SHORT LOOPS IN SURFACES WITH A CIRCLE BOUNDARY COMPONENT

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Abstract. It is a classical theorem of Loewner that the systole of a Riemannian torus can be bounded in terms of its area. We answer a question of a similar flavor of Robert Young showing that if S is a Riemannian surface with connected boundary in $\mathbb{R}^n$, such that the boundary curve is a standard unit circle, then the length of the shortest non-contractible loop in S is bounded in terms of the area of S.

1. Introduction

Robert Young in [10] conjectures the following: There is a constant $M > 0$ such that if $T \subset \mathbb{R}^n$ is an embedded torus with one boundary component and ∂T is a unit circle, then there is a closed curve of length ℓ in T which is not null-homotopic and satisfies $\ell^2 \leq M(area\, T - \pi)$.

Note that the area of a disk bounding the unit circle is π, so by a surgery one can get a torus with boundary T with area arbitrarily close to π. What the conjecture really says is that if the area is close to π then necessarily there is a 'short' non null homotopic curve in T. Clearly if the area is much bigger than π, say 2π, then the conjecture follows from the classical result of Loewner [9].

The purpose of this note is to show that this conjecture holds. In fact we show more generally that one can obtain such a bound for a surface of any genus and one could find a non-seprating curve satisfying the above inequality.

We note that these results are somewhat related to the classical isoperimetric inequality. Namely it is known that the least area filling surface for the unit circle in $\mathbb{R}^n$ is a disc. The above results quantify this: if we fill the circle with a surface which has area 'close' to the area of the disc then this surface is 'close' to the disc in the sense that it has a 'small' non separating closed curve.

I would like to thank S. Sabourau for suggesting that my proof applies to higher genus surfaces as well. I am grateful to the anonymous referee for pointing out some gaps in my initial argument and simplifying the proof.

2. Length area inequality and systoles

We will use the co-area formula [4, Theorem 13.4.2], which we state now in a simplified form:

Length area inequality. *Let M be a Riemannian 2-manifold and let $f : M \to \mathbb{R}$ be a 1-Lipschitz function. Then*

$$areaM \geq \int \text{length}\,(f^{-1}(t))dt.$$

We recall also Loewner's inequality

Loewner's inequality. *Let T be a Riemannian 2-torus. Then T has a non-contractible geodesic γ of length ℓ satisfying $\ell^2 \leq \frac{2}{\sqrt{3}}\,\text{area}\,T$.*

Loewner's inequality has been generalized by Gromov to surfaces of any genus g where one obtains a stronger bound as $g \to \infty$. Here we will only need use the following inequality that does not depend on the genus (see [7, sec. 2.C], [6, Cor. 5.2.B] and [9]):

Systolic inequality . *Let M be a nonsimply connected closed Riemannian surface. Then M has a non-contractible geodesic γ of length ℓ satisfying $\ell^2 \leq \frac{\pi}{2}\,\text{area}\,M$.*

We state now our main result:

Theorem 2.1. *Let S be an orientable Riemannian surface with a single boundary component embedded in $\mathbb{R}^n$. Assume that ∂S is the unit circle. Then there is a non-separating simple closed curve in S of length ℓ such that*

$$\ell^2 \leq 10^3(\text{area } S - \pi).$$

Proof. Let ℓ be the length of the shortest non-separating simple closed curve in S. We say that a simple closed curve in S is *essential* if it is not contractible.

Notation. We denote respectively by ℓ_{ess}, ℓ_{sep} the lengths of the shortest essential and the shortest essential separating simple closed curve on S. We set $\delta = \text{area}\,S - \pi$.

It turns out that if there is a short separating loop then there is a short non-separating loop as well:

Lemma 2.2. *The following inequality holds:*

$$\ell \leq 2\sqrt{\delta} + \ell_{sep}.$$

Proof. Let w be a separating curve of length ℓ_{sep}. Let's denote by S_1 the connected component of $S \setminus w$ containing ∂S and by S_2 the other connected component of $S \setminus w$. By the isoperimetric inequality

$$\mathrm{area}(S_1) \geq \pi - \frac{1}{4\pi}(\ell_{sep})^2.$$

It follows that

$$\mathrm{area}(S_2) = \mathrm{area}(S) - \mathrm{area}(S_1) \leq \delta + \frac{1}{4\pi}(\ell_{sep})^2 \qquad (*).$$

We fill w by a disk D of arbitrarily small area to obtain a surface $S' = D \cup S_2$ of area less or equal to $\mathrm{area}(S_2) + \epsilon$ for some arbitrarily small $\epsilon > 0$. By choosing carefully the metric on the gluing we can make sure that S' is a smooth Riemannian manifold.

