Time-Inconsistency: Performance of the Local Mean-Variance Optimal Portfolio

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1 Introduction

Portfolio selection is to seek the best allocation of wealth among a basket of securities. The mean variance formulation by Markowitz(1959,1989) provides a fundamental basis for portfolio selection analysis. Multiperiod portfolio selection problems have been studied later and the classical mean variance criteria was generalized. For example, maximizing the objective function $E(X_T) - wVar(X_T)$ $(w > 0)$ over all the possible asset allocation strategies is a generalization of the traditional mean variance formulation. Multiperiod problems can become complicated when different means of optimality are considered.

In multiperiod problems, typically an precommitment policy which specify the asset allocation strategies for all the periods beforehand maximize the objective function in the end. That is, at time zero, One need to decide the asset allocation strategies for all the subsequent periods which optimize the objective function. However, this policy is only optimal at time zero. If we stand at time $t(t > 0)$,then the precommitment policy may not be optimal in terms of maximizing the objective function.

This paper deals with a backwards policy which is different in terms of optimality for the multiperiod problem. Assume there are totally T periods, the initial wealth and the objective as a function of X_T are known. As usual, the statistical properties of the assets that can be invested in are known. The backwards policy is, at first, assuming X_{T-1} is known and we treat the last period as a single period problem. That is we find U_{T-1}, which represent the asset allocation strategy for the period $T-1$ to T which maximize the objective function for a given X_{T-1}. Secondly, given fixed U_{T-1} from above and X_{T-2}, find the optimal U_{T-2} which maximize the objective function. Repeat the above process until time zero, we get a sequence $U_i, i = 0, 1, ..., T-1$ which is the investment strategy of this backward policy.

The precommitment policy is not optimal dynamically. That is, for some integer t $(0 < t < T)$, the sequence of $U_i, i = t, ..., T-1$ taken from the precommitment policy may not be optimal in terms of maximizing the objective function. This kind of optimality in multiperiod portfolio selection problem is called time inconsistent, because the optimality fails as time changes. On the contrary, the backward policy is in some sense dynamically optimal.

It is obviously that the objective function from the backwards policy can not reach a larger value than the objective function from the precommitment policy. However, if the target function is in some special form, these two policies lead to the same result.

If we use $E(X_T) - wVar(X_T)$ as our objective function, then the precommitment policy and the backwards policy provide different results. Li and Ng(2000) gives an analytical solution of the precommitment policy for this objective function. This paper deals with the same problem through the backwards policy. The analytical results are given for two periods cases and the algorithm is given for multiperiod cases. At last, a simple numerical example is illustrated.

2 Market Settings for The Multiperiod Portfolio Selection Problem

We consider a capital market with $(n+1)$ risky asset, with random rates of returns. At time zero, An investor start with initial wealth X_0. The investor can allocate his wealth among the $(n+1)$ assets at the beginning of each of the following $T-1$ periods. The rates of return of the risky assets at the time period t within the planning horizon are denoted by a vector $r_t = [r_t^0, r_t^1, ..., r_t^n]'$, where r_t^i is the random return for the asset i at the time period t.

It is assumed in this paper that vectors $r_t, t = 1, 2, ...T-1$, are statistically independent and return r_t has a known mean

$E(r_t) = [E(r_t^0), E(r_t^1), ..., E(r_t^n)]'$

For convenience, we denote the covariance matrix of the asset returns during the time period t to be

$\Omega_t = cov(r_t) = E(r_t r_t') - E(r_t)E(r_t')$

Let X_t be the wealth at the beginning of the tth period, and let $U_t = [U_t^0, U_t^1, ..., U_t^n]'$ be the wealth allocation strategy of the t-th period. U_t^i If we denote $\mathbf{1} = [1, 1, ..., 1]'$ which has $(n+1)$ entries, then we have $\mathbf{1}^T U_0 = X_0$ as the initial wealth constraint.

3 The Classical Mean-Variance Solutions

The classical mean-variance solution for the precommitment policy is done by Li and Ng(2000). The market settings are in part 2 except here we denote the optimal solution to be $u_t = [u_t^1, u_t^2, ..., u_t^n]'$, u_t^1 $(i = 1, ..., n)$ is the amount invested in the i-th asset. Then the amount invested on the 0-th asset is $X_t - \sum_{i=1}^n u_t^i$.

