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## General Section

Upper bounds on large deviations of Dirichlet  $L$ -functions in the  $q$ -aspect

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## ABSTRACT

We prove a result on the large deviations of the central values of even primitive Dirichlet  $L$ -functions with a given modulus. For  $V \sim \alpha \log \log q$  with  $0 < \alpha < 1$ , we show that

$$\frac{1}{\varphi(q)} \#\left\{ \chi \text{ even, primitive mod } q : \log \left| L\left(\chi, \frac{1}{2}\right) \right| > V \right\} \ll \frac{e^{-\frac{V^2}{\log \log q}}}{\sqrt{\log \log q}}.$$

This yields the sharp upper bound for the fractional moments of central values of Dirichlet  $L$ -functions proved by Gao, upon noting that the number of even, primitive characters with modulus  $q$  is  $\frac{\varphi(q)}{2} + O(1)$ . The proof is an adaptation to the  $q$ -aspect of the recursive scheme developed by Arguin, Bourgade and Radziwiłł for the local maxima of the Riemann zeta function, and applied by Arguin and Bailey to the large deviations in the  $t$ -aspect. We go further and get bounds on the case where  $V = o(\log \log q)$ . These bounds are not expected to be sharp, but the discrepancy from the Central Limit Theorem estimate grows very slowly with  $q$ . The method involves a formula for the twisted mollified second

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moment of central values of Dirichlet  $L$ -functions, building on the work of Iwaniec and Sarnak.

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## 1. Introduction

This paper is on the central value of Dirichlet  $L$ -functions with fixed modulus  $q$ . The most famous result in the study of the distribution of the values of Dirichlet  $L$ -functions is Selberg’s Central Limit Theorem. In [Sel46], it is shown that the normalised values of the logarithm of the Riemann zeta function in a dyadic interval high up the critical line have a Gaussian limiting distribution. More precisely, for any measurable set  $E \subset \mathbb{C}$ ,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ t \in [T, 2T] : \frac{\log \left( \zeta \left( \frac{1}{2} + it \right) \right)}{\sqrt{\frac{1}{2} \log \log T}} \in E \right\} = \frac{1}{2\pi} \int_E e^{-\frac{x^2+y^2}{2}} dx dy. \quad (1.1)$$

A remarkable feature of Selberg’s Central Limit Theorem is that it gives information about the distribution of the imaginary part of the logarithm of the Riemann zeta function on the critical line, the fluctuation of which is connected to zeros of the Riemann zeta function, without assuming the Riemann Hypothesis.

Much work has been done to generalise Equation (1.1), both in extending the range of the Gaussian behaviour, and proving central limit theorems for other problems related to Dirichlet  $L$ -functions. For example, in [HW20], it is shown that the real parts of the logarithm of distinct Dirichlet  $L$ -functions in a dyadic interval high up the critical line behave like independent Gaussian variables. More precisely, for any integer  $N$ , if  $(\chi_j)_{j=1}^N$  is a sequence of distinct Dirichlet characters, then Theorem 1.3 of [HW20] states that for  $t$  uniformly distributed in the interval  $[T, 2T]$  with  $T$  large, the random vector

$$\left( \log \left| L \left( \chi_1, \frac{1}{2} + it \right) \right|, \dots, \log \left| L \left( \chi_N, \frac{1}{2} + it \right) \right| \right)$$

approximately has the distribution of a real Gaussian random variable with mean  $\mathbf{0}_N$  and variance  $\frac{1}{2} (\log \log T) \mathbf{I}_N$ .

