Adaptive Robust Model Predictive Control

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Abstract

Model Predictive Control (MPC) repeatedly solves a finite horizon optimal control problem subject to input and state constraints. The performance of MPC algorithms depends crucially on the quality of the model that is used to optimize the predicted future performance. To reduce system model uncertainties, adaptive controllers have been proposed for MPC enabling model parameters to be identified online. Adaptive robust MPC, combining adaptive control and robust MPC, therefore has the potential to estimate system parameters while satisfying operating constraints and providing stability guarantees. However, although adaptive robust MPC algorithms have been considered for several decades, robust, computationally tractable adaptive MPC remains a topic under research.

This thesis proposes several adaptive robust Model Predictive Control strategies for linear and nonlinear systems with constant unknown model uncertainties and bounded additive disturbances. By combining set membership parameter identification with robust tube MPC, the proposed algorithms guarantee robust constraint satisfaction and recursive feasibility. Computational tractability is ensured by using polytopic sets of fixed complexity to bound parameter sets and predicted states. The closed-loop system is proven to be input-to-state stable under a nominal cost.

For parameter identification, the thesis proposes various approaches that incorporate persistence of excitation conditions as computationally tractable constraints with feasibility guarantees within adaptive MPC. Methods of ensuring persistence of excitation include augmenting the cost function, exploiting the random nature of disturbance inputs inherent in the model, introducing additional excitation into the control inputs, and including both disturbance and additional input excitation in the algorithm. The performance of various approaches is compared in terms of parameter convergence and tracking or regulation performance. In addition, the thesis relates the probabilistic rates of parameter set convergence to persistence of excitation conditions and proves convergence of the estimated parameter sets to the true parameter values using the proposed algorithms.

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This thesis proposes several adaptive robust Model Predictive Control strategies for linear and nonlinear systems with constant unknown model uncertainties and bounded additive disturbances. By combining set membership parameter identification with robust tube MPC, the proposed algorithms guarantee robust constraint satisfaction and recursive feasibility. Computational tractability is ensured by using polytopic sets of fixed complexity to bound parameter sets and predicted states. The closed-loop system is proven to be input-to-state stable under a nominal cost.

For parameter identification, the thesis proposes various approaches that incorporate persistence of excitation conditions as computationally tractable constraints with feasibility guarantees within adaptive MPC. Methods of ensuring persistence of excitation include augmenting the cost function, exploiting the random nature of disturbance inputs inherent in the model, introducing additional excitation into the control inputs, and including both disturbance and additional input excitation in the algorithm. The performance of various approaches is compared in terms of parameter convergence and tracking or regulation performance. In addition, the thesis relates the probabilistic rates of parameter set convergence to persistence of excitation conditions and proves convergence of the estimated parameter sets to the true parameter values using the proposed algorithms.

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List of Notations and Abbreviations

Definition

A	matrices (uppercase letters)
a	vectors (lower case letter)
$\mathcal{A}$	sets (calligraphic letters)
$\mathbb{R}$	the set of real numbers
$\mathbb{N}$	the set of integers
$\mathbb{N}_{\geq 0}$	the set of positive real numbers, i.e. $\mathbb{N}_{\geq 0} = \{n \in \mathbb{N} : n > 0\}$
$\mathbb{N}_{[a,b]}$	the set of positive real numbers between a and b, i.e. $\mathbb{N}_{[a,b]} = \{n \in \mathbb{N}, a \leq n \leq b\}$
$\mathbf{1}$	A vector/matrix with all elements equal to 1 and a suitable dimension
$\mathbf{0}$	A vector/matrix with all elements equal to 0 and a suitable dimension
$\mathbb{I}$	an identity matrix with a suitable dimension
$\mathbb{B}^n$	A unit ball in n-dimensional space

Vectors and Matrices

$[A]_i$	ith row of a matrix A
$[a]_i$	ith element of a vector a
$[a]_{\geq 0}$	the maximum of the scalar a and 0, i.e. $[a]_{\geq 0} = \max\{0, a\}$
$\text{vec}(A)$	the vector formed by stacking the columns of A
$\|a\|$	the Euclidean norm
$\|a\|_P^2$	the scaled norm where $\|a\|_P^2 = a^\top P a$
$\|a\|$	the absolute value of a
$\lfloor a \rfloor$	the floor value of a

$\max a$ . . . . . the largest element of the vector a

$A \geq B$ . . . . . elementwise inequality

$A \succeq 0 / A \succ 0$ . . A is positive semidefinite (positive definite) matrix

$A \succeq B$ . . . . . for square matrices A and B, $x^\top Ax \geq x^\top Bx$ for all vectors x of conformal dimensions

$\underline{\lambda}(A)$ . . . . . The minimum eigenvalue of a real symmetric matrix A

Set Operation

$|\mathcal{S}|$ . . . . . the cardinality in a set $\mathcal{S}$

$A\mathcal{X}$ . . . . . Matrix - set multiplication $\{Ax : x \in \mathcal{X}\}$

$\mathcal{A} \oplus \mathcal{B}$ . . . . . Minkowski addition for sets $\mathcal{A}$ and $\mathcal{B}$ where $\mathcal{A} \oplus \mathcal{B} = \{a + b : a \in \mathcal{A}, b \in \mathcal{B}\}$

$\mathcal{A} \ominus \mathcal{B}$ . . . . . Pontryagin difference for sets $\mathcal{A}$ and $\mathcal{B}$ where $\mathcal{A} \ominus \mathcal{B} = \{a : a + b \in \mathcal{A}, \forall y \in \mathcal{B}\}$

$\rho(\mathcal{A})$ . . . . . the maximum 2-norm of any element of $\mathcal{A}$ for a compact set $\mathcal{A}$

Control Framework

z_k . . . . . the k-steps ahead predicted value of a variable z

$z_{k|t}$ . . . . . the k-steps ahead predicted value of a variable z at time t

$\mathbf{u}_{k|t}$ . . . . . The bold character u represents a sequence of control inputs $u_{k|t}$

$\max_{x \in \mathcal{X}} J(x)$ . . maximum value of J over the set $\mathcal{X}$ for a real scalar-valued function J

Functions

$\mathcal{K}$-function . . A continuous function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{K}$-function if it is strictly increasing with $\sigma(0) = 0$

$\mathcal{K}_\infty$-function . A $\mathcal{K}$-function that also satisfies $\sigma(s) \to \infty$ as $s \to \infty$

$\mathcal{KL}$-function . A continuous function $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{KL}$-function if, for all $t \geq 0$, $\phi(\cdot, t)$ is a $\mathcal{K}$-function, and, for all $s \geq 0$, $\phi(s, \cdot)$ is decreasing with $\phi(s, t) \to 0$ as $t \to \infty$

$\sigma_a \circ \sigma_b(\cdot)$ . . . . For functions σ_a and σ_b, $\sigma_a \circ \sigma_b(\cdot) = \sigma_a\Big(\sigma_b(\cdot)\Big)$

$\sigma_a^{k+1}(\cdot)$ . . . . . For function σ_a, $\sigma_a^{k+1}(\cdot) = \sigma_a \circ \sigma_a^k(\cdot)$ with $\sigma_a^1(\cdot) = \sigma_a(\cdot)$

Probabilities and Expectations

$\mathbb{E}_t[x]$	the conditional expectation of a variable x given the information available to the controller at time t
$\mathbb{E}[x]$	the expectation; also refers to the expectation conditioned on information at time $t = 0$, i.e. $\mathbb{E}[\cdot] = \mathbb{E}_0(\cdot)$
$\Pr(\epsilon)$	the probability of event ϵ
$\Pr(\epsilon\|x_t)$	the probability of event ϵ conditioned on the state x_t and $\Pr(\epsilon\|x_t) = \Pr_t(\epsilon)$

1 Introduction

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1.1 Motivation

Model Predictive Control (MPC) was first proposed about half a century ago [1]. Its key idea is to perform numerical optimization with a receding horizon implementation, which is an effective way of approximating the optimal control law for a problem with or without an analytical solution. With the development in hardware computational ability, MPC's ability to anticipate future events and take actions accordingly makes it a popular research area in the last two decades. It is widely applied in automotive, chemical and manufacturing industries [2].

Model Predictive Control (MPC) repeatedly solves a receding horizon optimisation problem online in the presence of constraints. At each sampling instant, the optimal control law is approximated based on a model of the plant, and the first step of the computed control sequence is applied to the plant [1]. In most cases, models are

only approximations to physical systems and uncertainties are inevitable. Methods of designing and analysing MPC laws are well developed for linear systems with known dynamics [3], additive uncertainties and multiplicative uncertainties [4], but the case of systems with parametric uncertainties is still under research. As MPC requires a model of the plant being controlled, its performance relies on model quality and any mismatch between model and plant can cause degradation of controller performance [5, 6]. As a result, the degree of uncertainty in the system model and unknown disturbances crucially affect the bounds on the achievable performance of a MPC strategy [4].

In response to this, researchers have developed different approaches, such as robust MPC, stochastic MPC and adaptive MPC/learning-based MPC. Robust MPC requires constraints to be satisfied for all realizations of model uncertainty. Stochastic MPC takes explicit account of the probability distribution of the stochastic model uncertainty in the optimisation process. It can allow systems to operate close to the constraint boundary and achieve better performance if constraints only need to be satisfied with a specified probability (less than 1). Adaptive MPC, on the other hand, improves the system model accuracy by reducing the size of the parameter uncertainty set while simultaneously controlling the system. Robust and stochastic forms of adaptive MPC allow the system to be subject to robust or stochastic constraints.

Therefore, adaptive robust MPC, the combination of robust MPC and adaptive control, has the potential to improve the system model accuracy while providing robust feasibility guarantees. Although several approaches have been proposed in the area, few papers prove theoretic properties for adaptive MPC [7]. In addition, the online optimisation in MPC could in principle be used to find a balance between the conflicting requirements to obtain good control performance at the same time as providing sufficiently informative system responses to enable model parameters to be identified. In addition, it is challenging to incorporate the Persistent Excitation (PE) conditions, which are nonconvex conditions on controller operation to ensure accurate parameter estimation [8], with feasibility guarantees in adaptive MPC.

A major challenge for adaptive MPC design is therefore the dual problem, in which the objectives of achieving sufficient excitation and satisfactory regulation of the controlled system are competing.

Different adaptive MPC algorithms proposed in the literature place varying emphases on these competing objectives. Many papers focus on constraint satisfaction and treat persistent excitation (PE) as a given assumption [9]. For example, Di Cairano [10] achieves robust constraint satisfaction and stability by constraint tightening, while Adetola et al. [11] use a min-max cost function. Some papers consider a nominal MPC problem and force the input to be persistently exciting, but fail to ensure constraint satisfaction and stability of the closed-loop system [12, 13]. Other papers aim at achieving the two objectives simultaneously but fail to guarantee properties such as robust constraint satisfaction, computational tractability or convergence of the estimated parameter set [9, 14, 15].

Another difficulty in adaptive MPC is that closed-loop learning is often not guaranteed because the predicted trajectory usually differs from the actual closed-loop trajectory due to the presence of uncertainty and disturbances. In other words, even if the predicted trajectory is optimized to guarantee uncertainty reduction, the closed-loop trajectory may fail to have the same guarantee unless disturbances are properly accounted for in the controller design.

1.2 Contribution

This thesis focuses on the challenges of adaptive robust MPC, including the computational complexity that arises from projecting the parametric uncertainties over predicted states, the compromise between conflicting objectives, the satisfaction of PE conditions in the closed-loop system and the controller stability guarantees. The thesis also investigates the relationship between PE conditions and the rate of convergence of the parameter set estimate, which is seldom discussed in the existing literature. By extending the control strategies for linear systems to nonlinear systems, the proposed algorithms cover a good range of potential applications.

Chapter 2 Background

This chapter first introduces the basic concepts of Model Predictive Control, including problem formulation, dual-mode control law, important control properties such as recursive feasibility, stability and convergence. It then formulates various forms of uncertainties, including additive disturbances, multiplicative disturbances and parameter uncertainties, and the robust MPC approaches to tackle these uncertainties. After that, it introduces adaptive control, various parameter estimation methods, Adaptive MPC and Persistent Excitation (PE) conditions by discussing the relevant literature for linear and nonlinear systems.

Chapter 3 Min-max robust MPC with online parameter identification

This chapter presents a robust MPC framework that combines set-based model identification method and the robust tube MPC formulation. Compared with existing work, the proposed algorithm provides flexibility in modelling the effects of model uncertainty on predicted state trajectories by using a fixed-complexity tube formulation with more degrees of freedom. The proposed algorithm achieves robust constraint satisfaction and recursive feasibility. It also incorporates a convex approximation for persistency of excitation conditions, and gives asymptotic bounds on the closed-loop states that depend explicitly on the identified parameter set when a min-max cost function is adopted.

Chapter 4 Adaptive robust MPC and convergence of model parameters

Building on Chapter 3, this chapter uses hyperplane and vertex representations interchangeably to further simplify computation for fixed-complexity tube MPC. Using a nominal performance objective, we prove the closed-loop system under the proposed algorithm robustly satisfies the constraints and is input-to-state stable (ISS). In addition, the persistence of excitation condition is linearised around a reference trajectory and a penalty term is included in the objective function to promote convergence of the estimated parameter sets.

Moreover, this chapter proves that estimated parameter sets under fixed-complexity polytopic update law converge with probability 1 to the true parameter values despite their approximate nature. Similar bounds on the convergence rates are proven for the case of inexact knowledge of the disturbance bounding set and for the case that model states are estimated in the presence of measurement noise.

Chapter 5 Closed-loop satisfaction of Persistent Excitation Condition

This chapter first derives the bounds on expected PE coefficients under a linear feedback law and under a linear feedback law with an artificial input disturbance consisting of random noise signals. For both cases, the resulting parameter set estimate converges to the true value with probability 1. One drawback of the algorithm proposed in Chapter 4 is that closed-loop satisfaction of the PE conditions is only promoted but not guaranteed. In this chapter, we propose three algorithms that incorporate recursively feasible PE conditions in order to ensure the satisfaction of PE conditions in the closed-loop operation and therefore the convergence of the estimated parameter sets. All of the algorithms proposed in this chapter achieve robust constraint satisfaction and input-to-state stability (ISS).

The first algorithm contains a nonconvex condition in the online implementation and we propose a method of checking this condition after solving a convex MPC optimisation problem. To reduce the computational load, we also propose a modified algorithm that achieves the same closed-loop properties under an additional assumption on its terminal control law. The third alternative adaptive MPC algorithm avoids the nonconvex condition by imposing the PE conditions on the linearized PE coefficients recursively.

Chapter 6 Adaptive MPC with PE conditions on nominal tubes

This chapter proposes an adaptive robust MPC strategy where the PE conditions are imposed on the nominal state tubes, rather than on the predicted tubes in Chapter 4 and 5. The nominal tube formulation separates the effects of control inputs

from artificially injected noise signals and additive disturbances, and therefore both injected noise and disturbances can be accounted for in the bounds of the expected PE coefficients. In addition, the proposed adaptive MPC has a lower computational load because it requires fewer constraints to enforce the PE conditions and avoids the nonconvex PE condition. It also achieves robust constraint satisfaction, recursive feasibility and input-to-state stability (ISS). We provide sufficient conditions for the resulting controller to be persistently exciting.

Chapter 7 Adaptive MPC for nonlinear systems

This chapter considers nonlinear systems with affine parameter uncertainty and additive disturbances. By combining the nonlinear tube MPC formulation with a set-based parameter identification method, the proposed algorithm guarantees robust constraint satisfaction, recursive feasibility and input-to-state stability. To facilitate system identification, Persistent Excitation condition is included in two forms in the two modified algorithms. The first modified algorithm convexifies the PE condition around a reference trajectory and augments the cost function with the PE coefficient. The second modified algorithm incorporates recursively feasible PE conditions by using multiple PE windows. This chapter ends with a Continuous Stirred Tank Reactor (CSTR) by comparing the performance of three proposed strategies. The simulations show a trade-off between parameter identification and tracking performance.

1.2.1 Published Work

A fraction of the work presented in this thesis has been presented in the following publications:

1. Adaptive MPC: combining robustness with online performance improvement X. Lu and M. Cannon Control 2018: The 12th International UKACC Conference on Control, 2018

2. Robust adaptive tube model predictive control X. Lu and M. Cannon American Control Conference, Philadelphia, USA, pp. 3695-3701, 2019

3. Robust adaptive model predictive control: performance and parameter estimation X. Lu, M. Cannon, and D. Koksal-Rivet Int. Journal of Robust and Nonlinear Control (Early Access), Vol. -, pp. 1-22, 2020

The material of first two papers are presented in Chapter 3 and the result of the third paper is presented in Chapter 4.

2

Background

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2.1 Introduction

This chapter provides an introduction to Model Predictive Control (MPC), Robust MPC and Adaptive MPC. It starts with classical Model Predictive Control,

explaining the basic concepts such as problem formulation, dual-mode control laws, recursive feasibility, stability and convergence. It then formulates various forms of uncertainties, including additive disturbances, multiplicative disturbances and parameter uncertainties, and the robust MPC approaches to tackle these uncertainties. It ends with adaptive control, various parameter estimation methods, Adaptive MPC, Persistence Excitation (PE) conditions and a discussion of the relevant literature covering linear and nonlinear systems.

2.2 Classical Model Predictive Control

This section presents the classical MPC algorithm, where the system does not have any disturbance or uncertainty term. It then introduces the conventional "dual-mode" control law and discusses key properties such as recursive feasibility, stability and convergence.

2.2.1 Problem description

Classical MPC usually involves a simple discrete-time Linear Time Invariant (LTI) system with/without linear constraints. The system dynamics can be described by the state-space model

$$x_{t+1} = Ax_t + Bu_t, \tag{2.1a}$$

$$y_t = Cx_t, \tag{2.1b}$$

where $x_t \in \mathbb{R}^{n_x}$, $u_t \in \mathbb{R}^{n_u}$, $y_t \in \mathbb{R}^{n_y}$ are the state, input and output respectively.

Linear constraints take the form:

$$G_x x_t + G_u u_t \leq \mathbf{1}, \qquad t = 0, 1, \ldots \tag{2.2}$$

For the case that these constraints consist of separate input and state constraints, they can also be written in set form:

$$u_t \in \mathbb{U}, x_t \in \mathbb{X}, \text{ for all } t = 0, 1, \ldots \tag{2.3}$$

where $\mathbb{U}$ and $\mathbb{X}$ represent the admissible sets of input and state respectively.

An infinite horizon problem is described as

$$\min_{u_{0|t},u_{1|t},\cdots} J = \sum_{k=0}^{\infty} l(x_{k|t}, u_{k|t}) \tag{2.4a}$$

$$\text{subject to } G_x x_{k|t} + G_u u_{k|t} \leq \mathbf{1}, \qquad k = 0, 1, \ldots \tag{2.4b}$$

$$x_{k+1|t} = Ax_{k|t} + Bu_{k|t}, \tag{2.4c}$$

where $l(x_{k|t}, u_{k|t})$ is the cost of each prediction time step and can take different forms, such as quadratic cost, infinity norm, one norm and so on. The optimal solution is $\mathbf{u}_t^* \in \text{argmin} J(x_t, \mathbf{u}_{k|t})$ and the optimal cost is $J^*(x_t) = J(x_t, \mathbf{u}_t^*)$.

2.2.2 Dual-mode predicted control law

A dual-mode predicted control law refers to the split of the predicted control sequence into two parts: the first N predicted time-steps (which are usually computed online at each sampling time) and those that apply to the remainder of the prediction horizon (which are determined by a terminal feedback law) [16]. The dual-mode control law effectively turns an infinite-dimension control problem into a finite-horizon control problem with a terminal cost and/or a terminal set. One commonly used dual-mode implementation is

$$u_{k|t} = \begin{cases} Kx_{k|t} + v_{k|t} & \text{if } k = 0, 1, \ldots, N-1 \\ Kx_{k|t} & \text{if } k \geq N \end{cases} \tag{2.5}$$

where the feedback gain K is determined offline and is assumed to stabilize the system in the absence of constraints. The input sequence $\boldsymbol{v}_{k|t} = \{v_{0|t}, \cdots, v_{N-1|t}\}$ are optimization variables computed online.

If the stage cost is quadratic, (2.4) can be re-written as

$$\min_{v_{0|t},\ldots,v_{N-1|t}} J = \sum_{k=0}^{N-1} \left(\|x_{k|t}\|_Q^2 + \|u_{k|t}\|_R^2 \right) + \|x_{N|t}\|_P^2 \tag{2.6a}$$

$$\text{subject to } G_x x_{k|t} + G_u u_{k|t} \leq \mathbf{1}, \quad k = 0, 1, \ldots, N-1 \tag{2.6b}$$

$$x_{k+1|t} = Ax_{k|t} + Bu_{k|t}, \quad k = 0, 1, \ldots, N-1 \tag{2.6c}$$

$$x_{N|t} \in \mathcal{X}_T, \tag{2.6d}$$

$$u_{k|t} = Kx_{k|t} + v_k, \tag{2.6e}$$

where Q and R are weighting matrices and $Q \succeq 0$ and $R \succ 0$. $\mathcal{X}_T$ is a suitably defined terminal set and P is the solution of the discrete time Riccatti equation $P = A^\top PA + Q - A^\top PB(B^\top PB + R)^{-1}B^\top PA$ where $(Q^{1/2}, A)$ is observable. The optimal input sequence to (2.6) is denoted as $\mathbf{v}_t^*$ and the corresponding optimal cost as $J^*(x_k) = J(x_k, \mathbf{v}_k^*)$.

2.2.3 Recursive feasibility, stability and convergence

An algorithm is recursively feasible means that the algorithm is feasible at all time instants if it is feasible at the initial time. The terminal set $\mathcal{X}_T$ is *a positively invariant admissible set* if and only if it satisfies the constraint (2.2) and $(A + BK)x \in \mathcal{X}_T$ for all $x \in \mathcal{X}_T$ under the dynamics (2.1a) and constraints (2.2). We define the *tail* solution, $\hat{\mathbf{v}}_{t+1}$ as

$$\hat{\mathbf{v}}_{t+1} = \{u_{1|t}, \ldots, u_{N-1|t}, Kx_{N|t}\},$$

which is calculated at time t and is a feasible and suboptimal solution for time $t+1$.

For stability analysis, we prove that the optimal value of the cost function is non-increasing along trajectories of the closed-loop system and guarantee asymptotic convergence of the state and Lyapunov stability[1]. Given the feasibility of the tail solution $\hat{\mathbf{v}}_{t+1}$, the optimal cost $J^*(x_{t+1}, \hat{v}_{t+1})$ after optimisation at time $k+1$ satisfies

$$J^*(x_{t+1}) \leq J(x_{t+1}, \hat{\mathbf{v}}_{t+1}^*) = J^*(x_t) - \|x_t\|_Q^2 - \|u_t\|_R^2 \tag{2.7}$$

Summing both sides over $t \geq 0$, we get

$$\sum_t^\infty (\|x_t\|_Q^2 + \|u_t\|_R^2) \leq J^*(x_0) - \lim_{t\to\infty} J^*(x_t)$$

where the LHS, the cost evaluated along the closed-loop trajectories, is no greater than the initial optimal cost $J^*(x_0)$ because $J^*(x_t) \geq 0$. In addition, since each term in the LHS of (2.7) is non-negative, the closed-loop bound in (2.7) implies

$$\lim_{k\to\infty} (\|x_t\|_Q^2 + \|u_t\|_R^2) = 0$$

which implies the origin is asymptotically stable (exponentially stable if $Q \succ 0$[17]).

2.3 Robust Model Predictive Control

2.3.1 Robust MPC Overview

In presence, the model is likely to be an inaccurate representation of the plant. There are various forms of uncertainties in the system model, including additive disturbances/noises, multiplicative uncertainty and parametric uncertainties. Also, a system can have a combination of disturbances/uncertainties.

One approach to tackle uncertainties is to use robust control [18], where a fully known nominal model is postulated with known uncertainty bounds and a fixed controller is expected to stabilize the whole admissible set of uncertainties. Another approach is adaptive control, which will be explained later in the chapter.

For systems with disturbances, there are two main alternative definitions of cost when evaluating the predicted performance: nominal cost and min-max cost. The nominal cost, denotes as $J_{\text{norm}}(x_t, \mathbf{v}_k)$, is in the form as that in classical MPC but uses nominal states and inputs instead, i.e.

$$J_{\text{nom}}(x_t, \mathbf{v}_k) = \min_{\mathbf{v}_t} \sum_{k=0}^{\infty} l(\bar{x}_{k|t}, \bar{u}_{k|t}).$$

where $l(x, u)$ denotes the stage cost and the nominal state $\bar{x}_{k|t}$ and nominal input $\bar{u}_{k|t}$ are given by

$$\bar{x}_{k+1|t} = A\bar{x}_{k|t} + B\bar{u}_{k|t}, \tag{2.8a}$$

$$\bar{u}_{k|t} = K\bar{x}_{k|t} + v_{k|t}, \tag{2.8b}$$

$$\bar{x}_{0|t} = x_t. \tag{2.8c}$$

Here the nominal state $\bar{x}_{k|t}$ and input $\bar{u}_{k|t}$ denote the k-step ahead prediction when the uncertainty is assumed to be zero.

For the case of worst-case cost, the controller minimizes the worst case value of a cost function, denoted as J_{worst}, over all possible uncertainties, i.e.

$$J_{\text{worst}} = \min_{\mathbf{v}_t} \max_{w \in \mathcal{W}} \sum_{k=0}^{\infty} l(x_{k|t}, u_{k|t}).$$

As a result, the cost J_{worst} is also called the min-max cost.

An additional term (such as $-\gamma\|w_{k|t}\|_2^2$ for a sufficiently long weight $\gamma > 0$) is usually included in the cost function in order to make the cost a concave function of the additive disturbance, in that case the cost function becomes $J_{\text{worst2}} = \min_{\mathbf{v}_t} \max_{w\in\mathcal{W}} \sum_{k=0}^{\infty} l(x_{k|t}, u_{k|t}) - \gamma\|w_{k|t}\|_2^2$.

For robust constraint satisfaction, various methods are developed, including tube formulation, augmented variable and state/disturbance feedback paramterisation, which will be explained in the following section. In addition, invariance, recursive feasibility and stability are more difficult to achieve because these properties must hold for all realizations of uncertainties. The computational load is also an issue for a large prediction horizon.

2.3.2 Robust MPC for systems with additive disturbances

Consider the system with an additive disturbance term

$$x_{t+1} = Ax_t + Bu_t + Fw_t \tag{2.9}$$

Here $w_t \in \mathbb{R}^{n_w}$ is a vector of disturbance inputs and $w_t \in \mathcal{W}$ for all t.
For the robust satisfaction of constraints, the image of disturbance set $\mathcal{W}$thorough F must be bounded and contain the origin[1]. The disturbance set is assumed to be a convex polytopic set, which can be represented by its vertices ($\mathcal{W} = \text{Co}\{w^{(1)}, \ldots, w^{(m)}\}$) or linear inequalities ($\mathcal{W} = \{w : \Pi_w w \leq \pi_w\}$).

Augmented variable with fixed gain K

Robust MPC for systems with additive disturbance can be converted to classical MPC problem by using augmented variables and taking the worst-case effect of

[1] If the set is unbounded in certain direction, robust satisfaction of constraint cannot be achieved in that direction. If the disturbance set does not contain the origin, it can be rewritten as a disturbance set that contains the origin and a constant offset.

the additive disturbances. Define an augmented state variable z_t as

$$z_t^\top = \begin{bmatrix} x_t^\top & v_{0|t}^\top & \dots & v_{N-1|t}^\top \end{bmatrix}. \tag{2.10}$$

The nominal component can be updated according to the autonomous dynamics

$$z_{k|t} = \Psi^k z_t \tag{2.11}$$

where $\Psi = \begin{bmatrix} A_K & BE \\ 0 & M \end{bmatrix}$, $E = \begin{bmatrix} I_{n_u} & 0 \dots 0 \end{bmatrix}$, $M = \begin{bmatrix} 0 & I_{n_u} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I_{n_u} \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Constraints (2.2) are satisfied for a system with uncertainty (2.9) if and only if the following condition is satisfied for all $k = 0, 1 \dots$

$$G_x x_{k|t} + G_u u_{k|t} \leq \mathbf{1} \quad \forall \{w_{0|t}, \dots, w_{i-1|t}\} \in \mathcal{W} \times \dots \times \mathcal{W}, \tag{2.12}$$

It is apparent that the number of constraints involved grows polynomially with the horizon length. The computational load can be reduced by taking a maximisation which obtains the worst-case future uncertainty h_k through

$$h_k = \sum_{j=0}^{k-1} \max_{w_j \in \mathcal{W}} (G_x + G_u K) A_K^j w_j, \quad k = 1, 2, \dots \tag{2.13}$$

so that the constraint (2.12) can be simplified into

$$\begin{bmatrix} G_x + G_u K & G_u E \end{bmatrix} \Psi^k z_t \leq \mathbf{1} - h_k \quad k = 0, 1, \dots \tag{2.14}$$

As a result, the constraint is in a form almost identical to the classical MPC, with an extra term h_k. Recursive feasibility can be proven and the state converges asymptotically to the minimal RPI set with a finite l_2 gain from the disturbance input [4].

Rigid Tube Formulation

Tube MPC uses a tube (a sequence of sets) $\{\mathcal{X}_k\}_{k=0,\dots,N}$ to bound the predicted state for the given sequence of control inputs. To account for the disturbance, a disturbance invariant set $\mathcal{Z}$ is defined. For the controlled uncertain system $x_{k+1} = (A + BK)x_k + w_k$, $\mathcal{Z}$ satisfies

$$A_K \mathcal{Z} \oplus \mathcal{W} \subseteq \mathcal{Z} \tag{2.15}$$

where $A_K = A + BK$.

To reduce conservativeness, the invariant set should be as small as possible. The minimal disturbance invariant set is $\mathcal{Z}_{min} = \sum_{i=0}^{\infty} {A_K}^i \mathcal{W}$[19]. However, $\mathcal{Z}_{min}$ is not necessarily polytopic and a simplified outer polytopic approximation of the minimal disturbance set is usually used.

For a system with additive disturbances, the set of control sequences needs to satisfy tighter control, state and terminal constraints so that the original constraints (2.3) are satisfied robustly.

$$\begin{aligned} v_{k|t} &\in \bar{\mathcal{U}} = \mathcal{U} \ominus K\mathcal{Z}, \quad \text{for all } k = 1, 2, \dots \\ z_{k|t} &\in \bar{\mathcal{X}} = \mathcal{X} \ominus \mathcal{Z}, \quad \text{for all } k = 1, 2, \dots \\ z_N &\in \mathcal{X}_T = \mathcal{X} \ominus \mathcal{Z}, \end{aligned} \tag{2.16}$$

where $\mathcal{X}_T$ is the terminal constraint set (2.6). With this modification, Mayne et al.[20] propose a MPC approach that optimizes the initial state of the model employed in the problem and the sequence of control inputs. The disturbance invariant set $\mathcal{Z}$ is regarded as the 'origin' of the system with disturbance and the associated value function is zero. Mathematically,

$$\begin{gathered} \min_{\mathbf{z}_t, \mathbf{v}_t} \sum_{k=0}^{N-1} l(x_k, \mathbf{v}_k) + V_f(x_N) \\ \text{subject to } z_{k+1} = Az_k + Bv_k, (2.16) \text{ and } x_t \in z_0 \oplus \mathcal{Z} \end{gathered} \tag{2.17}$$

The optimal control sequence is obtained $\mathbf{v}_k^*(x) = \{v_0^*(x), v_1^*(x), \dots, v_{N-1}^*(x)\}$ and the associated optimal state sequence is $\mathbf{z}_k^*(x) = \{z_0^*(x), z_1^*(x), \dots, z_{N-1}^*(x)\}$. The method is also called *rigid tube MPC* because the actual states and control inputs are bounded by tubes, which are sequences of sets $\{\mathcal{X}_k = z_k \oplus \mathcal{Z}\}$ and $\{\mathcal{U}_k = v_k \oplus K\mathcal{Z}\}$. The tube is *rigid* because it has fixed cross section $\mathcal{Z}$.

Since the optimal initial state $x_0^*(x)$ might not be identical to the current state x_t, the implemented control law is

$$u^*(x) = v_{0|t}^*(x) + K(x_t - z_0^*(x)) \tag{2.18}$$

and the algorithm is proven to be recursively feasible and exponentially stable to an outer approximation of the mRPI set [20].

Homothetic Tube Formulation

Rakovic et al.[21] build on the rigid tube formulation and propose homothetic tube MPC. The state and control tubes are

$$\{\mathcal{X}_k = z_k \oplus \alpha_k \mathcal{X}_0\},\ \{\mathcal{U}_k = u_k \oplus \alpha_k \mathcal{U}_0\},\ \text{for all } k = 1, \ldots, N-1 \tag{2.19}$$

where $\mathcal{X}_0$ and $\mathcal{U}_0$ are fixed shape sets and designed offline, and $\mathbf{z_k}$ and $\mathbf{u_k}$ are sequences of centers and $\boldsymbol{\alpha_k} \geq 0$ is a scaling factor, which is a decision variable in the MPC optimisation. The corresponding K-parameterized control policy is a sequence of control laws $\{\pi_k(\cdot, \mathcal{X}_k, \mathcal{U}_k)\}$ for all $k = 0, 1, \ldots, N-1$ with

$$\forall x \in \mathcal{X}_k, \quad \pi_k(x, \mathcal{X}_k, \mathcal{U}_k) = v_k + K(x - z_k) \tag{2.20}$$

With suitable assumptions on sets $\mathcal{X}_0$ and $\mathcal{U}_0$ and stage and terminal cost functions $l(\cdot)$ and $V_f(\cdot)$, the method demonstrates invariance and stabilizing properties.

Fixed-complexity Tube Formulation

Extending from rigid tube and homothetic tube formulations, a fixed-complexity tube [22] formulation has been introduced where the tube no longer has fixed cross-sections. In general, the cross section is assumed to be a compact polytopic set which is an approximation of the state

$$\mathcal{X}_k = \{x :\ Tx \leq \alpha_k\}^2 \tag{2.21}$$

where $T \in \mathbb{R}^{n_\alpha \times n_x}$ is a fixed matrix whose rows define the direction of the half-planes for which $\mathcal{X}_k$ is constructed and $\alpha_k \in \mathbb{R}^{n_\alpha}$ is a variable vector defining the size of the set. The evolution of the tube must satisfy

$$A\mathcal{X}_k \oplus \{Bu_k\} \oplus \mathcal{W} \subseteq \mathcal{X}_{k+1}. \tag{2.22}$$

The constraints will be applied to the tube

$$\mathcal{X}_k \subseteq \{x_k :\ G_x x_k + G_u u_k \leq \mathbf{1}\} \tag{2.23}$$

[2] For different problem formulations, x can be replaced by disturbance component e_k of state or augmented variable z_t (2.10).

And the final cross-section must satisfy

$$\mathcal{X}_N = \{x : Tx \leq \alpha_N\} \subseteq \mathcal{X}_T, \tag{2.24}$$

By repeatedly using set inclusion conditions [22] and update law (2.22), constraints (2.23) and (2.24) are translated into linear constraints in $\boldsymbol{\alpha}_k$ and $\mathbf{u}_k$.

State/disturbance feedback parameterization

A more general feedback control law is state feedback parameterisation, where the gain K is not assumed to be fixed. In other words, the state feedback parameterisation takes the form that control policy is affine state feedback policies of the predicted states,

$$u_{k|t} = g_{k|t} + \sum_{j=0}^{k} L_{k,j|t} x_{j|t}, \text{ for } k = 0, 1, \ldots, N-1 \tag{2.25}$$

where each $L_{k,j|t} \in \mathbb{R}^{n_u \times n_x}$ and $g_{k|t} \in \mathbb{R}^{n_u}$ are the optimization variables.
A similar parameterisation which avoids the nonconvexity inherent in (2.25) is disturbance feedback parameterisation, which has the form that control laws are affine functions of the sequence of predicted disturbances,

$$u_{k|t} = c_{k|t} + \sum_{j=0}^{k-1} M_{k,j|t} w_{j|t}, \text{ for } k = 0, 1, \ldots, N-1 \tag{2.26}$$

where each $M_{k,j|t} \in \mathbb{R}^{n_u \times n_x}$ and $c_{k|t} \in \mathbb{R}^{n_u}$ are the optimisation variables. If the system state is directly measurable, the past disturbance can be easily measured as the difference between the actual state and the nominal state

$$w_t = x_{t+1} - Ax_t - Bu_t \tag{2.27}$$

Goulart et al.[23] compare affine state feedback parameterisation and disturbance feedback parameterization and prove that the two parameterizations are equivalent. Recursive feasibility can be guaranteed if the algorithm is implemented with a receding horizon with appropriate terminal cost and terminal constraints.

Explicit MPC

For quadratic cost functions and linear constraints, the optimal control law of explicit MPC approach is a piecewise affine function of the initial state x_0. Each constituent affine feedback law depends on which constraints are active[3], and the state space can be divided into polytopic and convex regions, each of which has an associated affine feedback law. The drawback of the conventional explicit MPC approach is that the number of regions grows exponentially with the increase in the dimension of state and horizon length. In [25], the paper proposes a homotopy-based active set approach using dynamic programming, which adopts an iterative mechanism to compute the feedback law for the active set and update the active set. The algorithm results in a linear relationship between the computational load and length of the horizon.

2.3.3 Robust MPC for systems with multiplicative disturbances

Different forms of multiplicative uncertainties are considered in the literature. For example, Fleming et al.compare the approach of augmented variable (2.11) and tube MPC [26] for systems in the following structure

$$x_{t+1} = (A + \theta_t)x_t + Bu_t, \quad \theta_t \in \mathrm{Co}(\theta^{(1)}, \ldots, \theta^{(i)}) \tag{2.28}$$

For the augmented variable approach (called the 'lifted approach' in the paper), the number of constraints grows exponentially with the horizon length. Unlike the case for additive disturbance, the multiplicative uncertainty term cannot be grouped into a simpler term h_k. However, the tube MPC approach reduces computational load by using a tube to bound the predicted state. Using tube MPC, feasibility and asymptotic stability are guaranteed and the relationship between computational load and horizon length is linear.

[3] A constraint is active if the constraint holds with equality and removing the constraint causes a reduction in the optimal value of the cost. Using the Karush-Kuhn-Tucker (KKT) conditions and assuming strict complementarity, an inequality constraint is active if its associated Lagrange multiplier is greater than zero[24].

In [27], multiplicative uncertainties lies in both A and B matrix, where the system equation is given by

$$x_{t+1} = A_0 x_t + B_0 u_t + \sum_{j=1}^{q} (A_j x_t + B_j u_t)\theta_t^j. \tag{2.29}$$

where θ_t represents an independent disturbance variable with a known distribution.

Farina and Scattolini convert the probabilistic state and input constraints into deterministic constraints using Cantelli-Chebyshev inequality. For linear systems with multiplicative uncertainties, the proposed algorithm proves recursive feasibility and point-wise convergence of the state by choosing a suitable terminal cost and constraints. Although the nonlinear deterministic constraints can be replaced by the linear constraints, there is no easy rule on how to choose a suitable parameter ϵ in the linearised constraints.

2.3.4 Robust MPC for parameter uncertainty

As the information of an exact system model might not be available, a set of models can be used to describe the system behaviour. Consider the Linear Time Invariant (LTI) system with parameter uncertainty

$$x_{k+1} = A(\theta)x_t + B(\theta)u_t \tag{2.30}$$

where θ represents the unknown parameter, which is assumed to be either fixed or slowly varying and can be estimated online.

Fixed parameter uncertainty

In general, it is not straightforward to exploit the information that the parameters are constant rather than time-varying when predicting state trajectories. This is because system trajectories are contained in the convex hull predicted using the linear difference inclusion model, but not necessarily in the convex hull predicted using extreme points of the parameter uncertainty set (e.g. the vertices of a polytopic uncertainty set). In other words, the model parameter should have the flexibility to

take any value at each prediction time step, which means parameter uncertainty is treated like bounded multiplicative uncertainty without distribution information.

For FIR models, Campo and Morari propose a robust MPC controller that minimises the worse-case performance predicted by a set of linear models [28]. The coefficients of the system models can be conveniently estimated from step response experiments. Allwright and Papavasiliou work on the same problem and improve the result by deriving an equivalent linear program with fewer constraints [29]. However, both approaches require simple FIR models and do not apply to unstable systems.

Kothare et al.address model parameter uncertainties in two forms, namely linear difference inclusion ('polytopic model uncertainty' in the paper) and linear systems with a feedback uncertainty [30]. The systems in both cases can be expressed as linear time-varying systems with known dynamics. The input and output constraints can be converted to linear matrix inequalities (LMIs) and the resulting controller is robustly stabilising for all models in the uncertain model sets.

Another approach is to combine adaptive control with (robust) MPC for systems with parameter uncertainties. The combined adaptive MPC, which has the potential to improve the accuracy of the system model and regulate the system at the same time, will be explained in the next section.

Time-varying parameter uncertainty

For systems with time-varying parameter uncertainty, the main challenge is that measurements generated from changing dynamics result in empty model set. In other words, no static model can be consistent with all measurements. Theoretically, various approaches could be used to update the time-varying parameter, such as using a forgetting factor on the older data, inflating the model over time, and removing old data that would result in emptiness of the set [31]. However, most papers need an additional assumption which bounds the rate of parameter change in order to achieve recursive feasibility and robust satisfaction [32–34]. On the other hand,

Di Cairano and Danielson separate the parameter estimation from the controller design and maintain stability using a delay buffer for the parameter estimate[35].

Tanasokovic et al [32] convert the bound on the parameter change between consecutive steps to conditions on the estimated parameter set through Linear Programs (LPs). The proposed adaptive control is recursively feasible and satisfy the constraints robustly. Building on that, Bujarbaruah et al [33] uses a similar adaptive approach and proves Input to State Stability (ISS) of the proposed algorithm. Likewise, Lorenzen et al [34] introduce a dilation factor in computing the unfalsified parameter set. Stability can be obtained if the parameter change is slow and the input is persistently exciting. The same approach can be used in this thesis for systems with slowly varying parameters.

2.4 Adaptive Model Predictive Control

The performance of MPC depends crucially on the model quality because MPC relies on the system model to optimise the predicted future performance of the controlled system [1, 5]. To reduce model uncertainty, adaptive controllers have been combined with MPC to estimate the model parameters of the system model online while regulating the system subject to input and state constraints.

Adaptive control and recursive parameter identification date back to the 1960s. In [36], Astrom and Bohlin provide proof showing that persistent excitation ensures convergence of identified parameters to the true parameters of the plant. In 1980, the global stability of adaptive control was established for problems where no disturbances are present [37, 38]. The robustness of the controllers has been summarised in [39]. Later developments include systems with external disturbances (additive/multiplicative), time-varying systems [40] and the use of reduced-order models [41].

For adaptive MPC, the control input has the dual purpose of providing sufficient excitation for the identification of model parameters and regulating the system subject to constraints [13]. These two purposes are usually conflicting and the

optimisation of cost function over both the regulation and excitation requirements is generally intractable [42]. Despite various attempts in the field, robust adaptive MPC with tractable computational load remains an open topic under research [7].

2.4.1 System Identification Methods

The recent developments in adaptive MPC involve different strategies for system identification, such as a comparison set [43], set membership [15, 44, 45], recursive-least squares [46] and non-parametric methods [47–50].

Aswani et al use two models in the controller [43]. The nominal model provides a guarantee of robust feasibility and constraint satisfaction, while the learnt model uses statistical identification tools for system identification and performance improvement. The benefit of using a comparison model is that different types of statistical identification tools (parametric or nonparametric) can be used. However, the constraints cannot be relaxed to exploit the information from the updated learnt model because the quality of the learnt model is undeterminable in general.

Tanaskovic et al.consider a Finite Impulse Response (FIR) system[4] with state constraints, input constraints and disturbance [44]. By iteratively applying set membership identification, the approach provides a set of candidate plants and enforces constraints for all plants in the model. The complexity of the model set is bounded by taking an approximation of the polytopic set of unfalsified model parameters when the number of its faces reaches a predefined upper limit. In [15], a similar controller is designed for MIMO systems with guaranteed recursive feasibility. If no measurement noise is present, and no constraints are active, the steady-state output value is proven to be equal to the desired reference level. The paper demonstrates recursive feasibility but does not show convergence of the identified parameter set to the true parameters.

Heirung, Ydstie and Foss publish a series of papers that uses recursive least squares algorithm (RLS) for parameter estimation. In [46], a controller with

[4] A FIR system is an input-output model in which the output at time t is related to past inputs by $y(t) = \sum_{i=1}^{n} h_S(i)u(t-i)$

a two-stage cost function is proposed, which can control the process and probe the system for parameter information at the same time for a linear SISO system with additive disturbances. The first stage of the cost function represents the tradeoff between output control and input signal size, while the second stage uses the current parameter estimate in the prediction model to achieve a balance between output regulation and excitation. However, with the presence of constraints, the optimization problem is nonconvex and nonlinear. Heirung et al.build on the result and design controllers that converge to a standard certainty-equivalence MPC as uncertainty decreases or generated information is sufficient[51]. The optimization problem can be formulated into a quadratically-constrained quadratic programming (QCQP) problem. Heirung et al.also design a similar controller that transforms a stochastic problem with probability constraints into a deterministic problem that can be solved using QCQP[52]. However, in all three papers, proofs for recursive feasibility and stability are absent.

Maddalena et al.propose Kernel Predictive Control (KPC), where the system models are learnt through non-parametric kernel regression [50]. The optimised control input sequence can drive the system to the terminal set under constraints. However, recursive feasibility cannot be guaranteed because distinct system models are formulated for each prediction step.

Set-membership identification

We next explain the system identification method used in the thesis. Define Δ_t to be the unfalsifiable set of parameter estimates that includes state and input information from the current time step. We can update the parameter set estimate Θ_t using various methods, including minimal [53], fixed-complexity [34], and limited-complexity [15] update laws. In each case, Θ_t is non-increasing and $\Theta_t \subseteq \Theta_{t-1}$ for all $t \in \mathbb{N}_{>0}$. Here we use a polypotic description for the parameter set Θ_t.

Minimal Parameter set update: In this case, the parameter set estimate Θ_t at each time $t > 0$ is the intersection of Θ_{t-1} and the unfalsified set Δ_t,

$$\Theta_t = \Delta_t \cap \Theta_{t-1}. \tag{2.31}$$

This results in the smallest parameter set with given information. However, the Θ_t set can have unbounded number of faces, making computation intractable over long intervals.

Fixed-complexity parameter set update: Let $M_\Theta \in \mathbb{R}^{r\times p}$ be a matrix chosen *a priori*. In this case, the parameter set estimate Θ_t is defined for $t > 0$ as

$$\Theta_t = \Theta(\mu_t) = \{\theta : M_\Theta \theta \leq \mu_t\}$$

where $\mu_t \in \mathbb{R}^r$ is determined so that Θ_t is the smallest set containing the intersection of Θ_{t-1} and the unfalsified sets $\Delta_{t-N_u+1}, \ldots, \Delta_t$,

$$\mu_t = \min_{\mu\in\mathbb{R}^r} \mathrm{vol}\Big(\Theta(\mu)\Big) \text{ subject to } \bigcap_{j=t-N_u+1}^{t} \Delta_j \cap \Theta_{t-1} \subseteq \Theta(\mu) \tag{2.32}$$

where $\Delta_j := \mathbb{R}^p$ for $j \leq 0$ and N_u is the parameter update window length. Note that μ_t can be computed by solving a set of linear programs (see Section 3.3 in [22]).

Limited-complexity parameter set update: In this case, Θ_t is updated using (2.31) if it consists of the intersection of fewer half-spaces than some predefined limit M_1. Once this limit is reached, each new facet that is added to the polytope is parallel to a hyperplane that belongs to a predefined set of M_2 hyperplanes, so that the total number of half-spaces defining Θ_t is at all times bounded by M_1+M_2. In other words, this update law is a combination of minimal and fixed-complexity update laws.

Other descriptions for parameter sets are also available. For example, Adetola et al.use an n-dimensional ball description for the parameter uncertainty [11]. The ball description facilitates the derivation of state error bound using the sum of norms. Also, the proposed approach guarantees that the estimated parameter set is non-increasing in size by designing the parameter update law with additional constraints on centre and radius.

2.4.2 Persistent Excitation

Persistent excitation conditions, if satisfied by the control input, ensure that a given identification scheme converges to the true model parameters [8], [54].

For the general parameter identification problem $y_i = \Phi_i^\top \theta + w_i$, where y_i is the observed output, Φ_i is the regressor, θ is the unknown paramter and w_i is unmodelled noise/disturbance. The regressor Φ_k is persistently exciting (PE) if

$$\underline{\epsilon}^2 \mathbb{I} \preceq \sum_{k=t}^{t+N_u-1} \Phi_k^\top \Phi_k \preceq \bar{\epsilon}^2 \mathbb{I} \tag{2.33}$$

for some positive $\underline{\epsilon}$, $\bar{\epsilon}$ and N_u and for all $t \geq 0$.

In adaptive identification and control, the Persistent Excitation (PE) condition plays a crucial role in establishing convergence of parameter estimates [55, 56]. Although the literature on adaptive control has long acknowledged the need for persistently exciting inputs for system identification [8], few papers have explored how to incorporate Persistence of Excitation (PE) conditions with feasibility guarantee within adaptive MPC [7]. By incorporating constraints to ensure appropriate PE conditions, a constrained MPC strategy can impose a lower bound on the expected rate of parameter convergence.

Bai et al.prove that the diameter of the membership set under minimal volume update law converges with probability 1 to the actual parameter vector (assumed constant) if a tight bound on additive disturbances is known and the input sequence is persistently exciting (PE) [57]. However, the minimal parameter update law is restrictive in applications because the complexity of the parameter set grows without bound over time, resulting in intractable optimisation problems.

In [58], the Persistent Excitation condition is achieved by injecting random noise into the linear feedback law, where the variance of injected noises and the linear feedback gain are optimisation variables. The proposed algorithm minimises the worst-case cost attached by the new LQ controller, while reducing uncertainty in the model sets. However, the paper does not include any input or state constraints and cannot provide any guaranteed on recursive feasibility or closed-loop stability.

In addition, the PE condition is nonconvex. Marafioti, Bitmead and Hovd converted a nonlinear MPC problem with PE constraints into two Quadratic Program (QP) problems by decomposing the nonconvex quadratic constraints into two linear constraints [13]. The resulting control algorithm admits a feasible periodic solution although not necessarily optimal. However, the proposed approach fails to demonstrate closed-loop system stability or robust constraint satisfaction.

2.4.3 Adaptive robust MPC without PE conditions

The control input of adaptive MPC has the dual objectives of estimating system parameters and controlling the system subject to constraints. Different adaptive MPC algorithms place varying emphasis on the two competing objectives. Some papers focus on robust constraint satisfaction and stability through constraint tightening [10], min-max cost formulations [11, 59] or tube MPC [34, 60]. They either assume the input can be persistently exciting [34], or neglect that condition in the formulations [10, 11, 59]. In general, these algorithms might result in more accurate system descriptions, but this is not guaranteed.

Di Cairano combines a MPC algorithm with online parameter estimation and uses a parameter-dependent Lyapunov function for the terminal cost [10]. For uncertain systems that can be described using polytopic linear difference inclusions, the resulting algorithm is recursively feasible and input-to-state stable(ISS). The proposed algorithm enforces robust constraint satisfaction using a robust control invariant set approach. However, the parameter prediction is N-step delayed for stability requirement. In his recent work [35], the control strategy is extended to general nonlinear systems and the same closed-loop properties are proven under given assumptions. The new work separates controller and parameter estimator design, which means different learning methods can be incorporated with the control strategy. Although both works require only the online solution of Quadratic Programs (QPs), they fail to include any conditions on PE condition and therefore cannot guarantee convergence of the parameter estimation.

Lorenzen, Allgoewer and Cannon combine online set-membership identification with tube MPC to improve performance and reduce conservatism [45]. With a homothetic tube MPC formulation [21], the algorithm is made computationally tractable by bounding the state x with a polytope $\mathcal{X}$ which can translate and dilate. Specifically, the tube cross-sections are defined by $\mathcal{X}_{k|t} = \alpha_{k|t}\mathcal{X}_0 + \hat{x}_{k|t}$ where $\hat{x}_{k|t}$ is a nominal predicted state centre and $\alpha_{k|t}$ is a scalar optimization variable. As a result, robust satisfaction of constraints for all $x \in \mathcal{X}_{k|t}$ can be simplified into constraints on the vertices of $\mathcal{X}_{k|t}$. Recursive feasibility and practical stability are demonstrated. Building on [45], Lorenzen et al.propose an adaptive MPC strategy that uses set-based identification together with a point estimate [34]. Closed-loop properties such as robust constraint satisfaction, recursive feasibility and stability are proven. The same properties are demonstrated for systems with slowly varying parameter vectors when the rate of parameter variation is bounded. In addition, results on parameter convergence based on PE conditions are presented. However, both approaches suffer from a lack of flexibility the its robust MPC formulation, which is based on homothetic tubes [21], allowing only the centres and scalings of tube cross-sections to be optimised online. In addition, neither paper provides recursively feasible conditions to ensure persistently exciting control inputs.

2.4.4 Adaptive robust MPC with PE or similar conditions

Some approaches aim to achieve the dual objectives of system regulation and sufficient excitation at the same time. They either augment the excitation condition in the cost function [61], include it as a separate step in the online optimisation [15], or achieve one target at a time [14].

Weiss and Di Cairano balance the conflicting objectives of identifying the system parameters and regulating the systems using an augmented cost [61]. The proposed algorithms achieved robust constraint satisfaction and stability by selecting the control input from the Robust Admissible Invariant (RAI) set and imposing an additional constraint based on control Lyapunov functions. However, the proposed

optimisation is nonconvex and existing nonlinear solvers fail to guarantee global convergence. In addition, the resulting input is more likely, but not guaranteed to satisfy the PE condition.

Tanaskovic et al.combine an iterative set membership identification algorithm with model predictive control [15]. All input and state constraints are satisfied robustly by imposing the constraints for all the plant models inside the model sets. To facilitate parameter estimation, a separate optimisation step is proposed that minimise the size of possible model sets. However, the resulting control strategy requires the solution of a nonconvex and infinite-dimensional program and can only be simplified for specific examples.

Gonzalez et al.separate the two objectives via the use of a switching control strategy [14]. By defining an invariant region for identification and suitable cost function, the state is driven to the invariant set when the state is outside the region. Once the state enters the target region and identification can be performed without constraint violation, different strategies can be incorporated to inject excitation into the systems. Although the proposed approach guarantees stability and satisfaction of PE condition, it is only applicable for open-loop stable linear systems and the existence of the target region depends on the specific example.

Hernandez et al.propose an algorithm that robustly satisfies state and input constraints while guarantees convergence of parameter estimates [9]. By splitting the input into regulatory and exciting parts, the algorithm uses robust MPC for regulation and injects excitation via another finite-horizon nonconvex optimisation. However, the stability of the proposed algorithm relies on the nominal system parameters and therefore, the system model cannot be updated online.

2.5 Adaptive MPC for nonlinear systems

For nonlinear systems, algorithms that simultaneously achieve computational tractability and robust constraint satisfaction without restrictive assumptions are still under development [62]. As a result, many (perhaps even most) papers focus

more on simulation performance rather than theoretical properties. Existing work includes Min-max and Lipschitz-based approaches [11, 59, 63, 64], incremental Lyapunov functions [65], and the incorporation of nonparametric machine learning methods [47–50]. In particular, various machine learning techniques are combined with predictive controllers for nonlinear systems. However, due to the data-driven nature of the derived system model, the proposed algorithms usually fail to guarantee desirable control properties such as recursive feasibility or stability without unrealistic assumptions.

In [11], two adaptive MPC strategies, namely a min-max approach and a Lipschitz-based approach, are proposed for constrained nonlinear systems with affine parameter vectors. The min-max approach maximises the objective cost with respect to the parameter vector and minimises the cost over the feedback control law. However, the resulting optimisation problem is computationally intractable. On the other hand, the Lipschitz approach reduces computational complexity but introduces conservatism by controlling a nominal model with a state prediction error bounded by a suitable Lipschitz constant. Recursive feasibility and stability are proven for both cases under given assumptions. The parameter update law is driven by the system's excitation over the observed error, which means contraction of the parameter set can happen even if the tracking error is zero. However, the numerical example provided in the paper fails to satisfy the assumption that the terminal set is a robust invariant set for all possible uncertainties.

Building on [11], Goncalves and Guay propose a min-max approach and a Lipschitz-based approach for nonlinear uncertain systems with exogenous disturbances[63]. By combining set-based parameter identification and robust MPC, the proposed strategies regulate the system subject to input and state constraints and update the plant model when model improvement is guaranteed. Robust constraint satisfaction, recursive feasibility and stability are demonstrated for both approaches. The paper also proves the convergence of parameter estimates to their true values under the persistent excitation condition. In [64], Goncalves and Guay extend the

results to discrete-time nonlinear uncertain systems with exogenous disturbance and derive similar control properties.

In [65], Koehler et al.propose an adaptive robust MPC formulation by combining the results on incremental Lyapunov functions [66, 67] and set membership identification. The proposed approach assumes a nominal system that is exponentially stabilisable with an incremental Lyapunov function and a known contraction rate. The tube dynamics is constructed based on the contraction rate and a nonlinear function that bounds the difference between the nominal and uncertain system with bounded additive disturbances and affine parametric uncertainties. The proposed control strategy guarantees robust recursive feasibility and robust constraint satisfaction. Moreover, the strategy ensures finite-gain stability with respect to disturbances by using an additional mean-squared estimate in the objective function. However, choosing the incremental Lyapunov function and parameterisation are in general challenging. Also, the terminal condition is conservative and therefore the numerical example can only consider very small parametric uncertainties.

Di Cairano and Danielson propose an adaptive robust MPC framework using robust control invariant sets and prove robust constraint satisfaction and input-to-state stability [35]. However, for general nonlinear problems, the structure and complexity of the cost functions and constraint sets are difficult to predict. Also, the optimization problem may be non-convex, which means convergence of the numerical algorithm cannot be ensured.

Manzano et al.combine a machine learning technique, Kinky inference [68], with robust MPC for nonlinear systems subject to output constraints[49]. The proposed algorithm assumes the information of a tight error bound that accounts for the difference between the estimated output and the actual closed-loop output. As a result, the bound for the prediction error can be calculated and used for constraint tightening. Assuming the tightened constraint is non-empty along the prediction horizon, the proposed algorithm is proven to be input-to-state stable without a terminal constraint. As shown from the example, a large amount of data

is needed to train the system model and cross-validate the system model and the control parameters.

Akpan and Hassapic present an adaptive MPC algorithm that combines neural network(NN) based system identification with nonlinear MPC [47]. The NN identification model has a series-parallel architecture and is trained online using an adaptive recursive least squares algorithm. Based on the derived NN model, the system's output is predicted and optimised using Newton method. The two simulation examples presented in the paper show that the proposed method can adapt to time-varying nonlinear systems and outperform the existing algorithm at the expense of an increase in computational time.

Reese and Collins propose an adaptive sampling-based MPC that samples the input space using pseudo-random samplings [48] and represents the system model using a Radial Basis Function (RBF) neural network. Given the current state and input samples, a search graph with tree structure is produced by propagating the model. The optimisation problem is to find the node at the depth of the prediction horizon that achieves the lowest cumulative cost using graph search algorithms. Output constraints can be incorporated by removing the node that violates the constraint. As shown from the examples, the proposed algorithm avoids local minima successfully compared with gradient-based methods.

3

Min-max robust MPC with online parameter identification

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3.1 Introduction

In this chapter, we consider state space systems, set-based model identification and robust tube MPC. Compared with [15, 45], the proposed algorithm provides flexibility in modelling the effects of model uncertainty on predicted input and state trajectories, incorporates a convex approximation for persistence of excitation,

and gives asymptotic bounds on the closed-loop state that depend explicitly on the identified parameter set. In particular, a polytopic description is used so that the identified parameter set has pre-specified complexity. This is combined with a tube MPC formulation based on [69] with fixed complexity polytopic sets. As a result, the approach retains computational tractability but allows greater flexibility in the definition of predicted state tubes than [45] since the shapes of the tube cross-sections are optimised in addition to their centres and scalings. We also propose a convex constraint that ensures persistence of excitation (PE) conditions, and incorporate the conditions in an MPC algorithm with guaranteed stability. The PE condition facilitates the convergence of the identified model to the true system parameters, thus extending the result of [13] to MIMO systems. Finally, we show that the resulting MPC strategy is recursively feasible and give asymptotic bounds on the convergence of the closed-loop state that depend on the identified parameter set. This bounding set is used in the definition of the MPC cost and is updated online as the parameter set estimate is updated.

3.2 Problem formulation and preliminaries

This chapter considers a linear system with linear state and input constraints and unknown additive disturbance:

$$x_{t+1} = A(\theta^*)x_t + B(\theta^*)u_t + w_t, \tag{3.1}$$

Here $x_t \in \mathbb{R}^{n_x}$ is the system state, $u_t \in \mathbb{R}^{n_u}$ is the control input, $w_t \in \mathbb{R}^{n_x}$ is the disturbance input, and t is the discrete time index. The system matrices $A(\theta)$ and $B(\theta)$ depend on an unknown but constant parameter $\theta \in \mathbb{R}^p$.

The state and input constraints are given by

$$G_x x_t + G_u u_t \leq \mathbf{1} \quad \forall t \in \mathbb{N}_{\geq 0} \tag{3.2}$$

for given $G_x \in \mathbb{R}^{n_c \times n_x}$ and $G_u \in \mathbb{R}^{n_c \times n_u}$.

Assumption 3.1 (Additive disturbance)**.** *the disturbance w_t lies in a convex and compact polytope $\mathcal{W}$, where*

$$\mathcal{W} = \{w : \Pi_w w \leq \pi_w\} \tag{3.3}$$

with $\Pi_w \in \mathbb{R}^{q_w \times n_x}$, $\pi_w \in \mathbb{R}^{q_w}$ and $\pi_w > 0$ and $q_w \geq n_x$.

Assumption 3.2 (Parameter uncertainty)**.** *the system matrices are affine functions of the parameter vector $\theta \in \mathbb{R}^p$*

$$(A(\theta), B(\theta)) = (A_0, B_0) + \sum_{i=1}^{p} (A_i, B_i)[\theta]_i \tag{3.4}$$

for known matrices A_j, B_j, $j = 1, \ldots, p$, and true value θ^* lies in a known, bounded, convex polytope Θ_0 given by

$$\begin{aligned} &\textit{H-form:} \qquad \Theta_0 = \{\theta : M_\Theta \theta \leq \mu_0\} \\ &\textit{V-form:} \qquad \Theta_0 = \mathrm{co}\{\theta_0^{(1)}, \ldots, \theta_0^{(m)}\}. \end{aligned}$$

where M_Θ fixes the face directions of the parameter set Θ.

