

Coloured and directed designs

Peter Keevash*

July 17, 2018

Abstract

We give some illustrative applications of our recent result on decompositions of labelled complexes, including some new results on decompositions of hypergraphs with coloured or directed edges. For example, we give fairly general conditions for decomposing an edge-coloured graph into rainbow triangles, and for decomposing an r -digraph into tight q -cycles.

To László Lovász on his seventieth birthday

1 Introduction

When can we decompose an object into copies of some other object? This vague question suggests a number of mathematical problems. Within graph theory, a fundamental instance of this question asks for a decomposition (i.e. partition of the edge set) of the complete graph K_n into copies of K_q . We require $n \geq q^2 - q + 1$ by Fisher's inequality (see e.g. [27, Theorem 19.6]). If q is one more than a prime power then the lines of a projective plane give a construction with $n = q^2 - q + 1$, but we do not know any construction with $n = q^2 - q + 1$ when q is not of this form; the Prime Power Conjecture suggests that there are none. On the other hand, we may fix q and ask for conditions on n that guarantee a decomposition (perhaps only for large $n > n_0(q)$ so as to exclude the difficulties associated with the Prime Power Conjecture). The first such result, obtained by Kirkman in 1846 (see [29]), shows that K_n has a triangle decomposition iff n is 1 or 3 modulo 6.

These beginnings suggest several possible directions for further generalisation. From the combinatorial perspective (taken in this paper), one may ask for a decomposition of G by copies of H where G and H are any given graphs, or hypergraphs, or indeed other related structures (we will consider coloured and directed hypergraphs). On the other hand, the above questions also have natural interpretations in Design Theory, which suggests many further questions (some of which also have natural combinatorial interpretations). Perhaps the oldest topic in this area is that of Latin and Magic squares, which have their roots in antiquity (see [3, Chapter 2]); they were given prominence in the Western mathematical tradition by Euler in 1776, who posed the *36 officer's puzzle*, which was open until its solution by Tarry in 1900. In modern terminology, the result is that there is no pair of orthogonal Latin squares of order 6. A pair of orthogonal Latin squares of order 4 is illustrated in Figure 1, together with an associated magic square (obtained by assigning values 1, 2, 3, 4 to a, b, c, d and 0, 4, 8, 12 to $\alpha, \beta, \gamma, \delta$).

*Mathematical Institute, University of Oxford, Oxford, UK. Email: keevash@maths.ox.ac.uk.
Research supported in part by ERC Consolidator Grant 647678.

b	c	a	d
a	d	b	c
d	a	c	b
c	b	d	a

β	γ	α	δ
δ	α	γ	β
γ	β	δ	α
α	δ	β	γ

6	11	1	16
13	4	10	7
12	5	15	2
3	14	8	9

Figure 1: Orthogonal and magic squares

1	8	4	9	6	3	7	2	5
5	6	2	7	4	8	3	1	9
3	9	7	5	1	2	8	6	4
2	3	9	6	5	7	1	4	8
7	5	6	1	8	4	2	9	3
4	1	8	2	3	9	6	5	7
9	4	1	3	7	6	5	8	2
6	2	3	8	9	5	4	7	1
8	7	5	4	2	1	9	3	6

Figure 2: A completed Sudoku puzzle

In general, a Latin square of order n is a labelling of the cells of an n by n square with n symbols so that every symbol appears once in each row and once in each column. An equivalent combinatorial description is a triangle decomposition of $K_3(n)$, the complete tripartite graph with parts of size n . Indeed, we identify the three parts with the sets of rows, columns and symbols of the square, and then each cell corresponds to a triangle in the obvious way. For a pair of orthogonal Latin squares of order n we require two such squares with the extra condition that every pair of symbols appears together once; this is analogously equivalent to a K_4 -decomposition of $K_4(n)$ (and similarly for larger numbers of mutually orthogonal Latin squares). We have chosen the pair in Figure 1 with the extra property that both diagonals use all symbols in both squares, so as to obtain a magic square (all rows, columns and diagonals have the same sum). In Figure 2 we illustrate the popular puzzle of completing a partially filled Sudoku square, which is a Latin square of order 9 partitioned into 3 by 3 subsquares each of which uses every symbol once.

We now consider the generalisations of the above problems from graphs to r -graphs (hypergraphs in which every edge has size r). When does an r -multigraph G have a decomposition into copies of some fixed r -graph H ? The case that $H = K_q^r$ is the complete r -graph on q vertices is of particular interest, as a K_q^r -decomposition of K_n^r is equivalent to a Steiner (n, q, r) system, i.e. a collection of blocks of size q in a set of size n covering every set of size r exactly once. For example, if $(q, r) = (3, 2)$ a triangle decomposition of K_n is equivalent to a Steiner Triple System. More generally, giving each edge of K_n^r some fixed multiplicity λ , a K_q^r -decomposition of λK_n^r is equivalent to a (n, q, r, λ) design. Some necessary conditions for the existence of a K_q^r -decomposition of an r -multigraph G may be observed by considering the degrees. The degree of $e \subseteq V(G)$ is the number of edges of G containing

e , i.e. the size of the neighbourhood $G(e) = \{f \subseteq V(G) \setminus e : e \cup f \in G\}$. We say G is K_q^r -divisible if $|G(e)|$ is divisible by $\binom{q-|e|}{r-|e|}$ for all $e \subseteq V(G)$; this is a necessary condition for a K_q^r -decomposition, as every copy of K_q^r containing e contains $\binom{q-|e|}{r-|e|}$ edges that contain e . For example, a necessary condition for the existence of a (n, q, r, λ) design is $\binom{q-i}{r-i} \mid \lambda \binom{n-i}{r-i}$ for all $0 \leq i \leq r-1$. The Existence Conjecture, proved in [10], is that if $n > n_0(q, r, \lambda)$ is large and this divisibility condition holds then there is a (n, q, r, λ) design. More generally, we can find a K_q^r -decomposition in any K_q^r -divisible r -multigraph G that is sufficiently dense and quasirandom.

The Existence Conjecture has had a long history in Design Theory since 1853 when Steiner asked about the existence of Steiner (n, q, r) systems. Here we briefly mention a few highlights that are relevant to our discussion here. The case $r = 2$ was proved by Wilson [30, 31, 32] in the 1970's. Around the same time, Graver and Jurkat [6] and Wilson [33] showed that the divisibility condition suffices for an integral (n, q, r, λ) design, i.e. an assignment of integer weights w_Q to copies Q of K_q^r in K_n^r such that $\sum \{w_Q : e \in Q\} = \lambda$ for all $e \in K_n^r$. Rödl [23] showed the existence of approximate Steiner systems, i.e. that there are edge-disjoint copies of K_q^r in K_n^r such that only $o(n^r)$ edges are not covered; his semi-random (nibble) method is now an indispensable tool of modern Probabilistic Combinatorics. Teirlinck [25] was the first to show that there are *any* non-trivial (n, q, r, λ) designs for arbitrary r . Kuperberg, Lovett and Peled [13] gave an alternative probabilistic proof of this result (and the existence of many other regular combinatorial structures); their method was extended by Lovett, Rao and Vardy [18] to show the existence of ‘large sets’ of designs (for certain parameter sets). Glock, Kühn, Lo and Osthus [4] gave an alternative combinatorial proof of the Existence Conjecture (the proof in [10] used a randomised algebraic construction); they also weakened the typicality hypothesis of [10] (version 1) to an extendability hypothesis, similar to that subsequently used in [10] (version 2). Furthermore, in [5] they obtained analogous results on H -decompositions where H is any r -graph and G is an r -graph that is H -divisible, i.e. each degree $|G(e)|$ is divisible by the gcd of all degrees $|H(f)|$ with $|f| = |e|$.

Having discussed some hypergraph generalisations of Kirkman’s result on triangle decompositions of K_n (Steiner Triple Systems), let us now consider such generalisations for triangle decompositions of $K_3(n)$ (Latin Squares). Besides being a combinatorially natural direction, this also has practical applications. For example, in software testing (see [9]), a K_q^r -decomposition of $K_q^r(n)$ can be thought of as a sequence of tests to a program taking q inputs from $[n]$, so that for every r inputs all possible combinations are tested once (so an efficient K_q^r -covering of $K_q^r(n)$ suffices in this context). Another example is to a secret sharing scheme that distributes information to $q-1$ bank clerks so that any r of them can open the safe but any $r-1$ cannot: pick a random copy of K_q^r in the decomposition, give one vertex to each clerk, and make the final vertex the combination for the safe. High-dimensional permutations (also called Latin Hypercubes) are equivalent to K_{r+1}^r -decompositions of $K_{r+1}^r(n)$. In section 2 we will show how the result of [11] implies an approximate formula for the number of such decompositions, thus confirming a conjecture of Linial and Luria [15]. The method applies in greater generality: as an other illustration we will give an approximate formula for the number of generalised Sudoku squares, via H -decompositions of $H(n)$ for an auxiliary 4-graph H .

In section 3 we consider a common generalisation of the nonpartite and partite decompositions discussed above to a generalised partite setting in which the edges of H and G have the same intersection patterns with respect to some partitions of their vertex sets. This general setting encodes several further problems in Design Theory. For example, Kirkman’s Schoolgirl Problem (a popular puzzle in the 19th century) asks for the construction of a Steiner Triple System that is resolvable, meaning that its blocks can be partitioned into perfect matchings (sets of triples covering every vertex

exactly once). We will illustrate the generalisation to hypergraph decompositions given in [11]. We will also illustrate the construction in [11] of large sets of designs, i.e. decomposition of K_n^q into (n, q, r, λ) designs. An application of the latter (see [28]) is to the following ‘Russian Cards’ problem in information security. From a deck of n cards, we randomly deal cards so that Alice receives a cards, Eve $e < a$ cards and Bob $b = n - a - e$ cards. Alice wants to make a public announcement from which Bob can learn her cards (given the cards that he holds) while limiting the information that Eve receives (e.g. for any card that she does not hold she should not learn which of Alice or Bob holds it). A strategy for this problem can be identified with a partition of K_n^a , where edges represent the possible sets of cards for Alice, and Alice announces to which part her actual set belongs. An optimal (minimum number of parts) strategy such that Bob can learn Alice’s hand corresponds to a partition of K_n^a into Steiner $(n, a, a - e)$ systems; furthermore, if $n > n_0(a, e)$ is large then it is secure against Eve, as for any card x that she does not hold, among the blocks disjoint from her hand in any of the Steiner systems, at least one contains x and at least one does not.

We will explain the statement of the result of [11] in section 4, and illustrate it with two new applications in the subsequent two sections. In section 5 we generalise the results on hypergraph decomposition discussed above to decompositions of hypergraphs where edges have colours which must be respected by the decomposition. As well as being combinatorially natural, such generalisations encode other problems of Design Theory (e.g. Whist Tournaments) and also fit within the large literature on rainbow versions of classical combinatorial results, which can encode seemingly unrelated questions (see e.g. [21]). In section 6 we give a different generalisation, namely to decompositions of directed hypergraphs. This illustrates the following important feature of the result of [11]: it is fundamentally concerned with sets of functions (which we call labelled edges), so to apply it to sets of (unlabelled) edges (i.e. hypergraphs) we must encode an edge by a suitable set of labelled edges. This general setting has more applications, albeit at the expense of considerable effort required in setting up the theory in section 4. However, this seems unavoidable, as there are divisibility phenomena even for unlabelled coloured hypergraphs that require labels to analyse (see [11, section 1.5]). We conclude in section 7 by discussing some directions for potential future research.

2 Partite decompositions, hypermutations, Sudoku

Over the next three sections we will gradually move from examples to the general setting. We start with this section by illustrating some results on hypergraph decompositions and some of their applications discussed in introduction. First we consider the nonpartite setting with the typicality condition from [10], which describes an r -graph where the common neighbourhood of small set of $(r - 1)$ -sets behaves roughly as one would expect in a random r -graph of the same density.

Definition 2.1. Suppose G is an r -graph on $[n]$. The density of G is $d(G) = |G| \binom{n}{r}^{-1}$. We say that G is (c, s) -typical if for any set A of $(r - 1)$ -subsets of $V(G)$ with $|A| \leq s$ we have $|\cap_{f \in A} G(f)| = (1 \pm |A|c)d(G)^{|A|}n$.

The following result of [5] (see also [11, Theorem 1.5]) shows that any dense typical r -graph has an H -decomposition provided that it satisfies the necessary divisibility condition discussed above. Henceforth we fix parameters

$$h = 2^{50q^3} \quad \text{and} \quad \delta = 2^{-10^3q^5}.$$

Theorem 2.2. Let H be an r -graph on $[q]$ and G be an H -divisible (c, h^q) -typical r -graph on $[n]$, where $n > n_0(q)$ is large, $d(G) > 2n^{-\delta/h^q}$, $c < c_0d(G)^{h^{30q}}$ and $c_0 = c_0(q)$ is small. Then G has an H -decomposition.

Next we set up some notation for stating the partite analogue of the previous result.

Definition 2.3. Let H be an r -graph. We call an r -graph G an H -blowup if $V(G)$ is partitioned as $(V_x : x \in V(H))$ and each $e \in G$ is f -partite for some $f \in H$, i.e. $f = \{x : e \cap V_x \neq \emptyset\}$.

We write G_f for the set of f -partite $e \in G$. For $f \in H$ let $d_f(G) = |G_f| \prod_{x \in f} |V_x|^{-1}$. We call G a (c, s) -typical H -blowup if for any $s' \leq s$ and distinct $e_1, \dots, e_{s'}$ where each e_j is f_j -partite for some $f_j \in \binom{V(H)}{r-1}$, and any $x \in \cap_{j=1}^{s'} H(f_j)$ we have $|V_x \cap \cap_{j=1}^{s'} G(e_j)| = (1 \pm s'c) |V_x| \prod_{j=1}^{s'} d_{f_j+x}(G)$.

We say G has a partite H -decomposition if it has an H -decomposition using copies of H with one vertex in each part V_x .

We say G is H -balanced if for every $f \subseteq V(H)$ and f -partite $e \subseteq V(G)$ there is some n_e such that $|G_{f'}(e)| = n_e$ for all $f' \in H$.

Note in particular that the H -balance condition for $e = f = \emptyset$ implies equality of all $|G_{f'}|$ with $f' \in H$. If G has a partite H -decomposition then G must be H -balanced; the following result ([11, Theorem 1.7]) shows the converse for typical H -blowups.

Theorem 2.4. Let H be an r -graph on $[q]$ and G be an H -balanced (c, h^q) -typical H -blowup on $(V_x : x \in V(H))$, where each $n/h \leq |V_x| \leq n$ for some large $n > n_0(q)$ and $d_f(G) > d > 2n^{-\delta/h^q}$ for all $f \in H$ and $c < c_0 d^{h^{30q}}$, where $c_0 = c_0(q)$ is small. Then G has a partite H -decomposition.

In the previous result, we can not only show that G has a partite H -decomposition, but also give an approximate formula for the number of such decompositions. We will show some applications of this when G is a complete H -blowup. We start by considering the upper bound, which comes from the following result of Luria [19].

Theorem 2.5. Let R be fixed and $D \rightarrow \infty$. Suppose A is an R -graph on N vertices such that all vertex degrees are $^1 D + o(D)$ and all pair degrees are $o(D)$. Then the number of perfect matchings in A is at most $(De^{1-R} + o(D))^{N/R}$.