There are many similarities between the distribution in the  $q$ -aspect of central values of Dirichlet  $L$ -functions with a large modulus, and the distribution in the  $t$ -aspect of the values of a given Dirichlet  $L$ -function, such as the Riemann zeta function, high up the critical line. The analysis of the distribution of values of  $L$ -functions via their moments is a widely investigated area, and parallels can be drawn here between the  $q$ -aspect and the  $t$ -aspect. The Keating-Snaith Conjecture from [KS00] predicts asymptotics for the moments of the Riemann zeta function. It conjectures that, for all real  $\lambda > -\frac{1}{2}$ :

$$\lim_{T \rightarrow \infty} \frac{1}{T(\log T)^{\lambda^2}} \int_0^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^{2\lambda} dt = C_\lambda, \tag{1.2}$$

where  $C_\lambda$  is an explicit constant. For the  $q$ -aspect, an analogous result is conjectured in [RS05]. They report a widely believed conjecture, that for all  $k \in \mathbb{N}$ :

$$\lim_{\substack{q \rightarrow \infty \\ q \neq 2 \pmod 4}} \frac{1}{\varphi(q)(\log q)^{k^2}} \sum_{\chi \text{ primitive mod } q} \left| L \left( \chi, \frac{1}{2} \right) \right|^{2k} = C_{1,k}, \tag{1.3}$$

where  $C_{1,k}$  is an explicit constant.

In [Gao21], it is shown that

$$\frac{1}{\varphi(q)} \sum_{\chi \text{ primitive mod } q} \left| L \left( \chi, \frac{1}{2} \right) \right|^{2\beta} \asymp (\log q)^{\beta^2}, \tag{1.4}$$

for all  $\beta \in (0, 1)$ . This means that, up to a constant, these moments of  $|L(\chi, \frac{1}{2})|$ , for  $\chi$  ranging over the primitive characters with modulus  $q$ , match the moment generating function of a Gaussian random variable with variance  $\frac{1}{2} \log \log q$ .

Moreover, we have a lower bound for all the integer moments of the conjectured correct order from [RS05]. Here, it is shown that, for  $k$  any fixed natural number and  $q$  a large prime,

$$\frac{1}{\varphi(q)} \sum_{\chi \text{ primitive mod } q} \left| L \left( \chi, \frac{1}{2} \right) \right|^{2k} \gg_k (\log q)^{k^2}. \tag{1.5}$$

Equation (1.5) is proven in [RS05] for the case where  $q$  is a large prime, but the method of proof easily generalises to the general case for large values of  $q$  not congruent to 2 modulo 4. From Equation (1.5), we see that the integer moments of the central values are bounded below by the moment generating function of a Gaussian random variable. We prove further that the large deviations of the real part of the logarithm of the central value are bounded above by a Gaussian tail.

Further work on central limit theorems in the  $q$ -aspect can be found in [Das22]. For example, in their Theorem 4.2.1, a central limit theorem over the  $q$ -aspect is proven, averaging both over the character and a short interval around the central value. If  $V$  is a fixed positive real number, then uniformly for all  $v \in [-V, V]$ , for  $q$  large, we have:

$$\begin{aligned} & \frac{1}{\varphi(q)} \sum_{\chi \text{ primitive mod } q} \text{meas} \left\{ -1 \leq t \leq 1 : \log \left| L \left( \chi, \frac{1}{2} + it \right) \right| \geq v \sqrt{\frac{1}{2} \log \log q} \right\} \\ & \sim \frac{1}{\sqrt{2\pi}} \int_v^\infty e^{-\frac{u^2}{2}} du. \end{aligned} \tag{1.6}$$

The standard deviation of the random variable  $\log |L(\chi, \frac{1}{2})|$  is approximately  $\sqrt{\frac{1}{2} \log \log q}$ , so in Equation (1.6), an asymptotic is provided for deviations of the same order as the standard deviation. In this paper, we will instead consider large deviations of order  $\log \log q$ , which are of the same order as the variance. We will also be able to consider the values at the central point,  $\frac{1}{2}$ , rather than having to average over an interval of the critical line.