Here θ^* is assumed to be unique. If the true plant model is not uniquely representable by the parameter θ^*, we can always reduce multiplicity by redefining the system matrices (A_i, B_i) such that θ^* can be uniquely defined.

To obtain finite numbers of decision variables and constraints in the MPC optimisation problem, the predicted control sequences are assumed to have the dual mode form:

$$u_{k|t} = \begin{cases} Kx_{k|t} + v_{k|t} & \forall k \in \mathbb{N}_{[0,N-1]} \\ Kx_{k|t} & \forall k \geq N \end{cases} \tag{3.5}$$

where $v_{0|t}, v_{1|t} \ldots, v_{N-1|t}$ are optimisation variables at time t and N is the MPC prediction horizon. The feedback gain K is designed offline and is assumed to robustly stabilize the uncertain system $x_{t+1} = (A(\theta) + B(\theta)K)x_t$, $\theta \in \Theta_0$ in the absence of constraints.

Assumption 3.3 (Feedback gain and contractive set)**.** *There exists a polytopic set* $\mathcal{X} = \{x : Tx \leq \mathbf{1}\}$ *and a feedback gain* K *such that* $\mathcal{X}$ *is* λ*-contractive for some* $\lambda \in [0, 1)$*, i.e.*

$$T\Big(A(\theta) + B(\theta)K\Big)x \leq \lambda \mathbf{1} \quad \textit{for all } x \in \{x : Tx \leq \mathbf{1}\} \textit{ and } \theta \in \Theta_0. \tag{3.6}$$

for all $x \in \{x : Tx \leq \mathbf{1}\}$ *and* $\theta \in \Theta_0$*. The representation* $\mathcal{X} = \{x : Tx \leq \mathbf{1}\}$ *is assumed to be minimal in the sense that it contains no redundant inequalities.*

The existence of a λ-contractive set is a necessary and sufficient condition for positive invariance of polyhedral sets (See Theorem 4.40 and 4.41 in [22]). In other words, any gain K satisfying Assumption 3.3 is stabilising for the closed-loop system. The design of K and T will be explained in Section 3.3.4.

Given knowledge of $\mathcal{W}$ and an initial parameter set Θ_0, the control objective is to robustly solve an optimal regulation problem while respecting the constraints (3.2) and ensuring stability of the closed-loop system. Simultaneously with controlling the system, an additional objective is to update the model parameter set Θ_t online using measurements of the state x_t, and to use the updated parameter set to improve controller performance.

3.3 Adaptive Robust MPC

In this section, a suitable parameter estimation scheme based on [53, 70] is introduced. This is followed by the construction of a tube bounding predicted model states and corresponding constraints. Subsequently, the MPC cost is discussed, linearised sufficient conditions for persistent excitation are described, and a summary of the MPC algorithm is given. The section concludes with a summary of closed-loop system properties. Standard results on polyhedral set inclusion conditions (e.g. [22]) are employed. In particular, we make use of the following result on linearly parameterised sets.

Lemma 3.1. *Let* $\mathcal{P}_1 = \{x : F_1 x \leq f_1\}$ *and* $\mathcal{P}_2(\theta) = \{x : F_2(\theta)x \leq f_2\}$, *where* $F_1 \in \mathbb{R}^{n_1 \times n_x}$, $F_2(\theta) \in \mathbb{R}^{n_2 \times n_x}$ *and* $[F_2(\theta)]_i = \theta^\top \hat{F}_{2,i}$, $i \in \mathbb{N}_{[1,n_2]}$. *Then* $\mathcal{P}_1 \subseteq \mathcal{P}_2(\theta)$ *for all* $\theta \in \Theta \subset \mathbb{R}^p$ *if and only if there exists* $\hat{H}_i \in \mathbb{R}^{p \times n_1}$ *satisfying*

$$\theta^\top \hat{H}_i \geq 0 \quad \hat{H}_i F_1 = \hat{F}_{2,i} \quad \theta^\top \hat{H}_i f_1 \leq f_{2,i}$$

for all $i \in \mathbb{N}_{[1,n_2]}$ *and all* $\theta \in \Theta$.

Proof. Using standard results on polyhedral set inclusion conditions [22], $\mathcal{P}_1 \subseteq \mathcal{P}_2(\theta)$ holds for all $\theta \in \Theta$ if and only if there exists $H(\theta) \in \mathbb{R}^{n_2 \times n_1}$ satisfying $H(\theta) \geq 0$, $H(\theta)F_1 = F_2(\theta)$ and $H(\theta)f_1 \leq f_2$ for all $\theta \in \Theta$. Since $[F_2(\theta)]_i = \theta^\top \hat{F}_{2,i}$, the condition $H(\theta)F_1 = F_2(\theta)$ can be rewritten as $[H(\theta)]_i F_1 = \theta^\top \hat{F}_{2,i}$. Noting that Θ can be assumed without loss of generality to be a full-dimensional subset of $\mathbb{R}^p$ (i.e. Θ necessarily contains a p-dimensional ball of radius $r > 0$), it follows that $[H(\theta)]_i = \theta^\top \hat{H}_i$ where $\hat{H}_i F_1 = \hat{F}_{2,i}$. □

3.3.1 System parameter estimation

At time step t we use observations of the system state x_t, which is assumed to be known, to determine a polytopic set Δ_t of possible model parameters. This unfalsified parameter set is then combined with the current polytopic parameter set estimate to construct a new parameter set estimate Θ_t.

Unfalsified parameter set

Define $\Phi(x, u) \in \mathbb{R}^{n_x \times p}$ as

$$\Phi(x_t, u_t) = \Phi_t = \begin{bmatrix} A_1 x_t + B_1 u_t & \cdots & A_p x_t + B_p u_t \end{bmatrix}, \tag{3.7}$$

and let $\phi_t \in \mathbb{R}^{n_x}$ be defined

$$\phi(x_t, u_t) = \phi_t = A_0 x_t + B_0 u_t. \tag{3.8}$$

From the disturbance set $\mathcal{W}$ in (3.3), given x_t, x_{t-1} and u_{t-1}, the unfalsified parameter set at time t is given by

$$\Delta_t = \{\theta : x_t - (A(\theta)x_{t-1} + B(\theta)u_{t-1}) \in \mathcal{W}\}$$

$$= \{\theta : \Pi_w \Phi(x_{t-1}, u_{t-1})\theta \leq \pi_w + \Pi_w \phi(x_{t-1}, u_{t-1}) - x_t\} = \{\theta : P_t\theta \leq p_t\} \quad (3.9)$$

with $P_t = -\Pi_w \Phi(x_{t-1}, u_{t-1})$ and $p_t = \pi_w + \Pi_w \phi(x_{t-1}, u_{t-1}) - x_t$.

Parameter set update

Let $M_\Theta \in \mathbb{R}^{2p \times p}$ be an *a priori* chosen matrix defining the initial parameter set $\Theta_0 = \{\theta : M_\Theta \theta \leq \mu_0\}$. At each time step $t \in \mathbb{N}_{\geq 0}$ the estimated parameter set is defined as

$$\Theta_t = \Theta(\mu_t) = \{\theta : M_\Theta \theta \leq \mu_t\}, \quad (3.10)$$

where μ_t is determined online. The complexity of Θ_t is controlled by fixing M_Θ, which is equivalent to fixing the directions of the half-spaces defining Θ_t. The smallest set Θ_{t+1} satisfying $\Theta_{t+1} \supseteq \Theta_t \cap \Delta_t$ is obtained by solving, for each $i \in \mathbb{N}_{[1,2p]}$ the linear program (LP):

$$\begin{aligned} [\mu_{t+1}]_i = & \min_{\mu \in \mathbb{R},\, H_i \in \mathbb{R}^{1 \times (2p + n_w)}} \mu \\ \text{s.t. } & H_i \begin{bmatrix} M_\Theta \\ P_t \end{bmatrix} = [M_\Theta]_i, \; H_i \begin{bmatrix} \mu_t \\ p_t \end{bmatrix} \leq \mu, \; H_i \geq 0. \end{aligned} \quad (3.11)$$

Lemma 3.2. *If $M_\Theta \theta \leq \mu_0$ and μ_{t+1} is defined by (3.11) for all $t \in \mathbb{N}_{\geq 0}$, then $\theta^* \in \Theta_t$ and*

$$\Theta_t \supseteq \Theta_{t+1} \supseteq \Theta_t \cap \Delta_t$$

for all $t \in \mathbb{N}_{\geq 0}$. Furthermore μ_{t+1} defines the minimum volume set $\Theta_{t+1} = \{\theta : M_\Theta \theta \leq \mu_{t+1}\}$ containing $\Theta_t \cap \Delta_t$.

Proof. First, the constraints in (3.11) ensure that $\Theta_{t+1} \supseteq \Theta_t \cap \Delta_t$. As $\theta^* \in \Delta_t$ for all t and $\theta^* \in \Theta_0$, by induction we have $\theta \in \Theta_t$ for all t. In addition, $H_i = [\,\mathbb{I} \;\; 0\,]$ and $\mu_{t+1} = \mu_t$ provide a feasible solution for (3.11), and it follows that $\mu_{t+1} \leq \mu_t$ and hence $\Theta_{t+1} \subseteq \Theta_t$ for all t. Finally, note that the volume of Θ_{t+1} is $(2^p / \det(M_\Theta)) \prod_{i=1}^{p} ([\mu_{t+1}]_i + [\mu_{t+1}]_{p+i})$, which is minimized for any given M_Θ by the choice of objective in (3.11). □

Remark 3.3. *Instead of using only the current observation Δ_t to update the parameter set estimate, a tighter set estimate can be achieved at the expense of increased computation by block-processing all past observations or observations in a moving window of length n, see [53] for details.*

3.3.2 Polytopic tubes for constraint satisfaction

This subsection is concerned with predicted state and input trajectories. To simplify notation, we omit the subscript t indicating the time at which predictions are made; in particular the current state x_t and parameter set Θ_t are denoted x, Θ, and k steps-ahead predictions of x and u are denoted x_k, u_k. For given $T \in \mathbb{R}^{n_\alpha \times n_x}$ satisfying Assumption 3 and $\alpha_k \in \mathbb{R}^{n_\alpha}$, let

$$\mathcal{X}_k = \{x : Tx \leq \alpha_k\}. \tag{3.12}$$

The selection of T will be explained in Section 3.3.4.

To ensure that the predicted state and control sequences satisfy the operating constraints (3.2) robustly for the given uncertainty bounds we construct a tube (sequence of sets) $\mathcal{X}_0, \mathcal{X}_1, \ldots$ satisfying, for all $k \in \mathbb{N}_{\geq 0}$,

$$\begin{aligned} &A_K(\theta)x + B(\theta)v_k + w \in \mathcal{X}_{k+1} \\ &\text{for all } x \in \mathcal{X}_k,\ w \in \mathcal{W},\ \theta \in \Theta = \text{co}\{\theta^{(1)}, \ldots \theta^{(m)}\} \end{aligned} \tag{3.13}$$

where $A_K(\theta) = A(\theta) + B(\theta)K = \sum_{i=0}^{p} A_{K,i}[\theta]_i$ with $A_{K,i} = A_i + B_i K$ and $[\theta]_0 = 1$. For notational convenience we also define $\hat{\theta}^{(j)}$ as $\hat{\theta}^{(j)\top} = \begin{bmatrix} 1 & \theta^{(j)\top} \end{bmatrix}$ for all $j \in \mathbb{N}_{[1,m]}$.

Lemma 3.4. *For $x_k \in \mathcal{X}_k$ and $u_k = Kx_k + v_k$, the constraint (3.2) is satisfied at the kth prediction time step if*

$$H_c \alpha_k + G_u v_k \leq \mathbf{1} \tag{3.14}$$

where H_c is computed offline by solving the linear program (LP):

$$[H_c]_i = \underset{H \in \mathbb{R}^{1 \times n_\alpha}}{\text{argmin}}\ H\mathbf{1} \quad s.t. \quad H \geq 0,\ HT = [G_x + G_u K]_i. \tag{3.15}$$

for each $i \in \mathbb{N}_{[1,n_c]}$.

Proof. The constraint (3.2) applied to $x_k \in \mathcal{X}_k$ and $u_k = Kx_k + v_k$ is equivalent to

$$(G_x + G_u K)x + G_u v_k \leq \mathbf{1} \quad \text{for all } x \in \mathcal{X}_k, \tag{3.16}$$

and since $\mathcal{X}_k = \{x : Tx \leq \alpha_k\}$, this can be expressed

$$\{x : Tx \leq \alpha_k\} \subseteq \{x : (G_u + G_u K)x \leq \mathbf{1} - G_u v_k\}.$$

From [22], condition (3.16) is therefore equivalent to the existence of H_c satisfying:

$$H_c \geq 0 \tag{3.17a}$$

$$H_c T = G_x + G_u K \tag{3.17b}$$

$$H_c \alpha_k + G_u v_k \leq \mathbf{1} \tag{3.17c}$$

In the robust MPC implementation, the predicted control sequence $v_0, \ldots, v_{N-1}$ and state tube variables $\alpha_1, \ldots, \alpha_N$ are optimised online. To ensure that the online MPC optimisation is convex, matrices H_c satisfying (3.17a) and (3.17b) are computed offline. This is done by solving (3.15) for each row of H_c, so that (3.14) (which is linear in the online decision variables) is sufficient to ensure (3.16). □

Lemma 3.5. *The predicted state tube satisfies (3.13) if, for all $j \in \mathbb{N}_{[1,m]}$ and $i \in \mathbb{N}_{[1,n_\alpha]}$,*

$$\hat{\theta}^{(j)\top} \hat{H}_i \alpha_k + [T]_i B(\theta^{(j)}) v_k + \bar{w}_i \leq [\alpha_{k+1}]_i, \tag{3.18}$$

where $\bar{w}_i = \max_{w \in \mathcal{W}} [T]_i w$ and the matrices $\hat{H}_i$ are computed offline by solving an LP for each $i \in \mathbb{N}_{[1,n_\alpha]}$:

$$\begin{aligned} \hat{H}_i = \underset{H \in \mathbb{R}^{(p+1)\times n_\alpha}}{\operatorname{argmin}} \max_{j \in \mathbb{N}_{[1,m]}} \hat{\theta}^{(j)\top} H \mathbf{1} \\ \text{subject to} \quad \hat{\theta}^{(j)\top} H \geq 0,\ j \in \mathbb{N}_{[1,m]},\ HT = \begin{bmatrix} [T]_i A_{K,0} \\ \vdots \\ [T]_i A_{K,p} \end{bmatrix}. \end{aligned} \tag{3.19}$$

Proof. By a similar argument to the proof of Lemma 3.4, the condition for satisfaction of (3.13) is equivalent to

$$\{x : Tx \leq \alpha_k\} \subseteq \{x : TA_K(\theta)x + TB(\theta)v_k + \bar{w} \leq \alpha_{k+1}\}$$

for all $\theta \in \Theta$, where $\bar{w} = [\,\bar{w}_1 \;\cdots\; \bar{w}_{n_\alpha}\,]^\top$. Applying Lemma 3.1, the linear dependence of A_K and B on θ implies that condition (3.13) is equivalent to the existence of $\hat{H}_1, \ldots, \hat{H}_{n_\alpha}$ satisfying, for all $i \in \mathbb{N}_{[1,n_\alpha]}$ and all $j \in \mathbb{N}_{[1,m]}$:

$$\hat{\theta}^{(j)\top} \hat{H}_i \geq 0 \tag{3.20a}$$

$$\hat{H}_i T = \begin{bmatrix} [T]_i A_{K,0} \\ \vdots \\ [T]_i A_{K,p} \end{bmatrix} \tag{3.20b}$$

$$\hat{\theta}^{(j)\top} \hat{H}_i \alpha_k + [T]_i B(\theta^{(j)}) v_k + \bar{w}_i \leq [\alpha_{k+1}]_i. \tag{3.20c}$$

To obtain a convex online MPC optimisation in which v_k and α_k are decision variables, matrices $\hat{H}_i$ satisfying (3.20a) and (3.20b) are computed offline by solving the LP (3.19) for each $i \in \mathbb{N}_{[1,n_\alpha]}$. As a result, constraint (3.20c) (which is linear in the online decision variables) is sufficient to ensure satisfaction of (3.13). □

Initial and final conditions on state tubes are imposed as

$$\alpha_0 \geq Tx, \tag{3.21}$$

$$[\alpha_N]_i \geq \hat{\theta}^{(j)\top} \hat{H}_i \alpha_N + \bar{w}_i, \quad \forall j \in \mathbb{N}_{[1,m]}, i \in \mathbb{N}_{[1,n_\alpha]} \tag{3.22}$$

$$H_c \alpha_N \leq \mathbf{1}. \tag{3.23}$$

Here (3.21) implies $x \in \mathcal{X}_0$ whereas (3.22) and (3.23) ensure $\mathcal{X}_N$ is robustly invariant for the system $x_{k+1} = A_K(\theta)x_k + w_k$ with constraint $G_x x_k + G_u u_k \leq \mathbf{1}$, for all $\theta \in \Theta$ and $w_k \in \mathcal{W}$.

Remark 3.6. *The existence of some α_N satisfying (3.22) is ensured by the λ-contractivity condition (3.6), which implies that the solution of (3.19) satisfies $\hat{\theta}^{(j)\top} \hat{H}_i \mathbf{1} \leq \lambda$ for all $i \in \mathbb{N}_{[1,n_\alpha]}$. Therefore, with $\alpha_N = r\mathbf{1}$ the RHS of (3.22) has upper bound $\lambda r \mathbf{1} + \bar{w}$, and since $\lambda \in [0, 1)$, this choice of α_N is feasible for (3.22) if r is sufficiently large.*

3.3.3 Min-max objective function

Consider a min-max MPC strategy based on a piecewise-linear worst case cost index of the form

$$\max_{x_k \in \mathcal{X}_k,\, k=0,1,\ldots} \sum_{k=0}^{\infty} \Big(\max\{Qx_k\} + \max\{Ru_k\} \Big) \tag{3.24}$$

where $Q \in \mathbb{R}^{n_Q \times n_x}$ and $R \in \mathbb{R}^{n_R \times n_u}$ are weighting matrices and $\max\{Qx\}$ and $\max\{Ru\}$ denote the maximum elements of Qx and Ru respectively. We assume that Q and R are chosen so that $\max\{Qx\}$ and $\max\{Ru\}$ are positive definite functions of x and u, so that $\{x : Qx \leq \mathbf{1}\}$ and $\{u : Ru \leq \mathbf{1}\}$ are compact polytopic sets.

Lemma 3.7. *If s_k and t_k satisfy the linear constraints:*

$$s_k \mathbf{1} \geq H_Q \alpha_k \tag{3.25}$$

$$t_k \mathbf{1} \geq H_R \alpha_k + R v_k \tag{3.26}$$

where H_Q, H_R are computed offline by solving the LPs:

$$[H_Q]_i = \underset{H \in \mathbb{R}^{1 \times n_T}}{\operatorname{argmin}} \; H\mathbf{1} \quad s.t. \quad H \geq 0, \; HT = [Q]_i \tag{3.27}$$

$$[H_R]_j = \underset{H \in \mathbb{R}^{1 \times n_T}}{\operatorname{argmin}} \; H\mathbf{1} \quad s.t. \quad H \geq 0, \; HT = [R]_j K \tag{3.28}$$

for $i \in \mathbb{N}_{[1,n_Q]}$ and $j \in \mathbb{N}_{[1,n_R]}$, then the terms in (3.24) satisfy

$$\max_{x \in \mathcal{X}_k} \max\{Qx\} \leq s_k, \quad \max_{x \in \mathcal{X}_k} \max\{R(Kx_k + v_k)\} \leq t_k.$$

Proof. $\max\{Qx\}$ and $\max\{Ru\}$ are the gauge functions of the sets $\{x : Qx \leq \mathbf{1}\}$ and $\{u : Ru \leq \mathbf{1}\}$, i.e.

$$\max\{Qx\} = \min_{s \in \mathbb{R}} \; s \;\; \text{s.t.} \;\; Qx \leq s\mathbf{1},$$
$$\max\{Ru\} = \min_{t \in \mathbb{R}} \; t \;\; \text{s.t.} \;\; Ru \leq t\mathbf{1},$$

and hence the terms in the stage cost of (3.24) are given by

$$\max_{x \in \mathcal{X}_k} \max\{Qx\} = \min_{s \in \mathbb{R}} \; s \;\; \text{s.t.} \;\; \mathcal{X}_k \subseteq \{x : Qx \leq s\mathbf{1}\},$$
$$\max_{x \in \mathcal{X}_k} \max\{R(Kx + v_k)\} = \min_{t \in \mathbb{R}} \; t \;\; \text{s.t.} \;\; \mathcal{X}_k \subseteq \{x : R(Kx + v_k) \leq t\mathbf{1}\}.$$

The set inclusion constraints in these expressions are equivalent [22] to the existence of matrices H_Q and H_R satisfying

$$H_Q \geq 0, \qquad H_R \geq 0, \tag{3.29}$$

$$H_Q T = Q, \qquad H_R T = RK, \tag{3.30}$$

$$H_Q \alpha_k \leq s\mathbf{1}, \qquad H_R \alpha_k + R v_k \leq t\mathbf{1}. \tag{3.31}$$

To ensure convexity of the online MPC optimisation that has α_k and v_k as optimisation variables, matrices H_Q and H_R satisfying (3.29) and (3.30) are computed by solving the LPs (3.27) and (3.28) offline, while the constraints of (3.31) are imposed online via conditions (3.25) and (3.26). □

By Lemma 3.7, the piecewise-linear worst case cost (3.24) is bounded by $\sum_{k=0}^{\infty}(s_k+t_k)$. However this cannot be used as an objective function for MPC because: (a) it involves an infinite number of terms, and (b) its value is infinite in general due to the presence of persistent additive disturbances in the system model. To resolve (a), a terminal penalty term is introduced in the cost to bound $\sum_{k=N}^{\infty}(s_k+t_k)$. Problem (b) is avoided by penalizing only the amounts that s_k and t_k exceed upper bounds on asymptotic limits as $k \to \infty$. The required asymptotic bounds and an associated terminal penalty term are provided by Lemmas 3.8 and 3.9, which are based on the stabilizing feedback K and λ-contractivity condition (3.6).

Lemma 3.8. *For $k \geq N$, we have $v_k = 0, u_k = Kx_k$ and*

$$\max_{x\in\mathcal{X}_k} \max\{Qx\} \leq \bar{s}_k(\lambda), \quad \max_{x\in\mathcal{X}_k} \max\{RKx\} \leq \bar{t}_k(\lambda)$$

$$\textit{where } \bar{s}_k(\lambda) = \max\{H_Q\mathbf{1}\}\lambda^{k-N}\max\{\alpha_N\} + \tfrac{1-\lambda^{k-N}}{1-\lambda}\max\{H_Q\mathbf{1}\}\max\{\bar{w}\}, \tag{3.32}$$

$$\bar{t}_k(\lambda) = \max\{H_R\mathbf{1}\}\lambda^{k-N}\max\{\alpha_N\} + \tfrac{1-\lambda^{k-N}}{1-\lambda}\max\{H_R\mathbf{1}\}\max\{\bar{w}\}. \tag{3.33}$$

Proof. If $Tx_k \leq \alpha_k$, then $Tx_{k+1} \leq \alpha_{k+1}$ where, from (3.6), the maximum element of α_{k+1} satisfies, for all $k \geq N$,

$$\begin{aligned}\max\{\alpha_{k+1}\} &= \max_{\theta\in\Theta,\, x_k\in\mathcal{X}_k} \max\left\{TA_K(\theta)x_k + \bar{w}\right\} \\ &\leq \max_{\theta\in\Theta,\, x_k\in\mathcal{X}_k} \max\{TA_K(\theta)x_k\} + \max\{\bar{w}\}\end{aligned}$$

$$\leq \lambda \max\{\alpha_k\} + \max\{\bar{w}\},$$

and hence, for all $k \geq N$,

$$\max\{\alpha_k\} \leq \lambda^{k-N} \max\{\alpha_N\} + \frac{1-\lambda^{k-N}}{1-\lambda} \max\{\bar{w}\}.$$

Defining $s_k = \max\{H_Q \alpha_k\}$ and $t_k = \max\{H_R \alpha_k\}$ for all $k \geq N$, the bounds (3.32) and (3.33) then follow from $\max\{H_Q \alpha_k\} \leq \max\{H_Q \mathbf{1}\} \max\{\alpha_k\}$, $\max\{H_R \alpha_k\} \leq \max\{H_R \mathbf{1}\} \max\{\alpha_k\}$ and Lemma 3.7. □

Since $\lambda \in [0, 1)$, Lemma 3.8 implies the asymptotic bounds

$$\lim_{k\to\infty} \bar{s}_k(\lambda) = \bar{s}_\infty(\lambda) = \tfrac{1}{1-\lambda} \max\{H_Q \mathbf{1}\} \max\{\bar{w}\}$$
$$\lim_{k\to\infty} \bar{t}_k(\lambda) = \bar{t}_\infty(\lambda) = \tfrac{1}{1-\lambda} \max\{H_R \mathbf{1}\} \max\{\bar{w}\}$$

and it follows that $\bar{s}_\infty(\lambda) + \bar{t}_\infty(\lambda)$ provides an upper bound on the value of $\max\{Qx\} + \max\{Ru\}$ for all x in the minimal robust positively invariant set [4] for the uncertain system (3.1) with $\theta \in \Theta$ under $u = Kx$. Although the bound $\bar{s}_\infty(\lambda) + \bar{t}_\infty(\lambda)$ is conservative, the proposed MPC algorithm updates this bound as the estimated parameter set Θ and the contractive coefficient λ are updated online, reducing the conservativeness. Furthermore, convergence of the closed-loop system state to a set on which $\max\{Qx\} + \max\{Ru\} \leq \bar{s}_\infty(\lambda) + \bar{t}_\infty(\lambda)$ is ensured (as demonstrated in Proposition 3.17) by defining the MPC cost as the amount by which $s_k + t_k$ exceeds this bound. We therefore redefine the MPC cost as

$$\min_{x_k \in \mathcal{X}_k,\, k=0,1,\ldots} \sum_{k=0}^{\infty} (\sigma_k + \tau_k) \tag{3.34}$$

where σ_k, τ_k are the amounts by which s_k and t_k exceed the limits $\bar{s}_\infty(\lambda)$ and $\bar{t}_\infty(\lambda)$:

$$\sigma_k = [s_k - \bar{s}_\infty(\lambda)]_{\geq 0} \quad \tau_k = [t_k - \bar{t}_\infty(\lambda)]_{\geq 0} \tag{3.35}$$

(with $[s]_{\geq 0}$ denoting $\max\{0, s\}$ for any scalar s).

Lemma 3.9. *The value of $\sum_{k=N}^{\infty} (\sigma_k + \tau_k)$ is bounded by $\bar{\sigma}_N + \bar{\tau}_N$, where*

$$\bar{\sigma}_N = \left[\max\{H_Q \mathbf{1}\} \tfrac{1}{1-\lambda} \left(\max\{\alpha_N\} - \tfrac{1}{1-\lambda} \max\{\bar{w}\}\right)\right]_{\geq 0}$$
$$\bar{\tau}_N = \left[\max\{H_R \mathbf{1}\} \tfrac{1}{1-\lambda} \left(\max\{\alpha_N\} - \tfrac{1}{1-\lambda} \max\{\bar{w}\}\right)\right]_{\geq 0}.$$

Proof. From the definition of $\bar{s}_k(\lambda)$ in (3.32) we have

$$\begin{aligned}
&\sum_{k=N}^{\infty}\Big[\max_{x\in\mathcal{X}_k}\max\{Qx\}-\bar{s}_\infty(\lambda)\Big]_{\geq 0}\\
&\leq\sum_{k=N}^{\infty}[\bar{s}_k(\lambda)-\bar{s}_\infty(\lambda)]_{\geq 0}\\
&=\sum_{k=N}^{\infty}\Big[\max\{H_Q\mathbf{1}\}\lambda^{k-N}\Big(\max\{\alpha_N\}-\tfrac{1}{1-\lambda}\max\{\bar{w}\}\Big)\Big]_{\geq 0}\\
&=\Big[\max\{H_Q\mathbf{1}\}\tfrac{1}{1-\lambda}\Big(\max\{\alpha_N\}-\tfrac{1}{1-\lambda}\max\{\bar{w}\}\Big)\Big]_{\geq 0}\\
&=\bar{\sigma}_N,
\end{aligned}$$

and a similar argument demonstrates the bound $\bar{\tau}_N$. □

Remark 3.10. *From (3.34), (3.35) and Lemma 3.9, the MPC cost function can be expressed*

$$J(x,\mathbf{v},\boldsymbol{\alpha},\boldsymbol{\sigma},\boldsymbol{\tau})=\sum_{k=0}^{N-1}(\sigma_k+\tau_k)+\bar{\sigma}_N+\bar{\tau}_N \tag{3.36}$$

where $\mathbf{v}=\{v_0,\dots,v_{N-1}\}$, $\boldsymbol{\alpha}=\{\alpha_0,\dots,\alpha_N\}$, $\boldsymbol{\sigma}=\{\sigma_0,\ \dots,\sigma_{N-1},\bar{\sigma}_N\}$, $\boldsymbol{\tau}=\{\tau_0,\dots,\tau_{N-1},\bar{\tau}_N\}$ *are optimisation variables, with* $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$ *defined by the linear constraints*

$$\sigma_k\geq 0,\ \ \sigma_k\mathbf{1}\geq H_Q\alpha_k-\bar{s}_\infty(\lambda)\mathbf{1} \tag{3.37}$$

$$\tau_k\geq 0,\ \ \tau_k\mathbf{1}\geq H_R\alpha_k+Rv_k-\bar{t}_\infty(\lambda)\mathbf{1} \tag{3.38}$$

$$\bar{\sigma}_N\geq 0,\ \bar{\sigma}_N\geq\tfrac{1}{1-\lambda}\Big(\max\{H_Q\mathbf{1}\}\max\{\alpha_N\}-\bar{s}_\infty(\lambda)\Big) \tag{3.39}$$

$$\bar{\tau}_N\geq 0,\ \bar{\tau}_N\geq\tfrac{1}{1-\lambda}\Big(\max\{H_R\mathbf{1}\}\max\{\alpha_N\}-\bar{t}_\infty(\lambda)\Big). \tag{3.40}$$

3.3.4 Selection of tube shape T and feedback gain K

In this subsection, we introduce one way of selecting matrix T, which defines the tube cross section $\mathcal{X}_k$ in (3.12). We require $\mathcal{X}_k$ to be compact so that the maximum of any linear function of x subject to $x\in\mathcal{X}_k$ is finite, and hence the upper bounds on the stage cost and constrained variables are finite.

If a subset of the rows of T is equal to the rows of G_x+G_uK, Q and RK, then the conservativeness of cost bounds and constraints computed using the predicted

state tube can be reduced. The constraint (3.14), with H_c computed by solving (3.15), is non-conservative in the sense that it is necessary as well as sufficient for (3.16) if each row of H_c contains at most one non-zero element. Similarly, constraints (3.25) and (3.26) are non-conservative when H_Q and H_R are computed by solving (3.27) and (3.28) in the sense that the minimum values of s_k and t_k are equal to $\max_{x\in\mathcal{X}_k}\max\{Qx_k\}$ and $\max_{x\in\mathcal{X}_k}\max\{Rx_k\}$ respectively if each row of H_Q and H_R contains only one non-zero element.

Increasing the number of rows of T will result in a tighter bound on the predicted state tube. However, the number of rows, n_α, of T determines the dimensions of H_c, H_u, H_Q and H_R. Therefore n_α should be kept as small as possible in order that the computational requirements of the MPC algorithm remain reasonable. In the simulation, T is chosen so that $\{x : Tx \leq 1\}$ approximates the Lyapunov level of an unconstrained control problem with the given system dynamics by solving an LMI optimisation problem.

For a given T, a feedback gain K satisfying the λ-contractivity condition (3.6) for some $\lambda < 1$ can be computed whenever this condition is feasible by minimizing the value of λ subject to $\{x : Tx \leq \mathbf{1}\} \subseteq \{x : TA_K(\theta)x \leq \lambda\mathbf{1}\}$ for all $\theta \in \Theta$. If T is chosen independently of K, the constraints of this problem are linear in K and λ. This suggests choosing T (or a subset of the rows of T) as $[\,Q^\top\ G_{x,0}^\top\,]^\top$, where $G_{x,0}$ is a subset of the rows of G_x such that the corresponding rows of G_u are equal to zero, so that a subset of the constraints in (3.2) are equivalent to state constraints, $G_{x,0}x_k \leq \mathbf{1}$, with no dependence on the control input u_k.

Using Lemma 3.1, the computation of K and λ satisfying Assumption 3 for given T can be expressed as a linear program:

$$(\lambda, K, \hat{H}_1, \ldots, \hat{H}_{n_\alpha}) = \underset{\substack{\lambda\in\mathbb{R}\\ K\in\mathbb{R}^{n_u\times n_x}\\ \hat{H}_1,\ldots,\hat{H}_{n_\alpha}\in\mathbb{R}^{(p+1)\times n_\alpha}}}{\operatorname{argmin}} \quad \lambda \tag{3.41}$$

subject to

$$\lambda \geq \hat{\theta}^{(j)\top}\hat{H}_i\mathbf{1}, \quad \hat{\theta}^{(j)\top}\hat{H}_i \geq 0, \quad \forall j \in \mathbb{N}_{[1,m]}, \quad \forall i \in \mathbb{N}_{[1,n_\alpha]},$$

$$\hat{H}_i T = \begin{bmatrix} [T]_i(A_0 + B_0 K) \\ \vdots \\ [T]_i(A_p + B_p K) \end{bmatrix}, \qquad \forall i \in \mathbb{N}_{[1,n_\alpha]}.$$

3.3.5 Persistent Excitation

In [57] it is shown that, if the disturbance bound $w_t \in \mathcal{W}$ is tight and the regressor is persistently exciting, then the diameter of the parameter set estimated using set membership identification will converge asymptotically to zero with probability one.

In the current context, given a suitable PE window length, N_u, the PE condition over time steps $t - N_u + 1, \ldots, t$, where t is the current time, is

$$\sum_{k=t-N_u+1}^{t} \Phi_k^\top \Phi_k \succeq \epsilon \mathbb{I} \tag{3.42}$$

where $\Phi_k = \Phi(x_k, u_k)$. This condition is in general nonconvex in x and u, and we therefore derive a convex sufficient condition for (3.42) as follows.

Assume a control input, $\hat{u}_{0|t}$, that satisfies the state and input constraints has been obtained at time t. To obtain an input u_t that additionally satisfies the PE constraint (3.42), consider applying a perturbation, δu, to $\hat{u}_{0|t}$:

$$u_t = \hat{u}_{0|t} + \delta u.$$

Since $\Phi_t = \Phi(x_t, u_t)$ depends linearly on δu, we have

$$\sum_{k=t-N_u+1}^{t} \Phi_k^\top \Phi_k = M_t + L(\delta u)^\top \Phi(x_t, \hat{u}_{0|t}) + \Phi(x_t, \hat{u}_{0|t})^\top L(\delta u) + L(\delta u)^\top L(\delta u)$$

where M_t is independent of δu,

$$M_t = \sum_{k=t-N_u+1}^{t-1} \Phi_k^\top \Phi_k + \Phi(x_t, \hat{u}_{0|t})^\top \Phi(x_t, \hat{u}_{0|t}),$$

and $L(\delta u)$ is linear in δu

$$L(\delta u) = \begin{bmatrix} B_1 \delta u & \cdots & B_p \delta u \end{bmatrix}.$$

The quadratic term $L(\delta u)^\top L(\delta u)$ is necessarily positive semidefinite and the following condition

$$M_t + \Phi(x_t, \hat{u}_{0|t})^\top L(\delta u) + L(\delta u)^\top \Phi(x_t, \hat{u}_{0|t}) \succeq \beta' \mathbb{I}. \tag{3.43}$$

is therefore a sufficient condition for (3.42) whenever $\beta' \geq \epsilon$.

This is a linear matrix inequality (LMI) constraint on δu and β' which is necessarily convex in δu and which ensures the PE condition (3.42) with a degree of conservativeness that depends on δu as $O(\|\delta u\|^2)$.

Remark 3.11. *The only variable in (3.43) is a perturbation δu associated with the current time step, and there may not exist a solution that satisfies this condition simultaneously with the constraints. On the other hand, it is not necessary for condition (3.43), and by implication (3.42), to be satisfied at every time step t for convergence of the parameter set Θ_t.*

3.4 Proposed Algorithm

Algorithm 3.1 (Robust Adaptive MPC)**.**

***Offline**, given an initial parameter set estimate Θ_0:*

Choose T and compute K, λ and

$$\begin{array}{ll} H_c,\ \hat{H}_1, \ldots, \hat{H}_{n_\alpha} & \text{by solving (3.15), (3.19)} \\ H_Q,\ H_R & \text{by solving (3.27), (3.28).} \end{array}$$

***Online** at times $t \in \mathbb{N}_{\geq 0}$, given the state measurement x_t:*

1) *Update Θ_t using the parameter set identification algorithm (3.11) and compute its vertices $\theta_t^{(1)} \ldots \theta_t^{(m)}$.*

2) *Update λ_t by solving*

$$\lambda_t = \max_{j \in \mathbb{N}_{[1,m]},\, i \in \mathbb{N}_{[1,n_\alpha]}} \hat{\theta}_t^{(j)\top} \hat{H}_i \mathbf{1}.$$

3) *Compute the optimal solution: $\mathbf{v}_t = \{v_{0|t}, \ldots, v_{N-1|t}\}$, $\boldsymbol{\alpha}_t = \{\alpha_{0|t}, \ldots, \alpha_{N|t}\}$, $\boldsymbol{\sigma}_t = \{\sigma_{0|t}, \ldots, \sigma_{N-1|t}, \bar{\sigma}_{N|t}\}$, $\boldsymbol{\tau}_t = \{\tau_{0|t}, \ldots, \tau_{N-1|t}, \bar{\tau}_{N|t}\}$ of the LP*

$$\begin{aligned} J^o(x_t) = \min_{\mathbf{v}, \boldsymbol{\alpha}, \boldsymbol{\sigma}, \boldsymbol{\tau}} & \sum_{k=0}^{N-1} (\sigma_k + \tau_k) + \bar{\sigma}_N + \bar{\tau}_N \\ \text{subject to} \quad & \text{(3.14), (3.18), (3.37), (3.38) } \forall k \in \mathbb{N}_{[0,N-1]} \\ & \text{(3.21), (3.22), (3.23), (3.39), (3.40).} \end{aligned} \tag{3.44}$$

4) *Calculate the control law $\hat{u}_{0|t} = Kx_t + v_{0|t}$ and check the PE condition (3.42) with $\Phi_t = \Phi(x_t, \hat{u}_{0|t})$.*

(a) *If (3.42) is satisfied, implement* $u_t = \hat{u}_{0|t}$

(b) *Otherwise, recompute the optimisation (3.44) with an additional optimisation variable* δu *and with the convex PE condition (3.43),* $v_0 + Kx_t = \hat{u}_{0|t} + \delta u$ *and* $J(x_t, \mathbf{v}, \boldsymbol{\alpha}, \boldsymbol{\sigma}, \boldsymbol{\tau}) \leq J(x_t, \tilde{\mathbf{v}}_t, \tilde{\boldsymbol{\alpha}}_t, \tilde{\boldsymbol{\sigma}}_t, \tilde{\boldsymbol{\tau}}_t)$ *as additional constraints (where* $\tilde{\mathbf{v}}_t$, $\tilde{\boldsymbol{\alpha}}_t$, $\tilde{\boldsymbol{\sigma}}_t$, $\tilde{\boldsymbol{\tau}}_t$ *are defined in (3.45) below).*

(i) *If the optimisation in step 4(b) is infeasible, then implement* $u_t = \hat{u}_{0|t}$.

(ii) *Otherwise implement* $u_t = \hat{u}_{0|t} + \delta u$.

Remark 3.12. *The control input has the dual purpose of regulating the system via feedback and of providing the excitation necessary for the identification of the model parameters, which can be conflicting requirements [13]. Step 4(b)(i) of Algorithm 1 simply states that, if it is not possible to satisfy the convex PE condition (3.43) at the same time as robustly satisfying the constraints (3.2) and achieving a sufficiently rapid decrease in predicted cost, then robust constraint satisfaction takes priority over the PE condition.*

Remark 3.13. *An alternative approach to achieve both objectives at the same time is to augment the PE condition as a soft constraint in the cost function. This approach will be explained in detail in Chapter 4.*

Remark 3.14. *Under robust stabilizability condition implied by Assumption 3, the offline computation of* K *is necessarily feasible for any choice of* T *and* $\lambda \in [0, 1)$ *(see e.g. [22]).*

3.4.1 Recursive feasibility and closed-loop stability

Consider the suboptimal decision variables $(\tilde{\mathbf{v}}_t, \tilde{\boldsymbol{\alpha}}_t, \tilde{\boldsymbol{\sigma}}_t, \tilde{\boldsymbol{\tau}}_t)$ defined at time t in terms of the optimal solution of (3.44) at time $t-1$, $(\mathbf{v}_{t-1}, \boldsymbol{\alpha}_{t-1}, \boldsymbol{\sigma}_{t-1}, \boldsymbol{\tau}_{t-1})$, as

$$\begin{aligned} \tilde{\mathbf{v}}_t &= \{v_{1|t-1}, \dots, v_{N-1|t-1}, 0\}, \qquad \tilde{\boldsymbol{\alpha}}_t = \{\alpha_{1|t-1}, \dots, \alpha_{N|t-1}, \tilde{\alpha}_{N|t}\}, \\ \tilde{\boldsymbol{\sigma}}_t &= \{\tilde{\sigma}_{0|t}, \dots, \tilde{\sigma}_{N-1|t}, \tilde{\bar{\sigma}}_{N|t}\}, \qquad \tilde{\boldsymbol{\tau}}_t = \{\tilde{\tau}_{0|t}, \dots, \tilde{\tau}_{N-1|t}, \tilde{\bar{\tau}}_{N|t}\}. \end{aligned} \tag{3.45}$$

where

$$\tilde{\sigma}_{k|t} = \Big[\max\{H_Q \alpha_{k+1|t-1}\} - \bar{s}_\infty(\lambda_t)\Big]_{\geq 0} \tag{3.46}$$

$$\tilde{\tau}_{k|t} = \Big[\max\{H_R \alpha_{k+1|t-1} + R v_{k+1|t-1}\} - \bar{t}_\infty(\lambda_t)\Big]_{\geq 0} \tag{3.47}$$

for all $k \in \mathbb{N}_{[0,N-1]}$, and

$$[\tilde{\alpha}_{N|t}]_i = \max_{j\in\{1,\dots,m\}} {\theta_t^{(j)}}^\top \hat{H}_i \alpha_{N|t-1} + \bar{w}_i, \tag{3.48}$$

for all $i \in \mathbb{N}_{[1,n_\alpha]}$, and

$$\tilde{\bar{\sigma}}_{N|t} = \Big[\tfrac{1}{1-\lambda_t}\Big(\max\{H_Q \mathbf{1}\} \max\{\tilde{\alpha}_{N|t}\} - \bar{s}_\infty(\lambda_t)\Big)\Big]_{\geq 0} \tag{3.49}$$

$$\tilde{\bar{\tau}}_{N|t} = \Big[\tfrac{1}{1-\lambda_t}\Big(\max\{H_R \mathbf{1}\} \max\{\tilde{\alpha}_{N|t}\} - \bar{t}_\infty(\lambda_t)\Big)\Big]_{\geq 0}. \tag{3.50}$$

Proposition 3.15 (Recursive feasibility)**.** *The online MPC optimisation (3.44) is feasible at all times $t \geq 1$ if (3.44) is feasible at $t = 0$.*

Proof. If the optimisation (3.44) is feasible at time $t-1$, then $(\tilde{\mathbf{v}}_t, \tilde{\boldsymbol{\alpha}}_t, \tilde{\boldsymbol{\sigma}}_t, \tilde{\boldsymbol{\tau}}_t)$ is:

- feasible at time t for (3.14), (3.18) for $k \in \mathbb{N}_{[0,N-2]}$ due to feasibility of (3.14), (3.18) at $t-1$ and $\Theta_t \subseteq \Theta_{t-1}$;
- feasible at time t for (3.14), (3.18) for $k = N-1$ due to feasibility of (3.22), (3.23) at $t-1$ and $\Theta_t \subseteq \Theta_{t-1}$;
- feasible at time t for (3.37), (3.38) for $k \in \mathbb{N}_{[0,N-1]}$ because of the definitions (3.46), (3.47) of $\tilde{\sigma}_{k|t}$, $\tilde{\tau}_{k|t}$;
- feasible at time t for (3.21) because $Tx_t \leq \alpha_{1|t-1}$;
- feasible at time t for (3.22), since feasibility of (3.22) at $t-1$ and the definition (3.48) of $\tilde{\alpha}_{N|t}$ together imply that $\alpha_{N|t-1} \geq \tilde{\alpha}_{N|t}$ due to $\Theta_t \subseteq \Theta_{t-1}$;
- feasible at time t for (3.23) since the constraint is by assumption feasible at $t-1$ and $\tilde{\alpha}_{N|t} \leq \alpha_{N|t-1}$;
- feasible at time t for (3.39), (3.40) because of the definitions (3.49), (3.50) of $\tilde{\bar{\sigma}}_{N|t}$, $\tilde{\bar{\tau}}_{N|t}$.

It follows that $(\tilde{\mathbf{v}}_t, \tilde{\boldsymbol{\alpha}}_t, \tilde{\boldsymbol{\sigma}}_t, \tilde{\boldsymbol{\tau}}_t)$ is a feasible (possibly suboptimal) solution of (3.44) at time t. □

Lemma 3.16. *If (3.44) is feasible at $t-1$, then at time t:*

$$\tilde{\sigma}_{N-1|t} + \tilde{\bar{\sigma}}_{N|t} \leq \bar{\sigma}_{N|t-1} + \max\{H_Q\mathbf{1}\}\left(\tfrac{1}{(1-\lambda_{t-1})^2} - \tfrac{1}{(1-\lambda_t)^2}\right)\max\{\bar{w}\} \tag{3.51}$$

$$\tilde{\tau}_{N-1|t} + \tilde{\bar{\tau}}_{N|t} \leq \bar{\tau}_{N|t-1} + \max\{H_R\mathbf{1}\}\left(\tfrac{1}{(1-\lambda_{t-1})^2} - \tfrac{1}{(1-\lambda_t)^2}\right)\max\{\bar{w}\} \tag{3.52}$$

where $(1-\lambda_{t-1})^{-2} \geq (1-\lambda_t)^{-2} \geq 1$.

Proof. The definitions of $\tilde{\alpha}_{N|t}$ in (3.48) and λ_t in Algorithm 1 imply

$$\max\{\tilde{\alpha}_{N|t}\} \leq \lambda_t \max\{\alpha_{N|t-1}\} + \max\{\bar{w}\}.$$

From (3.49) and (3.46) we therefore obtain

$$\begin{aligned}
\tilde{\bar{\sigma}}_{N|t} &\leq \left[\max\{H_Q\mathbf{1}\}\tfrac{\lambda_t}{1-\lambda_t}\left(\max\{\alpha_{N|t-1}\} - \tfrac{1}{1-\lambda_t}\max\{\bar{w}\}\right)\right]_{\geq 0} \\
\tilde{\sigma}_{N-1|t} &\leq \left[\max\{H_Q\mathbf{1}\}\left(\max\{\alpha_{N|t-1}\} - \tfrac{1}{1-\lambda_t}\max\{\bar{w}\}\right)\right]_{\geq 0}.
\end{aligned}$$

But $\lambda_t \leq \lambda_{t-1}$ since $\Theta_t \subseteq \Theta_{t-1}$ and it follows that

$$\begin{aligned}
\tilde{\sigma}_{N-1|t} + \tilde{\bar{\sigma}}_{N|t} &\leq \left[\max\{H_Q\mathbf{1}\}\left(\tfrac{1}{1-\lambda_{t-1}}\max\{\alpha_{N|t-1}\} - \tfrac{1}{(1-\lambda_t)^2}\max\{\bar{w}\}\right)\right]_{\geq 0} \\
&= \bar{\sigma}_{N|t-1} + \max\{H_Q\mathbf{1}\}\left(\tfrac{1}{(1-\lambda_{t-1})^2} - \tfrac{1}{(1-\lambda_t)^2}\right)\max\{\bar{w}\},
\end{aligned}$$

which demonstrates (3.51). The bound in (3.52) can be demonstrated using a similar argument, whereas $(1-\lambda_{t-1})^{-2} \geq (1-\lambda_t)^{-2} \geq 1$ follows from $\lambda_{t-1} \geq \lambda_t \geq 0$. □

Proposition 3.17 (Asymptotic bounds on system state and control input)**.** *If the MPC optimisation (3.44) is feasible at $t=0$, then the closed-loop system satisfies the constraint $G_x x_t + G_u u_t \leq \mathbf{1}$ for all $t \in \mathbb{N}_{\geq 0}$ and the state and control input, x_t and u_t, satisfy the asymptotic bounds:*

$$\lim_{t\to\infty} \max\{Qx_t\} \leq \bar{s}_\infty(\lambda_\infty) \tag{3.53}$$

$$\lim_{t\to\infty} \max\{Ru_t\} \leq \bar{t}_\infty(\lambda_\infty) \tag{3.54}$$

where $\lambda_\infty = \lim_{t\to\infty} \lambda_t$.

Proof. First note that λ_t necessarily converges to a limit λ_∞, where $\lambda_\infty \leq \lambda_0$, since by assumption Θ_0 is compact and $\Theta_t \subseteq \Theta_{t-1}$ for all $t \geq 1$. To simplify notation, we use J_t^o to denote the optimal value, $J^o(x_t) = J(x_t, \mathbf{v}_t, \boldsymbol{\alpha}_t, \boldsymbol{\sigma}_t, \boldsymbol{\tau}_t)$, of the MPC cost (3.44) at time t. The convergence of closed-loop state and control sequences can be analysed by considering the rate of decrease in J_t^o with t. Thus

$$
\begin{aligned}
&J_t^o \leq J(x_t, \tilde{\mathbf{v}}_t, \tilde{\boldsymbol{\alpha}}_t, \tilde{\boldsymbol{\sigma}}_t, \tilde{\boldsymbol{\tau}}_t) \\
&\leq J_{t-1}^o - \Big[\max\{H_Q \alpha_{0|t-1}\} - \bar{s}_\infty(\lambda_{t-1})\Big]_{\geq 0} - \Big[\max\{H_R \alpha_{0|t-1} + R v_{0|t-1}\} - \bar{t}_\infty(\lambda_{t-1})\Big]_{\geq 0} \\
&\qquad + (\max\{H_Q \mathbf{1}\} + \max\{H_R \mathbf{1}\})\Big(\tfrac{1}{(1-\lambda_{t-1})^2} - \tfrac{1}{(1-\lambda_t)^2}\Big) \max\{\bar{w}\} \\
&= J_{t-1}^o - \Big[\max\{Q x_{t-1}\} - \bar{s}_\infty(\lambda_{t-1})\Big]_{\geq 0} - \Big[\max\{R u_{t-1}\} - \bar{t}_\infty(\lambda_{t-1})\Big]_{\geq 0} \\
&\qquad + (\max\{H_Q \mathbf{1}\} + \max\{H_R \mathbf{1}\})\Big(\tfrac{1}{(1-\lambda_{t-1})^2} - \tfrac{1}{(1-\lambda_t)^2}\Big) \max\{\bar{w}\}.
\end{aligned}
$$

Here the first inequality follows from the feasibility of $(\tilde{\mathbf{v}}_t, \tilde{\boldsymbol{\alpha}}_t, \tilde{\boldsymbol{\sigma}}_t, \tilde{\boldsymbol{\tau}}_t)$ at time t (by Proposition 3.15) together with the optimality of the solution of (3.44) in step 3 and the constraints introduced in step 4(b) of Algorithm 1; the second inequality follows directly from Lemma 3.16; and the final equality is due to $T x_{t-1} = \alpha_{0|t-1}$, which implies $H_Q \alpha_{0|t-1} = Q x_{t-1}$ and $H_R \alpha_{0|t-1} = RK x_{t-1}$. For any positive integer T we therefore obtain

$$
\begin{aligned}
&\sum_{t=0}^{T-1} \Big[\max\{Q x_t\} - \bar{s}_\infty(\lambda_t)\Big]_{\geq 0} + \sum_{t=0}^{T-1} \Big[\max\{R u_t\} - \bar{t}_\infty(\lambda_t)\Big]_{\geq 0} \\
&\qquad \leq J_0^o - J_T^o + (\max\{H_Q \mathbf{1}\} + \max\{H_R \mathbf{1}\})\Big(\tfrac{1}{(1-\lambda_0)^2} - \tfrac{1}{(1-\lambda_T)^2}\Big) \max\{\bar{w}\} \\
&\qquad \leq J_0^o - J_T^o + (\max\{H_Q \mathbf{1}\} + \max\{H_R \mathbf{1}\})\Big(\tfrac{1}{(1-\lambda_0)^2} - 1\Big) \max\{\bar{w}\}.
\end{aligned}
$$

The RHS of this inequality is necessarily finite since $J_T^o \geq 0$ for all T. It follows that the sum on the LHS converges as $T \to \infty$, which implies the asymptotic bounds in (3.53) and (3.54). □

3.5 Numerical example

This section presents an example to illustrate the operation of the proposed adaptive robust MPC scheme. We first demonstrate the online parameter set estimation and constraint satisfaction.

Consider the second-order discrete-time uncertain linear system from [45], with model parameters

$$A_0 = \begin{bmatrix} 0.5 & 0.2 \\ -0.1 & 0.6 \end{bmatrix}, \; A_1 = \begin{bmatrix} 0.042 & 0 \\ 0.072 & 0.03 \end{bmatrix} \; A_2 = \begin{bmatrix} 0.015 & 0.019 \\ 0.009 & 0.035 \end{bmatrix}, \; A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$B_0 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; B_3 = \begin{bmatrix} 0.0397 \\ 0.059 \end{bmatrix}$$

and with true system parameter $\theta^* = \begin{bmatrix} 0.8 & 0.2 & -0.5 \end{bmatrix}^\top$. The initial parameter set estimate is $\Theta_0 = \{\theta : \|\theta\|_\infty \leq 1\}$. The elements of the disturbance sequence $\{w_0, w_1, \ldots\}$ are independent and identically (uniformly) distributed on $\mathcal{W} = \{w \in \mathbb{R}^2 : \; \|w\|_\infty \leq 0.1\}$. The state and input constraints are $[x_t]_2 \geq 0.3$ and $u_t \leq 1$.

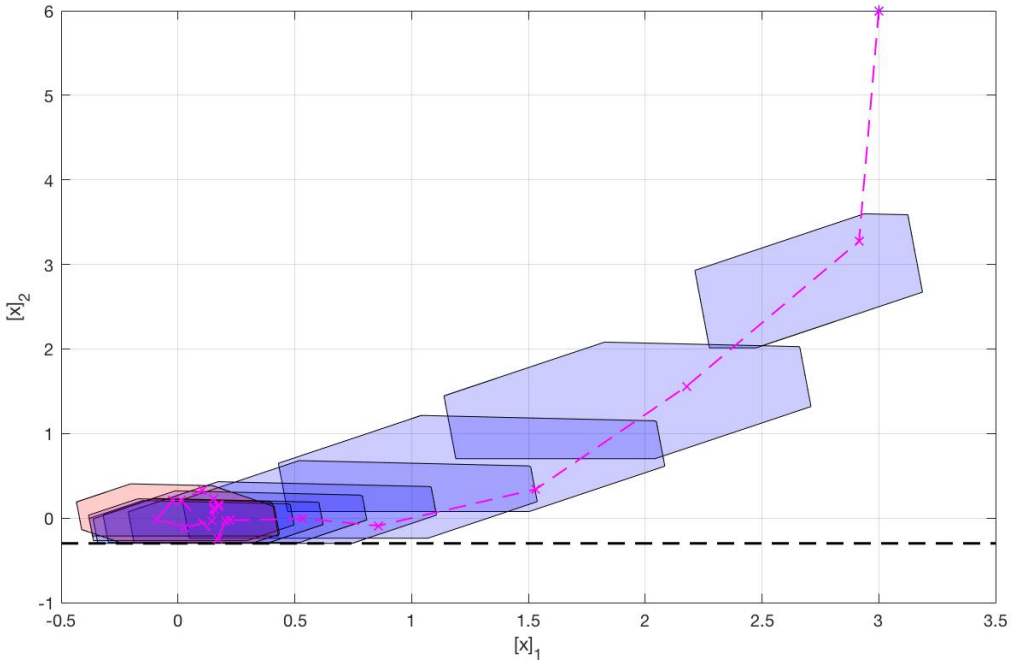


Figure 3.1: Closed-loop state trajectory (pink line) from initial condition $x_0 = (3, 6)$, and predicted state tube cross-sections $\{\mathcal{X}_0, \ldots, \mathcal{X}_N\}$ at time $t = 0$, with $\mathcal{X}_N$ shown in red.

Simulations were performed in Matlab on a 2.7 GHz Intel Core i5 processor. Algorithm 1 was solved using Yalmip[71] and Gurobi[72]. The scalar input allows the decomposition of the PE condition into two linear constraints[34]. For T with 9 rows and MPC prediction horizon $N = 10$, the average solver time is 0.0127 s, with a minimum 0.0098 s and maximum 0.0194 s over 5000 time steps. Figure 3.1 shows the cross-sections of the robust state tube predicted at $t = 0$, with initial

condition $x_0 = (3, 6)$. The state constraint $[x]_2 \leq -0.3$ is robustly satisfied by the state tube predicted at $t = 0$ and it can be seen that this constraint is satisfied by the closed-loop state trajectory.

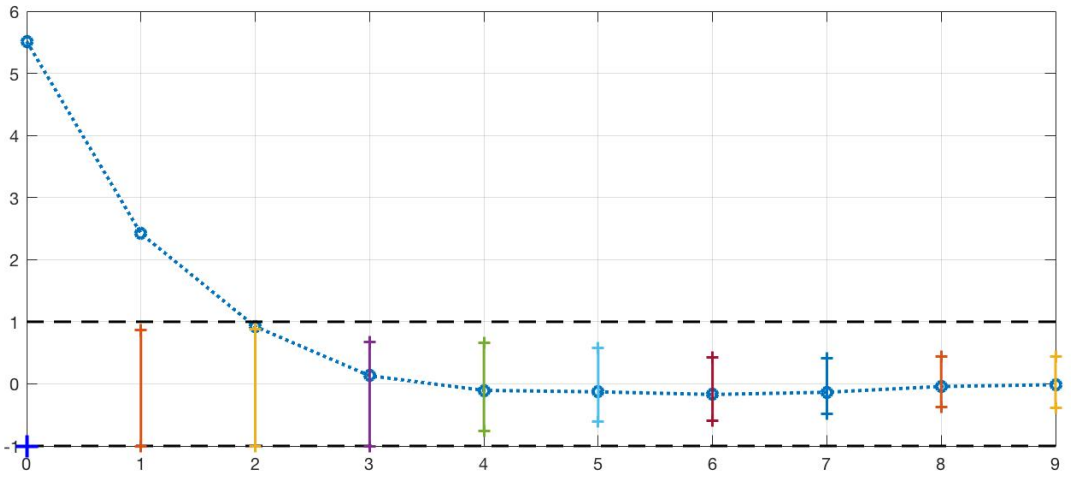


Figure 3.2: Predicted input sequence over the prediction horizon.
($\mathbf{v}_t$: dotted line and blue circles; $\mathbf{u}_t$: coloured lines between crosses)

Figure 3.2 illustrates the robust tube containing the predicted control input sequence $\mathbf{u}_0 = \{u_{0|0}, \ldots, u_{N-1|0}\}$ (sets containing $u_{k|t}$ are shown as coloured lines between crosses) and the predicted sequence $\mathbf{v}_0 = \{v_{0|0}, \ldots, v_{N-1|0}\}$. The input constraint $|u_k| \leq 1$ is active initially (blue cross at $(0, -1)$). Each $u_{k|t}$ lies within the constraints (black dashed lines), which means the predicted input sequence robustly satisfies the constraints over the prediction horizon.

The change in the estimated parameter set over time is shown in Figure 3.3. Table 3.1 gives the parameter set volume, $\text{vol}(\Theta_t)$ as a percentage of the initial volume $\text{vol}(\Theta_0)$, with and without the PE condition step. The parameter set shrinks by 74% at the first time step and 87% by the 10th time step. After 5000 time steps, the parameter set with step 4 is 0.00086% of its original size, implying that by this time it has been reduced almost to a point.

For this example, the algorithm reaches online step 4(a) 97.8% of the time, which means the optimal input from (3.44) satisfy the PE condition. About 2.2% of the

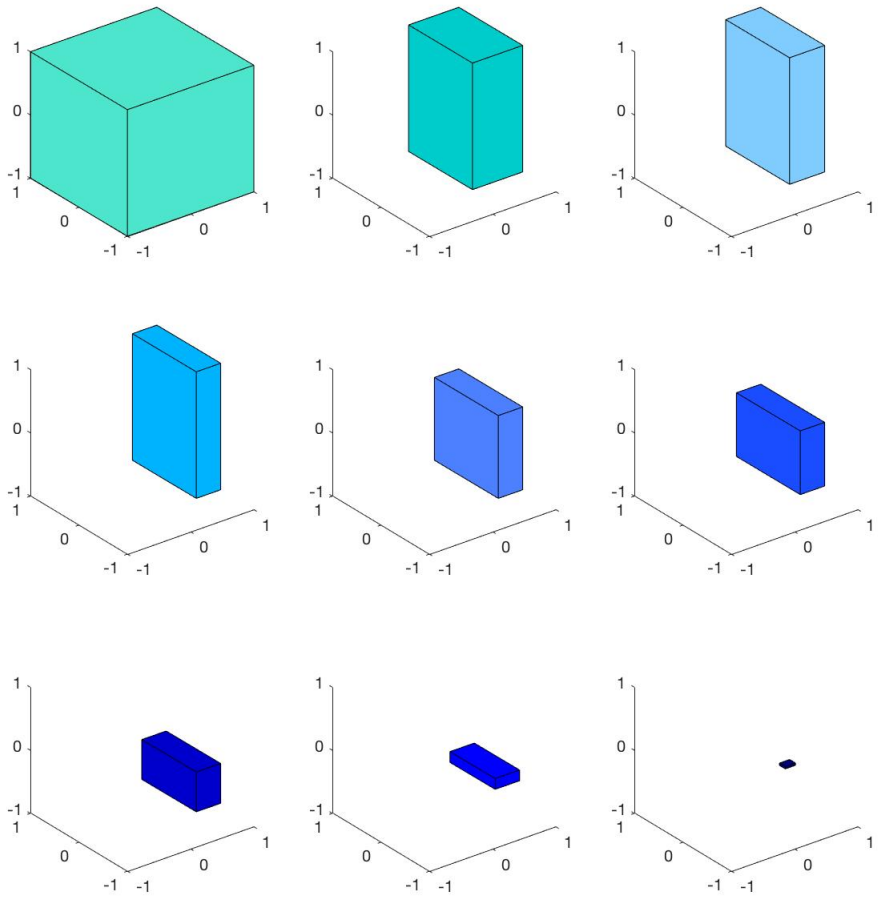


Figure 3.3: Parameter set Θ_t at time steps $t = 0, 1, 2$ (1st row) $t = 10, 25, 50$ (2nd row) $t = 100, 500, 5000$ (3rd row)

Table 3.1: Volume of parameter set Θ_t using Algorithm 3.1 (with and without step 4) at different time steps as a percentage of Θ_0

Time step t	1	2	10	100	200	500	5000
Volume (%) with step 4	26.1	18.3	12.7	1.64	0.83	0.29	0.00086
Volume (%) without step 4	26.1	18.3	12.7	3.43	1.91	0.77	0.0087

time the algorithm reaches 4(b)(ii), when the optimal solution from (3.44) does not satisfy the PE condition, but the optimisation with the extra PE constraint is feasible. For the rest 0.06% of time, the algorithm reaches 4(b)(i), when the extra PE condition makes the optimisation infeasible. In other words, the extra PE condition is active 2.2% of the time. Table 3.2 shows that the resulting parameter set size Θ_{5000} with extra PE condition is about 10% of that without the PE condition.