If ℓ_1 is the length of the shortest non-separating loop on S', since ∂S_2 has length less than ℓ_{sep}, we have that $\ell \leq \ell_1 + \frac{\ell_{sep}}{2}$. Applying the systolic inequality to S' we have that

$$\ell_1^2 \leq \frac{\pi}{2}(\mathrm{area}(S_2) + \epsilon)$$

and since this holds for any $\epsilon > 0$, using (*) we obtain

$$\ell \leq \sqrt{\frac{\pi}{2}\left(\delta + \frac{1}{4\pi}(\ell_{sep})^2\right)} + \frac{\ell_{sep}}{2} \leq \sqrt{\frac{\pi}{2}\delta} + \sqrt{\frac{1}{4\pi}(\ell_{sep})^2} + \frac{\ell_{sep}}{2} < 2\sqrt{\delta} + \ell_{sep}.$$

□

Since $\ell_{ess} = \min(\ell, \ell_{sep})$ It follows from this lemma that to prove the theorem it suffices to find an essential simple closed curve of length $\ell_{ess} \leq (\sqrt{1000} - 2)\sqrt{\delta}$ as then

$$\ell \leq 2\sqrt{\delta} + \ell_{ess} \leq \sqrt{1000\delta}.$$

For technical reasons we will show a slightly different version of the theorem than the one stated. We prove below that if S' is surface with boundary a square of side length 2 then the length l' of the shortest non separating loop in S' is bounded by $\sqrt{1000(\mathrm{area}\, S' - 2)}$.

To see that this implies the theorem note that if S is a surface with boundary the unit circle then we can inscribe this circle to a square with side length 2 and add to S the subset of the plane between the square and the circle to obtain a surface S'. We have then that $\ell' = \ell$ and $\mathrm{area}\, S - \pi = \mathrm{area}\, S' - 2$. We remark further that the inequality of the theorem is scale invariant so it is enough to prove it for a surface with boundary a square of side length 1.

To keep notation simple, henceforth we will denote by S a smooth orientable surface embedded in $\mathbb{R}^n$ with a single boundary component equal to a square of side length 1 contained in the xy-plane. We set $\delta = \text{area}\, S - 1$. To prove the theorem we will show that that S contains an essential simple closed curve of length $\ell_{ess} \leq (\sqrt{1000} - 1)\sqrt{\delta}$.

Without loss of generality we assume further assume that the functions $X : (x_1, ..., x_n) \to x_1$ and $Y : (x_1, ..., x_n) \to x_2$ are Morse functions for S. Indeed a slight linear perturbation of X, Y gives Morse functions [5, p.43], and this slight perturbation won't affect significantly the calculations that follow. Alternatively this can be obtained by slightly deforming S.

We note that for each $t \in [0, 1]$, $X^{-1}(t)$ contains a simple arc α_t spanning $X^{-1}(t) \cap \partial S$. Similarly $Y^{-1}(t)$ contains a simple arc β_t spanning $Y^{-1}(t) \cap \partial S$.

Clearly α_t has length at least 1. Using this notation we make some remarks that will be useful in the sequel.

Remark 1. Assume that the set of t for which

$$\text{length}\,(X^{-1}(t)) \geq 1 + \frac{\ell_{ess}}{20}$$

has measure greater or equal to $\ell_{ess}/40$. Then by the length-area inequality

$$\text{area}(S) \geq \int \text{length}\,(X^{-1}(t))dt \geq 1 + {\ell_{ess}}^2/800.$$

It follows that in this case the theorem holds as

$$\ell_{ess} \leq \sqrt{800\delta} < (\sqrt{1000} - 2)\sqrt{\delta}.$$

In particular the set of t such that $X^{-1}(t)$ contains an essential simple closed curve has measure less than $\ell_{ess}/40$.

Also for any s there is some t with $|t - s| < \ell_{ess}/40$ such that $\text{length}(\alpha_t) < 1 + \dfrac{\ell_{ess}}{20}$.

Clearly the above inequalities apply to β_t as well.

Remark 2. It is easy to see that the theorem holds when ℓ_{ess} is 'big enough' as by the systolic inequality the area of the surface is big as well. Specifically assume that $\ell_{ess} \geq 3$. We may obtain a closed surface M from S by gluing a Euclidea square (of area 1) to its boundary. If ℓ'_{ess} is the length of the shortest essential simple closed curve on S' clearly $\ell'_{ess} \geq \ell_{ess} - 1 \geq 2$ so by the systolic inequality

$$4 \leq (\text{area}\, S + 1)\frac{\pi}{2} \Rightarrow \text{area}\, S \geq \frac{3}{2}.$$

Since $\ell_{ess} \leq \ell'_{ess} + 1$ we have that

$$\ell_{ess}{}^2 \leq (\operatorname{area} S + 1)\frac{\pi}{2} + 1$$

On the other hand since $\operatorname{area} S \geq \frac{3}{2}$ we have that

$$800(\operatorname{area} S - 1) \geq (\operatorname{area} S + 1)\frac{\pi}{2} + 1.$$

so $\ell_{ess}{}^2 \leq 800\delta$ which implies the theorem as before.