Moreover, in [RS15], a weaker bound than a Central Limit Theorem proven for the degree 2  $L$ -functions is associated to elliptic curves over  $\mathbb{Q}$ . If  $E/\mathbb{Q}$  is an elliptic curve with conductor  $N$ , then the Hasse-Weil function associated to  $E$  is normalised in [RS15] as

$$L(E, s) = \sum_n a(n)n^{-s}, \tag{1.7}$$

where  $a(n)$  is normalised so that the Hasse bound reads  $|a(n)| \leq d(n)$ . Note that under this normalisation, the central point of this  $L$ -function is at  $\frac{1}{2}$ , whereas when the coefficients of the  $L$ -function are not normalised, the central point lies at 1. Using this normalisation convention, the authors prove an upper bound for the tail distribution of central values of the  $L$ -functions associated to the twisted elliptic curves. If  $d$  is a fundamental discriminant coprime to  $2N$ , then  $E_d$  is defined as the twist of the elliptic curve by  $d$ . The character  $\chi_d$  is defined as  $\chi_d = \left(\frac{d}{\cdot}\right)$ , so that the normalised  $L$ -function associated to the twisted elliptic curve, instead of Equation (1.7), reads

$$L(E_d, s) = \sum_n a(n)\chi_d(n)n^{-s}. \tag{1.8}$$

One sees that the completed  $L$ -function

$$\Lambda(E_d, s) := \left(\frac{\sqrt{N}|d|}{2\pi}\right)^s \Gamma\left(s + \frac{1}{2}\right) L(E_d, s) \tag{1.9}$$

is entire and satisfies the functional equation:

$$\Lambda(E_d, s) = \epsilon_E(d)\Lambda(E_d, 1 - s), \tag{1.10}$$

where the root number is defined as:

$$\epsilon_E(d) = \epsilon_E\chi_d(-N). \tag{1.11}$$

If  $\epsilon_E(d) = -1$ , then from Equation (1.10), one obtains  $L(E_d, \frac{1}{2}) = 0$ , and so the authors restrict their attention to twists with root number 1. They define this set with variable central value:

$$\mathcal{E} = \{d : d \text{ a fundamental discriminant with } (d, 2N) = 1 \text{ and } \epsilon_E(d) = 1\}. \tag{1.12}$$

Then Theorem 1 in [RS15] states that for  $0 < k < 1$  a real number, as  $X \rightarrow \infty$ ,

$$\frac{1}{X} \left| \left\{ d \in \mathcal{E}, |d| \leq X, L\left(E_d, \frac{1}{2}\right) \geq (\log |d|)^{k-\frac{1}{2}} \right\} \right| \ll (\log X)^{-\frac{k^2}{2}}. \tag{1.13}$$

This is not quite as sharp as Conjecture 3 in [CKRS06], that, for  $d \in \mathcal{E}$ , the quantity  $\log L\left(E_d, \frac{1}{2}\right)$  has a normal distribution with mean  $-\frac{1}{2} \log \log |d|$  and variance  $\log \log |d|$ . More concretely, adjusting for different normalisation conventions, the conjecture predicts:

$$\lim_{X \rightarrow \infty} \frac{\left| \left\{ d \in \mathcal{E}, |d| \leq X, \frac{\log L(E_d, \frac{1}{2}) + \log \log |d|}{\sqrt{\log \log |d|}} \in (\alpha, \beta) \right\} \right|}{|\{d \in \mathcal{E}, |d| \leq X\}|} = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} \exp\left(-\frac{u^2}{2}\right) du. \tag{1.14}$$

The most famous conjecture about the non-vanishing of Dirichlet  $L$ -functions is the Generalised Riemann Hypothesis, which predicts that all non-trivial zeros of primitive Dirichlet  $L$ -functions lie on the critical line with real part  $\frac{1}{2}$ . However, even on the critical line, one can consider whether it is possible for any Dirichlet  $L$ -function to vanish at a certain point. It is conjectured that no primitive Dirichlet  $L$ -function vanishes at the central point. In [Pra19], it is shown that for large values of  $q$ , at least 50.073% of the central values of  $L$ -functions associated to primitive characters of modulus  $q$  are non-vanishing. In this article, we are able to show that the large deviations of order  $\log \log q$  are bounded above by a Gaussian tail. If one were able to extend this to show that the large deviations beyond the order  $\sqrt{\log q \log \log q}$  were also bounded by the same Gaussian tail, one would be able to show the non-vanishing of all central values, however the authors were unable to extend their results to this range.

