$$(g \circ f(x), S_{[p]}(g \circ f(x)), J) = (g \circ f(x), \int \mathrm{d}(g \circ f)(x, S_{[p]}(x)) \mathrm{d}S_{[p]}(x), J)$$

Therefore

$$((g \circ f)_*)_{|G\Omega_1(U)} = (g_* \circ f_*)_{|G\Omega_1(U)}$$

As both $(g \circ f)_*$ and $g_* \circ f_*$ are continuous in the p-variation metric (theorem 4.48) and as $G\Omega_1(U)$ is dense in $G\Omega_p(U)$ for this metric then we deduce that $(g \circ f)_* = g_* \circ f_*$. The first assertion regarding the identity map can be proved using a similar argument. $\square$

4.7.3 A new definition of rough paths on manifolds

Based on our findings so far, we are now to able to give a minimal approach for defining rough paths on a manifold.

Definition 4.59. *Let $\gamma \geq 1$. Let M be a topological manifold. We say that M is a locally $Lip-\gamma$ n-manifold if it has an atlas such that any two charts in that atlas (U, ϕ) and (V, ψ) such that $U \cap V \neq \varnothing$, the map $\psi \circ \phi^{-1} : \phi(U \cap V) \to \mathbb{R}^n$ is locally $Lip - \gamma$.*

Example 4.60. *A Lip-γ manifold is a locally Lip-γ manifold.*

Example 4.61. *Any topological space homeomorphic to an open subset of the Euclidean space is a locally Lip-γ manifold. In particular, finite-dimensional vector spaces are locally Lip-γ manifolds.*

Definition 4.62. *Let $n \in \mathbb{N}^*$ and $p, \gamma \in [1, +\infty[$ such that $\gamma > p$. Let M be a locally $Lip - \gamma$ n-manifold. A local p-rough path on M over an interval J is a collection $(x_i, X_i, J_i, (\phi_i, U_i))_{i \in I}$ of p-rough paths on $\mathbb{R}^n$ satisfying the following conditions:*

- *$(J_i)_{i \in I}$ is a compact cover for J;*

- *For every $i \in I$, (ϕ_i, U_i) is a locally Lip $-\gamma$ chart on M.*

- $\forall i \in I : \quad (x_i, X_i, J_i) \in G\Omega_p(\phi_i(U_i))$;

- ***(Consistency condition)*** *If $i, k \in I$ such that $J_i \cap J_k \neq \varnothing$, then we have:*

$$(\phi_k \circ \phi_i^{-1})_*(x_i, X_i, J_i \cap J_k) = (x_k, X_k, J_i \cap J_k)$$

We identify similar local rough paths in the following way:

Definition 4.63. *Let $p, \gamma \in [1, +\infty[$ such that $\gamma > p$. Let M be a locally Lip $-\gamma$ manifold and J an interval. Two local p-rough paths on M over J, denoted by $A = (x_i, X_i, J_i, (\phi_i, U_i))_{i \in I}$ and $B = (x_i, X_i, J_i, (\phi_i, U_i))_{i \in J}$ are said to be equivalent if $A \cup B$ is also a local p-rough path.*

This, of course, defines an equivalence relation. We will be henceforth only considering the equivalence classes associated to this relation.

Definition 4.64. *Let $p, \gamma \in [1, +\infty[$ such that $\gamma > p$. Let M be a locally Lip $-\gamma$ manifold. A local rough path $(x_i, X_i, J_i, (\phi_i, U_i))_{i \in I}$ on an interval J is said to be a rough path extension for the path $x : J \to M$ if the following holds:*

- $\forall i \in I : x(J_i) \subseteq U_i$;

- $\forall i \in I : x_i = \phi_i \circ x_{|J_i}$.

If such a rough path exists, we say then that x admits a rough path extension.

In the same way as in definition 4.63, we will consider that two rough path extensions of a given path to be the same if their unions is also a rough path extension of that path.

4.7.4 Identification

Theorem 4.65. *Let M be a Lipschitz-γ manifold and J a compact interval. There is a one-to-one mapping between rough paths on M and equivalence classes of local rough paths on M.*

Proof.

- A rough path on a manifold has a localising sequence by theorem 4.25. The pushforwards of the elements of this localising sequence under the associated chart maps define a local rough path: every rough path in this sequence has an end point consistent with the starting point of the following rough path, which is equivalent in our case to the consistency condition in definition 4.62, where the intersections of successive intervals are reduced to single points.

- Conversely, the compact cover for a compact interval J defines a subdivision $(a_i)_{1\leq i\leq n}$. We can construct then a representative of a given equivalence class of local rough paths that is of the form $(x_i, X_i, [a_i, a_{i+1}], (\phi_i, U_i))_{1\leq i\leq n-1}$. For each $i \in [\![1, n-1]\!]$, define $Z_i = (\phi_i^{-1})_*(x_i, X_i)$. Now let $Z = Z_1 * \cdots Z_n$. Then Z is a rough path on M.

□

4.8 Coloured paths on manifolds

4.8.1 Definitions and examples

The procedure detailed in the previous sections is more general and can be extended to define any notion of *colouring* already existing on the Euclidean space to a manifold, assuming that we can find a suitable functorial rule. Indeed, the roughness of a path can be seen as a colour: an extra bit of information that cannot necessarily be learned by looking at the base path only.

Let $n \in \mathbb{N}^*$. Let $\mathcal{C}$ be a category whose objects are the open subsets of $\mathbb{R}^n$. We can define a notion of *coloured* charts and atlas over any n-topological manifold by assuming that the transition maps are arrows in the category $\mathcal{C}$. We will call such a manifold a *coloured manifold.*

Definition 4.66. *Let M be a topological manifold. We say that M is a coloured manifold if it has an atlas such that for any two charts in that atlas (U, ϕ) and (V, ψ) such that $U \cap V \neq \varnothing$, we have $\psi \circ \phi^{-1} \in \hom_C(\phi(U \cap V), \psi(U \cap V))$.*

Suppose now that we have a notion of *coloured paths* on $\mathbb{R}^n$ that have base paths underlying them which we will be calling traces (rough paths are an example). Denote by $T(U)$ the sets of couloured paths whose traces lie in an open subset U of $\mathbb{R}^n$. For each arrow f between the objects U and V of $\mathcal{C}$, assume there exists a map f_* between $T(U)$ and $T(V)$ such that the trace of the image of each coloured path on U is the image by f of its trace. Such a map will be called a *colouring map.* We can now define coloured paths on a coloured manifold in the same way as in definition 4.62 and the existence of coloured path extensions for manifold-based paths as in definition 4.64.

Definition 4.67. *Let $n \in \mathbb{N}^*$. Let M be a coloured n-manifold. A coloured path on M over an interval J is a collection $(X_i, J_i, (\phi_i, U_i))_{i \in I}$ of coloured paths on $\mathbb{R}^n$ satisfying the following conditions:*

- *$(J_i)_{i \in I}$ is a compact cover for J;*
- *For every $i \in I$, (ϕ_i, U_i) is in the coloured atlas of M.*
- *$\forall i \in I : \quad X_i \in T(\phi_i(U_i))$ and X_i is defined over J_i;*
- ***(Consistency condition)** If $i, k \in I$ such that $J_i \cap J_k \neq \varnothing$, then we have:*

$$(\phi_k \circ \phi_i^{-1})_*(X_{i|J_i \cap J_k}) = X_{k|J_i \cap J_k}$$

Definition 4.68. *Let $n \in \mathbb{N}^*$. Let M be a coloured n-manifold. A coloured path $(X_i, J_i, (\phi_i, U_i))_{i \in I}$ on an interval J is said to be a coloured path extension for the path $x : J \to M$ if the following holds:*

- $\forall i \in I : x(J_i) \subseteq U_i$;
- $\forall i \in I : \mathit{trace}(X_i) = \phi_i \circ x_{|J_i}$.

If such a coloured path exists, we say then that x admits a coloured path extension.

Finally, for these definitions to make sense on their own and be consistent with the definitions of coloured paths on the Euclidean space (seen now as a coloured manifold), the rule that assigns $T(U)$ to U and f_* to f, for U object in $\mathcal{C}$ and f arrow in $\mathcal{C}$, must be functorial.

Example 4.69. *In the study of rough paths made in the previous sections, rough paths can be seen as coloured paths, locally Lipschitz maps as arrows and locally Lipschitz manifolds as coloured manifolds. The maps that assign to each rough path its image by a locally Lipschitz map are colouring maps (as defined in subsection 4.7.2).*

Example 4.70. *On a more basic and trivial level, continuous paths can also be seen as coloured paths. Let M be a topological manifold. If we take continuous maps as arrows, then a continuous atlas will be a covering of M with charts such that the transition maps are continuous. As it happens, a topological manifold comes naturally with a continous atlas! The colouring map associated to an arrow f is a map that assigns to every continuous path x the path $f(x)$. Then our new definition of a continuous map on M can be identified with the classical one which relies only on the topology on M. Conversely, every continuous path on M in the classical sense can be seen as the concatenation of pushforwards of continuous paths on the Euclidean space.*

Example 4.71. *Building on the previous example, we now aim to give an analogous definition of smooth maps on a manifold. Classically defined, smooth atlases and smooth manifolds correspond exactly to coloured atlases and coloured manifolds, when the arrows are taken to be smooth maps. The associated functorial rule is the same as earlier by replacing continuity with smoothness. Finally, one can see the definitions of smooth maps in the classical sense and when using coloured paths are equivalent.*

4.8.2 Discussion

Obviously, one does not need the notion of local Lipschitzness to define rough paths on manifolds. Indeed, this is one of the achievements of [3] and we will be using Lipschitz manifolds ourselves in the remainder of this work. However, we preferred to use local Lipschitz maps here for two main reasons:

- Local Lipschitz maps constitute a more general class of maps than Lipschitz maps and allow one to directly use, for example, linear and polynomial maps without the need of detours via Whitney-type theorems. Moreover, this class of maps retains many of the advantages of working with the more classical $\mathcal{C}^\gamma$ maps without requiring as much. For example, to define a rough path-integration map (in the sense of subsection 4.7.2) that deals with all paths that have finite p-variation for any $p \in [1, 2[$ (realisations of the Brownian Motion are a good example), one needs a (locally) Lipschitz-2 map. $\mathcal{C}^2$ maps satisfy this condition (but no map in $\mathcal{C}^\gamma - \mathcal{C}^2$ does, for $\gamma \in [1, 2[$) but require more (a second derivative and the continuity of said derivative). In summary, local Lipschitz maps combine most advantages of both smooth and Lipschitz maps.

