On a problem of Erdős and Moser

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Abstract

A set A of vertices in an r-uniform hypergraph $\mathcal{H}$ is *covered in* $\mathcal{H}$ if there is some vertex $u \notin A$ such that every edge of the form $\{u\} \cup B$, $B \in A^{(r-1)}$ is in $\mathcal{H}$. Erdős and Moser (1970) determined the minimum number of edges in a graph on n vertices such that every k-set is covered. We extend this result to r-uniform hypergraphs on sufficiently many vertices, and determine the extremal hypergraphs. We also address the problem for directed graphs.

1 Introduction

Let $\mathcal{H}$ be an r-uniform hypergraph with vertex set X. We say that a set $A \subset X$ is *covered in* $\mathcal{H}$ if there is $u \notin A$ such that, for every $B \in A^{(r-1)}$ we have $B \cup \{u\} \in \mathcal{H}$; we say that u *covers* A *(in* $\mathcal{H}$*).* Let $f(n, k, r)$ be the minimum number of edges in an r-uniform hypergraph $\mathcal{H}$ with vertex set $[n] = \{1, \dots, n\}$ such that every k-set in $[n]$ is covered in $\mathcal{H}$.

The problem of determining $f(n, k, r)$ for graphs ($r = 2$) was raised by Erdős and Moser in [3]. Subsequently, they proved in [4] that, for $r = 2$ and all $n > k$,

$$f(n, k, 2) = (k-1)(n-1) - \binom{k-1}{2} + \left\lceil \frac{n-k+1}{2} \right\rceil,$$

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with a unique extremal graph. Our aim in this paper is to generalise this to larger values r: for fixed k, r and sufficiently large n, we will determine the value of $f(n,k,r)$ and find the extremal graphs.

We begin with a construction giving an upper bound on $f(n,k,r)$. Let $D(n,r)$ be the minimum number $|\mathcal{H}|$ of edges in an r-uniform hypergraph $\mathcal{H}$ with vertex set $[n]$ such that every $(r-1)$-set is contained in some element of $\mathcal{H}$. It is clear that $D(n,r) \geq \binom{n}{r-1}/\binom{r}{r-1}$. In fact, this bound is close to optimal: for fixed r, we have

$$D(n,r) = (1+o(1))\binom{n}{r-1}\Big/\binom{r}{r-1} \sim n^{r-1}/r!.$$

This follows as a (very) special case of an important result of Rödl [10]. There is also a simple construction (pointed out to us by Noga Alon): for $c \in [n]$, consider the set $\{A \in [n]^{(r)} : \sum_{a\in A} a \equiv c \pmod n\}$. This covers all $(r-1)$-sets except those B for which $c - \sum_{b\in B} \in B \pmod n$. There are only $O(n^{r-2})$ of these, so we can cover them one at a time.

Now for $n > k \geq r-1$, let $\mathcal{G}(n,k,r)$ be the family of r-uniform hypergraphs $\mathcal{G}$ on vertex set $[n]$ that can be constructed as follows:

- add all edges that meet any of the vertices $\{1, \ldots, k-r+1\}$;
- on vertices $\{k-r+2, \ldots, n\}$, we add a collection of $D(n-k+r-1, r)$ edges such that every $(r-1)$-set in $\{k-r+2, \ldots, n\}$ is contained in at least one of these edges.

Let us check that for $\mathcal{H} \in \mathcal{G}(n,k,r)$, every k-set B of vertices is covered in $\mathcal{H}$: if $[k-r+1] \not\subseteq B$ then any element i of $[k-r+1]\setminus B$ covers B. Otherwise, $B \supseteq [k-r+1]$, so $B \setminus [k-r+1]$ has size $r-1$ and is therefore contained in some edge C added in the second step of the construction: the unique element u of $C \setminus B$ then covers B. We note that our construction generalizes that of Erdős and Moser: for $r=2$, $\mathcal{G}(n,k,r)$ consists of a unique graph up to isomorphism.

Let $g(n,k,r)$ be the number of edges in each element of $\mathcal{G}(n,k,r)$, so

$$\begin{aligned} g(n,k,r) &= \binom{n}{r} - \binom{n-k+r-1}{r} + D(n-k+r-1, r) \qquad (1) \\ &= (k-r+1+1/r+o(1))\frac{n^{r-1}}{(r-1)!}. \end{aligned}$$

Furthermore, if there exists a Steiner system with parameters $(n-k+r-1, r, r-1)$ (in other words, a perfect covering of $(r-1)$-sets by r-sets) then

$$g(n,k,r) = \binom{n}{r} - \binom{n-k+r-1}{r} + \binom{n-k+r-1}{r-1}/r. \qquad (2)$$

By a recent major result of Keevash [5], such designs exist for infinitely many values of $n-k+r+1$, and so (2) holds for infinitely many values of n.

The main result of this paper is to show that, for fixed k, r and all sufficiently large n, the construction above is optimal (and that all hypergraphs of minimal size can be constructed in this way).