This paper is heavily motivated by the paper [AB23], where a large deviations result is proven for the Riemann zeta function; here we show an analogous result holds for the central values of Dirichlet  $L$ -functions with a given modulus. It was necessary to prove certain results used in the proof of the large deviations result in [AB23] for the context of central values of Dirichlet  $L$ -functions; most notably, the twisted mollifier formula for second moments of the Riemann zeta function needed to be adapted to the  $q$ -aspect, cf. Theorem 1.5. As in [IS99], we restricted our attention to looking at just the even primitive characters; the case of odd primitive characters yields the same results.

### 1.1. Main results

**Theorem 1.1.** *Let  $q$  be a large natural number, with  $q \not\equiv 2 \pmod{4}$ . Suppose that  $V \sim \alpha \log \log q$ , with  $0 < \alpha < 1$ . Then*

$$\frac{1}{\varphi(q)} \# \left\{ \chi \text{ even, primitive mod } q : \log \left| L\left(\chi, \frac{1}{2}\right) \right| > V \right\} \ll \frac{e^{-\frac{V^2}{\log \log q}}}{\sqrt{\log \log q}}. \tag{1.15}$$

In Section 5, we deduce an upper bound on the fractional moments of the central values of primitive Dirichlet  $L$ -functions, in line with Gao’s result, Equation (1.4).

**Corollary 1.2.** *Let  $0 < \beta < 1$  and  $q$  be a large natural number, with  $q \not\equiv 2 \pmod{4}$ . Then we have*

$$\frac{1}{\varphi(q)} \sum_{\chi \text{ even, primitive mod } q} \left| L\left(\chi, \frac{1}{2}\right) \right|^{2\beta} \ll (\log q)^{\beta^2}, \tag{1.16}$$

where the implicit constant is uniform for  $\beta$  lying in any interval  $(0, B)$ , for any  $B < 1$ .

Note that the range for  $\beta$  in Corollary 1.2 is controlled by the range of  $\alpha$  for which Theorem 1.1 is valid. Indeed, if we were able to extend the range of Theorem 1.1 to be valid for  $0 < \alpha < A$ , for some  $A > 1$ , then we would be able to extend our bound on the moments in Corollary 1.2 to the range  $0 < \beta < A$ .

In [AB23], they re-prove a result of Heap, Radziwiłł and Soundararajan from [HRS19], which is the analogous result to Corollary 1.2 in the  $t$ -aspect.

**Corollary 1.3** (Corollary 1.2 in [AB23]). *Let  $0 < \beta < 2$ . Then there exists a constant  $C_\beta$  such that*

$$\frac{1}{T} \int_T^{2T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2\beta} \leq C_\beta (\log T)^{\beta^2}, \tag{1.17}$$

for all  $T$  sufficiently large.

The bounds in [AB23] have  $C_\beta \rightarrow \infty$  as  $\beta \rightarrow 0$ . However, by tightening the bounds when we adapt their method, we are able to avoid our constant in the  $q$ -aspect becoming unbounded near  $\beta = 0$ . Because we are able to tighten our bounds, we are able to attain (weaker) bounds which include the case when  $V = o(\log \log q)$ .