- This class of maps makes it easier to construct rough paths on manifolds than Lipschitz maps without the need of secondary constructions. For example, the

> Euclidean space $\mathbb{R}^d$ endowed with the single chart $(\mathbb{R}^d, \mathrm{Id})$ defines a locally Lipschitz manifold but not a Lipschitz manifold. To achieve the latter, one needs to “break” the Euclidean space into precompact balls and construct suitable chart maps as explained in the example 4.2.

Finally, we opted for a presentation that relies on simple notions from category theory to show how one can simply translate similar notions as that of rough paths without any further particular considerations of the class of manifolds one is working on or without the need to exactly mimick the contructions on the Euclidean space (for example, one does not need the manifold to be Riemannian): all that is needed is a suitable functorial rule. We highlighted this by giving the abstract definition of coloured manifolds and coloured paths on them.

Chapter 5

The geometric structure of the reachability set

The aim of this chapter is to endow the reachability set with a Lipschitz manifold structure so that we make sense of rough paths and rough differential equations on it. We do this by studying the geometric structure of points reached by solving rough differential equations driven by axis paths (called orbits) in the same fashion as in [26] and adapting our study to the Lipschitz geometry. We identify a quantitative condition under which such a construction is possible and any points reached by solving an R.D.E. driven by a geometric rough path can also be reached by driving it by an axis path. By the same occasion, we develop a notion of Lipschitz submanifolds and show some consequences of this definition when working with rough paths.

5.1 Orbits of vector fields and distributions

In this section, we endow the reachability set in the Lipschitz case with an appropriate topology and identify a suitable tangent bundle as suggested by Sussmann on [26] for the smooth case.

5.1.1 Flows of vector fields and orbits

Lemma 5.1. *Let E be a normed vector space and $\mathcal{D}$ be a family of Lipschitz vector fields on E. Let $y, \tilde{y} \in E$. There exists a piece-wise integral curve of $\mathcal{D}$ joining y and $\tilde{y}$ if and only if $\mathbb{L}_y(\mathcal{D}) = \mathbb{L}_{\tilde{y}}(\mathcal{D})$.*

Let E be a normed vector space and $\mathcal{D}$ be a family of Lipschitz vector fields on E. For $\xi = (f^1, \dots, f^n) \in \mathcal{D}^n$ $(n \in \mathbb{N}^*)$, we denote by $\tilde{\xi}$ the map:

$$\begin{array}{ccc} \mathbb{R}^n \times E & \longrightarrow & E \\ (t_1, \dots, t_n, x) & \longmapsto & \tilde{f}^n_{t_n} \circ \dots \circ \tilde{f}^1_{t_1}(x) \end{array}$$

For $y \in E$, $\tilde{\xi}_y$ is the map $(t_1, \dots, t_n) \mapsto \tilde{\xi}(t_1, \dots, t_n, y)$ and for $\vec{s} = (s_1, \dots, s_n)$ we denote by $\tilde{\xi}_{\vec{s}}$ the map $x \mapsto \tilde{\xi}(s_1, \dots, s_n, x)$. More generally, for $p \in \mathbb{N}^*$, $n_1, \dots, n_p \in \mathbb{N}^*$, $(\xi^1, \dots, \xi^p) \in \mathcal{D}^{n_1} \times \dots \times \mathcal{D}^{n_p}$ and , if η is defined to be $(\xi^1, \dots, \xi^p)$, then $\tilde{\eta}$ is the map given by the identity:

$$\forall (\vec{t_1}, \dots, \vec{t_p}) \in \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_p}, \forall x \in E : \quad \tilde{\eta}(\vec{t_1}, \dots, \vec{t_p}, x) = \tilde{\xi}^p_{\vec{t_p}} \circ \dots \circ \tilde{\xi}^1_{\vec{t_1}}(x)$$

For $y \in E$, $\tilde{\eta}_y$ is the map $\tilde{\eta}(., y)$ and for $(\vec{s_1}, \dots, \vec{s_p})$, $\tilde{\eta}_{\vec{s}}$ is the map $\tilde{\xi}^p_{\vec{s_p}} \circ \dots \circ \tilde{\xi}^1_{\vec{s_1}}$. We will denote $\hat{\xi} = (f^n, \dots, f^1)$; and for $\vec{t} = (t_1, \dots, t_n)$, similarly $\hat{\vec{t}} = (t_n, \dots, t_1)$.

As suggested in [26], we endow $\mathbb{L}_y(\mathcal{D})$ with the largest topology for which all the maps in the set:

$$\{\tilde{\xi}_y | \exists n \in \mathbb{N}^*, \xi \in \mathcal{D}^n\}$$

are continuous (note that the $\tilde{\xi}_y$'s take their values in $\mathbb{L}_y(\mathcal{D})$). To avoid confusion between topologies, we denote by $\mathcal{X}$ the topology on E and the induced topologies on all its subsets and by $\mathcal{Y}_y$ the topology introduced above. We show below that $\mathcal{Y}_y$ is independent of y and subsequently it will simply be denoted $\mathcal{Y}$.

Lemma 5.2. *Let E be a normed vector space and $\mathcal{D}$ be a family of Lipschitz vector fields on E, $y \in E$ and $\tilde{y} \in \mathbb{L}_y(\mathcal{D})$. Then $(\mathbb{L}_y(\mathcal{D}), \mathcal{Y}_y) = (\mathbb{L}_{\tilde{y}}(\mathcal{D}), \mathcal{Y}_{\tilde{y}})$.*

Proof. By lemma 5.1, $\mathbb{L}_y(\mathcal{D}) = \mathbb{L}_{\tilde{y}}(\mathcal{D})$. We therefore only need to show that the topologies $\mathcal{Y}_y$ and $\mathcal{Y}_{\tilde{y}}$ are the same. Let $n \in \mathbb{N}^*$ and $\xi = (f^1, \dots, f^n) \in \mathcal{D}^n$. Let $p \in \mathbb{N}^*$, $f^{n+1}, \dots, f^{n+p} \in \mathcal{D}$ and $t_{n+1}, \dots, t_{n+p} \in \mathbb{R}$ such that $\tilde{f}^{n+p}_{t_{n+p}} \circ \dots \circ \tilde{f}^{n+1}_{t_{n+1}}(\tilde{y}) = y$. Finally, define $\eta = (f^{n+1}, \dots, f^{n+p}, f^1, \dots, f^n)$ and ρ the map:

$$(t_1, \dots, t_n) \mapsto (t_{n+1}, \dots, t_{n+p}, t_1, \dots, t_n)$$

It is then clear that $\tilde{\xi}_y = \tilde{\eta}_{\tilde{y}} \circ \rho$. As, by definition, $\tilde{\eta}_{\tilde{y}} : \mathbb{R}^{n+p} \to (\mathbb{L}_y(\mathcal{D}), \mathcal{Y}_{\tilde{y}})$ is continuous, then $\tilde{\xi}_y : \mathbb{R}^n \to (\mathbb{L}_y(\mathcal{D}), \mathcal{Y}_{\tilde{y}})$ is also continuous. Hence, as this holds for all $\tilde{\xi}_y$'s, $\mathcal{Y}_{\tilde{y}} \subseteq \mathcal{Y}_y$. The reverse holds too as y and $\tilde{y}$ play similar roles. □

Lemma 5.3. *Let E be a normed vector space and $\mathcal{D}$ be a family of Lipschitz vector fields on E and $y \in E$:*

- *Open subsets of $(\mathbb{L}_y(\mathcal{D}), \mathcal{X})$ are also open subsets of $(\mathbb{L}_y(\mathcal{D}), \mathcal{Y})$.*
- *$(\mathbb{L}_y(\mathcal{D}), \mathcal{Y})$ is Hausdorff.*

Proof.

- Let $n \in \mathbb{N}^*$ and $\xi = (f^1, \dots, f^n) \in \mathcal{D}^n$. The map $\tilde{\xi}_y : \mathbb{R}^n \to (E, \mathcal{X})$ is continuous and takes its values in $\mathbb{L}_y(\mathcal{D})$. Therefore $\tilde{\xi}_y : \mathbb{R}^n \to (\mathbb{L}_y(\mathcal{D}), \mathcal{X})$ is continuous. This proves the first assertion as this holds for all $\tilde{\xi}_y$'s.
- Two distinct points x and z in $\mathbb{L}_y(\mathcal{D})$ have, respectively, distinct neighborhoods in $(E, \mathcal{X})$, namely O_x and O_z. As open subsets of $(\mathbb{L}_y(\mathcal{D}), \mathcal{X})$ are also open subsets

of $(\mathbb{L}_y(\mathcal{D}), \mathcal{Y})$, then $O_x \cap \mathbb{L}_y(\mathcal{D})$ and $O_z \cap \mathbb{L}_y(\mathcal{D})$ are distinct neighborhoods in $(\mathbb{L}_y(\mathcal{D}), \mathcal{Y})$ of x and z.

□

Definition 5.4. *Let E be a normed vector space and $\mathcal{D}$ be a family of Lipschitz vector fields on E. Let $n \in \mathbb{N}^*$. Let $f^1, \ldots, f^n \in \mathcal{D}$ and $t_1, \ldots, t_n \in \mathbb{R}$. The E-valued map on E defined by $\tilde{f}^n_{t_n} \circ \cdots \circ \tilde{f}^1_{t_1}$ is called a $\mathcal{D}$-integral.*

Lemma 5.5. *Let $\gamma \geq 1$. Let E be a Banach space and $\mathcal{D}$ be a set of Lip-γ vector fields on E. Then the set of all $\mathcal{D}$-integrals constitutes a group of global Lip-γ diffeomorphisms on E.*

Proof. Let $n \in \mathbb{N}^*$. Let $f^1, \ldots, f^n \in \mathcal{D}$ and $t_1, \ldots, t_n \in \mathbb{R}$. The Lipschitzness of each of the maps $\tilde{f}^1_{t_1}, \ldots, \tilde{f}^n_{t_n}$ was established in Chapter 3 (Lipschitz flows). Therefore $\tilde{f}^n_{t_n} \circ \cdots \circ \tilde{f}^1_{t_1}$ is Lip-γ. It is bijective and its inverse $\tilde{f}^1_{-t_1} \circ \cdots \circ \tilde{f}^n_{-t_n}$ is also Lip-γ by the same previous argument. Hence, it is a Lip-γ diffeomorphism. □

Notation. *The group of all $\mathcal{D}$-integrals will be denoted by $\mathcal{G}_{\mathcal{D}}$.*

Lemma 5.6. *Let E and F be two Banach spaces. Let X be a Lipschitz-1 vector field on E and g a Lipschitz-1 diffeormorphism from E to F such that* $\mathrm{d}g$ *is also Lipschitz-1. Then g_*X is a Lipschitz-1 vector field on F.*

5.1.2 Distributions

After defining a natural topology on orbits in the previous subsection, we aim now to appropriately define their tangent spaces at each of their points. Most of the constructions and examples below are mainly due to Sussmann ([26]) and other control theorists such as Lobry ([19]) and Hermann ([14, 15, 16]), we merely adapt them to the Lipschitz geometry.