Theorem 1. *For every k, r there is an integer $n_0(k, r)$ such that for all $n > n_0$,*

$$f(n,k,r) = g(n,k,r).$$

Furthermore, if $\mathcal{H}$ is an r-uniform hypergraph with n vertices and $f(n,k,r)$ edges in which every k-set of vertices is covered, then $\mathcal{H}$ is isomorphic to some element of $\mathcal{G}(n,k,r)$.

We note that Theorem 1 implies that, for $n > n_0(k, r)$,

$$f(n,k,r) \geq \binom{n}{r} - \binom{n-k+r-1}{r} + \binom{n-k+r-1}{r-1}/r,$$

with equality for infinitely many values of n.

We will prove Theorem 1 in the next section; further discussion, and results for directed graphs, can be found in the final section. We use standard notation throughout: in particular, $[n] = \{1, \ldots, n\}$, $X^{(r)} = \{A \subset X : |A| = r\}$ and, for a hypergraph $\mathcal{H}$, we write $|\mathcal{H}|$ for the number of edges in $\mathcal{H}$.

2 Covering hypergraphs

The upper bound in Theorem 1 follows immediately from the construction given in the introduction. We will prove the lower bound by an induction on r, for which we will need two lemmas: the first lemma shows that, for fixed k, r and large n, an extremal hypergraph must have $k-r+1$ vertices with degree close to the maximum possible; the second will allow us to show that if an extremal hypergraph has $k-r+1$ vertices of almost maximum possible degree then it must in fact belong to $\mathcal{G}(n,k,r)$. We will state and prove both lemmas, and then complete the proof of Theorem 1 at the end of the section.

Recall that if $\mathcal{H}$ is an r-uniform hypergraph then the *shadow* $\partial\mathcal{H}$ of $\mathcal{H}$ is the $(r-1)$-uniform hypergraph consisting of all $(r-1)$-sets that are contained in some edge of $\mathcal{H}$. We will need a version of the Kruskal-Katona Theorem ([7], [6]; see also [2], p. 30). It will be convenient to use it in the simplified form due to Lovász [9]: for $x \in [r,\infty)$, if $\mathcal{H}$ is an r-uniform hypergraph with $\binom{x}{r}$ edges, then $\partial\mathcal{H}$ has at least $\binom{x}{r-1}$ edges.

We can now state the first lemma. This will be used in our inductive argument, so if $r > 2$ we will be able to assume that Theorem 1 holds for $r-1$.

Lemma 2. *Let $k \geq r \geq 2$ be fixed, and if $r > 2$ suppose that Theorem 1 holds with $r-1$ in place of r (we do not assume anything if $r = 2$). Let $\epsilon > 0$ and let n be sufficiently large (depending on k, r, ϵ). Let $\mathcal{H}$ be an r-uniform hypergraph with n vertices and $f(n,k,r)$ edges, and suppose that every k-set of vertices is covered in $\mathcal{H}$. Then $\mathcal{H}$ has $k-r+1$ vertices of degree at least $(1-\epsilon)\binom{n-1}{r-1}$.*

Proof. Let V and E be the vertex and edge sets of $\mathcal{H}$ respectively, and write $m = |E| = f(n,k,r)$. Note that $m \leq g(n,k,r)$, which by (1) is at most $(k-r+1+1/r+o(1))n^{r-1}/(r-1)! = O(n^{r-1})$. Let $d_1 \geq d_2 \geq \cdots \geq d_n$ be the degree sequence of $\mathcal{H}$, and suppose that $d_{k-r+1} < (1-\epsilon)\binom{n-1}{r-1}$. Define η by $(1-\eta)^{r-1} = 1-\epsilon$.

Let v be a vertex of minimal degree in $\mathcal{H}$, and let $\mathcal{H}_v$ be the neighbourhood hypergraph of v, that is the $(r-1)$-uniform hypergraph with vertex set $V\setminus v$, and edge set $\{e \setminus v : v \in e \text{ and } e \in E\}$. Since every k-set is covered in $\mathcal{H}$, it follows that if $r \geq 3$ then every $(k-1)$-set is covered in $\mathcal{H}_v$ (for any $(k-1)$-set $A \subset V\setminus v$, the set $A \cup \{v\}$ is covered in $\mathcal{H}$ by some u; then u covers A in $\mathcal{H}_v$). Thus $\mathcal{H}_v$ has at least $f(n-1,k-1,r-1)$ edges. It follows by assumption that $f(n-1,k-1,r-1) = g(n-1,k-1,r-1)$ for sufficiently large n, and so $\mathcal{H}$ has minimal degree

$$\begin{aligned}\delta(\mathcal{H}) &\geq g(n-1,k-1,r-1)\\ &\sim (k-r+1+1/(r-1)+o(1))\frac{n^{r-2}}{(r-2)!}. \qquad (3)\end{aligned}$$

If $r = 2$, we have directly that $\delta(\mathcal{H}) \geq k$ or else any k-set containing v and all its neighbours is not covered in $\mathcal{H}$. So (3) holds for all $r \geq 2$.