When applying Theorem 2.5 to the setting of Theorem 2.4, we consider the auxiliary R -graph A on $V(A) = E(G)$ where edges correspond to copies of H , so $N = |G|$ and $R = |H|$. If we let $G = H(n)$ be the complete H -blowup of size n then $N = |H|n^r$ and the degree conditions of Theorem 2.5 hold with $D = n^{q-r}$. In fact, all pair degrees are at most n^{q-r-1} . We deduce that the number of H -decompositions of $H(n)$ is at most $((e^{1-|H|} + o(1))n^{q-r})^{n^r}$. We will show below how a matching lower bound follows from Theorem 2.4. Before doing so, we discuss two applications.

First we consider the number $N_r(n)$ of r -dimensional permutations of order n , which is also the number of K_{r+1}^r -decompositions of $K_{r+1}^r(n)$. For $r = 2$ (Latin squares), Van Lint and Wilson [27, Theorem 17.3] obtained the approximate formula $N_2(n) = (n/e^2 + o(n))^{n^2}$; this was a short deduction from two celebrated breakthroughs on permanents (the proof of the Van der Waerden Conjecture by Falikman and by Egorychev and of the Minc Conjecture by Bregman). The upper bound can be obtained more simply by entropy inequalities, by which means Linial and Luria [19] showed $N_r(n) \leq (n/e^r + o(n))^{n^r}$, and Luria obtained the more general result in Theorem 2.5. However, the lower bound argument appeared not to generalise, even from Latin squares to Steiner Triple Systems, for which the approximate formula was a conjecture of Wilson [35], proved in [12]. In [11] we established the lower bound, thus giving the following approximate formula.

Theorem 2.6. The number of r -dimensional permutations of order n is $(n/e^r + o(n))^{n^r}$.

¹ The statement in [19] has D here, but the proof works with $D + o(D)$.

Our second application is to the number of generalised Sudoku squares, which are Latin squares of order n^2 partitioned into n by n subsquares each of which uses every symbol once (the usual Sudoku squares have $n = 3$). We encode these by the 4-graph H with $V(H) = \{x_1, x_2, y_1, y_2, z_1, z_2\}$ and $E(H) = \{x_1x_2y_1y_2, x_1x_2z_1z_2, y_1y_2z_1z_2, x_1y_1z_1z_2\}$. Then an H -decomposition of the complete n -blowup of H can be viewed as a Sudoku square, where we represent rows by pairs (a_1, a_2) , columns by (b_1, b_2) , symbols by (c_1, c_2) and boxes by (a_1, b_1) ; a copy of H with vertices $\{a_1, a_2, b_1, b_2, c_1, c_2\}$ represents a cell in row (a_1, a_2) and column (b_1, b_2) with symbol (c_1, c_2) . The following estimate then follows from the estimate for general H given below.

Theorem 2.7. *The number of Sudoku squares with n^2 boxes of order n is $(n^2/e^3 + o(n^2))^{n^4}$.*

We conclude this section with the general formula that implies the two examples discussed above.

Theorem 2.8. *For any r -graph H on $[q]$, the number of H -decompositions of $H(n)$ is $((e^{1-|H|} + o(1))n^{q-r})^{n^r}$.*

Proof. The upper bound comes from Theorem 2.5 applied to the auxiliary R -graph A described above. For the lower bound, we consider the random greedy matching process, in which we construct a sequence of vertex-disjoint edges e_0, e_1, \dots in A and subgraphs A_0, A_1, \dots , where $A_0 = A$, e_i is a uniformly random edge of A_i , and A_{i+1} is obtained from A_i by deleting the vertices of e_i and all edges that intersect e_i . We will estimate the number of runnings of this process, stopped at some subgraph A_t which is quite sparse, but sufficiently dense and typical that Theorem 2.4 applies to show that A_t has a perfect matching. This will give a lower bound on the number of perfect matchings of A , i.e. H -decompositions of $H(n)$, which matches Luria's upper bound.

Bennett and Bohman [1] showed if A is a D -regular R -graph on N vertices with all pair degrees at most $L = o(D \log^{-5} N)$ then whp² the process persists until the proportion of uncovered vertices is at most $(L/D)^{1/2(R-1)+o(1)}$. (Their proof applies verbatim under the weaker assumption that all vertex degrees are $D \pm \sqrt{DL}$.) Here we have $L/D = n^{-1}$ and $R = |H|$, so we could run the process until the uncovered proportion is e.g. $n^{-1/2|H|}$, but we stop it when the remaining r -graph $G_t = V(A_t)$ has density $d = 3n^{-\delta/h^q}$. Furthermore, one can show that whp throughout the process the r -graphs $G_i = V(A_i)$ are (c, h^q) -typical H -blowups with $c < c_0 d^{h^{30q}}$ (similar lemmas in the nonpartite setting are well-known, see e.g. [2]). Then Theorem 2.4 can be applied to G_t , and we have a good estimate for the number of choices at each step of the process: at step i when all densities $d_f(G_i)$ with $f \in H$ are $d(i) = 1 - in^{-r}$ there are $(1 \pm 2|H|c)d(i)^{|H|}n^q$ edges of A_i (i.e. copies of H in G_i).

Given the above results, a simple counting argument now gives the required lower bound on the number of H -decompositions of $H(n)$. For $0 \leq j \leq j' \leq t$, let us say that a running of the process from $A_0, \dots, A_{j'}$ is j -good if G_i is (c, h^q) -typical for $1 \leq i \leq j$. Let $R_{j'}^j$ be the number of such runnings. Then $R_{j+1}^j/R_j^j = (1 \pm 2|H|c)d(j)^{|H|}n^q$ by typicality and $R_{j+1}^{j+1}/R_{j+1}^j = 1 \pm c$ (say) as whp typicality does not first fail at step $j+1$. Multiplying these estimates, the number of t -good runnings is $R_t^t = \prod_{j=0}^t ((1 \pm 3|H|c)d(j)^{|H|}n^q)$. By Theorem 2.4, each t -good running can be completed to an H -decomposition of $H(n)$. Furthermore, the number of runnings giving rise to any fixed decomposition is at most $\prod_{j=0}^t (n^r - j)$. We deduce that the number of H -decompositions of $H(n)$ is at least $\prod_{j=0}^t ((1 \pm 3|H|c)d(j)^{|H|-1}n^{q-r}) = ((e^{1-|H|} + o(1))n^{q-r})^{n^r}$, where the last estimate follows by a short calculation using Stirling's estimate on factorials. \square

² We say that an event E holds *with high probability* (whp) if $\mathbb{P}(E) = 1 - e^{-\Omega(n^c)}$ for some $c > 0$ as $n \rightarrow \infty$; by union bounds we can assume that any specified polynomial number of such events all occur.

3 Generalised partite decompositions

In this section we state and give applications of a result that generalises both the nonpartite and partite decomposition results of the previous section to the generalised partite setting of the definition below (which is followed by some explanatory remarks).

Definition 3.1. Let H be an r -graph on $[q]$ and $\mathcal{P} = (P_1, \dots, P_t)$ be a partition of $[q]$. Let G be an r -graph on $[n]$ and $\mathcal{Q} = (Q_1, \dots, Q_t)$ be a partition of $[n]$. We say G has a \mathcal{P} -partite H -decomposition if it has an H -decomposition using copies $\phi(H)$ of H with all $\phi(P_i) \subseteq Q_i$.

For $S \subseteq [q]$ the \mathcal{P} -index of S is $i_{\mathcal{P}}(S) = (|S \cap P_1|, \dots, |S \cap P_t|)$; similarly, we define the \mathcal{Q} -index of subsets of $[n]$, and also refer to both as the ‘index’.

For $i \in \mathbb{N}^t$ we let H_i and G_i be the edges in H and G of index i . Let $I = I(H) = \{i : H_i \neq \emptyset\}$. We call G an (H, \mathcal{P}) -blowup if $G_i \neq \emptyset \Rightarrow i \in I$.

For $e \subseteq [n]$ we define the degree vector $G_I(e) \in \mathbb{N}^I$ by $G_I(e)_i = |G_i(e)|$ for $i \in I$. Similarly, for $f \subseteq [q]$ we define $H_I(f)$ by $H_I(f)_i = |H_i(f)|$. For $i' \in \mathbb{N}^t$ let $H_{i'}^I$ be the subgroup of \mathbb{Z}^I generated by $\{H_I(f) : i_{\mathcal{P}}(f) = i'\}$. We say G is (H, \mathcal{P}) -divisible if $G_I(e) \in H_{i_{\mathcal{Q}}(e)}^I$ whenever $i_{\mathcal{Q}}(e) = i'$.

For $i \in \mathbb{N}^t$ let $d_i(G) = |G_i| \prod_{j \in [t]} \binom{|V_j|}{i_j}^{-1}$. We call G a (c, s) -typical (H, \mathcal{P}) -blowup if for any $s' \leq s$, $\{f_1, \dots, f_{s'}\} \subseteq \binom{V(G)}{r-1}$, $j \in [t]$ we have³ $|V_j \cap \bigcap_{k=1}^{s'} G(f_k)| = (1 \pm s'c) |V_j| \prod_{k=1}^{s'} d_{i(f_k) + e_j}(G)$.

The simplest examples of the previous definition are given by the trivial partitions with $t = 1$ (non-partite decompositions) or $t = q$ (partite decompositions). The latter is instructive for understanding the divisibility condition. We will illustrate it in the case that H is a (graph) triangle on $[3]$, with parts $P_i = \{i\}$ for $i \in [3]$ and G is a tripartite graph with parts Q_i for $i \in [3]$. Then $I = \{i^1, i^2, i^3\}$ with $i^1 = (1, 1, 0)$, $i^2 = (1, 0, 1)$, $i^3 = (0, 1, 1)$. For each $i \in I$ we have $G(\emptyset)_i = |G_i|$ and $H(\emptyset)_i = |H_i| = 1$, so the 0-divisibility condition is that the three bipartite pieces of G all have the same number of edges. For the 1-divisibility condition, we note that $H(1)_{i^1} = H(1)_{i^2} = 1$, $H(1)_{i^3} = 0$ and $G(x_1)_i = |G_i(x_1)|$ for $x_1 \in Q_1$, so we require every vertex in Q_1 to have equal degrees into Q_2 and Q_3 (and similarly for each part). The 2-divisibility condition is trivially satisfied, so this completes the description. Our final remark on Definition 3.1 is that the typicality condition is a direct generalisation of that in Definition 2.3, allowing the possibility that both sides are zero if some $i(f_k) + e_j \notin I$.

Next we state a decomposition result in the generalised partite setting (a case of [11, Theorem 7.8]); the case $\mathcal{P} = ([q])$ implies Theorem 2.2 and the case $\mathcal{P} = (\{1\}, \dots, \{q\})$ implies Theorem 2.4.

Theorem 3.2. *Let H be an r -graph on $[q]$ and $\mathcal{P} = (P_1, \dots, P_t)$ be a partition of $[q]$. Let $n > n_0(q)$, $d > 2n^{-\delta/h^q}$ and $c < c_0 d^{h^{30q}}$, where $c_0 = c_0(q)$ is small. Suppose G is an (H, \mathcal{P}) -divisible (c, h) -typical (H, \mathcal{P}) -blowup wrt $\mathcal{Q} = (Q_1, \dots, Q_t)$, such that each $n/h \leq |Q_i| \leq n$ and $d_i(G) > d$ for all $i \in I(H)$. Then G has a \mathcal{P} -partite H -decomposition.*

In the remainder of this section we give two applications of the following simplified version of the preceding result (the case that G is complete).

Theorem 3.3. *Let H be an r -graph on $[q]$ and $\mathcal{P} = (P_1, \dots, P_t)$ be a partition of $[q]$. Suppose G is an (H, \mathcal{P}) -divisible complete (H, \mathcal{P}) -blowup wrt $\mathcal{Q} = (Q_1, \dots, Q_t)$ such that each $n/h \leq |Q_i| \leq n$ with $n > n_0(q)$. Then G has a \mathcal{P} -partite H -decomposition.*

As our first application we reprove the result of [22] in the case that n is large on the existence of resolvable Steiner Triple Systems (for a hypergraph generalisation see [11, Theorem 7.9]).

³Let $\{e_1, \dots, e_t\}$ be the standard basis of \mathbb{Z}^t .

Theorem 3.4. *Suppose $n = 6k + 3$ with $k \in \mathbb{N}$ is large. Then there is a resolvable Steiner Triple System of order n .*

Proof. Let $H = K_4$ be the complete graph on 4 vertices, with $V(H) = [4]$ partitioned as $\mathcal{P} = (P_1, P_2)$, where $P_1 = [3]$ and $P_2 = \{4\}$. Let Q_1 and Q_2 be disjoint sets with $|Q_1| = n$ and $|Q_2| = (n - 1)/2$. Let G be the graph with $V(G) = Q_1 \cup Q_2$ whose edges are all pairs in $Q_1 \cup Q_2$ not contained in Q_2 . Then G is a complete (H, \mathcal{P}) -blowup.

We claim that a resolvable Steiner Triple System of order n is equivalent to a \mathcal{P} -partite H -decomposition of G . To see this, suppose first that we have some \mathcal{P} -partite H -decomposition \mathcal{H} of G . This means that \mathcal{H} partitions $E(G)$, and each $\phi(H) \in \mathcal{H}$ has $\phi([3]) \subseteq Q_1$ and $\phi(4) \in Q_2$. Then $\mathcal{T} := \{\phi(H - 4) : \phi(H) \in \mathcal{H}\}$ is a triangle decomposition of the complete graph on P_1 , i.e. a Steiner Triple System of order n . We can partition \mathcal{H} as $(\mathcal{H}_y : y \in Q_2)$, where each $\mathcal{H}_y = \{\phi(H) : \phi(4) = y\}$. Note that each $T_y = \{\phi([3]) : \phi(H) \in \mathcal{H}_y\}$ is a perfect matching on P_1 ; indeed, for each $x \in P_1$, as \mathcal{H} partitions $E(G)$, there is a unique $\phi(H) \in \mathcal{H}$ containing xy , and then $\phi([3])$ is the unique triple in T_y containing x . Thus \mathcal{T} is a resolvable Steiner Triple System. Conversely, the same construction shows that any resolvable Steiner Triple System gives rise to a \mathcal{P} -partite H -decomposition of G . Indeed, given a Steiner Triple System \mathcal{T} on P_1 partitioned into perfect matchings, we arbitrarily label the perfect matchings as $(T_y : y \in Q_2)$ and form a \mathcal{P} -partite H -decomposition of G by taking all $\phi(H)$ with $\phi([3]) \in T_y$ and $\phi(4) = y$ for some $y \in Q_2$. This proves the claim.

To complete the proof of the theorem, we show that Theorem 3.3 applies to give a \mathcal{P} -partite H -decomposition of G . In the notation of Definition 3.1, we have $I = I(H) = \{(2, 0), (1, 1)\}$ and need to show that $G_I(e) \in H_{i'}^I$ whenever $i_Q(e) = i'$. First we consider $i_Q(e) = (0, 0)$, i.e. $e = \emptyset$. We have $H_I(\emptyset) = (3, 3)$, as H contains 3 edges of each of the indices $(2, 0)$ and $(1, 1)$. Thus $H_{(0,0)}^I \leq \mathbb{Z}^2$ is generated by $(3, 3)$. We have $G_I(\emptyset) = ((n, 2), (n, 2))$, as G contains $\binom{n}{2}$ edges inside P_1 and $\binom{n}{2}$ edges between P_1 and P_2 . As $3 \mid n$ we have $G_I(\emptyset) \in H_{(0,0)}^I$.