Definition 5.7 (Group of local diffeomorphisms). *Let M be a $\mathcal{C}^1$-manifold. A family $\mathcal{G}$ of M-valued maps defined on an open subset of M is said to be a group of local diffeomorphisms on M if the following conditions are satisfied:*

1. *$Id_U \in \mathcal{G}$, for all open subsets U of M;*
2. *For all g and h in $\mathcal{G}$, if the range of g is contained in the domain of h, then $h \circ g \in \mathcal{G}$;*
3. *For all g in $\mathcal{G}$, g is a $\mathcal{C}^1$-diffeomorphism onto its range and $g^{-1} \in \mathcal{G}$.*

Definition 5.8. *Let M be a $\mathcal{C}^1$-manifold. Let $\mathcal{G}$ be a group of local diffeomorphisms on M and Δ a distribution on M. We say that Δ is $\mathcal{G}$-invariant if, for every $g \in \mathcal{G}$ and p in the domain of g, $(g_*)_{|\Delta(p)}$ defines a diffeomorphism onto $\Delta(g(p))$.*

Remark 5.9. *Given a group of local diffeomorphisms $\mathcal{G}$ and a distribution Δ on a $\mathcal{C}^1$-manifold M, there exists a natural $\mathcal{G}$-invariant distribution $\Delta_{\mathcal{G}}$ that contains Δ and which is the smallest one as such. For every $p \in M$, $\Delta_{\mathcal{G}}(p)$ is spanned by the union of $\Delta(p)$ and the derivations of all the $g_*(\Delta(q))$'s, where $g \in \mathcal{G}$ and q is in the domain of g and is such that $g(q) = p$.*

Lemma 5.10. *Let M be a $\mathcal{C}^1$-manifold. Let $\mathcal{G}$ be a group of local diffeomorphisms on M and Δ a $\mathcal{G}$-invariant distribution on M. Let $g \in \mathcal{G}$ and p in the domain of g. Then $\Delta(p)$ and $\Delta(g(p))$ are of equal dimensions.*

Notation. *Given a set of Lipschitz vector fields $\mathcal{D}$ on a finite-dimensional vector space E, we denote by $P_{\mathcal{D}}$ the smallest distribution containing $\mathcal{L}(\mathcal{D})$ that is invariant under the action of all $\mathcal{D}$-integrals.*

Lemma 5.11. *Let E be a finite-dimensional vector space and $\mathcal{D}$ a set of Lipschitz vector fields on E. Then $P_{\mathcal{D}}$ is spanned by the set of vector fields under the form $g_*(f)$, where $f \in \mathcal{D}$ and g is a $\mathcal{D}$-integral.*

Notation. *We denote by $\tilde{\mathcal{D}}$ the vector space of vector fields spanning $P_{\mathcal{D}}$.*

With the notations of the previous lemma, $\mathcal{L}(\mathcal{D})$ will be, in general, nowhere near being rich enough to contain all the tangent directions to a $\mathcal{D}$-orbit. A typical example of this will be a situation where, for two points p_1 and p_2 that can be joined by a piece-wise integral curve of $\mathcal{D}$, $\mathcal{L}(\mathcal{D})(p_1)$ and $\mathcal{L}(\mathcal{D})(p_2)$ are of different dimensions. According to lemma 5.10, $P_{\mathcal{D}}$ doesn't have that fundamental pathology.

Example 5.12. *In $\mathbb{R}^2$, let $\mathcal{D} = \{\underline{f}^1, \underline{f}^2\}$, where:*

$$\underline{f}^1 = \phi \frac{\partial}{\partial x} \quad \textit{and} \quad \underline{f}^2 = \frac{\partial}{\partial y}$$

ϕ is any Lip-γ map ($\gamma > 1$) on $\mathbb{R}^2$ depending only on the second variable y and satisfying $\phi(y) = 0$ for $y \leq 0$ and $\phi(y) > 0$ for $y > 0$. It is clear that the $\mathcal{D}$-orbit at any point is the whole 2-dimensional space $\mathbb{R}^2$. But $\mathcal{L}(\mathcal{D})((0, -1))$ is only one-dimensional, whereas $\mathcal{L}(\mathcal{D})((0, 1))$ is two-dimensional.

As a nice illustration of the richness of the distribution $P_{\mathcal{D}}$, we show that it contains the directions of Lie brackets of vector fields in $\mathcal{D}$.

Lemma 5.13. *Let E be a finite-dimensional vector space and $\mathcal{D}$ a set of Lipschitz-γ ($\gamma > 1$) vector fields on E. Let $f^1, f^2 \in \mathcal{D}$, then, for every $y \in E$, $[f^1, f^2]_y \in P_{\mathcal{D}}(y)$.*

Proof. Define the following map:

$$\begin{array}{rrcl} F_y : & \mathbb{R} & \longrightarrow & P_{\mathcal{D}}(y) \\ & t & \longmapsto & ((\tilde{f}_t^2(y))_*)^{-1}(\tilde{f}^1(f_t^2(y))) \end{array}$$

We show first that F_y is well defined. Let $t \in \mathbb{R}$ and define $\tilde{y} = \tilde{f}_t^2(y)$. Then $f^1(\tilde{y}) \in \mathcal{L}(\mathcal{D}(\tilde{y}))$. Since $\tilde{f}^2_{-t} \in \mathcal{G}_{\mathcal{D}}$, then, by definition of $P_{\mathcal{D}}$, $F_y(t)$, which equals $(\tilde{f}^2_{-t}(\tilde{y}))_*(f^1(\tilde{y}))$, is in $P_{\mathcal{D}}(y)$.

F_y is differentiable and its derivative at all times must lie in the closed space $P_{\mathcal{D}}(y)$. This concludes the proof since $F'_y(0) = [f^1, f^2]_y$. □

Always with the notation of the above lemma, the distribution spanned by the closed algebra of $\mathcal{D}$ under Lie brackets (denote it $\mathcal{D}^*$) may still not be enough, especially in the case where $\mathcal{L}(\mathcal{D}) = \mathcal{L}(\mathcal{D}^*)$ as in the example explicited earlier.

Finally, we show that $\mathcal{D}$-orbits and $\tilde{\mathcal{D}}$-orbits give rise to the same points: a $\tilde{\mathcal{D}}$-integral curve corresponds to the image of a $\mathcal{D}$-integral curve by a $\mathcal{D}$-integral:

Lemma 5.14. *Let E be a finite-dimensional vector space. Let f be a $Lip-1$ vector field on E and g a $\mathcal{C}^1$-diffeomorphism on E. Define $h = g_*(f)$. Then:*

$$\forall y \in E, \forall t \in \mathbb{R} : \quad \tilde{h}(t, y) = g(\tilde{f}(t, g^{-1}(y)))$$

Proof. By lemma 5.6, h is Lipschitz-1 and therefore, its global flow on E is well defined. Fix $y \in E$. Define:

$$\begin{array}{rccc} \Gamma : & \mathbb{R} & \rightarrow & E \\ & t & \longmapsto & g(\tilde{f}(t, g^{-1}(y))) \end{array}$$

Let $t_0 \in \mathbb{R}$. Γ is differentiable in t_0. Using the definition of the flow of f starting at $g^{-1}(y)$, the chain rule gives:

$$\begin{aligned} \Gamma'(t_0) &= g_*(f(\tilde{f}(t_0, g^{-1}(y)))) \\ &= g_*(f(g^{-1}(\Gamma(t_0)))) \\ &= h(\Gamma(t_0)) \end{aligned}$$

And as $\Gamma(0) = y$, we conclude, by uniqueness of the flow, that $\Gamma = \tilde{h}(., y)$, which ends the proof. □

Definition 5.15. *Let M be a $\mathcal{C}^1$-manifold and Δ a distribution on M. Let S be a $\mathcal{C}^1$-submanifold of M. We say that S is an integral submanifold of Δ if the tangent space at any point $s \in S$ is equal to $\Delta(s)$.*

5.2 Rough paths on the reachability set

In this section, we show that once one puts a Lipschitz structure on the orbits of a family of vector fields, we can prove that these are exactly the reachability set of rough differential equations.

5.2.1 Lipschitz submanifolds

Definition 5.16. *Let $\gamma \geq 1$. Let E and F be two Lip-γ manifolds of dimensions d and n respectively and A be a subset of E. We say that a map $f : A \to F$ is Lipschitz-γ if there exists a constant C such that for every Lipschitz-γ chart (U, ϕ) on E such that $U \cap A \neq \varnothing$ and Lipschitz-γ chart (V, ψ) on F, the map:*

$$\psi \circ f \circ \varphi^{-1} : \varphi(U \cap A)(\subseteq B_{\mathbb{R}^d}(0,1)) \to \mathbb{R}^n$$

is Lipschitz-γ with a Lipschitz-γ norm less than or equal to C. If this is the case, the smallest constant C satisfying such a property is called the Lipschitz-γ norm of f.