Formally, the statement of the problem (P1) is

max $E(X_T) - wVar(X_T)$, $w > 0$

s.t. $X_{t+1} = r_t^0 X_t + P_t' u_t$, $t = 0, 1, ..., T-1$

where $P_t = [p_t^1, p_t^2, ..., p_t^n] = [(r_t^1 - r_t^0), (r_t^2 - r_t^0), ..., (r_t^n - r_t^0)]'$

Define

$$B_t^0 = E(P_t)E^{-1}(P_t P_t')E(P_t), t = 0, 1, ..., T-1 \tag{3.1}$$

$$A_t^1 = E(r_t^0) - E(P_t')E^{-1}(P_t P_t')E(r_t^0 P_t), t = 0, 1, ..., T-1 \tag{3.2}$$

$$A_t^2 = E((r_t^0)^2) - E(r_t^0 P_t')E^{-1}(P_t P_t')E(r_t^0 P_t), t = 0, 1, ..., T-1 \tag{3.3}$$

$$B_t^1 = B_t^0 \frac{\Pi_{k=t+1}^{T-1} A_k^1}{2\Pi_{k=t+1}^{T-1} A_k^2}, t = 0, 1, ..., T-1 \tag{3.4}$$

$$B_t^2 = B_t^0 (\frac{\Pi_{k=t+1}^{T-1} A_k^1}{2\Pi_{k=t+1}^{T-1} A_k^2})^2, t = 0, 1, ..., T-1 \tag{3.5}$$

Where in equations (3.4) and (3.5) $\Pi_{k=t+1}^{T-1} A_k^i, i = 1, 2$, are defined to equal to one.

Define further

$$\mu = \Pi_{k=0}^{T-1} A_k^1 \tag{3.6}$$

$$\nu = \Sigma_{t=0}^{T-1} (\Pi_{k=t+1}^{T-1} A_k^1) B_t^1 \tag{3.7}$$

$$\tau = \Pi_{k=0}^{T-1} A_k^2 \tag{3.8}$$

$$a = \frac{\nu}{2} - \nu^2 \tag{3.9}$$

$$b = \frac{\nu\mu}{a} \tag{3.10}$$

$$c = \tau - \mu^2 - ab^2 \tag{3.11}$$

the optimal multiperiod portfolio policy for the problem (P1) is specified by the following analytical form:

$$\begin{aligned} U_t &= -E^{-1}(P_t P_t') E(r_t^0 P_t) X_t \\ &+ \frac{1}{2}(bX_0 + \frac{\nu}{2wa})(\Pi_{k=t+1}^{T-1} \frac{A_k^1}{A_k^2}) E^{-1}(P_t P_t') E(P_t) \\ & t = 0, 1, ..., T-2 \end{aligned} \tag{3.12}$$

$$\begin{aligned} U_{T-1} &= -E^{-1}(P_{T-1} P_{T-1}') E(r_{T-1}^0 P_{T-1}) X_{T-1} \\ &+ \frac{1}{2}(bX_0 + \frac{\nu}{2wa}) E^{-1}(P_{T-1} P_{T-1}') E(P_{T-1}) \end{aligned} \tag{3.13}$$

The optimal solution to problem (P1) is a precommitment policy and it is time inconsistent. This paper deals with a time consistent problem (P2). The market settings for (P2) is in part 2. The statement of (P2) is

Find the sequence of U_t which satisfy the following properties:

(i)Given fixed X_{T-1}, U_{T-1} maximize $E(X_T) - wVar(X_T)$, $w > 0$

(ii)For fixed U_{T-1} which is a function of X_{T-1} and fixed X_{T-2}, U_{T-2} maximize $E(X_T) - wVar(X_T)$.

(iii)Continue the above steps backwardly until we find U_0

4 Time Inconsistency and The Mean-Variance Target Function

Although (P1) and (P2) have different definitions, we have not proved they will give different results. For certain objective functions, the precommitment policy and the backwards policy give the same results on the asset allocation strategy. That is, if we change the objective function in (P1) and (P2), they may give the same result. So the time consistency of the optimal asset allocation strategy from the precommitment policy is related to the objective function. For example, if the objective function can be written as $E(f(X_T))$ for some deterministic function f, then

i)Under the precommitment policy: The sequence of $U_i, i = 0, 1, ..., T-1$ maximize $E(f(X_T))$ at time zero.