Table 3.2 shows the reduction in the parameter set Θ_t causes a decrease in the

optimal value of the MPC cost, $J^o(x_t)$, for the same value of the state x_t and the same set of constraints. The initial parameter set Θ_0 gives an optimal predicted cost of $J^o(x) = 62.2$ for $x = x_0 = (3, 6)$, whereas with Θ_{500} the optimal cost is $J^o(x) = 56.17$ (9.7% reduction), and with Θ_{5000} the optimal cost is $J^o(x) = 53.1$ (14.7% reduction), and with no parameter uncertainty ($\Theta = \theta^*$) the optimal cost is $J^o(x) = 52.70$ (15.3% reduction).

Table 3.2: The predicted optimal cost when a particular parameter set is used, with the same initial x_0 and constraints.

Parameter set	Cost	Reduction in cost compared with that using original Θ_0
Original Θ_0	62.22	-
If $\Theta = \Theta_1$	61.13	1.8%
If $\Theta = \Theta_{10}$	60.96	2.0%
If $\Theta = \Theta_{50}$	60.77	2.3%
If $\Theta = \Theta_{500}$	56.17	9.7%
If $\Theta = \Theta_{5000}$	53.10	14.7%

3.6 Conclusion

This chapter proposes a control framework that combines fixed-complexity tube MPC and set membership identification. Robust constraint satisfaction and recursive feasibility are guaranteed for the closed-loop system. In addition, the state is shown to converge asymptotically to a set that depends explicitly on the uncertainty in model parameters when a min-max cost function is adopted. The offline and online optimisations are linear programming problems.

For parameter identification, the polytopic representation of uncertain parameter set Θ reduces the computational load while allowing sufficient flexibility. Conditions for persistent excitation are linearised and included whenever possible in the MPC optimisation to enforce parameter set convergence. A numerical example illustrates that the algorithm is able to reduce the parameter set size while satisfying the constraints robustly and steering the closed-loop state to a neighbourhood of the origin.

4

Adaptive robust MPC and convergence of model parameters

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4.1 Introduction

Building on Chapter 3, we take advantage of fixed complexity polytopic tube representations and use hyperplane and vertex representations interchangeably to further simplify computation. Similar to [46], we use a nominal performance objective, but we impose constraints robustly on all possible models within the identified model set. We prove that the closed-loop system is input-to-state stable (ISS). In comparison with the min-max approach of Chapter 3, the resulting performance bound is an l_2-gain bound, which provides an asymptotic bound on the 2-norm of the sequence of closed-loop states in terms of the 2-norms of the additive disturbance and parameter estimate error sequences.

In addition, we convexify the persistence of excitation (PE) condition around a reference trajectory and include a penalty term in the cost function to promote convergence of the parameter set. The convexification method is somewhat analogous to that proposed in [58], [73], where the uncertainty information of the parameter set is approximated using a nominal gain. However, the convexification here is obtained by direct linearization of PE constraints on predicted trajectories. The cost function modification proposed here allows the relative importance of the two objectives, namely controller performance and convergence of model parameters, to be specified.

Moreover, this chapter proves that estimated parameter sets under a fixed-complexity polytopic update law converge with probability 1 to the true parameter values despite their approximate nature. We also derive lower bounds on convergence rates for the case of inexact knowledge of the disturbance bounding set and for the case that model states are estimated in the presence of measurement noise.

4.2 Adaptive robust MPC with augmented cost

This chapter considers the same linear system(3.1) and linear constraints (3.2) as that in Chapter 3, reproduced below:

$$x_{t+1} = A(\theta^*)x_t + B(\theta^*)u_t + w_t, \quad G_x x_t + G_u u_t \leq \mathbf{1} \quad \forall t \in \mathbb{N}_{\geq 0}.$$

Assumption 4.1 (State and control constraints)**.** *The set*

$$\mathcal{Z} = \{(x, u) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} : G_x x + G_u u \leq \mathbf{1}\}$$

is compact and contains the origin in its interior.

In this chapter, the unfalsified set Δ_t follows the definition in (3.9) in Section 3.3. A block recursive polytopic update method [53] is used, where Θ_t is defined as the smallest set (3.10) containing the intersection of Θ_{t-1} and unfalsified sets Δ_j over a window of length N_u:

$$\mu_t = \min_{\mu \in \mathbb{R}^r} \mathrm{vol}\Big(\Theta(\mu)\Big) \;\text{ subject to }\; \Theta(\mu) \supseteq \bigcap_{j=t-N_u+1}^{t} \Delta_j \cap \Theta_{t-1} \tag{4.1}$$

(where $\Delta_j = \mathbb{R}$ for all $j \leq 0$). We refer to N_u as the PE window. Note that N_u is independent of the MPC prediction horizon N. Using linear conditions for polyhedral set inclusion [22], μ_t in (4.1) can be obtained by solving a linear program for each $i \in \mathbb{N}_{[1,r]}$:

$$[\mu_t]_i = \min_{\mu,\, H_i} \mu \;\text{ subject to }\; H_i \begin{bmatrix} M_\Theta \\ P_{t-N_u+1} \\ \vdots \\ P_t \end{bmatrix} = [M_\Theta]_i,\; H_i \begin{bmatrix} \mu_{t-1} \\ q_{t-N_u+1} \\ \vdots \\ q_t \end{bmatrix} \leq \mu,\; H_i \geq 0.$$

4.2.1 Polytopic tubes for robust constraint satisfaction

This chapter improves on the polytopic formulation in Section 3.3.2 and reduces the conservativeness introduced by the set-inclusion method. Given the definition of the predicted tube $\mathcal{X}_k = \{x : Tx \leq \alpha_k\}$ (3.12), the shape of the predicted tube is optimised online by allowing α_k to be an optimisation variable. The same tube progression relationship $A_K(\theta)x + B(\theta)v_k + w \in \mathcal{X}_{k+1}$(3.13) must be satisfied for all $w \in \mathcal{W}, x \in \mathcal{X}_t, \theta \in \Theta_t$. By using hyperplane and vertex representation interchangeably for the fixed-complexity tube, the new formulation describes the constraints less conservatively and reduces computational burden.

Hyperplane form: For given $T \in \mathbb{R}^{n_\alpha \times n_x}$ satisfying Assumption 3.3 and $\alpha_k \in \mathbb{R}^{n_\alpha}$, let $\mathcal{X}_k \subset \mathbb{R}^{n_x}$ denote the k steps ahead cross-section of the predicted state tube:

$$\mathcal{X}_k = \{x : Tx \leq \alpha_k\},$$

If, for a given α_k, the constraint $[T]_i x \leq [\alpha_k]_i$ is redundant for some $i \in \mathbb{N}_{[1,n_\alpha]}$ in the hyperplane description (3.12) (i.e. if the set $\mathcal{X}_k$ is unchanged by removing this constraint), we redefine (without loss of generality) $[\alpha_k]_i = \max_{x \in \mathcal{X}_k} [T]_i x$. Thus, for each $i \in \mathbb{N}_{[1,n_\alpha]}$, $[\alpha_k]_i = [T]_i x$ necessarily holds for some $x \in \mathcal{X}_k$. Then (3.13) is equivalent to, for all $x \in \mathcal{X}_k$ and $\theta \in \Theta_t$,

$$T\Big(A(\theta) + B(\theta)K\Big)x + TB(\theta)v_k + \bar{w} \leq \alpha_{k+1}, \quad \forall k \in \mathbb{N}_{\geq 0}$$

where $\bar{w}$ is the vector with ith element $[\bar{w}]_i = \max_{w \in \mathcal{W}} [Tw]_i$ for all $i \in \mathbb{N}_{[1,n_\alpha]}$. Substituting $\Phi(x,u)$ and $\phi(x,u)$ from (3.7), (3.8), this implies linear conditions on θ, for all $x \in \mathcal{X}_k$, $\theta \in \Theta_t$:

$$T\Big(\Phi(x, Kx + v_k)\theta + \phi(x, Kx + v_k)\Big) + \bar{w} \leq \alpha_{k+1}, \quad \forall k \in \mathbb{N}_{\geq 0}$$

and, for a given initial state x, the constraint $x \in \mathcal{X}_0$ requires

$$Tx \leq \alpha_0. \tag{4.2}$$

Vertex form: The tube $\mathcal{X}_k$ has an equivalent representation in terms of its vertices, which we denote as $x_k^{(j)}$, $j \in \mathbb{N}_{[1,m]}$:

$$\mathcal{X}_k = \mathrm{Co}\{x_k^{(1)}, \ldots, x_k^{(m)}\}. \tag{4.3}$$

Note that m, the number of vertices of $\mathcal{X}_k$, is fixed, and for given α_k in (3.12), we may have $x_k^{(i)} = x_k^{(j)}$ for some $i \neq j \in \mathbb{N}_{[1,m]}$ in (4.3) (i.e. the vertex description may contain repeated vertices). The presence of repeated vertices does not affect the formulation. For each $j \in \mathbb{N}_{[1,m]}$, define an index set $\mathcal{R}_j$ (with $|\mathcal{R}_j| = n_x$), such that the ith row of T and ith element of α_k satisfy

$$[T]_i x_k^{(j)} = [\alpha_k]_i \ \forall i \in \mathcal{R}_j.$$

The matrix T is chosen so that the polytope $\mathcal{X} = \{x : Tx \leq \mathbf{1}\}$ has no repeated vertices. Since T is constant, the index set $\mathcal{R}_j$ associated with active inequalities at the vertex $x_k^{(j)}$ is independent of α_k and can be computed offline. Therefore, for each $j \in \mathbb{N}_{[1,m]}$, we have

$$x_k^{(j)} = U_j \alpha_k, \tag{4.4}$$

where the matrix $U_j \in \mathbb{R}^{n_x \times n_\alpha}$ can be computed offline given knowledge of $\mathcal{R}_j$ using the property that

$$[T]_i U_j = [\mathbb{I}]_i, \quad \forall i \in \mathcal{R}_j,\ j \in \mathbb{N}_{[1,m]}. \tag{4.5}$$

Using the vertex representation (4.4), the condition that (3.2) is satisfied for all $x \in \mathcal{X}_k$ is equivalent to, for all $j \in \mathbb{N}_{[1,m]}$,

$$(G_x + G_u K) U_j \alpha_k + G_u v_k \leq \mathbf{1}. \tag{4.6}$$

Substituting $\Phi(x,u)$ and $\phi(x,u)$ from (3.7), (3.8), condition (3.13) can be expressed equivalently as

$$T\Big(\Phi(U_j\alpha_k, KU_j\alpha_k + v_k)\theta + \phi(U_j\alpha_k, KU_j\alpha_k + v_k)\Big) + \bar{w} \leq \alpha_{k+1}$$

for all $\theta \in \Theta_t$, $j \in \mathbb{N}_{[1,m]}$ and $k \in \mathbb{N}_{\geq 0}$.

By applying the set inclusion principle [22, Prop. 3.31] to the parameter vector θ, the condition is equivalent to the requirement that there exist matrices $\Lambda_{k,j}$ satisfying, for each prediction time step $k \in \mathbb{N}_{\geq 0}$ and each vertex $j \in \mathbb{N}_{[1,m]}$, the conditions

$$\Lambda_{k,j} M_\Theta = T\Phi(U_j\alpha_k, KU_j\alpha_k + v_k) \tag{4.7a}$$

$$\Lambda_{k,j} \mu_t \leq \alpha_{k+1} - T\phi(U_j\alpha_k, KU_j\alpha_k + v_k) - \bar{w} \tag{4.7b}$$

$$\Lambda_{k,j} \geq 0. \tag{4.7c}$$

Given the dual mode predicted control law (3.5), we introduce the terminal conditions that $\Big(A(\theta) + B(\theta)K\Big)x + w \in \mathcal{X}_N$ and $(G_x + G_u K)x \leq \mathbf{1}$ for all $x \in \mathcal{X}_N$, $w \in \mathcal{W}$ and $\theta \in \Theta$. Then (3.2) and (3.13) are satisfied if (4.6), (4.7)

hold for all $j \in \mathbb{N}_{[1,m]}$ and $k \in \mathbb{N}_{[0,N-1]}$, and there exist matrices $\Lambda_{N,j}$ satisfying the conditions, for all $j \in \mathbb{N}_{[1,m]}$

$$(G_x + G_u K) U_j \alpha_N \leq \mathbf{1} \tag{4.8a}$$

$$\Lambda_{N,j} M_\Theta = T\Phi(U_j \alpha_N, K U_j \alpha_N) \tag{4.8b}$$

$$\Lambda_{N,j} \mu_t \leq \alpha_N - T\phi(U_j \alpha_N, K U_j \alpha_N) - \bar{w} \tag{4.8c}$$

$$\Lambda_{N,j} \geq 0. \tag{4.8d}$$

4.2.2 Objective function

Consider the nominal cost defined for $Q, R \succ 0$ by

$$J(x, \mathbf{v}, \bar{\theta}_t) = \sum_{k=0}^{\infty} (\|\bar{x}_k\|_Q^2 + \|\bar{u}_k\|_R^2), \tag{4.9}$$

where $\bar{x}_k$ and $\bar{u}_k$ are elements of predicted state and control sequences generated by a nominal parameter vector $\bar{\theta}_t$:

$$\bar{x}_0 = x \tag{4.10a}$$

$$\bar{x}_{k+1} = \big(A(\bar{\theta}_t) + B(\bar{\theta}_t)K\big)\bar{x}_k + B(\bar{\theta}_t) v_k \tag{4.10b}$$

$$\bar{u}_k = \begin{cases} K\bar{x}_k + v_k & k < N \\ K\bar{x}_k & k \geq N \end{cases} \tag{4.10c}$$

for $k \in \mathbb{N}_{>0}$ and where $\mathbf{v} = \{v_0, \ldots, v_{N-1}\}$. Define $P(\theta)$ as the solution of the Lyapunov matrix equation

$$P(\theta) - A_K(\theta)^\top P(\theta) A_K(\theta) = Q + K^\top R K \tag{4.11}$$

where $A_K(\theta) = A(\theta) + B(\theta)K$. Note that $P(\theta) \succ 0$ is well-defined for all $\theta \in \Theta_0$ due to Assumption 3.3. Then (4.9) is equivalent to

$$J(x, \mathbf{v}, \bar{\theta}_t) = \sum_{k=0}^{N-1} (\|\bar{x}_k\|_Q^2 + \|\bar{u}_k\|_R^2) + \|\bar{x}_N\|_{P(\bar{\theta}_t)}^2. \tag{4.12}$$

We assume knowledge of an initial nominal parameter vector $\bar{\theta}_0 \in \Theta_0$, which could be estimated using physical modeling or offline system identification, alternatively

$\bar{\theta}_0$ could be defined as the Chebyshev centre of Θ_0. For $t > 0$, we assume that $\bar{\theta}_t$ is updated by projecting $\bar{\theta}_{t-1}$ onto the parameter set estimate Θ_t, i.e.

$$\bar{\theta}_t = \underset{\theta \in \Theta_t}{\operatorname{argmin}} \|\bar{\theta}_{t-1} - \theta\|. \tag{4.13}$$

Remark 4.1. *For the input-to-state stability analysis in Section 4.3 it is essential that $\bar{\theta}_t \in \Theta_t$. However, subject to this constraint, alternative update laws for $\bar{\theta}_t$ are possible; for example a Least Mean Squares (LMS) estimate projected onto Θ_t [34].*

4.2.3 Augmented objective function and Persistence of Excitation

The Persistent Excitation (PE) condition for the regressor Φ_t is as stated in (3.42), reproduced below:

$$\sum_{k=t-N_u+1}^{t} \Phi_k^\top \Phi_k \succeq \epsilon \mathbb{I} \tag{4.14}$$

where ϵ refers to the closed-loop PE coefficient. Relationship between the convergence rate of the parameter set Θ_t and PE coefficient ϵ will be investigated in Section 4.4.

Previously proposed MPC strategies that incorporate persistence of excitation (PE) constraints consider the PE condition to be defined on an interval such as $\{t - N_u + 1, \ldots, t\}$, where t is current time, which means that the PE constraint depends on only the first element of the predicted control sequence. Marafioti et al. [13] simplify the PE condition by expressing it as a nonconvex quadratic inequality in u_t. Likewise, Lorenzen et al. [34] show that the PE condition is equivalent to a nonconvex constraint on the current control input. In Chapter 3, we linearise the PE condition about a reference trajectory and thus obtain a sufficient condition for persistent excitation.

In this chapter, on the other hand, we define the PE condition over predicted trajectories (from $k = 0$ to $k = N_u - 1$ steps ahead), and we therefore require, at time t and for some $\beta > 0$,

$$\sum_{k=0}^{N_u-1} \Phi_{k|t}^\top \Phi_{k|t} \succeq \beta \mathbb{I}. \tag{4.15}$$

where β refers to the predicted PE coefficient. The inclusion of predicted future states and control inputs in this PE condition allows for greater flexibility in meeting the constraint. To avoid nonconvex constraints, we will derive a convex relaxation that provides a sufficient condition for (4.15).

Assume reference state and control predicted sequences, $\hat{\mathbf{x}} = \{\hat{x}_0, \dots \hat{x}_N\}$ and $\hat{\mathbf{u}} = \{\hat{u}_0, \dots \hat{u}_{N-1}\}$, approximating the optimal predicted state and control sequences, are available. To derive sufficient conditions for (4.15), we consider the difference between the reference and optimised sequences, denoted $\tilde{u}_k$ and $\tilde{x}_k$ (i.e. $\tilde{u}_k = u_k - \hat{u}_k = Kx_k + v_k - \hat{u}_k$ and $\tilde{x}_k = x_k - \hat{x}_k$). Since the prediction tube implies $x_k \in \mathcal{X}_k$, we therefore have $\tilde{x}_k \in \tilde{\mathcal{X}}_k$ where $\tilde{\mathcal{X}}_k = \mathcal{X}_k \oplus -\hat{x}_k$. Denote the vertices of $\tilde{\mathcal{X}}_k$ as $\tilde{x}_k^{(j)}$ and for $j \in \mathbb{N}_{[1,m]}$, we have

$$\tilde{x}_k^{(j)} = x_k^{(j)} - \hat{x}_k = U_j \alpha_k - \hat{x}_k.$$

Moreover Φ_k depends linearly on u_k and x_k, and hence for $j \in \mathbb{N}_{[1,m]}$,

$$\begin{aligned}\Phi(x_k^{(j)}, u_k^{(j)})^\top \Phi(x_k^{(j)}, u_k^{(j)}) =& \Phi(\hat{x}_k, \hat{u}_k)^\top \Phi(\hat{x}_k, \hat{u}_k) + \Phi(\tilde{x}_k^{(j)}, \tilde{u}_k^{(j)})^\top \Phi(\hat{x}_k, \hat{u}_k) \\ &+ \Phi(\hat{x}_k, \hat{u}_k)^\top \Phi(\tilde{x}_k^{(j)}, \tilde{u}_k^{(j)}) + \Phi(\tilde{x}_k^{(j)}, \tilde{u}_k^{(j)})^\top \Phi(\tilde{x}_k^{(j)}, \tilde{u}_k^{(j)}).\end{aligned}$$

Here $\Phi(\tilde{x}_k^{(j)}, \tilde{u}_k^{(j)})^\top \Phi(\tilde{x}_k^{(j)}, \tilde{u}_k^{(j)})$ is a positive semidefinite matrix, and by omitting this term we obtain sufficient conditions for (4.15) as a set of LMIs in α_k and v_k. The following convex conditions are thus sufficient to ensure (4.15)

$$\begin{aligned}&\Phi(\hat{x}_k, \hat{u}_k)^\top \Phi(\hat{x}_k, \hat{u}_k) + \Phi(U_j \alpha_k - \hat{x}_k, KU_j \alpha_k + v_k - \hat{u}_k)^\top \Phi(\hat{x}_k, \hat{u}_k) \\ &+ \Phi(\hat{x}_k, \hat{u}_k)^\top \Phi(U_j \alpha_k - \hat{x}_k, KU_j \alpha_k + v_k - \hat{u}_k) \succeq M_k, \quad \forall j \in \mathbb{N}_{[1,m]}, \ \forall k \in \mathbb{N}_{[0,N_u-1]} \end{aligned} \tag{4.16a}$$

$$\sum_{k=0}^{N_u-1} M_k \succeq \beta' \mathbb{I} \tag{4.16b}$$

where $M_k \in \mathbb{R}^{p \times p}$ $\forall k \in \mathbb{N}_{[0,N_u-1]}$ are intermediate variables and β' is the linearized predicted PE coefficient.

Another innovation introduced here is the inclusion of the PE coefficient in the cost function. Previous approaches [34, 60] face the difficulty of choosing a suitable β' value for the PE constraint in the implementation. A larger value of β' is generally

desirable, but a large β' might make the optimisation problem with the PE condition infeasible. In this paper we incentivise a large value of β' by modifying the MPC objective function as follows incentivise

$$\sum_{k=0}^{N-1}(\|\bar{x}_k\|_Q^2 + \|\bar{u}_k\|_R^2) + \|\bar{x}_N\|_{P(\bar{\theta}_t)}^2 - \gamma\beta' \tag{4.17}$$

where $\gamma \geq 0$ is a weight that controls the relative priority given to satisfaction of the PE condition (4.15) and tracking performance. This modification does not affect the feasibility of the optimisation.

Although incorporating a condition such as (4.15) or the convex relaxation (4.16)-(4.17) into a MPC strategy does not ensure that the closed-loop system satisfies a corresponding PE condition, its effect on the convergence rate of the estimated parameter set is significant, as shown in the numerical example in Section 4.5. Also, γ is a meaningful and straightforward coefficient to tune, and the enforcement of the PE constraint can be easily switched off by setting $\gamma = 0$.

4.2.4 Proposed algorithm

Algorithm 4.1 (Adaptive MPC with augmented cost)**.**

Offline:

1. *Choose suitable T defining the predicted state tube and compute the corresponding U_j in (4.5).*
2. *Obtain a nominal $\bar{\theta}_0$.*
3. *Minimise the contracitivity factor λ satisfying (3.6) and obtain a feedback gain K.*

Online: *For $t = 0, 1, 2, \ldots$*

1. *Obtain the current state x_t and set $x = x_t$.*
2. *Update $\Theta_t = \{\theta : M_\Theta \theta \leq \mu_t\}$ using (4.1) and the nominal parameter vector $\bar{\theta}_t$ using (4.13), and solve (4.11) for $P(\bar{\theta}_t)$.*

3. *At $t = 0$ compute the initial reference state and control sequences $\hat{\mathbf{x}} = \{\hat{x}_0, \ldots, \hat{x}_N\}$ and $\hat{\mathbf{u}} = \{\hat{u}_0, \ldots, \hat{u}_{N-1}\}$, for example by solving the nominal problem described in Remark 4.4.*

   At $t > 0$ compute the reference state and control sequences $\hat{\mathbf{x}}$ and $\hat{\mathbf{u}}$, using the solution $\mathbf{v}^*_{t-1} = \{v^*_{0|t-1}, \ldots, v^*_{N-1|t-1}\}$ at $t-1$, and

$$\hat{x}_0 = x \tag{4.18a}$$

$$\hat{x}_{k+1} = A(\bar{\theta}_t)\hat{x}_k + B(\bar{\theta}_t)\hat{u}_k \tag{4.18b}$$

$$\hat{u}_k = K\hat{x}_k + \hat{v}_k \tag{4.18c}$$

$$\hat{v}_k = \begin{cases} v^*_{k+1|t-1} & k = 0, \ldots, N-2 \\ 0. & k = N-1 \end{cases} \tag{4.18d}$$

4. *Compute $\mathbf{v}^*_t = \{v^*_0, \ldots, v^*_{N-1}\}$, $\boldsymbol{\alpha}^*_t = \{\alpha^*_0, \ldots, \alpha^*_N\}$, $\bar{\mathbf{x}}^*_t = \{\bar{x}^*_0, \ldots, \bar{x}^*_N\}$, $\bar{\mathbf{u}}^*_t = \{\bar{u}^*_0, \ldots, \bar{u}^*_{N-1}\}$, β'^*_t , $\boldsymbol{\Lambda}^* = \{\Lambda_{k,j},\ \forall\ k \in \mathbb{N}_{[0,\ldots,N]}, j \in \mathbb{N}_{[1,\ldots,p]}\}$ the solution of the semidefinite program (SDP):*

$$\mathcal{P}_{4.1} : \min_{\mathbf{v},\boldsymbol{\alpha},\bar{\mathbf{x}},\bar{\mathbf{u}},\beta',\boldsymbol{\Lambda}} \sum_{k=0}^{N-1} (\|\bar{x}_k\|_Q^2 + \|\bar{u}_k\|_R^2) + \|\bar{x}_N\|^2_{P(\bar{\theta}_t)} - \gamma\beta'$$

subject to (4.2), (4.6), (4.7), (4.8), (4.10), and (4.16).

5. *Implement the current control input $u_t = Kx_t + v^*_0$.*

Remark 4.2. *In offline step 1, T can be chosen so that the polytope $\mathcal{X} = \{x : Tx \leq \mathbf{1}\}$ in Assumption 3.3 approximates a Robust Control Invariant (RCI) set of the form $\{x : x^\top \tilde{P} x \leq \mathbf{1}\}$, where $\tilde{P} = \tilde{P}^\top \succ 0$ satisfies $\tilde{P} - (A(\theta) + B(\theta)\tilde{K})^\top \tilde{P}(A(\theta) + B(\theta)\tilde{K}) \succ 0$ for some $\tilde{K} \in \mathbb{R}^{n_u \times n_x}$. Using the vertex representation of Θ_0, the matrix $\tilde{P}$ can be computed by solving a semidefinite program [74, Chap. 5]. This approach allows the number of rows in T to be specified by the designer. However, T can alternatively be chosen so that $\mathcal{X}$ approximates the minimal Robust Positive Invariant (RPI) set or the maximal RPI set for the system (3.1)-(3.2) under a specified stabilizing feedback law [22].*

Remark 4.3. *In offline step 3, the computation of $\min_K \lambda$ subject to (3.6) for given T can be performed by solving a LP using the vertex representation of Θ_0 [22, Chap. 7]. The objective of minimizing λ is chosen to make the constraints of*

problem $\mathcal{P}$ easier to satisfy. In particular, choosing K so that $\lambda < 1$ in (3.6) ensures that α_N exists satisfying the terminal constraints (4.8b-d).

Remark 4.4. *At $t = 0$, the reference sequences $\hat{\mathbf{x}} = \{\hat{x}_0, \ldots, \hat{x}_N\}$ and $\hat{\mathbf{u}} = \{\hat{u}_0, \ldots, \hat{u}_{N-1}\}$ may be computed by solving*

$$\min_{\hat{\mathbf{v}}} \sum_{k=0}^{N-1} (\|\hat{x}_k\|_Q^2 + \|\hat{u}_k\|_R^2) + \|\hat{x}_N\|_{P(\bar{\theta}_0)}^2 \quad \textit{subject to} \quad \begin{aligned} &\hat{x}_0 = x_0 \\ &\hat{x}_{k+1} = A(\bar{\theta}_0)\hat{x}_k + B(\bar{\theta}_0)\hat{u}_k \\ &\hat{u}_k = K\hat{x}_k + \hat{v}_k \\ &G_x\hat{x}_k + G_u\hat{u}_k \leq 1 \end{aligned} \tag{4.19}$$

Remark 4.5. *The online computation of the proposed algorithm may be reduced by updating Θ_t only once every $N_u > 1$ time steps. For example, in Step 2, set $\Theta_t = \bigcap_{j=t-N_u+1}^{t} \Delta_j \cap \Theta_{t-N_u}$ for $t \in \{N_u, 2N_u, \ldots\}$ and $\Theta_t = \Theta_{t-1}$ at all times $t \notin \{N_u, 2N_u, \ldots\}$.*

4.3 Recursive feasibility and stability

In this section, we use the property that $\Theta_t \subseteq \Theta_{t-1}$ to show that the solution, $\mathbf{v}_{t-1}^*$, of $\mathcal{P}$ at time $t-1$ forms part of a feasible solution of $\mathcal{P}$ at time t. As a result, the reference sequences $\hat{\mathbf{x}}$, $\hat{\mathbf{u}}$ in Step 3 of the online stage are feasible for problem $\mathcal{P}$ at all times $t > 0$.

4.3.1 Recursive feasibility

At time $t \geq 1$, define a suboptimal set of decision variables $(\hat{\mathbf{v}}_t, \hat{\boldsymbol{\alpha}}_t)$ in terms of the optimal solution $(\mathbf{v}_{t-1}^*, \boldsymbol{\alpha}_{t-1}^*)$ of $\mathcal{P}$ at time $t-1$ by

$$\hat{\mathbf{v}}_t = \{v_{1|t-1}^*, \ldots, v_{N-1|t-1}^*, 0\}, \quad \hat{\boldsymbol{\alpha}}_t = \{\alpha_{1|t-1}^*, \ldots, \alpha_{N|t-1}^*, \alpha_{N|t-1}^*\}.$$

Proposition 4.6 (Recursive Feasibility)**.** *If $\mathcal{P}$ is feasible at $t = 0$ and $\Theta_t \subseteq \Theta_{t-1}$ for all time t, the online MPC optimisation $\mathcal{P}$ is feasible at all times $t \in \mathbb{N}_{>0}$.*

Proof. If $\mathcal{P}$ is feasible at $t-1$, then at time t, $(\mathbf{v}, \boldsymbol{\alpha}) = (\hat{\mathbf{v}}_t, \hat{\boldsymbol{\alpha}}_t)$ is:

- feasible for (4.2) because $x_t \in \mathcal{X}_{1|t-1}$;
- feasible for (4.6) and (4.7) for $k \in \mathbb{N}_{[0,N-2]}$ because $(\mathbf{v}, \boldsymbol{\alpha}) = (\mathbf{v}^*_{t-1}, \boldsymbol{\alpha}^*_{t-1})$ is feasible for (4.6) and (4.7) for $k \in \mathbb{N}_{[1,N-1]}$ and $\Theta_t \subseteq \Theta_{t-1}$;
- feasible for (4.6) and (4.7) for $k = N-1$ because of (4.8);
- feasible for (4.8) because $\alpha_N = \alpha^*_{N|t-1}$ is feasible for (4.8) and $\Theta_t \subseteq \Theta_{t-1}$;
- feasible for (4.10) because of the definition;
- (4.16) necessarily holds for some scalar β' if $(\mathbf{v}, \boldsymbol{\alpha}) = (\hat{\mathbf{v}}_t, \hat{\boldsymbol{\alpha}}_t)$.

□

4.3.2 Input-to-state stability (ISS)

Throughout this section we set $\gamma = 0$ in problem $\mathcal{P}$. Therefore the objective of $\mathcal{P}$ is $J(x, \mathbf{v}, \bar{\theta}_t)$ where J is the nominal cost (4.12). As a result of parameter adaption, the change of $\bar{\theta}_t$ online might increase the cost, but this is absorbed in the ISS terms. To simplify notation we define a stage cost $L(x, v)$ and terminal cost $L_T(x, \theta)$ as $L(x, v) = \|x\|^2_Q + \|Kx + v\|^2_R$ and $L_T(x, \theta) = \|x\|^2_{P(\theta)}$ so that (4.12) is equivalent to

$$J(x, \mathbf{v}, \bar{\theta}_t) = \sum_{k=0}^{N-1} L(\bar{x}_k, \bar{u}_k) + L_T(\bar{x}_N, \bar{\theta}_t).$$

Denoting the actual state at next time step as x^+, we define the function $f(x, v, w, \theta^*)$ as

$$x^+ = f(x, v, w, \theta^*) = \Big(A(\theta^*) + B(\theta^*)K\Big)x + B(\theta^*)v + w,$$

so that $x_{t+1} = f(x_t, v_t, w_t, \theta^*)$.

Lemma 4.7 (ISS-Lyapunov function [75])**.** *The system*

$$x^+ = f(x, v, w, \theta), \tag{4.20}$$

with control law $v = v(x, \bar{\theta}_t, t)$ is ISS with region of attraction $\mathcal{R} \subseteq \mathbb{R}^{n_x}$ if the following conditions are satisfied.

(i). $\mathcal{R}$ contains the origin in its interior, is compact, and is a robust positively invariant set for (4.20), i.e. $f\Big(x, v(x, \bar{\theta}_t, t), w, \theta\Big) \in \mathcal{R}$ for all $x \in \mathcal{R}$, $w \in \mathcal{W}$, $\theta \in \Theta$

and $t \in \mathbb{N}_{\geq 0}$.

(ii). There exist $\mathcal{K}_\infty$*- functions* $\varsigma_1, \varsigma_2, \varsigma_3$*,* $\mathcal{K}$*-functions* σ_1*,* σ_2 *and a function* $\mathcal{V} : \mathcal{R} \times \mathbb{N}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *such that for all* $t \in \mathbb{N}_{\geq 0}$*,* $\mathcal{V}(\cdot, t)$ *is continuous, and for all* $(x, t) \in \mathcal{R} \times \mathbb{N}_{\geq 0}$*,*

$$\varsigma_1(\|x\|) \leq \mathcal{V}(x, t) \leq \varsigma_2(\|x\|), \tag{4.21}$$

$$\mathcal{V}(x^+, t+1) - \mathcal{V}(x, t) \leq -\varsigma_3(\|x\|) + \sigma_1(\|w\|) + \sigma_2(\|\bar{\theta}_t - \theta^*\|). \tag{4.22}$$

In the following we define $X_{\mathcal{P}}$ as the set of states x such that problem $\mathcal{P}$ is feasible and assume that $X_{\mathcal{P}}$ is non-empty. In addition, for a given state x, nominal parameter vector $\bar{\theta}$ and parameter set Θ, we denote $\mathbf{v}^*(x, \bar{\theta}, \Theta)$ as the optimal solution of problem $\mathcal{P}$, and let $V^*(x, \bar{\theta}, \Theta)$ be the corresponding optimal value of the cost in (4.12), so that $V^*(x, \bar{\theta}, \Theta) = J(x, \mathbf{v}^*(x, \bar{\theta}, \Theta), \bar{\theta})$.

Theorem 4.8. *Assume that* $\gamma = 0$ *and the nominal parameter vector* $\bar{\theta}_t$ *is not updated, i.e.* $\bar{\theta}_t = \bar{\theta}_0$ *for all* $t \in \mathbb{N}_{\geq 0}$*. Then for all initial conditions* $x_0 \in X_{\mathcal{P}}$*, the system (3.1) with control law* $u_t = Kx_t + v^*_{0|t}$*, where* $v^*_{0|t}$ *is the first element of* $\mathbf{v}^*(x_t, \bar{\theta}_t, \Theta_t)$*, robustly satisfies the constraint (3.2) and is ISS with region of attraction* $X_{\mathcal{P}}$*.*

Proof. We first show that condition (i) of Lemma 4.7 is satisfied with $\mathcal{R} = X_{\mathcal{P}}$. If $\mathcal{P}$ is feasible at $t = 0$, then (4.8) implies that α_N exists such that $\mathcal{X}_N = \{x : Tx \leq \alpha_N\}$ satisfies

$$\begin{gathered} \Big(A(\theta) + B(\theta)K\Big)x + w \in \mathcal{X}_N \;\; \forall x \in \mathcal{X}_N,\; w \in \mathcal{W},\; \theta \in \Theta_0, \\ (G_x + G_u K)x \leq \mathbf{1} \;\; \forall x \in \mathcal{X}_N. \end{gathered} \tag{4.23}$$

Therefore $(\mathbf{v}, \boldsymbol{\alpha}) = (\{0, \ldots, 0\}, \{\alpha_N, \ldots, \alpha_N\})$ is feasible for $\mathcal{P}$ for all $x_0 \in \mathcal{X}_N$, and hence $\mathcal{X}_N \subseteq X_{\mathcal{P}}$. In addition, the robust invariance of $\mathcal{X}_N$ implied by (4.23) ensures that $\mathcal{W} \subseteq \mathcal{X}_N$, and since $0 \in \text{int}(\mathcal{W})$ due to Assumption 3.1, $X_{\mathcal{P}}$ must contain the origin in its interior. Furthermore, Proposition 4.6 shows that if $\mathcal{P}$ is initially feasible, then it is feasible for all $t \geq 0$, so that $X_{\mathcal{P}}$ is positively invariant for (4.20). Finally, $X_{\mathcal{P}}$ is necessarily compact by Assumption 4.1, and it follows that condition (i) of Lemma 4.7 is satisfied if $\mathcal{R} = X_{\mathcal{P}}$.

We next consider the bounds (4.21) in condition (ii) of Lemma 4.7. For a given state x, nominal parameter vector $\bar{\theta}$ and parameter set Θ, problem $\mathcal{P}$ with $\gamma = 0$ and $Q, R \succ 0$ is a convex quadratic program. Therefore $V^*(x, \bar{\theta}, \Theta)$ is a continuous positive definite, piecewise quadratic function of x [17] for each $\bar{\theta} \in \Theta_0$ and $\Theta \subseteq \Theta_0$. Furthermore Θ_0 is compact due to Assumption 3.2 and it follows that there exist $\mathcal{K}_\infty$-functions ς_1, ς_2 such that (4.21) holds with

$$\mathcal{V}(x, t) = V^*(x, \bar{\theta}_t, \Theta_t).$$

To show that the bound (4.22) in condition (ii) of Lemma 4.7 holds, let $\mathcal{F}_\mathcal{P}$ denote the set $\{v : (x, Kx + v) \in \mathcal{Z},\ x \in X_\mathcal{P}\}$. Then, given the linear dependence of the system (3.1), the model parameterisation (3.4) and the predicted control law (3.5) on the state x, disturbance w and parameter vector θ, and since $\mathcal{W}$, Θ_0, $X_\mathcal{P}$ and $\mathcal{F}_\mathcal{P}$ are compact sets by Assumptions 3.1, 3.2 and 4.1, there exist $\mathcal{K}_\infty$ functions σ_x, σ_w, σ_θ, σ_L, σ_T such that, $\forall x, x_1, x_2 \in X_\mathcal{P}$, $\forall v \in \mathcal{F}_\mathcal{P}$, $\forall w, w_1, w_2 \in \mathcal{W}$, $\forall \theta, \theta_1, \theta_2 \in \Theta_0$,

$$\begin{aligned}
\|f(x_1, v, w_1, \theta_1) - f(x_2, v, w_2, \theta_2)\| &\leq \sigma_x(\|x_1 - x_2\|) + \sigma_w(\|w_1 - w_2\|) + \sigma_\theta(\|\theta_1 - \theta_2\|), \\
|L(x_1, v) - L(x_2, v)| &\leq \sigma_L(\|x_1 - x_2\|), \\
|L_T(x_1, \theta) - L_T(x_2, \theta)| &\leq \sigma_T(\|x_1 - x_2\|).
\end{aligned}$$

Following the proof of Theorem 5 in Limon et al. [75] and using the weak triangle inequality for $\mathcal{K}$-functions [76], we obtain

$$J(x^+, \hat{\mathbf{v}}, \bar{\theta}_t) - \mathcal{V}(x, t) \leq -\varsigma_3(\|x\|) + \sigma_1(\|w\|) + \sigma_2(\|\bar{\theta}_t - \theta^*\|)$$

where $\sigma_1(\|s\|) = \left(\sum_{k=0}^{N-2} \sigma_L \circ \sigma_x^k + \sigma_T \circ \sigma_x^{N-1}\right) \circ 2\sigma_w(\|s\|)$ and $\sigma_2(\|s\|) = \left(\sum_{k=0}^{N-2} \sigma_L \circ \sigma_x^k + \sigma_T \circ \sigma_x^{N-1}\right) \circ 2\sigma_\theta((\|s\|))$, and both σ_1 and σ_2 are $\mathcal{K}$-functions. Since $\hat{\mathbf{v}}$ is a feasible but suboptimal solution of $\mathcal{P}$ at x^+, and since $\bar{\theta}_{t+1} = \bar{\theta}_t$ by assumption, the optimal cost function satisfies $\mathcal{V}(x^+, t+1) = V^*(x^+, \bar{\theta}_{t+1}, \Theta_{t+1}) \leq J(x^+, \hat{\mathbf{v}}, \bar{\theta}_t)$ and hence

$$\mathcal{V}(x^+, t+1) - \mathcal{V}(x, t) \leq -\varsigma_3(\|x\|) + \sigma_1(\|w\|) + \sigma_2(\|\bar{\theta}_t - \theta^*\|).$$

Thus all conditions of lemma 4.7 are satisfied. □

Corollary 4.9. *Assume that $\gamma = 0$ and the nominal parameter vector $\bar{\theta}_t$ is updated at each time $t \in \mathbb{N}_{\geq 0}$ using (4.13). Then for all initial conditions $x_0 \in X_{\mathcal{P}}$, the system (3.1) with control law $u_t = Kx_t + v^*_{0|t}$, where $v^*_{0|t}$ is the first element of $\mathbf{v}^*(x_t, \bar{\theta}_t, \Theta_t)$, robustly satisfies the constraint (3.2) and is ISS with region of attraction $X_{\mathcal{P}}$.*

Proof. It can be shown that condition (i) of Lemma 4.7 and the bounds (4.21) in condition (ii) of Lemma 4.7 are satisfied with $\mathcal{R} = X_{\mathcal{P}}$ and $\mathcal{V}(x,t) = V^*(x, \bar{\theta}_t, \Theta_t)$ for some $\mathcal{K}_\infty$-functions ς_1, ς_2 using the same argument as the proof of Theorem 4.8. To show that (4.22) is also satisfied and hence complete the proof we use an argument similar to the proof of Theorem 4.8. In particular, as before we define $\bar{\mathbf{x}}^* = \{\bar{x}^*_0, \dots, \bar{x}^*_N\}$ using the optimal solution of $\mathcal{P}$, $\mathbf{v}^*(x, \bar{\theta}_t, \Theta_t) = \{v^*_0, \dots, v^*_{N-1}\}$, and

$$\bar{x}^*_{k+1} = f(\bar{x}^*_k, v^*_k, 0, \bar{\theta}_t), \quad \bar{x}^*_0 = x.$$

However, here we define $\mathbf{z} = \{z_0, \dots, z_N\}$ as the sequence

$$z_{k+1} = f(z_k, \hat{v}_k, 0, \bar{\theta}_{t+1}), \quad z_0 = x^+$$

where $\hat{\mathbf{v}} = \{\hat{v}_0, \dots, \hat{v}_{N-1}\}$ has $\hat{v}_k = v^*_{k+1}$ for $k \in \mathbb{N}_{[0,N-2]}$ and $\hat{v}_{N-1} = 0$. Then $\mathcal{V}(x,t) = V^*(x, \bar{\theta}_t, \Theta_t)$ implies

$$\begin{aligned} J(x^+, \hat{\mathbf{v}}, \bar{\theta}_{t+1}) - \mathcal{V}(x,t) = -L(x, v^*_0) + \sum_{k=0}^{N-2} \Big(L(z_k, \hat{v}_k) - L(\bar{x}^*_{k+1}, v^*_{k+1}) \Big) + L(z_{N-1}, 0) \\ + L_T(z_N, \bar{\theta}_{t+1}) - L_T(z_{N-1}, \bar{\theta}_{t+1}) + L_T(z_{N-1}, \bar{\theta}_{t+1}) - L_T(\bar{x}^*_N, \bar{\theta}_t). \end{aligned} \tag{4.24}$$

The update law (4.13) ensures that $\|\bar{\theta}_{t+1} - \bar{\theta}_t\| \leq \|\bar{\theta}_t - \theta^*\|$ since $\theta^* \in \Theta_{t+1}$. Hence, for all $k \in \mathbb{N}_{[1,N-1]}$, we have

$$\begin{aligned} \|z_k - \bar{x}^*_{k+1}\| &\leq \sigma_x(\|z_{k-1} - \bar{x}^*_k\|) + \sigma_\theta(\|\bar{\theta}_{t+1} - \bar{\theta}_t\|) \\ &\leq \sigma_x(\|z_{k-1} - \bar{x}^*_k\|) + \sigma_\theta(\|\bar{\theta}_t - \theta^*\|) \end{aligned}$$

and it follows that, for all $k \in \mathbb{N}_{[1,N-1]}$,

$$\|z_k - \bar{x}^*_{k+1}\| \leq \tfrac{1}{2}(2\sigma_x)^k(\|x^+ - \bar{x}^*_1\|) + \sum_{j=0}^{k-1} \tfrac{1}{2}(2\sigma_x)^j \circ 2\sigma_\theta(\|\theta_t - \theta^*\|)$$

where $\|x^+ - \bar{x}_1^*\| \leq \sigma_\theta(\|\bar{\theta}_t - \theta^*\|) + \sigma_w(\|w_0\|)$. In addition we have, for all $k \in \mathbb{N}_{[0,N-2]}$

$$|L(z_k, \hat{v}_k) - L(\bar{x}_{k+1}^*, v_{k+1}^*)| \leq \sigma_L(\|z_k - \bar{x}_{k+1}^*\|),$$

and, since $Q \succ 0$, there exists a $\mathcal{K}_\infty$ function ς_3 such that $L(x, v_0^*) \geq \varsigma_3(\|x\|)$, while (4.11) implies

$$L(z_{N-1}, 0) + L_T(z_N, \bar{\theta}_{t+1}) - L_T(z_{N-1}, \bar{\theta}_{t+1}) = 0.$$

Furthermore (4.11) is linear in $P(\theta)$ and the solution $P(\theta)$ is unique for all $\theta \in \Theta_t$ (see e.g. [77]) since $A(\theta) + B(\theta)K$ is by assumption stable. Therefore, by the implicit function theorem [78], $P(\theta)$ is Lipschitz continuous and $|L_T(x, \theta_1) - L_T(x, \theta_2)| \leq \kappa_\phi \|\theta_1 - \theta_2\|$ for all $x \in X_\mathcal{P}$, $\theta_1, \theta_2 \in \Theta_0$, for some $\kappa_\phi > 0$. Hence

$$|L_T(z_{N-1}, \bar{\theta}_{t+1}) - L_T(\bar{x}_N^*, \bar{\theta}_t)| \leq \sigma_T(\|z_{N-1} - \bar{x}_N^*\|) + \kappa_\phi \|\bar{\theta}_{t+1} - \bar{\theta}_t\|.$$

Collecting the bounds derived above on individual terms in the expression for $J(x^+, \hat{\mathbf{v}}, \bar{\theta}_{t+1}) - \mathcal{V}(x, t)$ in (4.24), we obtain

$$J(x^+, \hat{\mathbf{v}}, \bar{\theta}_{t+1}) - \mathcal{V}(x, t) \leq -\varsigma_3(\|x\|) + \sigma_1(\|w\|) + \sigma_2(\|\bar{\theta}_t - \theta^*\|)$$

where σ_1, σ_2 are $\mathcal{K}$-functions. But by optimality we have $\mathcal{V}(x^+, t+1) = V^*(x^+, \bar{\theta}_{t+1}, \Theta_{t+1}) \leq J(x^+, \hat{\mathbf{v}}, \bar{\theta}_{t+1})$. Therefore (4.22) holds and hence all of the conditions of Lemma 4.7 are satisfied. □

Remark 4.10. *The input-to-state stability property implies that there exists a $\mathcal{KL}$-function $\eta(\cdot, \cdot)$ and $\mathcal{K}$-functions $\psi(\cdot)$ and $\zeta(\cdot)$ such that for all feasible initial conditions $x_0 \in X_\mathcal{P}$, the closed-loop system trajectories satisfy, for all $t \in \mathbb{N}_{\geq 0}$,*

$$\|x_t\| \leq \eta(\|x_0\|, t) + \psi\Big(\max_{k \in \mathbb{N}_{[0,t-1]}} \|w_k\|\Big) + \zeta\Big(\max_{k \in \mathbb{N}_{[0,t-1]}} \|\bar{\theta}_k - \theta^*\|\Big).$$

Remark 4.11. *Theorem 4.8 and Corollary 4.9 do not apply to the case that $\gamma \neq 0$. However, if γ is replaced by a time-varying weight γ_t in the objective of problem $\mathcal{P}$, then input-to-state stability (ISS) can be guaranteed by switching $\gamma_t = 0$ for all $t \geq t_0$, for some finite horizon t_0.*

4.4 Convergence of the estimated parameter set

In terms of Φ and d defined in (3.7) and (3.8), the system model $x_{t+1} = A(\theta^*)x_t + B(\theta^*)u_t + w_t$ can be rewritten as

$$x_{t+1} = \Phi(x_t, u_t)\theta^* + \phi(x_t, u_t) + w_t \tag{4.25}$$

where x_{t+1}, $\Phi(x_t, u_t)$ and $\phi(x_t, u_t)$ are known at time $t+1$. Thus, the system is linear with regressor Φ_t, uncertain parameter vector θ^* and additive disturbance $w_t \in \mathcal{W}$.

Bai et al. [57] show that, for such a system, the diameter of the parameter set constructed using a set-membership identification method converges to zero with probability 1 if the uncertainty bound $\mathcal{W}$ is tight and the regressor Φ_t is persistently exciting. We extend this result and prove convergence of the estimated parameter set in more general cases. Here we avoid the problem of computational intractability arising from a minimum volume update law of the form $\Theta_{t+1} = \Theta_t \cap \Delta_{t+1}$. Instead, we derive stochastic convergence results for parameter sets with fixed-complexity update law of the form $\Theta_{t+1} \supseteq \Theta_t \cap \Delta_{t+1}$.

In this section, we first discuss relevant results for an update law that gives a minimal parameter set estimate for a given sequence of states (but whose representation has potentially unbounded complexity), before considering the convergence under the fixed-complexity parameter set update law of Section 3.3.1. We then compute bounds on the parameter set diameter if the bounding set for the additive disturbances is overestimated. Lastly, we demonstrate that similar results can be achieved when errors are present in the observed state, as would be encountered for example if the system state are estimated from noisy measurements. In each case we relate the PE condition to the rate of parameter set convergence. We also prove that the parameter set converges to a point (or minimal uncertainty set) with probability one.

In common with Bai et al. [57, 79], we do not assume a specific distribution for the disturbance input. However, the set $\mathcal{W}$ bounding the model disturbance is assumed to be tight in the sense that there is non-zero probability of a realisation w_t lying arbitrarily close to any given point on the boundary, $\partial\mathcal{W}$, of $\mathcal{W}$.

Assumption 4.2 (Tight disturbance bounds)**.** *For all $w^0 \in \partial\mathcal{W}$ and any $\upsilon > 0$, there exists $p_w(\upsilon) > 0$ whenever $\upsilon > 0$ such that the disturbance sequence $\{w_0, w_1, \ldots\}$ satisfies $\Pr\Big\{\|w_t - w^0\| < \upsilon\Big\} \geq p_w(\upsilon)$, for all $t \in \mathbb{N}_{\geq 0}$.*

Assumption 4.3 (Persistent Excitation)**.** *There exist positive scalars τ, ϵ and an integer $N_u \geq \lceil p/n_x \rceil$ such that, for each $t \in \mathbb{N}_{\geq 0}$ we have $\|\Phi_t\| \leq \tau$ and*

$$\sum_{j=t}^{t+N_u-1} \Phi_j^\top \Phi_j \succeq \epsilon \mathbb{I}.$$

We further assume throughout this section that the rows of M_Θ are normalised so that $\|[M_\Theta]_i\| = 1$ for all i.

4.4.1 Minimal parameter set update law

In this subsection, the minimal parameter set is defined as

$$\Theta_{k+1} = \Theta_k \cap \Delta_k, \quad \forall k \in \mathbb{N}_+ \tag{4.26}$$

where Δ_k is defined in (3.9). Therefore $\Theta_k = \bigcap_{j=0}^{k-1} \Delta_j \cap \Theta_0$ is the smallest parameter set that can be determined starting from Θ_0 using the sequences $\{x_0, \ldots, x_k\}$ and $\{u_0, \ldots, u_{k-1}\}$.

The unfalsified parameter set at time t can be also expressed as

$$\Delta_t = \{\theta : \Phi_{t-1}(\theta^* - \theta) + w_{t-1} \in \mathcal{W}\}, \tag{4.27}$$

where w_t is the disturbance realisation at time t and $\Phi_t = \Phi(x_t, u_t)$. Let w^0 be an arbitrary point on the boundary $\partial\mathcal{W}$, then the normal cone $\mathcal{N}_\mathcal{W}(w^0)$ to $\mathcal{W}$ at w^0 is defined

$$\mathcal{N}_\mathcal{W}(w^0) := \{g : g^\top(w - w^0) \leq 0 \ \ \forall w \in \mathcal{W}\}. \tag{4.28}$$

Proposition 4.12. *For all $t \in \mathbb{N}_{\geq 0}$, all $\upsilon > 0$, and for any $\theta \in \mathbb{R}^p$ such that $\|\theta^* - \theta\| \geq \upsilon$, under Assumptions 3.1, 4.2 and 4.3 we have*

$$\Pr\{\theta \notin \Delta_j\} \geq p_w\Big(\upsilon\sqrt{\epsilon/N_u}\Big)$$

for some $j \in \mathbb{N}_{[t+1,t+N_u]}$.

Proof. Assumption 3.1 implies that there exists $w^0 \in \partial\mathcal{W}$ so that $\Phi_j(\theta^* - \theta) \in \mathcal{N}_{\mathcal{W}}(w^0)$ for any given $j \in \mathbb{N}_{[t,t+N_u-1]}$ and $\theta \in \Theta_t$ Therefore, if θ satisfies $(\theta^* - \theta)^\top \Phi_j^\top [\Phi_j(\theta^* - \theta) + w_j - w^0] > 0$, then the definition (4.28) of $\mathcal{N}_{\mathcal{W}}(w^0)$ implies $\Phi_j(\theta^* - \theta) + w_j \notin \mathcal{W}$, and hence $\theta \notin \Delta_{j+1}$ from (4.27). But

$$\begin{aligned}(\theta^* - \theta)^\top \Phi_j^\top \left[\Phi_j(\theta^* - \theta) + w_j - w^0\right] =& \|\Phi_j(\theta^* - \theta)\|^2 + (\theta^* - \theta)^\top \Phi_j^\top (w_j - w^0) \\ \geq& \|\Phi_j(\theta^* - \theta)\|^2 - \|\Phi_j(\theta^* - \theta)\| \, \|w_j - w^0\|.\end{aligned}$$

Therefore $\theta \notin \Delta_{j+1}$ whenever $\|w_j - w^0\| < \|\Phi_j(\theta^* - \theta)\|$. Furthermore, for all $t \in \mathbb{N}_{\geq 0}$ Assumption 4.3 implies

$$\sum_{j=t}^{t+N_u-1} \|\Phi_j(\theta^* - \theta)\|^2 \geq \epsilon \|\theta^* - \theta\|^2.$$

Hence, if $\|\theta^* - \theta\| \geq \upsilon$, then there must exist some $j \in \mathbb{N}_{[t,t+N_u-1]}$ such that

$$\|\Phi_j(\theta^* - \theta)\| \geq \upsilon \sqrt{\epsilon / N_u}.$$

If $\|w_j - w^0\| < \upsilon\sqrt{\epsilon/N_u}$, then it follows that $\|w_j - w^0\| < \upsilon\sqrt{\epsilon/N_u} \leq \|\Phi_j(\theta^* - \theta)\|$ and thus $\theta \notin \Delta_{j+1}$. Assumption 4.2 implies the probability of this event is at least $p_w\left(\upsilon\sqrt{\epsilon/N_u}\right)$. □

Theorem 4.13. *If $\Theta_t = \bigcap_{j=1}^t \Delta_j \cap \Theta_0$ and Assumptions 3.1, 4.2 and 4.3 hold, then for all $\theta \in \Theta_0$ such that $\|\theta - \theta^*\| \geq \upsilon$, for all $t \in \mathbb{N}_{\geq 0}$ and any $\upsilon > 0$, we have*

$$\Pr\{\theta \in \Theta_t\} \leq \left[1 - p_w\left(\upsilon\sqrt{\epsilon/N_u}\right)\right]^{\lfloor t/N_u \rfloor}.$$

Proof. For the non-trivial case of $t \geq N_u$ we have $\Pr\{\theta \in \Theta_t\} = \Pr\{\theta \in \Theta_t \,|\, \theta \in \Theta_{t-N_u}\} \Pr\{\theta \in \Theta_{t-N_u}\}$ since $\Theta_t \subseteq \Theta_{t-N_u}$. Also $\Theta_t = \bigcap_{j=t-N_u+1}^t \Delta_j \cap \Theta_{t-N_u}$ and Proposition 4.12 implies $\Pr\{\theta \in \Theta_t \,|\, \theta \in \Theta_{t-N_u}\} \leq 1 - p_w\left(\upsilon\sqrt{\epsilon/N_u}\right)$ if $\|\theta - \theta^*\| \geq \upsilon$. Therefore

$$\Pr\{\theta \in \Theta_t\} \leq \left(1 - p_w\left(\upsilon\sqrt{\epsilon/N_u}\right)\right) \Pr\{\theta \in \Theta_{t-N_u}\},$$

and the result follows by applying this inequality $\lfloor t/N_u \rfloor$ times. □

Corollary 4.14. *Under Assumptions 3.1, 4.2 and 4.3, $\Theta_t = \bigcap_{j=1}^t \Delta_j \cap \Theta_0$ converges to $\{\theta^*\}$ with probability 1.*

Proof. For any $\theta \in \Theta_0$ and $\upsilon > 0$ such that $\|\theta - \theta^*\| \geq \upsilon$, Theorem 4.13 implies that $\sum_{t=0}^{\infty} \Pr\{\theta \in \Theta_t\}$ is necessarily finite as it is bounded by a geometric series with a common factor smaller than 1. Also, $\theta \in \Theta_t$ requires that $\theta \in \Theta_{t-1}$, and therefore the Borel-Cantelli Lemma implies that $\Pr\Big\{\theta \in \bigcap_{t=0}^{\infty} \Theta_t\Big\} = 0$. It follows that $\Theta_t \to \{\theta^*\}$ as $t \to \infty$ with probability 1 since $\upsilon > 0$ is arbitrary. □

4.4.2 Fixed complexity parameter set update law

In order to reduce computational load and ensure numerical tractability, we assume that the parameter set Θ_t is defined by a fixed complexity polytope, as in (3.10) and (4.1). This section shows that, although a degree of conservativeness is introduced by fixing the complexity of Θ_t, asymptotic convergence of this set to the true parameter vector θ^* still holds with probability 1.

Theorem 4.15. *If Θ_t is updated according to (3.10), (4.1) and Remark 4.5, and Assumptions 3.1, 4.2 and 4.3 hold, then for all $\theta \in \Theta_0$ such that $[M_\Theta]_i(\theta - \theta^*) \geq \upsilon$ for some $i \in \mathbb{N}_{[1,r]}$, we have, for all $t \in \mathbb{N}_{\geq 0}$ and any $\upsilon > 0$,*

$$\Pr\{\theta \in \Theta_t\} \leq \left\{1 - \left[p_w\Big(\frac{\upsilon\epsilon}{N_u\tau}\Big)\right]^{N_u}\right\}^{\lfloor t/N_u \rfloor}.$$

Proof. For the non-trivial case of $t \geq N_u$ we have $\Pr\{\theta \in \Theta_t\} = \Pr\{\theta \in \Theta_t \,|\, \theta \in \Theta_{t-N_u}\} \Pr\{\theta \in \Theta_{t-N_u}\}$ since $\Theta_t \subseteq \Theta_{t-N_u}$ by Lemma 3.2. Consider therefore the probability that any given $\theta \in \Theta_{t-N_u}$ satisfying $[M_\Theta]_i(\theta - \theta^*) \geq \upsilon$ lies in $\Delta_{t-N_u+1} \cap \cdots \cap \Delta_t$. Define vectors g_j for $j \in \mathbb{N}_{[t-N_u,t-1]}$ by

$$g_j^\top = -[M_\Theta]_i \left(\sum_{k=t-N_u}^{t-1} \Phi_k^\top \Phi_k\right)^{-1} \Phi_j^\top. \tag{4.29}$$

Assumption 3.1 implies that, for any given $g_j \in \mathbb{R}^{n_x}$, there exists a $w_j^0 \in \partial\mathcal{W}$ such that $g_j \in \mathcal{N}_\mathcal{W}(w_j^0)$. Accordingly, choose $w_j^0 \in \partial\mathcal{W}$ so that g_j in (4.29) satisfies $g_j \in \mathcal{N}_\mathcal{W}(w_j^0)$ for each $j \in \mathbb{N}_{[t-N_u,t-1]}$. Then

$$g_j^\top \left[\Phi_j(\theta^* - \theta) + w_j - w_j^0\right] \leq 0 \tag{4.30}$$

is a necessary condition for $\theta \in \Delta_{j+1}$ due to (4.27) and (4.28). But (4.29) and Assumption 4.3 imply

$$\sum_{j=t-N_u}^{t-1} g_j^\top \left[\Phi_j(\theta^* - \theta) + w_j - w_j^0\right] = [M_\Theta]_i(\theta - \theta^*) + \sum_{j=t-N_u}^{t-1} g_j^\top (w_j - w_j^0)$$
$$\geq [M_\Theta]_i(\theta - \theta^*) - \frac{N_u \tau}{\epsilon} \max_{j \in \mathbb{N}_{[t-N_u, t-1]}} \|w_j - w_j^0\|.$$

where $[M_\Theta]_i(\theta - \theta^*) \geq \upsilon$ by assumption, and it follows from (4.30) that $\theta \notin \bigcap_{j=t-N_u+1}^{t} \Delta_j$ if $\|w_j - w_j^0\| < \upsilon\epsilon/(N_u\tau)$ for all $j \in \mathbb{N}_{[t-N_u,t-1]}$. From Assumption 4.2 and the independence of the sequence $\{w_0, w_1, \ldots\}$ we therefore conclude that

$$\Pr\{\theta \in \Theta_t\} \leq \left\{1 - \left[p_w\Big(\frac{\upsilon\epsilon}{N_u\tau}\Big)\right]^{N_u}\right\} \Pr\{\theta \in \Theta_{t-N_u}\},$$

and the result follows by applying this inequality $\lfloor t/N_u \rfloor$ times. □

Corollary 4.16. *Under Assumptions 3.1, 4.2 and 4.3, the fixed complexity parameter set estimate Θ_t converges to $\{\theta^*\}$ with probability 1.*

Proof. By applying the Borel-Cantelli Lemma to Theorem 4.15 it can be shown (analogously to the proof of Corollary 4.14) that $\Pr\{\theta \in \bigcap_{t=0}^{\infty} \Theta_t\} = 0$ if $[M_\Theta]_i(\theta - \theta^*) \geq \upsilon$ for some $i \in \mathbb{N}_{[1,r]}$ and $\upsilon > 0$. Since M_Θ is assumed to be chosen so that Θ_t is compact for all μ_t such that Θ_t is non-empty, it follows that $\Theta_t \to \{\theta^*\}$ as $t \to \infty$ with probability 1. □

4.4.3 Inexact disturbance bounds

We next consider the case in which the set $\mathcal{W}$ bounding w_t in Assumption 3.1 does not satisfy Assumption 4.2. Instead, we assume that a compact set Ω providing a tight bound on w_t exists but is either unknown or non-polytopic or nonconvex. We define the unit ball $\mathcal{B} = \{x : \|x\| \leq 1\}$ and use a scalar ρ to characterize the accuracy to which $\mathcal{W}$ approximates Ω.