Definition 5.17. *Let $\gamma \geq 1$. Let E and M be two Lip-γ manifolds such that M is a subset of E. We say that M is a Lipschitz-γ submanifold of E if $i : M \hookrightarrow E$ is Lip-γ and i_* is an immersion.*

Lemma 5.18. *Let $\gamma > 1$. Let E be a Lip-γ manifold and M be a Lip-γ submanifold of E. Then for every Banach space-valued Lip-$(\gamma - 1)$ one-form α on E, $\alpha_{|M}$ is a Lip-$(\gamma - 1)$ one-form on M, and there exists a constant $d_{\gamma,E,M}$ depending only on γ (and the*

Lipschitz structures defining E and M) such that:

$$\|\alpha_{|M}\|_{Lip\text{-}(\gamma-1)} \leq d_{\gamma,E,M}\|\alpha\|_{Lip\text{-}(\gamma-1)}$$

Proof. Let $n, p \in \mathbb{N}$ such that E (respectively M) is n-dimensional (p-dimensional). Suppose that E is Lip-γ with constants (R^E, δ^E, L^E). Denote by i the natural inclusion map from M to E. Let (ϕ, U) be a Lip-γ chart on M. We aim to show that the one-form $((\phi_{|U})^{-1})^*\alpha_{|M}$ is Lip-γ with well-controlled Lip-γ norm:
Let $r \in (0, \frac{\delta^E}{\|i\|_{\text{Lip-1}}})$. Let $x \in B_{\mathbb{R}^p}(0,1)$. Let (ψ, V) be a Lip-γ chart on E such that $(\phi_{|U})^{-1}(x) \in (\psi_{|V})^{-1}(B(0, 1-\delta^E))$. Then by lemma 4.7, we have:

$$(\phi_{|U})^{-1}(B_{\mathbb{R}^p}(x,r) \cap B_{\mathbb{R}^p}(0,1)) \subseteq (\psi_{|V})^{-1}(B_{\mathbb{R}^n}(\psi \circ (\phi_{|U})^{-1}(x), \delta^E))$$

We can then write, for $y \in B_{\mathbb{R}^p}(u,r) \cap B_{\mathbb{R}^p}(0,1)$ and $v \in \mathbb{R}^p$ that:

$$(i \circ (\phi_{|U})^{-1})(y) = ((\psi_{|V})^{-1} \circ (\psi \circ \phi_{|U})^{-1})(y) \text{ and } i_*(\phi_{|U})_*^{-1}(y)(v) = (\psi_{|V})_*^{-1}((\psi \circ \phi_{|U})^{-1})_*(y)(v))$$

By definition $\alpha_{|M} = i^*(\alpha)$. Therefore we have:

$$\begin{aligned}
((\phi_{|U})^{-1})^*\alpha_{|M}(y)(v) &= \alpha((i \circ (\phi_{|U})^{-1})(y))(i_*(\phi_{|U})_*^{-1}(y)(v)) \\
&= \alpha(((\psi_{|V})^{-1} \circ (\psi \circ \phi_{|U})^{-1})(y))((\psi_{|V})_*^{-1} \circ (\psi \circ \phi_{|U})^{-1})_*(y)(v)) \\
&= (\psi \circ (\phi_{|U})^{-1})^*(((\psi_{|V})^{-1})^*\alpha)(x)
\end{aligned}$$

Hence, using proposition 4.16 and the definition of a Lip-$(\gamma-1)$ one-form on a manifold, the restriction of $((\phi_{|U})^{-1})^*\alpha_{|M}$ to $B_{\mathbb{R}^p}(u,r) \cap B_{\mathbb{R}^p}(0,1)$ is Lip-$(\gamma-1)$ and its Lip-$(\gamma-1)$ norm is upper-bounded by $C_\gamma\|\alpha\|_{\text{Lip-}(\gamma-1)}\|i\|_{\text{Lip-}\gamma}\max(1, \|i\|_{\text{Lip-}\gamma}^{\gamma})$, where C_γ is a constant depending only on γ. By lemma 3.19, $((\phi_{|U})^{-1})^*\alpha_{|M}$ is Lip-$(\gamma-1)$ on $B_{\mathbb{R}^p}(0,1)$ and its

Lip-$(\gamma-1)$ norm is upper bounded by:

$$C_\gamma \|\alpha\|_{\text{Lip-}(\gamma-1)} \|i\|_{\text{Lip-}\gamma} \max(1, \|i\|^\gamma_{\text{Lip-}\gamma}) \max(1, \frac{2}{r^{\gamma-\lfloor\gamma\rfloor}})$$

which proves the claim. □

Theorem 5.19. *let $\gamma, p \in \mathbb{R}$ such that $\gamma > p \geq 1$. Let E be a Lip-γ manifold and M be a Lip-γ submanifold of E. Then every p-rough path X in M, with starting point x, defines a p-rough path $\tilde{X}$ in E with starting point x such that, for every compactly supported Banach space-valued Lip-$(\gamma-1)$ one-form α on E: $\tilde{X}(\alpha) = X(\alpha_{|M})$. Moreover, $\tilde{X}$ and X have the same support. If there is no confusion, we use the same notation for X and $\tilde{X}$.*

Proof. Let X be a p-rough path in M.

- By lemmas 5.18 and 4.15, we deduce that $i : M \hookrightarrow E$ induces a pushforward map of rough paths in M to rough paths in E (since for every compactly supported Banach space-valued Lip-$(\gamma-1)$ one-form α on E, $\alpha_{|M} = i^*(\alpha)$). Therefore $\tilde{X} = i_*(X)$ is a rough path in E.

- $\text{supp}(X)$ is a compact set in M. Since $i : M \hookrightarrow E$ is continuous, then $\text{supp}(X)$ is compact in E and $E - \text{supp}(X)$ is an open subset in E. If α is a compactly supported Lip-$(\gamma-1)$ one-form on E supported in $E - \text{supp}(X)$, then $\alpha_{|M}$ is a compactly supported Lip-$(\gamma-1)$ one-form on M supported in $M - \text{supp}(X)$. By definition of the support of a rough path, we have then $X(\alpha_{|M}) = 0$. Therefore $\tilde{X}(\alpha) = 0$ and we deduce that $\tilde{X}$ misses $E - \text{supp}(X)$ and

  $$\text{supp}(\tilde{X}) \subseteq \text{supp}(X) (\subseteq M)$$

  Conversely, let U be an open set of E and suppose that $\tilde{X}$ misses U. Since i is continuous, then $U \cap M$ is an open subset of M. Let β be a compactly supported Lip-$(\gamma - 1)$ one-form on M supported in $U \cap M$. Since $\mathrm{supp}(\beta)$ is a compact set of E, $\beta_{|U\cap M}$ can be extended to a compactly supported Lip-$(\gamma-1)$ one-form $\tilde{\beta}$ on E with support in U. We have then $X(\tilde{\beta}_{|M}) = \tilde{X}(\tilde{\beta}) = 0$. Since $\mathrm{supp}(\tilde{\beta}_{|M}) \subseteq U \cap M$ and $\tilde{\beta}_{|U\cap M} = \beta_{|U}$, then $\tilde{\beta}_{|M} = \beta$ and $X(\beta) = 0$. Therefore, X misses $U \cap M$. We get then the inclusion:

$$\mathrm{supp}(X) \subseteq \{x\} \bigcup_{\tilde{X} \text{ misses } U} (M - U)$$

where x is the starting point of X. Since $\mathrm{supp}(\tilde{X}) \subseteq M$, then:

$$\mathrm{supp}(\tilde{X}) = \{x\} \bigcup_{\tilde{X} \text{ misses } U} (M - U)$$

Thus $\mathrm{supp}(X) \subseteq \mathrm{supp}(\tilde{X})$, which concludes the proof.

□

Theorem 5.20. *let $\gamma, p \in \mathbb{R}$ such that $\gamma > p \geq 1$. Let E and F be two Lip-γ manifolds and M be a Lip-γ submanifold of E. Let g be a map defined on F, such that for all $x \in F$, $g(x,.)$ is a linear map from T_xF to $\tau(E)$, the space of vector fields on E. Assume that, for all $x \in F$, $v \in T_xF$ and $y \in M$, $g(x,y)(v)$ is tangent to M, i.e. $g(x,y)(v) \in T_yM$ and denote by $\tilde{g}$ the map on F that associates to every $x \in F$ and $v \in T_xF$ the vector field $(g(x.)(v))_{|M}$. Let Γ (respectively $\tilde{\Gamma}$) be the connection associated to g (resp. $\tilde{g}$). Then we have:*

1. *If, for $\tilde{\gamma} \leq \gamma - 1$, g is Lip-$\tilde{\gamma}$, then $\tilde{g}$ is also Lip-$\tilde{\gamma}$.*
2. *For any Lip-$(\gamma - 1)$ one-form α on $F \times E$, we have:*

$$(\alpha^{\Gamma})_{|F\times M} = (\alpha_{|F\times M})^{\tilde{\Gamma}}$$

3. *Assume that g is Lip-γ. Let X be a p-rough path on F and $y_0 \in M$. If Y is a solution to the rough differential equation on M:*

$$\begin{cases} \mathrm{d}Y_t = \tilde{g}(X_t, Y_t)\mathrm{d}X_t \\ Y(0) = y_0 \end{cases} \tag{5.1}$$

then it is also a solution (in the sense of theorem 5.19) to the rough differential equation on E:

$$\begin{cases} \mathrm{d}Y_t = g(X_t, Y_t)\mathrm{d}X_t \\ Y(0) = y_0 \end{cases} \tag{5.2}$$

Proof. Let $n, p, d \in \mathbb{N}$ such that E (respectively M, F) is n-dimensional (resp. p-dimensional, d-dimensional). Suppose that E is Lip-γ with constants (R^E, δ^E, L^E). Denote by i the natural inclusion map from M to E. Let us fist note that g and $\tilde{g}$ are linked in the following way: for every $(x, y) \in F \times M$ and $v_x \in T_xF$, we have:

$$g(x, y)(v_x) = (i(y))_* \tilde{g}(x, i(y))(v_x)$$

1. Let (ϕ, U) (respectively $(\tilde{\psi}, \tilde{V})$) be a Lip-$\gamma$ chart on F (resp. on M). Let $r \in (0, \frac{\delta^E}{\|i\|_{\text{Lip-1}}})$. Let $(x, y) \in B_{\mathbb{R}^d}(0, 1) \times B_{\mathbb{R}^p}(0, 1)$. Let (ψ, V) be a Lip-γ chart on E such that:

$$(\tilde{\psi}_{|\tilde{V}})^{-1}(y) \in (\psi_{|V})^{-1}(B_{\mathbb{R}^p}(0, 1 - \delta^E))$$