Next we consider $i_Q(e) = (1, 0)$, i.e. $e \in P_1$. We have $i_P(f) = (1, 0)$ iff $f \in [3]$, and for any such f we have $H_I(f) = (2, 1)$, as f is contained in 2 edges of index $(2, 0)$ and 1 edge of index $(1, 1)$. Thus $H_{(1,0)}^I \leq \mathbb{Z}^2$ is generated by $(2, 1)$. We have $G_I(e) = (n - 1, (n - 1)/2)$, as e has degree $n - 1$ in P_1 and degree $(n - 1)/2$ in P_2 . As n is odd, $G_I(e) \in H_{(1,0)}^I$. The only remaining non-trivial case is that $i_Q(e) = (0, 1)$, i.e. $e \in P_2$. We have $i_P(f) = (0, 1)$ iff $f = 4$, and $H_I(4) = (0, 3)$, as f is contained in no edges of index $(2, 0)$ and 3 edges of index $(1, 1)$. Thus $H_{(0,1)}^I \leq \mathbb{Z}^2$ is generated by $(0, 3)$. We have $G_I(e) = (0, n)$, as e has degree 0 in P_2 and degree n in P_1 . As $3 \mid n$ we have $G_I(e) \in H_{(0,1)}^I$. \square

Our second application is to reprove the existence of large sets of Steiner Triple Systems for large n (due to Lu, completed by Teirlinck, see [26]); see [11, Theorem 1.2] for the hypergraph version.

Theorem 3.5. *Suppose n is large and 1 or $3 \pmod{6}$. Then K_n^3 can be decomposed into Steiner Triple Systems.*

Proof. Let $H = K_4$ be the complete 3-graph on 4 vertices, with $V(H) = [4]$ partitioned as $\mathcal{P} = (P_1, P_2)$, where $P_1 = [3]$ and $P_2 = \{4\}$. Let Q_1 and Q_2 be disjoint sets with $|Q_1| = n$ and $|Q_2| = n - 2$. Let G be the 3-graph with $V(G) = Q_1 \cup Q_2$ whose edges are all triples $e \subseteq Q_1 \cup Q_2$ with $|e \cap Q_1| \geq 2$. Then G is a complete (H, \mathcal{P}) -blowup.

We claim that a decomposition of K_n^3 into Steiner Triple Systems is equivalent to a \mathcal{P} -partite H -decomposition of G . To see this, suppose we have some \mathcal{P} -partite H -decomposition \mathcal{H} of G . We can partition \mathcal{H} as $(\mathcal{H}_y : y \in Q_2)$, where each $\mathcal{H}_y = \{\phi(H) : \phi(4) = y\}$. Note that each $T_y = \{\phi([3]) : \phi(H) \in \mathcal{H}_y\}$ is a Steiner Triple System on P_1 ; indeed, for each pair xx' in P_1 , as \mathcal{H}

partitions $E(G)$, there is a unique $\phi(H) \in \mathcal{H}$ containing $xx'y$, and then $\phi([3])$ is the unique triple in T_y containing xx' . Furthermore, each triple in Q_1 belongs to exactly one element of \mathcal{H} , and so to exactly one T_y . Thus $\{T_y : y \in Q_2\}$ is a decomposition of K_n^3 into Steiner Triple Systems. Conversely, the same construction converts any decomposition of K_n^3 into Steiner Triple Systems into a \mathcal{P} -partite H -decomposition of G .

To complete the proof of the theorem, we show that Theorem 3.3 applies to give a \mathcal{P} -partite H -decomposition of G . We have $I = I(H) = \{(3, 0), (2, 1)\}$ and need to show that $G_I(e) \in H_{i'}^I$ whenever $i_Q(e) = i'$. First we consider $i_Q(e) = (a, 0)$ with $0 \leq a \leq 2$. For any $f \subseteq V(H)$ with $i_P(f) = (a, 0)$ we have $H_I(f) = (1, 3-a)$, as f is contained in 1 edge of H with index $(3, 0)$ and $3-a$ edges of H with index $(2, 1)$. Thus $H_{(a,0)}^I \leq \mathbb{Z}^2$ is generated by $(1, 3-a)$. We have $G_I(e) = ((\binom{n-a}{3-a}, (3-a)\binom{n-a}{3-a}))$, as e is contained in $\binom{n-a}{3-a}$ edges of G with index $(3, 0)$ and $\binom{n-a}{2-a}(n-2) = (3-a)\binom{n-a}{3-a}$ edges of G with index $(2, 1)$. Therefore $G_I(e) \in H_{(a,0)}^I$.

Next consider $i_Q(e) = (0, 1)$, i.e. $e \in P_2$. We have $i_P(f) = (0, 1)$ iff $f = 4$, and $H_I(4) = (0, 3)$, as 4 is contained in 0 edges of index $(3, 0)$ and 3 edges of index $(2, 1)$. Thus $H_{(0,1)}^I \leq \mathbb{Z}^2$ is generated by $(0, 3)$. We have $G_I(e) = (0, \binom{n}{2})$, as e is contained in no edges of G with index $(3, 0)$ and $\binom{n}{2}$ edges of G with index $(2, 1)$. As $3 \mid \binom{n}{2}$ we have $G_I(e) \in H_{(0,1)}^I$.

The only remaining non-trivial case is $i_Q(e) = (1, 1)$. We have $i_P(f) = (1, 1)$ iff $f = a4$ for some $a \in [3]$. Then $H_I(f) = (0, 2)$, as f is contained in 0 edges of index $(3, 0)$ and 2 edges of index $(2, 1)$. Thus $H_{(1,1)}^I \leq \mathbb{Z}^2$ is generated by $(0, 2)$. We have $G_I(e) = (0, n-1)$, as e is contained in no edges of G with index $(3, 0)$ and $n-1$ edges of G with index $(2, 1)$. As n is odd, $G_I(e) \in H_{(1,1)}^I$. \square

4 General theory

In this section we state the main result of [11], from which all the other results in this paper follow. Most of the section will be occupied with preparatory definitions for the statement of the result, which we will illustrate with the following running example. Consider a graph G with $V(G) = [n]$ partitioned as (V_1, V_2) , where there are no edges within V_2 , edges within V_1 are red, and edges between V_1 and V_2 are blue or green. When does G have a decomposition into rainbow triangles?

4.1 Labelled complexes and embeddings

All decomposition problems that fit in our general framework are encoded by labelled complexes, which are sets of functions (which we think of as labelled edges) closed under taking restriction; this is analogous to (simplicial) complexes, which are sets of sets closed under taking subsets. To apply the following definition in our example we take $V = V(G)$, $R = [3]$ and for each $B \subseteq [3]$ we let Φ_B consist of all injections $\phi : B \rightarrow V$ with $\phi(B \cap \{1, 2\}) \subseteq V_1$ and $\phi(B \cap \{3\}) \subseteq V_2$: we also call Φ the complete $(\{1, 2\}, 3)$ -partite $[3]$ -complex wrt (V_1, V_2) . We think of $\phi \in \Phi_3$ as an embedding of the triangle on $[3]$ where 12 is red, 13 is blue and 23 is green. It is useful to consider all such embeddings, even though the only ones that can appear in a decomposition of G are those that are contained in G with $\phi(12)$ red, $\phi(13)$ blue and $\phi(23)$ green.

Definition 4.1.

We call $\Phi = (\Phi_B : B \subseteq R)$ an R -system on V if $\phi : B \rightarrow V$ is injective for each $\phi \in \Phi_B$.

We call Φ an R -complex if whenever $\phi \in \Phi_B$ and $B' \subseteq B$ we have $\phi|_{B'} \in \Phi_{B'}$.

Let $\Phi_B^\circ = \{\phi(B) : \phi \in \Phi_B\}$, $\Phi_j^\circ = \bigcup\{\Phi_B^\circ : B \in \binom{[R]}{j}\}$, $\Phi^\circ = \bigcup\{\Phi_B^\circ : B \subseteq R\}$ and $V(\Phi) = \Phi_1^\circ$.

Next we consider the functional analogue of the subgraph notion for hypergraphs. Just as an embedding of a hypergraph H in a hypergraph G is an injection from $V(H)$ to $V(G)$ taking edges to edges, an embedding of labelled complexes is an injection taking labelled edges to labelled edges. In our example, Φ is as above, and H is the complete $(\{1, 2\}, 3)$ -partite $[3]$ -complex wrt $(\{1, 2\}, 3)$, i.e. each Φ_B with $B \subseteq [3]$ consists of all injections $\phi : B \rightarrow [3]$ with $\phi(B \cap \{1, 2\}) \subseteq \{1, 2\}$ and $\phi(B \cap \{3\}) \subseteq \{3\}$. We think of the edge 12 of the triangle on $[3]$ as encoded by the two labelled edges $(1 \mapsto 1, 2 \mapsto 2)$ and $(1 \mapsto 2, 2 \mapsto 1)$, the edge 13 by $(1 \mapsto 1, 3 \mapsto 3)$ and $(2 \mapsto 1, 3 \mapsto 3)$, and the edge 23 by $(2 \mapsto 2, 3 \mapsto 3)$ and $(1 \mapsto 2, 3 \mapsto 3)$. If ϕ is a Φ -embedding of H then the edge $\phi(12)$ of Φ_2° is encoded by the labelled edges $(1 \mapsto \phi(1), 2 \mapsto \phi(2))$ and $(1 \mapsto \phi(2), 2 \mapsto \phi(1))$, and similarly for the other two edges.

Definition 4.2. Let H and Φ be R -complexes. Suppose $\phi : V(H) \rightarrow V(\Phi)$ is injective. We call ϕ a Φ -embedding of H if $\phi \circ \psi \in \Phi$ for all $\psi \in H$.

4.2 Extensions and extendability

Next we will formulate our extendability condition. In our example, we could consider extending some fixed rainbow triangle to an octahedron in which every triangle is rainbow. To implement this in the following two definitions, we let $J = [3](2)$ and $F = [3] \times \{1\}$. We identify F with $[3]$ by identifying each $(i, 1)$ with i . Then $J[F]_B = \{id_B\}$ for $B \subseteq [3]$ and ϕ is a Φ -embedding of $J[F]$ iff $\phi \in \Phi_3$. We think of $Im(\phi)$ as our fixed rainbow triangle. Now consider any $\phi^+ \in X_E(\Phi)$ where $E = (J, F, \phi)$, i.e. ϕ^+ is a Φ -embedding of J that restricts to ϕ on F . For each $i \in [3]$ we have $(i \mapsto (i, 2)) \in J_1$, so $(i \mapsto \phi^+((i, 2))) \in \Phi_1$; thus $\phi^+((i, 2)) \in V_1$ if $i \in [2]$ or $\phi^+((i, 2)) \in V_2$ if $i = 3$. We think of $\{\phi^+((i, 1)), \phi^+((i, 2))\}$ for $i \in [3]$ as the opposite vertices of an octahedron extending the fixed triangle $Im(\phi)$. (We do not yet consider the colours; these will come into play when we consider Definition 4.5.) We have $X_E(\Phi) = (|V_1| - 3)(|V_1| - 4)(|V_2| - 2)$, so E is $\Omega(1)$ -dense if $|V_1|$ and $|V_2|$ are both $\Omega(n)$.

Definition 4.3. Let $R(S)$ be the R -complex of all partite maps from R to $R \times S$, i.e. whenever $i \in B \subseteq R$ and $\psi \in R(S)_B$ we have $\psi(i) = (i, x)$ for some $x \in S$. If $S = [s]$ we write $R(S) = R(s)$.

Definition 4.4. Suppose $J \subseteq R(S)$ is an R -complex and $F \subseteq V(J)$. Define $J[F] \subseteq R(S)$ by $J[F] = \{\psi \in J : Im(\psi) \subseteq F\}$. Suppose ϕ is a Φ -embedding of $J[F]$. We call $E = (J, F, \phi)$ a Φ -extension of rank $s = |S|$. We write $X_E(\Phi)$ for the set or number of Φ -embeddings of J that restrict to ϕ on F . We say E is ω -dense (in Φ) if $X_E(\Phi) \geq \omega|V(\Phi)|^{v_E}$, where $v_E := |V(J) \setminus F|$. We say Φ is (ω, s) -extendable if all Φ -extensions of rank s are ω -dense.

Next we augment our extendability condition to allow for various restrictions (coloured edges in our example). We continue the above example of extending a fixed rainbow triangle to an octahedron of rainbow triangles. We continue to ignore colours and first consider how the last paragraph of the following definition ensures that the octahedron is a subgraph of G . Indeed, if $\phi^+ \in X_{E, J \setminus J[F]}(\Phi, \Phi')$ with $\Phi' = \{\phi \in \Phi : Im(\phi) \in G\}$ then Φ'_B is only defined when $|B| = 2$, and for all $\psi \in J_2 \setminus J[F]$ we have $\phi^+ \circ \psi \in \Phi'$, i.e. $\phi^+(Im(\psi)) \in G$, as required.

Definition 4.5. Let Φ be an R -complex and $\Phi' = (\Phi^t : t \in T)$ with each $\Phi^t \subseteq \Phi$. Let $E = (J, F, \phi)$ be a Φ -extension and $J' = (J^t : t \in T)$ for some mutually disjoint $J^t \subseteq J \setminus J[F]$; we call (E, J') a (Φ, Φ') -extension.

We write $X_{E, J'}(\Phi, \Phi')$ for the set or number of $\phi^+ \in X_E(\Phi)$ with $\phi^+ \circ \psi \in \Phi_B^t$ whenever $\psi \in J_B^t$ and Φ_B^t is defined. We say (E, J') is ω -dense in (Φ, Φ') if $X_{E, J'}(\Phi, \Phi') \geq \omega|V(\Phi)|^{v_E}$. We say (Φ, Φ') is (ω, s) -extendable if all (Φ, Φ') -extensions of rank s are ω -dense in (Φ, Φ') .

When $|T| = 1$ we identify $\Phi' \subseteq \Phi$ with (Φ') . For $G \subseteq \Phi^\circ$ and $J' \subseteq J \setminus J[F]$ we write $X_{E,J'}(\Phi, G) = X_{E,J'}(\Phi, \Phi')$, where $\Phi' = \{\phi \in \Phi : \text{Im}(\phi) \in G\}$. We say that (Φ, G) is (ω, s) -extendable if (Φ, Φ') is (ω, s) -extendable.

To implement colours, we let $T = \{12, 13, 23\}$, and for $t \in T$ let G^t be the set of edges of G of the appropriate colour (red if $t = 12$, blue if $t = 13$, green if $t = 23$), $\Phi^t = \{\phi \in \Phi : \text{Im}(\phi) \in G^t\}$ and $J^t = J_t \setminus J[F]$ for $t \in T$. If $\phi^+ \in X_{E,J'}(\Phi, \Phi')$ then for each $t \in T$, $\psi \in J_t \setminus J[F]$ we have $\phi^+(\text{Im}(\psi)) \in G^t$, as required. The extendability condition says that there are at least ωn^3 such octahedra of rainbow triangles containing ϕ (and similarly for any other extension of bounded size).