For $i = 1, \ldots, n$, define a real number $x_i \in [r-1, \infty)$ by

$$d_i = \binom{x_i}{r-1}, \tag{4}$$

so $x_1 \geq \cdots \geq x_n$. By the Kruskal-Katona Theorem, a vertex v that covers $\binom{x}{r}$ r-sets must have degree at least $\binom{x}{r-1}$ (since $\mathcal{H}_v$ must contain all $(r-1)$sets in the shadow of the r-sets covered by v). So the ith vertex in $\mathcal{H}$ covers at most $\binom{x_i}{r}$ r-sets.

Each r-set R in V is covered at least $k - r + 1$ times in $\mathcal{H}$, or else we could choose a k-set S containing R and all vertices that cover R, and then S would not be covered in $\mathcal{H}$. Counting all pairs (R, u) such that R is an r-set and u covers R, we see that

$$\sum_{i=1}^{n} \binom{x_i}{r} \geq (k - r + 1)\binom{n}{r}. \tag{5}$$

We have $x_1 \leq n - 1$ and, as $d_{k-r+1} < (1-\epsilon)\binom{n-1}{r-1}$, we have from (4) and the definition of η that

$$x_{k-r+1} \leq (1 - \eta + o(1))n.$$

Let $t = \lceil \log n \rceil^{r-1}$. Then, as $\sum_{i=1}^{n} d_i = rm$, we have $d_t \leq rm/t = O(n^{r-1}/t)$ and so

$$x_t = O(n/t^{1/(r-1)}) = O(n/\log n).$$

Furthermore, for every i we have

$$\binom{x_i}{r} = \frac{x_i - r + 1}{r}\binom{x_i}{r-1} = \frac{x_i - r + 1}{r} d_i.$$

Since the sequence $(x_i)_{i=1}^{n}$ is decreasing, it follows that

$$\begin{aligned}
\sum_{i=1}^{n} \binom{x_i}{r} &= \sum_{i=1}^{n} \frac{x_i - r + 1}{r} d_i \\
&\leq \frac{x_1 - r + 1}{r} \sum_{i=1}^{k-r} d_i + \frac{x_{k-r+1} - r + 1}{r} \sum_{i=k-r+1}^{t-1} d_i + \frac{x_t - r + 1}{r} \sum_{i=t}^{n} d_i \\
&\leq \frac{n-r}{r} \sum_{i=1}^{k-r} d_i + \frac{(1 - \eta + o(1))n}{r} \sum_{i=k-r+1}^{t-1} d_i + O(nm/\log n)
\end{aligned}$$

Now $m = O(n^{r-1})$, so the last term is $o(n^r)$. Suppose $\sum_{i=1}^{k-r} d_i = \alpha\binom{n-1}{r-1}$ and $\sum_{i=k-r+1}^{t-1} d = \beta\binom{n-1}{r-1}$. Then by (5) and the bound on $\sum_{i=1}^{n}\binom{x_i}{r}$ above we must have

$$\begin{aligned}(k-r+1)\binom{n}{r} &\leq \frac{n-r}{r}\alpha\binom{n-1}{r-1} + \frac{(1-\eta+o(1))n}{r}\beta\binom{n-1}{r-1} + o(n^r)\\ &= \alpha\binom{n-1}{r} + (1-\eta+o(1))\beta\binom{n-1}{r} + o(n^r)\\ &= (\alpha + (1-\eta)\beta + o(1))\binom{n}{r},\end{aligned}$$

and so $\alpha + (1-\eta)\beta \geq k-r+1+o(1)$. However, $\alpha \leq k-r$ (as $d_1 \leq \binom{n-1}{r-1}$) and so $\beta \geq 1/(1-\eta) + o(1)$, giving

$$\alpha + \beta \geq k - r + \frac{1}{1-\eta} + o(1) \geq k - r + 1 + \eta,$$

for large enough n. So

$$\sum_{i=1}^{t-1} d_i \geq (k-r+1+\eta)\binom{n-1}{r-1}. \tag{6}$$

On the other hand, by (3),

$$\begin{aligned}\sum_{i=t}^{n} d_i &\geq (1+o(1))n\delta(\mathcal{H})\\ &\geq \big(k-r+1+1/(r-1)+o(1)\big)\frac{n^{r-1}}{(r-2)!}\\ &= \big((r-1)(k-r+1)+1\big)\frac{n^{r-1}}{(r-1)!} + o(n^{r-1}). \end{aligned}\tag{7}$$

Since $\sum_{i=1}^{n} d_i = mr$, (6) and (7) imply that

$$m \geq (k-r+1+1/r+\eta/r+o(1))\binom{n-1}{r-1},$$

which gives a contradiction for large enough n. We conclude that we must have $d_{k-r+1} \geq (1-\epsilon)\binom{n-1}{r-1}$ if n is sufficiently large, which implies the result immediately. □

Our second lemma will allow us to clean up the structure of a hypergraph that is close to extremal.