Then the minimal parameter set update (4.26) has to be replaced with

$$\Theta_{k+1} = \Theta_k \cap \hat{\Delta}_k \tag{4.31}$$

where $\hat{\Delta}_k$ is the unfalsified parameter set given knowledge of $\mathcal{W}$, x_{k+1}, Φ_k and ϕ_k, i.e.

$$\hat{\Delta}_k = \{\theta : x_{k+1} - \Phi_k\theta - \phi_k \in \mathcal{W}\},$$

which can be expressed in terms of the disturbance realisation $w_k \in \Omega$ at time k:

$$\hat{\Delta}_k = \{\theta : \Phi_k(\theta^* - \theta) + w_k \in \mathcal{W}\}. \tag{4.32}$$

Assumption 4.4 (Inexact disturbance bounds)**.** *Ω is a compact set such that $\Omega \oplus \rho\mathcal{B} \supseteq \mathcal{W} \supseteq \Omega$ for some $\rho > 0$, and, for all $w^0 \in \partial\Omega$ and $\upsilon > 0$, the disturbance sequence $\{w_0, w_1, \ldots\}$ satisfies, for all $t \in \mathbb{N}_{\geq 0}$, $w_t \in \Omega$ and $\Pr\Big\{\|w_t - w^0\| < \upsilon\Big\} \geq p_w(\upsilon)$, where $p_w(\upsilon) > 0$ whenever $\upsilon > 0$.*

Remark 4.17. *Assumption 4.4 implies that $\mathcal{W} \ominus \Omega \subseteq \rho\mathcal{B}$. As a result, every point in $\mathcal{W}$ can be a distance no greater than ρ from a point in Ω, i.e.*

$$\max_{\hat{w}\in\mathcal{W}} \min_{w\in\Omega} \|\hat{w} - w\| \leq \rho \tag{4.33}$$

Theorem 4.18. *If $\Theta_t = \bigcap_{j=1}^{t} \Delta_j \cap \Theta_0$ and Assumptions 3.1, 4.3 and 4.4 hold, then for all $\theta \in \Theta_0$ such that $\|\theta - \theta^*\| \geq \upsilon + \rho\sqrt{N_u/\epsilon}$, for all $t \in \mathbb{N}_{\geq 0}$ and any $\upsilon > 0$, we have*

$$\Pr\{\theta \in \Theta_t\} \leq \left[1 - p_w\Big(\upsilon\sqrt{\epsilon/N_u}\Big)\right]^{\lfloor t/N_u \rfloor}.$$

Proof. For $k \in \{0, N_u, 2N_u, \ldots\}$, a point $\theta^0 \in \Theta_t$ is excluded from Θ_{t+N_u} if $\theta^0 \notin \hat{\Delta}_j$ for some $j \in \{k, \ldots, k + N_u - 1\}$. Under Assumptions 3.1, 4.3 and 4.4, there necessarily exists $\hat{w}_j^0 \in \partial\mathcal{W}$ so that $\Phi_j(\theta^* - \theta^0) \in \mathcal{N}_{\mathcal{W}}(\hat{w}_j^0)$, and hence $\theta^0 \notin \hat{\Delta}_j$ if $(\theta^* - \theta^0)^\top \Phi_j^\top \Big[\Phi_j(\theta^* - \theta^0) + w_j - \hat{w}_j^0\Big] > 0$, but (4.33) implies

$$(\theta^* - \theta^0)^\top \Phi_j^\top (w_j - \hat{w}_j^0) \geq (\theta^* - \theta^0)^\top \Phi_j^\top (w_j - w_j^0) - \rho\|\Phi_j(\theta^* - \theta^0)\|$$

for some $w_j^0 \in \Omega$, and hence

$$\begin{aligned}
&(\theta^* - \theta^0)^\top \Phi_j^\top \Big[\Phi_j(\theta^* - \theta^0) + w_j - \hat{w}_j^0\Big] \\
\geq &\|\Phi_j(\theta^* - \theta^0)\|^2 + (\theta^* - \theta^0)^\top \Phi_j^\top (w_j - w_j^0) - \rho\|\Phi_j(\theta^* - \theta^0)\| \\
\geq &\|\Phi_j(\theta^* - \theta^0)\|^2 - \|\Phi_j(\theta^* - \theta^0)\|\,\|w_j - w_j^0\| - \rho\|\Phi_j(\theta^* - \theta^0)\|.
\end{aligned}$$

Therefore $\theta^0 \notin \hat{\Delta}_j$ if $\|\Phi_j(\theta^* - \theta^0)\| > \|w_j - w_j^0\| + \rho$. But Assumption 4.3 implies that

$$\|\Phi_j(\theta^* - \theta^0)\| \geq \upsilon\sqrt{\epsilon/N_u} + \rho$$

for some $j \in \{k, \ldots, k + N_u - 1\}$ whenever $\|\theta^* - \theta^0\| \geq \upsilon + \rho\sqrt{N_u/\epsilon}$. Hence if θ^0 satisfies this condition, then $\theta^0 \notin \bigcap_{l=t}^{t+N_u-1} \hat{\Delta}_l$ if $\|w_j - w_j^0\| < \upsilon\sqrt{\epsilon/N_u}$, so Assumption 4.2 implies

$$\Pr\left\{\theta^0 \notin \bigcap_{l=t}^{t+N_u-1} \hat{\Delta}_l\right\} \geq p_w\Big(\upsilon\sqrt{\epsilon/N_u}\Big).$$

The bound given in the theorem on the probability that θ^0 is not contained in Θ_k then follows from the independence of the disturbance sequence $\{w_0, w_1, \ldots\}$ and the definition of Θ_k. □

Corollary 4.19. *Under Assumptions 3.1, 4.3 and 4.4, $\Theta_t = \bigcap_{j=1}^{t} \Delta_j \cap \Theta_0$ converges to $\Theta_\infty \subseteq \{\theta^*\} \oplus \rho\sqrt{N_u/\epsilon}\,\mathcal{B}$ with probability 1.*

Proof. The update (4.31) ensures that $\Theta_{t+1} \subseteq \Theta_t$ for all $t \in \mathbb{N}_+$, and hence Θ_t necessarily converges to a limit Θ_∞ as $k \to \infty$. Furthermore, since $p_w\Big(\upsilon\sqrt{\epsilon/N_u}\Big) \geq \delta$ for some $\delta > 0$ whenever $\upsilon > 0$, Theorem 4.18 implies

$$\Pr\left\{\lim_{k\to\infty} \max_{\theta\in\Theta_k} \|\theta - \theta^*\| < \upsilon + \rho\sqrt{N_u/\epsilon}\right\} = 1$$

for all $\upsilon > 0$, which implies that $\Theta_\infty \subseteq \{\theta^*\} \oplus \rho\sqrt{N_u/\epsilon}\,\mathcal{B}$ with probability 1. □

As the accuracy to which $\mathcal{W}$ approximates Ω improves (i.e. $\rho \to 0$), the results of Theorem 4.18 and Corollary 4.19 approaches that of Theorem 4.13 and Corollary 4.14.

Theorem 4.20. *Let Assumptions 3.1, 4.3 and 4.4 hold and let Θ_t be the fixed complexity parameter set defined by (3.10), (4.1) with Remark 4.5. Then, for all $\theta \in \Theta_0$ such that $[M_\Theta]_i(\theta - \theta^*) \geq \upsilon + \rho N_u\tau/\epsilon$ for some $i \in \mathbb{N}_{[1,r]}$ and any $\upsilon > 0$, we have, for all $t \in \mathbb{N}_{\geq 0}$,*

$$\Pr\{\theta \in \Theta_t\} \leq \left\{1 - \left[p_w\Big(\frac{\upsilon\epsilon}{N_u\tau}\Big)\right]^{N_u}\right\}^{\lfloor t/N_u \rfloor}.$$

Proof. A bound on the probability that $\theta \in \Theta_{t-N_u}$ satisfying $[M_\Theta]_i(\theta - \theta^*) \geq \upsilon + \rho N_u \tau/\epsilon$ lies in $\Delta_{t-N_u+1} \cap \cdots \cap \Delta_t$ can be found using the same argument as the proof of Theorem 4.15. Thus choose $\hat{w}_j^0 \in \partial\mathcal{W}$, $j \in \mathbb{N}_{[t-N_u,t-1]}$ so that $g_j \in \mathcal{N}_\mathcal{W}(\hat{w}_j^0)$, where

$$g_j^\top = -[M_\Theta]_i \left(\sum_{k=t-N_u}^{t-1} \Phi_k^\top \Phi_k \right)^{-1} \Phi_j^\top,$$

and pick $w_j^0 \in \partial\Omega$ so that $\|\hat{w}_j^0 - w_j^0\| \leq \rho$ for each $j \in \mathbb{N}_{[t-N_u,t-1]}$. Then, from $[M_\Theta]_i(\theta - \theta^*) \geq \upsilon + \rho N_u \tau/\epsilon$ and Assumptions 4.3 and 4.4 we have

$$\begin{aligned}
&\sum_{j=t-N_u}^{t-1} g_j^\top \left[\Phi_j(\theta^* - \theta) + w_j - \hat{w}_j^0 \right] \\
=&[M_\Theta]_i(\theta - \theta^*) + \sum_{j=t-N_u}^{t-1} g_j^\top (w_j - w_j^0) + \sum_{j=t-N_u}^{t-1} g_j^\top (w_j^0 - \hat{w}_j^0) \\
\geq&\upsilon + \rho \frac{N_u \tau}{\epsilon} - \frac{N_u \tau}{\epsilon} \max_{j \in \mathbb{N}_{[t-N_u,t-1]}} \|w_j - w_j^0\| - \rho \frac{N_u \tau}{\epsilon}.
\end{aligned}$$

Therefore, if $\|w_j - w_j^0\| < \upsilon\epsilon/(N_u\tau)$ for all $j \in \mathbb{N}_{[t-N_u,t-1]}$, then $\sum_{j=t-N_u}^{t-1} g_j^\top \left[\Phi_j(\theta^* - \theta) + w_j - \hat{w}_j^0 \right] > 0$ which implies $\theta \notin \bigcap_{j=t-N_u+1}^{t} \Delta_j$. From Assumption 4.4 and the independence of the sequence $\{w_0, w_1, \ldots\}$ we therefore conclude that

$$\Pr\{\theta \in \Theta_t\} \leq \left\{ 1 - \left[p_w\left(\frac{\upsilon\epsilon}{N_u\tau} \right) \right]^{N_u} \right\} \Pr\{\theta \in \Theta_{t-N_u}\},$$

and the result follows by applying this inequality $\lfloor t/N_u \rfloor$ times. □

Corollary 4.21. *Under Assumptions 3.1, 4.3 and 4.4, the fixed complexity parameter set defined by (3.10), (4.1) and Remark 4.5 converges with probability 1 to a subset of* $\{\theta : M_\Theta(\theta - \theta^*) \leq \rho N_u \tau/\epsilon\}$.

Proof. The proof is analogous to that of Corollary 4.19. □

Similarly, as the accuracy to which $\mathcal{W}$ approximates Ω improves (i.e. $\rho \to 0$), the results of Theorem 4.20 and Corollary 4.21 approaches that of Theorem 4.15 and Corollary 4.16.

4.4.4 System with measurement noise

Consider the system model with an unknown parameter vector θ^* and measurement noise s_t:

$$x_{t+1} = A(\theta^*)x_t + B(\theta^*)u_t + w_t \tag{4.34a}$$

$$y_t = x_t + s_t \tag{4.34b}$$

where $y_t \in \mathbb{R}^{n_x}$ is a measurement (or state estimate) and the noise sequence $\{s_0, s_1, \ldots\}$ has independent elements satisfying $s_t \in \mathcal{S}$ for all $t \in \mathbb{N}_{\geq 0}$.

Assumption 4.5 (Measurement noise bounds)**.** *$\mathcal{S}$ is a compact convex polytope with vertex representation $\mathcal{S} = \mathrm{Co}\{s^{(1)}, \ldots, s^{(h)}\}$.*

Due to the measurement noise, the unfalsified parameter set must be constructed at each time $t \in \mathbb{N}_{\geq 0}$ using the available measurements y_t, y_{t-1}, the known control input u_{t-1}, and sets $\mathcal{W}$ and $\mathcal{S}$ bounding the disturbance and the measurement noise. To be consistent with (4.34), θ^* must clearly lie in the set $\{\theta : y_t - \Phi(y_{t-1} - s_{t-1}, u_{t-1})\theta - \phi(y_{t-1} - s_{t-1}, u_{t-1}) \in \mathcal{W} \oplus \mathcal{S}\}$, and the smallest unfalsified parameter set based on this information is given by

$$\Delta_t = \mathrm{Co}\{\Delta_t^{(1)}, \ldots, \Delta_t^{(h)}\}, \tag{4.35a}$$

$$\Delta_t^{(j)} = \{\theta : y_t - \Phi(y_{t-1} - s^{(j)}, u_{t-1})\theta - \phi(y_{t-1} - s^{(j)}, u_{t-1}) \in \mathcal{W} \oplus \mathcal{S}\} \quad \forall j \in \mathbb{N}_{[1,h]}. \tag{4.35b}$$

Thus Assumption 4.5 implies that the unfalsified set Δ_t is a convex polytope and the parameter set Θ_t can be estimated using, for example, the update law (3.10), (4.1) if $\mathcal{S}$ is known.

Assumption 4.6 (Tight measurement noise and disturbance bounds)**.** *For all $w^0 \in \partial\mathcal{W}$, $s^0 \in \partial\mathcal{S}$ and $\upsilon > 0$ we have*

$$\Pr\left\{\left\|\begin{bmatrix} w_t - w^0 \\ s_t - s^0 \end{bmatrix}\right\| < \upsilon\right\} \geq p_{w,s}(\upsilon)$$

where $p_{w,s}(\upsilon) > 0$ whenever $\upsilon > 0$.

Given Assumptions 4.5 and 4.6, the results of Sections 4.4.1 and 4.4.2 apply with minor modifications. Define $\xi_t = w_t + s_t$, then Assumption 4.6 implies

$$\Pr\Big\{\|\xi_t - \xi_t^0\| < \upsilon\Big\} \geq p_{w,s}\Big(\upsilon/\sqrt{2}\Big)$$

for any given $\xi_t^0 = w_t^0 + s_t^0$ with $w_t^0 \in \partial\mathcal{W}$ and $s_t^0 \in \partial\mathcal{S}$. This implies the following straightforward extensions of Theorems 4.13 and 4.15 and Corollaries 4.14 and 4.16.

Corollary 4.22. *Let Assumptions 3.1, 4.3, 4.5 and 4.6 hold and $\Theta_t = \bigcap_{j=1}^{t} \Delta_j \cap \Theta_0$, with Δ_j given by (4.35). Then for all $\theta \in \Theta_0$ such that $\|\theta - \theta^*\| \geq \upsilon$, for all $t \in \mathbb{N}_{\geq 0}$ and all $\upsilon > 0$, we have*

$$\Pr\{\theta \in \Theta_t\} \leq \left[1 - p_{w,s}\Big(\upsilon\sqrt{\frac{\epsilon}{2N_u}}\Big)\right]^{\lfloor t/N_u \rfloor}.$$

Corollary 4.23. *Let Assumptions 3.1, 4.3, 4.5 and 4.6 hold and let Θ_t be the fixed complexity parameter set defined by (3.10), (4.1) with Remark 4.5 and (4.35). Then for all $\theta \in \Theta_0$ such that $[M_\Theta]_i(\theta - \theta^*) \geq \upsilon$ for some $i \in \mathbb{N}_{[1,r]}$ and any $\upsilon > 0$, we have, for all $t \in \mathbb{N}_{\geq 0}$,*

$$\Pr\{\theta \in \Theta_t\} \leq \left\{1 - \left[p_{w,s}\Big(\frac{\upsilon\epsilon}{\sqrt{2}N_u\tau}\Big)\right]^{N_u}\right\}^{\lfloor t/N_u \rfloor}.$$

Corollary 4.24. *Under Assumptions 3.1, 4.3, 4.5, 4.6 the parameter set Θ_t defined in Corollary 4.22 or 4.23 converges to $\{\theta^*\}$ with probability 1.*

Remark 4.25. *If measurement noise is present, modifications to the proposed algorithm in section 4.2.4 are needed. If no additional output constraints are present, then, given the noisy measurement y_t, the constraint (4.2) should be replaced by*

$$Tx \leq \alpha_0, \ \forall x \in \{y_t\} \oplus (-\mathcal{S})$$

in order to ensure robust satisfaction of input and state constraints. In addition, the unfalsified parameter set Δ_t is in this case given by (4.35).

4.5 Numerical examples

This section presents simulations to illustrate the operation of the proposed adaptive robust MPC scheme. The section consists of two parts. The first part investigates the effect of additional weight γ in optimisation problem $\mathcal{P}$ by using the example of a second-order system from [45]. The second part demonstrates the relationship between the speed of parameter convergence and minimal eigenvalue ϵ from the PE condition.

4.5.1 Adaptive robust MPC with augmented cost

Consider the second-order discrete-time uncertain linear system from [45], with model parameters

$$A_0 = \begin{bmatrix} 0.5 & 0.2 \\ -0.1 & 0.6 \end{bmatrix},\ A_1 = \begin{bmatrix} 0.042 & 0 \\ 0.072 & 0.03 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.015 & 0.019 \\ 0.009 & 0.035 \end{bmatrix},\ A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$B_0 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ B_3 = \begin{bmatrix} 0.0397 \\ 0.059 \end{bmatrix},$$

and with true system parameter $\theta^* = \begin{bmatrix} 0.8 & 0.2 & -0.5 \end{bmatrix}^\top$. The initial parameter set estimate is $\Theta_0 = \{\theta : \|\theta\|_\infty \leq 1\}$, and for all $t \geq 0$, Θ_t is a hyperrectangle, with $\mathcal{M}_\theta = [\mathbb{I} \ -\mathbb{I}]^\top$. The elements of the disturbance sequence $\{w_0, w_1, \ldots\}$ are independent and identically (uniformly) distributed on $\mathcal{W} = \{w \in \mathbb{R}^2 : \ \|w\|_\infty \leq 0.05\}$. The state and input constraints are $[x_t]_2 \geq -0.3$ and $u_t \leq 1$. The MPC prediction horizon and PE window length are set to be $N = 10$ and $N_u = 2$ respectively. The matrix T is chosen according to Remark 4.2 and has 9 rows.

All simulations were performed in Matlab on a 3.4 GHz Intel Core i7 processor, and the online MPC optimisation $\mathcal{P}$ was solved using Mosek [80]. For purposes of comparison, the same parameter set update law and nominal parameter update law were used in all cases. Robust satisfaction of input and state constraints and recursive feasibility were observed in all simulations, in agreement with Proposition 4.6. To

illustrate satisfaction of the state constraint $[x]_2 \geq -0.3$, Figure 4.1 shows the cross-sections of the robust state tube predicted at $t = 0$, with initial condition $x_0 = (3, 2)$, together with the closed-loop state trajectories for 10 different initial conditions.

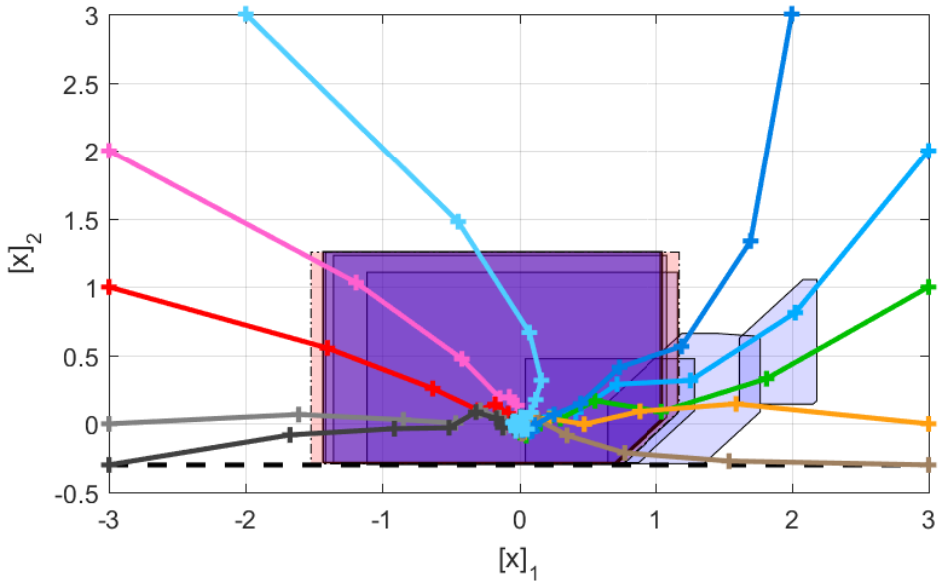


Figure 4.1: Closed-loop trajectories (solid lines) from different initial conditions, and predicted state tube cross-sections $\{\mathcal{X}_1, \ldots, \mathcal{X}_N\}$ at $t = 0$ for initial condition $x_0 = (3, 2)$, with $\mathcal{X}_N$ shown in red and enclosed by dashed line when $\gamma = 10^3$.

Table 4.1: Performance comparison of robust MPC algorithms, with and without PE conditions

Algorithm	(A) Homothetic tube (no PE)[34]	(B) Homothetic tube with PE [34]	(C) Proposed Algorithm (no PE)	(D) Proposed Algorithm with PE ($\gamma = 10^3$)
Solver time/s	0.1130	0.2065	0.4761	0.0766
Θ_{100} set size /%	18.26	18.51	18.26	16.56

Table 4.1 compares the the computational time and parameter sizes of the proposed algorithm and existing algorithms when the same initial conditions and disturbance sequences $\{w_0, w_1, \ldots\}$ are used. Algorithm (A) refers to the robust adaptive MPC in Section 3.4 of Lorenzen et al. [34]. Algorithm (B) is a modification of algorithm (A) that incorporates the PE constraint $\sum_{l=0}^{N_u} \mathbf{u}_{t-l}\mathbf{u}_{t-l}^\top \geq \beta' \mathbb{I}$, which is implemented as described in Lu and Cannon [60] with a fixed β' value: $\beta' = 10^{-4}$.

Algorithms (C) and (D) are the algorithm proposed in Section 4.2.4, with and without the PE condition, respectively. Namely, Algorithm (C) is an adaptive robust MPC that combines fixed-complexity tube MPC with set-based parameter estimation, while (D) is Algorithm 4.1, which has augmented PE term in the objective and additional PE constraints.

Consider first algorithms (A) and (C) in Table 4.1. Algorithm (C) uses a more flexible tube representation and therefore consumes more computational time relative to Algorithm (A). The more flexible tube representation employed in (C) provides a larger terminal set, as shown in Figure 4.2. Moreover, the formulation of (C) incorporates information on Θ_t in the constraints, and as a result, the terminal set increases in size over time as the parameter set Θ_t shrinks. On the other hand, the homothetic tube MPC employed in (A) employs a terminal set that is computed offline and is not updated online.

Comparing algorithms (B) and (D) in Table 4.1, it can be seen that implementing the PE condition using an augmented cost function and linearized constraint (as in (D)) results in lower computation and faster parameter convergence than using a PE constraint with a fixed value of β' (as in (B)). Tuning the value of β' in algorithm (B) is challenging, since a value that is too small results in slow convergence whereas choosing β' too large frequently causes the PE constraint to be infeasible. Moreover, whenever the PE constraint is infeasible in algorithm (B), the MPC optimisation is solved a second time without the PE constraint, thus increasing computation.

Figure 4.3 shows the effect of the weighting coefficient γ in the objective function (4.17) on the parameter set Θ_t when the same initial conditions ($x_0 = [3, 4]^\top$, Θ_0, $\bar{\theta}_0$) and disturbance sequences $\{w_0, w_1, \ldots\}$ are used. Larger values of γ place greater weighting on β' in the MPC cost (4.17), and thus on satisfaction of the PE condition (4.15). Therefore increasing γ results in a faster convergence rate in the parameter set volume. When the same weighting coefficient γ is used, performing the parameter set update periodically (as discussed in Remark 4.5) slows down the convergence rate of the parameter set, as shown by the green line. For this

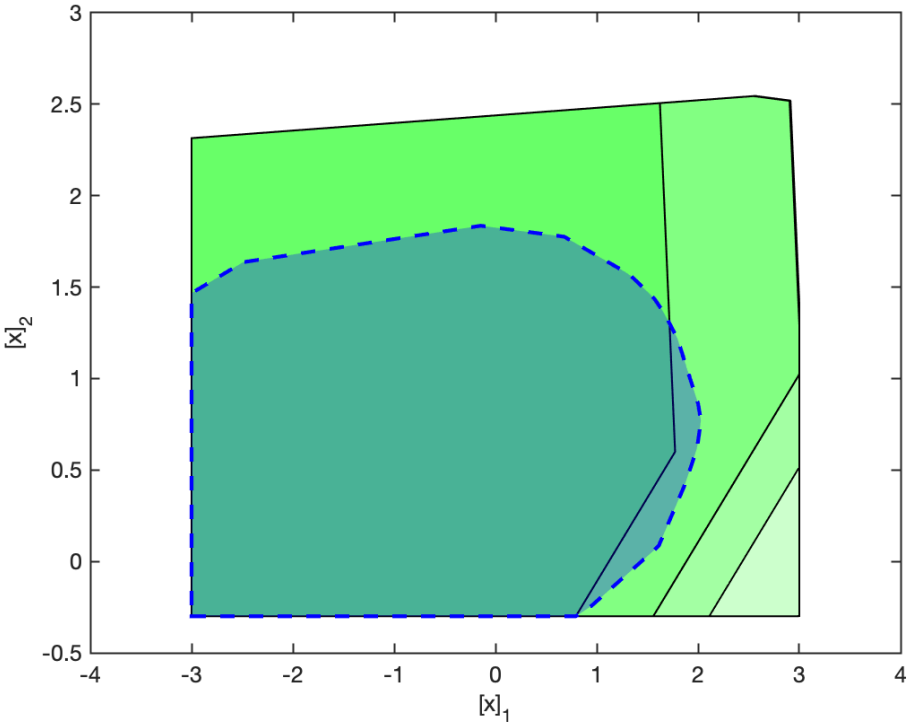


Figure 4.2: Terminal sets for Algorithm (C) at times $t = 0, 1, 100, 1000$ and the terminal set for (A) (which is computed offline and not updated online). The terminal sets for (C) are shown in green with solid boundaries, and are nested and increasing over time. The terminal set for (A) is shown in blue with dashed line boundary.

simulation, the parameter set update (online step 2 in Section 4.2.4) takes only 2% of the total computational time.

The relationship between weighting coefficient γ and volume of parameter set Θ_t is illustrated in Figure 4.4. For values of γ between 10^{-3} and 10^{3}, closed-loop simulations were performed with the same initial conditions, disturbance sequences, and initial nominal model and parameter set. The parameter set volume after 20 time-steps is shown. Figure 4.4 also shows that increasing γ results in a faster parameter set convergence rate, in agreement with Figure 4.3.

For the same set of simulations, Figure 4.5 shows the optimal value of β' in (4.16) and (4.14) against γ. From (4.17), it is expected that a larger γ value will increase the influence of the term $-\gamma\beta'$, thus pushing β' to be more positive. The left-hand figure shows the value of β' in the convexified constraint (4.16). As expected, the increase in γ leads to a smooth increase in β' initially, but after a certain point,

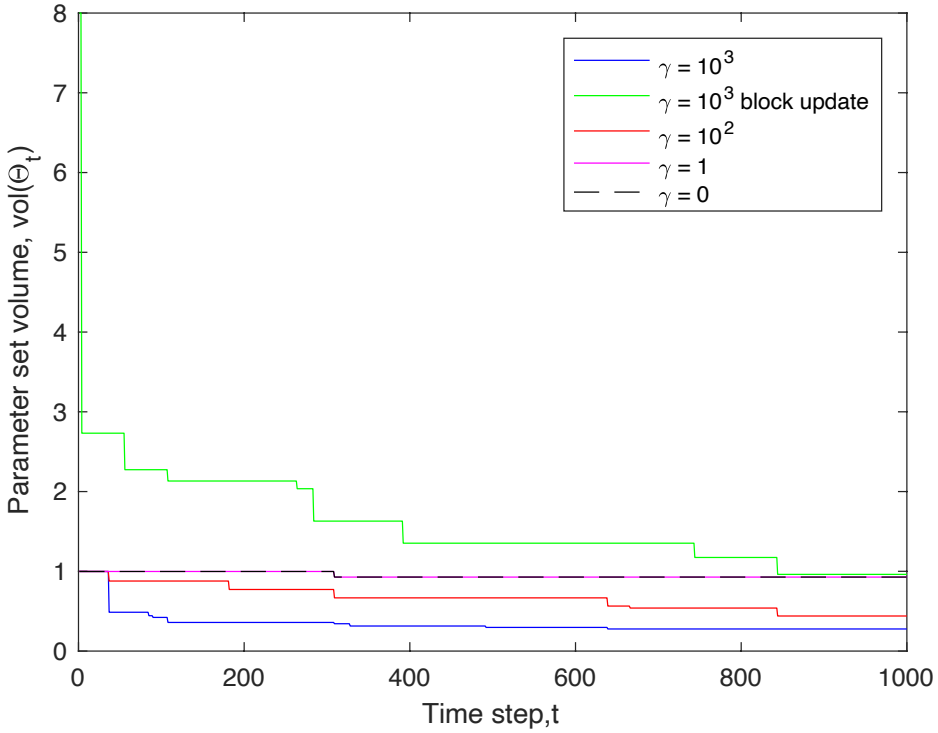


Figure 4.3: Volume of parameter set Θ_t over time for a range of weights γ in the MPC objective function (4.17).

any further increase in the weighting factor γ does not affect the calculated β' value. The right-hand figure shows the value of ϵ in the PE condition (4.15). The difference between β' and ϵ illustrates the conservativeness of the convexification and prediction proposed in Section 4.2.3. Note that this can be reduced by repeating steps (3) and (4) in the online part of the proposed algorithm, thus iteratively re-computing the reference sequences $\hat{\mathbf{x}}$, $\hat{\mathbf{u}}$ and reducing the conservativeness of linearisation to any desired level. It is interesting to note that, although the optimal value of β' in (4.15) levels off at $\gamma = 1$, the value of ϵ in the PE condition (4.14) increases monotonically between $\gamma = 10$ and $\gamma = 10^3$. The smaller β' values observed with (4.15) also explain the lower rates of parameter convergence for small values of γ in Figure 4.4. In practice, it can be used as a guideline for the tuning of γ.

Table 4.2 illustrates the convergence of the estimated parameter set over a large number of time steps for the initial condition $x_0 = [2, 3]^\top$ and a randomly generated disturbance sequence $\{w_0, w_1, \dots\}$. Here γ was chosen as 10^3 to speed up the

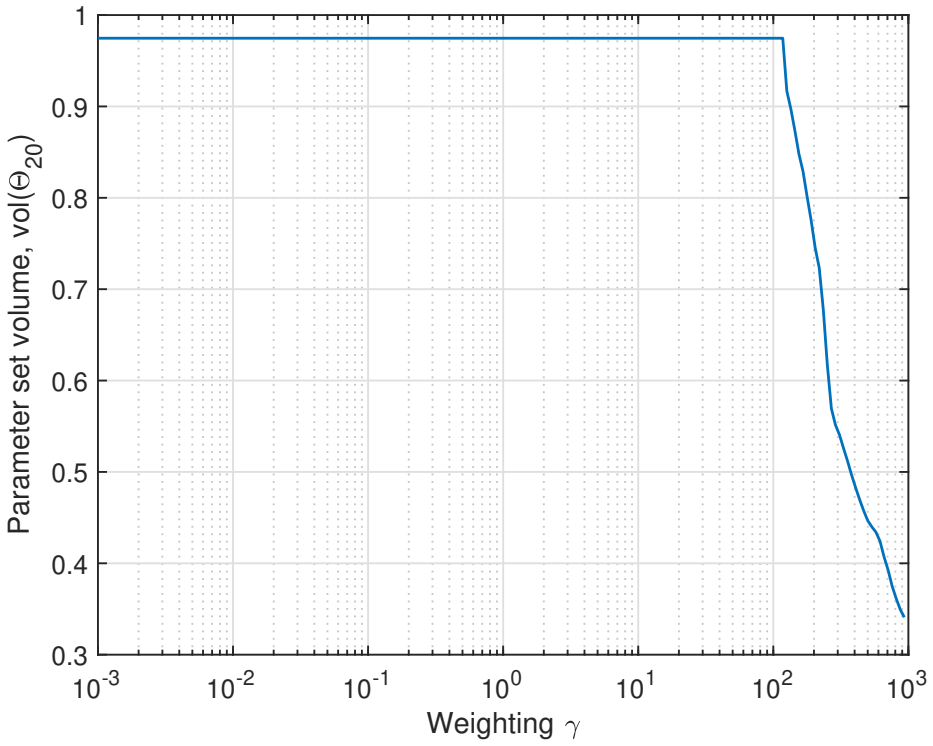


Figure 4.4: Volume of parameter set Θ_{20} against weighting γ in the MPC objective function (4.17).

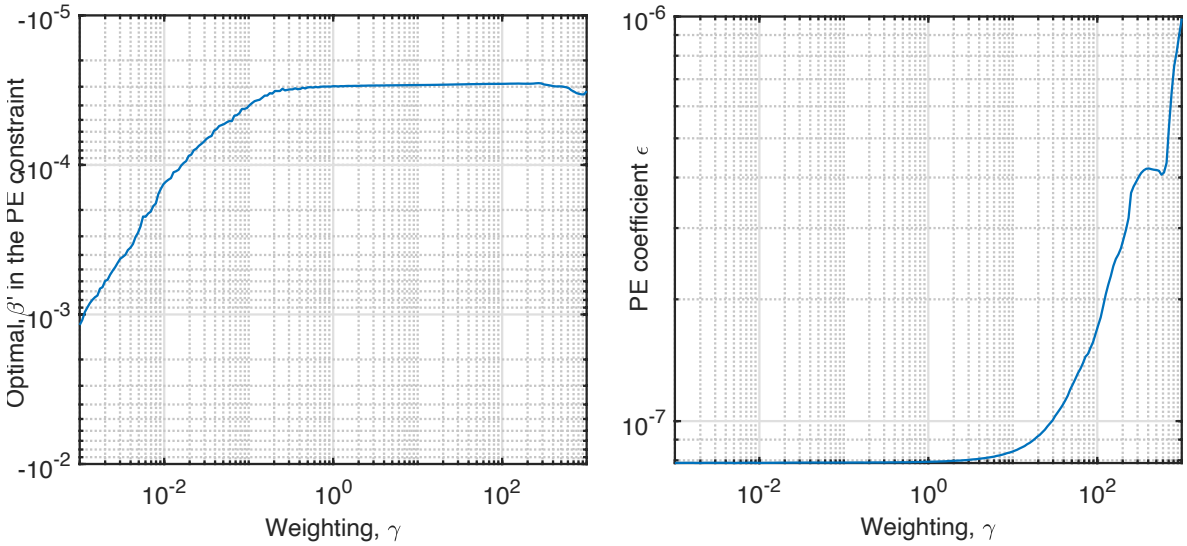


Figure 4.5: Degree of satisfaction of conditions for persistence of excitation as a function of the weighting γ in the MPC objective function (4.17). Left: optimal value of β' in the constraint (4.16). Right: computed value of the PE coefficient ϵ in (4.14).

convergence process. In agreement with Theorem 4.15, Θ_t has shrunk to a small region around the true parameter value at $t = 5000$.

Table 4.2: Asymptotic convergence of the estimated parameter set

Time Step /t	0	1	5	50	100	500	1000	5000
Θ_t set size /%	100	30.21	18.50	14.46	12.75	11.11	1.51	0.25

4.5.2 Relationship between the PE coefficient and the convergence rate of estimated parameter sets

We next consider third-order discrete-time linear systems given by (3.1) with $x \in \mathbb{R}^3$, $u \in \mathbb{R}^2$, $\theta \in \mathbb{R}^3$ and

$$\mathcal{W} = \{w : \|w\|_\infty \leq 0.1\}.$$

The system matrices $\Big(A(\theta), B(\theta)\Big)$ satisfy (3.4) with randomly generated A_i, B_i, θ^* parameters and initial parameter set $\Theta_0 = \{\theta : \|\theta\|_\infty \leq 0.25\}$. In each case, the estimated parameter sets Θ_t have fixed complexity, with face normals aligned with the coordinate axes in parameter space. A linear feedback law is applied, $u_t = Kx_t$, where K is a stabilizing gain. We use these systems to investigate the relationship between the coefficient ϵ in the PE condition (4.14) and the rate of convergence of the estimated parameter set.

Taking the window length in (4.14) to be $N_u = 10$, closed-loop trajectories were computed for 10 time steps and the parameter set Θ_t was updated according to (4.1). Simulations were performed for 500 different initial conditions, and the average value of ϵ was computed for each initial condition using 100 random disturbance sequences $\{w_0, w_1, \ldots\}$. Figure 4.6 illustrates the relationship between the average size of the identified parameter set Θ_t and the average value of ϵ in the PE condition (4.14). Clearly, increasing ϵ results in a smaller parameter set on average, and hence a faster rate of convergence of Θ_t, which is consistent with the analysis of Section 4.4.2. The inner and outer radii shown in the figure on the left are the radii of the smallest and largest spheres, respectively, that contain and are contained within the parameter set estimate after 10 time steps. A similar trend can also be seen between the average volume of the parameter set Θ and the ensemble average value of ϵ.

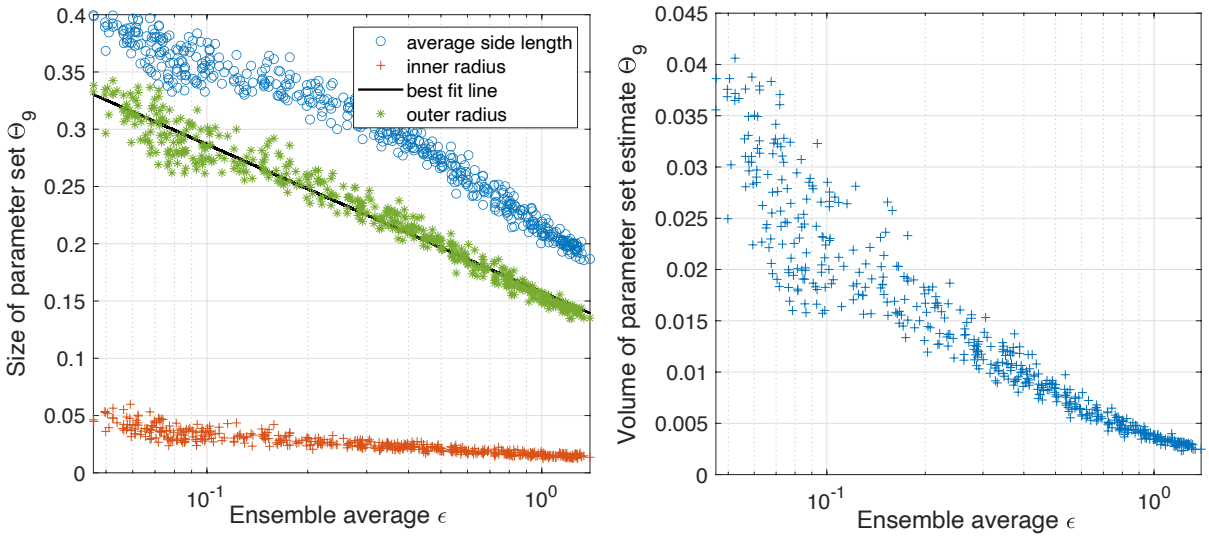


Figure 4.6: Average size of parameter set after 10 time steps against average value of ϵ in the PE condition (4.14) with $N_u = 10$. The parameter set size and ϵ were computed for 500 different initial conditions. Left: mean side length and inner and outer radii of Θ_9. Right: volume of Θ_9.

4.6 Conclusion

In this chapter, we use a nominal performance index and guarantee robust constraint satisfaction, recursive feasibility and input-to-state stability (ISS) for the proposed algorithm. The MPC objective includes a convexified persistent excitation condition via a weighting coefficient, and the relationship between this weight and the convergence rate of the estimated parameter set is investigated. In addition, the chapter proves that the parameter set under fixed-complexity update law converge to the actual system parameter vector with probability 1 despite the approximation. Conditions for convergence of the estimated parameter set are also derived for the case of inexact disturbance bounds and noisy measurements.

5

Closed-loop satisfaction of Persistent Excitation conditions

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5.1 Introduction

In this chapter, we derive closed-loop bounds on the expectation of Persistent Excitation (PE) coefficients. Starting from the simple case of a linear feedback law, we derive the bounds on expected PE coefficients and prove that the parameter set estimate converges to the true value with probability 1. We include an artificial random disturbance signal in the predicted control law to provide excitation whenever this is compatible with robust satisfaction of input and state constraints. The PE conditions in the proposed algorithm in Chapter 4 apply only to predicted trajectories, and therefore the algorithm does not provide any guarantees of convergence of estimated parameters in closed-loop operation. Building on Chapter 4, we propose adaptive MPC algorithms that combine set-based parameter identification, robust tube MPC, and recursively feasible PE condition.

Three algorithms are proposed in this chapter. The proposed algorithms achieve robust constraint satisfaction, input-to-state stability (ISS) and persistence of excitation in closed-loop operation. The first contains a nonconvex condition in the online implementation. The nonconvex condition can be checked using numerical methods for low dimensional systems, and is computationally intractable for high dimensional systems. To reduce the computational load, we propose a modified algorithm that achieves the same closed-loop properties under an additional assumption on its terminal control law.

The third alternative adaptive MPC algorithm avoids the nonconvex condition by imposing the conditions on the linearised PE coefficients directly. The algorithm is proven to be recursive feasible and input-to-state stable. We provide conditions under which persistence of excitation is ensured in closed-loop operation and show that these hold if the system satisfies certain reachability conditions and the PE window is sufficiently large.

5.2 PE condition under fixed linear feedback laws

In this section, we consider the case of linear feedback law, which provides useful insight into properties of predicted control laws, and the application to adaptive MPC will be discussed in the later sections. We first introduce a set of assumptions on the system model before discussing the modified persistence of excitation conditions and show that the conditions are satisfied under linear feedback law with mild assumptions.

This chapter considers a linear system subject to additive disturbances and parameter uncertainty.

$$x_{t+1} = A(\theta^*)x_t + B(\theta^*)u_t + Fw_t. \tag{5.1}$$

The system equation is slightly different from (3.1) in Chapter 3 as it contains an additional matrix F. The linear system is subject to the same linear constraint (3.2).

Assumption 5.1 (Additive disturbance)**.** *The disturbance sequence* $\{w_t,\, t \in \mathbb{N}_{\geq 0}\}$ *is independent and identically distributed (i.i.d.),* $\mathbb{E}(w_t) = 0$, $\mathbb{E}(w_t w_t^\top) \succeq \epsilon_w \mathbb{I} \succ 0$, *and* w_t *lies in a known, convex polytope* $\mathcal{W}$ *for all* t.

Remark 5.1. *Compared with Assumption 3.1, this assumption also assumes the additive disturbance is i.i.d with zero mean.*

Assumption 5.2 (Model parameters)**.** *(a). The pair* $\Big(A(\theta^*), B(\theta^*)\Big)$ *is reachable. (b).* $(A(\theta), B(\theta)) = (A(\theta^*), B(\theta^*))$ *if and only if* $\theta = \theta^*$.

The pair (A, B) is reachable if and only if $\text{rank}[A^{n_x-1}B\ A^{n_x-2}B\ \dots\ B] = n_x$ or equivalently $\text{rank}\Big(\sum_{i=0}^{n_x-1} A^i B B^\top (A^i)^\top\Big) = n_x$ [81]. If A is a full rank matrix, the reachability condition is equivalent to the controllability condition.

5.2.1 Set-based parameter estimate

The system model (5.1) can be rewritten as

$$x_{t+1} = \Phi(x_t, u_t)\theta^* + \phi(x_t, u_t) + Fw_t$$

where the definition of Φ_t and ϕ_t are in (3.7) and (3.8). Given x_t, x_{t-1}, u_{t-1} and the disturbance set $\mathcal{W}$, the unfalsified parameter set at time t is given by

$$\Delta_t = \{\theta : x_t - A(\theta)x_{t-1} - B(\theta)u_{t-1} \in F\mathcal{W}\}$$

We briefly recap the definition of persistent excitation (PE). A regressor Φ_t is persistently exciting if it satisfies (3.42), reproduced below

$$\sum_{t=k}^{k+N_u-1} \Phi_t^\top \Phi_t \succeq \epsilon \mathbb{I}.$$

for some PE window N_u and some scalar $\epsilon > 0$, and for all $k \in \mathbb{N}_{\geq 0}$. In this chapter however, we define persistent excitation using a condition on expectations:

$$\sum_{t=k}^{k+N_u-1} \mathbb{E}\Big\{\Phi_t^\top \Phi_t\Big\} \succeq \epsilon \mathbb{I}, \tag{5.2}$$

for some $\epsilon > 0$ and all $k \in \mathbb{N}_{\geq 0}$. We refer to the interval $\{k, \ldots, k + N_u - 1\}$ as a PE window. Compared with (3.42), PE condition on expectation (5.2) is a relaxed condition because the constraint is probabilistic not deterministic.

Corollary 5.2. *Under Assumptions 4.2 and 5.1, if the regressor in (3.7) satisfies the PE condition (5.2), then the minimal and fixed complexity parameter set estimates Θ_t in (2.31) and (4.1) converge to $\{\theta^*\}$ with probability 1.*

Proof. If (5.2) holds for all $k \in \mathbb{N}_{\geq 0}$, then with probability 1 there is an infinite sequence of times $\{\kappa_i,\, i \in \mathbb{N}_{\geq 0}\}$ such that

$$\sum_{t=\kappa_i}^{\kappa_i+N_u-1} \Phi_t^\top \Phi_t \succeq \epsilon \mathbb{I}. \tag{5.3}$$

Indeed, if this were not true, then κ would exist such that the probability of $\sum_{t=k}^{k+N_u-1} \Phi_t^\top \Phi_t \not\succeq \epsilon_\Phi \mathbb{I}$ for all $k \geq \kappa$ is non-zero, but this contradicts (5.2), which implies

$$\mathbb{P}\Big\{\sum_{t=k}^{k+N_u-1} \Phi_t^\top \Phi_t \succeq \epsilon_\Phi \mathbb{I}\Big\} > 0$$

for all $k \geq 0$. Therefore (5.3) holds with probability 1. The proof is completed by choosing κ_i so that $\kappa_{i+1} \geq \kappa_i + N_u - 1$ and hence $\Theta_{\kappa_i+N_u-1} \supseteq \Theta_{\kappa_{i+1}}$, and applying the arguments of Theorems 4.13 and 4.15 and Corollaries 4.14 and 4.16 in Chapter 4. □

Assumption 5.1 on the disturbance sequence $\{w_t, t \in \mathbb{N}_{\geq 0}\}$ is a common assumption. Assumption 4.2 may be more difficult to verify, but we note that this assumption can be relaxed at the expense of some residual uncertainty in the parameter set estimate (see Section 4.4.3 for details). In the next section we show that the PE condition (3.42) can be satisfied under a given linear feedback law.

5.2.2 Regressor under a fixed linear feedback law

Consider the system (5.1) under the action of a linear feedback law $u_t = Kx_t$. To simplify notation we define $A_K(\theta) = A(\theta) + B(\theta)K$ for all $\theta \in \Theta_0$ and $A_{K,i} = A_i + B_iK$ for all $i \in \mathbb{N}_{[0,p]}$.

Assumption 5.3. *The feedback gain K is chosen so that*
(a) $z_{k+1} \in \mathrm{Co}\{A_K(\theta)z_k, \theta \in \Theta_0\}$ is quadratically stable,
(b) the pair $\Big(A_K(\theta^*), F\Big)$ is reachable.

Assumption 5.3(a) is equivalent to feasibility of the following linear matrix inequalities (LMIs) in $P = P^\top \succ 0$:

$$P - A_K(\theta^{(j)})^\top P A_K(\theta^{(j)}) \succ 0,\ j = 1, \ldots, v.$$

See Section 6.3 in [22] for more detail.

Satisfaction of Assumption 5.3(b) is more difficult to check, since the information of θ^* is inaccessible and the reachability condition is therefore checked for the pair $\Big(A_K(\theta), F\Big)$ for all $\theta \in \Theta_0$. The reachability of $(A_K(\theta), F)$ for all $\theta \in \Theta_0$ is equivalent to a semi-infinite LMI feasibility problem that $X(\theta) = X(\theta)^\top \succ 0$ exists satisfying, for all $\theta \in \Theta_0$,

$$X(\theta) - A_K(\theta)X(\theta)A_K(\theta)^\top - FF^\top \succ 0.$$

Assumption 5.4. *$A_K(\theta) = A_K(\theta^*)$ if and only if $\theta = \theta^*$.*

Theorem 5.3. *If $N_u > n_x$, then under Assumptions 3.2, 5.1, 5.3 and 5.4, the regressor Φ_t defined in (3.7) of the system (5.1) with $u_t = Kx_t$ satisfies, for some $\epsilon_\Phi > 0$ and all $k \in \mathbb{N}_{\geq 0}$,*

$$\sum_{t=k}^{k+N_u-1} \mathbb{E}\Big(\Phi_t^\top \Phi_t\Big) \succeq \epsilon_\Phi \mathbb{I}. \tag{5.4}$$

Proof. Let $N_u > n_x$. Under the feedback law $u_t = Kx_t$, the system (5.1) becomes $x_{t+1} = A_K(\theta^*)x_t + Fw_t$, and the definition of the regressor Φ_t yields, for all $\theta \in \Theta_0$,

$$\sum_{t=k}^{k+N_u-1} \theta^\top \mathbb{E}\Big[\Phi(x_t, Kx_t)^\top \Phi(x_t, Kx_t)\Big]\theta = \operatorname{tr}\Big\{\Big(\sum_{i=1}^{p} A_{K,i}^\top [\theta]_i\Big)\Big(\sum_{j=1}^{p} A_{K,j}\,[\theta]_j\Big)\Big(\sum_{t=k}^{k+N_u-1} \mathbb{E}(x_t x_t^\top)\Big)\Big\}.$$

But w_t is i.i.d. and $\mathbb{E}(w_t w_t^\top) \succeq \epsilon_w \mathbb{I}$ (Assumption 5.1), so

$$\mathbb{E}(x_t x_t^\top) \succeq \epsilon_w \sum_{j=0}^{t-1} \Big[A_K(\theta^*)\Big]^j F F^\top \Big[A_K(\theta^*)^\top\Big]^j$$

for all $t > 0$. In addition, the pair $\Big(A_K(\theta^*), F\Big)$ is reachable (Assumption 5.3), so a positive scalar $\epsilon_{\mathcal{R}}$ exists such that $\sum_{j=0}^{n_x-1} \Big[A_K(\theta^*)\Big]^j F F^\top \Big[A_K(\theta^*)^\top\Big]^j \succeq \epsilon_{\mathcal{R}} \mathbb{I}$. It follows that $\mathbb{E}(x_t x_t^\top) \succeq \epsilon_w \epsilon_{\mathcal{R}} \mathbb{I}$ for all $t \geq n_x$ and hence

$$\sum_{t=k}^{k+N_u-1} \mathbb{E}(x_t x_t^\top) \succeq \epsilon_{\mathcal{F}} \epsilon_w \mathbb{I}$$

for all $k \geq 0$, where $\epsilon_{\mathcal{F}} = (N_u - n_x)\epsilon_{\mathcal{R}}$. Therefore

$$\begin{aligned}\sum_{t=k}^{k+N_u-1} \theta^\top \mathbb{E}\Big[\Phi(x_t, Kx_t)^\top \Phi(x_t, Kx_t)\Big]\theta &\geq \epsilon_w \epsilon_{\mathcal{F}} \operatorname{tr}\Big\{\sum_{i=1}^{p} A_{K,i}^\top [\theta]_i \sum_{j=1}^{p} A_{K,j}\,[\theta]_j\Big\} \\ &= \epsilon_w \epsilon_{\mathcal{F}} \Big\|\operatorname{vec}\Big(\sum_{i=1}^{p} A_{K,i}\,[\theta]_i\Big)\Big\|^2 \geq \epsilon_w \epsilon_{\mathcal{F}} \sigma^2 \|\theta\|^2\end{aligned}$$

where we have used the identities $\operatorname{tr}(A^\top A) = \|\operatorname{vec}(A)\|^2$ and $\operatorname{vec}\Big(\sum_{i=1}^{p} A_{K,i}\,[\theta]_i\Big) = \Big[\operatorname{vec}(A_{K,1}) \;\; \cdots \;\; \operatorname{vec}(A_{K,p})\Big]\theta$. Here σ is the minimum singular value of the matrix $[\operatorname{vec}(A_{K,1}) \;\; \cdots \;\; \operatorname{vec}(A_{K,p})]$, which is non-zero due to Assumption 5.4. It follows that (5.4) holds for all $k \geq 0$ with $\epsilon_\Phi = \epsilon_w \epsilon_{\mathcal{F}} \sigma^2 > 0$ for any $N_u > n_x$. □

From Corollary 5.2 and Theorem 5.3, the minimal and fixed-complexity parameter set estimates under linear feedback therefore converge to $\{\theta^*\}$ with probability 1 if Assumptions 4.2, 5.1-5.4 hold.

5.2.3 Regressor under a fixed linear feedback with injected noise

The main contribution of Theorem 5.3 is to link the convergence of parameter set estimates directly to properties of the disturbance sequence $\{w_0, w_1, w_2, \ldots\}$

(via Assumptions 4.2 and 5.1) and system matrices (A_i, B_i) and feedback gain K (via Assumptions 3.2, 5.2, 5.3 and 5.4). We next consider a feedback law with injected noise:

$$u_t = Kx_t + s_t \tag{5.5}$$

where $s_t \in \mathbb{R}^{n_u}$ is a random noise sequence that is known to the controller. The noise s_t provides additional excitation to encourage convergence of parameter estimates.

Assumption 5.5. *The sequence $\{s_t,\, t \in \mathbb{N}_{\geq 0}\}$ is i.i.d., s_t is independent of w_t with $\mathbb{E}(s_t) = 0$, $\mathbb{E}(s_t s_t^\top) \succeq \epsilon_s \mathbb{I} \succ 0$, and s_t lies in a known, convex polytope $\mathcal{S}$ for all t.*

Remark 5.4. *The condition $\mathbb{E}(s_t s_t^\top) \succeq \epsilon_s \mathbb{I} \succ 0$ can be replaced with a more generic covariance condition $\mathbb{E}(s_t s_t^\top) \succeq \mathbb{S} \succ 0$, where $\mathbb{S} \in \mathbb{R}^{n_u \times n_u}$ is the covariance matrix. Compared with covariance coefficient ϵ_s, the covariance matrix $\mathbb{S}$ provides design flexibility and improves exploration performance in desired directions. We can easily derive similar results in Chapter 5 to 7 using covariance matrix $\mathbb{S}$.*

Similarly to Theorem 5.3, we derive a lower bound on the expectation of PE condition (3.42) under this feedback law.

Theorem 5.5. *If $N_u > n_x$, then under Assumptions 3.2, 5.1, 5.2, 5.3(a) and 5.5, the regressor Φ_t in (3.7) of the system (5.1) with $u_t = Kx_t + s_t$ satisfies, for some $\epsilon_\Phi > 0$ and all $k \in \mathbb{N}_{\geq 0}$,*

$$\sum_{t=k}^{k+N_u-1} \mathbb{E}\Big(\Phi_t^\top \Phi_t\Big) \succeq \epsilon_\Phi \mathbb{I}. \tag{5.6}$$

Proof. Let $N_u > n_x$ be given. The control law (5.5) implies $x_{t+1} = A_K(\theta^*)x_t + B(\theta^*)s_t + Fw_t$, and since $\Big(A_K(\theta^*), B(\theta^*)\Big)$ is reachable (Assumption 5.2(a)) we have

$$\mathbb{E}[x_t x_t^\top] \succeq \epsilon_s \sum_{j=0}^{t-1} A_K(\theta^*)^j B(\theta^*) B(\theta^*)^\top {A_k(\theta^*)^j}^\top \succeq \epsilon_s \epsilon_{\mathcal{R}} \mathcal{I}$$

for all $t \geq n_x$, where $\epsilon_{\mathcal{R}} > 0$. Hence Assumption 5.5 implies

$$\sum_{t=k}^{k+N_u-1} \mathbb{E}\left\{ \begin{bmatrix} x_k \\ s_k \end{bmatrix} \begin{bmatrix} x_k^\top & s_k^\top \end{bmatrix} \right\} \succeq \sum_{t=0}^{N_u-1} \begin{bmatrix} \mathbb{E}[x_t x_t^\top] & 0 \\ 0 & \epsilon_s \mathbb{I} \end{bmatrix} \succeq \epsilon_s \epsilon_{\mathcal{F}} \mathbb{I}$$

where $\epsilon_{\mathcal{F}} = \min\{(N_u - n_x)\epsilon_{\mathcal{R}}, 1\}$. Furthermore, since

$$\Phi(x_t, u_t) = \Phi(x_t, Kx_t + s_t) = \sum_{i=1}^{p} \begin{bmatrix} A_{K,i} & B_i \end{bmatrix} [\theta]_i \begin{bmatrix} x_t \\ s_t \end{bmatrix},$$

an argument similar to the proof of Theorem 5.3 gives

$$\begin{aligned} &\sum_{t=t_0}^{t_0+N_u-1} \theta^\top \mathbb{E}\Big[\Phi(x_t, Kx_t + s_t)^\top \Phi(x_t, Kx_t + s_t)\Big]\theta \\ &\geq \epsilon_s \epsilon_{\mathcal{F}} \operatorname{tr}\Big\{\sum_{i=1}^{p} \begin{bmatrix} A_{K,i}^\top \\ B_i^\top \end{bmatrix} [\theta]_i \sum_{j=1}^{p} \begin{bmatrix} A_{K,j} & B_j \end{bmatrix} [\theta]_j \Big\} \geq \epsilon_s \epsilon_{\mathcal{F}} \sigma^2 \|\theta\|^2 \end{aligned}$$

where σ is the minimum singular value of the matrix with $[\operatorname{vec}(A_{K,i})^\top \ \operatorname{vec}(B_i)^\top]^\top$ as its ith column, for $i \in \mathbb{N}_{[1,p]}$, and $\sigma > 0$ by Assumption 5.2(b). Hence (5.6) holds with $\epsilon_\Phi = \epsilon_s \epsilon_{\mathcal{F}} \sigma^2 > 0$. □

Remark 5.6. *In comparison to Theorem 5.3, Theorem 5.5 can be applied to more systems because it does not require Assumptions 5.3(b) and 5.4. (Assumption 5.4 is more restrictive than Assumption 5.2.) At the same time, the additional Assumption 5.5 can be easily satisfied because s_t is user-defined.*

Applying Corollary 5.2 it can be seen that Assumptions 3.2, 4.2, 5.1, 5.2, 5.3(a) and 5.5 ensures that the minimal and fixed-complexity parameter set estimates under $u_t = Kx_t + s_t$ converge to $\{\theta^*\}$ with probability 1.

5.3 Adaptive robust MPC

To obtain a finite-dimensional MPC optimisation problem the predicted control sequence at time t is expressed in terms of decision variables $\mathbf{v}_t = \{v_{0|t}, \ldots, v_{N-1|t}\}$ as

$$u_{k|t} = \begin{cases} Kx_{k|t} + v_{k|t}, \ 0 \leq k \leq N-1 \\ Kx_{k|t} + s_{k|t}, \ k \geq N \end{cases} \tag{5.7}$$

where N is the prediction horizon. Given that the injected noise sequence $\{s_{0|t}, \ldots, s_{N-1|t}\}$ is known, $v_{k|t}$ absorbs the injected noise $s_{k|t}$ during the first N steps of the prediction horizon. For notational purposes, define $v_{k|t} = 0$ for all $k \in \mathbb{N}_{\geq N}$.

The gain K is designed offline and is assumed to be robustly stabilizing (Assumption 5.3(a)). We further assume that a polytopic terminal set $\mathcal{X}_T$ is determined offline and is consistent with the state and input constraints.

Assumption 5.6 (Terminal set)**.** *The terminal set $\mathcal{X}_T$ satisfies*

$$\begin{aligned}(G_x + G_u K)x \oplus \mathcal{S} &\leq \mathbf{1} \\ A_K(\theta)x \oplus B(\theta)\mathcal{S} \oplus F\mathcal{W} &\in \mathcal{X}_T\end{aligned} \tag{5.8}$$

for all $x \in \mathcal{X}_T$ and for all $\theta \in \Theta_t$.