4.3 Adapted complexes

A common feature of the decomposition results obtained from our main theorem is that they are implemented by a labelled complex equipped with a permutation group action, and the decomposition respects the orbits of the action. The simplest example is when the permutation group is the entire symmetric group, e.g. if $R = [3]$ and $\Sigma = S_3$ then any $\phi \in \Phi_3$ has an orbit consisting of all six bijections from $[3]$ to $e = \text{Im}(\phi)$, which we would think of as encoding the edge e in a 3-graph. In our running example, we have $\Sigma = \{id, (12)\} \leq S_3$. We recall that if ϕ is a Φ -embedding of H then the edge $\phi(12)$ of Φ_2° is encoded by the labelled edges $(1 \mapsto \phi(1), 2 \mapsto \phi(2))$ and $(1 \mapsto \phi(2), 2 \mapsto \phi(1))$, and note that these form an orbit (and similarly for the other edges).

Definition 4.6. Suppose Σ is a permutation group on R . For $B, B' \subseteq R$ we write $\Sigma_B^{B'} = \{\sigma \mid_B : \sigma \in \Sigma, \sigma(B) = B'\}$, $\Sigma^{B'} = \cup_B \Sigma_B^{B'}$ and $\Sigma^\leq = \cup_{B, B'} \Sigma_B^{B'}$.

Definition 4.7. Suppose Φ is an R -complex and Σ is a permutation group on R . For $\sigma \in \Sigma$ and $\phi \in \Phi_{\sigma(B)}$ let $\phi\sigma = \phi \circ \sigma \mid_B$. We say Φ is Σ -adapted if $\phi\sigma \in \Phi$ for any $\phi \in \Phi$, $\sigma \in \Sigma$.

Definition 4.8. For $\psi \in \Phi_B$ with $B \subseteq R$ we define the orbit of ψ by $\psi\Sigma := \psi\Sigma^B = \{\psi\sigma : \sigma \in \Sigma^B\}$. We denote the set of orbits by Φ/Σ . We write $\text{Im}(O) = \text{Im}(\psi)$ for $\psi \in O \in \Phi/\Sigma$.

Definition 4.9. Let Γ be an abelian group. For $J \in \Gamma^{\Phi_r}$ and $O \in \Phi_r/\Sigma$ we define J^O by $J_\psi^O = J_\psi 1_{\psi \in O}$. The orbit decomposition of J is $J = \sum_{O \in \Phi_r/\Sigma} J^O$.

4.4 Decompositions

Now we set up the general framework for decompositions. To apply the following definition to our example, $\mathcal{A} = \{A\}$ consists of a single copy of the $[3]$ -complex Σ^\leq on $[3]$, which is identical with H as above, i.e. the complete $(\{1, 2\}, 3)$ -partite $[3]$ -complex wrt $(\{1, 2\}, 3)$. We let $\Gamma = \mathbb{Z}^3$ and denote the standard basis by e_{12}, e_{13}, e_{23} , which we think of as the colours red, blue and green. We define $\gamma \in \Gamma^{A_2}$ by $\gamma_\theta = e_{\text{Im}(\theta)}$. The constituent parts of our decompositions are γ -molecules $\gamma(\phi)$, which encode rainbow triangles in Φ : we have $\phi \in A(\Phi)$ (which can be identified with Φ_3), i.e. $\phi \circ \theta \in \Phi$ for all $\theta \in A = \Sigma^\leq$, and e.g. the blue edge $\phi(1)\phi(3)$ is encoded by the coordinates $\gamma(\phi)_{\phi \circ \theta} = \gamma_\theta = e_{13}$ for $\theta \in A_2$ with $\text{Im}(\theta) = \{1, 3\}$, i.e. $\theta = (1 \mapsto 1, 3 \mapsto 3)$ and $\theta = (2 \mapsto 1, 3 \mapsto 3)$. We encode any coloured graph G by $G^* \in (\mathbb{Z}^3)^{\Phi_2}$ defined by $G_\psi^* = e_{12}$ if $\text{Im}(\psi)$ is a red edge, $G_\psi^* = e_{13}$ if $\text{Im}(\psi)$ is a blue edge, $G_\psi^* = e_{23}$ if $\text{Im}(\psi)$ is a green edge. Then a $\gamma(\Phi)$ -decomposition of G^* encodes a rainbow triangle decomposition of G .

Definition 4.10. Let \mathcal{A} be a set of R -complexes; we call \mathcal{A} an R -complex family. If each $A \in \mathcal{A}$ is a copy of Σ^\leq we call \mathcal{A} a Σ^\leq -family. For $r \in \mathbb{N}$ we write $A_r = \bigcup \{A_B : B \in \binom{R}{r}\}$ and $\mathcal{A}_r = \cup_{A \in \mathcal{A}} A_r$.

We let $A(\Phi)$ denote the set of Φ -embeddings of A . We let $A(\Phi)^\leq$ denote the $V(A)$ -complex where each $A(\Phi)_F^\leq$ for $F \subseteq V(A)$ is the set of Φ -embeddings of $A[F]$.

We let $\mathcal{A}(\Phi)^\leq$ denote the $V(A)$ -complex family $(A(\Phi)^\leq : A \in \mathcal{A})$.

Let $\gamma \in \Gamma^{\mathcal{A}_r}$ for some abelian group Γ .

Let Φ be an R -complex. For $\phi \in A(\Phi)^\leq$ with $A \in \mathcal{A}$ we define $\gamma(\phi) \in \Gamma^{\Phi_r}$ by $\gamma(\phi)_{\phi \circ \theta} = \gamma_\theta$ for $\theta \in A_r$ (zero otherwise). We call $\gamma(\phi)$ a γ -molecule and let $\gamma(\Phi)$ be the set of γ -molecules.

Given $\Psi \in \mathbb{Z}^{\mathcal{A}(\Phi)}$ we define $\partial\Psi = \partial^\gamma\Psi = \sum_\phi \Psi_\phi \gamma(\phi) \in \Gamma^{\Phi_r}$. We also call Ψ an integral $\gamma(\Phi)$ -decomposition of $\partial\Psi$ and call $\langle\gamma(\Phi)\rangle$ the decomposition lattice. If furthermore $\Psi \in \{0,1\}^{\mathcal{A}(\Phi)}$ (i.e. $\Psi \subseteq \mathcal{A}(\Phi)$) we call Ψ a $\gamma(\Phi)$ -decomposition.

Now we formalise in general the objects (atoms) that are being decomposed into molecules. In our example, atoms represent coloured edges. To see this, consider again the encoding of the blue edge $\phi(1)\phi(3)$ described above. The relevant orbit $O \in \Phi_2/\Sigma$ consists of the two labelled edges $(1 \mapsto \phi(1), 3 \mapsto \phi(3))$ and $(2 \mapsto \phi(1), 3 \mapsto \phi(3))$, and the relevant γ -atom at O is $\gamma(\phi)^O$ which is a vector supported on O with both coordinates equal to e_{13} . There are two other γ -atoms at O , which are vectors supported on O with both coordinates equal to e_{12} (meaning red edge), or both coordinates equal to e_{23} (meaning green edge). Thus γ is elementary, which is an important assumption in our main theorem, ensuring that our decomposition problems do not exhibit arithmetic peculiarities (as seen e.g. in the Frobenius coin problem).

Definition 4.11. (atoms) For any $\phi \in \mathcal{A}(\Phi)$ and $O \in \Phi_r/\Sigma$ such that $\gamma(\phi)^O \neq 0$ we call $\gamma(\phi)^O$ a γ -atom at O . We write $\gamma[O]$ for the set of γ -atoms at O . We say γ is elementary if all γ -atoms are linearly independent. We define a partial order \leq_γ on Γ^{Φ_r} where $H \leq_\gamma G$ iff $G - H$ can be expressed as the sum of a multiset of γ -atoms.

4.5 Lattices

We conclude with a characterisation of the decomposition lattice $\langle\gamma(\Phi)\rangle$, with conditions that are somewhat analogous to the degree-based divisibility conditions considered above, but also account for the labels on the edges and the orbits of the group action.

Definition 4.12. For $J \in \Gamma^{\Phi_r}$ we define $J^\# \in (\Gamma^Q)^\Phi$ by $(J^\#_\psi)_B = \sum\{J_\psi : \psi' \subseteq \psi \in \Phi_B\}$ for $B \in Q := \binom{[q]}{r}$, $\psi' \in \Phi$. We define $\gamma^\# \in (\Gamma^Q)^{\cup\mathcal{A}}$ by $(\gamma^\#_\theta)_B = \sum\{\gamma_\theta : \theta' \subseteq \theta \in A_B\}$ for $B \in Q$, $\theta' \in A \in \mathcal{A}$. We let $\mathcal{L}_\gamma(\Phi)$ be the set of all $J \in \Gamma^{\Phi_r}$ such that $(J^\#)^O \in \langle\gamma^\#[O]\rangle$ for any $O \in \Phi/\Sigma$.

We illustrate Definition 4.12 with our running example. We start with the orbit $O = \{\emptyset\}$, where \emptyset denotes the unique function with domain \emptyset (also denoting the empty set). Recall that we encode our coloured graph G by $G^* \in (\mathbb{Z}^3)^{\Phi_2}$ and write G^{ij} for the edges of G with colour corresponding to ij . Then $((G^*)^\#_\emptyset)_{ij} = \sum_{\psi \in \Phi_{ij}} G^*_\psi$ equals $2|G^{12}|e_{12}$ if $ij = 12$ or $|G^{13}|e_{13} + |G^{23}|e_{23}$ otherwise. Similarly, $(\gamma^\#_\emptyset)_{ij} = \sum_{\theta \in \Sigma_{ij}^\leq} \gamma_\theta$ equals $2e_{12}$ if $ij = 12$ or $e_{13} + e_{23}$ otherwise. The 0-divisibility condition is that $(2|G^{12}|e_{12}, |G^{13}|e_{13} + |G^{23}|e_{23}, |G^{13}|e_{13} + |G^{23}|e_{23})$ is an integer multiple of $(2e_{12}, e_{13} + e_{23}, e_{13} + e_{23})$, i.e. G has an equal number of edges of each colour.

Next consider the 1-divisibility condition for any orbit $O = \{1 \rightarrow x, 2 \rightarrow x\}$ with $x \in V_1$. For $i, i' \in [2]$, $j \neq i$ we have $((G^*)^\#_{i \rightarrow x})_{ij} = \sum\{G^*_\psi : \psi \in \Phi_{ij}, \psi(i) = x\}$, which equals $|G^{12}(x)|e_{12}$ if $j \in [2]$ or $|G^{13}(x)|e_{13} + |G^{23}(x)|e_{23}$ if $j = 3$. Also, $(\gamma^\#_{i' \rightarrow x})_{ij} = (\gamma^\#_{i \rightarrow i'})_{ij} = \sum\{\gamma_\theta : \theta \in \Sigma_{ij}^\leq, \theta(i) = i'\}$, which equals e_{12} if $j \in [2]$ or $e_{i'3}$ if $j = 3$. Thus we need $(|G^{12}(x)|e_{12}, |G^{13}(x)|e_{13} +$

$|G^{23}(x)|_{e_{23}}, |G^{13}(x)|_{e_{13}} + |G^{23}(x)|_{e_{23}}$ to lie in the group generated by $(e_{12}, e_{13}, 0)$, $(e_{12}, e_{23}, 0)$, $(e_{12}, 0, e_{13})$ and $(e_{12}, 0, e_{23})$, which holds iff $|G(x) \cap V_1| = |G(x) \cap V_2|$, i.e. each $x \in V_1$ has equal degrees in V_1 and in V_2 .

The other 1-divisibility conditions are for orbits $O = \{3 \rightarrow x\}$ with $x \in V_2$. For $i \in [2]$ we have $((G^*)_{3 \rightarrow x}^\#)_{i3} = \sum \{G_\psi^* : \psi \in \Phi_{i3}, \psi(3) = x\} = |G^{13}(x)|_{e_{13}} + |G^{23}(x)|_{e_{23}}$ and $(\gamma^\#(3 \rightarrow x)_{3 \rightarrow x})_{i3} = (\gamma_{3 \rightarrow 3}^\#)_{i3} = \sum \{\gamma_\theta : \theta \in \Sigma_{i3}^\leq, \theta(3) = 3\} = e_{13} + e_{23}$, so we need $|G^{13}(x)| = |G^{23}(x)|$, i.e. each $x \in V_2$ has blue degree equal to green degree. There are no further conditions, as the 2-divisibility conditions hold trivially (we leave this verification to the reader).

Returning to the general setting, it is not hard to see $\langle \gamma(\Phi) \rangle \subseteq \mathcal{L}_\gamma(\Phi)$. The following result ([11, Lemma 5.19]) shows that the converse inclusion holds under an extendability assumption on Φ .

Lemma 4.13. *Let $\Sigma \leq S_q$, \mathcal{A} be a Σ^\leq -family and $\gamma \in (\mathbb{Z}^D)^{\mathcal{A}_r}$. Let Φ be a Σ -adapted (ω, s) -extendable $[q]$ -complex with $s = 3r^2$, $n = |V(\Phi)| > n_0(q, D)$ large and $\omega > n^{-1/2}$. Then $\langle \gamma(\Phi) \rangle = \mathcal{L}_\gamma(\Phi)$.*

4.6 Types and regularity

Next we will formulate our regularity assumption, which can be thought of as robust fractional decomposition. In the following definition we give two notations for atoms. We illustrate both in our example to describe the atom $\gamma(\phi)^O$ representing a blue edge $\phi(1)\phi(3)$ as above. For the second way we can write $\gamma(\phi)^O = \gamma(\phi')$ where $\phi' = \phi|_{\{1,3\}}$ has domain $\{1, 3\}$, so if $\theta \in \mathcal{A}_2$ with $\text{Im}(\theta) \subseteq \text{Dom}(\phi')$ then $\theta = (1 \mapsto 1, 3 \mapsto 3)$ or $\theta = (2 \mapsto 1, 3 \mapsto 3)$. For the first way, we can write $\gamma(\phi)^O = \gamma[\phi']^\theta$ with $\theta = (1 \mapsto 1, 3 \mapsto 3)$, as $\gamma[\phi']^\theta$ is supported on $\phi' = (1 \mapsto \phi(1), 3 \mapsto \phi(3))$ with value $\gamma_\theta = e_{13}$ and on $\phi' \circ (12) = (2 \mapsto \phi(1), 3 \mapsto \phi(3))$ with value $\gamma_{\theta \circ (12)} = e_{13}$.

Definition 4.14. For $\psi \in \Phi_B$ and $\theta \in \mathcal{A}_B$ we define $\gamma[\psi]^\theta \in \Gamma^{\psi\Sigma}$ by $\gamma[\psi]_{\psi\sigma}^\theta = \gamma_{\theta\sigma}$.

For $\phi \in A(\Phi)^\leq = \Phi$ we define $\gamma(\phi) \in \Gamma^{\Phi_r}$ by $\gamma(\phi)_{\phi\theta} = \gamma_\theta$ whenever $\theta \in \mathcal{A}_r$ with $\text{Im}(\theta) \subseteq \text{Dom}(\phi)$.