Lemma 3. *For every $r \geq 2$ and $\lambda > 0$ there is $\epsilon > 0$ such that the following holds for all sufficiently large n. Let $\mathcal{F}$ and $\mathcal{H}$ be two hypergraphs with vertex set $[n]$ such that*

- *$\mathcal{F}$ is $(r-1)$-uniform and $|\mathcal{F}| < \epsilon n^{r-1}$;*
- *$\mathcal{H}$ is r-uniform;*
- *every $A \in [n]^{(r)}$ that contains an element of $\mathcal{F}$ is covered in $\mathcal{H}$.*

Then there is an r-uniform hypergraph $\mathcal{H}'$ with vertex set $[n]$ such that

- *$|\mathcal{H}'| + \lambda|\mathcal{F}| < |\mathcal{H}|$;*
- *$\partial\mathcal{H}' \supset \partial\mathcal{H}$.*

Proof. The proof hinges on the fact that the presence of $\mathcal{F}$ forces $\mathcal{H}$ to cover many r-sets, and this in turn means that that $\mathcal{H}$ is not efficiently structured to have a large shadow (as some $(r-1)$ sets are contained in multiple elements of $\mathcal{H}$). We will obtain $\mathcal{H}'$ from $\mathcal{H}$ by deleting some edges, and possibly adding a smaller number of new edges to make sure that the shadow does not shrink.

We choose a large positive integer $M = M(r,\lambda) > 0$ and then a small constant $\epsilon = \epsilon(M,r,\lambda) > 0$. Let $m = |\mathcal{F}|$. Let us say that an ordered $(r+1)$-tuple $(x_1,\ldots,x_{r-1},w,u)$ is *good* if $\{x_1,\ldots,x_{r-1}\} \in \mathcal{F}$ and u covers $\{x_1,\ldots,x_{r-1},w\}$ in $\mathcal{H}$. Since there are $(r-1)!$ ways of ordering an element of $\mathcal{F}$, and $n-r+1$ choices for an additional vertex w (with at least one choice of u for each of these), there are $(r-1)!(n-r+1)m$ sequences $(x_1,\ldots,x_{r-1},w)$ that extend to a good $(r+1)$-tuple of form $(x_1,\ldots,x_{r-1},w,u)$.

Fix a partition $X_1 \cup \cdots \cup X_{r+1}$ of $[n]$, and let $\mathcal{B}_0$ be the set of good $(r+1)$-tuples $(a_1,\ldots,a_{r+1})$ such that $a_i \in X_i$ for each i. Let $\mathcal{A}_0$ be the set of r-tuples $(a_1,\ldots,a_r)$ such that $(a_1,\ldots,a_{r+1}) \in \mathcal{B}_0$ for some $a_{r+1} \in X_{r+1}$; and let $\mathcal{F}_0$ be the the set of $(r-1)$-tuples $(a_1,\ldots,a_{r-1})$ such that $(a_1,\ldots,a_r) \in \mathcal{A}_0$ for some $a_r \in X_r$. We will also describe this as follows: we consider $\mathcal{B}_0$ as a subset of the cartesian product $X_1 \times \cdots \times X_{r+1}$, and then $\mathcal{F}_0$ and $\mathcal{A}_0$ are the projections of $\mathcal{B}_0$ on to the first $r-1$ coordinates and the first r coordinates respectively.

We choose $X_1, \dots, X_{r+1}$ so that $|\mathcal{A}_0|$ is as large as possible: by considering a random partition, we see that

$$|\mathcal{A}_0| \geq (r-1)!(n-r+1)m/(r+1)^{r+1}. \tag{8}$$

For each $F = (a_1, \dots, a_{r-1}) \in \mathcal{F}_0$, let $\alpha(F)$ be the number of distinct elements $u \in X_{r+1}$ such that $(a_1, \dots, a_{r-1}, w, u) \in \mathcal{B}_0$ for some $w \in X_r$. Note that $\alpha(F) \geq 1$ for all $F \in \mathcal{F}_0$.

Suppose first that

$$\sum_{F \in \mathcal{F}_0} \max\{\alpha(F) - 1, 0\} > \lambda m. \tag{9}$$

In this case we construct our set system $\mathcal{H}'$ as follows. For each $F = (a_1, \dots, a_{r-1}) \in \mathcal{F}_0$ with $\alpha(F) > 1$, we choose $\alpha(F)$ elements of $\mathcal{B}_0$ that have F as an initial segment and contain distinct elements of X_{r+1}, say

$$(a_1, \dots, a_{r-1}, w_1, u_1), \dots, (a_1, \dots, a_{r-1}, w_{\alpha(F)}, u_{\alpha(F)})$$