5.3.1 Tube MPC formulation

To ensure robust satisfaction of the constraints (3.2) we construct a tube (a sequence of sets) $\{\mathcal{X}_{0|t}, \ldots, \mathcal{X}_{N|t}\}$ satisfying, for all $\theta \in \Theta_t$ and $k \in \mathbb{N}_{[0,N-1]}$,

$$\mathcal{X}_{k|t} \subseteq \mathbb{X}, \quad K\mathcal{X}_{k|t} \oplus \{v_{k|t}\} \subseteq \mathbb{U}, \tag{5.9}$$

$$A_K(\theta)\mathcal{X}_{k|t} \oplus \{B(\theta)v_{k|t}\} + F\mathcal{W} \subseteq \mathcal{X}_{k+1|t}. \tag{5.10}$$

The initial and terminal conditions are given by

$$\mathcal{X}_{0|t} = \{x_{0|t}\}, \tag{5.11}$$

$$\mathcal{X}_{N|t} \subseteq \mathcal{X}_T, \tag{5.12}$$

where $\mathcal{X}_T$ is a terminal set satisfying Assumption 5.6. Depending on the computational resources available, various tube representations, such as homothetic [34] and fixed-complexity polytopic tubes [60] may be employed. These tubes have computationally convenient vertex representations, denoted as

$$\mathcal{X}_{k|t} = \mathrm{Co}\{x_{k|t}^{(1)}, \ldots, x_{k|t}^{(m)}\}.$$

where m is the number of vertices of the tube cross-sections.

Here we consider a nominal cost defined for a given nominal parameter $\bar{\theta} \in \Theta_t$ in the following form, where the nominal parameter $\bar{\theta}$ is time-varying,

$$J(x, \mathbf{v}, \bar{\theta}) = \sum_{k=0}^{N-1} l(\bar{x}_k, v_k) + V_N(\bar{x}_N)$$

with $\bar{x}_0 = x$, $\bar{x}_{k+1} = A_K(\bar{\theta})\bar{x}_k + B(\bar{\theta})v_k$, $k \in \mathbb{N}_{[0,N-1]}$. Here $l(\cdot,\cdot)$ and $V_N(\cdot)$ are quadratic stage and terminal cost functions.

Assumption 5.7 (Terminal cost)**.** *The terminal and stage costs $V_N(\cdot)$ and $l(\cdot,\cdot)$ satisfy, for all $\theta \in \Theta_t$,*

$$V_N\Big(A_K(\theta)x, \theta\Big) - V_N(x,\theta) + l(x, Kx) \leq 0, \ \forall x \in \mathcal{X}_T.$$

5.3.2 The PE condition

Define the number of windows in the PE condition as N_{pe}. The number of PE windows initially increases with time, i.e.

$$N_{pe} = \begin{cases} t & \text{if } t < N + N_u - 1 \\ N + N_u - 1 & \text{if } t \geq N + N_u - 1 \end{cases} \tag{5.13}$$

The PE condition is thus, for all $\kappa \in \mathbb{N}_{[N-N_{pe},\ldots,N-1]}$,

$$\sum_{k=\kappa}^{\kappa+N_u-1} \Phi(x_{k|t}, u_{k|t})^\top \Phi(x_{k|t}, u_{k|t}) \succeq \beta_{\kappa|t}\mathbb{I}, \quad \forall x_{k|t} \in \mathcal{X}_{k|t}, u_{k|t} \in \{v_{k|t}\} \oplus K\mathcal{X}_{k|t}, \tag{5.14}$$

where κ denotes the start of the PE windows and $\beta_{\kappa|t}$ is the corresponding predicted PE coefficient. With slight abuse of notation, we can denote the past trajectories as $x_{k|t} = x_{k+t}$ and $u_{k|t} = u_{k+t}$ if $k \leq 0$.

As the regressor $\Phi(x_k, u_k)$ is linear in x_k and u_k and $u_k^{(j)} = Kx_k^{(j)} + v_k$, we can convexify the condition in (5.14) using the Taylor series expansion around the reference trajectory,

$$\begin{aligned} &\Phi(x_k^{(j)}, u_k^{(j)})^\top \Phi(x_k^{(j)}, u_k^{(j)}) \\ =&\Phi(\hat{x}_k, \hat{u}_k)^\top \Phi(\hat{x}_k, \hat{u}_k) + \Phi(x_k^{(j)} - \hat{x}_k, u_k^{(j)} - \hat{u}_k)^\top \Phi(\hat{x}_k, \hat{u}_k) \\ +&\Phi(\hat{x}_k, \hat{u}_k)^\top \Phi(x_k^{(j)} - \hat{x}_k, u_k^{(j)} - \hat{u}_k) + \Phi(x_k^{(j)} - \hat{x}_k, u_k^{(j)} - \hat{u}_k)^\top \Phi(x_k^{(j)} - \hat{x}_k, u_k^{(j)} - \hat{u}_k). \end{aligned} \tag{5.15}$$

where the reference state $\hat{\mathbf{x}} = \{\hat{x}_0, \ldots \hat{x}_N\}$ and input sequences $\hat{\mathbf{u}} = \{\hat{u}_0, \ldots \hat{u}_{N-1}\}$ are defined using the optimal solution $\mathbf{v}_{t-1}^* = \{v_{0|t-1}^*, \ldots, v_{N-1|t-1}^*\}$ at $t-1$ in the following manner,

$$\hat{x}_0 = x \tag{5.16a}$$

$$\hat{x}_{k+1} = A(\bar{\theta}_t)\hat{x}_k + B(\bar{\theta}_t)\hat{u}_k \tag{5.16b}$$

$$\hat{u}_k = K\hat{x}_k + \hat{v}_k \tag{5.16c}$$

$$\hat{v}_k = \begin{cases} v^*_{k+1|t-1}, & k = 0, \dots, N-2, \\ s_{k|t}, & k \geq N-1. \end{cases} \tag{5.16d}$$

PE condition (5.14), which is nonconvex in the predicted states and control inputs is linearised around the reference trajectory $(\hat{\mathbf{x}}, \hat{\mathbf{u}})$ using the same approach in Section 4.2.3. Denote the predicted PE coefficient vector after linearisation as $\boldsymbol{\beta'_t} = [\beta'_{N-N_{pe}|t}, \dots, \beta'_{N-1|t}]$. The linearised PE conditions are imposed for all $\kappa \in \mathbb{N}_{[N-N_{pe},N-1]}$.

For $\kappa \geq 0$, $k \in \mathbb{N}_{[\kappa,\kappa+N_u-1]}$, the linearised conditions are

$$\begin{aligned} \sum_{k=0}^{\kappa+N_u-1} & \Phi(\hat{x}_k, \hat{u}_k)^\top \Phi(\hat{x}_k, \hat{u}_k) + \Phi(x_k^{(j)} - \hat{x}_k, Kx_k^{(j)} + v_k - \hat{u}_k)^\top \Phi(\hat{x}_k, \hat{u}_k) \\ & + \Phi(\hat{x}_k, \hat{u}_k)^\top \Phi(x_k^{(j)} - \hat{x}_k, Kx_k^{(j)} + v_k - \hat{u}_k) \succeq \beta'_{\kappa|t}\mathbb{I}, \ \forall j \in \mathbb{N}_{[1,m]} \end{aligned} \tag{5.17a}$$

For $\kappa < 0$, $k \in \mathbb{N}_{[\kappa,\kappa+N_u-1]}$, the linearised conditions are

$$\begin{aligned} & \Phi(\hat{x}_k, \hat{u}_k)^\top \Phi(\hat{x}_k, \hat{u}_k) + \Phi(x_k^{(j)} - \hat{x}_k, Kx_k^{(j)} + v_k - \hat{u}_k)^\top \Phi(\hat{x}_k, \hat{u}_k) \\ & + \Phi(\hat{x}_k, \hat{u}_k)^\top \Phi(x_k^{(j)} - \hat{x}_k, Kx_k^{(j)} + v_k - \hat{u}_k) \succeq M_{k,\kappa}, \ \forall j \in \mathbb{N}_{[1,m]} \end{aligned} \tag{5.17b}$$

$$\hat{M}_\kappa + \sum_{k=0}^{\kappa+N_u-1} M_{k,\kappa} \succeq \beta'_{\kappa|t}\mathbb{I}, \tag{5.17c}$$

where $M_{k,\kappa} \in \mathbb{R}^{p\times p}$ are intermediate variables and $\hat{M}_\kappa = \sum_{k=\kappa}^{-1} \Phi(x_{k+t}, u_{k+t})^\top \Phi(x_{k+t}, u_{k+t})$.

We can define the reference predicted PE coefficient $\hat{\boldsymbol{\beta}}'_{\boldsymbol{t}} = [\hat{\beta}'_{N-N_{pe}|t}, \dots, \hat{\beta}'_{N-1|t}]$ using the reference control input $\hat{v}_k$ given by (5.16d). In other words, for all $\kappa \in \mathbb{N}_{[N-N_{pe},N-1]}$,

$$\hat{\beta}'_{\kappa|t} = \min_{x_k \in \mathcal{X}_k} \max_{\beta'_{\kappa|t}} \beta'_{\kappa|t} \ \text{ subject to (5.17) and } \mathbf{v_t} = \hat{\mathbf{v}}_\mathbf{t}, \tag{5.18}$$

and the constraint on $\beta'_{\kappa|t}$ is thus, for all $\kappa \in \mathbb{N}_{[N-N_{pe},N-1]}$,

$$\beta'_{\kappa|t} \geq \hat{\beta}'_{\kappa|t}. \tag{5.19}$$

Note that condition (5.19) approximates the nonconvex condition

$$\beta_{k|t} \geq \hat{\beta}_{k|t}, \qquad \forall k \in \mathbb{N}_{[N-N_{pe},N-1]} \tag{5.20}$$

where $\beta_{k|t}$ is given by (5.14). The two conditions (5.18) and (5.20) are equivalent when the last (quadratic) term in (5.15) can be neglected. The implementation of (5.20) is discussed in detail in Remark 5.7.

Algorithm 5.1 (Adaptive Robust MPC with recursive PE constraints)**.**

Offline Stage:

1. *Choose a suitable tube representation* $\mathcal{X}$.
2. *Compute the terminal set* $\mathcal{X}_T$ *and feedback gain* K *that satisfy Assumptions 5.3 and 5.6 .*
3. *Obtain a nominal parameter estimate* $\bar{\theta}_0 \in \Theta_0$.
4. *Choose suitable objective functions* $l(x,u)$ *and* $V(x)$ *that satisfy Assumption 5.7.*
5. *Choose the length of PE windows* N_u *where* $N_u \geq n_x$.
6. *Compute the control sequence* $\mathbf{v}_t^* = \{v_0^*, \ldots, v_{N-1}^*\}$*, the solution of the Quadratic Program (QP):*

$$\begin{aligned} \mathcal{P}_{initialization}: \ \min_{\mathbf{v}_t} \ & \sum_{k=0}^{N-1} l(x_k, v_k) + V_N(x_N) \\ \text{subject to } & (3.2), (5.10), (5.11), (5.12) \end{aligned} \tag{5.21}$$

Implement the current control input $u_t = Kx_t + v_0^*$.

Online Stage

(a) Obtain the current state x_t *and set* $x = x_t$*. Update the number of PE windows* N_{pe} *using (5.13).*

(b) Update Θ_t *and* $\bar{\theta}_t$ *using (4.1) and (4.13).*

(c) Compute the reference sequences $\hat{\mathbf{x}}$*,* $\hat{\mathbf{u}}$ *and* $\hat{\boldsymbol{\beta}}_{\boldsymbol{t}}'$*, using (5.16) and (5.18).*

(d) Given the reference PE coefficient $\hat{\boldsymbol{\beta}}_{\boldsymbol{t}}'$*, compute the control sequence* $\mathbf{v}_t^* = \{v_0^*, \ldots, v_{N-1}^*\}$ *and the predicted PE coefficients* $\boldsymbol{\beta}_{\boldsymbol{t}}' = [\beta'_{N-N_{pe}|t}, \ldots, \beta'_{N-1|t}]$ *as the solution of the semidefinite program (SDP):*

$$\begin{aligned} \mathcal{P}_{recursive}: \ \min_{\mathbf{v_t}, \boldsymbol{\beta}_{\boldsymbol{t}}'} \ & \sum_{k=0}^{N-1} l(x_k, v_k) + V_N(x_N) \\ \text{subject to } & (3.2), (5.10), (5.11), (5.12), (5.17), (5.19) \end{aligned} \tag{5.22}$$

(e) Check the condition (5.20)

- *If the condition is satisfied, implement the optimal control input* $u_t = Kx_t + v_0^*$ *from (5.22) .*
- *If not, implement the suboptimal solution* $u_t = Kx_t + v_{1|t-1}^*$ *from the previous time step.*

Remark 5.7. *Condition (5.20) in step (e) is generally difficult to verify because it is nonconvex. Since it is impossible to check the condition over all possible uncertainties, we can take random samples of additive disturbance* $w^{(n)}$ *from the disturbance set* $\mathcal{W}$ *and random samples of parametric uncertainty* $\theta^{(m)}$ *from the estimated parameter set* Θ_t *and check wether the condition is satisfied with the drawn samples. This allows us to guarantee the satisfaction of (5.20) with a certain confidence level [82]. In the result section, we show that (5.20) is not ensured by (5.19) because of the presence of the nonconvex term.*

Remark 5.8. *PE constraints are applied to* N_{pe} *number of windows at each time step, and each constraint (5.17) is a LMI. Therefore, as time increases, the computational load of online optimisation initially increases over time linearly and it flattens off at time* $t = N + N_u - 1$.

Remark 5.9. *In place of (4.1) and (4.13), other parameter and nominal parameter update laws can be used. The rest of the results will not be affected as long as* $\Theta_{t+1} \subseteq \Theta_t$ *and* $\bar{\theta} \in \Theta_t$.

5.3.3 Recursive feasibility and input-to-state stability

Proposition 5.10 (Recursive feasibility)**.** *The online MPC optimisation in Algorithm 5.1 is feasible at all times* $t \geq 1$ *if (5.22) is feasible at* $t = 1$ *and* $\Theta_t \subseteq \Theta_{t-1}$ *for all* t.

Proof. Let the suboptimal decision variables $\tilde{\mathbf{v}}_t$ be defined in terms of the optimal solution $\mathbf{v}_{t-1}^*$ of (5.22) at time $t-1$ as

$$\tilde{\mathbf{v}}_t = \{v_{1|t-1}^*, \ldots, v_{N-1|t-1}^*, 0\}$$

If the optimisation (5.22) is feasible at time $t-1$, then $(\tilde{\mathbf{v}}_t, \hat{\boldsymbol{\beta}}_t')$ is:

- feasible at time t for (3.2), (5.10) for $k \in \{0, 1, \ldots, N-2\}$ due to feasibility of (3.2), (5.10) at $t-1$ and $\Theta_t \subseteq \Theta_{t-1}$;
- f feasible at time t for (3.2), (5.10) for $k = N-1$ due to feasibility of (5.12) at $t-1$ and $\Theta_t \subseteq \Theta_{t-1}$;
- feasible at time t for (5.11) since $x_{0|t+1} \in \mathcal{X}_{1|t}$;
- feasible at time t for (5.12) since (5.12) is by Assumption 5.6 feasible at $t-1$;
- feasible at time t for (5.17), (5.19) for $k \in \{0, 1, \ldots, N-1\}$ because of the definitions (5.16), (5.18) of $\hat{\boldsymbol{\beta}}_t'$;

It follows that $(\tilde{\mathbf{v}}_t, \hat{\boldsymbol{\beta}}_t')$ is a feasible solution of (5.22) at time t. □

Proposition 5.11 (Stability)**.** *The closed-loop system (5.1) under Algorithm 5.1 is input-to-state stable (ISS).*

Proof. For a given realisation of the injected noise sequence $\{s_t, s_{t+1}, \ldots\}$, let $\zeta_{k|t}$ be a new input variable that combines the decision variable $v_{k|t}$ with the injected noise s_{k+t} so that, for all $t \in \mathbb{N}_{\geq 0}$, $v_{k|t} = \zeta_{k|t} + s_{k+t}$ and $\zeta^*_{k|t} = v^*_{k|t} - s_{k+t}$ for $k \in \mathbb{N}_{[0,N-1]}$, and let $\hat{\zeta}_{k|t+1} = \zeta^*_{k+1|t}$ for $k \in \mathbb{N}_{[0,N-2]}$ and $\hat{\zeta}_{N-1|t+1} = -s_{N+t}$.

Given recursive feasibility (Proposition 5.10), input-to-state stability (ISS) can be demonstrated using the proofs of Theorem 4.8 and Corollary 4.9, with all instances of $v_{k|t}$ and $\hat{v}_{k|t}$ replaced respectively by $\zeta_{k|t}$ and $\hat{\zeta}_{k|t}$, and with the disturbance input w_t in (3.1) replaced by $Fw_t + B(\theta^*)s_t$. This analysis shows that the bounds on the ISS-Lyapunov function in (4.22) and Remark 4.10 can be derived for Algorithm 5.1 despite the presence of injected noise and additional PE conditions in Algorithm 5.1.

□

5.3.4 Closed-loop satisfaction of the PE condition

Define ϵ_t to be the closed-loop PE coefficient at time t, where

$$\sum_{l=t}^{t+N_u-1} \Phi_l^\top \Phi_l \succeq \epsilon_t \mathbb{I} \tag{5.23}$$

where $\Phi_l = \Phi(x_l, u_l)$ and (x_l, u_l) are closed-loop system state and control input. The closed-loop PE coefficient ϵ_t is a random variable and depends on disturbance w and state x.

For closed-loop satisfaction of the PE condition, a more restrictive version of Assumption 5.2(a) is needed.

Assumption 5.8. *The pair* $\Big(A(\theta), B(\theta)\Big)$ *is reachable for all* $\theta \in \Theta_0$.

The reachability assumption can be verified by checking whether the following inequality holds for all $\theta \in \Theta_0$,

$$\mathcal{R}_{n_x} = \sum_{j=0}^{n_x-1} [A_K(\theta)]^j B(\theta) B(\theta)^\top [A_K(\theta)^\top]^j \succ 0$$

which is equivalent to checking a set of LMIs that depends linearly on θ, where $\mathcal{R}_{n_x}$ is given by

$$\begin{aligned}
\theta_0^* &= \operatorname*{argmin}_{\theta\in\Theta} \max \operatorname{tr}(\mathcal{R}) \;\; \text{s.t. } \mathcal{R} - B(\theta)B(\theta)^\top \succ 0, \\
\mathcal{R}_0 &= B(\theta_0^*)B(\theta_0^*)^\top, \\
\text{For } k &= 0, 1, \ldots, n_x - 1, \\
\theta_k^* &= \operatorname*{argmin}_{\theta\in\Theta} \max \operatorname{tr}(\mathcal{R}) \;\; \text{s.t. } \mathcal{R} - [A_K(\theta)]\mathcal{R}_k[A_K(\theta)^\top] - B(\theta)B(\theta)^\top \succ 0, \\
\mathcal{R}_{k+1} &= [A_K(\theta_k^*)]\mathcal{R}_k[A_K(\theta_k^*)^\top] + B(\theta_k^*)B(\theta_k^*)^\top.
\end{aligned}$$

The finite set of LMI conditions can be checked using the vertices of the parameter set Θ_0.

Lemma 5.12. *Given Assumptions 3.2, 5.2, 5.4, 5.5 and 5.8, the closed-loop PE coefficient* ϵ_t *in (5.23) under the Algorithm 5.1 satisfy, for* $t \geq N_{pe}$

$$\mathbb{E}(\epsilon_t) \succeq \epsilon_\Phi \mathbb{I}. \tag{5.24}$$

where $\epsilon_\Phi > 0$.

Proof. Given the optimal control sequence $\mathbf{v}_t^* = \{v_{1|t}^*, \dots, v_{N|t}^*, 0, 0, \cdots\}$ at time t, we can rewrite the predicted PE coefficient $\beta_{N|t}$ in (5.14), in the mode 2 control law region in the form of optimisation

$$\begin{aligned} \beta_{N|t} = & \min_{x_{k|t} \in \mathcal{X}_{k|t}} \max_{\beta} \beta \\ \text{s.t.} \quad & \sum_{k=N}^{N+N_u-1} \Phi(x_{k|t}, Kx_{k|t} + v_{k|t}^* + s_{k|t})^\top \Phi(x_{k|t}, Kx_{k|t} + v_{k|t}^* + s_{k|t}) \succeq \beta \mathbb{I}. \end{aligned} \tag{5.25}$$

Similarly, the definition of the reference PE coefficient $\hat{\beta}_{N-1|t+1}$ regarding the same window at time $t+1$ can be expressed as

$$\begin{aligned} \hat{\beta}_{N-1|t+1} = & \min_{x_{k|t+1} \in \mathcal{X}_{k|t+1}} \max_{\beta} \beta \\ \text{s.t. } & \Phi(x_{N-1|t+1}, Kx_{N-1|t+1} + \hat{v}_{N-1|t+1})^\top \Phi(x_{N-1|t+1}, Kx_{N-1|t+1} + \hat{v}_{N-1|t+1}) \\ & \sum_{k=N}^{N+N_u-2} \Phi(x_{k|t+1}, Kx_{k|t+1} + \hat{v}_{k|t+1} + s_{k|t+1})^\top \Phi(x_{k|t+1}, Kx_{k|t+1} + \hat{v}_{k|t+1} + s_{k|t+1}) \succeq \beta \mathbb{I}. \end{aligned} \tag{5.26}$$

By substituting the values of reference input $\hat{v}_{k|t+1}$ in (5.16d) and defining $z_k = x_{k-1|t+1}$ and $\mathcal{Z}_k = \mathcal{X}_{k-1|t+1}$ respectively, (5.26) can be rewritten as

$$\begin{aligned} \hat{\beta}_{N-1|t+1} = & \min_{z_k \in \mathcal{Z}_k} \max_{\beta} \beta \\ \text{s.t.} \quad & \sum_{k=N}^{N+N_u-1} \Phi(z_k, Kz_k + v_{k|t}^* + s_{k-1|t+1})^\top \Phi(z_k, Kz_k + v_{k|t}^* + s_{k-1|t+1}) \succeq \beta \mathbb{I}, \end{aligned} \tag{5.27}$$

Comparing (5.25) and (5.27), the difference lies in the constraint sets and the injected noise.

Next, we consider the relationship between constraint sets $\mathcal{X}_{k|t}$ and $\mathcal{Z}_k$. Given that (5.11), we know $\mathcal{X}_{0|t+1} = \{x_{0|t+1}\}$. Also, $x_{0|t+1} = x_{t+1} \in \mathcal{X}_{1|t}$ by definition in (5.10). Therefore, we have

$$\mathcal{X}_{0|t+1} \subseteq \mathcal{X}_{1|t}.$$

In addition, set-based parameter estimation methods guarantee that $\Theta_{t+1} \subseteq \Theta_t$ and we can infer that

$$\mathcal{X}_{k|t+1} \subseteq \mathcal{X}_{k+1|t}, \quad \forall k \in \mathbb{N}_{[N-1,\dots,N+N_u-2]}$$

Substituting $\mathcal{X}_{k-1|t+1}$ with $\mathcal{Z}_k$, we have

$$\mathcal{Z}_k \subseteq \mathcal{X}_{k|t}, \quad \forall k \in \mathbb{N}_{[N,\dots,N+N_u-1]}. \tag{5.28}$$

If we define the optimal solution to (5.27) as $\mathbf{z}' = \{z'_N, \dots, z'_{N+N_u-1}\}$. Given (5.28), $\mathbf{z}'$ is also a feasible solution to (5.25) and

$$\begin{aligned}
&\mathbb{E}\Big[\hat{\beta}_{N-1|t+1}\Big] \\
&= \underline{\lambda}\left(\sum_{k=N}^{N+N_u-1} \mathbb{E}\Big[\Phi(z'_k, Kz'_k + v^*_{k|t}+s_{k-1|t+1})^\top \Phi(z'_k, Kz'_k + v^*_{k|t}+s_{k-1|t+1})\Big]\right) \\
&= \underline{\lambda}\left(\min_{z_{k|t}\in\mathcal{Z}_{k|t}} \mathbb{E}\Big[\sum_{k=N}^{N+N_u-1} \Phi(z_k, Kz_k + v^*_{k|t}+s_{k|t})^\top \Phi(z_k, Kz_k + v^*_{k|t}+s_{k|t})\Big]\right) \\
&\succeq \underline{\lambda}\left(\min_{x_{k|t}\in\mathcal{X}_{k|t}} \mathbb{E}\Big[\sum_{k=N}^{N+N_u-1} \Phi(x_{k|t}, Kx_{k|t} + v^*_{k|t}+s_{k|t})^\top \Phi(x_{k|t}, Kx_{k|t} + v^*_{k|t}+s_{k|t})\Big]\right) \\
&\succeq \mathbb{E}\Big[\beta_{N|t}\Big].
\end{aligned} \tag{5.29}$$

because $\mathbf{s_{t+1}}$ and $\mathbf{s_t}$ are i.i.d sequences with zero mean and the regressor $\Phi(x,u)$ is affine in (x,u). The notation $\underline{\lambda}(A)$ denotes the minimum eigenvalue of a real symmetric matrix A.

In addition, step (e) in Algorithm 5.1 guarantees

$$\beta_{N-1|t+1} \geq \hat{\beta}_{N-1|t+1}. \tag{5.30}$$

Collecting (5.29) and (5.30) results in

$$\mathbb{E}\Big[\beta_{N-1|t+1}\Big] \geq \mathbb{E}\Big[\hat{\beta}_{N-1|t+1}\Big] \geq \mathbb{E}\Big[\beta_{N|t}\Big].$$

If we apply the same argument recursively, we can show that

$$\mathbb{E}\Big[\beta_{N-N_{pe}|t+N_{pe}}\Big] \geq \mathbb{E}\Big[\hat{\beta}_{N-N_{pe}|t+N_{pe}}\Big] \geq \cdots \geq \mathbb{E}\Big[\beta_{N-1|t+1}\Big] \geq \mathbb{E}\Big[\hat{\beta}_{N-1|t+1}\Big] \geq \mathbb{E}\Big[\beta_{N|t}\Big]. \tag{5.31}$$

where $\beta_{N-N_{pe}|t+N_{pe}}$ only consists one optimisation variable $v_{0|t+N_{pe}}$.

When the first element of the optimized input sequence $\mathbf{v}_{t+N_{pe}}$ is implemented, i.e. $v_{0|t+N_{pe}} = v^*_{0|t+N_{pe}}$, the closed-loop PE coefficient $\epsilon_{t+N_{pe}}$ will be equal to the predicted PE coefficient $\beta_{N-N_{pe}|t+N_{pe}}$, namely

$$\epsilon_{t+N_{pe}} = \beta_{N-N_{pe}|t+N_{pe}}.$$

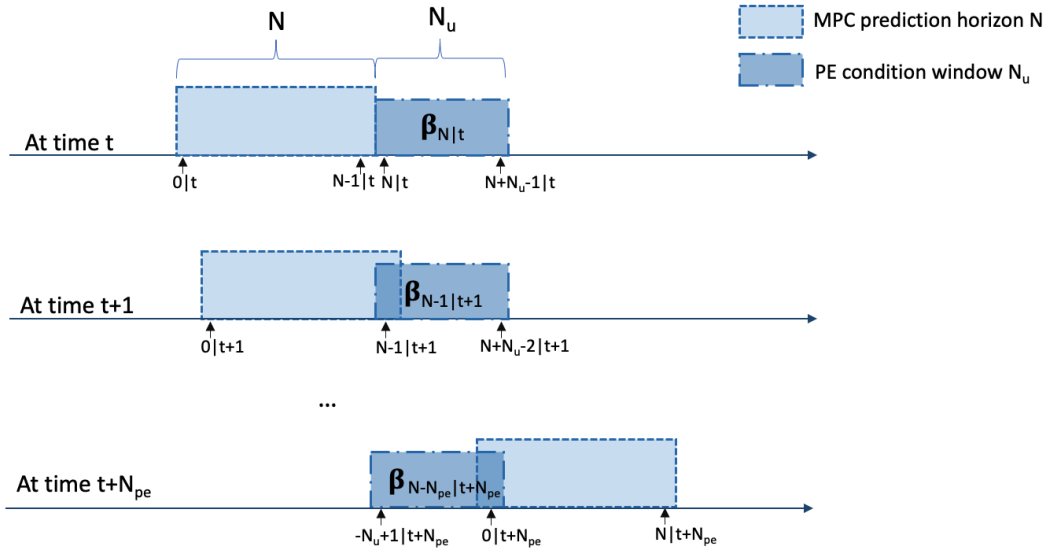


Figure 5.1: The progression of predicted PE windows compared with MPC prediction horizon over time

Combing with (5.31), we can relate the closed-loop PE coefficient $\epsilon_{t+N_{pe}}$ with the predicted PE coefficient $\beta_{N|t}$,

$$\mathbb{E}\Big[\epsilon_{t+N_{pe}}\Big] \geq \mathbb{E}\Big[\beta_{N|t}\Big].$$

Next, we consider the lower bound of the predicted regressor product at time steps on the prediction horizon that are more than N steps ahead. The proof is analogous to that of Theorem 5.3 and 5.5. The main difference is the regressor $\Phi(x,u)$ is for all x_k in the set $\mathcal{X}_k$, not a particular trajectory.

Under a linear control law and Assumption 5.5, we have

$$\mathbb{E}\Big[x_{k|t}x_{k|t}^\top\Big] \geq \epsilon_s \sum_{j=0}^{k-N+1} A_K(\theta)^j B(\theta)B(\theta)^\top A_K(\theta)^{j^\top}.$$

Given Assumption 5.8, $(A_K(\theta), B(\theta))$ is reachable for all $\theta \in \Theta$, therefore a positive scalar $\epsilon_{\mathcal{B}}$ exists such that $\sum_{j=0}^{n_x-1}[A_K(\theta)]^j B(\theta)B(\theta)^\top[A_K(\theta)^\top]^j \succeq \epsilon_{\mathcal{B}}$. It follows that

$$\sum_{k=N}^{N+N_u-1} \mathbb{E}\Big[x_{k|t}x_{k|t}^\top\Big] \geq (N_u - n_x)\epsilon_{\mathcal{B}}\mathbb{E}(s_k s_k^\top) \tag{5.32}$$

For all $x_{k|t} \in \mathcal{X}_k$, we want to derive the lower bound of the regressor product

$$
\begin{aligned}
&\hat{\theta}^\top \sum_{k=N}^{N+N_u-1} \mathbb{E}\Big\{\Phi(x_{k|t}, Kx_{k|t}+s_{k|t})^\top \Phi(x_{k|t}, Kx_{k|t}+s_{k|t})\Big\}\hat{\theta} \\
&= \mathbb{E}\left[\sum_{k=N}^{N+N_u-1} \Big(A_K(\hat{\theta})x_{k|t}\Big)^\top \Big(A_K(\hat{\theta})x_{k|t}\Big) + 2\Big(A_K(\hat{\theta})x_{k|t}\Big)^\top B(\hat{\theta})s_k + s_k^\top B(\hat{\theta})^\top B(\hat{\theta})s_k\right] \\
&\overset{\text{(E1)}}{=} \sum_{k=N}^{N+N_u-1} \mathbb{E}\left[\Big(A_K(\hat{\theta})x_{k|t}\Big)^\top \Big(A_K(\hat{\theta})x_{k|t}\Big) + s_k^\top B(\hat{\theta})^\top B(\hat{\theta})s_k\right] \\
&\overset{\text{(I2)}}{\geq} \mathrm{tr}\left\{\Big(A_K(\hat{\theta})\Big)^\top \Big(A_K(\hat{\theta})\Big)(N_u-n_x)\epsilon_{\mathcal{B}}\mathbb{E}\Big[s_k s_k^\top\Big] + B(\hat{\theta})^\top B(\hat{\theta}) \sum_{k=N}^{N+N_u-1} \mathbb{E}\Big[s_k s_k^\top\Big]\right\} \\
&\overset{\text{(I3)}}{\geq} \mathrm{tr}\left\{\Big(\sum_i^p \begin{bmatrix}\epsilon'_B A_{K,i}^\top \\ \sqrt{N_u}B_i^\top\end{bmatrix}[\hat{\theta}]_i\Big)\Big(\sum_i^p \begin{bmatrix}\epsilon'_B A_{K,i} & \sqrt{N_u}B_i\end{bmatrix}[\hat{\theta}]_i\Big)\Big(\mathbb{E}\Big[s_k s_k^\top\Big]\Big)\right\} \\
&\overset{\text{(I4)}}{\geq} \|\mathrm{vec}\Big(\sum_i^p \begin{bmatrix}\epsilon'_B A_{K,i} & \sqrt{N_u}B_i\end{bmatrix}[\hat{\theta}]_i\Big)\|^2 \epsilon_s \\
&\overset{\text{(I5)}}{\geq} \sigma_{ABK}^2 \epsilon_s \|\hat{\theta}\|^2
\end{aligned}
$$

Equality (E1) holds because $x_{k|t}$ and $s_{k|t}$ are independent and $\mathbb{E}[s_k] = 0$. Inequality (I2) follows from substitution of (5.32). Inequality (I3) defines $\epsilon'_B = \sqrt{(N_u - n_x)\epsilon_B}$ and applies affine dependency of parameter matrices on parameter vector $\hat{\theta}$ in Assumption 3.2. Equality (I4) uses Assumption 5.5. In (I5), σ_{ABK} denotes the minimum singular value of $\begin{bmatrix}\mathrm{vec}(\epsilon'_B A_{K,1}) & \mathrm{vec}(\sqrt{N_u}B_1) & \cdots \mathrm{vec}(\epsilon'_B A_{K,p}) & \mathrm{vec}(\sqrt{N_u}B_p)\end{bmatrix}$ and by Assumptions 5.2 and 5.4, $\sigma_{ABK} > 0$.

In short, we have

$$
\mathbb{E}(\beta_{N|t}) = \underline{\lambda}\left(\sum_{k=N}^{N+N_u-1} \mathbb{E}\Big\{\Phi(x_{k|t}, Kx_{k|t}+s_{k|t})^\top \Phi(x_{k|t}, Kx_{k|t}+s_{k|t})\Big\}\right) \geq \epsilon_\Phi
$$

where $\epsilon_\Phi = \sigma_{ABK}^2 \epsilon_s > 0$.

Combining with 5.31, we can conclude that

$$
\mathbb{E}(\epsilon_{t+N_{pe}}) \geq \mathbb{E}(\beta_{N|t}) \geq \epsilon_\Phi.
$$

□

Corollary 5.13. *Under Assumptions 3.2, 4.2, 5.2, 5.4, 5.5 and 5.8, the parameter estimate under Algorithm 5.1 converges to $\{\theta^*\}$ with probability 1.*

Remark 5.14. *For closed-loop satisfaction of PE condition, Algorithm 5.1 is not reliant on an external disturbance for providing the excitation signal that allows model parameters to be estimated.*

5.3.5 Extension: adaptive robust MPC via terminal region parameter exploration

This section presents a modified algorithm with a lower computational load by decreasing the the number of PE conditions in the constraint. The modified algorithm contain two phases, performance phase and exploration phase. In the performance phase, when the state is outside the terminal region (i.e. $x \notin \mathcal{X}_T$), a standard robust MPC (without PE condition) is performed, and the optimisation is quadratic program (QP). Once the state enters the terminal region $\mathcal{X}_T$ (i.e. $x \in \mathcal{X}_T$), PE constraints are enforced to facilitate parameter identification and the resulting algorithm is semi-definite program (SDPs). The algorithm achieve the same closed-loop properties as that of Algorithm 5.1.

A new variable $N_m \in \mathbb{N}_{[0,N]}$ is introduced to account for a buffer window length included in PE conditions. The reference input (5.16d) is defined as,

$$\hat{v}_k = \begin{cases} v^*_{k+1|t-1} & k = 0, \dots, N_m - 1 \\ s_{k|t} & k \geq N_m - 1 \end{cases} \tag{5.33}$$

And therefore, the total number of windows in the PE condition N_{pe} is

$$N_{pe} = \begin{cases} t & \text{if } t < N_m + N_u - 1 \\ N_m + N_u - 1 & \text{if } t \geq N_m + N_u - 1 \end{cases} \tag{5.34}$$

Algorithm 5.2 (Adaptive MPC with parameter exploration in terminal region)**.**

Offline Stage:

1. *Choose a suitable tube representation $\mathcal{X}$.*
2. *Compute the terminal set $\mathcal{X}_T$ and feedback gain K that satisfy Assumptions 5.3 and 5.6.*
3. *Obtain a nominal parameter estimate $\bar{\theta}_0 \in \Theta_0$.*

4. *Choose suitable objective functions $l(x,u)$ and $V(x)$ that satisfy Assumption 5.7.*

5. *Choose the length of PE window N_u and length of buffer window N_m.*

Online Stage

A. *While x is outside the terminal set $\mathcal{X}_T$ (i.e. $x \notin \mathcal{X}_T$)*

   (a) *Obtain the current state x_t and set $x = x_t$.*

   (b) *Update Θ_t and $\bar{\theta}_t$ using (4.1) and (4.13).*

   (c) *Compute the control sequence $\mathbf{v}_t^* = \{v_0^*, \dots, v_{N-1}^*\}$, the solution of the Quadratic Program (QP),*

$$\begin{aligned} \mathcal{P}_{x\notin\mathcal{X}_T}: \ & \min_{\mathbf{v}} \sum_{k=0}^{N-1} l(x_k, v_k) + V_N(x_N) \\ & \text{subject to } (3.2), (3.13), (5.11), (5.12) \end{aligned} \tag{5.35}$$

   (d) *Implement the current control input $u_t = Kx_t + v_{0|t}^*$.*

B. *If x enters the terminal set $\mathcal{X}_T$ (i.e. $x \in \mathcal{X}_T$)*

   (a) *Obtain the current state x_t and set $x = x_t$. Update the number of PE windows N_{pe} using (5.34).*

   (b) *Update Θ_t and $\bar{\theta}_t$ using (4.1) and (4.13).*

   (c) *Compute the reference sequences $\hat{\mathbf{x}}$, $\hat{\mathbf{u}}$ and $\hat{\boldsymbol{\beta}}_t'$, using (5.16a-5.16c), (5.33) and (5.18).*

   (d) *Given the reference PE coefficient $\hat{\boldsymbol{\beta}}_t'$, compute the control sequence $\mathbf{v}_t^* = \{v_0^*, \dots, v_{N-1}^*\}$ and the predicted PE coefficients $\hat{\boldsymbol{\beta}}_t'$ as the solution of the Semidefinite Program (SDP):*

$$\begin{aligned} \mathcal{P}_{x\in\mathcal{X}_T}: \ & \min_{\mathbf{v},\boldsymbol{\beta}_t'} \sum_{k=0}^{N-1} l(x_k, v_k) + V_N(x_N) \\ & \text{subject to } (3.2), (3.13), (5.11), (5.12), (5.17), (5.19) \end{aligned} \tag{5.36}$$

   (e) *Implement the current control input $u_t = Kx_t + v_0^*$.*

The system (5.1) reaches the terminal region $\mathcal{X}_T$ in finite time under control law $u_t = Kx_t + v^*_{0|t}$ where $v^*_{0|t}$ is the first element of the optimal solution $\mathbf{v}_t$ to (5.35) because the closed-loop system is input-to-state stable.

Let t_T denote the time instance when x enters the terminal region $\mathcal{X}_T$. We can prove recursive feasibility, input-to-state stability and satisfaction of PE condition in the similar manner to Sections 5.3.3 and 5.3.4.

Proposition 5.15 (Recursive feasibility)**.** *The MPC optimisation in Algorithm 5.2 is feasible at all times if feasible at $t = 1$ and $\Theta_t \subseteq \Theta_{t-1}$ for all t.*

Proposition 5.16. *The closed-loop system (5.1) under Algorithm 5.2 is input-to-state stable (ISS).*

Lemma 5.17. *Given Assumptions 3.2, 5.2, 5.4, 5.5 and 5.8, the closed-loop PE coefficient ϵ_t in (5.23) under the Algorithm 5.2 satisfy, for all $t \geq t_T$,*

$$\mathbb{E}(\epsilon_t) \succeq \epsilon_\Phi \mathbb{I}. \tag{5.37}$$

where $\epsilon_\Phi > 0$.

5.4 Adaptive MPC with linearised PE coefficient comparison

This section considers an alternative method of reducing online computation, by removing the nonconvex condition check in Algorithm 5.1 and 5.2. We provide conditions under which constraints on the linearised PE coefficients similar to (5.19) are sufficient (without recourse to the nonconvex conditions in 5.20) to ensure that the closed-loop system satisfies the PE condition in (5.24).

In this algorithm, the reference trajectory for the same PE window remains the same over time. The superscript ν is the window index and $\nu \in \mathbb{N}_{[t-N_u+1,N+t]}$ for time t.

The reference state for a certain PE window is denoted as $\hat{\mathbf{x}}^\nu_t = \{\hat{x}^\nu_{\nu-t|t}, \ldots, \hat{x}^\nu_{\nu-t+N_u-1|t}\}$ and the input sequence is $\hat{\mathbf{u}}^\nu_t = \{\hat{u}^\nu_{\nu-t|t}, \ldots, \hat{u}^\nu_{\nu-t+N_u-1|t}\}$, which carry the values of the reference trajectory in the previous time step

$$\hat{x}^\nu_{\nu-t|t} = \hat{x}^\nu_{\nu-t+1|t-1} \tag{5.38a}$$

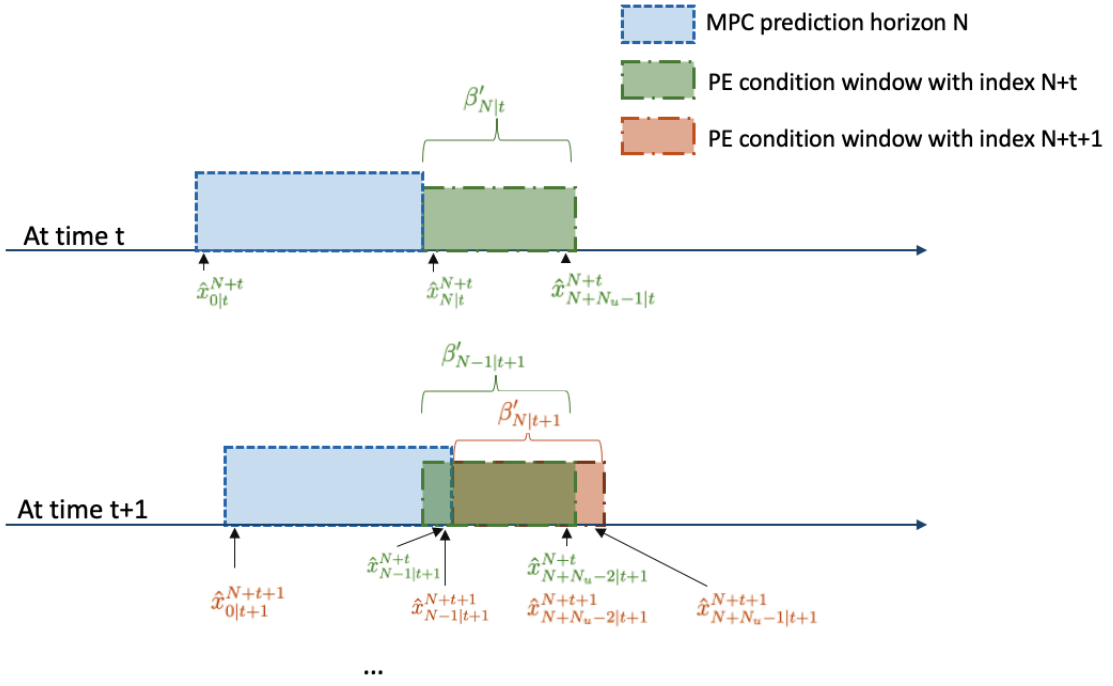


Figure 5.2: Illustration of reference trajectories and PE windows across the prediction horizon

$$\hat{u}^{\nu}_{\nu-t|t} = \hat{u}^{\nu}_{\nu-t+1|t-1} \tag{5.38b}$$

As shown in Figure 5.2, the green $\hat{x}$ indicates the reference state values for the green shaded PE window with index $N+t$. As time advances, the reference states for the same green shaded window remain unchanged. The red $\hat{x}$ indicates the reference state values for the red shaded PE window with index $N+t+1$. Note that reference state with the same subscript might be different (i.e $\hat{x}^{N+t}_{N-1|t+1}$ probably does not equal to $\hat{x}^{N+t+1}_{N-1|t+1}$) because they belongs to different PE windows.

The last window $(\hat{\mathbf{x}}^{N+t}_t,\ \hat{\mathbf{u}}^{N+t}_t)$, is defined by the state x_t, parameter estimate $\bar{\theta}_t$ and the optimal solution $\mathbf{v}^*_{t-1} = \{v^*_{0|t-1}, \ldots, v^*_{N-1|t-1}\}$ at $t-1$,

$$\hat{x}^{N+t}_{0|t} = x_t \tag{5.39a}$$

$$\hat{x}^{N+t}_{k+1|t} = A(\bar{\theta}_t)\hat{x}^{N+t}_{k|t} + B(\bar{\theta}_t)\hat{u}^{N+t}_{k|t} \tag{5.39b}$$

$$\hat{u}^{N+t}_{k|t} = K\hat{x}^{N+t}_{k|t} + \hat{v}_k \tag{5.39c}$$

$$\hat{v}_k = v^*_{k+1|t-1} \tag{5.39d}$$

The nonconvex PE condition (3.42) is linearised around the reference trajectory $\hat{\mathbf{x}}_t^\nu$ and $\hat{\mathbf{u}}_t^\nu$ in a similar manner as that in Section 4.2.3. For $\nu \in \mathbb{N}_{[t-N_u+1,t-1]}$, the linearised condition is

$$\begin{aligned}\sum_{k=\nu-t}^{-1} \Phi(x_{k+t},u_{k+t})^\top \Phi(x_{k+t},u_{k+t}) + \min_{x_k\in\mathcal{X}_k} \sum_{k=0}^{\nu-t+N_u-1} \Big\{ &\Phi(\hat{x}_k^\nu,\hat{u}_k^\nu)^\top \Phi(\hat{x}_k^\nu,\hat{u}_k^\nu) \\ &+ \Phi(x_k^{(j)}-\hat{x}_k^\nu, Kx_k^{(j)}+v_k-\hat{u}_k^\nu)^\top \Phi(\hat{x}_k^\nu,\hat{u}_k^\nu) \\ &+ \Phi(\hat{x}_k^\nu,\hat{u}_k^\nu)^\top \Phi(x_k^{(j)}-\hat{x}_k^\nu, Kx_k^{(j)}+v_k-\hat{u}_k^\nu) \Big\} \succeq \beta'_{\nu-t|t}\mathbb{I}\end{aligned} \tag{5.40}$$

for all $j = 1,\dots,m$ where $\mathcal{X}_{k|t} = \mathrm{co}\{x_k^j, j = 1,\dots,m\}$.

For $\nu \in \mathbb{N}_{[t,N+t-1]}$, the linearised condition is

$$\begin{aligned}\min_{x_k\in\mathcal{X}_k} \sum_{k=\nu-t}^{\nu-t+N_u-1} &\Phi(\hat{x}_k^\nu,\hat{u}_k^\nu)^\top \Phi(\hat{x}_k^\nu,\hat{u}_k^\nu) + \Phi(x_k^{(j)}-\hat{x}_k^\nu, Kx_k^{(j)}+v_k-\hat{u}_k^\nu)^\top \Phi(\hat{x}_k^\nu,\hat{u}_k^\nu) \\ &+ \Phi(\hat{x}_k^\nu,\hat{u}_k^\nu)^\top \Phi(x_k^{(j)}-\hat{x}_k^\nu, Kx_k^{(j)}+v_k-\hat{u}_k^\nu) \succeq \beta'_{\nu-t|t}\mathbb{I}\end{aligned} \tag{5.41}$$

for all $j = 1,\dots,m$ where $\mathcal{X}_{k|t} = \mathrm{co}\{x_k^j, j = 1,\dots,m\}$.

The recursive PE condition is imposed by direct comparison of β'

$$\beta'_{\nu|t} \geq \beta'_{\nu+1|t-1} \tag{5.42}$$

for $\nu \in [N-N_{pe}, N-1]$, where N_{pe} is given by (5.13).

For the last window, which uses a linear feedback law, the associated PE coefficient $\beta'_{N|t}$ can be calculated as

$$\begin{aligned}\beta'_{N|t} = \underline{\lambda}\Bigg(\min_{x_k\in\mathcal{X}_k} \sum_{k=N}^{N+N_u-1} &\Phi(\hat{x}_k^{N+t}, K\hat{x}_k^{N+t})^\top \Phi(\hat{x}_k^{N+t}, K\hat{x}_k^{N+t}) \\ &+ \Phi(x_k^{(j)}-\hat{x}_k^{N+t}, Kx_k^{(j)}-K\hat{x}_k^{N+t})^\top \Phi(\hat{x}_k^{N+t}, K\hat{x}_k^{N+t}) \\ &+ \Phi(\hat{x}_k^{N+t}, K\hat{x}_k^{N+t})^\top \Phi(x_k^{(j)}-\hat{x}_k^{N+t}, Kx_k^{(j)}-K\hat{x}_k^{N+t}) \Bigg)\end{aligned} \tag{5.43}$$

Algorithm 5.3 (Adaptive MPC with reduced computation via linearised PE coefficient)**.**

Offline Stage:

1. *Choose a suitable tube representation $\mathcal{X}$.*

2. *Compute the terminal set $\mathcal{X}_T$ and feedback gain K that satisfy Assumptions 5.3 and 5.6 .*

3. *Obtain a nominal parameter estimate $\bar{\theta}_0 \in \Theta_0$.*

4. *Choose suitable objective functions $l(x,u)$ and $V(x)$ that satisfy Assumption 5.7.*

5. *Choose the length of PE windows N_u where $N_u \geq n_x$.*

6. *Compute the control sequence $\mathbf{v}_t^* = \{v_0^*, \ldots, v_{N-1}^*\}$, the solution of the Quadratic Program (QP):*

$$\begin{aligned} \mathcal{P}_{initialization}: \ \min_{\mathbf{v}_t} \sum_{k=0}^{N-1} l(x_k, v_k) + V_N(x_N) \\ subject\ to\ (3.2), (5.10), (5.11), (5.12) \end{aligned} \tag{5.44}$$

Implement the current control input $u_t = Kx_t + v_0^*$.

Online Stage

(a) Obtain the current state x_t and set $x = x_t$. Update the number of PE windows N_{pe} using (5.13).

(b) Update Θ_t and $\bar{\theta}_t$ using (4.1) and (4.13).

(c) Compute the reference sequences $\hat{\mathbf{x}}$, $\hat{\mathbf{u}}$ and $\beta'_{N|t-1}$, using (5.38) , (5.39) and (5.43).

(d) Compute the control sequence $\mathbf{v}_t^* = \{v_0^*, \ldots, v_{N-1}^*\}$ and the predicted PE coefficients $\boldsymbol{\beta}'_t = [\beta'_{N-N_{pe}|t}, \ldots, \beta'_{N-1|t}]$ as the solution of the semidefinite program (SDP):

$$\begin{aligned} \mathcal{P}_{direct}: \ \min_{\mathbf{v_t}, \boldsymbol{\beta}'_t} \sum_{k=0}^{N-1} l(x_k, v_k) + V_N(x_N) \\ subject\ to\ (3.2), (5.10), (5.11), (5.12), (5.40), (5.41), (5.42) \end{aligned} \tag{5.45}$$

(e) Implement the optimal control input $u_t = Kx_t + v_{0|t}^*$.

5.4.1 Recursive feasibility and input-to-state stability

Proposition 5.18 (Recursive feasibility)**.** *The online MPC optimisation in Algorithm 5.3 is feasible at all times $t \geq 1$ if (5.45) is feasible at $t = 1$ and $\Theta_t \subseteq \Theta_{t-1}$ for all t.*

Proof. Let the suboptimal decision variables $\tilde{\mathbf{v}}_t$ and $\tilde{\boldsymbol{\beta}}'_t$ be defined in terms of the optimal solution $\mathbf{v}^*_{t-1}$ and $\boldsymbol{\beta}'^*_t$ of (5.45) at time $t-1$ as

$$\tilde{\mathbf{v}}_t = \{v^*_{1|t-1}, \ldots, v^*_{N-1|t-1}, 0\}, \quad \tilde{\boldsymbol{\beta}}'_t = [\beta'_{N-N_{pe}+1|t-1}, \ldots, \beta'_{N|t-1}]$$

where $\beta'_{N|t-1}$ is calculated in online step (c) using (5.43).

If the optimisation (5.45) is feasible at time $t-1$, then $(\tilde{\mathbf{v}}_t, \tilde{\boldsymbol{\beta}}'_t)$ is:

- feasible at time t for (3.2), (5.10) for $k \in \{0, 1, \ldots, N-2\}$ due to feasibility of (3.2), (5.10) at $t-1$ and $\Theta_t \subseteq \Theta_{t-1}$;
- feasible at time t for (3.2), (5.10) for $k = N-1$ due to feasibility of (5.12) at $t-1$ and $\Theta_t \subseteq \Theta_{t-1}$;
- feasible at time t for (5.11) since $x_{0|t+1} \in \mathcal{X}_{1|t}$;
- feasible at time t for (5.12) since (5.12) is by Assumption 5.6 feasible at $t-1$;
- feasible at time t for (5.40), (5.41) and (5.42) for $k \in \{0, 1, \ldots, N-2\}$ because $\hat{\boldsymbol{x}}^\nu_{t+1} = \hat{\boldsymbol{x}}^\nu_t$, $\hat{\boldsymbol{u}}^\nu_{t+1} = \hat{\boldsymbol{u}}^\nu_t$ and $\mathcal{X}_{k+1|t} \subseteq \mathcal{X}_{k|t+1}$ when the tail solution is used;
- feasible at time t for (5.41) and (5.42) for $k = N-1$ because of (5.43).

It follows that $(\tilde{\mathbf{v}}_t, \tilde{\boldsymbol{\beta}}'_t)$ is a feasible solution of (5.45) at time t. □

Proposition 5.19 (Stability)**.** *The closed-loop system (5.1) under Algorithm 5.3 is input-to-state stable (ISS).*

The proof is the analogous to that in Proposition 5.11.

5.4.2 Closed-loop satisfaction of the PE conditions

Theorem 5.20. *Given Assumptions 3.2, 5.2, 5.3 and 5.5 the closed-loop PE coefficient ϵ_t in (5.23) under the Algorithm 5.3 satisfies, for $t \geq N + N_u$,*

$$\mathbb{E}(\epsilon_t) \succ \sum_{k=0}^{N_u-1} r_k + q_k + \min_{\hat{\theta}\in\mathbb{B}_1} \hat{\theta}^\top P_k(\bar{\theta}_t, \Theta_t)\hat{\theta}. \tag{5.46}$$

where $P_k(\bar{\theta}_t, \Theta_t)$ is given by

$$\left.\begin{aligned}
&S_0(\theta) = B(\theta)\epsilon_s B(\bar{\theta}_t)^\top,\\
&T_0 = B(\bar{\theta}_t)\epsilon_s B(\bar{\theta}_t)^\top,\\
&\theta_k^* = \arg\min_{\theta\in\Theta_t} \max_{\kappa} \quad \kappa \ \text{ s.t. } \ S_k(\theta) + S_k(\theta)^\top - T_k \succeq \kappa I,\\
&P_{k+1}(\bar{\theta}_t, \Theta) = S_k(\theta_k^*) + S_k(\theta_k^*)^\top - T_k,\\
&S_{k+1}(\theta) = A_K(\theta) S_k(\theta_k^*) A_K(\bar{\theta}_t)^\top + B(\theta)\epsilon_s B(\bar{\theta}_t)^\top,\\
&T_{k+1} = A_K(\bar{\theta}_t) T_k A_K(\bar{\theta}_t)^\top + B(\bar{\theta}_t)\epsilon_s B(\bar{\theta}_t)^\top,
\end{aligned}\right\} \quad k = 0, 1, \dots$$

for all $\theta \in \Theta_t$, $\bar{\theta}_t \in \Theta_t$, and r_k is given by

$$r_k = \min_{\hat{\theta}\in\mathbb{B}_1} \min_{z\in\mathcal{Z}_{N+k|t}} \hat{\theta}^\top \Big\{ z\, \hat{x}_{N|t}^\top \Big(A_K(\bar{\theta}_t)\Big)^{k^\top} + \Big(A_K(\bar{\theta}_t)\Big)^k \hat{x}_{N|t}\, z^\top - \Big(A_K(\bar{\theta}_t)\Big)^k \hat{x}_{N|t}\hat{x}_{N|t}^\top \Big(A_K(\bar{\theta}_t)\Big)^{k^\top} \Big\} \hat{\theta},$$

where $z_{N+k|t} \in \mathcal{Z}_{N+k|t}$ and $\mathcal{Z}_{N+k|t} \supseteq A_K(\theta_{N+k-1}) \cdots A_K(\theta_N)\mathcal{X}_{N|t}$, for all $\theta_{N+i} \in \Theta_t$, $i = 0, \dots, k-1$ and q_k is given by

$$q_k = \min_{\hat{\theta}\in\mathbb{B}_1} \min_{\substack{\bar{w}_i\in\bar{\mathcal{W}}_i \\ (\theta_N,\dots,\theta_{N+k-1})\in(\Theta_t\times\dots\times\Theta_t)}} \hat{\theta}^\top \Big\{ \Big(\sum_{i=0}^{k-1} \bar{w}_i\Big) \hat{x}_{N|t}^\top \Big(A_K(\bar{\theta}_t)\Big)^{k^\top} + \Big(A_K(\bar{\theta}_t)\Big)^k \hat{x}_{N|t} \Big(\sum_{i=0}^{k-1} \bar{w}_i\Big)^\top \Big\} \hat{\theta},$$

where $\bar{W}_i \supseteq A_K(\theta_{N+k-1}) \cdots A_K(\theta_{N+k-i}) F\mathcal{W}$ and $\bar{W}_0 = F\mathcal{W}$, for all $\theta_{N+i} \in \Theta_t$, $i = 0, \dots, k-1$.

Proof. First, we remind the reader of the definition of the linearised PE coefficient $\beta'_{N|t}$ in the mode 2 window

$$\beta'_{N|t} = \min_{\hat{\theta}\in\mathbb{B}^1} \min_{x_{k|t}\in\mathcal{X}_{k|t}} \sum_{k=N}^{N+N_u-1} \hat{\theta}^\top \Big[\hat{\Phi}_{k|t}^\top \hat{\Phi}_{k|t} + \hat{\Phi}_{k|t}^\top \Phi(x_{k|t}-\hat{x}_{k|t}, u_{k|t}-\hat{u}_{k|t}) + \Phi(x_{k|t}-\hat{x}_{k|t}, u_{k|t}-\hat{u}_{k|t})^\top \hat{\Phi}_{k|t}\Big]\hat{\theta}$$

where $\hat{\Phi}_{k|t} = \Phi(\hat{x}_{k|t}, \hat{u}_{k|t})$. The form of the regressor product allows us to write

$$\sum_k \mathbb{E}\Big(\hat{\theta}^\top \Phi(x_{k|t}, u_{k|t}^\top \Phi(x_{k|t}, u_{k|t})\hat{\theta}\Big) = \sum_k \operatorname{tr}\left\{Q(\hat{\theta})\mathbb{E}\Big(\begin{bmatrix}x_{k|t}\\ s_{k|t}\end{bmatrix}\begin{bmatrix}x_{k|t}\\ s_{k|t}\end{bmatrix}^\top\Big)\right\}$$

where $Q(\hat{\theta}) = \Big(\sum_i \begin{bmatrix}A_{K,i} & B_i\end{bmatrix}[\hat{\theta}]_i\Big)^\top \Big(\sum_i \begin{bmatrix}A_{K,i} & B_i\end{bmatrix}[\hat{\theta}]_i\Big)$. Similar to Theorem 5.5, $Q(\hat{\theta}) \succeq 0$ given Assumption 5.2(b) for all $\hat{\theta} \neq 0$. Therefore, the expectation of the linearised PE coefficient $\beta'_{N|t}$ can be expressed as

$$\begin{aligned}\mathbb{E}[\beta'_{N|t}] &\geq \min_{\hat{\theta}\in\mathbb{B}_1} \min_{x_{k|t}\in\mathcal{X}_{k|t}} \sum_{k=N}^{N+N_u-1} \operatorname{tr}\left\{Q(\hat{\theta})\mathbb{E}\Big(\begin{bmatrix}\hat{x}_{k|t}\\ s_{k|t}\end{bmatrix}\begin{bmatrix}\hat{x}_{k|t}\\ s_{k|t}\end{bmatrix}^\top + \begin{bmatrix}\hat{x}_{k|t}\\ s_{k|t}\end{bmatrix}\begin{bmatrix}x_{k|t}-\hat{x}_{k|t}\\ 0\end{bmatrix}^\top \right.\\ &\qquad\left. + \begin{bmatrix}x_{k|t}-\hat{x}_{k|t}\\ 0\end{bmatrix}\begin{bmatrix}\hat{x}_{k|t}\\ s_{k|t}\end{bmatrix}^\top\Big)\right\}\\ &= \min_{\hat{\theta}\in\mathbb{B}_1} \min_{x_{k|t}\in\mathcal{X}_{k|t}} \sum_{k=N}^{N+N_u-1} \operatorname{tr}\left\{Q(\hat{\theta})\mathbb{E}\Big(\begin{bmatrix}x_{k|t}\hat{x}_{k|t}^\top + \hat{x}_{k|t}x_{k|t}^\top - \hat{x}_{k|t}\hat{x}_{k|t}^\top & 0\\ 0 & s_{k|t}s_{k|t}^\top\end{bmatrix}\Big)\right\}.\end{aligned}$$

Next, we want to demonstrate that $\sum_{k=N}^{N+N_u-1}\mathbb{E}\Big(x_{k|t}\hat{x}_{k|t}^\top + \hat{x}_{k|t}x_{k|t}^\top - \hat{x}_{k|t}\hat{x}_{k|t}^\top\Big) \succ 0$ for a given $\bar{\theta}_t \in \Theta_t$ and all $x_{k|t} \in \mathcal{X}_{k|t}$, $k = N, \ldots, N+N_u-1$.