We think of the first notation for atoms in Definition 4.14 as ‘an atom of type θ on ψ ’. In the following definition illustrated on the above example of $\gamma[\phi']^\theta$ with $\theta = (1 \mapsto 1, 3 \mapsto 3)$ we think of $\{\theta\} \in T_{13}$ as the ‘blue edge’ type with $(\gamma[\phi']_{\phi'}^\theta, \gamma[\phi']_{\phi' \circ (12)}^\theta) = (\gamma_{id}^\theta, \gamma_{(12)}^\theta) = (\gamma_{1 \mapsto 1, 3 \mapsto 3}, \gamma_{2 \mapsto 1, 3 \mapsto 3}) = (e_{13}, e_{13})$. The possibility of a zero type is not relevant to our example, as it allows for non-edges when decomposing into copies of a non-complete graph. The ‘red edge’ type in T_{12} is $\{(1 \mapsto 1, 2 \mapsto 2), (1 \mapsto 2, 2 \mapsto 1)\}$, as $(\gamma_{id}^{1 \mapsto 1, 2 \mapsto 2}, \gamma_{(12)}^{1 \mapsto 1, 2 \mapsto 2}) = (\gamma_{1 \mapsto 1, 2 \mapsto 2}, \gamma_{1 \mapsto 2, 2 \mapsto 1}) = (e_{12}, e_{12})$ and $(\gamma_{id}^{1 \mapsto 2, 2 \mapsto 1}, \gamma_{(12)}^{1 \mapsto 2, 2 \mapsto 1}) = (\gamma_{1 \mapsto 2, 2 \mapsto 1}, \gamma_{1 \mapsto 1, 2 \mapsto 2}) = (e_{12}, e_{12})$.

Definition 4.15. (types) For $\theta \in \mathcal{A}_B$ we define $\gamma^\theta \in \Gamma^{\Sigma^B}$ by $\gamma_\sigma^\theta = \gamma_{\theta\sigma}$.

A type $t = [\theta]$ in γ is an equivalence class of the relation \sim on any \mathcal{A}_B with $B \in Q = \binom{[q]}{r}$ where $\theta \sim \theta'$ iff $\gamma^\theta = \gamma^{\theta'}$. We write T_B for the set of types in \mathcal{A}_B .

For $\theta \in t \in T_B$ and $\psi \in \Phi_B$ we write $\gamma^t = \gamma^\theta$ and $\gamma[\psi]^t = \gamma[\psi]^\theta$.

If $\gamma^t = 0$ call t a zero type and write $t = 0$.

If $\phi \in \mathcal{A}(\Phi)$ with $\gamma(\phi)^{\psi\Sigma} = \gamma[\psi]^t$ we write $t_\phi(\psi) = t$.

Now we formulate our regularity assumption. The following definition can be roughly understood as saying that the vector J can be approximated by a non-negative linear combination of molecules, where all molecules that can be used (in that J contains all their atoms) are used with comparable weights (up constant factors). For example, suppose $J = G^* \in (\mathbb{Z}^3)^{\Phi_2}$ encodes G as above. An atom decomposition expresses J as a sum where each summand encodes a coloured edge of G by

some atom $\gamma[\psi^O]^t$ as discussed above. We have $\phi \in \mathcal{A}(\Phi, J)$ iff the molecule $\gamma(\phi)$ encodes a rainbow triangle in G . Then G^* is (γ, c, ω) -regular if we can assign each rainbow triangle in G a weight between ωn^{-1} and $\omega^{-1} n^{-1}$ so that the total weight of triangles on any edge is $1 \pm c$.

Definition 4.16. (regularity)

Suppose γ is elementary and $J \in (\mathbb{Z}^D)^{\Phi_r}$ with $J^O \in \langle \gamma[O] \rangle$ for all $O \in \Phi_r/\Sigma$. For $\psi \in \Phi_B$ with $|B| = r$ we define integers J_ψ^t for all nonzero $t \in T_B$ by $J^{\psi\Sigma} = \sum_{0 \neq t \in T_B} J_\psi^t \gamma[\psi]^t$. Any choice of orbit representatives $\psi^O \in \Phi_{BO}$ for each orbit $O \in \Phi_r/\Sigma$ defines an atom decomposition $J = \sum_{O \in \Phi_r/\Sigma} \sum_{0 \neq t \in T_{BO}} J_{\psi^O}^t \gamma[\psi^O]^t$.

Let $\mathcal{A}(\Phi, J) = \{\phi \in \mathcal{A}(\Phi) : \gamma(\phi) \leq_\gamma J\}$. We say J is (γ, c, ω) -regular (in Φ) if there is $y \in [\omega n^{r-q}, \omega^{-1} n^{r-q}]^{\mathcal{A}(\Phi, J)}$ such that for all $B \in Q$, $\psi \in \Phi_B$, $0 \neq t \in T_B$ we have

$$\partial^t y_\psi := \sum \{y_\phi : t_\phi(\psi) = t\} = (1 \pm c) J_\psi^t.$$

We require one further definition, used in the extendability hypothesis of Theorem 4.18 below; in our example it says that for any Φ -extension $E = (J, F, \phi)$ of rank h there are many $\phi^+ \in X_E(\Phi)$ such that all edges $Im(\phi^+ \psi)$ with $\psi \in J_2 \setminus J[F]$ are edges of G with the correct colour (red if $\psi \in J_{12}$, blue if $\psi \in J_{13}$, green if $\psi \in J_{23}$).

Definition 4.17. For $J \in \Gamma^{\Phi_r}$ we let $\gamma[J] = (\gamma[J]^A : A \in \mathcal{A})$ where each $\gamma[J]^A$ is the set of $\psi \in A(\Phi)_r^\leq = \Phi_r$ such that $\gamma(\psi) \leq_\gamma J$.

Finally we can state the main result (Theorem 3.1) of [11] (recall $h = 2^{50q^3}$ and $\delta = 2^{-10^3 q^5}$).

Theorem 4.18. For any $q \geq r$ and D there are ω_0 and n_0 such that the following holds for $n > n_0$, $n^{-\delta} < \omega < \omega_0$ and $c \leq \omega^{h^{20}}$. Let \mathcal{A} be a Σ^\leq -family with $\Sigma \leq S_q$. Suppose $\gamma \in (\mathbb{Z}^D)^{\mathcal{A}_r}$ is elementary. Let Φ be a Σ -adapted $[q]$ -complex on $[n]$. Let $G \in \langle \gamma(\Phi) \rangle$ be (γ, c, ω) -regular in Φ such that $(\Phi, \gamma[G]^A)$ is (ω, h) -extendable for each $A \in \mathcal{A}$. Then G has a $\gamma(\Phi)$ -decomposition.

5 Coloured hypergraphs

When can an edge-coloured graph be decomposed into rainbow triangles? In this section we illustrate the application of Theorem 4.18 to this question, and a hypergraph generalisation thereof. We start by formulating the general problem of decomposing an edge-coloured r -multigraph G by an edge-coloured r -graph H . For simplicity we assume that H is simple (one could allow multiple copies of edges in H provided they have distinct colours, but not multiple edges of a given colour, as then the associated γ in Definition 5.8 below is not elementary).

Definition 5.1. Suppose H is an r -graph on $[q]$, edge-coloured as $H = \cup_{d \in [D]} H^d$. We identify H with a vector $H \in (\mathbb{N}^D)^Q$, where each $(H_f)_d = 1_{f \in H^d}$.

Let Φ be an S_q -adapted $[q]$ -complex on $[n]$. For $\phi \in \Phi_q$ we define $\phi(H) \in (\mathbb{N}^D)^{\Phi_r^\circ}$ by $\phi(H)_{\phi(f)} = H_f$. Let \mathcal{H} be an family of $[D]$ -edge-coloured r -graphs on $[q]$. Let $\mathcal{H}(\Phi) = \{\phi(H) : \phi \in \Phi_q, H \in \mathcal{H}\}$.

Let $G \in \mathbb{N}^{\Phi_r^\circ}$ be an r -multigraph $[D]$ -edge-coloured as $G = \cup_{d \in [D]} G^d$, identified with $G \in (\mathbb{N}^D)^{\Phi_r^\circ}$. We call $\mathcal{H}' \subseteq \mathcal{H}(\Phi)$ with $\sum \mathcal{H}' = G$ an H -decomposition of G in Φ . We call $\Phi \in \mathbb{Z}^{\mathcal{H}(\Phi)}$ with $\sum_{H'} \Phi_{H'} H' = G$ an integral H -decomposition of G in Φ .

Note that copies of H in an integral H -decomposition of G can use edges $e \in \Phi_r^\circ$ with $G_e = 0$ or with the wrong colour, but all such terms must cancel. Before considering the general setting of the

previous definition, we warm up by specialising to graphs ($r = 2$) and the case that Φ is the complete $[q]$ -complex on $[n]$. We formulate a typicality condition for coloured graphs and a result on rainbow triangle decompositions analogous to that given in [12] for triangle decompositions of typical graphs.

Definition 5.2. Let G be a $[D]$ -edge-coloured graph on $[n]$. For $\gamma \in [D]$, the γ -density of G is $d(G^\gamma) = |G^\gamma| \binom{n}{2}^{-1}$. The density of G is $d(G) = |G| \binom{n}{2}^{-1}$. The density vector of G is $d(G)^* \in [0, 1]^D$ with $d(G)_\gamma^* = d(G^\gamma)$. Given vertices (x_1, \dots, x_t) with each $x_i \in [n]$ and colours $\gamma \in [D]^t$ we define the γ -degree $d_G^\gamma(x)$ of x in G as the number of vertices y such that $x_i y \in G^{\gamma_i}$ for all $i \in [t]$.

We say G is (c, h) -typical if $d_G^\gamma(x) = (1 \pm tc)n \prod_{i=1}^t d(G^{\gamma_i})$ for any such x and γ with $t \leq h$.

Theorem 5.3. Suppose G is a tridivisible (c, h) -typical $[D]$ -edge-coloured graph on $[n]$, where $D \geq 4$, $n > n_0(D)$ is large, $h = 2^{10^3}$, $\delta = 2^{-10^6}$, $c < c_0 d(G)^{h^{90}}$ where $c_0 = c_0(D)$ is small, and each $n^{-\delta/2h^3} < d(G^\gamma) < (1/3 - n^{-\delta/2h^3})d(G)$. Then G has a rainbow triangle decomposition.

Note that the tridivisibility condition (G has all degrees even and $3 \mid e(G)$) in Theorem 5.3 is necessary, as if we ignore the colours then we obtain a triangle decomposition of G ; it is perhaps surprising that the colours do not impose any additional condition. We will deduce Theorem 5.3 from a more general result on typical r -multigraphs, as in the following definition.

Definition 5.4. Let G be a $[D]$ -edge-coloured r -multigraph on $[n]$. For $\gamma \in [D]$, the γ -density of G is $d(G^\gamma) = |G^\gamma| \binom{n}{r}^{-1}$. The density of G is $d(G) = |G| \binom{n}{r}^{-1}$. The density vector of G is $d(G)^* \in \mathbb{R}^D$ with $d(G)_\gamma^* = d(G^\gamma)$.

For $e \subseteq [n]$, the degree of e in G is $|G(e)|$; the degree vector is $G(e)^* \in \mathbb{N}^D$ with $G(e)_\gamma^* = |G^\gamma(e)|$.

Given $f = (f_1, \dots, f_t)$ with each $f_i \in \binom{[n]}{r-1}$ and colours $\gamma \in [D]^t$ we define the γ -degree of f in G as $d_G^\gamma(f) = \sum_{v \in [n]} \prod_{i=1}^t G_{f_i+v}^{\gamma_i}$.

We say G is (c, h) -typical if $d_G^\gamma(f) = (1 \pm tc)n \prod_{i=1}^t d(G^{\gamma_i})$ for any such f and γ with $t \leq h$.

Given a family \mathcal{H} of $[D]$ -edge-coloured r -graphs on $[q]$, we say G is (b, c) -balanced wrt \mathcal{H} if there is $p \in [b, b^{-1}]^{\mathcal{H}}$ with $d(G)^* = (1 \pm c) \sum_H p_H d(H)^*$.

We say G is \mathcal{H} -divisible if each $G(e)^* \in \langle H(f)^* : f \in \binom{[q]}{e}, H \in \mathcal{H} \rangle$.

In the next lemma we show that in the case of rainbow triangles, the conditions in Definition 5.4 follow from the assumptions of Theorem 5.3.

Lemma 5.5. Let \mathcal{H} be the family of $[D]$ -edge-coloured rainbow triangles and G be a $[D]$ -edge-coloured graph on $[n]$, with $D \geq 4$. Then

- i. G is \mathcal{H} -divisible iff G is tridivisible, and
- ii. If each $bD^2 < d(G^\gamma) < (1/3 - bD^3)d(G)$ then G is $(b, 0)$ -balanced wrt \mathcal{H} .

Proof. For (i), we need to know the integer span $Z(r, s)$ of the rows of a matrix $M(r, s)$ whose rows are indexed by $\binom{[s]}{r}$ and columns by $[s]$, with $M(r, s)_{e,i} = 1_{i \in e}$. It follows from [36, Theorem 2] (and is not hard to show directly) that $Z(r, s) = \{x \in \mathbb{Z}^s : r \mid \sum_i x_i\}$ for $s > r$. To apply this to the divisibility conditions, first consider $G(\emptyset)^* = (|G^1|, \dots, |G^D|)$ and note that $H(\emptyset)^* = (|H^1|, \dots, |H^D|)$ for $H \in \mathcal{H}$ are the rows of $M(3, D)$. We have $G(\emptyset)^* \in \langle H(\emptyset)^* : H \in \mathcal{H} \rangle$ iff $3 \mid \sum_\gamma |G^\gamma| = |G|$. Next, for any $v \in [n]$ we have $G(v)^* = (|G^1(v)|, \dots, |G^D(v)|)$. As $H(x)^* = (|H^1(x)|, \dots, |H^D(x)|)$ for $x \in [q]$, $H \in \mathcal{H}$ are the rows of $M(2, D)$ we have $G(v)^* \in \langle H(x)^* : x \in [q], H \in \mathcal{H} \rangle$ iff $2 \mid \sum_\gamma |G^\gamma(v)| = |G(v)|$. Finally, for any $uv \in \binom{[n]}{2}$ we have $G(uv)^* = (G_{uv}^1, \dots, G_{uv}^D)$ and $H(xy)^*$ for $xy \in \binom{[q]}{2}$, $H \in \mathcal{H}$ is the standard basis, so the 2-divisibility condition is trivial. Thus G is \mathcal{H} -divisible iff G is tridivisible.

For (ii), we note that the set of density vectors $d(H)^*$ for $H \in \mathcal{H}$ consists of all probability distributions on $[D]$ with 3 coordinates equal to $1/3$ and the rest zero. By [8, Theorem 46], any probability distribution x on $[D]$ is a convex combination of the vectors $d(H)^*$ iff $x_\gamma \leq 1/3$ for all $\gamma \in [D]$. Thus for any $x \in [0, 1]^D$ with each $3x_{\gamma'} \leq \sum_\gamma x_\gamma \leq 1$ there is some $p \in [0, 1]^{\mathcal{H}}$ with $x = \sum_H p_H d(H)^*$ and $\sum_H p_H = \sum_\gamma x_\gamma$. We apply this to $x = d(G)^* - b \sum_H d(H)^*$, noting that $\sum_\gamma x_\gamma = d(G) - b \binom{D}{3}$ and each $0 \leq x_\gamma = d(G^\gamma) - b \binom{D-1}{2} \leq \frac{1}{3} \sum_\gamma x_\gamma$. Then $p' = p + b \in [b, b^{-1}]^{\mathcal{H}}$ has $d(G)^* = \sum_H p'_H d(H)^*$. \square

Next we consider how to encode decompositions of coloured multigraphs in the labelled edge setting of Theorem 4.18; this is similar to the running example used in the previous section.