ii)Under the backwards policy: U_{T-1} is the solution of the single period problem which maximize $E(f(X_T))$ for fixed X_{T-1}. If we fix this U_{T-1}, under a fixed value of X_{T-2}, then $E(f(X_T) \mid X_{T-1})$ become a random variable since X_{T-1} is random. Here we have U_{T-2} maximize $E(E(f(X_T) \mid X_{T-1}) \mid X_{T-2})$ by the definition of the backwards policy. Since the random returns of the assets in different time periods are assumed to be independent and by the tower property of conditional expectation, we have

$$E(E(f(X_T) \mid X_{T-1}) \mid X_{T-2}) = E(f(X_T) \mid X_{T-2}) \tag{4.1}$$

This means U_{T-1}, U_{T-2} under the backwards policy maximize $E(f(X_T))$ for fixed X_{T-2}. So if there are only two periods, the asset allocation strategy for backwards policy and the precommitment policy are the same. If the number of periods is larger than two, we just take conditional expectation more times, by the same reason, we can find that the sequence of $U_i, i = 0, 1, ..., T-1$ from the backwards policy indeed maximize $E(f(X_T))$ at time zero.

However, the result from the precommitment policy become time inconsistent if we choose the target function to be $E(X_T) - wVar(X_T)$. Here we have

$$E(X_T) - wVar(X_T) = E(X_T) - w[E(X_T^2) - E(X_T)^2] \tag{4.2}$$

The right hand side can not be written in the form of $E(f(X_T))$ because of the term $E(X_T)^2$. Assume for any random variable X, there is a function f, such that $E(X_T)^2 = E(f(X_T))$. Then, if we let X to be 1 with probability one, we must have $f(1) = 1$. Second, let X be 1 or -1 with probability 0.5 each. Here we get $f(-1) = -1$.

However, if we choose X to be -1 with probability one, we must have $f(-1) = 1$. The contradiction shows that there is no function f such that $E(X_T)^2 = E(f(X_T))$.

To show that the results from the two different policies for this mean-variance objective function are not the same, it is enough to consider the case when T = 2.

i)For the precommitment policy: $U_i, i = 0, 1$ maximize $E(X_T) - wVar(X_T)$ at time zero.

ii)For the backwards policy: U_1 maximize $E(X_T) - wVar(X_T)$ at time 1 and U_2 maximize $E(E(X_T) - wVar(X_T) \mid X_1)$ at time zero.

Since for positive w

$$E(E(X_T) - wVar(X_T) \mid X_1) \neq E(X_T) - wVar(X_T) \tag{4.3}$$

So the asset allocation strategy is different from precommitment one. Essentially, the difference is because the time-t variance exceeds the expected variance at time $t + \tau$ ($\tau > 0$) by the law of total volatility.(Basak 2008 , Weiss 2005)

$$Var_t[X_T] = E_t[Var_{t+\tau}(X_T)] + Var_t[E_{t+\tau}(W_T)], \ \tau > 0 \tag{4.4}$$

That is, as times changes, the expected variance of the terminal wealth also changes. So the result from (P1) is not optimal to the criterias in (P2).

5 Solution Under The Time Consistency Framework

5.1 mean-variance efficient frontier

To find our the optimal asset allocation strategy of the backward policy with target function $E(X_T) - wVar(X_T)$, we solve it in a backward way and use the law of total volatility. One advantage of this target function is that once we solve this problem, we can get the solution of the traditional mean-variance formulation problem. That is, minimize $var(X_T)$ for fixed $E(X_T)$ or maximize $E(X_T)$ for fixed $var(X_T)$. For convenience, here are some notations. for $i = 0, 1, ..., T-1$

$$A_i = \mathbf{1}^T \Omega_i^{-1} \mathbf{1} \tag{5.1}$$

$$B_i = \mathbf{1}^T \Omega_i^{-1} E(r_i) \tag{5.2}$$

$$C_i = E(r_i') \Omega_i^{-1} E(r_i) \tag{5.3}$$

$$\Delta_i = A_i C_i - B_i^2 \tag{5.4}$$

Firstly, we need the mean-variance efficient frontier of the single period problem in our settings.