We argue by contradiction. Assuming that the theorem does not hold we construct an essential simple closed curve w of length strictly less than ℓ_{ess}.

We will construct the curve w using the Morse functions X, Y. We construct first two pairs of arcs and then we use these arcs to define w.

We explain how to construct two arcs χ_1, χ_2 using the Morse function X.

Lemma 2.3. *There are $s_1, s_2 \in [0,1]$ and curves $\chi_1 \subset X^{-1}(s_1), \chi_2 \subset X^{-1}(s_2)$ such that:*

1. The endpoints a_1, a_2 and b_1, b_2 of χ_1, χ_2 respectively lie on opposite sides of the boundary square of S. There is a subsurface S_1 of S, of genus at least 1, with boundary $\chi_1 \cup [a_1, b_1] \cup \chi_2 \cup [a_2, b_2]$ (where $[a_1, b_1], [a_2, b_2] \subset \partial S$).

2. $\operatorname{length}(\chi_i) \leq 1 + \frac{\ell_{ess}}{20}$ *for $i = 1, 2$.*

3. $\frac{\ell_{ess}}{20} \leq |s_2 - s_1| \leq \frac{\ell_{ess}}{10}$.

Proof. Since S is is not a disk, for some s, $X^{-1}(s)$ contains an essential simple closed curve.

We distinguish two cases:

Case 1. $s > \frac{\ell_{ess}}{20}$ and $1 - s > \frac{\ell_{ess}}{20}$.

By remark 1 there is some $s_1 \in [s - \frac{\ell_{ess}}{20}, s - \frac{\ell_{ess}}{40}]$ such that

$$\operatorname{length}(X^{-1}(a)) \leq 1 + \frac{\ell_{ess}}{20}.$$

We set χ_1 to be the arc in $X^{-1}(s_1)$ with endpoints a_1, a_2 on ∂S. Clearly $X^{-1}(s_1)$ does not contain any essential simple closed curves so χ_1 separates the surface S. We construct χ_2 similarly by picking

$s_2 \in [s + \frac{\ell_{ess}}{40}, s + \frac{\ell_{ess}}{20}]$ such that

$$\text{length}(X^{-1}(s_2)) \leq 1 + \frac{\ell_{ess}}{20}.$$

We denote the endpoints of χ_2 by b_1, b_2. Conditions 2,3 are immediate. Since χ_1, χ_2 are separating $\chi_1 \cup [a_1, b_1] \cup \chi_2 \cup [a_2, b_2]$ bounds a subsurface S_1. Since $s_1 < s < s_2$, S_1 has genus ≥ 1 so condition 1 holds.

Case 2. $s \leq \frac{\ell_{ess}}{20}$ or $1 - s \leq \frac{\ell_{ess}}{20}$.

Note that by remark 2, $\ell_{ess} \leq 3$ so s satisfies exactly one of the inequalities above.

Let's assume first that $s \leq \frac{\ell_{ess}}{20}$. We define χ_2 as in case 1 by picking $s_2 \in [s + \frac{\ell_{ess}}{40}, s + \frac{\ell_{ess}}{20}]$ such that $\text{length}(X^{-1}(s_2)) \leq 1 + \frac{\ell_{ess}}{20}$. We pick $s_1 = 0$, so we take χ_1 to be the side of ∂S (which is a square) contained in $X^{-1}(0)$. As χ_2 is separating as before, conditions 1,2,3 are satisfied.

If $1 - s \leq \frac{\ell_{ess}}{20}$ we define the arcs similarly. We take $s_2 = 1$ and χ_2 to be the side of ∂S contained in $X^{-1}(1)$. We define s_1, χ_1 as in case 1.

□

Lemma 2.4. *Let s_1, s_2, χ_1, χ_2, S_1 be as in lemma 2.3. There are $t_1, t_2 \in [0,1]$ and arcs $\psi_1 \subset Y^{-1}(t_1), \psi_2 \subset Y^{-1}(t_2)$ with endpoints c_1, c_2 and d_1, d_2 respectively such that:*

1. c_1, d_1 lie in χ_1 and bound a subarc χ_1', c_2, d_2 lie in χ_2 and bound a subarc χ_2', and the curve

$$w = \psi_1 \cup \chi_1' \cup \psi_2 \cup \chi_2'$$

is the boundary of a subsurface S_2 of S_1 of genus at least 1.

2. $\text{length}(\psi_i) \leq |s_2 - s_1| + \frac{\ell_{ess}}{20}$ *for $i = 1, 2$.*

3. $|t_2 - t_1| \leq \frac{\ell_{ess}}{10}$.