Definition 5.6. Given a set e of size r , we write $e^{r \rightarrow q}$ for the set of all π^{-1} where $\pi : e \rightarrow [q]$ is injective. Given a $[D]$ -edge-coloured r -multigraph $G = (G^d : d \in [D])$ we define $G^{r \rightarrow q} = ((G^{r \rightarrow q})^d : d \in [D])$ where each G^d is the (disjoint) union of all $e^{r \rightarrow q}$ with $e \in G^d$.

Lemma 5.7. Let H and G be $[D]$ -edge-coloured r -multigraphs, $H^* = H^{r \rightarrow q}$ and $G^* = G^{r \rightarrow q}$. Then an (integral) H -decomposition of G is equivalent to an (integral) H^* -decomposition of G^* .

Proof. We associate any H -decomposition \mathcal{H} of G with an H^* -decomposition \mathcal{H}^* of G^* , associating each $\phi(H) \in \mathcal{H}$ with $\phi H^* := \{\phi \circ \theta : \theta \in H^*\} \in \mathcal{H}^*$. Then $e \in \phi(H^d)$ iff $e^{r \rightarrow q} \subseteq \phi H^{*d}$, as if $e = \phi(f)$ for some $f \in H^d$ and $\pi^{-1} \in e^{r \rightarrow q}$ then $\pi^{-1} = \phi\theta$, where $\theta = \phi^{-1}\pi^{-1} \in H^{*d}$, and conversely. The same proof applies to integral decompositions. \square

Definition 5.8. Given a family \mathcal{H} of $[D]$ -edge-coloured r -graphs on $[q]$, let $\mathcal{A} = \mathcal{A}^{\mathcal{H}} = \{A^H : H \in \mathcal{H}\}$ with each $A^H = S_q^{\leq}$ and $\gamma = \gamma^{\mathcal{H}} \in (\mathbb{Z}^D)^{\mathcal{A}^{\mathcal{H}}}$ with $\gamma_\theta = e_d$ if $\theta \in A_r^H$, $H \in \mathcal{H}$, $d \in [D]$ with $\text{Im}(\theta) \in H^d$ or $\gamma_\theta = 0$ otherwise.

Lemma 5.9. With notation as in Definitions 5.1, 5.6 and 5.8, an (integral) \mathcal{H} -decomposition of G is equivalent to an (integral) $\gamma(\Phi)$ -decomposition of G^* .

Furthermore, if Φ is (ω, s) -extendable with $s = 3r^2$, $\omega > n^{-1/2}$ and $n > n_0(q)$ large then G has an integral \mathcal{H} -decomposition in Φ_q iff G is \mathcal{H} -divisible.

Proof. The first statement is immediate from Lemma 5.7 and Definition 5.8. For the second statement, by Lemma 4.13 we have $\langle \gamma(\Phi) \rangle = \mathcal{L}_\gamma(\Phi)$. By Definition 4.12 we need to show that G is \mathcal{H} -divisible iff $((G^*)^\sharp)^O \in \langle \gamma^\sharp[O] \rangle$ for any $O \in \Phi/S_q$.

Fix any $O \in \Phi/S_q$, write $e = \text{Im}(O) \in \Phi^\circ$ and $i = |e|$. Then $((G^*)^\sharp)^O \in ((\mathbb{Z}^D)^Q)^O = (\mathbb{Z}^D)^{Q \times O}$ is a vector supported on the coordinates (B, ψ') with $B' \subseteq B \in Q$ and $\psi' \in O \cap \Phi_{B'}$ with each $((G^*)^\sharp_{\psi'})_B = \sum \{G_{\psi'}^* : \psi' \subseteq \psi \in \Phi_B\} = (r - i)! G(e)^* \in \mathbb{N}^D$.

Also, $\langle \gamma^\sharp[O] \rangle$ is generated by γ^\sharp -atoms $\gamma^\sharp(v)$ at O , each of which is supported on the same coordinates (B, ψ') as $((G^*)^\sharp)^O$, with each $(\gamma^\sharp(v)_{\psi'})_B$ equal to some $(r - i)! H(f)^*$ with $f \in \binom{[q]}{|e|}$, $H \in \mathcal{H}$. The lemma follows. \square

Now we state our theorem on decompositions of typical coloured r -multigraphs. By Lemma 5.5 it implies Theorem 5.3. We will deduce it from Theorem 5.13 below.

Theorem 5.10. Let \mathcal{H} be a family of $[D]$ -edge-coloured r -graphs on $[q]$. Suppose G is a (c, h^q) -typical $[D]$ -edge-coloured r -multigraph on $[n]$ with all $G^d < b^{-1}$ that is (b, c) -balanced wrt \mathcal{H} , where $n > n_0(q, D)$ is large, $d(G) > b := n^{-\delta/h^q}$, $c < c_0 d(G)^{h_{30q}^q}$ and $c_0 = c_0(q)$ is small. Then G has an \mathcal{H} -decomposition iff G is \mathcal{H} -divisible.

The next definition formulates the extendability and regularity conditions for coloured hypergraph decompositions; we will see below that they both follow from typicality.

Definition 5.11. With notation as in Definition 5.1, we say $G \in (\mathbb{N}^D)^{\Phi_r^\circ}$ is (\mathcal{H}, c, ω) -regular in Φ if there are $y_\phi^H \in [\omega n^{r-q}, \omega^{-1} n^{r-q}]$ for each $H \in \mathcal{H}$, $\phi \in \Phi_q$ with $\phi(H) \leq G$ so that $\sum \{y_\phi^H \phi(H)\} = (1 \pm c)G$. We say that (Φ, G) is (ω, h) -extendable if (Φ, G') is (ω, h) -extendable, where $G' = (G^1, \dots, G^D)$.

The next theorem shows extendability and regularity suffice for the equivalence of decomposition and integral decomposition. For wider applicability we formulate it in the setting of exactly adapted complexes, as in the following definition, which allows for an S_q -adapted $[q]$ -complex (such as the complete $[q]$ -complex, suppressed in the statement of Theorem 5.10), or a generalised partite complex, which is exactly Σ -adapted for some subgroup Σ of S_q (such as that in the running example of the previous section).

Definition 5.12. We say that an R -complex Φ is exactly Σ -adapted if whenever $\phi \in \Phi_B$ and $\tau \in \text{Bij}(B', B)$ we have $\phi \circ \tau \in \Phi_{B'}$ iff $\sigma \in \Sigma_{B'}$. We say Φ is exactly adapted if Φ is exactly Σ -adapted for some Σ .

Theorem 5.13. Let \mathcal{H} be an family of $[D]$ -edge-coloured r -graphs on $[q]$. Let Φ be an (ω, h) -extendable exactly adapted $[q]$ -complex on $[n]$ where $n > n_0(q, D)$ is large, $n^{-\delta} < \omega < \omega_0(q, D)$ is small and $c = \omega^{h^{20}}$. Suppose $G \in (\mathbb{N}^D)^{\Phi_r^\circ}$ is (\mathcal{H}, c, ω) -regular in Φ and (Φ, G) is (ω, h) -extendable. Then G has an \mathcal{H} -decomposition in Φ_q iff G has an integral \mathcal{H} -decomposition in Φ_q .

Proof. By Lemma 5.9, it is equivalent to consider $\gamma(\Phi)$ -decompositions of G^* , with notation as in Definitions 5.6 and 5.8. There are $D + 1$ types in γ for each $B \in Q$: the colour d type $\{\theta \in A_B^H : \text{Im}(\theta) \in H^d, H \in \mathcal{H}\}$ for each $d \in [D]$, and the nonedge type $\{\theta \in A_B^H : \text{Im}(\theta) \notin H \in \mathcal{H}\}$. Each γ^θ is e_d in all coordinates for θ in a colour d type or 0 in all coordinates for θ in a nonedge type, so γ is elementary. The atom decomposition of G^* is $G^* = \sum_{f \in \Phi_r^\circ} \sum_{d \in [D]} (G_f)_d f^d$, where $f_\psi^d = e_d$ for all $\psi \in \Phi_r$ with $\text{Im}(\psi) = f$.

As G is (\mathcal{H}, c, ω) -regular in Φ we have $\sum \{y_\phi^H \phi(H)\} = (1 \pm c)G$ for some $y_\phi^H \in [\omega n^{r-q}, \omega^{-1} n^{r-q}]$ for each $H \in \mathcal{H}$, $\phi \in \Phi_q$ with $\phi(H) \leq G$. For any such $\phi \in H(\Phi)$ we have $\gamma(\phi) \leq_\gamma G^*$, so $\phi \in \mathcal{A}(\Phi, G^*)$. Let $y_\phi = y_\phi^H$ for $\phi \in A^H(\Phi)$. For any $B \in Q$, $\psi \in \Phi_B$, $d \in [D]$, writing $t_d \in T_B$ for the colour d type, $\partial^{t_d} y_\psi = \sum \{y_\phi : t_\phi(\psi) = t_d\} = \sum \{y_\phi^H : \text{Im}(\psi) \in \phi(H^d), H \in \mathcal{H}\} = (1 \pm c)(G^*)_{\psi}^{t_d}$, so G^* is (γ, c, ω) -regular.

To apply Theorem 4.18, it remains to show that each $(\Phi, \gamma[G]^H)$ is (ω, h) -extendable. If $B \notin H$ then $\gamma[G]_B^H = \Phi_B$ and if $B \in H^d$ for $d \in [D]$ then $\gamma[G]_B^H = \{\psi \in \Phi_B : \text{Im}(\psi) \in G^d\}$. Consider any Φ -extension $E = (J, F, \phi)$ of rank s and $J' \subseteq J_r \setminus J[F]$. Let $J'' = (J^d : d \in [D])$ with each $J^d = \bigcup \{J'_B : B \in H^d\}$. As (Φ, G) is (ω, h) -extendable we have $X_{E, J''}(\Phi, G) > \omega n^{v_E}$. Consider any $\phi^+ \in X_{E, J''}(\Phi, G)$. For any $\psi \in J^d$ we have $\phi^+ \psi \in \Phi$ and $\text{Im}(\phi^+ \psi) \in G^d$, so $\phi^+ \psi \in \gamma[G]^H$. Thus $\phi^+ \in X_{E, J'}(\Phi, \gamma[G]^H)$, so $(\Phi, \gamma[G]^H)$ is (ω, h) -extendable. \square

Now we show that the extendability and regularity conditions follow from typicality, thus deducing our decomposition result for typical coloured r -multigraphs.

Proof of Theorem 5.10. Suppose G is an \mathcal{H} -divisible (c, h^q) -typical $[D]$ -edge-coloured r -multigraph on $[n]$ that is (b, c) -balanced wrt \mathcal{H} , where $n > n_0(q, D)$ is large, $d(G) > b := 2n^{-\delta/h^q}$, $c < c_0 d(G)^{h^{30q}}$ and $c_0 = c_0(q)$ is small. We need to show that G has an \mathcal{H} -decomposition.

Let Φ be the complete $[q]$ -complex on $[n]$. By Lemma 5.9 and \mathcal{H} -divisibility, G has an integral \mathcal{H} -decomposition in Φ_q . Let $p \in [b, b^{-1}]^{\mathcal{H}}$ with $d(G)^* = (1 \pm c) \sum_H p_H d(H)^*$. We can assume each

colour $\gamma \in [D]$ is used at least once by \mathcal{H} , so $d(G^\gamma) \geq b/2Q$, where $Q = \binom{q}{r}$. To apply Theorem 5.13, it remains to check extendability and regularity.

We claim that (Φ, G) is (ω, h) -extendable with $\omega > n^{-\delta}$. To see this, consider any Φ -extension $E = (J, F, \phi)$ with $J \subseteq [q](h)$ and $J' = (J^d : d \in [D])$ for some mutually disjoint $J^d \subseteq J^\circ \setminus J^\circ[F]$. Let $V(J) \setminus F = \{x_1, \dots, x_{v_E}\}$. For $i \in [v_E]$ we list the neighbourhood $J'(x_i)$ of x_i as $f^i = (f_1^i, \dots, f_{t_i}^i)$ and let $\gamma^i \in [D]^{[t_i]}$ be such that each $f_j^i + x_i \in J^{\gamma_j^i}$. Then the number of choices for x_i (weighted by edge-multiplicities) given any previous choices $\phi' \upharpoonright_{\{x_j : j < i\}}$ is $d_G^{\gamma^i}(\phi'(f^i)) = (1 \pm t_i c) n \prod_{j=1}^{t_i} d(G^{\gamma_j^i})$. As each $d(G^d) > b/2Q$ with $b = n^{-\delta/h^q}$, we deduce $X_{E, J'}(\Phi, G) = \sum_{\phi \in X_E(\Phi)} \prod_{d \in [D]} \prod_{f \in J^d} G_{\phi(f)}^d > n^{v_E - \delta}$.

As for regularity, the above for $J = [q](1)$, $J' = (H^d : d \in [D])$, $F = f \in H^\gamma$ with $H \in \mathcal{H}$, $\gamma \in [D]$, and $\psi \in \text{Bij}(f, e)$ with $e \in G^\gamma$ gives $X_{E, J'}(\Phi, G) = (1 \pm Qc) d(G^\gamma)^{-1} n^{q-r} \prod_{d \in [D]} d(G^d)^{|H^d|}$. Let $Z = n^{q-r} \prod_{d \in [D]} d(G^d)^{|H^d|}$ and $y_\phi = p_H(q)_r^{-1} Z^{-1} \prod_{d \in [D]} \prod_{f \in H^d} G_{\phi(f)}^d$ for each $\phi \in \mathcal{H}(\Phi)$ with $H \in \mathcal{H}$. Then each such $y_\phi \in [\omega n^{r-q}, \omega^{-1} n^{r-q}]$, as all $d(G^d) > b/2Q$, $p_H < b^{-1}$ and $G_{\phi(f)}^d < b^{-1}$. Letting f vary over H^γ , we have

$$\begin{aligned} \sum_H \sum_\phi y_\phi(\phi(H)_e)_\gamma &= \sum_H p_H r! (q)_r^{-1} \sum_{f \in H^\gamma} Z^{-1} \sum_{\phi \in X_E(\Phi)} \prod_{d \in [D]} \prod_{f \in H^d} G_{\phi(f)}^d \\ &= \sum_H p_H Q^{-1} \sum_{f \in H^\gamma} (1 \pm 2Qc) d(G^\gamma)^{-1} G_e^\gamma = (1 \pm q^r c) G_e^\gamma. \end{aligned}$$

Thus G is $(\mathcal{H}, q^r c, \omega)$ -regular in Φ . \square

We conclude with a theorem on coloured generalised partite decompositions, which can be used (we omit the details) to obtain a common generalisation of Theorems 3.2 and 5.10.