**Theorem 1.4.** *Let  $q$  be a large natural number, with  $q \not\equiv 2 \pmod{4}$  and  $V > 0$  with  $V = o(\log \log q)$ . Let  $\mathcal{L}$  be as in Equation (1.21). Then*

$$\frac{1}{\varphi(q)} \left\{ \chi \text{ even, primitive mod } q : \log \left| L\left(\chi, \frac{1}{2}\right) \right| > V \right\} \ll \frac{\mathcal{L} e^{-\frac{V^2}{\log \log q}}}{\sqrt{\log \log q}}. \tag{1.18}$$

Note that  $\mathcal{L}$  grows very slowly in  $q$ . Indeed, for a given natural number  $n$ , the maximum value of  $q$  for which  $\mathcal{L} = n$  grows like the Ackermann function  $F(4, n)$ .

1.2. *Notation and structure of the paper*

We will be taking expectations and sums over different sets of characters with modulus  $q$ :

- The set of primitive even characters with modulus  $q$ , denoted  $+$ .
- The set of all even characters with modulus  $q$ , denoted  $\oplus$ .
- The set of all primitive characters with modulus  $q$ , denoted  $*$ .

Where we take probabilities,  $\mathbf{P}$  will always denote the probability where a character  $\chi$  is drawn uniformly at random from the space of primitive even characters with modulus  $q$ .

1.3. *Structure of the proof*

Theorems 1.1 and 1.4 are proved in Section 4. The proof of Corollary 1.2 follows easily and the details are given in Section 5. Theorem 1.1 is derived from the recursive scheme first developed for the  $t$ -aspect in [ABR20]. The idea is to approximate  $\log |L(\chi, \frac{1}{2})|$  by Dirichlet polynomials of increasing length and whose values are restricted.

To this end, we split up the primes contributing most to the Dirichlet  $L$ -functions into intervals. We iteratively define steps by setting  $q_0 = 1.5$ , and setting

$$q_l = \exp\left(\frac{\log q}{\log_{l+1}(q)^{\mathbf{s}}}\right) \tag{1.19}$$

for  $l \geq 1$ , where  $\log_l$  means the natural logarithm iterated  $l$  times. Here,  $\mathbf{s}$  is a variable taking large values which depend only on  $\frac{V}{\log \log q}$ , and taking  $\mathbf{s} > 10^5$  as in Equation (4.10) will suffice. We then set

$$n_l = \log \log q_l, \quad l \geq 1, \tag{1.20}$$

which turns out to be the suitable scale for considering the variance of Dirichlet polynomials supported on primes up to  $q_l$ . We halt our steps at  $\mathcal{L}$ , the largest value of  $l$  such that

$$\exp(10^6(\log \log q - n_l)10^5 e^{n_{l+1}}) = q^{\frac{10^6}{(\log_{l+2} q)10^5 - \mathbf{s}}} \leq q^{\frac{1}{100}}. \tag{1.21}$$

For  $k > 0$ , we define a truncated sum of the formal logarithm of the Dirichlet  $L$ -series at  $\frac{1}{2}$ :

$$\tilde{S}_k = \sum_{p \leq e^{e^k}} \frac{\chi(p)}{p^{\frac{1}{2}}} + \frac{\chi(p)^2}{2p}. \tag{1.22}$$

The absolute value of the  $L$ -function at  $\frac{1}{2}$  may be controlled by the real part of the above series. We further define

$$S_k = \sum_{p \leq e^k} \Re \left( \frac{\chi(p)}{p^{\frac{1}{2}}} + \frac{\chi(p^2)}{2p} \right). \tag{1.23}$$

We expect for  $k$  large,  $S_k$  to be close to  $\Re(\log L(\chi, \frac{1}{2}))$ . Indeed, we only consider the values of  $S_k$  for  $k$  close to  $\log \log q$ , at the time steps  $k = n_l$ , for  $1 \leq l \leq \mathcal{L}$ .