Recall the definition of the reference state ($\hat{x}_{k|t}$) and state tube cross-section ($\mathcal{X}_{k|t}$)

$$\begin{aligned}\hat{x}_{k+1|t} &= A_K(\bar{\theta}_t)\hat{x}_{k|t} + B(\bar{\theta}_t)s_{k|t},\\ \mathcal{X}_{k+1|t} &\supseteq A_K(\theta)\mathcal{X}_{k|t} \oplus B(\theta)\{s_{k|t}\} \oplus F\mathcal{W},\ \ \forall\theta\in\Theta_t,\end{aligned}$$

for all $k \geq N$, where $s_{k|t} = s_{k+t}$. Therefore, for all $k \geq 1$ we have

$$\hat{x}_{N+k|t} = \Big(A_K(\bar{\theta}_t)\Big)^k \hat{x}_{N|t} + \Big(A_K(\bar{\theta}_t)\Big)^{k-1} B(\bar{\theta}_t)s_{N|t} + \cdots + B(\bar{\theta}_t)s_{N+k-1|t}$$

and, for all $\theta_{N+i} \in \Theta_t$, $i = 0, \ldots, k-1$,

$$\begin{aligned}\mathcal{X}_{N+k|t} \supseteq\ & A_K(\theta_{N+k-1})\cdots A_K(\theta_N)\mathcal{X}_{N|t} \oplus \Big(A_K(\theta_{N+k-1})\cdots A_K(\theta_{N+1})B(\theta_N)\Big)\{s_{N|t}\}\\ & \oplus \Big(A_K(\theta_{N+k-1})\cdots A_K(\theta_{N+2})B(\theta_{N+1})\Big)\{s_{N+1|t}\} \oplus \cdots \oplus B(\theta_{N+k-1})\{s_{N+k-1|t}\}\end{aligned}$$

$$\oplus\Big(A_K(\theta_{N+k-1})\cdots A_K(\theta_{N+1})\Big)F\mathcal{W}\oplus\Big(A_K(\theta_{N+k-1})\cdots A_K(\theta_{N+2})\Big)F\mathcal{W}\oplus\cdots\oplus F\mathcal{W}.$$

Recall that Assumption 5.6 in Algorithm 5.3 step 2 guarantees robust constraint satisfaction when the control input $\mathbf{v}_t$ is computed without the knowledge of the realisation of $s_{k|t}$ for $k \geq N$. In addition, Assumption 5.5 states that $\{s_{N|t}, s_{N+1|t}, \dots\}$ is a random i.i.d sequence that is independent of $x_{N|t}$. In addition, since the sequence has zero mean (i.e. $\mathbb{E}[s_t] = 0$), we have $\mathbb{E}[s_i s_j^\top] = 0$ if $i \neq j$ and $\mathbb{E}[x_{N|t} s_{k+N}^\top] = \mathbb{E}[x_{N|t}]\mathbb{E}[s_{k+N}^\top] = 0$. As a result, for all $k \geq 1$,

$$\begin{aligned}
&\min_{\hat{\theta}\in\mathbb{B}_1}\min_{x_{N+k|t}\in\mathcal{X}_{N+k|t}} \hat{\theta}^\top\mathbb{E}\Big(x_{N+k|t}\hat{x}_{N+k|t}^\top + \hat{x}_{N+k|t}x_{N+k|t}^\top - \hat{x}_{N+k|t}\hat{x}_{N+k|t}^\top\Big)\hat{\theta}\\
&\geq \min_{\hat{\theta}\in\mathbb{B}_1}\min_{\substack{z\in\mathcal{Z}_{N+k|t}\ \bar{w}_i\in\bar{\mathcal{W}}_i\\(\theta_N,\dots,\theta_{N+k-1})\in\Theta_t^k}} \hat{\theta}^\top\Bigg\{ z\,\hat{x}_{N|t}^\top\Big(A_K(\bar{\theta}_t)\Big)^{k^\top} + \Big(A_K(\bar{\theta}_t)\Big)^k\hat{x}_{N|t}\,z^\top\\
&\quad - \Big(A_K(\bar{\theta}_t)\Big)^k\hat{x}_{N|t}\hat{x}_{N|t}^\top\Big(A_K(\bar{\theta}_t)\Big)^{k^\top} + B(\theta_{N+k-1})\epsilon_s B(\bar{\theta}_t)^\top + B(\bar{\theta}_t)\epsilon_s B(\theta_{N+k-1})^\top\\
&\quad + \sum_{i=1}^{k-1} A_K(\theta_{N+k-1})\cdots A_K(\theta_{N+k-i})B(\theta_{N+k-1-i})\epsilon_s B(\bar{\theta}_t)^\top\Big(A_K(\bar{\theta}_t)\Big)^{i^\top}\\
&\quad + \sum_{i=1}^{k-1}\Big(A_K(\bar{\theta}_t)\Big)^i B(\bar{\theta}_t)\epsilon_s B(\theta_{N+k-1-i})^\top\Big(A_K(\theta_{N+k-1})\cdots A_K(\theta_{N+k-i})\Big)^\top\\
&\quad - \sum_{i=0}^{k-1}\Big(A_K(\bar{\theta}_t)\Big)^i B(\bar{\theta}_t)\epsilon_s B(\bar{\theta}_t)^\top\Big(A(\bar{\theta}_t)\Big)^{i^\top} \qquad (5.47)\\
&\quad + \Big(\sum_{i=0}^{k-1}\bar{w}_i\Big)\hat{x}_{N|t}^\top\Big(A_K(\bar{\theta}_t)\Big)^{k^\top} + \Big(A_K(\bar{\theta}_t)\Big)^k\hat{x}_{N|t}\Big(\sum_{i=0}^{k-1}\bar{w}_i\Big)^\top\Bigg\}\hat{\theta} \qquad (5.48)
\end{aligned}$$

for some set $\mathcal{Z}_{N+k|t}$ such that $\mathcal{Z}_{N+k|t} \supseteq A_K(\theta_{N+k-1})\cdots A_K(\theta_N)\mathcal{X}_{N|t}$ and $\bar{W}_i \supseteq A_K(\theta_{N+k-1})\cdots A_K(\theta_{N+k-i})F\mathcal{W}$ and $\bar{W}_0 = F\mathcal{W}$, for all $\theta_{N+i} \in \Theta_t$, $i = 0, \dots, k-1$. Given Assumption 5.3, $\mathcal{Z}_{N+k|t}$ can be chosen so that $\rho(\mathcal{Z}_{N+k|t}) \leq \lambda^k\rho(\mathcal{X}_{N|t})$ for some $\lambda \in [0, 1)$. Therefore, despite not necessarily being positive, the value of

$$\begin{aligned}
r_k = \min_{\hat{\theta}\in\mathbb{B}_1}\min_{\substack{z\in\mathcal{Z}_{N+k|t}\\(\theta_N,\dots,\theta_{N+k-1})\in(\Theta_t\times\dots\times\Theta_t)}} \hat{\theta}^\top\Bigg\{ & z\,\hat{x}_{N|t}^\top\Big(A_K(\bar{\theta}_t)\Big)^{k^\top} + \Big(A_K(\bar{\theta}_t)\Big)^k\hat{x}_{N|t}\,z^\top\\
& - \Big(A_K(\bar{\theta}_t)\Big)^k\hat{x}_{N|t}\hat{x}_{N|t}^\top\Big(A_K(\bar{\theta}_t)\Big)^{k^\top}\Bigg\}\hat{\theta}
\end{aligned}$$

can be made arbitrarily small by choosing k sufficiently large. Likewise,

$$q_k = \min_{\hat{\theta}\in\mathbb{B}_1} \min_{\substack{\bar{w}_i\in\bar{\mathcal{W}}_i \\ (\theta_N,\ldots,\theta_{N+k-1})\in(\Theta_t\times\ldots\times\Theta_t)}} \hat{\theta}^\top \Bigg\{ \Big(\sum_{i=0}^{k-1}\bar{w}_i\Big)\hat{x}_{N|t}^\top \Big(A_K(\bar{\theta}_t)\Big)^{k^\top} + \Big(A_K(\bar{\theta}_t)\Big)^k \hat{x}_{N|t}\Big(\sum_{i=0}^{k-1}\bar{w}_i\Big)^\top \Bigg\}\hat{\theta},$$

can be approximated given $\mathcal{W}$ and Θ_t.

The remaining terms on the RHS of the inequality (5.48) are lower-bounded by the minimum eigenvalue of $P_k(\bar{\theta}_t, \Theta_t)$, which is the solution of:

$$\left.\begin{aligned} &S_0(\theta) = B(\theta)\epsilon_s B(\bar{\theta}_t)^\top \\ &T_0 = B(\bar{\theta}_t)\epsilon_s B(\bar{\theta}_t)^\top \\ &\theta_k^* = \arg\min_{\theta\in\Theta_t}\max_{\kappa} \quad \kappa \ \text{ s.t. } \ S_k(\theta) + S_k(\theta)^\top - T_k \succeq \kappa I \\ &P_{k+1}(\bar{\theta}_t,\Theta) = S_k(\theta_k^*) + S_k(\theta_k^*)^\top - T_k \\ &S_{k+1}(\theta) = A_K(\theta)S_k(\theta_k^*)A_K(\bar{\theta}_t)^\top + B(\theta)\epsilon_s B(\bar{\theta}_t)^\top \\ &T_{k+1} = A_K(\bar{\theta}_t)T_k A_K(\bar{\theta}_t)^\top + B(\bar{\theta}_t)\epsilon_s B(\bar{\theta}_t)^\top \end{aligned}\right\} \quad k = 0, 1, \ldots$$

Therefore, the RHS of (5.48) is bounded by $r_k + \min_{\hat{\theta}\in\mathbb{B}_1}\hat{\theta}^\top P_k(\bar{\theta}_t,\Theta_t)\hat{\theta}$ and

$$\mathbb{E}[\beta'_{N|t}] \geq \sum_{k=N}^{N+N_u-1} r_k + q_k + \min_{\hat{\theta}\in\mathbb{B}_1}\hat{\theta}^\top P_k(\bar{\theta}_t,\Theta_t)\hat{\theta} \tag{5.49}$$

In addition, (5.42) guarantees

$$\beta'_{-N_u+1|t+N+N_u-1} \geq \beta'_{-N_u+2|t+N+N_u-2} \geq \cdots \geq \beta'_{N-1|t+1} \geq \beta'_{N|t} \tag{5.50}$$

By definition, $\epsilon_{t+N+N_u-1} \geq \beta'_{-N_u+1|t+N+N_u-1}$ because $\epsilon_{-N_u+1|t+N+N_u-1}$ contains an extra quadratic term that is positive semidefinite. Collecting (5.49) and (5.50), we can derive that

$$\mathbb{E}[\epsilon_{t+N+N_u-1}] \geq \mathbb{E}[\beta'_{N|t}] \geq \sum_{k=0}^{N_u-1} r_k + q_k + \min_{\hat{\theta}\in\mathbb{B}_1}\hat{\theta}^\top P_k(\bar{\theta}_t,\Theta_t)\hat{\theta}$$

□

Remark 5.21. *The condition $P_k(\bar{\theta}_t,\Theta_t) \succ 0$, for given $\bar{\theta}_t$ and Θ_t, can be formulated in terms of LMI conditions and checked using semidefinite programming (SDP). Since the nominal parameter vector $\bar{\theta}_t$ is time-varying, we need to check the positivity*

of the RHS of (5.46) for all possible choices of $\bar{\theta}_t \in \Theta_t$. Clearly this may not be possible in practice because it requires checking feasibility of an infinite number of LMIs. Therefore Theorem 5.20 is intended as a tool to guide the choice of PE window length and nominal parameter vector.

Remark 5.22. *The condition $r_k + \min_{\hat{\theta}\in\mathbb{B}_1} \hat{\theta}^\top P_k(\bar{\theta}_t, \Theta_t)\hat{\theta} \succeq 0$ will be satisfied when Θ_t is sufficiently small. If we denote $\Theta_t \subset \rho\mathbb{B}_1 \oplus \{\bar{\theta}_t\}$, for sufficiently small ρ, then $P_k(\bar{\theta}_t, \Theta_t) \succ 0$ and $r_k \succ 0$ since $S_k(\theta_k^*) + S_k(\theta_k^*)^\top \to 2T_k$ and $z\,\hat{x}_{N|t}^\top \Big(A_K(\bar{\theta}_t)\Big)^{k^\top} + \Big(A_K(\bar{\theta}_t)\Big)^k \hat{x}_{N|t}\, z^\top \to \Big(A_K(\bar{\theta}_t)\Big)^k \hat{x}_{N|t}\hat{x}_{N|t}^\top \Big(A_K(\bar{\theta}_t)\Big)^{k^\top}$ as $\rho \to 0$.*

Remark 5.23. *If N_u is sufficiently large and $P_k(\bar{\theta}_t, \Theta_t) \succ 0$ for all $k \geq \check{k}$, for some $\check{k}$, then for any $\bar{\theta}_t \in \Theta_t$ and all $x_{N+i|t} \in \mathcal{X}_{N+i|t}$, $i = 0, \ldots, N_u - 1$ we obtain*

$$\mathbb{E}[\epsilon_{t+N+N_u-1}] \succeq \sum_{k=N}^{N+N_u-1} q_k$$

which can be computed to provide a suitable bound for β'.

This term is similar to $\sum_{i=0}^{\infty} A_K^i \mathcal{W}$ in the rigid tube formulation.

5.5 Numerical examples

In this section, we adopt the example in [13] to illustrate how the proposed adaptive MPC algorithms can produce persistently exciting input signals while satisfying the system input and state constraints, and providing performance close to the optimal performance for MPC without PE constraints.

The plant is a first-order system given by (5.1), and its model parameters are given by

$$A_0 = 0.5, \quad A_1 = 1, \quad A_2 = 0, \quad B_0 = 0.5, \quad B_1 = 0, \quad B_2 = 1.$$

Its true system parameter θ^* is given by $\theta^* = \begin{bmatrix}\theta_1 & \theta_2\end{bmatrix}^\top = \begin{bmatrix}0.5 & 0\end{bmatrix}^\top$ and the parameter vector lies in the initial set estimate $\Theta_0 = \{|\theta_1| \leq 0.5, |\theta_2| \leq 0.25\}$. The bounded disturbance sequence $\{w_0, w_1, \ldots\}$ follows atruncated normal distribution with zero mean and covariances $\sigma_w = 0.06$ and $w_t \in \mathcal{W} = \{w : |w| \leq 0.008\}$. The bounded injected noise sequence $\{s_0, s_1, \ldots\}$ follows a uniform distribution with zero mean and $s_t \in \mathcal{S} = \{s : |s| \leq 0.005\}$. The initial condition for the state is $x_0 = 0.001$.

For comparison, we first define a robust MPC with online parameter set update.

Algorithm 5.4 (Robust MPC)**.**

Offline Stage:

1. *Choose a suitable tube representation* $\mathcal{X}$*.*
2. *Compute the terminal set* $\mathcal{X}_T$ *and feedback gain* K *that satisfy Assumptions 5.3 and 5.6 .*
3. *Obtain a nominal parameter estimate* $\bar{\theta}_0 \in \Theta_0$*.*
4. *Choose suitable stage cost* $l(x, u)$ *and terminal cost* $V_N(x)$ *that satisfy Assumption 5.7.*

Online Stage

(a) Obtain the current state x_t *and set* $x = x_t$*.*

(b) Update Θ_t *and* $\bar{\theta}_t$ *using (4.1) and (4.13).*

(c) compute the control sequence $\mathbf{v}_t^* = \{v_0^*, \ldots, v_{N-1}^*\}$*, the solution of Quadratic Program (QP):*

$$\begin{aligned} \mathcal{P}_{QP}: \ \min_{\mathbf{v}_t} \sum_{k=0}^{N-1} l(x_k, v_k) + V_N(x_N) \\ \text{subject to } (3.2), (5.10), (5.11), (5.12) \end{aligned} \tag{5.51}$$

(d) Implement the optimal control input $u_t = Kx_t + v_0^*$ *.*

Algorithm 5.4 is a robust MPC algorithm with known random noise s_t and unknown additive disturbance w_t. Compared with Algorithm 5.1, 5.2 and 5.3, Algorithm 5.4 does not contain any PE constraints or associated variables. Note that injected noise s_t is known for the prediction horizon and remains a variable in the terminal condition.

5.5.1 Linear feedback laws with/without injected noise

In the first simulation, a fixed linear feedback law is adopted, and no constraints are considered. In Figure 5.3, the estimated parameter set size under the linear feedback law (blue) remains almost constant after 360 time steps while that under linear feedback law with injected noise (red) shrinks continuously. This is because the

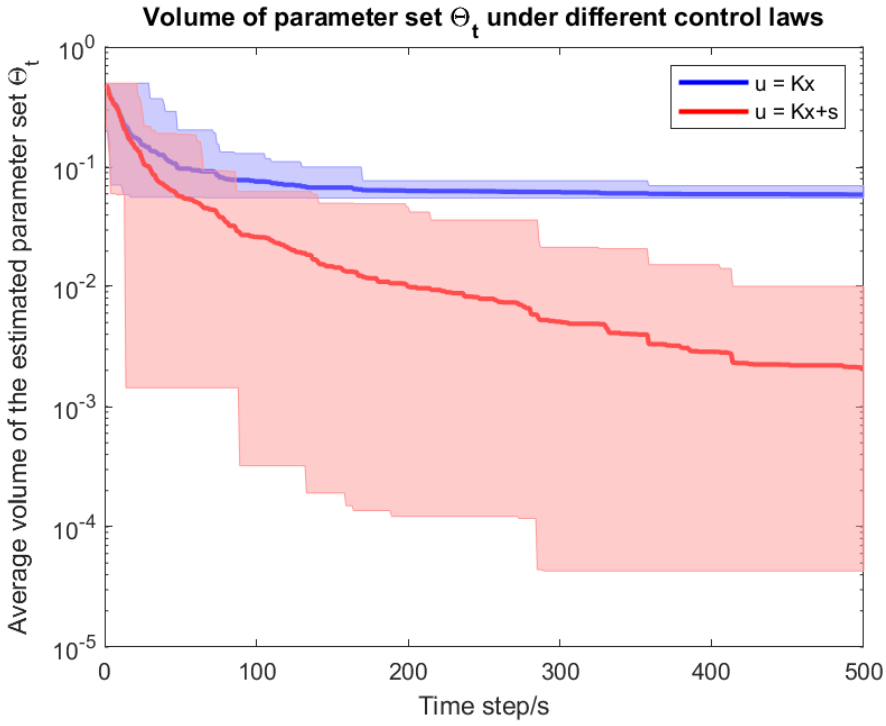


Figure 5.3: The volume of estimated parameter sets in logarithmic scale using linear feedback law $u = Kx$ (blue) and linear feedback law with injected noise $u = Kx + s$ (red). (Solid line: ensemble average. Shaded area: region for 30 disturbance sequences.)

injected noise provides extra excitation for the system exploration. In this example, for the linear feedback law, Assumption 5.4 is not satisfied (i.e. $\epsilon_\Phi = 0$ in Theorem 5.3), and therefore closed-loop satisfaction of the PE condition is not guaranteed. On the other hand, for a linear feedback with injected noise, all assumptions in Theorem 5.5 are satisfied and $\epsilon_\Phi = 7.41 \times 10^{-7}$. As a result, the parameter set estimates converge to the true value with probability 1.

5.5.2 Adaptive robust MPC with recursively feasible PE constraints

This section compares the performance of Algorithm 5.1, Algorithm 5.4 and linear feedback law with injected noise.

The state constraint and input constraint are $x_t \geq -0.1$ and $|u_t| \leq 0.5$. The MPC prediction horizon and PE window length are set to be $N = 10$ and $N_u = 3$

respectively. Given the system matrices and injected noise characteristics, we obtain $\sigma^2_{ABK} = 0.1209$ and $\epsilon_\Phi = 8.47 \times 10^{-7}$ in Lemma 5.12. The objective cost and polytopic tube description used are the same as those in Chapter 4.

All simulations were performed in Matlab on a 2.9 GHz Intel Core i7-10700F processor, and the online MPC optimisation was formulated using Yalmip [71]. Quadratic Programs (QPs) are solved using Gurobi [72] and Semi-definite Programs (SDPs) are solved using Mosek [80]. For the purpose of comparison, the identical disturbance sequences and the same parameter update law are used. Under Algorithm 5.1, input and state constraints are robustly satisfied in all simulations.

Table 5.1: Comparison of Algorithm 5.4 , Algorithm 5.1 (with/without the nonconvex condition check) and the linear feedback law (with the injected noise s_t)

Algorithm Name	Algorithm 5.4	Algorithm 5.1 with (5.20)	Algorithm 5.1 without (5.20)	Linear feedback with s_t (5.5)
$\Theta_{10}/\%$	50.68	49.76	50.58	51.55
$\Theta_{100}/\%$	11.54	7.61	11.29	5.54
$\Theta_{500}/\%$	8.45	1.14	7.66	0.34
$\mathbb{E}(\epsilon_t)$	1.238×10^{-11}	3.909×10^{-6}	1.500×10^{-7}	8.315×10^{-6}
Step(c) Solver time/s	-	0.0133	0.0133	-
Step(d) Solver time/s	0.0039	0.0044	0.0042	-

Table 5.1 shows the estimated parameter set sizes and the closed-loop PE coefficient ϵ_t (defined in (3.42)) averaged over 500 time steps and 30 ensemble disturbance sequences using Algorithm 5.1 (with and without the nonconvex check in step (e)), linear feedback law (5.5) and Algorithm 5.4. As shown, both cases of Algorithm 5.1 outperform Algorithm 5.4 in terms of the convergence rates of parameter sets. The size of the closed-loop PE coefficient ϵ_t also demonstrates that Algorithm 5.1, which includes the nonconvex condition check, satisfies the bound given in Lemma 5.12. In contrast, that without the nonconvex condition check and the Algorithm 5.4 do not satisfy the bound. For this simulation, the nonconvex condition is violated 60% of time-steps on average. As shown in the table, the linear feedback

law has the fastest rate of parameter set convergence, but a much higher closed-loop cost (shown in Figure 5.6). In addition, the linear feedback law ignores the system input and state constraints.

Table 5.1 also compares the computational time for different algorithms. Regarding step (d) (the optimisation step), the solver time for Algorithm 5.1 (with/without the nonconvex condition check) are higher than those for Algorithm 5.4, because in the optimisation step, Algorithm 5.1 solves a semi-definite program (SDP) with additional PE constraints. In addition, Algorithm 5.1 consumes significant time in order to compute the reference $\hat{\boldsymbol{\beta}}'_t$ (in Step (c)). As shown in (5.18), $\hat{\beta}'_{\kappa|t}$ is computed for each $\kappa \in [N-N_{pe}, N-1]$, and therefore the computational time for Step (d) scales linearly with N_{pe}. Including step (e), Algorithm 5.1 with the nonconvex condition check takes 0.022 seconds per time step more that without the nonconvex condition.

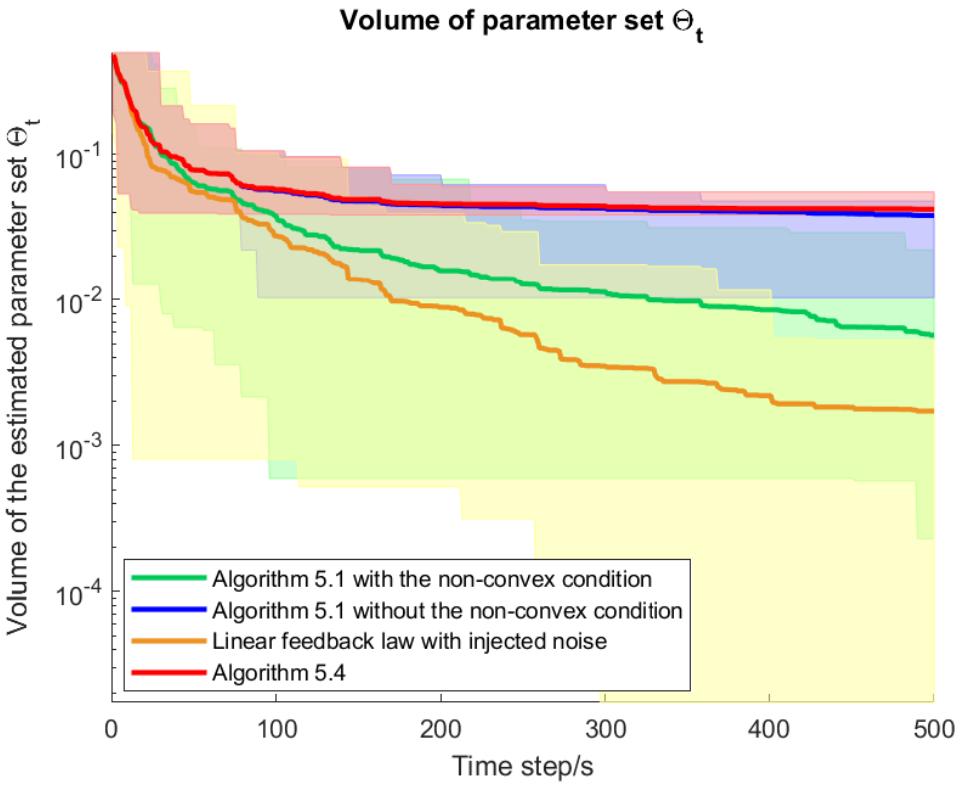


Figure 5.4: The volume of estimated parameter sets in logarithmic scale using Algorithm 5.1 (with and without the nonconvex condition check), Algorithm 5.4 and linear feedback law with injected noise s_t (Solid line: ensemble average. Shaded area: region for 30 disturbance sequences.)

Figure 5.4 compares the volume of the estimated parameter sets in logarithmic scale using Algorithm 5.1 (without and without the nonconvex condition check), Algorithm 5.4 and linear feedback law with injected noise. The solid lines plot the ensemble average of 30 disturbance sequences and the shaded areas indicate the region for 30 disturbance sequences. Algorithm 5.4 (red line) is almost constant after 300 time steps, which shows Algorithm 5.4 cannot identify an accurate description of the parameter estimate. Algorithm 5.1 without the nonconvex condition check performs slightly better than Algorithm 5.4 in terms of rate of parameter set convergence. Comparatively, Algorithm 5.1 with the nonconvex condition converges much faster. This is because the closed-loop PE condition is guaranteed by the nonconvex condition check. As a result, Algorithm 5.1 with the nonconvex condition is able to regulate the system and excite the system simultaneously, as shown by the decreasing green line. On the other hand, the parameter set convergence under linear feedback law is the fastest because no constraints or cost functions are considered.

Figure 5.5 shows the closed-loop ϵ_t value obtained using Algorithm 5.1 (with and without the nonconvex condition check), linear feedback law (5.5) and Algorithm 5.4. The closed-loop value of ϵ_t from (5.23) is an important indicator for Persistence of Excitation and its effect on convergence is explained in Chapter 4. The figure shows that the extra PE condition used in Algorithm 5.1, especially the one with the nonconvex condition check, effectively injects excitation into the system and therefore aids the parameter identification.

Figure 5.6 compares the accumulated cost given by $V_{t_i} = \sum_{t=1}^{t_i} \|x_t\|_Q + \|u_t\|_R$, averaged over 30 disturbance sequences. It shows that Algorithm 5.4 has the lowest cost and linear feedback law has the highest cost. Algorithm 5.1 without the nonconvex condition has a slightly (almost negligible) higher cost than Algorithm 5.4. On the other hand, the cost using Algorithm 5.1 with the nonconvex condition is between that of Algorithm 5.4 and linear feedback law. Combining with Figure 5.4, it is obvious that faster parameter convergence rates are linked with higher performance costs. In other words, there is always a compromise between parameter identification and performance.

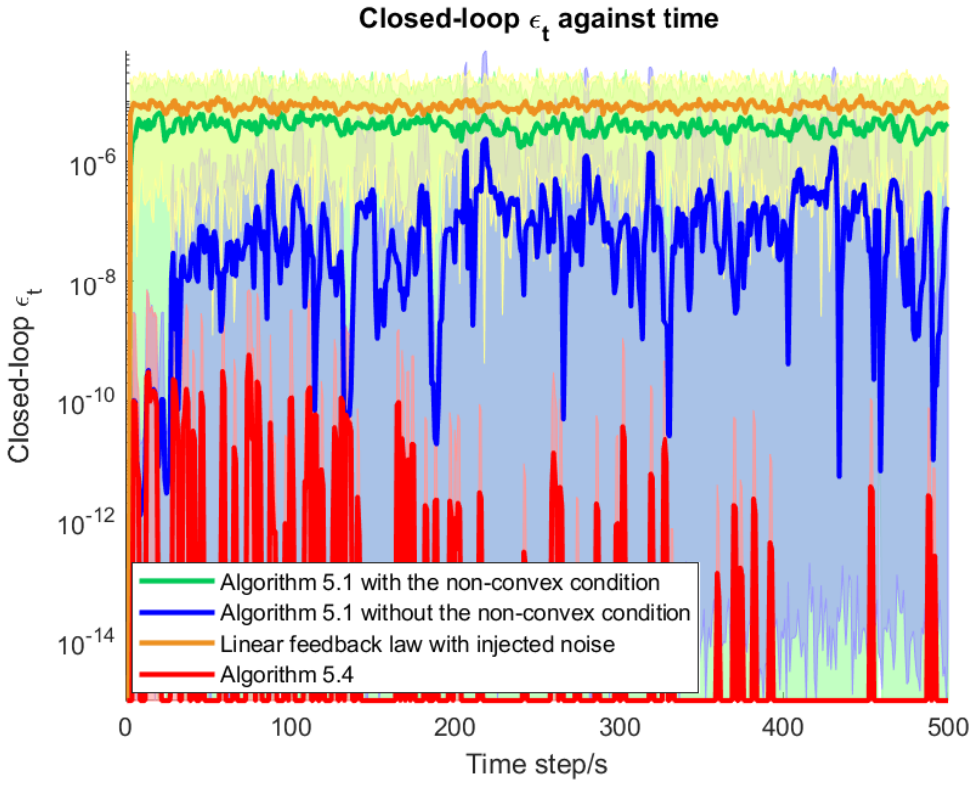


Figure 5.5: The closed-loop ϵ_t in logarithmic scale using Algorithm 5.1 (with and without the nonconvex condition check), Algorithm 5.4 and linear feedback law with injected noise s_t (Solid line: ensemble average. Shaded area: region for 30 disturbance sequences.)

5.5.3 Adaptive MPC with reduced computation via terminal region parameter exploration

Table 5.2: Comparison of Algorithm 5.2 (with different N_m) and Algorithm 5.1

Algorithm Name	Algorithm 5.2 $N_m = 4$	Algorithm 5.2 $N_m = 8$	Algorithm 5.1 (same as $N_m = 10$)
$\Theta_{10}/\%$	50.67	50.97	49.76
$\Theta_{100}/\%$	10.71	9.51	7.61
$\Theta_{500}/\%$	6.64	3.85	1.14
$\mathbb{E}(\epsilon_t)/*10^{-6}$	0.459	1.280	3.903
Step(c) Solver time/s	0.0057	0.0107	0.0133
Step(d) Solver time/s	0.0030	0.0038	0.0044

Table 5.2 tabulates the performance of Algorithm 5.2 and Algorithm 5.1. As N_m increases, the volume of the parameter set decreases at a faster rate, as supported

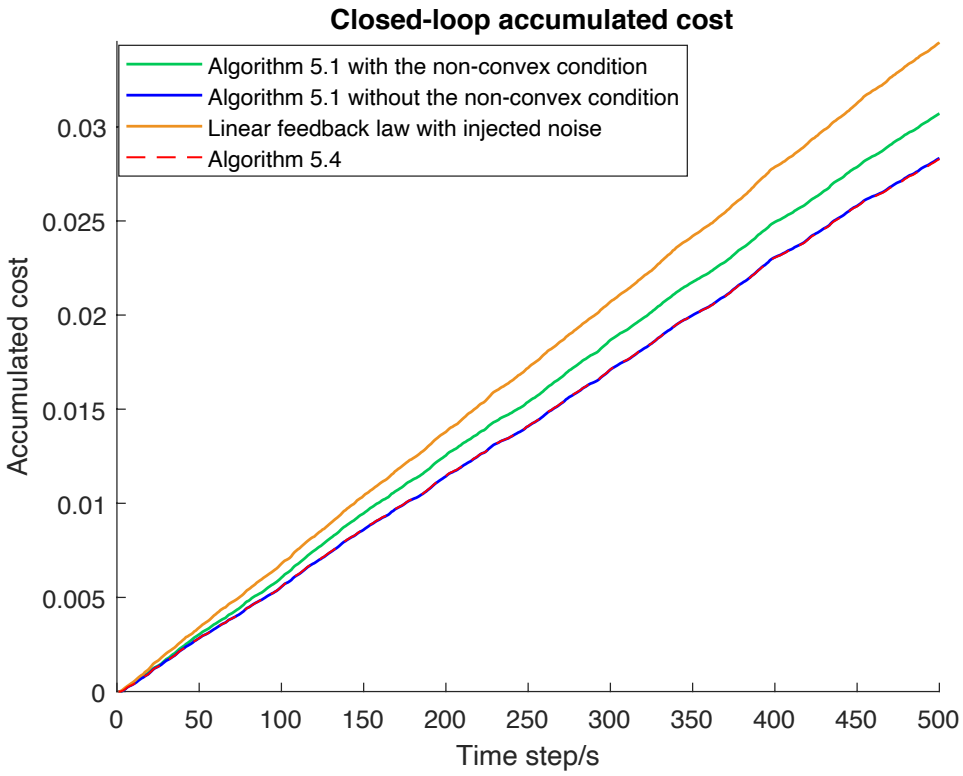


Figure 5.6: The closed-loop cost using Algorithm 5.1 (with and without the nonconvex condition check), Algorithm 5.4 and linear feedback law with injected noise s_t, averaged over 30 disturbance sequences

by the increasing PE coefficient ϵ_t. On the other hand, the computational time increases as N_m increases when Algorithm 5.2 is used. The solver time in step (c) is almost linear with N_m because the number of SDPs solved in that step depends linearly on N_m. In general, Algorithm 5.2 shows that there is a compromise between the accuracy of parameter identification and computational load.

Figure 5.7 shows the ensemble average of the estimated parameter set volumes over time when different N_m are used in Algorithm 5.2. For the purpose of comparison, estimated parameter volumes using Algorithm 5.1 and Algorithm 5.4 are also plotted in the same figure. As we reduce the computational load by decreasing N_m from 10 (Algorithm 5.1) to 4, the convergence rate of the estimated parameter set slows down.

Figure 5.8 compares the accumulated cost given by $V_{t_i} = \sum_{t=1}^{t_i} \|x_t\|_Q + \|u_t\|_R$, averaged over 30 disturbance sequences. The figure shows that the costs for Algorithm 5.2 are between that of Algorithm 5.4 and Algorithm 5.1. In other

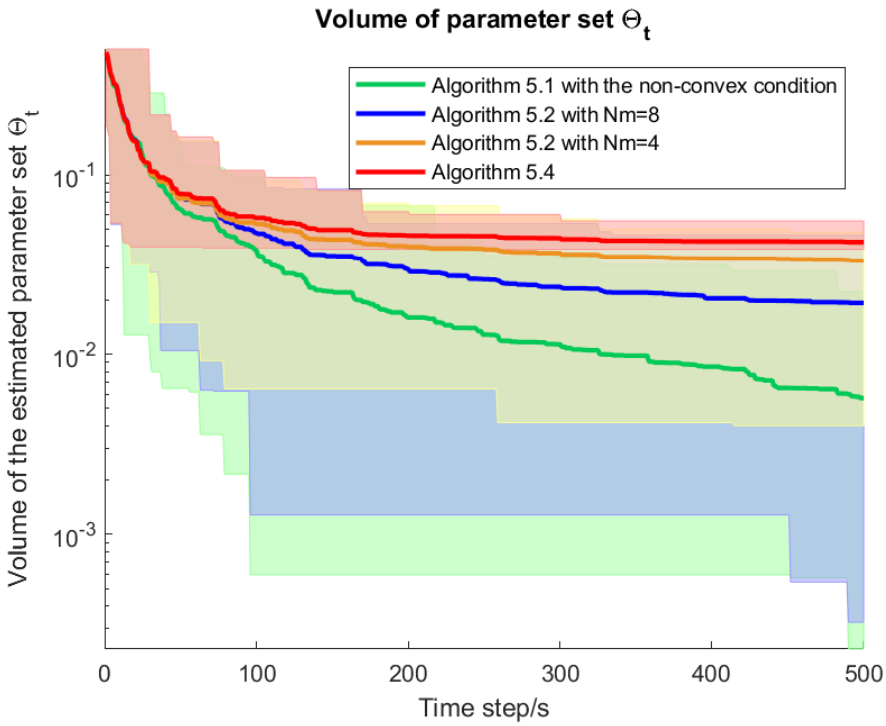


Figure 5.7: The volume of estimated parameter set using Algorithm 5.2 with varying N_m

words, the modified algorithm with the reduced computation via terminal region parameter exploration can achieve a compromise between the rate of parameter convergence and performance cost.

We further consider different initial conditions to evaluate the convergence of parameter estimates of Algorithm 5.2 during the transient phase before the state enters the terminal set. However, the state enters the terminal set in at most 10 time steps but the optimisation results for the first 10 time steps are the same with/without the PE constraint. Therefore, these experiments do not show any significant advantages of excitation before the state enters the terminal set. In addition, the input and state constraints are satisfied in all simulations using Algorithm 5.2.

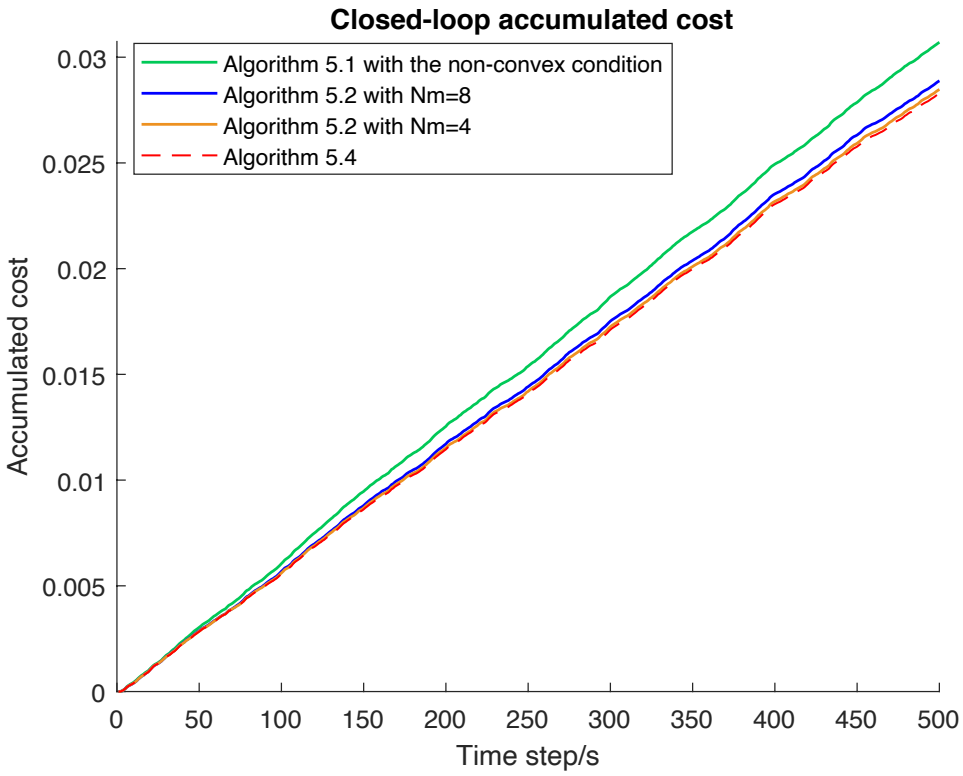


Figure 5.8: The closed-loop cost using Algorithm 5.1, Algorithm 5.2 and Algorithm 5.4, averaged over 30 disturbance sequences

5.5.4 Adaptive MPC with reduced computation via linearised PE coefficient

This section compares the performance of Algorithm 5.3 and Algorithm 5.4 using 30 randomly generated disturbance and noise sequences. The prediction horizon is taken as $N = 10$ and the PE window size $N_u = 3$. The initial Θ_0 is set to be $\Theta_0 = \{(\theta_1, \theta_2 | 0.1 < \theta_1 < 0.5, -0.1 < \theta_2 < 0.05\}$. The bounded injected noise sequence $\{s_0, s_1, \ldots\}$ follows uniform distribution with zero mean and $s_t \in \mathcal{S} = \{s : |s| \leq 0.005\}$. The initial state x_0 is set to be $x_0 = 2$ and the initial nominal parameter estimate is taken to be the centre of the initial parameter set Θ_0 where $\bar{\theta}_0 = \{0.3 \quad -0.025\}^\top$.

The bound given by Theorem 5.20 is hard to evaluate because r_k depends on $x_{N|t}$ and $\mathcal{X}_{N|t}$. Using a set of samples for $\bar{\theta}$ on a uniform grid over Θ, we can show that $\min_{\hat{\theta} \in \mathbb{B}} \hat{\theta}^\top P_k(\bar{\theta}, \Theta_0)\hat{\theta}$ in (5.46) is greater than zero for all $\bar{\theta} \in \Theta_0$. To facilitate

parameter convergence, we replace (5.43) with $\beta'_{N|t} \geq -0.0003$ whenever feasible and the coefficient -0.0003 is derived from past $\beta'_{N|t}$ values.

Table 5.3 shows the volume of the estimated parameter set under Algorithm 5.3 and Algorithm 5.4. In this simulation, the estimated parameter sets using both algorithms converge fast because the initial condition is outside the terminal region and the optimal input is exciting. Comparatively, the averaged volume of the estimated parameter sets using Algorithm 5.3 is always smaller than using Algorithm 5.4. Also, the difference in the estimated parameter set size increases as the parameter set size decreases, which demonstrates that the proposed algorithm is able to inject additional excitation into the system when the parameter set is small. In addition, the closed-loop ϵ_t value of Algorithm 5.3 is much higher than that of Algorithm 5.4.

Table 5.3: Comparison of Algorithm 5.4 and Algorithm 5.3

Algorithm Name	Algorithm 5.4	Algorithm 5.3
$\Theta_5/\%$	2.23	2.23
$\Theta_{10}/\%$	1.08	0.67
$\Theta_{1000}/\%$	1.04	0.13
$\Theta_{5000}/\%$	1.01	0.04
$\mathbb{E}(\epsilon_t)$	0.0004	0.0043

Figure 5.9 compares the volume of the estimated parameter sets in logarithmic scale using Algorithm 5.3 and Algorithm 5.4. The solid lines represent the averaged volume of the estimated parameter sets and the shaded areas indicate the region for 30 disturbance sequences. In general, Algorithm 5.3 results in smaller parameter sets than that using Algorithm 5.4. In addition, the red line stagnates after 15 time steps, which means Algorithm 5.4 fails to excite the system effectively, but the blue line continues to decreases and shows that Algorithm 5.3 with the additional PE constraints is able to inject excitation into the system.

Figure 5.10 shows the closed-loop ϵ_t using Algorithm 5.3 and Algorithm 5.4. As shown in Chapter 4, closed-loop ϵ_t provides a lower bound on the rate of parameter

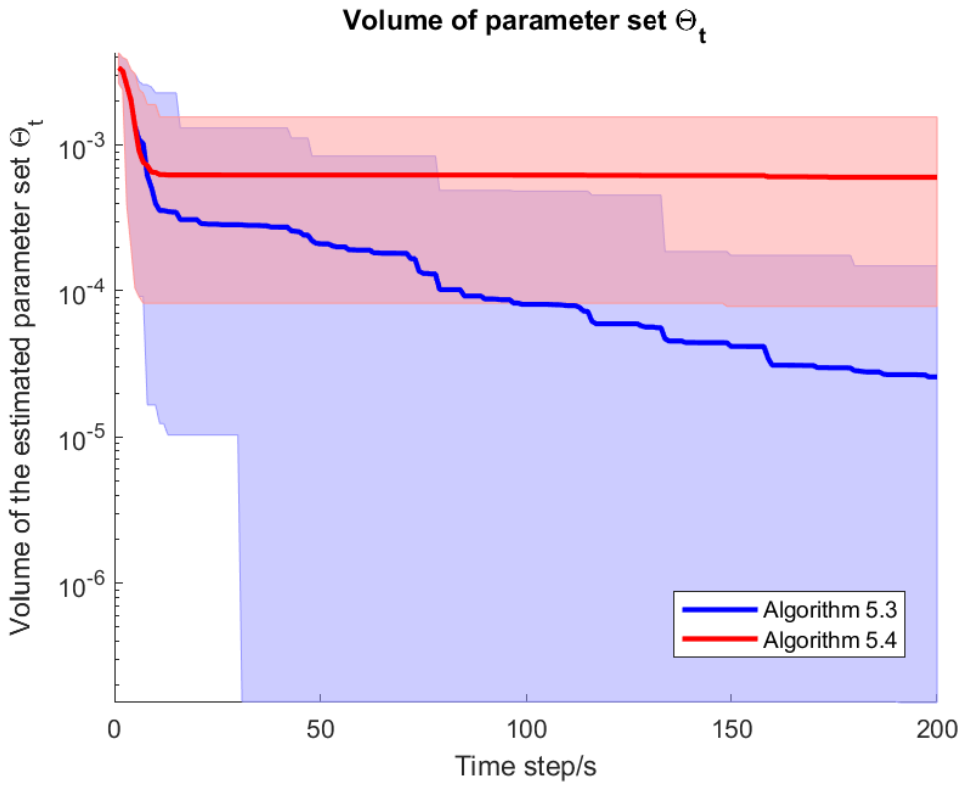


Figure 5.9: The volume of estimated parameter sets in logarithmic scale using Algorithm 5.3 and Algorithm 5.4

set convergence and therefore acts as an important indicator for Persistent Excitation conditions. The figure shows Algorithm 5.4 fails to excite the system most of the time for this example. Comparatively, the PE coefficient ϵ_t using Algorithm 5.3 is significantly larger and therefore aids parameter identification.

Figure 5.11 compares the accumulated cost given by $V_{t_i} = \sum_{t=1}^{t_i} \|x_t\|_Q + \|u_t\|_R$ using Algorithm 5.3 and Algorithm 5.4. Algorithm 5.3 has a higher cost than that of 5.4. Combining with the results from Figure 5.9, we can conclude faster parameter convergence is achieved at the expense of an increase in cost. Lastly, all constraints are satisfied robustly when Algorithm 5.3 is adopted.

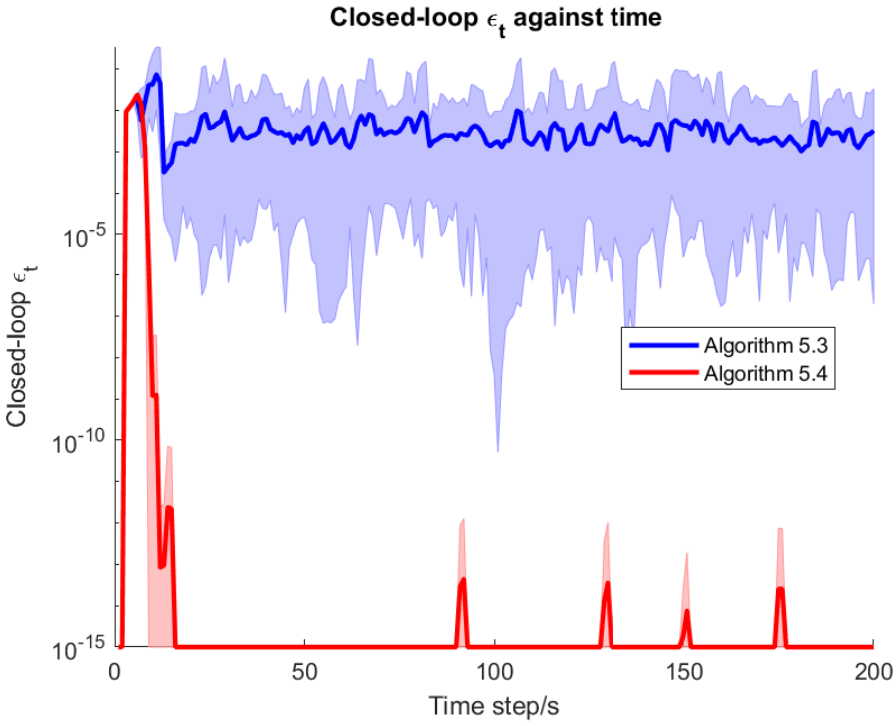


Figure 5.10: The closed-loop ϵ_t in logarithmic scale using Algorithm 5.3 and Algorithm 5.4

5.6 Conclusion

In this chapter, we first derived closed-loop bounds on the expectation of Persistent Excitation (PE) coefficients under a linear feedback law and a linear feedback law with injected noise. Building on that, we proposed three adaptive MPC algorithms that incorporate recursive feasible PE conditions with set-based parameter estimation and robust tube formulation. The proposed algorithms satisfy constraints robustly and are input-to-state stable (ISS). Closed-loop bounds on the expectation of the PE coefficients are also derived for each algorithm under different assumptions. The first algorithm contains a nonconvex condition in the online implementation and could be computationally intractable for high dimensional systems. The second algorithm reduces the computational load of the first algorithm by reducing the number of PE conditions needed. It achieves the same closed-loop properties under an additional assumption on the terminal control law. The

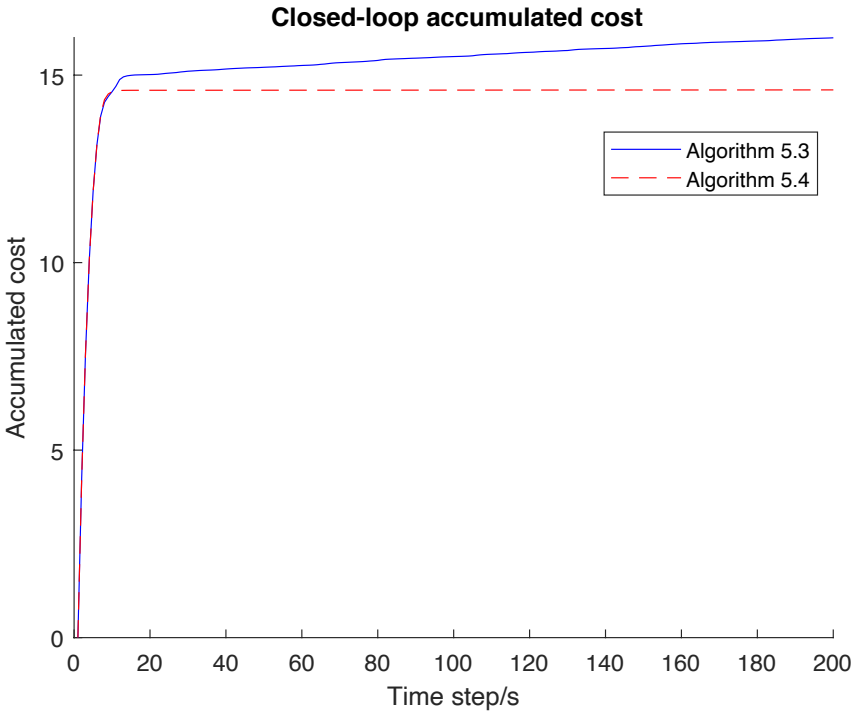


Figure 5.11: The closed-loop cost using Algorithm 5.3 and Algorithm 5.4

third algorithm avoids the nonconvex PE conditions by imposing the conditions on the linearised PE coefficients directly.

6

Adaptive MPC with PE conditions on nominal tubes

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6.1 Introduction

In this chapter, we consider the same system dynamics but we assume the injected noise sequence $\mathbf{s}_t$ is unknown to the controller. Moreover, we impose PE conditions on the nominal state tubes, rather than on the predicted state tubes in Chapter 5. The nominal tube formulation allows the separation of additive disturbances and injected noises from the state tube, and therefore the effects of disturbance and noises can be accounted for in the bounds on the expectation of the PE coefficients.

Compared with the algorithms in Chapter 5, the adaptive MPC algorithm proposed in this chapter reduces computational load because it can require fewer PE conditions and avoids nonconvex condition checking. The proposed algorithm also achieves robust constraint satisfaction, recursive feasibility, input-to-state stability (ISS) and closed-loop satisfaction of the PE condition under certain conditions.

6.2 Adaptive robust MPC with PE conditions imposed on nominal tubes

This chapter considers a linear system subject to additive disturbance and parameter uncertainty, shown in (5.1) and repeated here for convenience,

$$x_{t+1} = A(\theta^*)x_t + B(\theta^*)u_t + Fw_t.$$

We define a predicted control law similar to that in Chapter 5,

$$u_{k|t} = \begin{cases} Kx_{k|t} + v_{k|t} + s_{k|t}, & 0 \le k \le N-1 \\ Kx_{k|t} + s_{k|t}, & k \ge N \end{cases} \tag{6.1}$$

However, an important difference with (5.7) is that here the injected noise sequence $\{s_{0|t}, \dots, s_{N-1|t}\}$ is unknown to the controller when the decision variables $\{v_{0|t}, \dots, v_{N-1|t}\}$ are computed. Therefore the injected noise $s_{k|t}$ for $k \le N-1$ cannot be absorbed into the variable $v_{k|t}$.

To ensure robust satisfaction of the constraints, we construct a robust tube (a sequence of sets) $\{\mathcal{X}_{0|t}, \dots, \mathcal{X}_{N|t}\}$ satisfying, for all $\theta \in \Theta_t$ and $k \in \mathbb{N}_{[0,N-1]}$,

$$\mathcal{X}_{k|t} \subseteq \mathbb{X}, \quad K\mathcal{X}_{k|t} \oplus \{v_{k|t}\} \subseteq \mathbb{U} \ominus \mathcal{S}, \tag{6.2}$$

$$A_K(\theta)\mathcal{X}_{k|t} \oplus \{B(\theta)v_{k|t}\} \oplus \{B(\theta)\mathcal{S}\} + F\mathcal{W} \subseteq \mathcal{X}_{k+1|t}. \tag{6.3}$$

The initial and terminal conditions are given by

$$\mathcal{X}_{0|t} = \{x_{0|t}\}, \tag{6.4}$$

$$\mathcal{X}_{N|t} \subseteq \mathcal{X}_T, \tag{6.5}$$

where $\mathcal{X}_T$ is a terminal set satisfying Assumption 5.6. The details of tube calculation have been explained in Section 4.2.1 and 5.3.1.

The state $x_{k|t}$ is split into two parts, $x^o_{k|t}$ and $x^s_{k|t}$, where $x^o_{k|t}$ is driven by the input v_k and x^s_k is driven by the noise $s_{k|t}$ and disturbance $w_{k|t}$. Thus

$$\begin{aligned} x_{k|t} &= x^o_{k|t} + x^s_{k|t}, \\ x^o_{k|t} &= A_K(\theta^*)x^o_{k-1|t} + B(\theta^*)v_{k-1|t}, & x^o_{0|t} &= x_0, \\ x^s_{k|t} &= A_K(\theta^*)x^s_{k-1|t} + B(\theta^*)s_{k-1|t} + Fw_{k-1|t}, & x^s_{0|t} &= 0. \end{aligned}$$

Note that the noise $s_{k-1|t} = s_{k+t-1}$ and disturbance $w_{k-1|t} = w_{k+t-1}$ are unknown at time t.

Similarly, we decompose the control input $u_{k|t}$ into two parts

$$\begin{aligned} u_{k|t} &= u^o_{k|t} + u^s_{k|t}, \\ u^o_{k|t} &= Kx^o_{k|t} + v_{k|t}, \\ u^s_{k|t} &= Kx^s_{k|t} + s_{k|t}. \end{aligned}$$

so that $u^o_{k|t}$ is determined by the control variable $v_{k|t}$ and $u^s_{k|t}$ is governed by noise and disturbance inputs $s_{k|t}$ and $w_{k|t}$.

We define a nominal state tube $\{\mathcal{X}^o_{0|t}, \ldots, \mathcal{X}^o_{N|t}\}$, which is driven by the decision variable sequence $\{v_{0|t}, \ldots, v_{N-1|t}\}$. The tube satisfies for all $\theta \in \Theta_t$ and $k \in \mathbb{N}_{[0,N-1]}$,

$$x^o_{k|t} \in \mathcal{X}^o_{k|t}, \quad A_K(\theta)\mathcal{X}^o_{k|t} + B(\theta)v_{k|t} \subseteq \mathcal{X}^o_{k+1|t} \tag{6.6}$$

and the initial condition is given by

$$\mathcal{X}^o_{0|t} = \{x^o_{0|t}\} = \{x_t\}. \tag{6.7}$$

Next, we define the nominal reference state and input trajectories similarly to (5.16) and (5.39). A reference sequence $\hat{\mathbf{v}}_t$ is defined at any time $t > 0$ in terms of the optimal sequence $\mathbf{v}^*_{t-1} = \{v^*_{0|t-1}, \ldots, v^*_{N-1|t-1}\}$ computed at time $t-1$, as

$$\hat{\mathbf{v}}_t = \{v^*_{1|t-1}, \ldots, v^*_{N-1|t-1}, 0\}. \tag{6.8}$$

The nominal reference state and input sequences, $\hat{\mathbf{x}}_t^o = \{\hat{x}_{0|t}^o, \dots \hat{x}_{N_u-1|t}^o\}$, $\hat{\mathbf{u}}_t^o = \{\hat{u}_{0|t}^o, \dots \hat{u}_{N_u-1|t}^o\}$ are defined by

$$\hat{x}_{0|t}^o = x_t, \tag{6.9a}$$

$$\hat{v}_{k|t} = v_{k+1|t-1}^* \tag{6.9b}$$

$$\hat{u}_{k|t}^o = K\hat{x}_{k|t}^o + \hat{v}_{k|t}, \tag{6.9c}$$

$$\hat{x}_{k+1|t}^o = A(\bar{\theta}_t)\hat{x}_{k|t}^o + B(\bar{\theta}_t)\hat{u}_{k|t}^o. \tag{6.9d}$$

and for notational convenience we define $\hat{x}_{k|t}^o = x_{k+t}$ and $\hat{u}_{k|t}^o = u_{k+t}$ whenever $k < 0$.

The PE condition is implemented on the nominal state tube. Specifically, for all $\kappa \in \mathbb{N}_{[0,N_u-1]}$, we require

$$\sum_{k=-\kappa}^{-1} \Phi(x_{k+t}, u_{k+t})^\top \Phi(x_{k+t}, u_{k+t}) + \sum_{k=0}^{N_u-1-\kappa} \Phi(x_{k|t}^o, u_{k|t}^o)^\top \Phi(x_{k|t}^o, u_{k|t}^o) \succeq \beta_{-\kappa|t}\mathbb{I},$$
$$\forall x_{k|t}^o \in \mathcal{X}_{k|t}^o, u_{k|t}^o \in \{v_{k|t}\} \oplus K\mathcal{X}_{k|t}^o,\ k \in \mathbb{N}_{[0,N_u-1-\kappa]}, \tag{6.10}$$

where $t-\kappa$ denotes the start of the PE window and $\beta_{-\kappa|t}$ is the corresponding predicted PE coefficient. Note that the nominal states and control inputs corresponding to time-steps in the past are simply given by $x_{k|t}^o = x_{k+t}$ and $u_{k|t}^o = u_{k+t}$ whenever $k < 0$.

A PE condition of the form $\beta_{k|t} > 0$ is nonconvex in the state variable x_t and decision variable $\mathbf{v}_t$, and hence is difficult to implement in general. We can convexify the condition in (6.10) using the Taylor series expansion around the reference trajectory, and obtain a sufficient condition for persistent excitation (PE). We define a sequence of linearised predicted PE coefficients $\boldsymbol{\beta}'_t = \{\beta'_{-N_u+1|t}, \dots, \beta'_{0|t}\}$ as

$$\sum_{k=-\kappa}^{-1} \Phi(x_{k+t}, u_{k+t})^\top \Phi(x_{k+t}, u_{k+t})$$
$$+ \sum_{k=0}^{N_u-1-\kappa} \Big[\hat{\Phi}_{k+\kappa|t-\kappa}^\top \hat{\Phi}_{k+\kappa|t-\kappa} + \hat{\Phi}_{k+\kappa|t-\kappa}^\top \Phi(x_{k|t}^o - \hat{x}_{k+\kappa|t-\kappa}^o, u_{k|t}^o - \hat{u}_{k+\kappa|t-\kappa}^o)$$
$$+ \Phi(x_{k|t}^o - \hat{x}_{k+\kappa|t-\kappa}^o, u_{k|t}^o - \hat{u}_{k+\kappa|t-\kappa}^o)^\top \hat{\Phi}_{k+\kappa|t-\kappa}\Big] \succeq \beta'_{-\kappa|t}\mathbb{I},$$
$$\forall x_{k|t}^o \in \mathcal{X}_{k|t}^o,\ u_{k|t}^o \in \{v_{k|t}\} \oplus K\mathcal{X}_{k|t}^o,\ k \in \mathbb{N}_{[0,N_u-1-\kappa]}, \tag{6.11}$$

for all $\kappa \in \mathbb{N}_{[0,N_u-1]}$, where $\hat{\Phi}_{k|t} = \Phi(\hat{x}^o_{k|t}, \hat{u}^o_{k|t})$.

We also define the sequence of reference predicted PE coefficients $\hat{\boldsymbol{\beta}}'_t = \{\hat{\beta}'_{-N_u+1|t}, \ldots, \hat{\beta}'_{0|t}\}$ using the reference control sequence $\hat{v}_k$ given by (6.8). In other words, for all $\kappa \in \mathbb{N}_{[0,N_u-1]}$,

$$\hat{\beta}'_{-\kappa|t} = \min_{\substack{x^o_{k|t} \in \mathcal{X}^o_{k|t} \\ k=0,\ldots,N_u-1-\kappa}} \max_{\beta'_{-\kappa|t}} \beta'_{-\kappa|t} \text{ subject to (6.11) and } \mathbf{v}_t = \hat{\mathbf{v}}_t, \tag{6.12}$$

which are a series of Semi-definite Programs(SDPs) online.

In addition, $\beta'_{\kappa|t}$ is required to satisfy the following constraint, for all $\kappa \in \mathbb{N}_{[-N_u+1,0]}$,

$$\beta'_{-\kappa|t} \geq \hat{\beta}'_{-\kappa|t}. \tag{6.13}$$

Algorithm 6.1 (Adaptive MPC with PE conditions on nominal tubes)**.**

Offline Stage:

1. *Choose a suitable tube representation $\mathcal{X}$.*
2. *Compute the terminal set $\mathcal{X}_T$ and feedback gain K that satisfy Assumptions 5.3 and 5.6 .*
3. *Obtain a nominal parameter estimate $\bar{\theta}_0 \in \Theta_0$.*
4. *Choose suitable objective functions $l(x,u)$ and $V(x)$ that satisfy Assumption 5.7.*
5. *Compute the control sequence $\mathbf{v}_0^* = \{v^*_{0|0}, \ldots, v^*_{N-1|0}\}$, the solution of the Quadratic Program (QP):*

$$\begin{aligned} \mathcal{P}_{initialization}: \ & \min_{\mathbf{v}_0} \sum_{k=0}^{N-1} l(x_{k|0}, v_{k|0}) + V_N(x_{N|0}) \\ & \text{subject to } (6.2), (6.3), (6.4), (6.5), (6.6), (6.7) \end{aligned} \tag{6.14}$$

   Implement the current control input $u_0 = Kx_0 + v^*_{0|0} + s_0$.

Online Stage

(a) Obtain the current state x_t and set $x = x_t$.

(b) Update Θ_t and $\bar{\theta}_t$ using (4.1) and (4.13).