Theorem: For the market settings in part 2, consider only the first period. Among all the possible asset allocation strategies, we choose U_0 such that for a given value of $E(X_1)$, $Var(X_1)$ is minimized. If we implement this U_0, $E(X_1)$, $Var(X_1)$ must have the following relationship

$$Var(X_1) = U_0' \Omega_0 U_0 = (A_0 E^2(X_1) - 2B_0 X_0 E(X_1) + C_0 X_0^2)/\Delta_0 \tag{5.5}$$

Proof of the theorem:

The two wealth constraints are: $X_0 = \mathbf{1}^T U_0$, $X_1 = r_0 U_0$

Assume $E(X_1)$ is fixed, we minimize $Var(X_1)$

Use the Lagrangian method and let λ_1 andλ_2 be the Lagrangian multipliers.

$$L = Var(X_1) - \lambda_1(X_0 - \mathbf{1}^T U_0) - \lambda_2(X_1 - r_0 U_0) \tag{5.6}$$

$$\partial L/\partial \lambda_1 = 0 \ \Rightarrow \ \mathbf{1}^T \Omega_0^{-1}(\lambda_1 \mathbf{1} + \lambda_2 E(r_0)) = X_0 \tag{5.7}$$

$$\partial L/\partial \lambda_2 = 0 \ \Rightarrow \ E(r_0)\Omega_0^{-1}(\lambda_1 \mathbf{1} + \lambda_2 E(r_0)) = E(X_1) \tag{5.8}$$

$$\partial L/\partial U_0^i = 0, i = 0, 1, ..., n \ \Rightarrow \ U_0 = \Omega_0^{-1}(\lambda_1 \mathbf{1} + \lambda_2 E(r_0)) \tag{5.9}$$

Solve equations (5.7) and (5.8), we have

$$\lambda_1 = (C_0 X_0 - B_0 E(X_1))/\Delta_0 \tag{5.10}$$

$$\lambda_2 = (A_0 E(X_1) - B_0 X_0)/\Delta_0 \tag{5.11}$$

Take the expressions of λ_1 and λ_2 into (5.9), we get the efficient frontier

$$Var(X_1) = U_0' \Omega_0 U_0 = (A_0 E^2(X_1) - 2B_0 X_0 E(X_1) + C_0 X_0^2)/\Delta_0 \tag{5.12}$$

5.2 solution to the last period

Consider the last step as a single period problem: For given X_{T-1}, we find U_{T-1} such that $E(X_T) - wVar(X_T)$ is maximized. We have $Var(X_T) = U_{T-1}' \Omega_{T-1} U_{T-1}$ and $X_T = U_{T-1}' r_{T-1}$. The wealth constraint is $\mathbf{1}^T U_{T-1} = X_{T-1}$.

Let λ be the Lagrangian multiplier. We have

$$L = E(X_T) - wU_{T-1}' \Omega_{T-1} U_{T-1} - \lambda(\mathbf{1}^T U_{T-1} - X_{T-1}) \tag{5.13}$$

$$\partial L/\partial U_{T-1}^i = 0, \ i = 0, 1, ..., n$$

$$\Rightarrow E(r_{T-1}) - 2w\Omega_{T-1} U_{T-1} - \lambda \mathbf{1} = 0$$

$$\Rightarrow \ U_{T-1} = \frac{1}{2w} \Omega_{T-1}^{-1}(E(r_{T-1}) - \lambda \mathbf{1}) \tag{5.14}$$

$$\partial L/\partial \lambda = 0 \ \Rightarrow \ \mathbf{1}^T U_{T-1} = X_{T-1}$$

$$\Rightarrow\ \mathbf{1}^T \frac{1}{2w}\Omega_{T-1}^{-1}(E(r_{T-1}) - \lambda\mathbf{1}) = X_{T-1}$$

$$\Rightarrow\ \lambda = \frac{B_{T-1} - 2wX_{T-1}}{A_{T-1}} \tag{5.15}$$

Take the value of λ into the expression of U_{T-1}, we have

$$U_{T-1} = \frac{1}{2w}\Omega_{T-1}^{-1}(E(r_{T-1}) - \frac{B_{T-1} - 2wX_{T-1}}{A_{T-1}}\mathbf{1}) \tag{5.16}$$

And we can write it as

$$U_{T-1} = a_{T-1}X_{T-1} + b_{T-1} \tag{5.17}$$

where

$$a_{T-1} = \frac{\Omega_{T-1}^{-1}\mathbf{1}}{A_{T-1}} \tag{5.18}$$

$$b_{T-1} = \frac{1}{2w}\Omega_{T-1}^{-1}(E(r_{T-1}) - \frac{B_{T-1}\mathbf{1}}{A_{T-1}}) \tag{5.19}$$