   Then we have by lemma 4.7:

$$(\tilde{\psi}_{|\tilde{V}})^{-1}(B_{\mathbb{R}^p}(y, r) \cap B_{\mathbb{R}^p}(0, 1)) \subseteq (\psi_{|V})^{-1}(B_{\mathbb{R}^n}(\psi \circ (\tilde{\psi}_{|\tilde{V}})^{-1}(y), \delta^E))$$

Let $(a,b) \in B_{\mathbb{R}^d \times \mathbb{R}^p}((x,y),r) \cap B_{\mathbb{R}^d}(0,1) \times B_{\mathbb{R}^p}(0,1)$. Then:

$$i \circ (\tilde{\psi}_{|\tilde{V}})^{-1}(b) = (\psi_{|V})^{-1} \circ \left(\psi \circ (\tilde{\psi}_{|\tilde{V}})^{-1}\right)(b)$$

Since $(\psi \circ (\tilde{\psi}_{|\tilde{V}})^{-1})(b) \in B_{\mathbb{R}^n}(0,1)$, we have, by definition of $\tilde{g}$:

$$\tilde{g}((\phi_{|U})^{-1}(a), (\tilde{\psi}_{|\tilde{V}})^{-1}(b)) = (i^{-1})_* g\left((\phi_{|U})^{-1}(a), (\psi_{|V})^{-1}\left(\psi \circ (\tilde{\psi}_{|\tilde{V}})^{-1}(b)\right)\right)$$

Thus:

$$\tilde{g}_{(\phi,\tilde{\psi})}(a,b) = (\tilde{\psi} \circ i^{-1} \circ (\psi_{|V})^{-1})_* g_{(\phi,\psi)}(a, (\psi \circ (\tilde{\psi}_{|\tilde{V}})^{-1})(b))$$

which is Lip-γ by the properties of i and i_*.

2. Let α be a compactly supported Lip-$(\gamma-1)$ one-form on $F \times E$. Let π be the natural projection from $F \times E$ to F. Let $(x,y) \in F \times M$. Let $(v_x, v_y) \in T_{(x,y)}F \times M$ such that v_y is tangent to M. By definition of α^Γ, we have:

$$\begin{aligned}
(\alpha^\Gamma)_{|F\times M}(x,y)(v_x,v_y) &= \alpha^\Gamma(x,i(y))(v_x,(i(y))_* v_y) \\
&= \alpha(x,y)(\Gamma(x,y)\pi_*((v_x,(i(y))_* v_y))) \\
&= \alpha(x,y)(v_x, g(x,y)(v_x))
\end{aligned}$$

Similarly, by definition of α^Γ and $\tilde{g}$:

$$\begin{aligned}
(\alpha_{|F\times M})^{\tilde{\Gamma}}(x,y)(v_x,v_y) &= (\alpha_{|F\times M})(x,y)(\tilde{\Gamma}(x,y)\pi_*((v_x,v_y))) \\
&= (\alpha_{|F\times M})(x,y)(v_x, \tilde{g}(x,y)(v_x)) \\
&= \alpha(x,y)(v_x,(i(y))_* \tilde{g}(x,y)(v_x)) \\
&= \alpha(x,y)(v_x, g(x,y)(v_x))
\end{aligned}$$

Therefore $(\alpha^\Gamma)_{|F\times M} = (\alpha_{|F\times M})^{\tilde{\Gamma}}$.

3. Let Z be a rough path on $F \times M$. Let $\tilde{Z}$ be the extension of Z to a rough path on $F \times E$. Let α be a compactly supported Lip-$(\gamma - 1)$ one-form on $F \times E$. Note that since g is Lip-γ, both α^{Γ} and $(\alpha_{|F\times M})^{\tilde{\Gamma}}$ are Lip-γ. By definition:

$$\tilde{Z}(\alpha^{\Gamma}) = Z((\alpha^{\Gamma})_{|F\times M})$$

And by the previous result, this is equal to $Z((\alpha_{|F\times M})^{\tilde{\Gamma}})$. Assume that Z is a solution to the RDE (5.1). Then

$$Z((\alpha_{|F\times M})^{\tilde{\Gamma}}) = Z(\alpha_{|F\times M})$$

which is, by definition, equal to $\tilde{Z}(\alpha)$. Z and $\tilde{Z}$ having the same starting points and the same projection onto F, $\tilde{Z}$ is then a solution to the RDE (5.2).

$\square$

5.2.2 Application to reachability sets

Let $\gamma > 1$. Let $d \in \mathbb{N}^*$ and let $\mathcal{D} = \{f^1, \dots, f^d\}$ be a family of vector fields on a finite-dimensional vector space E. Let $y_0 \in E$. We assume that $\mathbb{L}_{y_0}(\mathcal{D})$ is endowed with a structure of a Lip-γ submanifold of E such that the vector fields of $\mathcal{D}$ are tangent to it. We define the two following maps:

$$\begin{array}{rrcl} f: & E & \to & \mathcal{L}(\mathbb{R}^d, E) \\ & y & \mapsto & ((v_1, \dots, v_d) \mapsto \sum v_i f^i(y)) \end{array} \quad \text{and} \quad \begin{array}{rrcl} \tilde{g}: & \mathbb{R}^d & \to & \mathcal{L}(\mathbb{R}^d, \tau(\mathbb{L}_{y_0}(\mathcal{D}))) \\ & x & \mapsto & (v \mapsto f(.)(v)) \end{array}$$

Theorem 5.21. *Let $p \in (1, \gamma)$. Let X be a geometric p-rough path in $\mathbb{R}^d$. Assume that $f^1, \dots, f^d$ are Lip-γ. Then:*

1. *The map:*

$$\begin{array}{rccl} g: & \mathbb{R}^d & \to & \mathcal{L}(\mathbb{R}^d, \tau(E)) \\ & x & \mapsto & (v \mapsto f(.)(v)) \end{array}$$

   is Lip-γ*.*

2. *A geometric p-rough path* Y *on* $\mathbb{L}_{y_0}(\mathcal{D})$ *solution to the RDE:*

$$\begin{cases} \mathrm{d}Y_t = \tilde{g}(X_t, Y_t)\mathrm{d}X_t \\ Y(0) = y_0 \end{cases} \tag{5.3}$$

   is also a solution to the RDE:

$$\begin{cases} \mathrm{d}Y_t = f(Y_t)\mathrm{d}X_t \\ Y(0) = y_0 \end{cases} \tag{5.4}$$

Proof. It is a straightforward exercise to show that g is Lip-γ (since $f^1, \ldots, f^d$ are assumed to be Lip-γ). Therefore, by theorem 5.20, a solution to (5.3) is also a solution to the RDE:

$$\begin{cases} \mathrm{d}Y_t = g(X_t, Y_t)\mathrm{d}X_t \\ Y(0) = y_0 \end{cases}$$

which, by the consistency of the classical and the rough path definitions of RDEs on a vector space, is the same as the solution to (5.4). $\square$

Theorem 5.22. *Let* $p \in [1, \gamma)$ *and assume that* $f^1, \ldots, f^d$ *are Lip-*γ*. Let* $y_0 \in M$*. Then:*

$$\mathcal{R}(y_0, \mathcal{D}, G\Omega_p(\mathbb{R}^d)) \subseteq M$$

Proof. Let $X \in G\Omega_p(\mathbb{R}^d)$. Then, by theorem 5.21 the differential equations (5.3) and (5.4) have the same solutions (the existence of which are granted by the theorems 1.56

and 4.29), denote it by Y. By theorem 5.19, $\mathrm{supp}(Y) \subseteq M$. Since the supports of Y as a classical or a manifold rough path are the same, we deduce then that $y_0 + \tilde{Y}^1_{0,T} \in M$, which proves our claim. □

Corollary 5.23. *Let $p \in [1, \gamma)$ and assume that $f^1, \ldots, f^d$ are Lip-γ. Let $y_0 \in E$. Let G be a family of geometric p-rough paths in $\mathbb{R}^d$ and assume that $\mathcal{R}(y_0, f, G)$ has a structure of a Lip-γ submanifold of E such that the vector fields of $\mathcal{D}$ are tangent to it. Then:*

$$\mathcal{R}(y_0, \mathcal{D}, G\Omega_p(\mathbb{R}^d)) = \mathcal{R}(y_0, f, G)$$

Corollary 5.24. *Let $p \in [1, \gamma)$ and assume that $f^1, \ldots, f^d$ are Lip-γ. Let $y_0 \in E$. Assume that $\mathbb{L}_{y_0}(\mathcal{D})$ has a structure of a Lip-γ submanifold of E such that the vector fields of $\mathcal{D}$ are tangent to it. Then:*

$$\mathcal{R}(y_0, \mathcal{D}, G\Omega_p(\mathbb{R}^d)) = \mathbb{L}_{y_0}(\mathcal{D})$$

Proof. It is clear that $\mathbb{L}_{y_0}(\mathcal{D}) \subseteq \mathcal{R}(y_0, \mathcal{D}, G\Omega_p(\mathbb{R}^d))$. Indeed, every point in $\mathbb{L}_{y_0}(\mathcal{D})$ can be reached by solving the RDE (5.4) using as a control a concatenation of finitely many axis paths. We conclude using the previous corollary. □

5.3 Geometric structure of the reachability set

For this section, let E be a finite dimensional vector space and $\gamma > 1$. Lemmas 5.25 to 5.30 are due to Sussmann ([26]) in the smooth case.