(c) *Compute the reference sequences* $\hat{\mathbf{x}}_t$, $\hat{\mathbf{u}}_t$ *and* $\hat{\boldsymbol{\beta}}'_t$, *using (6.8), (6.9) and (6.12).*

(d) *Given the reference PE coefficient* $\hat{\boldsymbol{\beta}}'_{\boldsymbol{t}}$, *compute the control sequence* $\mathbf{v}_t^* = \{v_{0|t}^*, \ldots, v_{N-1|t}^*\}$ *and the predicted PE coefficients* $\boldsymbol{\beta}'_{\boldsymbol{t}} = [\beta'_{-N_u+1|t}, \ldots, \beta'_{0|t}]$ *as the solution of the semidefinite program (SDP):*

$$\begin{aligned} \mathcal{P}_{online}: \ & \min_{\mathbf{v_t}, \boldsymbol{\beta}'_{\boldsymbol{t}}} \sum_{k=0}^{N-1} l(x_{k|t}, v_{k|t}) + V_N(x_{N|t}) \\ \text{subject to } & (6.2), (6.3), (6.4), (6.5), (6.6), (6.7), (6.11), (6.13) \end{aligned} \tag{6.15}$$

(e) *Implement the current control input* $u_t = Kx_t + v_{0|t}^* + s_t$.

Remark 6.1. *Besides (4.1) and (4.13), other parameter and nominal parameter update laws can be used. The rest of the results will not be affected as long as* $\Theta_{t+1} \subseteq \Theta_t$ *and* $\bar{\theta} \in \Theta_t$.

Remark 6.2. *The proposed algorithm uses two sets of tube formulations simultaneously: the predicted state tube* $\{\mathcal{X}_{0|t}, \ldots, \mathcal{X}_{N|t}\}$ *satisfying (6.3) for robust constraint satisfaction and the nominal tube* $\{\mathcal{X}^o_{0|t}, \ldots, \mathcal{X}^o_{N|t}\}$ *satisfying (6.6) for PE conditions. Other approaches can also be used, such as constraint tightening for the constraint satisfaction [10], or a disturbance-driven tube for* x_k^s *and* u_k^s.

Remark 6.3. *Algorithm 6.1 uses a robust state tube (6.3) for constraint satisfaction and a nominal state tube (6.6) for PE conditions. One alternative approach is to use the nominal state tube only by tightening the input and state constraints using the disturbance bound* $\mathcal{W}$, *the injected noise bound* $\mathcal{S}$ *and the parameter uncertainty set* Θ_t. *In other words, tightened constraints and the PE conditions are imposed on the nominal state tube. However, since the parameter set* Θ_t *is updated online, the tightening sets should be updated online using a series of linear programs(LPs), for example. In addition, the combination of system uncertainties and disturbances might make the tightening conditions conservative. As a result, although the alternative approach is conceptually intuitive, it can be computationally more expensive and more conservative.*

6.3 Recursive feasibility and input-to-state stability

Proposition 6.4 (Recursive feasibility)**.** *The online MPC optimisation in Algorithm 6.1 is feasible at all times $t \geq 1$ if (6.15) is feasible at $t = 0$ and $\Theta_t \subseteq \Theta_{t-1}$ for all t.*

Proof. For any $t > 0$, suppose that Algorithm 6.1 is feasible at time $t-1$. Then the sequence of suboptimal decision variables, $\hat{\mathbf{v}}_t = \{v^*_{1|t-1}, \ldots, v^*_{N-1|t-1}, 0\}$, is feasible at time t for:

- feasible at time t for (6.2), (6.3) at prediction time-steps $k \in \{0, 1, \ldots, N-2\}$ due to feasibility of (6.2), (6.3) at $t-1$ and $\Theta_t \subseteq \Theta_{t-1}$;
- feasible at time t for (6.2), (6.3) at prediction time $k = N-1$ due to feasibility of the terminal constraint (6.5) at $t-1$ and $\Theta_t \subseteq \Theta_{t-1}$;
- feasible at time t for (6.4) since $x_t \in \mathcal{X}_{1|t-1}$;
- feasible at time t for (6.5) by Assumption 5.6;
- feasible at time t for (6.6) and (6.7) for some $\mathcal{X}^o_k$;
- feasible at time t for (6.11), (6.13) for $\kappa \in \{0, 1, \ldots, N_u - 1\}$ because of the definition of $\hat{\boldsymbol{\beta}}'_t$ in (6.9) and (6.12)

It follows that $\hat{\mathbf{v}}_t$ is a feasible solution of (6.15) at time t. □

Proposition 6.5 (Stability)**.** *The closed-loop system (5.1) under Algorithm 6.1 is input-to-state stable (ISS).*

Proof. Given the recursive feasibility property of Proposition 6.4, input-to-state stability can be demonstrated using the proofs of Theorem 4.8 and Corollary 4.9 if the disturbance input w_t in (3.1) is replaced by $Fw_t + B(\theta^*)s_t$. □

6.4 Closed-loop satisfaction of the PE conditions

The main contribution of Theorem 6.6 is to link the convergence of parameter set estimates directly to properties of the disturbance sequence $\{w_0, w_1, w_2, \ldots\}$ (via Assumptions 4.2 and 5.1), injected noise sequence $\{s_0, s_1, s_2, \ldots\}$(via Assumption 5.5) and (A_i, B_i) and K (via Assumptions 3.2, 5.3 and 5.4).

Theorem 6.6. *If $N_u > n_x$, then Assumptions 3.2, 5.1, 5.3, 5.4 and 5.5, the closed loop system (5.1) under Algorithm 6.1 satisfies, for all $t \in \mathbb{N}_{\geq 0}$*

$$\min_{\hat{\theta}\in\mathbb{B}^p} \hat{\theta}^\top \left\{ \sum_{k=0}^{N_u-1} \mathbb{E}_t\Big[\Phi(x_{k+t}, u_{k+t})^\top \Phi(x_{k+t}, u_{k+t})\Big] \right\} \hat{\theta} \geq \beta'_{0|t} + \eta_{N_u} \tag{6.16}$$

where

$$\begin{aligned} \eta_{N_u} &:= \sum_{k=0}^{N_u-1} \min_{\hat{\theta}\in\mathbb{B}^p} \min_{\theta\in\Theta_0} \operatorname{tr} \left\{ \begin{bmatrix} \mathbb{E}_t[x^s_{k|t} {x^s_{k|t}}^\top] & 0 \\ 0 & \epsilon_s \mathbb{I} \end{bmatrix} \sum_{i=1}^{p} \begin{bmatrix} A_{K,i}^\top \\ B_i^\top \end{bmatrix} [\hat{\theta}]_i \sum_{i=1}^{p} \begin{bmatrix} A_{K,i} & B_i \end{bmatrix} [\hat{\theta}]_i \right\} \\ &\geq (N_u - n_x - 1) \min\{\epsilon_s \epsilon^s_{\mathcal{R}} + \epsilon_w \epsilon^w_{\mathcal{R}}, \epsilon_s\} \sigma^2 \mathbb{I} \end{aligned} \tag{6.17}$$

Proof. From (6.11) the linearised PE coefficient $\beta'_{-\kappa|t+\kappa}$ for any $\kappa \in \mathbb{N}_{[0,N_u-1]}$ is given by

$$\begin{aligned} \beta'_{-\kappa|t+\kappa} = \min_{\substack{x^o_{k|t+\kappa} \in \mathcal{X}^o_{k|\kappa+t} \\ k=0,\ldots,N_u-1-\kappa}} \min_{\hat{\theta}\in\mathbb{B}^p} \hat{\theta}^\top \Bigg\{ & \sum_{k=0}^{\kappa-1} \Phi(x_{k+t}, u_{k+t})^\top \Phi(x_{k+t}, u_{k+t}) \\ & + \sum_{k=0}^{N_u-1-\kappa} \Big[\hat{\Phi}^\top_{k+\kappa|t} \hat{\Phi}_{k+\kappa|t} + \hat{\Phi}^\top_{k+\kappa|t} \Phi(x^o_{k|\kappa+t} - \hat{x}^o_{k+\kappa|t}, u^o_{k|\kappa+t} - \hat{u}^o_{k+\kappa|t}) \\ & \qquad + \Phi(x^o_{k|\kappa+t} - \hat{x}^o_{k+\kappa|t}, u^o_{k|\kappa+t} - \hat{u}^o_{k+\kappa|t})^\top \hat{\Phi}_{k+\kappa|t} \Big] \Bigg\} \hat{\theta}. \end{aligned}$$

Hence for $\kappa > 0$, $\beta'_{-\kappa|t+\kappa}$ contains terms $\Phi(x_{k+t}, u_{k+t})^\top \Phi(x_{k+t}, u_{k+t})$, $k = 0, \ldots, \kappa-1$, which are quadratic functions of the injected noise inputs s_{k+t}, $k = 0, \ldots, \kappa-1$ (and quadratic in w_{k+t}, $k = 0, \ldots, \kappa-2$ if $\kappa > 1$). The bound in (6.16) can be derived by taking the expectation of these terms and using the property that the remaining terms in $\beta'_{-\kappa|t+\kappa}$ depend linearly on the vertices of $\mathcal{X}^o_{k|\kappa+t}$, $k = 0, \ldots, N_u - 1 - \kappa$, and are therefore linear functions of the injected noise and disturbance inputs s_{k+t}, w_{k+t}, $k = 0, \ldots, \kappa - 1$.

To derive the bound in (6.16) we proceed as follows. With $\kappa = 0$, $\beta'_{0|t}$ is evaluated over a window consisting of sampling instants $\{t, \dots, t + N_u - 1\}$:

$$\beta'_{0|t} = \min_{\hat{\theta} \in \mathbb{B}^p} \hat{\theta}^\top \Bigg\{ \sum_{k=0}^{N_u-1} \min_{x^o_{k|t} \in \mathcal{X}^o_{k|t}} \Big[\hat{\Phi}^\top_{k|t} \hat{\Phi}_{k|t} + \hat{\Phi}^\top_{k|t} \Phi(x^o_{k|t} - \hat{x}^o_{k|t}, u^o_{k|t} - \hat{u}^o_{k|t}) \\ + \Phi(x^o_{k|t} - \hat{x}^o_{k|t}, u^o_{k|t} - \hat{u}^o_{k|t})^\top \hat{\Phi}_{k|t} \Big] \Bigg\} \hat{\theta}.$$

For the same window, the expectation (conditioned on x_t) of $\hat{\beta}'_{-1|t+1}$ has the lower bound

$$\begin{aligned}
\mathbb{E}_t(\hat{\beta}'_{-1|t+1}) &= \min_{\hat{\theta} \in \mathbb{B}^p} \hat{\theta}^\top \Bigg\{ \mathbb{E}_t \Big[\Phi^\top(x_t, u_t) \Phi(x_t, u_t) \Big] \\
&\quad + \sum_{k=0}^{N_u-2} \mathbb{E}_t \Big[\min_{x^o_{k|t+1} \in \hat{\mathcal{X}}^o_{k|t+1}} \Big\{ \hat{\Phi}^\top_{k+1|t} \hat{\Phi}_{k+1|t} + \hat{\Phi}^\top_{k+1|t} \Phi(x^o_{k|t+1} - \hat{x}^o_{k+1|t}, u^o_{k|t+1} - \hat{u}^o_{k+1|t}) \\
&\quad + \Phi(x^o_{k|t+1} - \hat{x}^o_{k+1|t}, u^o_{k|t+1} - \hat{u}^o_{k+1|t})^\top \hat{\Phi}_{k+1|t} \Big\} \Big] \Bigg\} \hat{\theta} \\
&\geq \min_{\hat{\theta} \in \mathbb{B}^p} \hat{\theta}^\top \Bigg\{ \mathbb{E}_t \Big[\Phi^\top(x_t, u_t) \Phi(x_t, u_t) \Big] \\
&\quad + \sum_{k=0}^{N_u-2} \min_{x^o_{k+1|t} \in \mathcal{X}^o_{k+1|t}} \mathbb{E}_t \Big[\hat{\Phi}^\top_{k+1|t} \hat{\Phi}_{k+1|t} + \hat{\Phi}^\top_{k+1|t} \Phi(x^o_{k+1|t} - \hat{x}^o_{k+1|t}, u^o_{k+1|t} - \hat{u}^o_{k+1|t}) \\
&\quad + \Phi(x^o_{k+1|t} - \hat{x}^o_{k+1|t}, u^o_{k+1|t} - \hat{u}^o_{k+1|t})^\top \hat{\Phi}_{k+1|t} \Big] \Bigg\} \hat{\theta}
\end{aligned}$$

where $\{\hat{\mathcal{X}}^0_{0|t+1}, \dots, \hat{\mathcal{X}}^0_{N|t+1}\}$ denotes the tube generated by (6.6) at time $t+1$ with $\mathbf{v}_{t+1} = \hat{\mathbf{v}}_{t+1}$. The inequality holds because $\bar{\theta}^\top \Big\{ \min_{x^o \in \mathcal{X}^o} \Big[\hat{\Phi}^\top \Phi(x^o - \hat{x}, u^o - \hat{u}) + \Phi(x^o - \hat{x}, u^o - \hat{u})^\top \hat{\Phi} \Big] \Big\} \bar{\theta}$ depends linearly on the vertices of $\mathcal{X}^o$, and because the expectation (conditioned on x_t) of any vertex of $\hat{\mathcal{X}}^o_{k|t+1}$ necessarily belongs to $\mathcal{X}^o_{k+1|t}$ since $\Theta_{t+1} \subseteq \Theta_t$. Redefining the indices of the sum and using $x_t = x^o_{0|t} + x^s_{0|t}$ and $u_t = u^o_{0|t} + u^s_{0|t}$ we obtain

$$\begin{aligned}
\mathbb{E}_t(\hat{\beta}'_{-1|t+1}) \geq \min_{\hat{\theta} \in \mathbb{B}^p} \hat{\theta}^\top \Bigg\{ & \mathbb{E}_t \Big[\Phi^\top(x_t, u_t) \Phi(x_t, u_t) \Big] \\
& + \sum_{k=1}^{N_u-1} \min_{x^o_{k|t} \in \mathcal{X}^o_{k|t}} \mathbb{E}_t \Big[\hat{\Phi}^\top_{k|t} \hat{\Phi}_{k|t} + \hat{\Phi}^\top_{k|t} \Phi(x^o_{k|t} - \hat{x}^o_{k|t}, u^o_{k|t} - \hat{u}^o_{k|t}) \\
& + \Phi(x^o_{k|t} - \hat{x}^o_{k|t}, u^o_{k|t} - \hat{u}^o_{k|t})^\top \hat{\Phi}_{k|t} \Big] \Bigg\} \hat{\theta}
\end{aligned}$$

$$
\begin{aligned}
&\geq \min_{\hat{\theta}\in\mathbb{B}^p} \hat{\theta}^\top \Bigg\{ \Phi^\top(x^o_{0|t}, u^o_{0|t})\Phi(x^o_{0|t}, u^o_{0|t}) \\
&\quad + \sum_{k=1}^{N_u-1} \min_{x^o_{k|t}\in\mathcal{X}^o_{k|t}} \mathbb{E}_t\Big[\hat{\Phi}^\top_{k|t}\hat{\Phi}_{k|t} + \hat{\Phi}^\top_{k|t}\Phi(x^o_{k|t} - \hat{x}^o_{k|t}, u^o_{k|t} - \hat{u}^o_{k|t}) \\
&\quad + \Phi(x^o_{k|t} - \hat{x}^o_{k|t}, u^o_{k|t} - \hat{u}^o_{k|t})^\top \hat{\Phi}_{k|t}\Big]\Bigg\}\hat{\theta} \\
&\quad + \min_{\hat{\theta}\in\mathbb{B}^p} \operatorname{tr}\Bigg\{ \begin{bmatrix} x^s_{0|t}{x^s_{0|t}}^\top & 0 \\ 0 & \epsilon_s\mathbb{I} \end{bmatrix} \sum_{i=1}^{p} \begin{bmatrix} A^\top_{K,i} \\ B^\top_i \end{bmatrix} [\hat{\theta}]_i \sum_{i=1}^{p} \begin{bmatrix} A_{K,i} & B_i \end{bmatrix} [\hat{\theta}]_i \Bigg\}
\end{aligned}
$$

The term $\Phi(x^o_{0|t}, u^o_{0|t})^\top\Phi(x^o_{0|t}, u^o_{0|t})$ can be bounded from below by its linearisation, resulting in

$$
\begin{aligned}
\mathbb{E}_t(\hat{\beta}'_{-1|t+1}) &\geq \min_{\hat{\theta}\in\mathbb{B}^p} \hat{\theta}^\top \Bigg\{ \sum_{k=0}^{N_u-1} \min_{x^o_{k|t}\in\mathcal{X}^o_{k|t}} \mathbb{E}_t\Big[\hat{\Phi}^\top_{k|t}\hat{\Phi}_{k|t} + \hat{\Phi}^\top_{k|t}\Phi(x^o_{k|t} - \hat{x}^o_{k|t}, u^o_{k|t} - \hat{u}^o_{k|t}) \\
&\quad + \Phi(x^o_{k|t} - \hat{x}^o_{k|t}, u^o_{k|t} - \hat{u}^o_{k|t})^\top \hat{\Phi}_{k|t}\Big]\Bigg\}\hat{\theta} \\
&\quad + \min_{\hat{\theta}\in\mathbb{B}^p} \operatorname{tr}\Bigg\{ \begin{bmatrix} x^s_{0|t}{x^s_{0|t}}^\top & 0 \\ 0 & \epsilon_s\mathbb{I} \end{bmatrix} \sum_{i=1}^{p} \begin{bmatrix} A^\top_{K,i} \\ B^\top_i \end{bmatrix} [\hat{\theta}]_i \sum_{i=1}^{p} \begin{bmatrix} A_{K,i} & B_i \end{bmatrix} [\hat{\theta}]_i \Bigg\} \\
&= \mathbb{E}_t[\beta'_{0|t}] + \eta_1,
\end{aligned}
$$

where the definition of η_1 in (6.17) has been used. Since constraint (6.13) implies $\mathbb{E}_t(\beta'_{-1|t+1}) \geq \mathbb{E}_t(\hat{\beta}'_{-1|t+1})$ we therefore have

$$
\mathbb{E}_t(\beta'_{-1|t+1}) \geq \mathbb{E}_t(\hat{\beta}'_{-1|t+1}) \geq E_t(\beta'_{0|t}) + \eta_1.
$$

Next we apply the same sequence of steps to the case in which κ has any value in $\mathbb{N}_{[0,N_u-1]}$:

$$
\begin{aligned}
\mathbb{E}_t(\hat{\beta}'_{-\kappa|t+\kappa}) &= \min_{\hat{\theta}\in\mathbb{B}^p} \hat{\theta}^\top \Bigg\{ \sum_{k=0}^{\kappa-1} \mathbb{E}_t\Big[\Phi(x_{k+t}, u_{k+t})^\top\Phi(x_{k+t}, u_{k+t})\Big] \\
&\quad + \sum_{k=0}^{N_u-1-\kappa} \mathbb{E}_t\Bigg[\min_{x^o_{k|t+\kappa}\in\hat{\mathcal{X}}^o_{k|t+\kappa}} \Big\{ \hat{\Phi}^\top_{k+\kappa|t}\hat{\Phi}_{k+\kappa|t} + \hat{\Phi}^\top_{k+\kappa|t}\Phi(x^o_{k|t+\kappa} - \hat{x}^o_{k+\kappa|t}, u^o_{k|t+\kappa} - \hat{u}^o_{k+\kappa|t}) \\
&\quad + \Phi(x^o_{k|t+\kappa} - \hat{x}^o_{k+\kappa|t}, u^o_{k|t+\kappa} - \hat{u}^o_{k+\kappa|t})^\top \hat{\Phi}_{k+\kappa|t} \Big\}\Bigg]\Bigg\}\hat{\theta} \\
&\overset{I_1}{\geq} \min_{\hat{\theta}\in\mathbb{B}^p} \hat{\theta}^\top \Bigg\{ \sum_{k=0}^{\kappa-1} \mathbb{E}_t\Big[\Phi(x_{k+t}, u_{k+t})^\top\Phi(x_{k+t}, u_{k+t})\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{k=0}^{N_u-1-\kappa} \mathbb{E}_t\Big[\min_{x^o_{k+1|t+\kappa-1} \in \mathcal{X}^o_{k+1|t+\kappa-1}} \Big\{ \hat{\Phi}^\top_{k+\kappa|t} \hat{\Phi}_{k+\kappa|t} \\
&\qquad + \hat{\Phi}^\top_{k+\kappa|t} \Phi(x^o_{k+1|t+\kappa-1} - \hat{x}^o_{k+\kappa|t}, u^o_{k+1|t+\kappa-1} - \hat{u}^o_{k+\kappa|t}) \\
&\qquad + \Phi(x^o_{k+1|t+\kappa-1} - \hat{x}^o_{k+\kappa|t}, u^o_{k+1|t+\kappa-1} - \hat{u}^o_{k+\kappa|t})^\top \hat{\Phi}_{k+\kappa|t} \Big\} \Big] \Big\} \hat{\theta} \\
&\overset{E_2}{=} \min_{\hat{\theta} \in \mathbb{B}^p} \hat{\theta}^\top \Bigg\{ \sum_{k=0}^{\kappa-1} \mathbb{E}_t\Big[\Phi(x_{k+t}, u_{k+t})^\top \Phi(x_{k+t}, u_{k+t}) \Big] \\
&+ \sum_{k=1}^{N_u-\kappa} \mathbb{E}_t\Big[\min_{x^o_{k|t+\kappa-1} \in \mathcal{X}^o_{k|t+\kappa-1}} \Big\{ \hat{\Phi}^\top_{k+\kappa-1|t} \hat{\Phi}_{k+\kappa-1|t} \\
&\qquad + \hat{\Phi}^\top_{k+\kappa-1|t} \Phi(x^o_{k|t+\kappa-1} - \hat{x}^o_{k+\kappa-1|t}, u^o_{k|t+\kappa-1} - \hat{u}^o_{k+\kappa-1|t}) \\
&\qquad + \Phi(x^o_{k|t+\kappa-1} - \hat{x}^o_{k+\kappa-1|t}, u^o_{k|t+\kappa-1} - \hat{u}^o_{k+\kappa-1|t})^\top \hat{\Phi}_{k+\kappa-1|t} \Big\} \Big] \Big\} \hat{\theta} \\
&\overset{I_3}{\geq} \min_{\hat{\theta} \in \mathbb{B}^p} \hat{\theta}^\top \Bigg\{ \sum_{k=0}^{\kappa-2} \mathbb{E}_t\Big[\Phi(x_{k+t}, u_{k+t})^\top \Phi(x_{k+t}, u_{k+t}) \Big] \\
&+ \mathbb{E}_t\Big[\Phi(x^o_{0|t+\kappa-1}, u^o_{0|t+\kappa-1})^\top \Phi(x^o_{0|t+\kappa-1}, u^o_{0|t+\kappa-1}) \Big] \\
&+ \sum_{k=1}^{N_u-\kappa} \mathbb{E}_t\Big[\min_{x^o_{k|t+\kappa-1} \in \mathcal{X}^o_{k|t+\kappa-1}} \Big\{ \hat{\Phi}^\top_{k+\kappa-1|t} \hat{\Phi}_{k+\kappa-1|t} \\
&\qquad + \hat{\Phi}^\top_{k+\kappa-1|t} \Phi(x^o_{k|t+\kappa-1} - \hat{x}^o_{k+\kappa-1|t}, u^o_{k|t+\kappa-1} - \hat{u}^o_{k+\kappa-1|t}) \\
&\qquad + \Phi(x^o_{k|t+\kappa-1} - \hat{x}^o_{k+\kappa-1|t}, u^o_{k|t+\kappa-1} - \hat{u}^o_{k+\kappa-1|t})^\top \hat{\Phi}_{k+\kappa-1|t} \Big\} \Big] \Big\} \hat{\theta} \\
&+ \min_{\hat{\theta} \in \mathbb{B}^p} \min_{\theta \in \Theta_0} \mathrm{tr} \Bigg\{ \begin{bmatrix} \mathbb{E}_t[x^s_{\kappa-1|t} {x^s_{\kappa-1|t}}^\top] & 0 \\ 0 & \epsilon_s \mathbb{I} \end{bmatrix} \sum_{i=1}^{p} \begin{bmatrix} A_{K,i}^\top \\ B_i^\top \end{bmatrix} [\hat{\theta}]_i \sum_{i=1}^{p} \begin{bmatrix} A_{K,i} & B_i \end{bmatrix} [\hat{\theta}]_i \Bigg\} \\
&\overset{I_4}{\geq} \min_{\hat{\theta} \in \mathbb{B}^p} \hat{\theta}^\top \Bigg\{ \sum_{k=0}^{\kappa-2} \mathbb{E}_t\Big[\Phi(x_{k+t}, u_{k+t})^\top \Phi(x_{k+t}, u_{k+t}) \Big] \\
&+ \sum_{k=0}^{N_u-\kappa} \mathbb{E}_t\Big[\min_{x^o_{k|t+\kappa-1} \in \mathcal{X}^o_{k|t+\kappa-1}} \Big\{ \hat{\Phi}^\top_{k+\kappa-1|t} \hat{\Phi}_{k+\kappa-1|t} \\
&\qquad + \hat{\Phi}^\top_{k+\kappa-1|t} \Phi(x^o_{k|t+\kappa-1} - \hat{x}^o_{k+\kappa-1|t}, u^o_{k|t+\kappa-1} - \hat{u}^o_{k+\kappa-1|t}) \\
&\qquad + \Phi(x^o_{k|t+\kappa-1} - \hat{x}^o_{k+\kappa-1|t}, u^o_{k|t+\kappa-1} - \hat{u}^o_{k+\kappa-1|t})^\top \hat{\Phi}_{k+\kappa-1|t} \Big\} \Big] \Big\} \hat{\theta} \\
&+ \min_{\hat{\theta} \in \mathbb{B}^p} \min_{\theta \in \Theta_0} \mathrm{tr} \Bigg\{ \begin{bmatrix} \mathbb{E}_t[x^s_{\kappa-1|t} {x^s_{\kappa-1|t}}^\top] & 0 \\ 0 & \epsilon_s \mathbb{I} \end{bmatrix} \sum_{i=1}^{p} \begin{bmatrix} A_{K,i}^\top \\ B_i^\top \end{bmatrix} [\hat{\theta}]_i \sum_{i=1}^{p} \begin{bmatrix} A_{K,i} & B_i \end{bmatrix} [\hat{\theta}]_i \Bigg\} \\
&\overset{E_5}{=} \mathbb{E}_t(\beta'_{-\kappa+1|t+\kappa-1})
\end{aligned}
$$

$$+\min_{\hat{\theta}\in\mathbb{B}^p}\min_{\theta\in\Theta_0}\operatorname{tr}\left\{\begin{bmatrix}\mathbb{E}_t[x^s_{\kappa-1|t}{x^s_{\kappa-1|t}}^\top] & 0\\ 0 & \epsilon_s\mathbb{I}\end{bmatrix}\sum_{i=1}^{p}\begin{bmatrix}A_{K,i}^\top\\ B_i^\top\end{bmatrix}[\hat{\theta}]_i\sum_{i=1}^{p}\begin{bmatrix}A_{K,i} & B_i\end{bmatrix}[\hat{\theta}]_i\right\}. \tag{6.18}$$

Here the first inequality (I_1) follows from the observations that: (a) each term in the second summation in the expression for $\mathbb{E}_t(\hat{\beta}_{-\kappa|t+\kappa})$ depends linearly on the expectation (conditioned on x_t) of a vertex of $\hat{\mathcal{X}}^o_{k|t+\kappa}$, and (b) $\mathbb{E}_t(\hat{\mathcal{X}}^o_{k|t+\kappa}) \subseteq \mathbb{E}_t(\mathcal{X}^o_{k+1|t+\kappa-1})$ since $\Theta_{t+\kappa} \subseteq \Theta_{t+\kappa-1}$ and (6.6) enforces a set of linear inequalities on the vertices of $\hat{\mathcal{X}}^o_{k|t+\kappa}$ with $\mathbf{v}_{t+\kappa} = \hat{\mathbf{v}}_{t+\kappa}$. Equality ($E_2$) is a straightforward redefinition of the indices of the second summation, and inequality (I_3) is obtained by expressing the last term of the first summation as a quadratic function of $x^o_{0|t+\kappa-1}+x^s_{\kappa-1|t}$ and $u^o_{0|t+\kappa-1}+u^s_{\kappa-1|t}$, and by exploiting the independence of the s_k and w_k sequences. Inequality (I_4) is obtained by bounding the term $\mathbb{E}_t[\Phi(x^o_{0|t+\kappa-1}, u^o_{0|t+\kappa-1})^\top\Phi(x^o_{0|t+\kappa-1}, u^o_{0|t+\kappa-1})]$ using its linearisation. The final equality (E_5) follows from the definition of $\beta'_{-\kappa+1|t+\kappa-1}$.

In addition, since the first element of the input sequence is applied to the controller (i.e. $u_{N_u+t-1} = u^*_{0|N_u+t-1}$), we have

$$\min_{\hat{\theta}\in\mathbb{B}^p}\hat{\theta}^\top\left\{\sum_{k=0}^{N_u-1}\mathbb{E}_t\Big[\Phi(x_{k+t}, u_{k+t})^\top\Phi(x_{k+t}, u_{k+t})\Big]\right\}\hat{\theta} \geq \mathbb{E}_t(\beta'_{-N_u+1|t+N_u-1}) + \eta_1$$

by expressing the last term $\Phi(x_{N_u+t-1}, u_{N_u+t-1})^\top\Phi(x_{N_u+t-1}, u_{N_u+t-1})$ as a quadratic function of $x^o_{0|t+N_u-1} + x^s_{N_u-1|t}$ and $u^o_{0|t+N_u-1} + u^s_{N_u-1|t}$, and by exploiting the independence of the s_k and w_k sequences.

We then obtain the inequality in (6.16):

$$\min_{\hat{\theta}\in\mathbb{B}^p}\hat{\theta}^\top\left\{\sum_{k=0}^{N_u-1}\mathbb{E}_t\Big[\Phi(x_{k+t}, u_{k+t})^\top\Phi(x_{k+t}, u_{k+t})\Big]\right\}\hat{\theta} \geq \beta'_{0|t} + \eta_{N_u}$$

by combining the constraint (6.13), which implies $\mathbb{E}_t(\beta'_{-\kappa|t+\kappa}) \geq \mathbb{E}_t(\hat{\beta}'_{-\kappa|t+\kappa})$, with the bounds on $\mathbb{E}_t(\hat{\beta}'_{-\kappa|t+\kappa})$ given by (6.18), for $\kappa = 1, \ldots, N_u - 1$.

To complete the proof, we derive the lower bound on η_{N_u} in (6.17) using the properties of the system matrices $(A_K(\theta), B(\theta))$, the additive disturbance w_k and

the injected noise s_k. Given that w_k and s_k are i.i.d and $\mathbb{E}(w_k w_k^\top) \succeq \epsilon_w \mathbb{I}$, and $\mathbb{E}(s_k s_k^\top) \succeq \epsilon_s \mathbb{I}$ by Assumptions 5.1 and 5.5, we have

$$\mathbb{E}(x^s_{k|t} {x^s_{k|t}}^\top) = \epsilon_s \sum_{j=0}^{k-1} A_K(\theta^*)^j B(\theta^*) B(\theta^*)^\top A_K^\top(\theta^*)^j + \epsilon_w \sum_{j=0}^{k-1} A_K(\theta^*)^j F F^\top A_K^\top(\theta^*)^j \tag{6.19}$$

for all $k > 0$.

In addition, the pairs $(A_K(\theta^*), B(\theta^*))$ and $(A_K(\theta^*), F)$ are reachable (Assumption 5.3), so positive scalars $\epsilon^s_{\mathcal{R}}$ and $\epsilon^w_{\mathcal{R}}$ exist such that $\sum_{j=0}^{n_x-1} A_K(\theta^*)^j B(\theta^*) B(\theta^*)^\top A_K^\top(\theta^*)^j \succeq \epsilon^s_{\mathcal{R}}$ and $\sum_{j=0}^{n_x-1} A_K(\theta^*)^j F F^\top A_K^\top(\theta^*)^j \succeq \epsilon^w_{\mathcal{R}}$ respectively. It follows that, for $k \geq n_x$,

$$\mathbb{E}(x^s_{k|t} {x^s_{k|t}}^\top) \geq \epsilon_s \epsilon^s_{\mathcal{R}} + \epsilon_w \epsilon^w_{\mathcal{R}}.$$

Moreover,

$$\min_{\hat{\theta} \in \mathbb{B}^p} \operatorname{tr}\Big\{ \sum_{i=1}^{p} \begin{bmatrix} A_{K,i}^\top \\ B_i^\top \end{bmatrix} [\hat{\theta}]_i \sum_{i=1}^{p} \begin{bmatrix} A_{K,i} & B_i \end{bmatrix} [\hat{\theta}]_i \Big\} = \min_{\hat{\theta} \in \mathbb{B}^p} \|\operatorname{vec}\Big(\sum_{i=1}^{p} \begin{bmatrix} A_{K,i} & B_i \end{bmatrix} \Big) [\hat{\theta}]_i\|^2 \geq \sigma^2$$

where we have used the identity $\operatorname{tr}(A^\top A) = \|\operatorname{vec}(A)\|^2$ and where σ is minimum singular value of the matrix $[\operatorname{vec}[(A_{k,1}\ B_1]), \ldots, \operatorname{vec}[(A_{k,p}\ B_p])]$, which is non-zero due to Assumptions 3.2 and 5.4.

$$\begin{aligned} \eta_{N_u} &= \sum_{k=0}^{N_u-1} \min_{\hat{\theta} \in \mathbb{B}^p} \min_{\theta \in \Theta} \operatorname{tr}\Bigg\{ \begin{bmatrix} \mathbb{E}[x^s_k {x^s_k}^\top] & 0 \\ 0 & \epsilon_s \mathbb{I} \end{bmatrix} \sum_{i=1}^{p} \begin{bmatrix} A_{K,i}^\top \\ B_i^\top \end{bmatrix} [\hat{\theta}]_i \sum_{i=1}^{p} \begin{bmatrix} A_{K,i} & B_i \end{bmatrix} [\hat{\theta}]_i \Bigg\} \\ &\geq (N_u - n_x - 1) \min\{\epsilon_s \epsilon^s_{\mathcal{R}} + \epsilon_w \epsilon^w_{\mathcal{R}}, \epsilon_s\} \sigma^2 \mathbb{I} \end{aligned}$$

The conclusion follows by simple substitution.

□

Remark 6.7. *Assumption 5.3 can be relaxed in certain directions as long as* $\epsilon_s \sum_{j=0}^{k-1} A_K(\theta^*)^j B(\theta^*) B(\theta^*)^\top A_K^\top(\theta^*)^j + \epsilon_w \sum_{j=0}^{k-1} A_K(\theta^*)^j F F^\top A_K^\top(\theta^*)^j \succ 0$, *which means reachability condition needs to be jointly satisfied for the pairs* $(A_K(\theta^*), B(\theta^*))$ *and* $(A_K(\theta^*), F)$.

Remark 6.8. *The closed-loop satisfaction of the PE conditions depends on the value of* $\mathbb{E}[\beta'_{0|t} + \eta_{N_u}]$, *which might be negative for some examples. In those cases, parameter identification can be promoted by including an optional constraint* $\beta'_{0|t} + \eta_{N_u} \geq 0$ *in the optimisation whenever feasible.*

6.5 Numerical examples

This section compares the performance of Algorithm 6.1, Algorithm 6.2 (defined below) and Algorithm 5.4. The simulation example is the same as that in Chapter 5. The prediction horizon is taken as $N = 10$ and the PE window size $N_u = 3$. The initial Θ_0 is set to be $\Theta_0 = \{(\theta_1, \theta_2) | 0.1 < \theta_1 < 0.5, -0.1 < \theta_2 < 0.05\}$. The bounded injected noise sequence $\{s_0, s_1, \ldots\}$ has a uniform distribution with zero mean and $s_t \in \mathcal{S} = \{s : |s| \leq 0.005\}$. For this simulation, $\beta'_{0|0}$ is about -1.353×10^{-5} and η_{N_u} is about 3.54×10^{-6}. In other words, the RHS of Theorem 6.6 is less than zero. Therefore, the PE condition (5.2) is not satisfied in the closed-loop at time $t = 0$ (note that it could be satisfied at a later time, because the value of $\beta'_{0|t}$ for $t > 0$ is a random variable since it depends on the $\mathbf{s}$ and $\mathbf{w}$ sequences). Input and state constraints are satisfied robustly all the time.

We first define an adaptive robust MPC law with unknown random noise s_t and unknown additive disturbance w_t, but without the PE condition. In short, Algorithm 6.2 is a modification of Algorithm 6.1 by removing any PE constraints and associated variables. Unlike with Algorithm 5.4, the controller in Algorithm 6.2 cannot access the information of the injected noise s_t.

Algorithm 6.2 (Robust MPC with unknown but bounded noise and disturbance)**.**

Offline Stage:

1. *Choose a suitable tube representation $\mathcal{X}$.*
2. *Compute the terminal set $\mathcal{X}_T$ and feedback gain K that satisfy Assumptions 5.3 and 5.6 .*
3. *Obtain a nominal parameter estimate $\bar{\theta}_0 \in \Theta_0$.*
4. *Choose suitable stage cost $l(x, u)$ and terminal cost $V_N(x)$ that satisfy Assumption 5.7.*

Online Stage

(a) Obtain the current state x_t and set $x = x_t$.

(b) Update Θ_t *and* $\bar{\theta}_t$ *using (4.1) and (4.13).*

(c) compute the control sequence $\mathbf{v}_t^* = \{v_0^*, \ldots, v_{N-1}^*\}$*, the solution of Quadratic Program (QP):*

$$\begin{aligned} \mathcal{P}_{QP}: \ \min_{\mathbf{v}_t} \sum_{k=0}^{N-1} l(x_k, v_k) + V_N(x_N) \\ \textit{subject to } (6.2), (6.3), (6.4), (6.5) \end{aligned} \tag{6.20}$$

(d) Implement the optimal control input $u_t = Kx_t + v_0^* + s_t$.

All simulations were performed in Matlab on a 2.9 GHz Intel Core i7-10700F processor, and the online MPC optimisation was formulated using Yalmip [71]. Quadratic Programs (QPs) are solved using Gurobi [72] and Semi-definite Programs (SDPs) are solved using Mosek [80]. For the purpose of comparison, identical disturbance sequences and the same parameter update law are used. Under Algorithm 6.1 and Algorithm 6.2, input and state constraints are robustly satisfied in all simulations.

Table 6.1: Comparison of Algorithm 6.1, Algorithm 6.2 and Algorithm 5.4

Algorithm Name	Algorithm 6.1	Algorithm 6.2	Algorithm 5.4
$\Theta_{10}/\%$	67.42	67.44	68.89
$\Theta_{100}/\%$	14.63	14.94	21.27
$\Theta_{200}/\%$	7.94	8.02	14.38
$\Theta_{500}/\%$	2.69	2.79	12.59
$\Theta_{1000}/\%$	0.69	0.69	11.24
$\mathbb{E}(\epsilon_t)$	9.845×10^{-6}	9.690×10^{-6}	1.705×10^{-9}

Table 6.1 shows the average volume of the estimated parameter set over 30 realisations of **s** and **w** sequences using Algorithm 6.1, Algorithm 6.2 and Algorithm 5.4 respectively. For $t < 500$, the estimated parameter sets using Algorithm 6.1 (with the PE conditions) converge faster than those using Algorithm 6.2 (without the PE condition). For $t > 500$, the difference in the estimated volume using Algorithm 6.1 and Algorithm 6.2 becomes negligible. This shows the additional PE conditions are able to inject some extra excitation in addition to the disturbance w_t and noise s_t and speed up the parameter identification process. In addition,

both Algorithm 6.1 and Algorithm 6.2 have a faster convergence rate than that of Algorithm 5.4. This is because the injected noise s_t is unknown to the controller in the two algorithms and, therefore, provides more excitation into the system.

From the table, we can also see Algorithm 6.1 has a slightly higher closed-loop PE coefficient ϵ_t than Algorithm 6.2, with a 1.6% difference. On the other hand, the closed-loop PE coefficient ϵ_t for Algorithm 5.4 is much lower, which means w_t alone cannot effectively excite the system. The large parameter set volume Θ_{1000} using Algorithm 5.4 also corresponds to the small ϵ value.

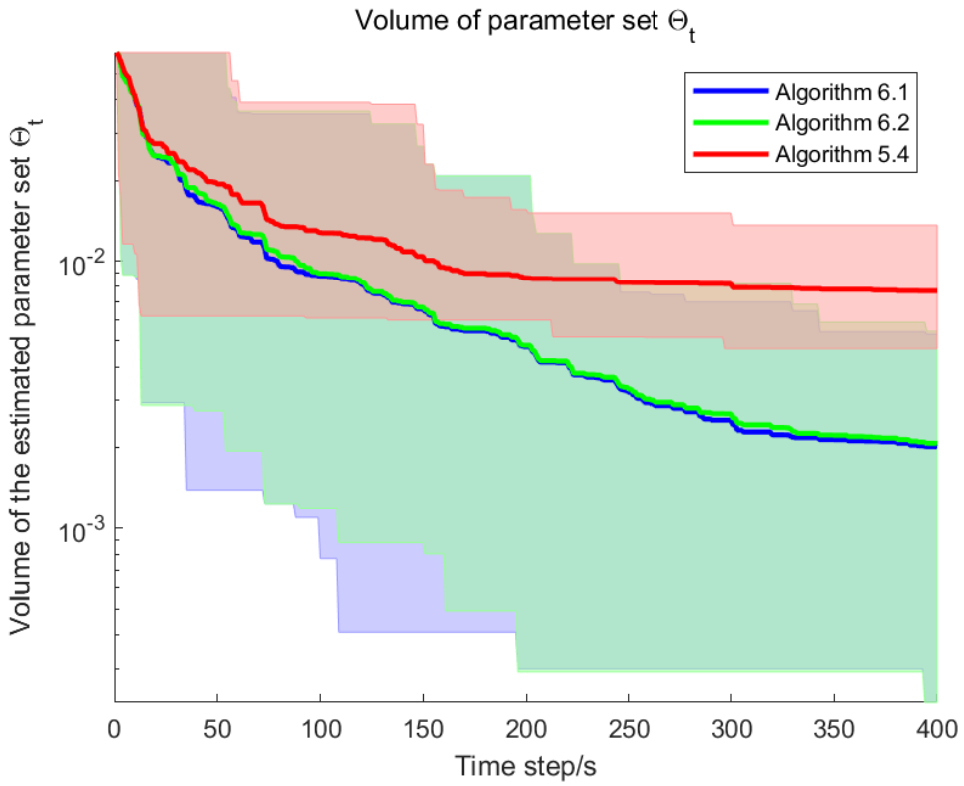


Figure 6.1: The volume of estimated parameter sets in logarithmic scale using Algorithm 6.1, Algorithm 6.2 and Algorithm 5.4

Figure 6.1 compares the volume of the estimated parameter sets in logarithmic scale using Algorithm 6.1, Algorithm 6.2 and Algorithm 5.4. The shaded areas represent observed ranges of parameter set volume for 30 ensemble disturbance sequences and the solid lines represent the mean volume over 30 ensemble disturbance sequences. The solid lines show that the mean volume using Algorithm 6.1 converges faster than

that using Algorithm 6.2. Compared with those of Algorithm 5.4, the estimated parameter sets using Algorithm 6.1 and Algorithm 6.2 shrink faster and continue to shrink even when the estimated parameter set using Algorithm 5.4 has stagnated. The shaded areas also show that the range of volume values that estimated sets can take using 6.1 is lower than that using Algorithm 6.2.

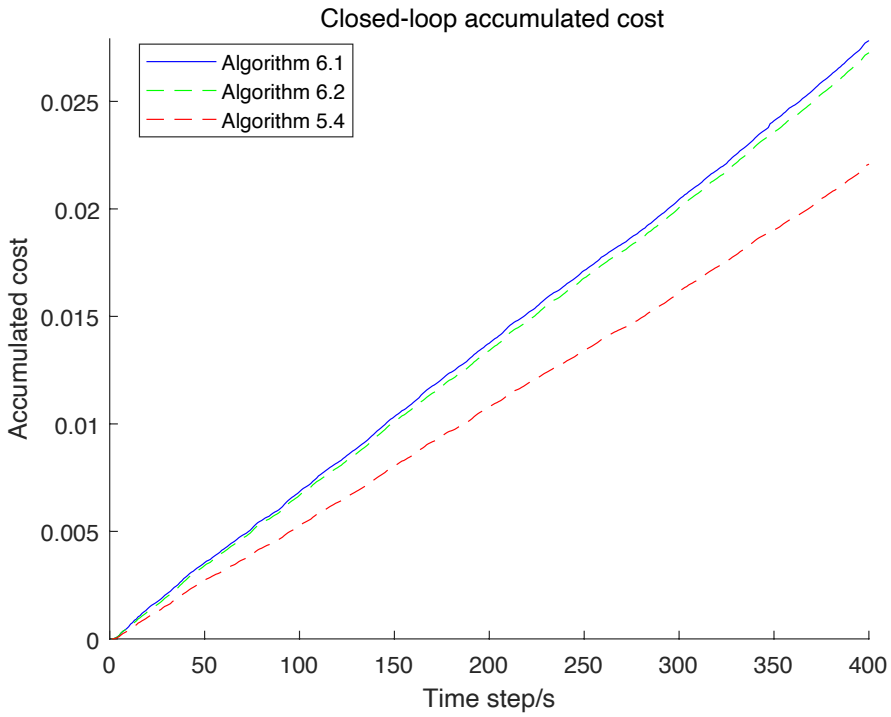


Figure 6.2: The closed-loop cost of Algorithm 6.1, Algorithm 6.2 and Algorithm 5.4

Figure 6.2 plots the accumulated cost given by $V_{t_i} = \sum_{t=1}^{t_i} \|x_t\|_Q + \|u_t\|_R$ using Algorithm 6.1, Algorithm 6.2 and Algorithm 5.4 respectively. The average cost after 400 time steps using Algorithm 6.1 is about 1.8% higher than that using Algorithm 6.2. Also, the averaged costs for Algorithm 6.1 and 6.2 are higher than that of Algorithm 5.4 because the injected noise is unknown and therefore it is not rejected by the controller in order to reduce the cost, as is the case for Algorithm 5.4. Combining with results from Figure 6.1, the higher closed-loop is associated with faster parameter convergence rates. In other words, there is always a compromise between parameter identification and performance.

6.6 Conclusion

This chapter proposes an adaptive robust MPC algorithm where the PE conditions are imposed on the nominal state tubes. We drive a lower bound on the expected value of the resulting PE coefficient based on both the additive disturbances in the state and injected noise in the control input. In addition, the proposed algorithm has a lower computational load than the algorithms considered in Chapter 5 because it requires fewer PE conditions and avoids nonconvex condition checking. The proposed algorithm achieves robust constraint satisfaction, recursive feasibility and input-to-state stability (ISS). Sufficient conditions for the resulting controller to be persistently exciting are also derived.

7

Adaptive robust MPC for nonlinear systems

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7.1 Introduction

In this chapter, we consider nonlinear systems with affine parameter uncertainty and additive disturbances. We propose a tube MPC formulation for nonlinear systems and combine it with a set-based parameter estimation method. The proposed algorithm guarantees robust constraint satisfaction, recursive feasibility and input-to-state stability.

To facilitate system identification, we include persistence of excitation in two forms. The first approach convexifies the PE condition around a reference trajectory and augments the cost function with the PE term. The weighting coefficient of the augmented term adjusts the relative importance of two control objectives, namely, parameter identification and tracking performance. However, stability of the proposed algorithm can only be proved by setting the coefficient of the augmented term to be zero, similar to the case of linear systems considered in Section 4.3.

The second approach incorporates recursively feasible PE conditions by using multiple PE windows. However, unlike the linear case, closed-loop satisfaction of the PE condition cannot be guaranteed, but only promoted. This chapter ends with a Continuous Stirred Tank Reactor (CSTR) example that compares the performance of three control strategies. The simulation results show that PE conditions are effective in increasing the convergence rate of the estimated parameter sets, at the expense of an increase in closed-loop cost.

7.2 Problem formulation and preliminaries

Consider a nonlinear system with unknown additive disturbance

$$x_{t+1} = f(x_t, u_t, \theta^*) + w_t \tag{7.1}$$

where $x_t \in \mathbb{R}^{n_x}$ is the system state, $u_t \in \mathbb{R}^{n_u}$ is the control input, $w_t \in \mathbb{R}^{n_x}$ is the unknown disturbance input. The model is defined in discrete time and t is the

discrete time index. The system function $f(x_t, u_t, \theta^*)$ depends on an unknown but constant parameter $\theta^* \in \mathbb{R}^p$. The disturbance sequence w_t follows Assumption 3.1.

State and control inputs are subject to constraints,

$$x_t \in \mathbb{X}, \quad u_t \in \mathbb{U} \quad \forall t \in \mathbb{N}_{\geq 0}. \tag{7.2}$$

Assumption 7.1 (State and input constraints)**.** *The set*

$$\mathcal{Z} = \{(x, u) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} : x \in \mathbb{X}, u \in \mathbb{U}\}$$

is compact and contains the origin in its interior.

Assumption 7.2 (Model continuity)**.** *The function $f(x, u, \theta^*)$ is uniformly continuous in (x, u) for all $(x, u) \in \mathcal{Z}$.*

Assumption 7.3 (Parameter uncertainty)**.** *The function $f(x, u, \theta)$ has affine dependence on the parameter vector $\theta \in \mathbb{R}^p$*

$$f(x, u, \theta) = \sum_{i=1}^{p} f_i(x, u)\theta_i + f_0(x, u) \tag{7.3}$$

for known functions $f_i(x, u)$, and $f_i(x, u)$ which are uniformly continuous in x and u for every $i \in \mathbb{N}_{[0,p]}$. The true parameter vector θ^* lies in a known, bounded, convex polytope Θ_0 given by

$$\Theta_0 = \{\theta : M_\Theta \theta \leq \mu_0\}.$$

To obtain finite numbers of decision variables in the MPC optimization problem, a dual-mode parameterised control law is used, where the decision variables are the sequence of the parameters of the control law. Denote the sequence of control input perturbations as $\mathbf{v}_t = \{v_{0|t}, v_{1|t}, \dots v_{N-1|t}\}$, we have

$$u_{k|t} = \begin{cases} Kx_{k|t} + v_{k|t} & k = 0, \dots, N-1 \\ Kx_{k|t} & k \geq N \end{cases} \tag{7.4}$$

and the first element of the optimal predicted control input sequence is used to define the control law at time t.

Assumption 7.4 (Terminal set)**.** *There exists a terminal control gain K such that the terminal set $\mathcal{X}_T \subseteq \mathbb{X}$, $K\mathcal{X}_T \subseteq \mathbb{U} \ominus \mathcal{S}$ and $\mathcal{X}_T$ is robustly positive invariant (RPI), i.e.*

$$f(x, Kx, \theta) \oplus \mathcal{W} \subseteq \mathcal{X}_T \qquad \forall x \in \mathcal{X}_T, \theta \in \Theta. \tag{7.5}$$

Remark 7.1. *Since the terminal controller gain K only needs to stabilise the nonlinear system around the equilibrium point, we can approximate the nonlinear system in the neighbourhood of the equilibrium point using a set of linear systems, and compute a robust terminal controller for the linearised systems with uncertainties.*

Remark 7.2. *As the parameter uncertainty Θ_t reduces, the terminal controller K could be updated online to increase the size of the terminal region $\mathcal{X}_T$, i.e. $K(\Theta_t)$ and $\mathcal{X}_T(\Theta_t)$ can be Θ_t dependent. However, the linearisation of the nonlinear dynamics and online computation of $K(\Theta_t)$ and $\mathcal{X}_T(\Theta_t)$ are computationally heavy.*

7.3 Adaptive robust MPC for nonlinear systems

7.3.1 Set-based parameter estimation

Define $\Phi(x, u) \in \mathbb{R}^{n_x \times p}$ as

$$\Phi(x, u) = [f_1(x, u) \cdots f_p(x, u)] \tag{7.6}$$

and $\phi_t \in \mathbb{R}^{n_x}$ as

$$\phi_t = f_0(x_{t-1}, u_{t-1}) - x_t \tag{7.7}$$

At time t we use the observation x_t to determine a set Δ_t of unfalsified model parameters. The set Δ_t is then combined with the parameter set estimate Θ_{t-1} to construct a new parameter estimate Θ_t.

$$\begin{aligned}\Delta_t &= \{\theta : x_t - \sum_{i=1}^{p} f(x_{t-1}, u_{t-1})\theta_i - f_0(x_{t-1}, u_{t-1}) \in \mathcal{W}\} \\ &= \{\theta : -\Pi_w \Phi(x_{t-1}, u_{t-1})\theta \leq \pi_w + \Pi_w \phi_t\} = \{\theta : P_t\theta \leq q_t\}\end{aligned}$$

The rest of the set-membership identification method follows that in Section 4.2.1.

7.3.2 Tube MPC formulation

Due to the presence of additive disturbance and unknown system parameters, the exact value of states in the future cannot be predicted. Instead, we use nominal predictions to define a cost function, and at the same time, we ensure the constraints are satisfied for all possible realizations of the uncertainties.

To ensure that the predicted state and control sequences satisfy the operating constraints (7.2) robustly for the given uncertainty bounds, we construct a tube (sequence of sets) $\mathcal{X}_{0|t}, \mathcal{X}_{1|t}, \ldots$ satisfying, for all $x_{k|t} \in \mathcal{X}_{k|t}$, $w_t \in \mathcal{W}$, $\theta \in \Theta_t$,

$$f(x_{k|t}, u_{k|t}, \theta) + w_t \in \mathcal{X}_{k+1|t} \quad \forall k \in \mathbb{N}_{\geq 0}. \tag{7.8}$$

The initial and terminal conditions are given by

$$\mathcal{X}_{0|t} = \{x_{0|t}\} \tag{7.9}$$

$$\mathcal{X}_{N|t} \subseteq \mathcal{X}_T \tag{7.10}$$

where $\mathcal{X}_T$ is a terminal set satisfying Assumption 7.4.

Here we consider a nominal cost defined for a given nominal parameter $\bar{\theta}_t \in \Theta_t$ in the following form, where the nominal parameter $\bar{\theta}_t$ is time-varying,

$$J(x, \mathbf{v}, \bar{\theta}_t) = \sum_{k=0}^{N-1} L(\bar{x}_{k|t}, \bar{u}_{k|t}) + V_f(\bar{x}_{N|t}, \Theta_t)$$

where $L(\cdot, \cdot)$ is the stage cost and $V_f(\cdot, \cdot)$ is the terminal cost. The nominal predicted state $\bar{x}_k$ and $\bar{u}_{k|t}$ is given by

$$\bar{x}_{0|t} = x_t, \tag{7.11a}$$

$$\bar{u}_{k|t} = K\bar{x}_{k|t} + v_{k|t}, \tag{7.11b}$$

$$\bar{x}_{k+1|t} = f(\bar{x}_{k|t}, \bar{u}_{k|t}, \bar{\theta}_t). \tag{7.11c}$$

for all $t \in \mathbb{N}_{[0,N-1]}$.

Assumption 7.5 (Stage cost)**.** *The stage cost $L(x, u)$ is such that $L(0, 0) = 0$ and there exists a $\mathcal{K}_\infty$ function $\varsigma_L(\cdot)$ such that $L(x, u) \geq \varsigma_L(|x|)$ for all $(x, u) \in \mathcal{Z}$. In addition, the stage cost $L(x, u)$ is uniformly continuous in x and u for all $(x, u) \in \mathcal{Z}$.*

Remark 7.3. *A common choice of cost is quadratic function of nominal state and input, i.e.* $L(\bar{x}, \bar{u}) = \|\bar{x}\|_Q^2 + \|\bar{u}\|_R^2$ *where* $Q \succ 0$ *and* $R \succ 0$.

Assumption 7.6 (Terminal cost)**.** *Given that the terminal control gain* K *and the terminal set* $\mathcal{X}_T$ *satisfy Assumption 7.4, there exists a terminal cost* $V_f(x, \theta)$ *such that*

$$V_f\Big(f(x, Kx, \theta), \theta\Big) - V_f(x, \theta) \leq -L(x, Kx)$$

hold for any $x \in \mathcal{X}_T$, $\theta \in \Theta$. *The function* $V_f(x, \theta)$ *is uniformly continuous in* x *and* θ. *In addition,* $V_f(x, \theta)$ *is a control Lyapunov function for (7.1) such that for all* $x \in \mathcal{X}_T$, *there exist two* $\mathcal{K}_\infty$ *functions* $\varsigma_{V_f,a}(|x|)$ *and* $\varsigma_{V_f,b}(|x|)$ *satisfying* $\varsigma_{V_f,a} \leq V_f(x) \leq \varsigma_{V_f,b}$.

Remark 7.4. *For linear terminal gain* K *and quadratic cost (i.e.* $L(x, u) = \|x\|_Q^2 + \|u\|_R^2$*), the terminal cost function can be chosen for example as the associated quadratic Lyapunov function where* $V_f(x) = \|x_{N|t}\|^2_{P(\bar{\theta}_t)}$ *and* $P(\bar{\theta}_t)$ *is the solution of the Lyapunov matrix equation (4.11) of the linearised system around the equilibrium point.*

Algorithm 7.1 (Adaptive robust MPC for nonlinear systems)**.**

Offline:

1. *Choose a suitable tube representation* $\mathcal{X}$
2. *Compute the terminal set* $\mathcal{X}_T$ *and feedback gain* K *that satisfy Assumption 7.4.*
3. *Obtain a nominal parameter vector* $\bar{\theta}_0 \in \Theta_0$.
4. *Choose suitable stage cost and terminal cost functions* $L(x, u)$ *and* $V(x)$ *that satisfy Assumptions 7.5 and 7.6.*

Online:

For $t = 0, 1, 2, \ldots$

1. *Obtain the current state* x_t *and set* $x = x_t$.
2. *Update* Θ_t *using (4.1) and the nominal parameter vector* $\bar{\theta}_t$ *using (4.13).*

3. Compute $\mathbf{v}_t^* = \{v_{0|t}^*, \ldots, v_{N-1|t}^*\}$ *by solving the nonlinear program (NLP):*

$$\mathcal{P} : \min_{\mathbf{v}_t} J(x, \mathbf{v}, \bar{\theta}_t) = \sum_{k=0}^{N-1} L(\bar{x}_{k|t}, \bar{u}_{k|t}) + V_f(\bar{x}_{N|t})$$
$$\text{subject to (7.2), (7.8), (7.9), (7.10), (7.11)} \tag{7.12}$$

4. Implement the first element of the optimal control sequence $u_t = Kx_t + v_{0|t}^*$.

Remark 7.5. *Algorithm 7.1 is generally computationally challenging because constraints (7.8) could be nonlinear and nonconvex. Several approaches can be used in order to tackle this issue, including sequential quadratic programming (SQP), approximating the cost and constraints by Affine Difference Inclusion (ADI) and using simple tube formulations such as cubes.*

7.3.3 Recursive feasibility

Proposition 7.6 (Recursive feasibility)**.** *The online MPC optimisation in Algorithm 7.1 is feasible at all times* $t \geq 1$ *if (7.12) is feasible at* $t = 0$ *and* $\Theta_t \subseteq \Theta_{t-1}$ *for all* t.

Proof. For any $t > 0$, suppose that Algorithm 7.1 is feasible at time $t-1$. Then the sequence of suboptimal decision variables, $\hat{\mathbf{v}}_t = \{v_{1|t-1}^*, \ldots, v_{N-1|t-1}^*, 0\}$,

- feasible at time t for (7.2), (7.8) at prediction time-steps $k \in \{0, 1, \ldots, N-2\}$ due to feasibility of (7.2), (7.8) at $t-1$ and $\Theta_t \subseteq \Theta_{t-1}$;
- feasible at time t for (7.2), (7.8) at prediction time $k = N-1$ due to feasibility of the terminal constraint (7.10) at $t-1$ and $\Theta_t \subseteq \Theta_{t-1}$;
- feasible at time t for (7.9) since $x_t \in \mathcal{X}_{1|t-1}$;
- feasible at time t for (7.10) by Assumption 7.4;
- feasible at time t for (7.11) for some value of $\bar{x}_{k|t}$ and $\bar{u}_{k|t}$;

It follows that $\hat{\mathbf{v}}_t$ is a feasible solution of (7.12) at time t. □

7.3.4 Input to State Stability

In this section, we prove that the closed-loop system (7.1) under Algorithm 7.1 is input-to-state stable (ISS). The ISS stability definition is stated in Lemma 4.7 and the proof is analogous to Theorem 4.8 and Corollary 4.9 in Chapter 4.

We define $X_{\mathcal{P}}$ as the set of states x such that problem $\mathcal{P}$ is feasible and assume that $X_{\mathcal{P}}$ is non-empty. In addition, for a given state x, nominal parameter vector $\bar{\theta}$ and parameter set Θ, we denote $\mathbf{u}^*(x, \bar{\theta}, \Theta)$ as the optimal solution of problem $\mathcal{P}$, and let $V^*(x, \bar{\theta}, \Theta)$ be the corresponding optimal value of the cost, so that $V^*(x, \bar{\theta}, \Theta) = J(x, \mathbf{v}^*(x, \bar{\theta}, \Theta), \bar{\theta})$.

We define the function $g(x, u, w, \theta)$ as

$$g(x, u, w, \theta) = \sum_{i}^{p} f_i(x, u)\theta_i + f_0(x, u) + w$$

and therefore the actual state at next time step x^+ is

$$x^+ = g(x, u, \theta^*, w).$$

Theorem 7.7. *Assume $\bar{\theta}_t$ is updated each time $t \in \mathbb{N}_{\geq 0}$ using (4.13). Then for all initial conditions $x_0 \in X_{\mathcal{P}}$, the system (7.1) with the control law of Algorithm 7.1, where $v^*_{0|t}$ is the first element of $\mathbf{v}^*$, robustly satisfies the constraint (7.2) and is ISS with region of attraction $X_{\mathcal{P}}$.*

Proof. First, we show that condition (i) of Lemma 4.7 is satisfied with $\mathcal{R} = X_{\mathcal{P}}$. If $\mathcal{P}$ is feasible at $t = 0$, then (7.10) and Assumption 7.4 implies that $u = Kx$ provides a feasible solution for $\mathcal{P}$ for all $x_0 \in \mathcal{X}_N$ and hence $\mathcal{X}_N \subseteq X_{\mathcal{P}}$. In addition, the robust invariance of $\mathcal{X}_N$ implies by (7.5) ensures that $\mathcal{W} \subseteq \mathcal{X}_N$, and since $0 \in \text{int}(\mathcal{W})$ due to Assumption 3.1, $X_{\mathcal{P}}$ must contain the origin in its interior. Furthermore, Proposition 7.6 shows that if $\mathcal{P}$ is initially feasible, then it is feasible for all $t \geq 0$, so that $X_{\mathcal{P}}$ is positively invariant for (4.20). Finally, $X_{\mathcal{P}}$ is necessarily compact by Assumption 7.1, and it follows that condition (i) of Lemma 4.7 is satisfied if $\mathcal{R} = X_{\mathcal{P}}$.