Then we have

$$\begin{aligned} E_{T-1}(X_T) &= E(r'_{T-1})U_{T-1} \\ &= E(r'_{T-1})a_{T-1}X_{T-1} + E(r'_{T-1})b_{T-1} \end{aligned} \tag{5.20}$$

and

$$\begin{aligned} Var_{T-1}(X_T) &= U'_{T-1}\Omega_{T-1}U_{T-1} \\ &= a'_{T-1}\Omega_{T-1}a_{T-1} + 2a'_{T-1}\Omega_{T-1}b_{T-1} + b'_{T-1}\Omega_{T-1}b_{T-1} \end{aligned} \tag{5.21}$$

Here U_{T-1}, $E_{T-1}(X_T)$ are linear on X_{T-1} and $Var_{T-1}(X_T)$ is quadratic on X_{T-1}. Later we will show that for any integer k ($k < T$), U_k and $E_k(X_T)$ are linear on X_k, $Var_k(X_T)$ is quadratic on X_k.

5.3 backwards recursive relationship

In order to get all the solutions $U_k,\ k = 0, 1, ..., T-1$, we will investigate the recursive relationship between U_k and U_{k-1}. Since we know U_{T-1}, $E_{T-1}(X_T)$ and $Var_{T-1}(X_T)$, we can calculate these value at time $T-2$. For convenience, let

$$E_{T-1}(X_T) = m_{T-1}X_{T-1} + n_{T-1} \tag{5.22}$$

and

$$Var_{T-1}(X_T) = \alpha_{T-1}X_{T-1}^2 + \beta_{T-1}X_{T-1} + \gamma_{T-1}. \tag{5.23}$$

Then we have

$$\begin{aligned} E_{T-2}(X_T) &= E_{T-2}(E_{T-1}(X_T)) \\ &= m_{T-1}E_{T-2}(X_{T-1}) + n_{T-1} \end{aligned} \tag{5.24}$$

and

$$\begin{aligned} Var_{T-2}(X_T) &= E_{T-2}(Var_{T-1}(X_T)) + Var_{T-2}(E_{T-1}(X_T)) \\ &= E_{T-2}(\alpha_{T-1}X_{T-1}^2 + \beta_{T-1}X_{T-1} + \gamma_{T-1}) + m_{T-1}^2 Var_{T-2}(X_{T-1}) \\ &= \alpha_{T-1}E_{T-2}^2(X_{T-1}) + \beta_{T-1}E_{T-2}(X_{T-1}) + \gamma_{T-1} \\ &\quad +(m_{T-1}^2 + \alpha_{T-1})Var_{T-2}(X_{T-1}) \end{aligned} \tag{5.25}$$

Our aim is to maximize $E_{T-2}(X_T) - wVar_{T-2}(X_T)$

$$\begin{aligned} & E_{T-2}(X_T) - wVar_{T-2}(X_T) \\ = \; & m_{T-1}E_{T-2}(X_{T-1}) + n_{T-1} - w[\alpha_{T-1}E_{T-2}^2(X_{T-1}) + \beta_{T-1}E_{T-2}(X_{T-1}) \\ & +\gamma_{T-1} + (m_{T-1}^2 + \alpha_{T-1})Var_{T-2}(X_{T-1})] \end{aligned} \tag{5.26}$$

The optimal solution can be obtained by using the mean variance efficient frontier. (5.26) is quadratic on $E_{T-2}(X_{T-1})$ if we take the expression of $Var_{T-2}(X_{T-1})$ into it. Where

$$Var_{T-2}(X_{T-1}) = \frac{A_{T-2}E_{T-2}^2(X_{T-1}) - 2B_{T-2}X_{T-2}E_{T-2}(X_{T-1}) + C_{T-2}X_{T-2}^2}{\Delta_{T-2}} \tag{5.27}$$

Thus we can get the optimal value of $E_{T-2}(X_{T-1})$ by finding the maximum point of the quadratic equation (5.26). Then we can get the optimal solution U_{T-2} by taking $E_{T-2}(X_{T-1})$ back into equation(5.9). It is easy to verify that $E_{T-2}(X_{T-1})$ is linear on X_{T-2} and thus U_{T-2} is linear on X_{T-2}. From (5.25), (5.27) we can see that $Var_{T-2}(X_T)$ is quadratic on X_{T-2}. Thus by induction, we can see that the conditional mean and the optimal policy are in linear form of the wealth, the conditional variance is in quadratic form of wealth. So we can make the following assumptions, for $k = 0, 1, ..., T-1$