Definition 5.14. Let \mathcal{H} be a family of $[D]$ -edge-coloured r -graphs on $[q]$ and $\mathcal{P} = (P_1, \dots, P_t)$ be a partition of $[q]$. Let $I^d = \{i : \cup_H H_i^d \neq \emptyset\}$ and $I = \cup_d I^d$.

Let Σ be the group of all $\sigma \in S_q$ with all $\sigma(P_i) = P_i$. Let Φ be an exactly Σ -adapted $[q]$ -complex with parts $\mathcal{Q} = (Q_1, \dots, Q_t)$, where each $Q_i = \{\psi(j) : j \in P_i, \psi \in \Phi_j\}$. Let $G \in (\mathbb{N}^D)^{\Phi_r^\circ}$. We call G an $(\mathcal{H}, \mathcal{P})$ -blowup if $G_i^d \neq \emptyset \Rightarrow i \in I^d$.

For $e \subseteq [n]$, $f \subseteq [q]$ we define $G(e)^*, H(f)^* \in (\mathbb{N}^D)^I$ by $(G(e)_i^*)_d = G_i^d(e)$, $(H(f)_i^*)_d = H_i^d(f)$. We say G is $(\mathcal{H}, \mathcal{P})$ -divisible if each $G(e)^* \in \langle H(f)^* : f \in \binom{[q]}{e}, H \in \mathcal{H} \rangle$.

In the following extendability hypothesis we consider G_i^d undefined for $i \notin I(H^d)$.

Theorem 5.15. *With notation as in Definition 5.14, suppose $n/h \leq |Q_i| \leq n$ with $n > n_0(q, D)$, G is an $(\mathcal{H}, \mathcal{P})$ -divisible $(\mathcal{H}, \mathcal{P})$ -blowup, G is (\mathcal{H}, c, ω) -regular in Φ , and (Φ, G) is (ω, h) -extendable, where $n^{-\delta} < \omega < \omega_0(q, D)$ and $c = \omega^{h^{20}}$. Then G has a \mathcal{P} -partite \mathcal{H} -decomposition.*

Proof. By Theorem 5.13 it suffices to show that G has an integral \mathcal{H} -decomposition in Φ_q , i.e. $G^* \in \langle \gamma(\Phi) \rangle = \mathcal{L}_\gamma(\Phi)$ (by Lemmas 5.9 and 4.13). Consider any $i \in I$ and $i' \in \mathbb{N}^t$ with all $i'_j \leq i_j$. Let $m_{i'}^i = \prod_{j \in [t]} (i_j - i'_j)!$. For any $B' \subseteq B \in \mathcal{Q}$ with $i_{\mathcal{P}}(B') = i'$ and $i_{\mathcal{P}}(B) = i$ and $\psi' \in \Phi_{B'}$ with $\text{Im}(\psi') = e$ we have $((G^*)_{\psi'}^\sharp)_B = \sum \{G_{\psi'}^* : \psi' \subseteq \psi \in \Phi_B\} = m_{i'}^i G_i(e)^* \in \mathbb{N}^D$. Writing $O = \psi' \Sigma$, for any $\psi \in O$ we have $((G^*)_{\psi}^\sharp)_B = m_{i'}^i G_i(e)^*$. Thus we obtain $((G^*)^\sharp)^O$ from $G(e)^*$ by copying coordinates and multiplying all copies of each i -coordinate by $m_{i'}^i$. Similarly, for any $H \in \mathcal{H}$, $\theta' \in A_{B'}^H$, $f = \text{Im}(\theta')$ we have $(\gamma_{\theta'}^\sharp)_B = \sum \{\gamma_\theta : \theta' \subseteq \theta \in A_B^H\} = m_{i'}^i H_i(f)^*$, so $\langle \gamma^\sharp[O] \rangle$ is generated by vectors $v^{Hf} \in (\mathbb{Z}^Q)^O$ where $H \in \mathcal{H}$, $f \subseteq [q]$ with $i_{\mathcal{P}}(f) = i'$ and for each $\psi \in O$, $B \in \mathcal{Q}$ we have

$(v_\psi^{Hf})_B = m_i^i H_i(f)^*$ where $i = i_{\mathcal{P}}(B)$. Thus all vectors in $\langle \gamma^\sharp[O] \rangle$ are obtained from vectors $H(f)^*$ with $H \in \mathcal{H}$ and $i_{\mathcal{P}}(f) = i_{\mathcal{Q}}(e)$ by the same transformation that maps $G(e)^*$ to $((G^*)^\sharp)^O$. As G is $(\mathcal{H}, \mathcal{P})$ -divisible we deduce $((G^*)^\sharp)^O \in \langle \gamma^\sharp[O] \rangle$ for any $O \in \Phi/\Sigma$, as required. \square

6 Directed hypergraphs

Our second illustration of Theorem 4.18 will be to decompositions of directed hypergraphs.

Definition 6.1. Let R be a set. An R -graph on V is a set G of injections from R to V . We call the elements of G arcs. If $R = [r]$ we call G an r -digraph. We say G is simple if $(Im(e) : e \in G)$ are all distinct. A copy of an R -graph H in an R -graph G is defined by an injection $\phi : V(H) \rightarrow V(G)$ such that $\phi H := \{\phi \circ e : e \in H\} \subseteq G$. An H -decomposition of G is a partition of G into copies of H .

Note that if $r = 2$ then a 2-digraph is equivalent to a digraph in the usual sense: we can think of an injection $f : [2] \rightarrow V$ as an arc directed from $f(1)$ to $f(2)$.

We will restrict our attention to H -decomposition problems in which H is simple; otherwise we obtain a non-elementary functional decomposition problem, which has arithmetic structure, and to which Theorem 4.18 does not apply.

Next we will state an example of our later theorem on r -digraph decompositions. Let KD_n^r denote the complete r -digraph on $[n]$, i.e. each of the $(n)_r = r!(\binom{n}{r})$ injections from $[r]$ to $[n]$ is an arc. The r -digraph tight q -cycle \circlearrowright_q^r has vertex set $[q]$ and arc set $\{\phi_j : j \in [q]\}$ with each $\phi_j(i) = i + j$, where addition wraps (we identify $q + i$ with i).

Theorem 6.2. Suppose $q > r \geq 2$ and $n > n_0(q)$ with $q \mid (n)_r$. Then KD_n^r has a \circlearrowright_q^r -decomposition.

Now we will describe the divisibility conditions in the general setting, and then illustrate them in the case $H = \circlearrowright_q^r$.

Definition 6.3. Let G be an r -digraph on $[n]$ and H be an r -digraph on $[q]$.

Given an injection $f : R' \rightarrow [n]$ with $R' \subseteq R$, we let $G|_f = \{e \in G : e|_{R'} = f\}$. The neighbourhood of f in G is the $(R \setminus R')$ -graph $G(f) = \{e|_{R \setminus R'} : e \in G|_f\}$. The degree of f in G is $|G(f)|$.

We write I_t^s for the set of injections $\pi : [s] \rightarrow [t]$. For $\psi \in I_n^i$ we define the degree vector $G(\psi)^* \in \mathbb{N}^{I_r^i}$ by $G(\psi)_\pi^* = |G(\psi\pi^{-1})|$.

We say G is H -divisible if $G(\psi)^* \in \langle H(\theta)^* : \theta \in I_q^i \rangle$ for all $0 \leq i \leq r$, $\psi \in I_n^i$.

Now we illustrate Definition 6.3 in the case $H = \circlearrowright_q^r$. For example, suppose $r = 2$, so H and G are digraphs. Writing \emptyset for the element of I_n^0 , we have $G(\emptyset)^* = (|G|)$ and $H(\emptyset)^* = (|H|) = (q)$, so the 0-divisibility condition is $q \mid |G|$. Next, for $\psi \in I_n^1$, writing $x = \psi(1) \in [n]$, we have $G(\psi)^* = (d_G^+(x), d_G^-(x))$, where $d_G^+(x) = |G(\psi)|$ is the number of arcs with $1 \mapsto x$ and $d_G^-(x) = |G(\psi \circ (1 \mapsto 2)^{-1})|$ is the number of arcs with $2 \mapsto x$. Also, for $\theta \in I_q^1$, writing $a = \theta(1) \in [q]$, we have $H(\theta)^* = (d_H^+(a), d_H^-(a)) = (1, 1)$, so the 1-divisibility condition is that G is vertex-regular, i.e. $d_G^+(x) = d_G^-(x)$ for all $x \in [n]$. Finally, for $\psi \in I_n^2$, $\theta \in I_q^2$ writing $x_i = \psi(i)$, $a_i = \theta(i)$, we have $G(\psi)^* = (1_{x_1 x_2 \in G}, 1_{x_2 x_1 \in G})$ and $H(\theta)^* = (1_{a_1 a_2 \in H}, 1_{a_2 a_1 \in H})$, so the 2-divisibility condition holds trivially. Next we describe the \circlearrowright_q^r -divisibility conditions in general.

Definition 6.4. We define an equivalence relation \sim on each I_r^i with $i \leq r$ by $\theta \sim \theta'$ if for some $c \in \mathbb{Z}$ we have $\theta'(j) = \theta(j) + c$ for all $j \in [i]$ (where addition does not wrap). We say that G is shift regular if $G(\psi)_\theta^* = G(\psi)_{\theta'}^*$ whenever $\theta \sim \theta'$.

We note that $G = KD_n^r$ is shift regular, indeed $G(\psi)_\theta^* = (n)_r/(n)_i$ for any $\theta \in I_r^i$, $\psi \in I_n^i$. We also note that there is redundancy (symmetry) in the above definitions. Indeed, for $\psi \in I_n^i$, $\sigma \in S_i$, $\pi \in I_r^i$ we have $G(\psi\sigma)_\pi^* = |G(\psi\sigma\pi^{-1})| = G(\psi)_{\pi\sigma^{-1}}^*$, i.e. $G(\psi\sigma)^* = G(\psi)^*\sigma$, where S_i acts on I_n^i by $\psi \mapsto \psi\sigma = \psi \circ \sigma$ and on I_r^i by $(v\sigma)_\pi = v_{\pi\sigma^{-1}}$. Note that the latter is a right action as $(v(\sigma\tau))_\pi = v_{\pi(\sigma\tau)^{-1}} = v_{\pi\tau^{-1}\sigma^{-1}} = (v\sigma)_{\pi\tau^{-1}} = ((v\sigma)\tau)_\pi$. For any expression $G(\psi)^* = \sum_\theta n_\theta H(\theta)^*$ with $n \in \mathbb{Z}^{I_q^i}$ we have $G(\psi\sigma)^* = G(\psi)^*\sigma = \sum_\theta n_\theta H(\theta)^*\sigma = \sum_\theta n_\theta H(\theta\sigma)^*$, so it suffices to check H -divisibility on a system of coset representatives for the action of S_i on I_n^i . Furthermore, as $\theta \sim \theta'$ iff $\theta\sigma \sim \theta'\sigma$, and as $G(\psi)_{\theta\sigma}^* = |G(\psi(\theta\sigma)^{-1})| = G(\psi\sigma^{-1})_\theta^*$, it suffices to check shift regularity on a system of coset representatives for the action of S_i on I_q^i , e.g. all order-preserving elements.

Lemma 6.5. G is \circlearrowleft_q^r -divisible iff G is shift regular and $q \mid |G|$.

Proof. The 0-divisibility condition is $q \mid |G|$. Fix $0 < i \leq r$. We classify the degree vectors $H(\theta)^*$ with $\theta \in I_q^i$. Note that $H(\theta)^*$ is the all-0 vector unless $\text{Im}(\theta)$ is contained in a cyclic interval of length r . By the cyclic symmetry of \circlearrowleft_q^r we have $H(\theta)^* = H(\theta+c)^*$ for any $c \in [q]$, defining $\theta+c \in I_q^i$ by $\theta(j) = \theta'(j) + c$ (where addition wraps). Thus we can assume $R := \text{Im}(\theta) \subseteq [r]$, i.e. $\theta \in I_r^i$. Note that $\text{id}_{[r]}$ is the unique arc of H containing id_R , so $1 = |H(\text{id}_R)| = H(\theta)_\theta^*$. Similarly, for each $c \in \mathbb{Z}$ such that $R+c \subseteq [r]$ (where addition does not wrap), $\text{id}_{[r]} - c$ is the unique arc of H containing $\text{id}_{R+c} - c$, so $1 = |H(\text{id}_{R+c} - c)| = H(\theta)_{\theta+c}^*$. All other coordinates of $H(\theta)^*$ are zero. We deduce that $H(\theta)^* = H(\theta')^*$ if $\theta \sim \theta'$, or otherwise $H(\theta)$ and $H(\theta')^*$ have disjoint support. Thus $G(\psi)^* \in \langle H(\theta)^* : \theta \in I_q^i \rangle$ iff G is constant on the support of each $H(\theta)^*$, i.e. G is shift regular. \square

Given Lemma 6.5, the case $H = \circlearrowleft_q^r$ of the following result implies Theorem 6.2.

Theorem 6.6. Suppose H is a simple r -digraph on $[q]$ and $n > n_0(q)$ is large. Then KD_n^r has an H -decomposition iff it is H -divisible.

We will deduce Theorem 6.6 from a more general result in which we replace KD_n^r by any r -digraph satisfying certain extendability and regularity conditions. First we encode r -digraph decompositions in the labelled complex setting of Theorem 4.18.

Definition 6.7. Given an injection $f : [r] \rightarrow X$, we write $f^{r \rightarrow q}$ for the set of all $f \circ \pi^{-1}$ where $\pi : [r] \rightarrow [q]$ is order-preserving. Given an r -digraph G , we let $G^{r \rightarrow q}$ be the (disjoint) union of all $f^{r \rightarrow q}$ with $f \in G$.

Lemma 6.8. Let H and G be r -digraphs, $H^* = H^{r \rightarrow q}$ and $G^* = G^{r \rightarrow q}$. Then an (integral) H -decomposition of G is equivalent to an (integral) H^* -decomposition of G^* .

Proof. We associate any H -decomposition \mathcal{H} of G with an H^* -decomposition \mathcal{H}^* of G^* , associating each $\phi H \in \mathcal{H}$ with $\phi H^* \in \mathcal{H}^*$. Then $e \in \phi H$ iff $e^{r \rightarrow q} \subseteq \phi H^*$, as if $e = \phi\theta$ for some $\theta \in H$ and $e\pi^{-1} \in e^{r \rightarrow q}$ then $e\pi^{-1} = \phi\theta^*$, where $\theta^* = \theta\pi^{-1} \in H^*$, and conversely. The same proof applies to integral decompositions. \square

Next we formulate the regularity condition.

Definition 6.9. Let Φ be a $[q]$ -complex on $[n]$, H be an r -digraph on $[q]$, G be an r -digraph on $[n]$, $H^* = H^{r \rightarrow q}$ and $G^* = G^{r \rightarrow q}$. We say G is (H, c, ω) -regular in Φ if there are $y_\phi \in [\omega n^{r-q}, \omega^{-1} n^{r-q}]$ for each $\phi \in \Phi_q$ with $\phi H^* \subseteq G^*$ so that $\sum_\phi y_\phi \phi H^* = (1 \pm c)G^*$.