We first start by showing that the directions tangent to piece-wise integral curves of a family of Lipschitz vector fields $\mathcal{D}$ all lie in the distribution $P_{\mathcal{D}}$:

Lemma 5.25. *Let $y \in E$ and $\mathcal{D}$ be a family of Lip-γ vector fields on E. Let $n \in \mathbb{N}^*$. Let $\xi \in \mathcal{D}^n$, $y_0 \in E$ and $\vec{T} \in \mathbb{R}^n$ such that $\tilde{\xi}(\vec{T}, y_0) = y$. Then:*

$$(\tilde{\xi}_{y_0}(\vec{T}))_*(\mathbb{R}^n) \subseteq P_{\mathcal{D}}(y)$$

Proof. We do this by induction on n:

For $n = 1$: let $f \in \mathcal{D}$, $y, y_0 \in E$ and $T \in \mathbb{R}$ such that $\tilde{f}(T, y_0) = y$. From the definition of the flow $\tilde{f}_{y_0}$, we get that

$$(\tilde{f}_{y_0})_* \left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_T \right) = f(\tilde{f}_{y_0}(T)) = f(y)$$

Since $P_{\mathcal{D}}$ contains $\mathcal{L}(\mathcal{D})$, we conclude that:

$$\tilde{f}_{y_0}(T)_*(\mathbb{R}) = \mathrm{Span}(f(y)) \subseteq P_{\mathcal{D}}(y)$$

Let $n \in \mathbb{N}^*$. We suppose the result true for n and we show it for $n+1$. Let $\xi \in \mathcal{D}^{n+1}$, $y, y_0 \in E$ and $\vec{T} \in \mathbb{R}^{n+1}$ such that $\tilde{\xi}(\vec{T}, y_0) = y$. Let $(\vec{T'}, t) \in \mathbb{R}^n \times \mathbb{R}$ and $(\eta, f) \in \mathcal{D}^n \times \mathcal{D}$ such that $\vec{T} = (\vec{T'}, t)$ and $\xi = (\eta, f)$. We will identify, in the usual way, the tangent space to a product of manifolds and the product of the tangent spaces to each manifold. Finally, define the following map:

$$\begin{array}{rccc} A: & \mathbb{R}^n \times \mathbb{R} & \longrightarrow & \mathbb{R} \times E \\ & (\vec{s}, t) & \longmapsto & (t, \tilde{\eta}_{y_0}(\vec{s})) \end{array}$$

It is then clear that: $\tilde{\xi}_{y_0} = \tilde{f} \circ A$. Given $(\vec{v}, u)$ in $\mathbb{R}^n \times \mathbb{R}$, the following identity is straightforward:

$$A_*(\mathrm{D}_{(\vec{T'},t),(\vec{v},u)}) = \left(u\frac{\mathrm{d}}{\mathrm{d}x}\Big|_t, (\tilde{\eta}_{y_0})_*(\mathrm{D}_{\vec{T'},\vec{v}}) \right)$$

And for $\vec{y} \in E$ and $Y \in T_{\vec{y}}E$:

$$\tilde{f}_*(\frac{\mathrm{d}}{\mathrm{d}x}\Big|_t, Y) = f(\tilde{f}(t,\vec{y})) + (\tilde{f}_t)_*(Y)$$

From which we finally get, by applying the chain rule, that for $(\vec{v}, u) \in \mathbb{R}^n \times \mathbb{R}$ we have:

$$\begin{aligned}(\xi_{y_0})_*(\mathrm{D}_{(\vec{T'},t),(\vec{v},u)}) &= uf(\tilde{f}(t, \tilde{\eta}_{y_0}(\vec{T'}))) + (\tilde{f}_t)_*((\tilde{\eta}_{y_0})_*(\mathrm{D}_{(\vec{T'},\vec{v})})) \\ &= uf(y) + (\tilde{f}_t)_*((\tilde{\eta}_{y_0})_*(\mathrm{D}_{(\vec{T'},\vec{v})}))\end{aligned}$$

By the induction assumption $(\tilde{\eta}_{y_0})_*(\mathrm{D}_{(\vec{T'},\vec{v})}) \in P_{\mathcal{D}}(\tilde{\eta}_{y_0}(\vec{T'}))$. Using the invariance property of the distribution $P_{\mathcal{D}}$, we have then $(\tilde{f}_t)_*((\tilde{\eta}_{y_0})_*(\mathrm{D}_{(\vec{T'},\vec{v})})) \in P_{\mathcal{D}}(\tilde{f}_t(\tilde{\eta}_{y_0}(\vec{T'})))$. Since $\tilde{f}_t(\tilde{\eta}_{y_0}(\vec{T'})) = y$ and $P_{\mathcal{D}}$ contains $\mathcal{L}(\mathcal{D})$, we conclude then that:

$$(\tilde{\xi}_{y_0})_*(\mathrm{D}_{(\vec{T'},t),(\vec{v},u)}) \in P_{\mathcal{D}}(y)$$

□

Now we show that $P_{\mathcal{D}}$ is not too big a distribution to be the suitable "candidate" tangent space to $\mathbb{L}_{y_0}(\mathcal{D})$: given any derivation in $P_{\mathcal{D}}$, there always exists a piece-wise integral curve of $\mathcal{D}$ which derivative at a certain point gives the required derivation.

Lemma 5.26. *Let $y \in E$, $\mathcal{D}$ be a family of Lip-γ vector fields on E and $f \in \mathcal{D}$. Let $y_0 \in E$, $n \in \mathbb{N}^*$, $f \in \mathcal{D}$, $\eta \in \mathcal{D}^n$ and $\vec{t} \in \mathbb{R}^n$ such that $\tilde{\eta}(\vec{t}, y_0) = y$. Then, there exist $\xi \in \mathcal{D}^{n+1}$ and $\vec{T} \in \mathbb{R}^{n+1}$ such that $\tilde{\xi}(\vec{T}, y_0) = y$ and $(\tilde{\eta}_{\vec{t}})_*(f(y_0)) \in (\tilde{\xi}_{y_0}(\vec{T}))_*(\mathbb{R}^{n+1})$.*

Proof. Let $\xi = (f, \eta)$ and $\vec{T} = (0, \vec{t})$. Then $\tilde{\xi}(\vec{T}, y_0) = y$. Let $(t, \vec{s}), (u, \vec{v}) \in \mathbb{R} \times \mathbb{R}^n$. Using the same techniques as in the proof of the previous lemma, we can show that:

$$(\tilde{\xi}_{y_0})_*(\mathrm{D}_{(t,\vec{s}),(u,\vec{v})}) = u(\tilde{\eta}_{\vec{t}})_*(f(\tilde{f}(t, y_0))) + (\tilde{\eta}_{\tilde{f}(t,y_0)})_*(\mathrm{D}_{\vec{s},\vec{v}})$$

In particular $(\tilde{\xi}_{y_0})_*(\mathrm{D}_{\vec{T},(1,\vec{0})}) = (\tilde{\eta}_{\vec{t}})_*(f(y_0))$, which ends the proof. $\square$

Lemma 5.27. *Let $y \in E$ and $\mathcal{D}$ be a family of Lip-γ vector fields on E. Let $y_1, y_2 \in E$, $n_1, n_2 \in \mathbb{N}^*$, $\eta \in \mathcal{D}^{n_1}$, $\sigma \in \mathcal{D}^{n_2}$ and $(\vec{T_1}, \vec{T_2}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ such that $\tilde{\eta}_{y_1}(\vec{T_1}) = \tilde{\sigma}_{y_2}(\vec{T_2}) = y$. Then there exist $y_3 \in E$, $n_3 \in \mathbb{N}^*$, $\xi \in \mathcal{D}^{n_3}$ and $\vec{T_3} \in \mathbb{R}^{n_3}$ such that:*

$$\begin{cases} \tilde{\xi}(\vec{T_3}, y_3) = y \\ (\tilde{\eta}_{y_1}(\vec{T_1}))_*(\mathbb{R}^{n_1}) + (\tilde{\sigma}_{y_2})_*(\mathbb{R}^{n_2}) \subseteq (\tilde{\xi}_{y_3}(\vec{T_3}))_*(\mathbb{R}^{n_3}) \end{cases}$$

Proof. Let $\xi = (\eta, \hat{\sigma}, \sigma)$ and $\vec{T_3} = (\vec{T_1}, -\overrightarrow{\hat{T_2}}, \vec{T_2})$. Then, for $(\vec{t_1}, \vec{t_2}, \vec{t_3})$ and $(\vec{u}, \vec{v}, \vec{w})$ in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_2}$, an easy computation using the chain rule gives:

$$\begin{aligned} (\tilde{\xi}_{y_0})_*(\mathrm{D}_{(\vec{t_1},\vec{t_2},\vec{t_3}),(\vec{u},\vec{v},\vec{w})}) \quad = \quad & (\tilde{\sigma}_{\vec{t_3}})_* \circ (\tilde{\hat{\sigma}}_{\vec{t_2}})_* \circ (\tilde{\eta}_{y_0})_*(\mathrm{D}_{(\vec{t_1},\vec{u})}) + \\ & (\tilde{\sigma}_{\vec{t_3}})_* \circ (\tilde{\hat{\sigma}}_{\tilde{\eta}(\vec{t_1},y_0)})_*(\mathrm{D}_{(\vec{t_2},\vec{v})}) + (\tilde{\sigma}_{\tilde{\hat{\sigma}}(\vec{t_2},\tilde{\eta}(\vec{t_1},y_0))})_*(\mathrm{D}_{(\vec{t_3},\vec{w})}) \end{aligned}$$

This identity, in the special case where $(\vec{t_1}, \vec{t_2}, \vec{t_3}) = \vec{T_3}$ and $\vec{v} = \vec{0}$ becomes:

$$(\tilde{\xi}_{y_0})_*(\mathrm{D}_{\vec{T_3},(\vec{w},\vec{0},\vec{u})}) = (\tilde{\eta}_{y_0})_*(\mathrm{D}_{(\vec{T_1},\vec{u})}) + (\tilde{\sigma}_{y_2})_*(\mathrm{D}_{(\vec{T_2},\vec{w})})$$

which implies the sought result. $\square$

Corollary 5.28. *Let $y \in E$, $\mathcal{D}$ be a family of Lip-γ vector fields on E and $\vec{w} \in P_{\mathcal{D}}(y)$. Then there exist $y_0 \in E$, $n \in \mathbb{N}^*$, $\xi \in \mathcal{D}^n$ and $\vec{T} \in \mathbb{R}^n$ such that $\tilde{\xi}(\vec{T}, y_0) = y$ and $\vec{w} \in (\tilde{\xi}_{y_0}(\vec{T}))_*(\mathbb{R}^n)$.*

Proof. Let $p \in \mathbb{N}^*$, $y_1, \ldots, y_p \in E$, $n_1, \ldots, n_p \in \mathbb{N}^*$, $f^1, \ldots, f^p \in \mathcal{D}$, $a_1, \ldots, a_p \in \mathbb{R}$, $(\eta^1, \ldots, \eta^p) \in \mathcal{D}^{n_1} \times \cdots \times \mathcal{D}^{n_p}$ and $(\vec{t_1}, \ldots, \vec{t_p}) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_p}$ such that, for all $i \in [\![1, p]\!]$, we have $\tilde{\eta}^i(\vec{t_i}, y_i) = y$ and $\vec{w} = \sum_1^p a_i (\tilde{\eta}^i_{\vec{t_i}})_*(f^i(y_i))$.