There are constant parameters

$a_k, b_k, m_k, n_k, \alpha_k, \beta_k, \gamma_k$, such that

$$U_k = a_k X_k + b_k \tag{5.28}$$

$$E_k(X_T) = m_k X_k + n_k \tag{5.29}$$

$$Var_k(X_T) = \alpha_k X_T^2 + \beta_k X_T + \gamma_k \tag{5.30}$$

We know the expression of (5.28), (5.29), (5.30) for $k = T-1$. If we can find the recursive relationship for the set of parameters

$a_k, b_k, m_k, n_k, \alpha_k, \beta_k, \gamma_k$, We can get the optimal strategies U_k, $k = 0, 1, ..., T-1$. For any positive integer k ($k < T-1$), by (5.26) we have

$$\begin{aligned} & E_{k-1}(X_T) - wVar_{k-1}(X_T) \\ = \; & m_k E_{k-1}(X_k) + n_k - w[\alpha_k E_{k-1}^2(X_k) + \beta_k E_{k-1}(X_k) + \\ & \gamma_k + (m_k^2 + \alpha_k) Var_{k-1}(X_k)] \end{aligned} \tag{5.31}$$

By the efficient frontier, we can further write (5.31) into

$$\begin{aligned} & E_{k-1}(X_T) - wVar_{k-1}(X_T) \\ = \; & m_k E_{k-1}(X_k) + n_k - w[\alpha_k E_{k-1}^2(X_k) + \beta_k E_{k-1}(X_k) + \gamma_k + \\ & (m_k^2 + \alpha_k)(A_{k-1} E_{k-1}^2(X_k) - 2B_{k-1} X_{k-1} E_{k-1}(X_k) + C_{k-1} X_{k-1}^2)/\Delta_{k-1}] \end{aligned} \tag{5.32}$$

To get maximum value,

$$\begin{aligned} E_{k-1}(X_k) & = \frac{m_k - w\beta_k + w(m_k^2 + \alpha_k) 2B_{k-1} X_{k-1}/\Delta_{k-1}}{2w(\alpha_k + (m_k^2 + \alpha_k)) A_{k-1}/\Delta_{k-1})} \\ & \equiv p_k X_{k-1} + q_k \end{aligned} \tag{5.33}$$

where

$$p_k = \frac{(m_k^2 + \alpha_k) B_{k-1}/\Delta_{k-1}}{\alpha_k + (m_k^2 + \alpha_k)) A_{k-1}/\Delta_{k-1}},$$
$$q_k = \frac{m_k - w\beta_k}{2w(\alpha_k + (m_k^2 + \alpha_k)) A_{k-1}/\Delta_{k-1})}$$

Then, by (5.9)

$$\begin{aligned} U_{k-1} & = \Omega_{k-1}^{-1}[\frac{(C_{k-1} X_{k-1} - B_{k-1} E_{k-1}(X_k))}{\Delta_{k-1}} \mathbf{1} + \frac{A_{k-1} E_{k-1}(X_k) - B_{k-1} X_{k-1}}{\Delta_{k-1}} E(r_{k-1})] \\ & \equiv c_k^1 E_{k-1}(X_k) + c_k^2 X_{k-1} \end{aligned} \tag{5.34}$$

where

$$c_k^1 = \frac{\Omega_{k-1}^{-1}}{\Delta_{k-1}}[A_{k-1}E(r_{k-1}) - B_{k-1}\mathbf{1}]$$

$$c_k^2 = \frac{\Omega_{k-1}^{-1}}{\Delta_{k-1}}[C_{k-1}\mathbf{1} - B_{k-1}E(r_{k-1}]$$