We also need to control the difference  $\log |L(\chi, \frac{1}{2})| - S_k$ . To do so, we define a mollifier of the contribution of the primes in the interval  $(q_{l-1}, q_l]$ . Given  $1 \leq l \leq \mathcal{L}$ , set

$$M_l(\chi, s) = \sum_{\substack{p|n \implies p \in (q_{l-1}, q_l] \\ \Omega_l(n) \leq 10(n_l - n_{l-1})^{10^5}}} \frac{\mu(n)\chi(n)}{n^s}, \tag{1.24}$$

where  $\Omega_l(n)$  denotes the number of prime factors with multiplicity of  $n$  in the interval  $(q_{l-1}, q_l]$ . The idea is that the mollifiers are long enough that  $M_1 \dots M_l(\chi, \frac{1}{2})$  should mollify all primes up to  $q_l$ . We want to show  $|M_1 \dots M_l|$  to be a good approximation of  $\exp(-S_{n_l})$ . Then, we will have  $|M_1 \dots M_l L(\chi, \frac{1}{2})|$  not too small or large, so we will have mollified the  $L$ -function at  $\frac{1}{2}$  successfully.

The mollifier described above is part of a larger family of Dirichlet polynomials which vary corresponding to the even primitive characters mod  $q$  by a scaling of their coefficients. Namely, if  $\chi$  is an even primitive character mod  $q$ , then its associated Dirichlet  $L$ -function is

$$L(\chi, s) = \sum_n \frac{\chi(n)}{n^s}. \tag{1.25}$$

For a sequence of real coefficients  $(a_m)$ , which is uniform for all even primitive Dirichlet characters with modulus  $q$  and supported on the integers  $1 \leq m \leq M$ , we consider the associated Dirichlet polynomial

$$Q(\chi, s) = \sum_n \frac{a_n \chi(n)}{n^s}. \tag{1.26}$$

We say  $Q = Q(\chi)$  is a *degree  $e^{n_l}$  well-factorable polynomial* if we can write

$$Q = \prod_{\lambda=1}^l Q_\lambda, \tag{1.27}$$

where  $Q_\lambda$  is a Dirichlet polynomial with the support of the coefficients restricted by their prime factors, and with the coefficients having the same dependence on  $\chi$  as in Equation (1.26), say

$$Q_\lambda(\chi, s) = \sum_{\substack{p|m \implies p \in (q_{\lambda-1}, q_\lambda] \\ \Omega_\lambda(m) \leq 10(n_\lambda - n_{\lambda-1})^{10^4}}} \frac{\gamma_m \chi(m)}{m^s}, \tag{1.28}$$

i.e., each factor  $Q_\lambda(\chi, s)$  has coefficients supported on integers with all prime factors in the interval  $(q_{\lambda-1}, q_\lambda]$ , and is not too long. We are only interested in the central values of the Dirichlet polynomials; for a Dirichlet polynomial  $Q$  with such a property we will write  $Q$  for  $Q(\chi, \frac{1}{2})$ .

We show that if  $Q$  is a degree  $e^{n_l}$  well-factorable Dirichlet polynomial, then  $M_1 \dots M_l$  mollifies  $L$  successfully, after a twist by  $Q$ . We consider the Euler product of  $L$ , which doesn't converge at  $\frac{1}{2}$ , but can still give good intuition. Since  $M_1 \dots M_l$  is a good mollifier of  $L$ , we would expect only the primes greater than  $q_l$  to affect the value of  $M_1 \dots M_l L$ . Hence, if  $Q$  is a Dirichlet polynomial which only depends on the primes up to  $q_l$ , then we would expect  $LM_1 \dots M_l$  and  $Q$  to be weakly dependent. This is a crucial step of the recursive scheme. We show this in Section 3, where we prove the following theorem:

**Theorem 1.5.** *Let  $Q$  be a degree  $e^{n_l}$  well-factorable polynomial and  $L$  and the mollifiers  $M_1, \dots, M_l$  be as defined above. Then the mollified twisted second moment satisfies*

$$\mathbb{E}_+ [ |LM_1 \dots M_l Q|^2 ] \ll \frac{\log q}{\log q_l} \mathbb{E}_\oplus [ |Q|^2 ]. \tag{1.29}$$