(so the u_i are distinct, but the w_i may contain repetitions). We now delete from $\mathcal{H}$ the edges

$$\{a_1, \dots, a_{r-1}, u_1\}, \dots, \{a_1, \dots, a_{r-1}, u_{\alpha(F)-1}\};$$

these belong to $\mathcal{H}$ as $(a_1, \dots, a_r, w_i, u_i)$ is a good $(r+1)$-tuple for each i and so u_i covers $\{a_1, \dots, a_r, w_i\}$. Repeating for each $F \in \mathcal{F}_0$ with $\alpha(F) > 1$, it follows from (9) that we delete a total of more than λm edges from $\mathcal{H}$. Let $\mathcal{H}'$ be the resulting hypergraph. Clearly $|\mathcal{H}'| + \lambda|\mathcal{F}| < |\mathcal{H}|$, so it is enough to show that $\partial\mathcal{H}' = \partial\mathcal{H}$. Suppose that $I \in \partial\mathcal{H}$ (so $|I| = r-1$). If $I \notin \partial H'$ then there must be an edge of form $A = \{a_1, \dots, a_{r-1}, u_i\}$ that contains I and that we deleted from $\mathcal{H}$. If $I = A \setminus a_j$ for some j, then $I \subset (A \setminus a_j) \cup \{w_i, u_i\}$, which belongs to $\mathcal{H}$ as u_i covers $\{a_1, \dots, a_r, w_i\}$ and it was not one of the edges we deleted; otherwise $I = \{a_1, \dots, a_{r-1}\}$, and is contained in $\{a_1, \dots, a_{r-1}, u_{\alpha(F)}\}$, which we did not delete from $\mathcal{H}$. We conclude that $\partial\mathcal{H}' = \partial\mathcal{H}$.

We may therefore assume that (9) does not hold. It follows that there are at most $\lambda m/M$ elements $F = (a_1, \dots, a_{r-1}) \in \mathcal{F}_0$ with $\alpha(F) \geq M + 1$. Let $\mathcal{F}_1 = \{F \in \mathcal{F}_0 : \alpha(F) \leq M\}$, and let $\mathcal{A}_1$ be the set of elements $(a_1, \dots, a_{r-1}, w) \in \mathcal{A}_0$ such that $(a_1, \dots, a_{r-1}) \in \mathcal{F}_1$. Since each $F \in \mathcal{F}_0$

extends to at most $n-r+1$ elements of $\mathcal{A}_0$, it follows from (8) that for all sufficiently large n

$$|\mathcal{A}_1| \geq (r-1)!(n-r+1)m/(r+1)^{r+1} - (n-r+1)(\lambda m/M) \geq mn/M,$$

provided M is large enough.

For each $F = (a_1, \ldots, a_{r-1}) \in \mathcal{F}_1$, define

$$\beta(F) = \max_{u \in X_{r+1}} |\{w \in X_r : (a_1, \ldots, a_{r-1}, w, u) \in \mathcal{B}_0\}|,$$

and choose $u_F \in X_{r+1}$ such that there are $\beta(F)$ elements $w \in X_r$ such that $(a_1, \ldots, a_{r-1}, w, u_F) \in \mathcal{B}_0$. Each $F \in \mathcal{F}_1$ extends to at most $\alpha(F)\beta(F)$ elements of $\mathcal{B}_0$, and so F extends to at most $\alpha(F)\beta(F)$ elements of $\mathcal{A}_1$. Therefore

$$|\mathcal{A}_1| \leq \sum_{F \in \mathcal{F}_1} \alpha(F)\beta(F) \leq M \sum_{F \in \mathcal{F}_1} \beta(F),$$

and so $\sum_{F \in \mathcal{F}_1} \beta(F) \geq mn/M^2$.

Let $\mathcal{F}_2 = \{F \in \mathcal{F}_1 : \beta(F) \geq n/2M^2\}$. So for each $F = (a_1, \ldots, a_{r-1}) \in \mathcal{F}_2$ we have chosen $u_F \in X_{r+1}$ such that there are at least $n/2M^2$ elements of $\mathcal{B}_0$ of form $(a_1, \ldots, a_{r-1}, w, u_F)$ with $w \in X_r$. By counting the number of ways of extending elements from $\mathcal{F}_1$ to $\mathcal{A}_1$, we see that

$$|\mathcal{A}_1| \leq (n-r+1)|\mathcal{F}_2| + (n/2M^2)|\mathcal{F}_1 \setminus \mathcal{F}_2| \leq n|\mathcal{F}_2| + mn/2M^2.$$

Since $|\mathcal{A}_1| \geq mn/M$, this implies that $|\mathcal{F}_2| \geq m/2M^2$.

Now $\mathcal{F}_2$ is a subset of $X_1 \times \cdots \times X_{r-1}$. For $i = 1, \ldots, r-1$, let $\mathcal{P}_i$ be the projection of $\mathcal{F}_2$ onto the coordinates other than i, so $\mathcal{P}_i \subset \prod_{j \in [r-1] \setminus i} X_j$. It follows from the Loomis-Whitney Inequality [8] (see also [1]) that

$$\prod_{i=1}^{r-1} |\mathcal{P}_i| \geq |\mathcal{F}_2|^{r-2},$$

and so in particular there is some i such that

$$|\mathcal{P}_i| \geq |\mathcal{F}_2|^{(r-2)/(r-1)} \geq (m/2M^2)^{(r-2)/(r-1)} > 5M^2\lambda m/n,$$

as $m \leq \epsilon n^{r-1}$ by assumption and we may also assume $\epsilon < 1/(10\lambda M^4)^r$.