Proof. Since S_1 is a surface of genus ≥ 1 there is some $t \in [0,1]$ such that $Y^{-1}(t)$ contains an essential closed curve in S_1. The choice of t_1, t_2 proceeds now as before. We distinguish two cases:

Case 1. $t > \frac{\ell_{ess}}{20}$ and $1 - t > \frac{\ell_{ess}}{20}$.

We define then t_1, t_2 as in lemma 2.3.

So $t \in [t_1, t_2]$ and

$$\operatorname{length} Y^{-1}(t_i) \leq 1 + \frac{\ell_{ess}}{20} \ , \ i = 1, 2.$$

Since $s_2 - s_1 \geq \dfrac{\ell_{ess}}{20}$ there is a single subarc ψ_i of $Y^{-1}(t_i) \cap S_1$ connecting χ_1, χ_2.

We note that every component of $(Y^{-1}(t_i) \cap S_1) \setminus \psi_i$ is either an inessential simple closed curve or an arc with both its endpoints on one of χ_1, χ_2. If γ is such an arc and λ is the subarc with the same endpoints on χ_1 or χ_2 then

$$\operatorname{length}(\gamma \cup \lambda) \leq \frac{\ell_{ess}}{10}$$

so $\gamma \cup \lambda$ is contractible. It follows that ψ_i separates the surface S_1 and the curve w bounds a surface of genus ≥ 1.

Case 2. $t \leq \dfrac{\ell_{ess}}{20}$ or $1 - t \leq \dfrac{\ell_{ess}}{20}$.

As in the previous lemma exactly one of these inequalities holds. Both cases are similar so we assume that $t \leq \dfrac{\ell_{ess}}{20}$. As in the previous lemma we pick $t_2 \in [t + \dfrac{\ell_{ess}}{40}, t + \dfrac{\ell_{ess}}{20}]$ such that $\operatorname{length}(Y^{-1}(t_2)) \leq 1 + \dfrac{\ell_{ess}}{20}$. We pick $t_1 = 0$, so we take ψ_1 to be

$$\partial S \cap Y^{-1}(0) \cap S_1.$$

As ψ_2 is separating as before, conditions 1,2,3 are satisfied.

□

We can now prove the theorem. Let w be the closed curve given by 1 of lemma 2.4. Then w is essential as it bounds a surface S_2 of genus ≥ 1. We remark now that since $|t_2 - t_1|, |s_2 - s_1| \leq \dfrac{\ell_{ess}}{10}$, by the definition of $X^{-1}(s_i), Y^{-1}(t_i)$ $(i = 1, 2)$ we have that the lengths of each of $\psi_1, \chi_1', \psi_2, \chi_2'$ is at most $\dfrac{\ell_{ess}}{10} + \dfrac{\ell_{ess}}{20}$. It follows that

$$\operatorname{length} w \leq 4\frac{\ell_{ess}}{10} + 4\frac{\ell_{ess}}{20} < \ell_{ess}$$

which is a contradiction.

□

3. Discussion

Theorem 2.1 'quantifies' the defect of a filling of S^1 by a general surface rather than a disc. It says that if the filling by a surface is 'close' to the optimal filling by a disc then the surface is 'close' to a disc as it has a short non-separating geodesic. This has a similar flavor to the classical Bonnesen inequality [3] on isoperimetric defect quantifying how far is a region from being optimal for the isoperimetric inequality. A strengthening of Loewner's inequality in this spirit is given in [8]. We note also that Babenko in [1] has studied systoles of manifolds with boundary.

One would expect the following stronger form of theorem 2.1 to hold:

Let S be an orientable Riemannian surface of genus g with a single boundary component embedded in $\mathbb{R}^n$. Assume that ∂S is isometric to the unit circle. Then there are g non-isotopic disjoint non separating loops on S of lengths $\ell_1, ..., \ell_g$ such that

$$\sum_{i=1}^{g} \ell_i^2 \leq C(\text{area } S - \pi)$$

for some constant C that does not depend on g.

We note that theorem 2.1 implies that this inequality holds for some constant C_g that depends on g. To see this cut S along a shortest non separating loop, cup off the holes by hemispheres and repeat g times.

A related well known question to theorem 2.1 is the conjecture of Gromov on the filling volume (area) of S^1 [6, sec.5.5, p.60]. One wonders if the analog of theorem 2.1 holds in this case, namely whether if T is a torus with boundary filling S^1 then there is a non-separating curve of length ℓ in T satisfying

$$\ell^2 \leq C(\text{area } T - 2\pi)$$

for some universal constant C. Note that the Gromov's conjecture is known to hold for tori with boundary [2] but is still open for higher genus surfaces.

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