For each $i \in [\![1, p]\!]$, by lemma 5.26, let $\xi^i \in \mathcal{D}^{n_i+1}$ and $\vec{T_i} \in \mathbb{R}^{n_i+1}$ such that $\tilde{\xi}^i(\vec{T_i}, y_i) = y$ and $(\tilde{\eta}^i_{\vec{t_i}})_*(f^i(y_0)) \in (\tilde{\xi}^i_{y_0}(\vec{T_i}))_*(\mathbb{R}^{n_i+1})$. By using lemma 5.27 finitely many times, we find that there exist $\tilde{y} \in E$, $m \in \mathbb{N}^*$, $\xi \in \mathcal{D}^m$ and $\vec{T} \in \mathbb{R}^m$ such that:

$$\begin{cases} \tilde{\xi}(\vec{T}, \tilde{y}) = y \\ \sum_{i=1}^{p} (\tilde{\xi}^i_y(\vec{T}_i))_*(\mathbb{R}^{n_i+1}) \subseteq (\tilde{\xi}_{\tilde{y}}(\vec{T}))_*(\mathbb{R}^m) \end{cases}$$

In particular

$$\vec{w} = \sum_{i=1}^{p} a_i (\tilde{\eta}^i_{\vec{t}_i})_*(f^i(y_0)) \in \sum_{i=1}^{p} (\tilde{\xi}^i_y(\vec{T}_i))_*(\mathbb{R}^{n_i+1}) \subseteq (\tilde{\xi}_{\tilde{y}}(\vec{T}))_*(\mathbb{R}^m)$$

□

Finally, we prove that one can always find a piece-wise integral curve of $\mathcal{D}$ whose pushforward generate the whole of $P_{\mathcal{D}}$ at a given point.

Lemma 5.29. *Let $y \in E$ and $\mathcal{D}$ be a family of Lip-γ vector fields on E. Then, there exists $n \in \mathbb{N}^*$, $\xi \in \mathcal{D}^n$, $y_0 \in E$ and $\vec{T} \in \mathbb{R}$ such that:*

$$\tilde{\xi}(\vec{T}, y_0) = y \text{ and } (\tilde{\xi}_{y_0}(\vec{T}))_*(\mathbb{R}^n) = P_{\mathcal{D}}(y)$$

Proof. This is a straightforward corollary of lemma 5.28 applied to a basis of $P_{\mathcal{D}}(y)$ then lemma 5.27 is used finitely many times. □

Lemma 5.30. *Let $\mathcal{D}$ be a family of Lip-γ vector fields on E ($\gamma \geq 2$). Let M be a connected integral $\mathcal{C}^1$-submanifold of $P_{\mathcal{D}}$ and let $y_0 \in M$. Then M is an open subset of $(\mathbb{L}_{y_0}(\mathcal{D}), \mathcal{Y})$.*

Proof. 1. Let $y \in M$. Let $f^1, \dots, f^p \in \mathcal{D}$, $g^1, \dots, g^p \in \mathcal{G}_{\mathcal{D}}$ such that $(g^1_*(f^1)(y), \dots, g^p_*(f^p)(y))$ is a basis for $P_{\mathcal{D}}(y)$. Define $h^i = g^i_*(f^i)$, for $i \in [\![1, p]\!]$ and $\xi = (h^1, \dots, h^p)$. As $h^1, \dots, h^p$ are tangent to M, there exists a neighborhood U_0 of 0 in $\mathbb{R}^p$ such that

$(\tilde{\xi}_y)_{|U_0}$ takes its values in M. As a direct consequence of lemma 5.14:

$$\forall \vec{T} \in U_0 : \tilde{\xi}(\vec{T}, y) \in \mathbb{L}_y(\mathcal{D})$$

$\tilde{\xi}_y$ is of full rank at 0 (since $(h^1(y), \ldots, h^p(y))$ is a basis for $P_{\mathcal{D}}(y)$). Therefore, there exists a neighborhood U_1 of 0 in $\mathbb{R}^p$ contained in U_0 such that $(\tilde{\xi}_y)_*$ is of full rank on U_1 and $\tilde{\xi}_y(U_1)$ is a neighborhood of y in M.

We have thus proved that for every point in M, there exists a neighborhood (in M) that is also a subset of the $\mathcal{D}$-orbit of that point. As M is path-connected, it is then easy to prove, by a compactness argument over a piece-wise integral curve of $\mathcal{D}$, that any two points of M belong to the same $\mathcal{D}$-orbit. Hence $M \subseteq \mathbb{L}_{y_0}(\mathcal{D})$.

2. Let $p \in \mathbb{N}^*$. Let $\xi = (f^1, \ldots, f^p) \in \mathcal{D}^p$. We want to show that $\tilde{\xi}_{y_0}^{-1}(M)$ is open in $\mathbb{R}^p$. Suppose there exists $\vec{t} = (t_1, \ldots, t_p) \in \mathbb{R}^p$ such that $y = \tilde{\xi}(\vec{t}, y_0) \in M$. Let $\vec{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_p) \in \mathbb{R}^p$. We can rewrite $\tilde{\xi}_{y_0}(\vec{t} + \vec{\varepsilon})$ in the following way, using lemma 5.14:

$$\begin{aligned}
\tilde{\xi}_{y_0}(\vec{t} + \vec{\varepsilon}) &= \tilde{f}^p_{t_p+\varepsilon_p} \circ \cdots \circ \tilde{f}^1_{t_1+\varepsilon_1}(y_0) \\
&= \tilde{f}^p_{t_p+\varepsilon_p} \circ \cdots \circ \tilde{f}^1_{t_1+\varepsilon_1} \circ \tilde{f}^1_{-t_1} \circ \tilde{f}^2_{-t_2} \circ \cdots \circ \tilde{f}^p_{-t_p}(y) \\
&= \tilde{f}^p_{t_p+\varepsilon_p} \circ \cdots \circ \tilde{f}^2_{t_2+\varepsilon_2} \circ \tilde{f}^1_{\varepsilon_1} \circ \tilde{f}^2_{-t_2} \circ \cdots \circ \tilde{f}^p_{-t_p}(y) \\
&= \tilde{f}^p_{t_p+\varepsilon_p} \circ \cdots \circ \tilde{f}^2_{t_2+\varepsilon_2} \circ \tilde{f}^2_{-t_2} \circ \cdots \circ \tilde{f}^p_{-t_p} \circ \tilde{f}^p_{t_p} \circ \cdots \circ \tilde{f}^2_{t_2} \circ \\
&\quad \tilde{f}^1_{\varepsilon_1} \circ \tilde{f}^2_{-t_2} \circ \cdots \circ \tilde{f}^p_{-t_p}(y) \\
&= \tilde{f}^p_{t_p+\varepsilon_p} \circ \cdots \circ \tilde{f}^2_{t_2+\varepsilon_2} \circ \tilde{f}^2_{-t_2} \circ \cdots \circ \tilde{f}^p_{-t_p} \circ \tilde{h}^1_{\varepsilon_1}(y)
\end{aligned}$$

where $h^1 = g^1_*(f^1)$ and $g^1 = \tilde{f}^p_{t_p} \circ \cdots \circ \tilde{f}^2_{t_2}$. Repeating the same process finitely many times, we write:

$$\tilde{\xi}_{y_0}(\vec{t} + \vec{\varepsilon}) = \tilde{h}^p_{\varepsilon_p} \circ \cdots \circ \tilde{h}^1_{\varepsilon_1}(y)$$

where:

$$\begin{cases} h^2 = g^2_*(f^2) & , \quad g^2 = \tilde{f}^p_{t_p} \circ \dots \circ \tilde{f}^3_{t_3} \\ \vdots & , \quad \vdots \\ h^{p-1} = g^{p-1}_*(f^{p-1}) & , \quad g^{p-1} = \tilde{f}^p_{t_p} \\ h^p = f^p \end{cases}$$

As $y \in M$ and $h^1, \dots, h^p$ are tangent to M, there exists $h > 0$ such that, for all $(\varepsilon_1, \dots, \varepsilon_p) \in B(0,h)$, $\tilde{h}^p_{\varepsilon_p} \circ \dots \circ \tilde{h}^1_{\varepsilon_1}(y) \in M$, i.e. $B(\vec{t}, h) \subseteq \xi^{-1}_{y_0}(M)$. Therefore, M is open in $(\mathbb{L}_{y_0}(\mathcal{D}), \mathcal{Y})$.

□

Definition 5.31. *Let $\mathcal{D}$ be a family of Lip-γ vector fields on E ($\gamma \geq 2$).*
For $y \in E$, define:

$$H_y = \Big\{(\xi, \vec{T}, \tilde{y}, n) |\, n \in \mathbb{N}, \xi \in \mathcal{D}^n, \vec{T} \in \mathbb{R}^n, \tilde{y} \in E : \tilde{\xi}_{\tilde{y}}(\vec{T}) = y,\, (\tilde{\xi}_{\tilde{y}}(\vec{T}))_*(\mathbb{R}^n) = P_{\mathcal{D}}(y)\Big\}$$

For a matrix M of rank $p \in \mathbb{N}^*$, define:

$$G(M,p) = \{R \in GL_p(\mathbb{R}) |\, R \text{ is extracted from } M\}$$

We will say that $\mathcal{D}$ is well controlled at $y_0 \in E$ with constants (M_1, M_2) if, for all $y \in \mathbb{L}_{y_0}(\mathcal{D})$, there exists $(\xi, \vec{T}, \tilde{y}, n) \in H_y$ such that

$$\|(\tilde{\xi}_{\tilde{y}})_{|B(\vec{T},1)}\|_{Lip-\gamma} \leq M_1$$

and there exists

$$M \in G(Mat(\mathrm{d}\tilde{\xi}_{\tilde{y}}(\vec{T})), dim(P_{\mathcal{D}}(y)))$$

such that $\|M^{-1}\| \leq M_2$.