Without loss of generality, we may assume that $i = r-1$. Let $\mathcal{P} = \mathcal{P}_{r-1}$. For each element of $P = (a_1, \ldots, a_{r-2}) \in \mathcal{P}$, choose an element $F \in \mathcal{F}_2$ that projects to P. Let $\mathcal{F}_3$ be the collection of $|\mathcal{P}|$ elements of $\mathcal{F}_2$ that are chosen. Thus the elements of $\mathcal{F}_3$ have pairwise distinct projections on to the first $r-2$ coordinates.

We are finally ready to construct our modification of $\mathcal{H}$. For $F = (a_1, \ldots, a_{r-1}) \in \mathcal{F}_3$, let $W_F = \{w \in X_r : (a_1, \ldots, a_{r-1}, w, u_F) \in \mathcal{B}_0\}$, say $W_F = \{w_1, \ldots, w_t\}$. Since $F \in \mathcal{F}_3 \subset \mathcal{F}_2$, we have $t \geq n/2M^2$; if t is odd, we reduce it by one (and do not use the final element of W_F). Now delete from $\mathcal{H}$ the edges

$$\{a_1, \ldots, a_{r-2}, w_1, u_F\}, \ldots, \{a_1, \ldots, a_{r-2}, w_t, u_F\} \tag{10}$$

and add (if not already present) new edges

$$\{a_1, \ldots, a_{r-2}, w_1, w_2\}, \{a_1, \ldots, a_{r-2}, w_3, w_4\}, \ldots, \{a_1, \ldots, a_{r-2}, w_{t-1}, w_t\}.$$

Thus we decrease the number of edges by at least $t/2 \geq n/5M^2$. Repeating for every $F \in \mathcal{F}_3$, we decrease the number of edges by at least $|\mathcal{F}_3|(n/5M^2) > (5M^2\lambda m/n)(n/5M^2) = \lambda m$ (note that we delete disjoint sets of edges for distinct $F, F' \in \mathcal{F}_3$, as F and F' extend distinct elements of $\mathcal{P}$ and so have distinct projections on to the first $r-2$ coordinates). Let $\mathcal{H}'$ be the resulting hypergraph.

All that remains is to check that $\partial H \subset \partial H'$. Given $I \in \partial H \setminus \partial H'$, we know that I must be contained in one of the edges we deleted. So suppose without loss of generality that $I \subset \{a_1, \ldots, a_{r-2}, w_1, u_F\}$ in (10). Since u_F covers $\{a_1, \ldots, a_{r-1}, w_1\}$ in $\mathcal{H}$ we cannot have $u_F \in I$, as then $I \cup \{a_{r-1}\}$ is an edge of both $\mathcal{H}$ and $\mathcal{H}'$. But then $I \subset \{a_1, \ldots, a_{r-2}, w_1, w_2\} \in \mathcal{H}'$. This gives a contradiction and proves the lemma. □

We can now complete the proof of our main theorem.

Proof of Theorem 1. We prove the theorem by induction on r. The case $r = 2$ is immediate from the Erdős-Moser theorem, so we may assume that $r \geq 3$ and we have proved smaller cases.

We first note that, since Theorem 1 holds for $r-1$, it follows that Lemma 2 holds for r. Let $\mathcal{H}$ be a hypergraph with n vertices and $f(n, k, r)$ edges in which every k-set is covered. Let $v_1, \ldots, v_n$ be the vertices of $\mathcal{H}$ in decreasing order of degree, let $V = \{v_1, \ldots, v_n\}$, and let $A = \{v_1, \ldots, v_{k-r+1}\}$. Let $\mathcal{H}_1$ be the restriction of $\mathcal{H}$ to $V \setminus A$.

Let $\mathcal{G} \subset V^{(r)}$ be the collection of all r-sets of vertices that meet A but are not present in $\mathcal{H}$, and suppose $\mathcal{G}$ is nonempty. It follows from Lemma 2 that, for any $\epsilon > 0$, if n is sufficiently large then $|\mathcal{G}| < \epsilon n^{r-1}$.

We split $\mathcal{G}$ into sets $\mathcal{G}_0, \ldots, \mathcal{G}_{r-1}$, where $\mathcal{G}_i = \{G \in \mathcal{G} : |G \setminus A| = i\}$. Let $\mathcal{G}_i$ be the largest of these sets, and let $\mathcal{G}_i^* = \{G \setminus A : G \in G_i\}$. Let $\phi : (V \setminus A)^{(i)} \to (V \setminus A)^{(r-1)}$ be an injection such that $\phi(S) \supset S$ for every S (the existence of such a mapping follows easily from Hall's Theorem) and let $\mathcal{F} = \{\phi(S) : S \in \mathcal{G}_i^*\}$. Note that $|\mathcal{G}_i| \geq |\mathcal{G}|/r$ and for each $B \in \mathcal{G}_i^*$ there are at most k^r sets $G \in \mathcal{G}_i$ with $G \setminus A = B$. Thus $\mathcal{F}$ is a collection of $(r-1)$-sets in $V \setminus A$, such that $|\mathcal{F}| \geq |\mathcal{G}|/k^r r$ and every set in $\mathcal{F}$ contains $G \setminus A$ for some $G \in \mathcal{G}$.