By (5.33), (5.34) can be written as

$$U_{k-1} = (c_k^1 p_k + c_k^2)X_{k-1} + c_k^1 q_k$$

By definition, $U_{k-1} = a_{k-1}X_{k-1} + b_{k-1}$. So

$$a_{k-1} = c_k^1 p_k + c_k^2 \tag{5.35}$$

$$b_{k-1} = c_k^1 q_k \tag{5.36}$$

Also we have

$$m_{k-1} = E(r_{k-1})a_{k-1} \tag{5.37}$$

$$n_{k-1} = E(r_{k-1})b_{k-1} \tag{5.38}$$

To get the values of $\alpha_{k-1}, \beta_{k-1}, \gamma_{k-1}$, we have to find $Var_{k-1}(X_T)$. By (5.25)

$$Var_{k-1}(X_T) = \alpha_k E_{k-1}^2(X_k) + \beta_k E_{k-1}(X_k) + \gamma_k + (m_k^2 + \alpha_k)Var_{k-1}(X_k)$$

substitute the value of $E_{k-1}(X_k)$ and $Var_{k-1}(X_k)$ and organize the coefficients. Finally we have

$$\alpha_{k-1} = \alpha_k p_k^2 + \frac{m_k^2 + \alpha_k}{\Delta_{k-1}}(A_{k-1}p_k^2 - 2B_{k-1}p_k + C_{k-1}) \tag{5.39}$$

$$\beta_{k-1} = 2\alpha_k p_k q_k + p_k \beta_k + \frac{m_k^2 + \alpha_k}{\Delta_{k-1}}(2A_{k-1}p_k q_k - 2B_{k-1}q_k) \tag{5.40}$$

$$\gamma_{k-1} = \alpha_k q_k^2 + q_k \beta_k + \gamma_k + \frac{m_k^2 + \alpha_k}{\Delta_{k-1}} A_{k-1} q_k^2 \tag{5.41}$$

The above recursive relationship and the initial values at $T-1$ enable us to get all asset allocation strategies over time. Since the recursive relationship is already quite complicated in expression, it is not easy to find simple analytical expressions for $U_i, i = 0, 1, ..., T-1$ as in Li and Ng (2000) where they give out an analytical solution for the precommitment policy.

6 Numerical Illustration

Consider the case studied by Li and Ng (2000). We assume there is a stationary multiperiod process with $T = 3$. An investor has one unit of wealth at beginning. The investor is trying to find the best allocation of his wealth among three risky assets, A, B and C, in order to find the time consistent optimal strategy which maximize the objective function $E(X_T) - wVar(X_T)$. $E(r_t^A) = 1.162$, $E(r_t^B) = 1.246$, $E(r_t^C) = 1.228$, $t = 0, 1, 2, 3$. $r_t = [r_t^A, r_t^B, r_t^C]$, the covariance matrix $Cov(r_t)$ is

$$\begin{pmatrix} 0.0146 & 0.0187 & 0.0145 \\ 0.0187 & 0.0854 & 0.0104 \\ 0.0145 & 0.0104 & 0.0289 \end{pmatrix}$$

We take the constant w to be 0.5 in our calculation.

The results of the parameters in section 5.3, $a_k, b_k, m_k, n_k, \alpha_k, \beta_k, \gamma_k$, are

$a_2 = [-7.88032.12625.7541]'$, $b_2 = [1.1040 - 0.0709 - 0.0332]'$, $m_2 = 0.5584$, $n_2 = 1.1539$, $\alpha_2 = 0.0143$, $\beta_2 = 0$, $\gamma_2 = 0.5584$

$a_1 = [4.2177 - 0.9109 - 2.3067]'$, $b_1 = [-15.89494.288611.6063]'$, $m_1 = 0.9332$, $n1 = 1.1263$, $\alpha_1 = 0.0456$, $\beta_1 = -0.2601$, $\gamma_1 = 1.3173$

$a_0 = [8.4928 - 2.0644 - 5.4284]'$, $b_0 = [-15.84464.275011.5696]'$, $m_0 = 0.6303$, $n_0 = 1.1227$, $\alpha_0 = 0.4811$, $\beta_0 = -2.0290$, $\gamma_0 = 3.1515$

So we can get all the information we need from (5.28), (5.29), (5.30). For example, the optimal strategy at the first period is $U_0 = a_0 + b_0 = [-7.3518, 2.2106, 6.1412]'$.

7 Conclusion

In this paper, the time consistency issue in multperiod portfolio selection problem is discussed and the solution algorithm is given for a mean-variance objective function within the time consistency framework. In practice, it is very reasonable that the time consistency optimality should be considered in the portfolio selection process. The concept of time consistency optimality can enhance investors' understanding of the optimal decision process in multiperiod portfolio selection scenario. However, this paper only deals with the special mean-variance objective functions. If one choose a different objective function, the algorithm will be very different in general. But the methodology of deriving the algorithm can be useful for other objective functions.

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