Remark 5.32. *By lemma 5.29, $H_y \neq \varnothing$ and by theorem 3.48, $(\tilde{\xi}_{\tilde{y}})_{|B(\vec{T},1)}$ is Lip-γ.*

Lemma 5.33. *Let $\mathcal{D}$ be a family of Lip-γ vector fields on E ($\gamma \geq 2$). Let $y_0 \in E$. Let $M_1, M_2 \in (0,\infty)$ and suppose that $\mathcal{D}$ is well controlled at $y_0 \in E$ with constants (M_1, M_2). For every $y \in \mathbb{L}_{y_0}(\mathcal{D})$, there exist a subset V_y of E containing y and a map $\varphi_y : \mathbb{L}_{y_0}(\mathcal{D}) \to \mathbb{R}^p$, where p is the dimension of $P_{\mathcal{D}}(y)$, satisfying:*

1. *V_y is a precompact open subset of $(\mathbb{L}_{y_0}(\mathcal{D}), \mathcal{Y})$;*
2. *$(\varphi_y)_{|V_y}$ is a Lip-γ diffeomorphism onto $B(0,1)$;*
3. *$V_y \subseteq B(y, M_1)$.*

Moreover, there exists a constant C depending only on M_1 and M_2 such that for all $y_1, y_2 \in \mathbb{L}_{y_0}(\mathcal{D})$:

$$\varphi_{y_2} \circ (\varphi_y)^{-1}_{|V_y} : B_{\mathbb{R}^p}(0,1) \to \mathbb{R}^p$$

is Lip-γ with norm less than or equal to C.

Proof. Let $p = \dim(P_{\mathcal{D}}(y_0))$ and $k = \dim(E)$.

- Let $y \in \mathbb{L}_{y_0}(\mathcal{D})$. Let $(\xi, \vec{T}, \tilde{y}) \in H_y$ such that $\|(\xi_{\tilde{y}})_{|B(\vec{T},1)}\|_{\text{Lip}-\gamma} \leq M_1$ and $M \in G(\text{Mat}(\text{d}\xi_{\tilde{y}}(\vec{T})), p)$ such that $\|M^{-1}\| \leq M_2$. Let $n \in \mathbb{N}^*$ such that $\xi \in \mathcal{D}^n$ and $\vec{T} \in \mathbb{R}^n$. Let $\vec{t} \in \mathbb{R}^n$, by lemma 5.25,

  $$(\xi_{\tilde{y}}(\vec{t}))_*(\mathbb{R}^n) \subseteq P_{\mathcal{D}}(\xi_{\tilde{y}}(\vec{t}))$$

  Since $\dim(P_{\mathcal{D}}(\xi_{\tilde{y}}(\vec{t}))) \leq p$, then $\text{rank}(\xi_{\tilde{y}}(\vec{t}))_* \leq p$ Therefore the map $\xi_{\tilde{y}}$ is of maximal rank p at $\vec{T}$. By lemma 3.50, let $\delta \in (0,1]$ depending only on γ and M_1M_2 such that $(\xi_{\tilde{y}})_*$ is invertible on $B_{\mathbb{R}^n}(\vec{T},\delta)$. By theorem 3.55, there exists an open subset U_0 of $B_{\mathbb{R}^n}(\vec{T},\delta)$ containing $\vec{T}$, an open subset V of E containing $\xi_{\tilde{y}}(U_0)$ and a constant

$\alpha > 0$ depending only on γ and $M_1 M_2$ such that the following diagram commutes:

$$\begin{tikzcd}
U_0(\subseteq \mathbb{R}^n) \arrow[r, "\xi_{\tilde{y}}"] \arrow[d, "f"'] & V(\subseteq E) \arrow[d, "g"] \\
B_{\mathbb{R}^n}(0,\alpha) \arrow[r, "\pi_{n,p,k}"] & B_{\mathbb{R}^k}(0,\alpha)
\end{tikzcd}$$

where f and g are Lip-γ diffeomorphisms and:

$$\pi_{n,p,k} : (t_1, \ldots, t_n) \mapsto (t_1, \ldots, t_p, 0, \ldots, 0)$$

Define $V_y = \xi_{\tilde{y}}(U_0)$. Then $V_y = g^{-1} \circ \pi_{n,p,k}(B_{\mathbb{R}^n}(0,\alpha))$ and as such $(V_y, \mathcal{X})$ is naturally endowed with a structure of a smooth p-submanifold of E. According to lemma 5.25, for $\vec{t} \in U_0$:

$$T_{\xi_{\tilde{y}(\vec{t})}} V_y = \xi_{\tilde{y}}(\vec{t})_*(\mathbb{R}^n) \subseteq P_{\mathcal{D}}(\xi_{\tilde{y}}(\vec{t}))$$

As $\xi_{\tilde{y}(\vec{t})}(\vec{t}) \in \mathbb{L}_{y_0}(\mathcal{D})$, then $P_{\mathcal{D}}(\xi_{\tilde{y}(\vec{t})}(\vec{t}))$ is of the same dimension as $P_{\mathcal{D}}(y_0)$, i.e. p, which is exactly the rank of $\xi_{\tilde{y}|U_0}$. Therefore, we conclude that:

$$T_{\xi_{\tilde{y}}(\vec{t})} V_y = P_{\mathcal{D}}(\xi_{\tilde{y}}(\vec{t}))$$

Hence, V_y is a connected integral submanifold of $P_{\mathcal{D}}$. And by lemma 5.30, V_y is an open subset of $(\mathbb{L}_{y_0}(\mathcal{D}), \mathcal{Y})$.

- Let $\phi_y : V_y \to B_{\mathbb{R}^p}(0,\alpha)$ be defined as the composition of the following maps:

$$\phi_y : V_y \hookrightarrow V \xrightarrow{g} B_{\mathbb{R}^k}(0,\alpha) \xrightarrow{\pi_{k,p,p}} B_{\mathbb{R}^p}(0,\alpha)$$

Then ϕ_y is a local inverse of the Lip-γ map $\xi_{\tilde{y}} \circ f^{-1} \circ \pi_{p,p,n}$ and is Lip-γ with norm depending only on M_1 and M_2. Let $\tilde{\varphi}_y$ be a Lip-γ extension of it to E that vanishes off $B(y, M_1 + 1)$ and define $\varphi_y = (\tilde{\varphi}_y)_{|\mathbb{L}_{y_0}(\mathcal{D})}$. Then up to a rescaling, the family $((\varphi_y, V_y))_{y\in\mathbb{L}_{y_0}(\mathcal{D})}$ satisfy all the required conditions.

□

Notation. *With the notations of the previous lemma, for each* $y \in \mathbb{L}_{y_0}(\mathcal{D})$*, we denote by* $V_y^{1/2}$ *the set* $(\varphi_y)^{-1}_{|V_y}(B(0, 1/2))$*.*

We will use the following sufficient condition for a family to be locally finite:

Lemma 5.34. *Let* $(T_i)_{i\in I}$ *be an open cover to a topological manifold such that, for every* $i \in I$*,* $\{j \in I | T_j \cap T_i \neq \varnothing\}$ *is finite. Then* $(T_i)_{i\in I}$ *is locally finite.*

Lemma 5.35. *Let* E *be a normed vector space,* M *be a subset of* E *endowed with a topology such that the natural inclusion map* $i : M \hookrightarrow E$ *is continuous. Let* $R > 0$ *and assume that for every* $y \in M$*, there exist two open subsets of* M *containing* y *namely* V_y *and* $\tilde{V}_y$ *such that* $\tilde{V}_y \subseteq V_y \subseteq B_E(y, R)$*. Then there exists a countable subset* I *of* M *such that* $(\tilde{V}_i)_{i\in I}$ *covers* M *and such that* $(V_i)_{i\in I}$ *is locally finite.*

Proof. For every $n \in \mathbb{N}^*$, let I_n be finite subset of $(\overline{B(0, n+1)} - B(0, n)) \cap M$ such that $(\tilde{V}_y)_{y\in I_n}$ covers $(\overline{B(0, n+1)} - B(0, n)) \cap M$. Let $I = \cup_n I_n$. Then I is countable and $(\tilde{V}_y)_{y\in I}$ cover M. Let $y \in I$ and let $n \in \mathbb{N}$, then for every $N \in \mathbb{N}$ such that $N > R+n+1$ or $N < n - R$ and $x \in I_N$, $V_x \cap V_y = \varnothing$. Therefore $(V_i)_{i\in I}$ is locally finite by lemma 5.34. □

The following final theorem comes then as a natural conclusion of all the above (using the two families $(V_y^{1/2})_{\mathbb{L}_{y_0}(\mathcal{D})}$ and $(V_y)_{\mathbb{L}_{y_0}(\mathcal{D})}$ in lemma 5.35 to get a suitable countable cover for $\mathbb{L}_{y_0}(\mathcal{D})$):

Theorem 5.36. *Let* $\mathcal{D}$ *be a family of Lip-*γ *vector fields on* E *(*$\gamma \geq 2$*). Let* $y_0 \in E$*. Let* $M_1, M_2 \in (0, \infty)$ *and suppose that* $\mathcal{D}$ *is well controlled at* $y_0 \in E$ *with constants* (M_1, M_2)*. Then* $\mathbb{L}_{y_0}(\mathcal{D})$ *is a Lipschitz-*γ *submanifold of* E*.*

5.4 Perspectives

We identified in this chapter a quantitative condition under which the reachability set is a Lipschitz manifold. However, we have not made yet full use of the power of the Lipschitz geometry which we believe can give an answer to the following important open question: given $x \in G\Omega_p(\mathbb{R}^d)$ and an axis path $\tilde{x}$ such that $y_T(x) = y_T(\tilde{x})$, what can we say about the p-variation of $\tilde{x}$? It is our belief that solving such a problem requires working in the Lipschitz framework. In a Lipschitz manifold, a rough path can be broken into finitely many pieces (theorem 4.25) so that every piece is a rough path living in the domain of a single chart that is homeomorphic to the unit ball. Our problem is then amenable to working in a single chart and finding a "simple" smooth path living in the ball that gives the same terminal value to the solution as the rough path in question.

Another perspective is to develop a bit more the concept of locally Lipschitz manifolds so that one can solve RDEs on them. Two possible routes are offered to us: either doing this from scratch or to build a Lipschitz manifold structure on any given locally Lipschitz manifold -which would certainly be a non-canonical choice.

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