Now for any $F \in \mathcal{F}$, we can pick $G \in \mathcal{G}$ such that $G \setminus A \subset F$ and choose $v \in A \cap G$. For any $w \in V \setminus (F \cup A)$, the set $(A \setminus v) \cup F \cup \{w\}$ has size k and so must be covered in $\mathcal{H}$ by some u. We do not have $u = v$, as G is not an edge of $\mathcal{H}$. Thus $u \notin A$: it follows that $F \cup \{w\}$ is covered by u in $\mathcal{H}_1$.

We can now apply Lemma 3 to the hypergraphs $\mathcal{F}$ and $\mathcal{H}_1$ (both on vertex set $V \setminus A$) with $\lambda = k^r r$ to deduce that there is some $\mathcal{H}_1'$ such that $|\mathcal{H}_1'| + k^r r|\mathcal{F}| < |\mathcal{H}_1|$ and $\partial H_1' \supset \partial H_1$. Let $\mathcal{H}'$ be the hypergraph with vertex set V obtained from $\mathcal{H}_1'$ by adding all r-sets incident with A. Then $|\mathcal{H}'| < |\mathcal{H}|$, as $k^r r|\mathcal{F}| \geq |\mathcal{G}|$.

Finally, we claim that every k-set in V is covered in $\mathcal{H}'$. Let $B \in V^{(k)}$. If $A \setminus B$ is nonempty, then any vertex of $A \setminus B$ covers B in $\mathcal{H}'$. Otherwise, $A \subset B$: let $C = B \setminus A$, so C is an $(r-1)$-set. There must be some $u \notin B$ that covers B in $\mathcal{H}$, and so $\{u\} \cup C \in \mathcal{H}$. Then we also have $\{u\} \cup C \in \mathcal{H}_1$, and so $C \in \partial\mathcal{H}_1$. By construction, we also have $C \in \partial\mathcal{H}_1'$, so there is some r-set $D \in \mathcal{H}_1'$ that contains C. Let u' be the single element of $D \setminus C$: then u' covers B in $\mathcal{H}'$, as $D \in \mathcal{H}'$, and every other r-set contained in $\{u'\} \cup B$ meets A and so belongs to $\mathcal{H}'$. This gives a contradiction, as $|\mathcal{H}'| < |\mathcal{H}|$, and every k-set is covered in $\mathcal{H}'$.

We conclude that every edge incident with A must be present in $\mathcal{H}$. But now consider any $(r-1)$-set $B \subset V \setminus A$: arguing as above, the set $A \cup B$ must be covered by some u in $\mathcal{H}$, and so $B \cup \{u\} \in \mathcal{H}_1$. It follows that $\partial\mathcal{H}_1 = (V \setminus A)^{(r-1)}$, and so $|\mathcal{H}_1| \geq D(n - |A|, r) = D(n - k + r - 1, r)$. Thus $f(n, k, r) = |\mathcal{H}| \geq g(n, k, r)$. Furthermore, since $\mathcal{H}$ contains all r-sets meeting A, and a collection of $D(n - k + r - 1, r)$ r-sets in $V \setminus A$ containing all $(r-1)$-sets in $V \setminus A$ in their shadow, we see that $\mathcal{H} \in \mathcal{G}(n, k, r)$. $\square$

3 Further directions

There are a number of interesting related problems. In their original paper [3], Erdős and Moser raised the question of covering oriented graphs. A digraph with at least $k+1$ vertices has *property* S_k (named after Schütte) if for every k-set B of vertices there is a vertex u such that B is contained in the outneighbourhood of u. Erdős and Moser [3] asked the following.

Problem 4. *What is the minimum number $F(n,k)$ of edges in a graph G of order n that has an orientation with property S_k?*

Erdős and Moser [4] noted that, for fixed k and large enough n, $f(k-1)n \leq F(n,k) \leq f(k)n$, where $f(k)$ is the minimum number of vertices in an oriented graph with property S_k; it is known that $f(k) = 2^{(1+o(1))k}$.

For $k \geq 2$, let $\delta(k)$ be the smallest positive integer δ such that some oriented graph with property S_k has a vertex with indegree δ. We shall show that the asymptotic behaviour of $F(n,k)$ depends primarily on $\delta(k)$.

Lemma 5. *For every $k \geq 2$ there is are integers $c = c(k)$ and $n_0 = n_0(k)$ such that $F(n,k) = \delta(k)n + c$ for all $n \geq n_0$.*

Proof. Note first that if G is an oriented graph with t vertices that has property S_k and v is any vertex of G then we can generate another oriented graph with property S_k by adding a new vertex v' with no outedges, and the same inneighbourhood as v. If G has a vertex with indegree $\delta(k)$ then applying this construction repeatedly to a vertex of minimal indegree gives, for all $n \geq t$ and some $c_1 = c_1(G)$, oriented graphs with n vertices, $\delta(k)n + c_1$ edges and property S_k.