We next consider the bounds (4.21) in condition (ii) of Lemma 4.7. Given Assumptions 7.5 and 7.6, there exist $\mathcal{K}_\infty$-functions ς_1, ς_2 such that (4.21) holds with

$$\mathcal{V}(x,t) = V^*(x,\bar{\theta}_t,\Theta_t).$$

To show that the bound (4.22) in condition (ii) of Lemma 4.7 holds, let $\mathcal{F}_\mathcal{P}$ denote the set $\{u : (x,u) \in \mathcal{Z},\ x \in X_\mathcal{P}\}$. Then, given that the control u is linear in state x and input v, $g(x,u,\theta,w)$, $L(x,u)$ and V_f are uniformly continuous and $\mathcal{W}, \Theta_0, X_\mathcal{P}, \mathcal{F}_\mathcal{P}$ are compacts sets by Assumptions 7.2, 7.3, 7.5 and 7.6, there exist $\mathcal{K}_\infty$ functions σ_x, σ_w, σ_θ, σ_L, $\sigma_{V_{f_x}}$, $\sigma_{V_{f_\theta}}$ such that, $\forall x, x_1, x_2 \in X_\mathcal{P}$, $\forall u \in \mathcal{F}_\mathcal{P}$, $\forall w, w_1, w_2 \in \mathcal{W}$, $\forall \theta, \theta_1, \theta_2 \in \Theta_0$,

$$\begin{aligned}
\|g(x_1,u,\theta,w) - g(x_2,u,\theta,w)\| &\le \sigma_x(\|x_1 - x_2\|),\\
\|g(x,u,\theta,w_1) - g(x,u,\theta,w_2)\| &\le \sigma_w(\|w_1 - w_2\|),\\
\|g(x,u,\theta_1,w) - g(x,u,\theta_2,w)\| &\le \sigma_\theta(\|\theta_1 - \theta_2\|),\\
|L(x_1,u) - L(x_2,u)| &\le \sigma_L(\|x_1 - x_2\|),\\
|V_f(x_1,\theta) - V_f(x_2,\theta)| &\le \sigma_{V_{f_x}}(\|x_1 - x_2\|).\\
|V_f(x,\theta_1) - V_f(x,\theta_2)| &\le \sigma_{V_{f_\theta}}(\|\theta_1 - \theta_2\|).
\end{aligned}$$

Let $\bar{\mathbf{x}}^* = \{\bar{x}_0^*, \dots, \bar{x}_N^*\}$ be the sequence defined in terms of the optimal solution $\mathbf{v}^*(x,\bar{\theta}_t,\Theta_t) = \{v_0^*, \dots, v_{N-1}^*\}$ by

$$\bar{x}_{k+1}^* = g(\bar{x}_k^*, K\bar{x}_k^* + v_k^*, \bar{\theta}_t, 0), \quad \bar{x}_0^* = x,$$

and define $\mathbf{z} = \{z_0, \dots, z_N\}$ as the sequence

$$z_{k+1} = g(z_k, Kz_k + \hat{v}_k, \bar{\theta}_{t+1}, 0), \quad z_0 = x^+$$

where $\hat{\mathbf{v}} = \{\hat{v}_0, \dots, \hat{v}_{N-1}\}$ has $\hat{v}_k = v_{k+1}^*$ for $k \in \mathbb{N}_{[0,N-2]}$ and $\hat{v}_{N-1} = 0$. Then

$$\begin{aligned}
&J(x^+,\hat{\mathbf{v}},\bar{\theta}_{t+1}) - J(x,\mathbf{v}^*,\bar{\theta}_t)\\
&= -L(x,u_0^*) + \sum_{k=0}^{N-2}\Big(L(z_k, Kz_k + \hat{v}_k) - L(\bar{x}_{k+1}^*, K\bar{x}_{k+1}^* + v_{k+1}^*)\Big) + L(z_{N-1}, Kz_{N-1})\\
&\quad + V_f(z_N,\bar{\theta}_{t+1}) - V_f(z_{N-1},\bar{\theta}_{t+1}) + V_f(z_{N-1},\bar{\theta}_{t+1}) - V_f(\bar{x}_N^*,\bar{\theta}_t).
\end{aligned} \tag{7.13}$$

The update law (4.13) ensures that $\|\bar{\theta}_{t+1} - \bar{\theta}_t\| \leq \|\bar{\theta}_t - \theta^*\|$ since $\theta^* \in \Theta_{t+1}$. Hence, for all $k \in \mathbb{N}_{[1,N-1]}$, we have

$$\begin{aligned}\|z_k - \bar{x}^*_{k+1}\| &\leq \sigma_x(\|z_{k-1} - \bar{x}^*_k\|) + \sigma_\theta(\|\bar{\theta}_{t+1} - \bar{\theta}_t\|) \\ &\leq \sigma_x(\|z_{k-1} - \bar{x}^*_k\|) + \sigma_\theta(\|\bar{\theta}_t - \theta^*\|)\end{aligned}$$

and it follows that, for all $k \in \mathbb{N}_{[1,N-1]}$,

$$\|z_k - \bar{x}^*_{k+1}\| \leq \tfrac{1}{2}(2\sigma_x)^k(\|x^+ - \bar{x}^*_1\|) + \sum_{j=0}^{k-1} \tfrac{1}{2}(2\sigma_x)^j \circ 2\sigma_\theta(\|\theta_t - \theta^*\|)$$

where $\|x^+ - \bar{x}^*_1\| \leq \sigma_\theta(\|\bar{\theta}_t - \theta^*\|) + \sigma_w(\|w_0\|)$. In addition we have, for all $k \in \mathbb{N}_{[0,N-2]}$,

$$|L(z_k, \hat{v}_k) - L(\bar{x}^*_{k+1}, v^*_{k+1})| \leq \sigma_L(\|z_k - \bar{x}^*_{k+1}\|).$$

Given Assumption 7.6, we have

$$\begin{aligned}V_f(z_{N-1}, \bar{\theta}_{t+1}) - V_f(\bar{x}^*_N, \bar{\theta}_t) &\leq \sigma_{V_{f_x}}(\|z_{N-1} - \bar{x}^*_N\|) + \sigma_{V_{f_\theta}}(\|\bar{\theta}_t - \theta^*\|) \\ V_f\big(z_N, \bar{\theta}_{t+1}\big) - V_f(z_{N-1}, \bar{\theta}_{t+1}) &\leq -L(z_{N-1}, Kz_{N-1})\end{aligned}$$

and there exists a $\mathcal{K}_\infty$ function ς_3 such that $L(x, u^*_0) \geq \varsigma_3(\|x\|)$ by Assumption 7.5. Collecting the bounds derived above on individual terms in the expression for $J(x^+, \hat{\mathbf{v}}, \bar{\theta}_{t+1}) - J(x, \mathbf{v}^*, \bar{\theta}_t)$ in (7.13), we obtain

$$J(x^+, \hat{\mathbf{v}}, \bar{\theta}_{t+1}) - J(x, \mathbf{v}^*, \bar{\theta}_t) \leq -\varsigma_3(\|x\|) + \sigma_1(\|w\|) + \sigma_2(\|\bar{\theta}_t - \theta^*\|)$$

where σ_1, σ_2 are $\mathcal{K}$-functions. Given the definition of $\mathbf{v}^*$, we have $\mathcal{V}(x, t) = V^*(x, \bar{\theta}_t, \Theta_t) = J(x, \mathbf{v}^*, \bar{\theta}_t)$. In addition, by optimality we have $\mathcal{V}(x^+, t+1) = V^*(x^+, \bar{\theta}_{t+1}, \Theta_{t+1}) \leq J(x^+, \hat{\mathbf{v}}, \bar{\theta}_{t+1})$. Therefore (4.22) holds and hence all of the conditions of Lemma 4.7 are satisfied. □

7.4 Adaptive robust MPC with augmented cost

This section proposes a nonlinear adaptive robust MPC algorithm with persistent excitation (PE) conditions imposed through constraints on predicted states and control inputs and linearised PE coefficients in the cost function. By adjusting

the weight of the linearised PE coefficient term in the cost function, the algorithm can control the relative priority given to the satisfaction of PE condition and tracking performance.

7.4.1 The PE conditions and augmented cost function

The Persistent Excitation (PE) condition for the regressor Φ_t is as stated in (3.42), reproduced below:

$$\sum_{j=t-N_u+1}^{t} \Phi(x_j, u_j)^\top \Phi(x_j, u_j) \succeq \epsilon \mathbb{I} \tag{7.14}$$

Similar to Chapter 4, we define the PE condition over predicted trajectories as

$$\sum_{k=0}^{N_u-1} \Phi(x_{k|t}, u_{k|t})^\top \Phi(x_{k|t}, u_{k|t}) \succeq \beta \mathbb{I} \tag{7.15}$$

for all $x_{k|t} \in \mathcal{X}_{k|t}$. Expressing $\Phi(x, u)$ with its definition in (7.6), we have

$$\sum_{k=0}^{N_u-1} \begin{bmatrix} f_1(x_k, u_k)^\top \\ \vdots \\ f_p(x_k, u_k)^\top \end{bmatrix} \begin{bmatrix} f_1(x_k, u_k) & \cdots & f_p(x_k, u_k) \end{bmatrix} \succeq \beta \mathbb{I} \tag{7.16}$$

for all $x_{k|t} \in \mathcal{X}_{k|t}$. However, enforcing the constraint (7.16) for all state in the the predicted tube $\mathcal{X}$ is equivalent to a set of nonconvex constraints that are generally intractable.

To reduce computational complexity, we convexify the regressor $\Phi(x)$ using the Taylor series expansion around a reference trajectory $(\hat{x}, \hat{u})$,

$$\sum_{k=0}^{N_u-1} \begin{bmatrix} \bar{f}_1(x_k, u_k)^\top \\ \vdots \\ \bar{f}_p(x_k, u_k)^\top \end{bmatrix} \begin{bmatrix} \bar{f}_1(x_k, u_k) & \cdots & \bar{f}_p(x_k, u_k) \end{bmatrix} \succeq \bar{\beta} \mathbb{I} \tag{7.17}$$

where $\bar{\beta}$ is an approximation of β in (7.16) and $\bar{f}_i$ is a first order approximation of function f_i by ignoring the higher order terms as follows

$$\bar{f}_i(x, u) = f_i(\hat{x}, \hat{u}) + \frac{\partial f_i(x, u)}{\partial x}\Big|_{(x,u)=(\hat{x},\hat{u})} (x - \hat{x}) + \frac{\partial f_i(x, u)}{\partial u}\Big|_{(x,u)=(\hat{x},\hat{u})} (u - \hat{u}) \tag{7.18}$$

We further simplify the condition by imposing the conditions over the vertices of the polytopic sets $\mathcal{X}_{k|t}$ defining the predicted tube.

$$\sum_{k=0}^{N_u-1} \begin{bmatrix} \bar{f}_1(x_k^{(j_k)}, u_k^{(j_k)})^\top \\ \vdots \\ \bar{f}_p(x_k^{(j_k)}, u_k^{(j_k)})^\top \end{bmatrix} \begin{bmatrix} \bar{f}_1(x_k^{(j_k)}, u_k^{(j_k)}) & \cdots & \bar{f}_p(x_k^{(j_k)}, u_k^{(j_k)}) \end{bmatrix} \succeq \beta' \mathbb{I} \tag{7.19}$$

for all $j_k = 1, \ldots, m$ where $\mathcal{X}_{k|t} = \text{co}\{x_k^{j_k}, j_k = 1, \ldots, m\}$.

However, in general the simplified PE condition (7.19) is only an approximation of the PE condition (7.15). Therefore, the relative sizes of β, $\bar{\beta}$ and β' depend on the choice of the reference trajectories $\hat{\mathbf{x}}$ and $\hat{\mathbf{u}}$ and properties of functions f_i.

In order to facilitate system parameter identification, we incentivize a large value of β' by defining the MPC objective function as follows

$$\sum_{k=0}^{N-1} L(\bar{x}_{k|t}, \bar{u}_{k|t}) + V_f(\bar{x}_{N|t}) - \gamma\beta' \tag{7.20}$$

where $\gamma \geq 0$ is a weight that controls the relative priority given to satisfaction of PE condition (7.19) and tracking performance.

7.4.2 The Proposed algorithm

Algorithm 7.2 (Adaptive robust MPC with augmented cost)**.**

Offline:

1. *Choose a suitable tube representation $\mathcal{X}$*
2. *Compute the terminal set $\mathcal{X}_T$ and feedback gain K that satisfy Assumption 7.4.*
3. *Obtain a nominal parameter vector $\bar{\theta}_0 \in \Theta_0$.*
4. *Choose suitable stage cost and terminal cost functions $L(x, u)$ and $V(x)$ that satisfy Assumptions 7.5 and 7.6.*

Online:

For $t = 0, 1, 2, \ldots$

1. *Obtain the current state x_t and set $x = x_t$.*

2. *Update Θ_t using (4.1) and the nominal parameter vector $\bar{\theta}_t$ using (4.13).*

3. *At $t = 0$, compute the initial reference state and input sequences $\hat{\mathbf{x}}$ and $\hat{\mathbf{u}}$, for example by solving the nominal problem described in Remark 7.8.*
   At $t > 0$, compute the reference state and control sequences $\hat{\mathbf{x}}$ and $\hat{\mathbf{u}}$, using the optimal solution $\mathbf{v}^*_{t-1} = \{v^*_{0|t-1}, \dots, v^*_{0|t-1}\}$ and time $t-1$, and

$$\hat{x}_{0|t} = x_t, \tag{7.21a}$$
$$\hat{v}_{k|t} = v^*_{k+1|t-1}, \tag{7.21b}$$
$$\hat{u}_{k|t} = K\hat{x}_{k|t} + \hat{v}_{k|t}, \tag{7.21c}$$
$$\hat{x}_{k+1|t} = f(\hat{x}_{k|t}, \hat{u}_{k|t}, \bar{\theta}_t). \tag{7.21d}$$

4. *Compute $\mathbf{v}^*_t = \{v^*_{0|t}, \dots, v^*_{N-1|t}\}$ and β' by solving the nonlinear program (NLP):*

$$\begin{aligned} \mathcal{P} : \min_{\mathbf{v}_t, \beta'} \; & J(x, \mathbf{v}, \bar{\theta}_t, \beta') = \sum_{k=0}^{N-1} L(\bar{x}_{k|t}, \bar{u}_{k|t}) + V_f(\bar{x}_{N|t}) - \gamma\beta' \\ & \textit{subject to (7.2), (7.8), (7.9), (7.10), (7.11), (7.19)} \end{aligned} \tag{7.22}$$

5. *Implement the optimal control input $u^*_t = Kx_t + v^*_{0|t}$.*

Remark 7.8. *At $t = 0$, the reference state $\hat{\mathbf{x}} = \{\hat{x}_0, \dots, \hat{x}_N\}$ and $\hat{\mathbf{u}} = \{\hat{u}_0, \dots, \hat{u}_{N-1}\}$ can be computed using*

$$\begin{aligned} & \min_{\hat{\mathbf{u}}_t} \sum_{k=0}^{N-1} L(\hat{x}_{k|0}, \hat{u}_{k|0}) + V_f(\hat{x}_{N|0}) \\ & \textit{subject to} \;\; \hat{x}_{0|0} = x_0, \;\; \hat{x}_N \in \mathcal{X}_T \\ & \qquad \hat{x}_{k+1|0} = f(\hat{x}_{k|0}, \;\; \hat{u}_{k|0}, \bar{\theta}_0), \;\; \hat{x}_{k|0} \in \mathcal{X}, \hat{u}_{k|0} \in \mathcal{U} \;\; \forall k \in \mathbb{N}_{[0,N-1]}. \end{aligned}$$

Remark 7.9. *Although Algorithm 7.2 is generally similar to Algorithm 4.1 in Chapter 4.1, there are differences between the PE conditions included. Firstly, the linearised PE condition (4.15) is linear in state x and input u, but the simplified PE condition (7.19) is still quadratic in state x and input u. Secondly, the linearised PE condition (4.16) in Algorithm 4.1 is a sufficient condition for the PE condition (4.15) if $\beta' \geq 0$, because the omitted term is a positive semidefinite matrix. However, the simplified PE condition (7.19) is an approximation of (7.15) by omitting the*

higher order terms. In other words, satisfaction of (7.19) and $\beta' \geq 0$ does not guarantee the satisfaction of (7.15).

Remark 7.10. *Incorporating (7.19) does not ensure the closed-loop satisfaction of a PE condition, although its effect on convergence rate of the estimated parameter set is significant, as shown in Section 7.6.*

Remark 7.11. *Algorithm 7.2 is input-to-state stable by setting the coefficient $\gamma = 0$. The proof is identical to that of Theorem 7.7.*

7.4.3 Recursive feasibility

Proposition 7.12 (Recursive feasibility)**.** *The online MPC optimisation in Algorithm 7.2 is feasible at all times $t \geq 1$ if (7.22) is feasible at $t = 0$ and $\Theta_t \subseteq \Theta_{t-1}$ for all t.*

Proof. For any $t > 0$, suppose that Algorithm 7.2 is feasible at time $t-1$. Then the sequence of suboptimal decision variables, $\hat{\mathbf{v}}_t = \{v^*_{1|t-1}, \ldots, v^*_{N-1|t-1}, 0\}$ and β', is

- feasible at time t for (7.2), (7.8) at prediction time-steps $k \in \{0, 1, \ldots, N-2\}$ due to feasibility of (7.2), (7.8) at $t-1$ and $\Theta_t \subseteq \Theta_{t-1}$;
- feasible at time t for (7.2), (7.8) at prediction time $k = N-1$ due to feasibility of the terminal constraint (7.10) at $t-1$ and $\Theta_t \subseteq \Theta_{t-1}$;
- feasible at time t for (7.9) since $x_t \in \mathcal{X}_{1|t-1}$;
- feasible at time t for (7.10) by Assumption 7.4;
- feasible at time t for (7.11) for some value of $\bar{x}_{k|t}$ and $\bar{u}_{k|t}$;
- feasible at time t for (7.19) for some scalar β'

It follows that $\hat{\mathbf{v}}_t$ is a feasible solution of (7.12) at time t. □

7.5 Adaptive robust MPC with recursively feasible PE conditions

This section proposes an adaptive robust MPC framework with recursively feasible PE conditions. Similar to Algorithm 5.3, we impose the PE conditions on the linearised PE coefficients over multiple PE windows. The algorithm achieves robust constraint satisfaction, recursive feasibility and input-to-state stability.

7.5.1 PE condition

Define the number of windows in the PE condition as N_{pe}. The number of PE windows initially increases with time, i.e.

$$N_{pe} = \begin{cases} t & \text{if } t < N + N_u - 1 \\ N + N_u - 1 & \text{if } t \geq N + N_u - 1 \end{cases} \tag{7.23}$$

The PE condition is thus, for all $\kappa \in \mathbb{N}_{[N-N_{pe},\dots,N-1]}$,

$$\sum_{k=\kappa}^{\kappa+N_u-1} \Phi(x_{k|t}, u_{k|t})^\top \Phi(x_{k|t}, u_{k|t}) \succeq \beta_{\kappa|t}\mathbb{I}, \quad \forall x_{k|t} \in \mathcal{X}_{k|t}, \tag{7.24}$$

where κ denotes the start of the PE windows and $\beta_{\kappa|t}$ is the corresponding predicted PE coefficient. With a slight abuse of notation, we can denote the past trajectories as $x_{k|t} = x_{k+t}$ and $u_{k|t} = u_{k+t}$ if $k \leq 0$.

Next, we define the reference trajectories. The reference state for a certain PE window is denoted as $\hat{\mathbf{x}}_t^\nu = \{\hat{x}^\nu_{\nu-t|t}, \dots, \hat{x}^\nu_{\nu-t+N_u-1|t}\}$ and the input sequence is $\hat{\mathbf{u}}_t^\nu = \{\hat{u}^\nu_{\nu-t|t}, \dots, \hat{u}^\nu_{\nu-t+N_u-1|t}\}$, where the superscript ν is the window index and $\nu \in \mathbb{N}_{[t-N_u+1,N+t]}$.

In this algorithm, the reference trajectory for the same PE window remains the same over the time, in other words,

$$\hat{\mathbf{x}}^\nu_{t+1} = \hat{\mathbf{x}}^\nu_t, \quad \hat{\mathbf{u}}^\nu_{t+1} = \hat{\mathbf{u}}^\nu_t. \tag{7.25}$$

The last window $(\hat{\mathbf{x}}_t^{N+t}, \hat{\mathbf{u}}_t^{N+t})$, is defined by the state x_t, parameter estimate $\bar{\theta}_t$ and the optimal solution $\mathbf{v}^*_{t-1} = \{v^*_{0|t-1}, v^*_{1|t-1}, \dots\}$ at $t-1$,

$$\hat{x}^{N+t}_{0|t} = x_t \tag{7.26a}$$

$$\hat{v}_{k|t}^{N+t} = v_{k+1|t-1}^{*} \tag{7.26b}$$

$$\hat{u}_{k|t}^{N+t} = K\hat{x}_{k|t} + \hat{v}_{k|t}^{N+t} \tag{7.26c}$$

$$\hat{x}_{k+1|t}^{N+t} = f(\hat{x}_{k|t}^{N+t}, \hat{u}_{k|t}^{N+t}, \bar{\theta}_t) \tag{7.26d}$$

Similar to Section 7.4.1, we simplify the PE conditions around the reference trajectories $\hat{\mathbf{x}}_t^{\nu}$ and $\hat{\mathbf{u}}_t^{\nu}$. For $\nu \in \mathbb{N}_{[t-N_u+1,t-1]}$, the simplified condition is

$$\begin{aligned}
&\sum_{k=\nu-t}^{-1} \begin{bmatrix} \bar{f}_1(x_{k+t}, u_{k+t})\Big|_{(\hat{x}_k^{\nu}, \hat{u}_k^{\nu})} \\ \vdots \\ \bar{f}_p(x_{k+t}, u_{k+t})\Big|_{(\hat{x}_k^{\nu}, \hat{u}_k^{\nu})} \end{bmatrix}^{\top} \begin{bmatrix} \bar{f}_1(x_{k+t}, u_{k+t})\Big|_{(\hat{x}_k^{\nu}, \hat{u}_k^{\nu})} & \cdots & \bar{f}_p(x_{k+t}, u_{k+t})\Big|_{(\hat{x}_k^{\nu}, \hat{u}_k^{\nu})} \end{bmatrix} \\
&+ \min_{x_k \in \mathcal{X}_k} \sum_{k=0}^{\nu-t+N_u-1} \begin{bmatrix} \bar{f}_1(x_k^{(j)}, u_k^{(j)})\Big|_{(\hat{x}_k^{\nu}, \hat{u}_k^{\nu})} \\ \vdots \\ \bar{f}_p(x_k^{(j)}, u_k^{(j)})\Big|_{(\hat{x}_k^{\nu}, \hat{u}_k^{\nu})} \end{bmatrix}^{\top} \begin{bmatrix} \bar{f}_1(x_k^{(j)}, u_k^{(j)})\Big|_{(\hat{x}_k^{\nu}, \hat{u}_k^{\nu})} & \cdots & \bar{f}_p(x_k^{(j)}, u_k^{(j)})\Big|_{(\hat{x}_k^{\nu}, \hat{u}_k^{\nu})} \end{bmatrix} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \succeq \beta'_{\nu-t|t}\mathbb{I}
\end{aligned} \tag{7.27}$$

for $j \in \mathbb{N}_{[1,m]}$. Function $\bar{f}_i$ is given by (7.18) and represents the linear approximation of f_i obtained from the Taylor's series expansion around the reference $(\hat{x}, \hat{u})$.

Note that, we cannot use the closed loop regressor $\Phi(x_{k+t}, u_{k+t})^{\top}\Phi(x_{k+t}, u_{k+t})$ in the first summation although the x_{k+t} and u_{k+t} are known because of the approximation error in the first order Taylor series expansion of f.

For $\nu \in \mathbb{N}_{[t,N+t-1]}$, the simplified condition is

$$\begin{aligned}
&\min_{x_k \in \mathcal{X}_k} \sum_{k=\nu-t}^{\nu-t+N_u-1} \begin{bmatrix} \bar{f}_1(x_k^{(j)}, u_k^{(j)})\Big|_{(\hat{x}_k^{\nu}, \hat{u}_k^{\nu})} \\ \vdots \\ \bar{f}_p(x_k^{(j)}, u_k^{(j)})\Big|_{(\hat{x}_k^{\nu}, \hat{u}_k^{\nu})} \end{bmatrix}^{\top} \begin{bmatrix} \bar{f}_1(x_k^{(j)}, u_k^{(j)})\Big|_{(\hat{x}_k^{\nu}, \hat{u}_k^{\nu})} & \cdots & \bar{f}_p(x_k^{(j)}, u_k^{(j)})\Big|_{(\hat{x}_k^{\nu}, \hat{u}_k^{\nu})} \end{bmatrix} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \succeq \beta'_{\nu-t|t}\mathbb{I}
\end{aligned} \tag{7.28}$$

for $j \in \mathbb{N}_{[1,m]}$.

The recursively feasible PE condition is imposed by direct comparison of β',

$$\beta'_{\nu-t|t} \geq \beta'_{\nu-t+1|t-1} \tag{7.29}$$

for $\nu \in [N - N_{pe} + t, N - 1 + t]$.

For the last window, which uses a terminal linear feedback law, the associated PE coefficient $\beta'_{N|t}$ can be calculated as

$$\beta'_{N|t} = \underline{\lambda}\left(\min_{x_k \in \mathcal{X}_k} \sum_{k=N}^{N+N_u-1} \begin{bmatrix} \bar{f}_1(x_k, u_k)\Big|_{(\hat{x}_k^{N+t}, \hat{u}_k^{N+t})} \\ \vdots \\ \bar{f}_p(x_k, u_k)\Big|_{(\hat{x}_k^{N+t}, \hat{u}_k^{N+t})} \end{bmatrix}^\top \begin{bmatrix} \bar{f}_1(x_k, u_k)\Big|_{(\hat{x}_k^{N+t}, \hat{u}_k^{N+t})} & \cdots \end{bmatrix} \right) \tag{7.30}$$

7.5.2 The proposed algorithm

Algorithm 7.3 (Adaptive robust MPC with recursively feasible PE conditions)**.**

Offline:

1. *Choose a suitable tube representation $\mathcal{X}$*
2. *Compute the terminal set $\mathcal{X}_T$ and feedback gain K that satisfy Assumption 7.4.*
3. *Obtain a nominal parameter vector $\bar{\theta}_0 \in \Theta_0$.*
4. *Choose suitable stage cost and terminal cost functions $L(x, u)$ and $V(x)$ that satisfy Assumptions 7.5 and 7.6.*
5. *Compute the control sequence $\mathbf{v}_0^* = \{v_{0|0}^*, \ldots, v_{N-1|0}^*\}$ for the optimisation without the PE condition by solving the nonlinear program (NLP):*

$$\begin{aligned} \mathcal{P}_{initialisation} : \min_{\mathbf{v}_t, \beta'} & J(x, \mathbf{v}, \bar{\theta}_t) = \sum_{k=0}^{N-1} L(\bar{x}_{k|t}, \bar{u}_{k|t}) + V_f(\bar{x}_{N|t}) \\ & \textit{subject to (7.2), (7.8), (7.9), (7.10), (7.11)} \end{aligned} \tag{7.31}$$

Online:

For $t = 1, 2, \ldots$

(a) Obtain the current state x_t and set $x = x_t$. Update the number of PE windows N_{pe} using (7.23).

(b) Update Θ_t using (4.1) and the nominal parameter vector $\bar{\theta}_t$ using (4.13).

(c) Compute the reference state and control sequences $\hat{\mathbf{x}}$, $\hat{\mathbf{u}}$ *and* $\beta'_{N|t-1}$, *using (7.25), (7.26) and (7.30)*

(d) Compute $\mathbf{v}_t^* = \{v_{0|t}^*, \ldots, v_{N-1|t}^*\}$ *and* $\boldsymbol{\beta}'_t = [\beta'_{N-N_{pe}|t}, \ldots, \beta'_{N-1|t}]$ *by solving the nonlinear program (NLP):*

$$\begin{aligned}\mathcal{P}: \min_{\mathbf{v}_t, \boldsymbol{\beta}'_t} & J(x, \mathbf{v}, \bar{\theta}_t) = \sum_{k=0}^{N-1} L(\bar{x}_{k|t}, \bar{u}_{k|t}) + V_f(\bar{x}_{N|t}) \\ & \textit{subject to (7.2), (7.8), (7.9), (7.10), (7.11), (7.27), (7.28), (7.29)}\end{aligned} \tag{7.32}$$

(e) Implement the optimal control input $u_t^* = Kx_t + v_{0|t}^*$.

7.5.3 Recursive feasibility and input-to-state stability

Proposition 7.13 (Recursive feasibility)**.** *The online MPC optimisation in Algorithm 7.3 is feasible at all times* $t \geq 1$ *if (7.32) is feasible at* $t = 1$ *and* $\Theta_t \subseteq \Theta_{t-1}$ *for all* t.

Proof. Let the suboptimal decision variables $\tilde{\mathbf{v}}_t$ and $\tilde{\boldsymbol{\beta}}'_t$ be defined in terms of the optimal solution $\mathbf{v}_{t-1}^*$ and $\boldsymbol{\beta}'^*_t$ of (7.32) at time $t-1$ as

$$\tilde{\mathbf{v}}_t = \{v_{1|t-1}^*, \ldots, v_{N-1|t-1}^*, 0\}, \tilde{\boldsymbol{\beta}}'_t = [\beta'_{N-N_{pe}+1|t-1}, \ldots, \beta'_{N-1|t-1}, \beta'_{N|t}]$$

where $\beta'_{N|t}$ is calculated in online step (c) using (7.30).

If the optimisation (7.32) is feasible at time $t-1$, then $(\tilde{\mathbf{v}}_t, \tilde{\boldsymbol{\beta}}'_t)$ is:

- feasible at time t for (7.2), (7.8) for $k \in \{0, 1, \ldots, N-2\}$ due to feasibility of (7.2), (7.8) at $t-1$ and $\Theta_t \subseteq \Theta_{t-1}$;
- feasible at time t for (7.2), (7.8) for $k = N-1$ due to feasibility of (7.10) at $t-1$ and $\Theta_t \subseteq \Theta_{t-1}$;
- feasible at time t for (7.9) since $x_{0|t+1} \in \mathcal{X}_{1|t}$;
- feasible at time t for (7.10) since (7.10) is by Assumption 7.4 feasible at $t-1$;
- feasible at time t for (7.27), (7.28) and (7.29) for $k \in \{0, 1, \ldots, N-2\}$ because $\hat{\boldsymbol{x}}_{t+1}^{\nu} = \hat{\boldsymbol{x}}_t^{\nu}$, $\hat{\boldsymbol{u}}_{t+1}^{\nu} = \hat{\boldsymbol{u}}_t^{\nu}$ and $\mathcal{X}_{k+1|t} \subseteq \mathcal{X}_{k|t+1}$ when the tail solution is used;

- feasible at time t for (7.28) and (7.29) for $k = N-1$ because of (7.30).

It follows that $(\tilde{\mathbf{v}}_t, \tilde{\boldsymbol{\beta}}'_t)$ is a feasible solution of (7.32) at time t. □

Proposition 7.14 (Stability)**.** *The closed-loop system (7.1) under Algorithm 7.3 is input-to-state stable (ISS).*

The proof is the analogous to that in Proposition 7.7.

7.6 Numerical examples

This section uses a Continuous Stirred Tank Reactor (CSTR) example where a first-order, irreversible exothermic reaction is regulated. The reaction can be described using the following dynamic model [11]

$$\begin{aligned}\dot{C}_A =& \frac{q}{V}(C_{Ain} - C_A) - k_o \exp\Big(\frac{-E}{RT_r}\Big)C_A + w_{1,t} \\ \dot{T}_r =& \frac{q}{V}(T_{in} - T_r) - \frac{\Delta H}{\rho c_p} k_0 \exp\Big(\frac{-E}{RT_r}\Big)C_A + \frac{UA}{\rho c_p V}(T_c - T_r) + w_{2,t}\end{aligned} \tag{7.33}$$

where $w_t \in \mathcal{W}$ is an unknown but bounded additive disturbance and $\mathcal{W} = \{(w_1, w_2) : |w_1| \leq 0.0001, |w_2| \leq 0.001\}$.

The system has an unstable equilibrium point where $C_A^{\text{eq}} = 0.5\text{mol/l}$, $T_r^{\text{eq}} = 350\text{K}$ and $T_c^{\text{eq}} = 350\text{K}$. The system has state constraint $0 \leq C_A \leq 1$ and $280 \leq T_r \leq 370$ and the input constraint $280 \leq T_c \leq 370$. The reaction kinetic constant k_0 and heat of reaction ΔH are only nominally known and parameterised as $k_0 = \theta_1 \times 10^{10}\text{min}^{-1}$ and $\Delta H k_0 = -\theta_2 \times 10^{15}\text{J/mol min}$ with the parameters satisfying $\Theta_0 = \{(\theta_1, \theta_2) : 6 \leq \theta_1 \leq 8,\ 3 \leq \theta_2 \leq 4\}$. The true parameter vector is $\theta^* = [7.2\ 3.6]^\top$.

The constants are taken as: $q = 100\text{l/min}$, $V = 100\text{l}$, $\rho = 1000\text{g/l}$, $c_p = 0.239\text{J/g K}$, $E/R = 8750\text{K}$, $UA = 5 \times 10^4\text{J/min K}$, $C_{Ain} = 1\text{mol/l}$ and $T_{in} = 350\text{K}$.

We derive a discrete time model using the Euler approximation to the state derivative and the resulting discretisation errors are included in the disturbance term w_t. The sample rate is chosen to be 1.2 seconds.

Define $[x_1, x_2]' = [\frac{C_A - C_A^{eq}}{0.5}, \frac{T_r - T_r^{eq}}{20}]'$ and $u = \frac{T_C - T_C^{eq}}{20}$. The system regressor $\Phi = [f_1 \ f_2]$ is therefore

$$f_1 = \begin{bmatrix} \Delta t \ exp(-\frac{E}{20x_{2,k}R + {T_r}^{eq}R})(x_{1,k} + 2C_A^{eq}) \\ 0 \end{bmatrix},$$

$$f_2 = \begin{bmatrix} 0 \\ \frac{\Delta t}{\rho C_p} exp(-\frac{E}{20x_{2,k}R + {T_r}^{eq}R})\big(\frac{C_A^{eq}}{20} + \frac{x_{1,k}}{40}\big) \end{bmatrix}$$

The MPC prediction horizon and PE window length are set to be $N = 3$ and $N_u = 3$ respectively.

The stage cost $L(x, u)$ is selected as a nominal quadratic cost in the form of $L(\bar{x}, \bar{u}) = \bar{x}^\top Q\bar{x} + \bar{u}^\top R\bar{u}$ where

$$Q = \begin{bmatrix} 0.5 & 0 \\ 0 & 1.1429 \end{bmatrix}, \quad R = 1.333.$$

For a small region around the equilibrium point $(x_1, x_2) \in (-0.2, 0.2) \times (-0.25, 0.25)$, we obtain a terminal linear feedback gain $K = [-21 \ -26]$ that stabilise the linearised dynamics for all $\theta \in \Theta_0$, $w \in \mathcal{W}$ and linearisation errors. The corresponding terminal region is as shown in Figure 7.1.

All simulations were performed in Matlab on a 2.9 GHz Intel Core i7-10700F processor, and the online MPC optimisation was solved using the Matlab NLP solver fmincon. For the purpose of comparison, the identical disturbance sequences and the same parameter update law are used in each simulation run. Input and state constraints are robustly satisfied in all simulations.

Table 7.1 compares the parameter set sizes and the closed-loop PE coefficient ϵ_t using Algorithm 7.1 and Algorithm 7.2 with different weights γ. The values shown in the table are averaged from 30 ensemble disturbance sequences. The estimated parameter set using Algorithm 7.1 converges at the slowest rate and has the smallest closed-loop PE coefficient ϵ_t. As the weight γ increases, the rate of parameter set convergence increases and the closed-loop PE coefficient ϵ_t increases. This is because a higher value of γ prioritizes parameter identification over regulation and therefore

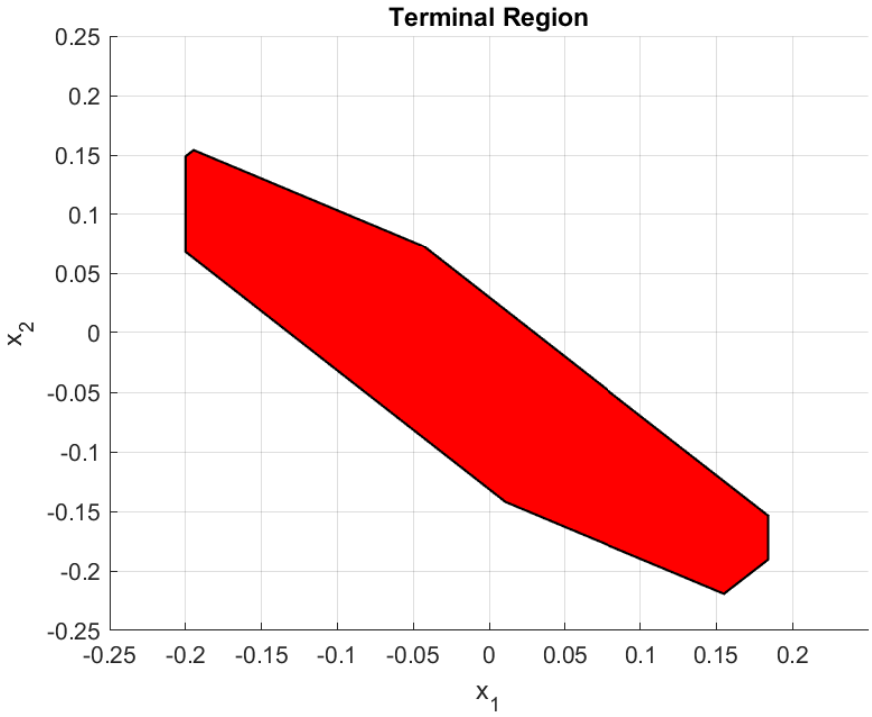


Figure 7.1: The terminal region using linear feedback law for all $\theta \in \Theta_0$, $w \in \mathcal{W}$ and linearisation errors

Table 7.1: Comparison of Algorithm 7.1, Algorithm 7.2

Algorithm Name	$\Theta_{10}/\%$	$\Theta_{100}/\%$	$\Theta_{200}/\%$	$\Theta_{500}/\%$	$\mathbb{E}(\epsilon_t)$
Algorithm 7.1	92.29	9.21	3.68	1.31	2.266×10^{-5}
Algorithm 7.2 with $\gamma = 10^2$	71.70	8.69	3.53	1.27	2.352×10^{-5}
Algorithm 7.2 with $\gamma = 10^3$	69.15	8.14	3.33	1.20	2.492×10^{-5}
Algorithm 7.2 with $\gamma = 10^4$	59.53	7.14	2.92	1.05	2.836×10^{-5}
Algorithm 7.2 with $\gamma = 10^5$	71.70	6.96	2.85	1.03	2.909×10^{-5}
Algorithm 7.2 with $\gamma = 5\times10^5$	71.70	6.96	2.85	1.03	2.909×10^{-5}

speed up the rate of convergence. When γ exceeds 10^5, the PE term dominates the cost function and any further increase in γ does not lead to any change.

Figure 7.2 shows the change of estimated parameter set over time using Algorithm 7.1 and Algorithm 7.2 with different weights γ. The volume of the estimated parameter set Θ_t at each time step t was computed by averaging over 30 ensemble disturbance sequences. From the figure, we can see that a greater weight γ leads to

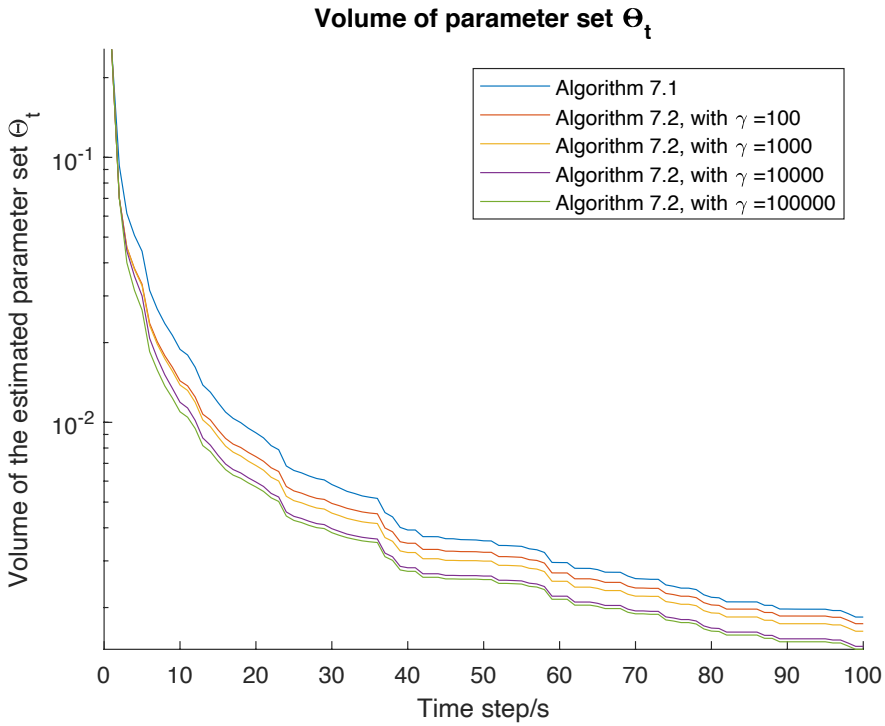


Figure 7.2: The volume of estimated parameter sets in logarithmic scale using Algorithm 7.1 and Algorithm 7.2 with different PE weights γ

a faster rate of convergence, since more emphasis is placed on system identification rather than performance.

Figure 7.3 shows the closed-loop ϵ_t using Algorithm 7.1 and Algorithm 7.2 with different weights γ. The plots start from $t = 3$ because we set the PE window length $N_u = 3$. The figure also shows a greater weight γ leads to larger ϵ_t, and hence faster parameter set convergence, which is consistent with the analysis of Section 4.4.2.

Figure 7.4 plots the accumulated cost given by $V_{t_i} = \sum_{t=1}^{t_i} \|x_t\|_Q + \|u_t\|_R$ using Algorithm 7.1 and Algorithm 7.2 with different weights γ. Note that the figure does not plot the optimal predicted cost, which in the case of Algorithm 7.2 is augmented by the predicted PE coefficient. The figure shows that greater weight γ leads to higher cost, because parameter identification is prioritized over tracking. The main difference in accumulated cost is due to an exciting signal at $t = 2$. For $t \geq 3$, the rate of increase in closed-loop cost increases with larger γ values. In

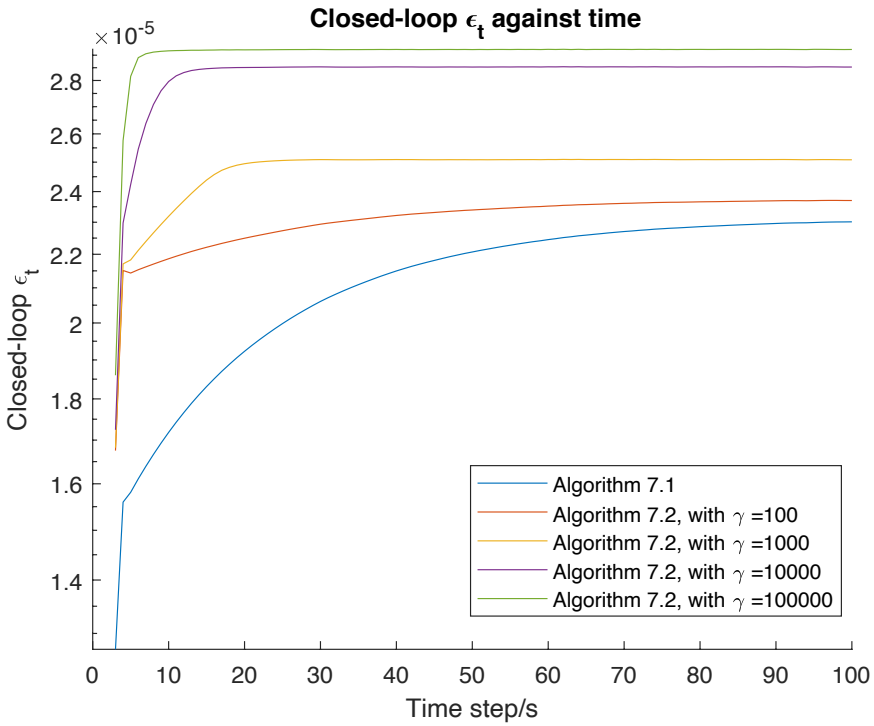


Figure 7.3: The closed-loop cost of Algorithm 7.1 and Algorithm 7.2 with different PE weights γ

short, we can conclude the additional PE conditions and the augmented cost can excite the system effectively but at the expense of reduced tracking performance.

Next, we compare Algorithm 7.3 which contains recursively feasible PE conditions with Algorithm 7.1 which does not contain any PE conditions or related coefficients. Figure 7.5 shows the change of the estimated parameter sets over time using Algorithm 7.1 and Algorithm 7.3. The solid lines represent the average over 30 disturbance sequences and the shaded regions represent range of observed parameter set volumes for 30 disturbance sequences. The figure shows that the PE conditions facilitate parameter set convergence and result in lower parameter set volumes, especially in the first 20 time steps. The difference diminishes as the estimated volume decreases.

Figure 7.6 shows the closed-loop ϵ_t using Algorithm 7.1 and Algorithm 7.3. The solid lines represent the average over 30 disturbance sequences and the shaded

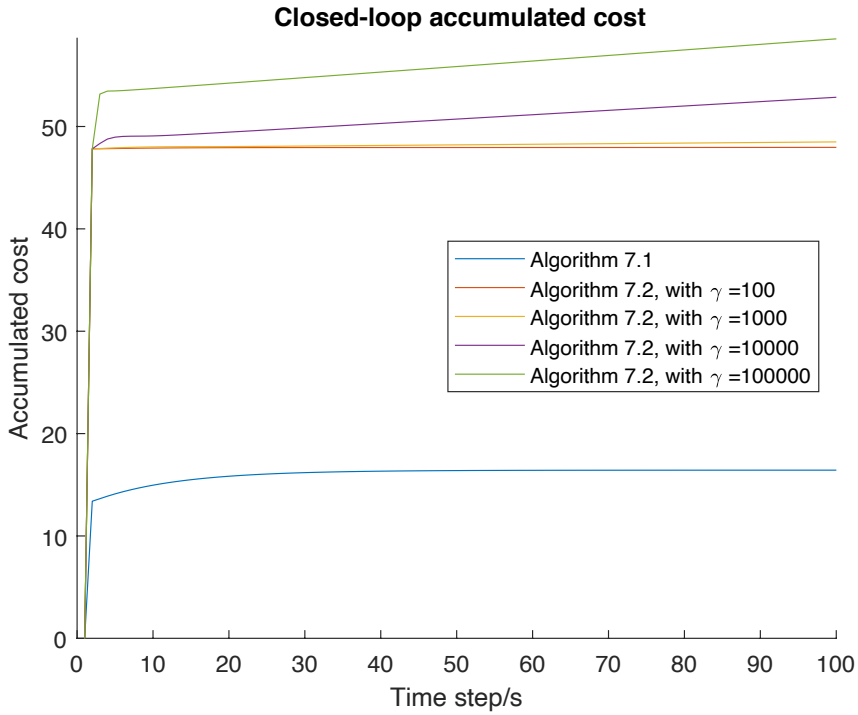


Figure 7.4: The closed-loop cost of Algorithm 7.1 and Algorithm 7.2 with different PE weights γ

regions represent the range of observed values for 30 disturbance sequences. The figure shows that the additional PE conditions result in slightly higher closed-loop ϵ_t, which shows the additional PE constraints inject extra excitation into the system.

Figure 7.7 plots the accumulated cost given by $V_{t_i} = \sum_{t=1}^{t_i} \|x_t\|_Q + \|u_t\|_R$ using Algorithm 7.1 and Algorithm 7.3. The figure shows that additional PE constraints result in higher costs. Combining with Figure 7.5, we can see that tracking performance and excitation are two conflicting objectives in adaptive robust MPC.

7.7 Conclusion

This chapter extends the results of Chapter 4 and 5 to nonlinear systems with affine parameter uncertainties and additive disturbances. It first proposes a nonlinear adaptive MPC algorithm that combines set membership identification with robust

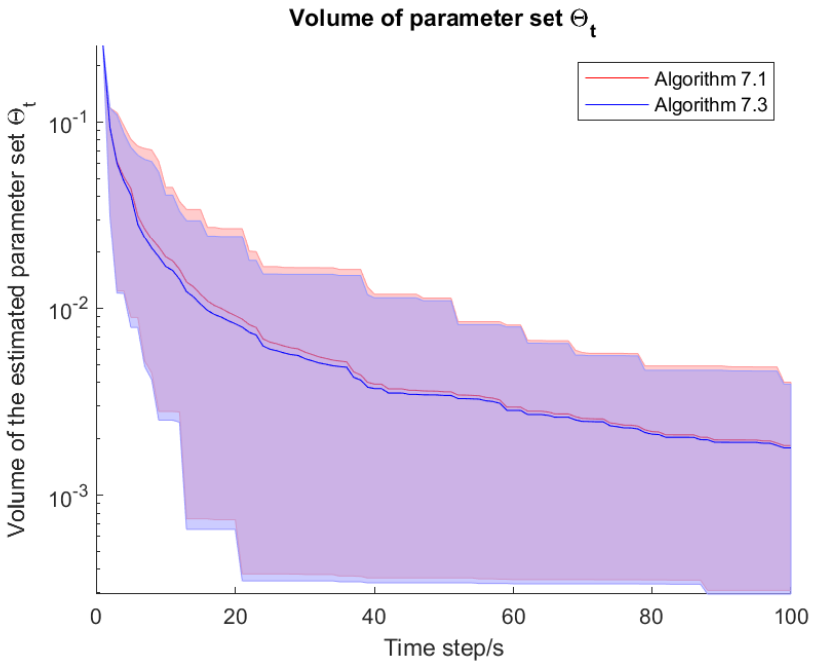


Figure 7.5: The volume of estimated parameter sets in logarithmic scale using Algorithm 7.1 and Algorithm 7.3

tube MPC formulation. The proposed algorithm guarantees robust constraint satisfaction, recursive feasibility, input-to-state stability and computational tractability. To facilitate parameter identification, we proposes two modified algorithms that incorporate PE conditions. The first approach approximates the PE condition using a Taylor's series expansion around a reference trajectory and augments the cost function with the predicted PE coefficient. The second approach incorporates recursively feasible PE conditions. Both approaches satisfy the constraints robustly and are recursively feasible with tractable computational loads. The algorithm with recursively feasible PE conditions is also input-to-state stable. The chapter ends with a simulation using the Continuous Stirred Tank Reactor (CSTR) example and the simulation show that the proposed algorithms can identify the system parameter and regulate the nonlinear system subject to the constraints. The simulation also illustrates the conflicting nature of the control objectives.

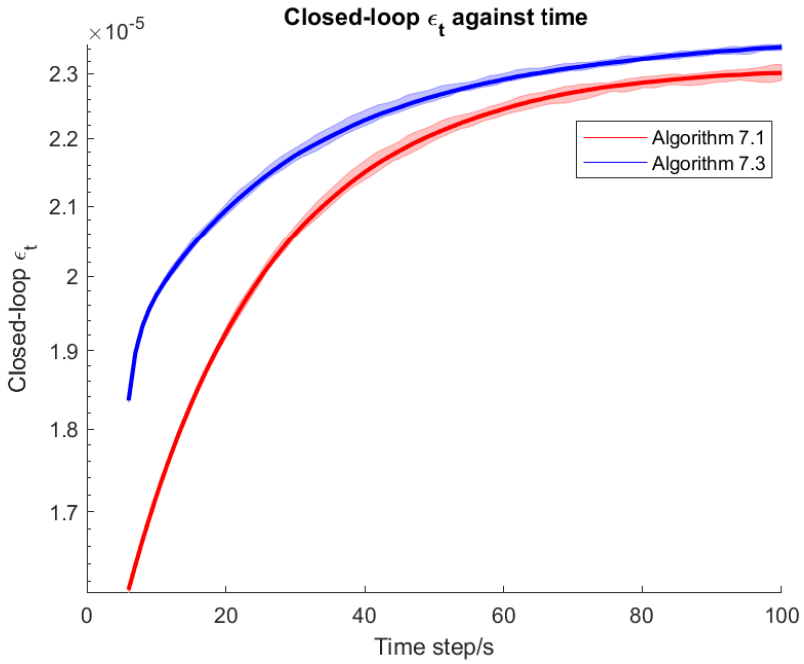


Figure 7.6: The closed-loop cost of Algorithm 7.1 and Algorithm 7.3

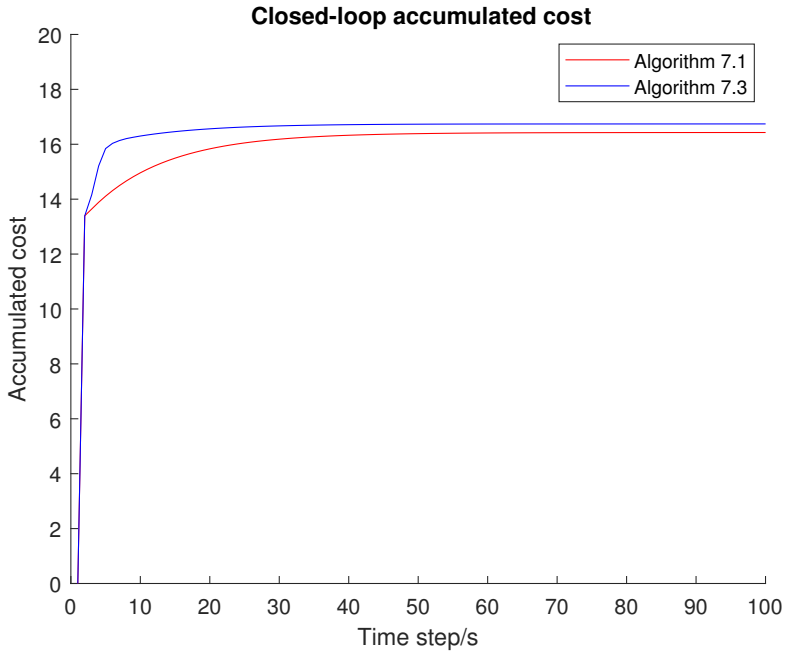


Figure 7.7: The closed-loop cost of Algorithm 7.1 and Algorithm 7.3

8 Conclusion

The thesis proposes several robust MPC strategies for uncertain linear systems with affine model uncertainties and bounded additive disturbance. By combining robust tube MPC formulation, set membership identification and persistence of excitation conditions, the proposed algorithms are able to estimate system parameters online while satisfying state and input constraints and providing stability guarantees. All algorithms proposed guarantee robust constraint satisfaction, recursive feasibility and computational tractability. The approaches are extended to nonlinear systems and similar closed-loop properties are demonstrated.

The thesis also presents an in-depth investigation into the persistence of excitation conditions within adaptive MPC. It proposes various approaches that incorporate persistence of excitation conditions as computationally tractable constraints, including augmenting the cost function, exploiting the random nature of disturbance inputs inherent in the model, introducing additional excitation into the control inputs, and including both additive disturbance and additional input excitation in the algorithm. The performance of various approaches is compared in terms of parameter convergence and tracking or regulation performance. In general, the thesis shows that the control inputs have the dual objectives of achieving good tracking performance and allowing asymptotic parameter set estimation, and the

proposed algorithms can balance these conflicting objectives. However, the proposed adaptive robust MPC algorithms scale badly with increase in dimension because the online optimisation problems are semi-definite programs (SDPs).

Chapter 3 proposes an adaptive robust Model Predictive Control strategy by combining set-based parameter identification and robust tube MPC. Compared with existing approaches, the proposed algorithm uses polytopic sets of fixed complexity to bound predicted state tubes, which allows greater flexibility in the definition of the predicted state tubes while retaining computational tractability. The proposed controller reduces the conservativeness of constraint handling, guarantees recursive feasibility and robust constraint satisfaction. It also provides asymptotic bounds on the closed-loop system state that depend explicitly on the identified parameter set. Moreover, this chapter proposes a convex approximation for the persistence of excitation (PE) condition, and the PE condition is included in the MPC algorithm with guaranteed stability.

Chapter 4 proposes an adaptive robust MPC algorithm that identifies sets of model parameters online. The algorithm is proven to be recursively feasible and input-to-state stable using a nominal cost function. Building on Chapter 3, the proposed algorithm uses hyperplane and vertex representations interchangeably to further reduce computational load for fixed-complexity tube MPC formulation. By including a penalty term in the cost function, the proposed algorithm can adjust the relative importance of the conflicting objectives, namely controller performance and system parameter convergence. In addition, this chapter relates the convexified PE conditions to probabilistic rates of convergence of asymptotic bounds on parameter set estimates when a computational tractable parameter set update law is adopted. The estimated parameter set is proven to converge to the true parameter values with probability 1 and similar results are derived for the case of inexact knowledge of the disturbance bounding set and for the case that model states are estimated in the presence of measurement noise.

Chapter 5 analyses the convergence rate of parameter estimates and incorporates recursively feasible PE conditions into adaptive robust MPC framework. Starting from the simplest case of a linear feedback law, the chapter derives closed-loop bounds on the expectation of the PE coefficients and proves that the estimated parameter set converges to the true parameter value with probability 1. The chapter then proposes three adaptive robust MPC strategies that guarantee robust constraint satisfaction, recursive feasibility, input-to-state stability and satisfaction of PE conditions in the closed-loop. The first algorithm contains linearised PE conditions in the optimisation and a nonconvex condition check after the optimisation. The second algorithm reduces computational load by including an additional assumption on the terminal control law. The third algorithm avoids the nonconvex condition check by imposing the PE conditions on the linearised PE coefficients recursively.

Chapter 6 proposes an adaptive robust MPC algorithm where the recursively feasible PE conditions are imposed on the nominal tubes. Additional excitation is injected into the system by artificially adding unknown but bounded noises to the input signals. The nominal tube formulation allows the separation of injected noise and additive disturbances from the control inputs, and therefore the effects of the noise and disturbances can be accounted for in the bounds of the expected PE coefficients. The proposed algorithm is proven to be recursively feasible and input-to-state stable. Sufficient conditions for the resulting controller to be persistently exciting are also derived.

Chapter 7 extends the results for linear systems to nonlinear systems with affine parameter uncertainty and bounded additive disturbances. Building on the linear case, this chapter proposes three adaptive robust MPC algorithms that combine set membership identification and nonlinear tube MPC. The proposed algorithms achieve robust constraint satisfaction, recursive feasibility and computational tractability. The first algorithm, without any PE condition, is proven to be input-to-state stable. The second algorithm includes PE constraints in the optimisation and a penalty term in the cost function to facilitate the convergence of estimated parameter sets. The third algorithm includes recursively feasible PE conditions

and is also input-to-state stable. The performance of three algorithms is illustrated using the Continuous Stirred Tank Reactor (CSTR) example.

Future Work

The presented research could be extended in the following directions:

Approximation of nonlinear dynamics using convex function(s)

For nonlinear systems, the thesis uses the nonlinear system dynamics for state tube update in the online optimisation. Depending on the nonlinear dynamics, calculating the state tube can be computationally challenging or even intractable. One possible approach is to approximate the actual system dynamics with a group of convex (nonlinear) basis functions, i.e. $x_{t+1} = f(x_t, u_t, \theta^*) \in \sum_{i=1}^{q} g_i(x_t, u_t, \theta^*)$ where g_j are convex basis functions. Compared with affine difference inclusion (ADI), where a set of linear equations is used to bound the system dynamics, the proposed approach is more general and therefore has wider applications. The other way is to approximate the actual system dynamics with a convex nonlinear function with bounded approximation error, i.e. $x_{t+1} = f(x_t, u_t, \theta^*) \in g(x_t, u_t, \theta^*) + h$, where h denotes the approximation error.

By transforming the system dynamics into convex function(s), we can take advantage of various properties, such as convexity, continuity and monotonicity, depending on the specific application and chosen function(s). For example, for monotonic function, we can compute the $k+1$ step ahead state tube $\mathcal{X}_{k+1|t} = \{x^1_{k+1|t}, \ldots, x^m_{k+1|t}\}$ using the vertices of the k-step ahead state tube $\mathcal{X}_{k|t} = \{x^1_{k|t}, \ldots, x^m_{k|t}\}$ and this reduces computational requirement for tube propagation significantly.

Closed-loop satisfaction of Persistent Excitation (PE) conditions for nonlinear systems

In chapter 7, the thesis presents algorithms with stability guarantees and recursively feasible PE conditions for nonlinear systems. However, the bounds on expected PE coefficients were not derived, unlike the linear cases. To derive the bound, the approximation error of the PE conditions between the reference trajectories and the actual/predicted trajectories must be considered. One possible approach to reduce the approximation error to a desirable level by iteratively re-computing the reference sequences $\hat{\mathbf{x}}$ and $\hat{\mathbf{u}}$. In addition, the current implementation convexifies the regressors using the Taylor series expansion around a reference trajectory, and for recursively feasible conditions, the regressors with past states and inputs are also convexified around the same trajectory. This leads to a gap between the actual closed-loop regressor products and the approximated regressor products around the reference trajectory. This gap is due to nonlinear system dynamics and can be addressed by using intermediate reference PE coefficients.

Probabilistic information of disturbance w

The thesis assumes the additive disturbance w belongs to a bounded set $\mathcal{W}$ and this set is used to construct the unfalsified parameter sets Δ_t. The thesis does not assume a specific distribution for the disturbance input. However, in real-life applications, it is more common to have unbounded disturbances with known distributions. In this case, we cannot construct the unfalsified parameter set Δ_t using set membership identification.

Instead, we can combine the prior information of the parameter set with state information and disturbance distribution to improve the system model description via Bayesian inference. The updated parameter set distribution can be used to regulate the system subject to probabilistic constraints.

Besides, we can extend our work by applying the proposed adaptive robust MPC algorithms to real life control problems. Limited by scalability and computational

requirement, suitable applications could be energy network and building heating/cooling systems. We can also convert the problem into a constant set-point tracking problem by redefining cost functions to penalize the deviation from the set-point[30]. For time-varying set-point tracking problems, it is generally difficult to guarantee the decrease of the optimal cost even for systems with perfect information. Building on [35, 83], this problem can be overcome using velocity form of the system model and control properties could be derived.

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