The following theorem when Φ and G are complete implies Theorem 6.6. Indeed, extendability is clear, and for regularity we let $y_\phi = |H^*|^{-1}(n)_r/(n)_q$ for each $\phi \in I_n^q$, so that for each $\psi \in I_n^r$ we have $\sum_\phi y_\phi (\phi H^*)_\psi = \sum_{\theta \in H^*} |H^*|^{-1}(n)_r(n)_q^{-1} |\{\phi : \psi = \phi\theta\}| = 1$.

Theorem 6.10. *Let H be a simple r -digraph on $[q]$, G be an r -digraph on $[n]$ and Φ be an (ω, h) -extendable S_q -adapted $[q]$ -complex on $[n]$ where $n > n_0(q)$ is large, $n^{-\delta} < \omega < \omega_0(q)$ is small and $c = \omega^{h^{20}}$. Suppose G is (H, c, ω) -regular in Φ and (Φ, G^*) is (ω, h) -extendable. Then G has an H -decomposition in Φ_q iff G is H -divisible.*

Proof. Let $H^* = H^{r \rightarrow q}$ and $G^* = G^{r \rightarrow q}$. Let $\mathcal{A} = \{A\}$ with $A = S_q^{\leq}$ and $\gamma \in \mathbb{Z}^{A_r}$ where each $\gamma_\theta = 1_{\theta \in H^*}$. Then a $\gamma(\Phi)$ -decomposition of G^* is equivalent to an H^* -decomposition of G^* , and so (by Lemma 6.8) to an H -decomposition of G .

Next we note that each type vector γ^θ is either all-0 or equal to 1 exactly on its order-preserving coordinates. Indeed, for any $B \in Q$, $\theta \in A_B$ and $\sigma \in \Sigma^B$, as H is simple we have $\gamma_{\theta\sigma} = 0$ unless $\theta \in H^*$ and σ is order-preserving, in which case $\gamma_{\theta\sigma} = 1$. Thus γ is elementary, with two types for each $B \in Q$: the arc type $\{\theta \in A_B : \theta \in H^*\}$ and the nonarc type $\{\theta \in A_B : \theta \notin H^*\}$. The atom decomposition is $G^* = \sum_{e \in G} e^*$, where $e^* = e^{r \rightarrow q}$.

As G is (H, c, ω) -regular in Φ , we have $\sum_\phi y_\phi \phi H^* = (1 \pm c)G^*$ for some $y_\phi \in [\omega n^{r-q}, \omega^{-1} n^{r-q}]$ for each $\phi \in \Phi_q$ with $\phi H^* \subseteq G^*$. For any such ϕ we have $\gamma(\phi) \leq_\gamma G^*$, so $\phi \in \mathcal{A}(\Phi, G^*)$. Also, for any $B \in Q$ and $\psi \in \Phi_B$, writing $1_B \in T_B$ for the arc type we have $\partial^{1_B} y_\psi = \sum \{y_\phi : t_\phi(\psi) = 1_B\} = \sum \{y_\phi : \psi \in \phi H^*\} = (1 \pm c)(G^*)_{\psi}^{1_B}$, so G^* is (γ, c, ω) -regular.

Next we consider extendability. We have $\gamma[G^*] = \{\psi \in \Phi_r : \gamma(\psi) \leq_\gamma G^*\}$, so $\psi \in \Phi_B$ is in $\gamma[G^*]$ iff (a) no arc in H has image B , or (b) $\psi\theta \in G$ for the unique arc θ in H with $Im(\theta) = B$. Let $E = (J, F, \phi)$ be any Φ -extension of rank s and $J' \subseteq J_r \setminus J[F]$. Let Q' be the set of $B \in Q$ such that there is some $\theta_B \in H$ with $Im(\theta_B) = B$. Let $\theta_B^* \in H^*$ with $Im(\theta_B^*) = Dom(\theta_B^*) = B$, i.e. $\theta_B^* = \theta_B \circ \pi_B$ where $\pi_B : B \rightarrow Dom(\theta_B)$ is order-preserving. Let $E^0 = ([q](s)[V(J)], F, \phi)$ and $J^0 = \bigcup \{\psi\theta_B^* : \psi \in J'_B, B \in Q'\}$. As (Φ, G^*) is (ω, h) -extendable we have $X_{E^0, J^0}(\Phi, G^*) > \omega n^{v_E}$. Consider any $\phi^+ \in X_{E^0, J^0}(\Phi, G^*)$. For any $\psi\theta_B^* \in J^0$ we have $\phi^+ \psi\theta_B^* \in G^*$, so $\phi^+ \psi\theta_B \in G$, so $\phi^+ \psi \in \gamma[G^*]$. Thus $\phi^+ \in X_{E, J'}(\Phi, \gamma[G^*])$, so $(\Phi, \gamma[G^*])$ is (ω, h) -extendable.

To deduce the theorem from Theorem 4.18, it remains to consider divisibility. By Lemma 4.13 we have $\langle \gamma(\Phi) \rangle = \mathcal{L}_\gamma(\Phi)$. By Definition 4.12 we need to show that G is H -divisible iff $((G^*)^\sharp)^O \in \langle \gamma^\sharp[O] \rangle$ for any orbit $O \in \Phi/S_q$. To describe $((G^*)^\sharp)^O \in (\mathbb{Z}^Q)^O$, recall that if $\psi' \in O \cap \Phi_{B'}$ then $((G^*)^\sharp_{\psi'})_B$ is the number of $\psi \in G^* \cap \Phi_B$ with $\psi|_{B'} = \psi'$. We can assume $B' \subseteq B$, otherwise this number is 0. Let $\pi_B : [r] \rightarrow B$ be order-preserving and $R = \pi_B^{-1}(B')$. Then $\psi \in G^* \cap \Phi_B$ iff $\psi\pi_B \in G$, and $\psi|_{B'} = \psi'$ iff $(\psi\pi_B)|_R = \psi'\pi_B$, so $((G^*)^\sharp_{\psi'})_B = |G(\psi'\pi_B)|$. Similarly, to describe $\langle \gamma^\sharp[O] \rangle$, recall that it is generated by vectors $\gamma^\sharp(\phi) \in (\mathbb{Z}^Q)^O$ where if $\psi' = \phi\theta'$ with $\theta' \in A_{B'}$ then $(\gamma^\sharp(\phi)_{\psi'})_B = (\gamma^\sharp_{\theta'})_B$ is the number of $\theta \in H_B^*$ with $\theta|_{B'} = \theta'$, which is $|H(\theta'\pi_B)|$.

Now fix $\psi \in O \cap \Phi_{[i]}$, where $O \in \Phi_i/S_q$. As G is H -divisible, there is $n \in \mathbb{Z}^{I_i}$ with $G(\psi)^* = \sum_\theta n_\theta H(\theta)^*$. Writing $\phi = \psi\theta^{-1}$, we claim that $((G^*)^\sharp)^O = \sum_\theta n_\theta \gamma^\sharp(\phi)$. To see this, note that it suffices to prove $((G^*)^\sharp)_{[r]}^O = \sum_\theta n_\theta \gamma^\sharp(\phi)_{[r]}$, as $((G^*)^\sharp_{\psi'})_B = |G(\psi'\pi_B)| = ((G^*)^\sharp_{\psi'\pi_B})_{[r]}$ and $(\gamma^\sharp(\phi)_{\psi'})_B = (\gamma^\sharp_{\phi^{-1}\psi'})_B = |H(\phi^{-1}\psi'\pi_B)| = (\gamma^\sharp(\phi)_{\psi'\pi_B})_{[r]}$. Now for any $\psi' \in O \cap \Phi_R$ with $R \subseteq [r]$, writing $\pi = (\psi')^{-1}\psi \in I_r^i$, we have $((G^*)^\sharp_{\psi'})_{[r]} = |G(\psi')| = G(\psi)_\pi^* = \sum n_\theta H(\theta)_\pi^*$, where each $H(\theta)_\pi^* = |H(\theta\pi^{-1})| = (\gamma^\sharp_{\theta\pi^{-1}})_{[r]} = (\gamma^\sharp(\phi)_{\psi'})_{[r]}$, so $((G^*)^\sharp_{\psi'})_{[r]} = \sum n_\theta (\gamma^\sharp(\phi)_{\psi'})_{[r]}$. \square

7 Perspectives

The existence of designs established in [10] has seen several subsequent applications, some of which are particularly instructive as they require not only the existence but also that designs can be

‘almost entirely random’, in that the semi-random (nibble) construction of approximate designs by Rödl [23] can be completed to an actual design by an absorption process (Randomised Algebraic Construction in [10] or Iterative Absorption in [4]). In this vein, we mention the proof by Kwan [14] that almost all Steiner triple systems have perfect matchings, results on discrepancy of high-dimensional permutations by Linial and Luria [16], and the existence of bounded degree coboundary expanders of every dimension by Lubotzky, Luria and Rosenthal [17]. These results suggest that the new results in [11] may create more fruitful connections with the theory of high-dimensional expanders and other topics in high-dimensional combinatorics.

In Design Theory, the most fundamental problems that remain open are those concerning designs with large block sizes. Here we recall from the introduction the Prime Power Conjecture on projective planes, where we know that the divisibility conditions do not always suffice; the conjecture seems to reflect a philosophy that a combinatorial description of a sufficient rich structure somehow implies an algebraic characterisation. On the other hand, a conjecture that reflects the opposite philosophy is that Hadamard matrices (see [7]) of order n should exist whenever the trivially necessary conditions are satisfied (i.e. n is 1, 2 or divisible by 4). It is not clear how the methods of [4, 5, 10, 11] could apply to such problems, where a more fruitful direction may be the development of the approach of [13], which can allow for large block sizes. There are also many well-known open problems in Design Theory that do not involve large block sizes, and so may be more approachable by absorption techniques. Here we mention Ryser’s Conjecture [24] that every Latin square of odd order should have a transversal; equivalently, any triangle decomposition of $K_3(n)$ for n odd should contain a triangle factor (perfect matching of triangles).

In Combinatorics, there are several natural directions in which one may seek to generalise the existence of various types of design, from extremal and/or probabilistic perspectives. A basic class of extremal questions is to determine the minimum degree threshold (which has various possible definitions) for decompositions (see e.g. [5, 20]). Natural probabilistic directions are thresholds for the existence of certain designs in random hypergraphs (e.g. Steiner Triple Systems in $G^3(n, p)$) or a theory of Random Designs analogous to the rich theory of Random Graphs.

References

- [1] P. Bennett and T. Bohman, A natural barrier in random greedy hypergraph matching, arXiv:1210.3581.
- [2] T. Bohman, A. Frieze and E. Lubetzky, Random triangle removal, *Adv. Math.* 280:379–438 (2015).
- [3] C. J. Colbourn and J. H. Dinitz, *Handbook of Combinatorial Designs*, 2nd ed. Chapman & Hall / CRC, Boca Raton, 2006.
- [4] S. Glock, D. Kühn, A. Lo and D. Osthus, The existence of designs via iterative absorption, arXiv:1611.06827.
- [5] S. Glock, D. Kühn, A. Lo and D. Osthus, Hypergraph F -designs for arbitrary F , arXiv:1706.01800.
- [6] J. E. Graver and W. B. Jurkat, The module structure of integral designs, *J. Combin. Theory Ser. A* 15:75–90, 1973.

- [7] J. Hadamard, Résolution d'une question relative aux déterminants, *Bull. des Sciences Math.* 17:240–246, 1893.
- [8] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press, 1952.
- [9] A. Hartman, Software and hardware testing using combinatorial covering suites, in: *Graph Theory, Combinatorics and Algorithms: Interdisciplinary Applications*, Springer, 237–266, 2005.
- [10] P. Keevash, The existence of designs, arXiv:1401.3665.
- [11] P. Keevash, The existence of designs II, arXiv:1802.05900.
- [12] P. Keevash, Counting designs, to appear in *J. Eur. Math. Soc.*
- [13] G. Kuperberg, S. Lovett and R. Peled, Probabilistic existence of regular combinatorial objects, *Geom. Funct. Anal.* 27:919–972 (2017). Preliminary version in *Proc. 44th ACM STOC* (2012).
- [14] M. Kwan, Almost all Steiner triple systems have perfect matchings, arXiv:1611.02246.
- [15] N. Linial and Z. Luria, An upper bound on the number of high-dimensional permutations, *Combinatorica*, 34:471–486, 2014.
- [16] N. Linial and Z. Luria, Discrepancy of high-dimensional permutations, *Discrete Analysis* 2016:11, 8pp.
- [17] A. Lubotzky, Z. Luria and R. Rosenthal, Random Steiner systems and bounded degree coboundary expanders of every dimension, arXiv:1512.08331.
- [18] S. Lovett, S. Rao and A. Vardy, Probabilistic Existence of Large Sets of Designs, arXiv:1704.07964.
- [19] Z. Luria, New bounds on the number of n-queens configurations, arXiv:1705.05225.
- [20] R. Montgomery, Fractional clique decompositions of dense graphs, arXiv:1711.03382.
- [21] R. Montgomery, A. Pokrovskiy and B. Sudakov, Embedding rainbow trees with applications to graph labelling and decomposition, arXiv:1803.03316.
- [22] D.K. Ray-Chaudhuri and R.M. Wilson, Solution of Kirkman's schoolgirl problem, *Proc. Sympos. Pure Math.*, American Mathematical Society, XIX:187–203 (1971).
- [23] V. Rödl, On a packing and covering problem, *Europ. J. Combin.* 6:69–78 (1985).
- [24] H. Ryser, Neuere Probleme in der Kombinatorik, Vorträge über Kombinatorik, Oberwolfach, 69–91 (1967).
- [25] L. Teirlinck, Non-trivial t -designs without repeated blocks exist for all t , *Discrete Math.* 65:301–311 (1987).
- [26] L. Teirlinck, A completion of Lu's determination of the spectrum for large sets of disjoint Steiner triple systems, *J. Combin. Theory Ser. A* 57:302–305, 1991.
- [27] J. H. van Lint and R. M. Wilson, *A course in combinatorics*, Cambridge University Press, 2001.

- [28] C.M. Swanson and D.R. Stinson, Combinatorial solutions providing improved security for the generalized Russian cards problem, *Des. Codes Cryptogr.* 72:345–367, 2014.
- [29] R. Wilson, The early history of block designs, *Rend. del Sem. Mat. di Messina* 9:267–276 (2003).
- [30] R. M. Wilson, An existence theory for pairwise balanced designs I. Composition theorems and morphisms, *J. Combin. Theory Ser. A* 13:220–245 (1972).
- [31] R. M. Wilson, An existence theory for pairwise balanced designs II. The structure of PBD-closed sets and the existence conjectures, *J. Combin. Theory Ser. A* 13:246–273 (1972).
- [32] R. M. Wilson, An existence theory for pairwise balanced designs III. Proof of the existence conjectures, *J. Combin. Theory Ser. A* 18:71–79 (1975).
- [33] R. M. Wilson, The necessary conditions for t -designs are sufficient for something, *Utilitas Math.* 4:207–215 (1973).
- [34] R. M. Wilson, Signed hypergraph designs and diagonal forms for some incidence matrices, *Des. Codes Cryptogr.* 17:289–297 (1999).
- [35] R. M. Wilson, Nonisomorphic Steiner Triple Systems, *Math. Zeit.* 135:303–313 (1974).
- [36] R. M. Wilson, A diagonal form for the incidence matrices of t -subsets *vs.* k -subsets, *Europ. J. Combin* 11:609–615 (1990).