Any oriented graph G with property S_k must have at least $\delta(k)|V(G)|$ edges, and so we must have $\delta(k)n \leq F(k,n) \leq \delta(k)n + c_1$ for all sufficiently large n. Let $c \geq 0$ be minimal such that there are infinitely many graphs G with $\delta(k)|V(G)| + c$ edges and property S_k: if $|V(G)|$ is large enough and G has $\delta(k)|V(G)| + c$ edges then it has minimal degree at most $\delta(k)$, so we can apply the construction to generate a sequence of oriented graphs with $\delta(k)n + c$ edges and property S_k for all sufficiently large n. If n is large enough, these are extremal. □

The covering problem is also natural for the more general class of digraphs (so we allow edges in both directions). In this case, we can solve the problem completely.

Let $\mathcal{A}(n,k)$ be the family of digraphs on vertex set $[n]$ that can be constructed as follows:

- Add all $k(k+1)$ directed edges that join any two elements from $[k+1]$
- For each $i > k+1$, add edges from exactly k elements of $[k+1]$ to i.

If $G \in \mathcal{A}(n,k)$ then every vertex of G has indegree exactly k. Let us check that every k-set B is covered in G. If an element $i \in [k+1]$ does not cover B then either $i \in B$, or $i \notin B$ and some vertex of B is not an outneighbour of i. There are $|B \cap [k+1]|$ elements of the first type and at most $|B \setminus [k+1]|$ elements of the second type (as each element of $B \setminus [k+1]$ has edges from all but one element of $[k+1]$); so there is at least one element of $[k+1]$ left over to cover B.

Lemma 6. *Let $k \geq 2$, and let D be a digraph of order n that has property S_k. Then D has at least kn edges, and if D has exactly kn edges then D is isomorphic to some element of $\mathcal{A}(n,k)$.*

Proof. Every vertex has indegree at least k, or else any k-set that contains a vertex of minimal indegree and all its inneighbours is not covered. Thus D has at least kn edges.

If D has exactly kn edges then every vertex has indegree exactly k. Given a vertex v, let $\Gamma^-(v)$ be the set of inneighbours of v. For any $w \in \Gamma^-(v)$, the only vertex that can cover $\{v\} \cup \Gamma^-(v) \setminus w$ is w (as it is the only inneighbour of v outside the set). It follows that $\Gamma^-(v)$ induces a complete directed graph. Now pick any $u \in \Gamma^-(v)$: we have $|\Gamma^-(u) \cap \Gamma^-(v)| = |\Gamma^-(v) \setminus u| = k-1$. Let $A = \Gamma^-(u) \cup \Gamma^-(v)$, so $|A| = k+1$ and every pair of elements from A is joined in both directions except possibly u, u', where u' is the unique element of $\Gamma^-(u) \setminus \Gamma^-(v)$. Since $k \geq 2$, we have $|A| \geq 3$ and so we can pick $w \neq u, u'$ in A: then $\Gamma^-(w) = A \setminus w$ induces a complete directed graph, and so u, u' are also joined in both directions. Thus A is complete and, as $|A| = k+1$, no further edges can enter A.

Finally, for any $x \notin A$ and any $a, b \in A$, consider the k-set $\{x\} \cup A \setminus \{a, b\}$: this can only be covered by a or b, and so one of a and b must send an edge to x. Since this holds for every pair of elements in A, it follows that k elements of A must direct edges to x. We conclude that $D \in \mathcal{A}(n,k)$. $\square$

Erdős and Moser [4] also considered a generalization of the graph problem, where every vertex must be covered by many other vertices. Let $h(n,k,r,s)$

be the minimum number of edges in an r-uniform hypergraph $\mathcal{H}$ with vertex set $[n]$ such that every k-set in $[n]$ is covered by at least s distinct vertices in $\mathcal{H}$ (so $h(n,k,r,1) = f(n,k,r)$). Erdős and Moser [4] considered the case $r = 2$ and noted that, for $n > n_0(k,s)$,

$$h(n,k,2,s) = f(n,k+s-1,2); \tag{11}$$

they further noted that this equality does not hold for every n, k, s (so the assumption that $n > n_0$ is necessary). It seems likely that our methods should be useful for this problem, although we note that (11) does not hold when 2 is replaced by r if k is close to r: for instance with $k = r = 3$ and $s = 3$ the graph $\mathcal{G}(n, k+s-1, r) = \mathcal{G}(n,5,3)$ has 3 vertices of large degree, but is not extremal as the edge containing these three vertices can be removed.

In a similar vein, we can consider a multicolour version of the problem: let $b(n,k,r,c)$ be the minimum number of edges in a graph on n vertices, in which the edges are partitioned into c classes such that every k-set is covered by the edges in each class. How does $b(n,k,r,c)$ behave for fixed k, r, c and large n?

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