The expectation over  $+$  is the expectation where  $\chi$  is drawn uniformly at random from the primitive even characters mod  $q$ , whilst the expectation over  $\oplus$  is where  $\chi$  is drawn uniformly from all even characters mod  $q$  (so includes the imprimitive even characters), and has simpler orthogonality relations. These relations are discussed in Section 2. We see that the ratio of the expectations in Theorem 1.5 is bounded by a factor independent of the twist  $Q$ , which suggests  $LM_1 \dots M_l$  and  $Q$  are weakly dependent. We need a slightly stronger mollified twisted moment than in Theorem 1.5, to handle twists by polynomial functions of the real parts of Dirichlet polynomials, such as  $S_k - S_{k-1}$ .

We require a similar version to being well-factorable in the real case. We say a function  $F = F(\chi)$  is *l-sufficient* if we may write:

$$F = \prod_{j=1}^l F_j, \tag{1.30}$$

where for each  $1 \leq j \leq l$ , we have

$$F_j = K_j \left( \Re \left( \sum_m \chi(m) b_{(m)}^j \right) \right), \tag{1.31}$$

for some sequence of complex numbers  $b_{(m)}^j$  and polynomial  $K_j$  of degree  $d_j$  with real coefficients. The coefficients  $b_{(m)}^j$  are assumed to be 0 unless  $p|m \implies p \in (q_{j-1}, q_j]$  and

$\Omega_j(m) \leq \nu_j$  for  $\nu_j = \frac{10(n_j - n_{j-1})^{10^4}}{d_j}$ . The bound on the number of prime factors ensures that the length of  $F_j$  is less than  $10(n_j - n_{j-1})^{10^4}$ . In Section 3, we will prove:

**Theorem 1.6.** *Let  $1 \leq l \leq \mathcal{L}$  and let  $F$  be  $l$ -sufficient. Then the twisted second moment mollified by  $F$  satisfies*

$$\mathbb{E}_+ [ |LM_1 \dots M_l F|^2 ] \ll \frac{\log q}{\log q_l} \mathbb{E}_\oplus [ F^2 ]. \tag{1.32}$$

By applying Theorem 1.6 for suitable choices of  $F$ , we will deduce Theorem 1.5. In order to prove Theorem 1.6, we require bounds on the twisted moments of Dirichlet  $L$ -functions by real parts of Dirichlet polynomials. We will follow the formulae for twists by the whole Dirichlet polynomial (not just the real part), i.e.,  $\mathbb{E}_+ [ |LM|^2 ]$ , where  $M$  is a short Dirichlet polynomial. In [BPRZ20], the authors give a bound where  $M$  has length up to  $q^{\frac{51}{101}}$ . However, for our purposes it suffices to follow the proof of the simpler formula in [IS99] for Dirichlet polynomials of length smaller than  $q^{\frac{1}{2}}$ .

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## 2. Moments in the $q$ -aspect

In this section we collate some simple bounds on the moments of even Dirichlet polynomials with a fixed modulus, which will be needed in the proof of Theorems 1.1, 1.5 and 1.6.

### 2.1. Moments over all even characters with a fixed modulus

The following identity is exact, as opposed to the  $t$ -aspect, where the diagonal terms only give a leading order approximation.

**Lemma 2.1.** *Let  $q$  be a modulus,  $N < \frac{q}{2}$  a positive integer, and  $(a_n)_{n=1}^N$  be a sequence of complex numbers. Then*

$$\mathbb{E}_\oplus \left[ \left| \sum_{n=1}^N a_n \chi(n) \right|^2 \right] = \sum_{\substack{n=1 \\ (n,q)=1}}^N |a(n)|^2, \tag{2.1}$$

where the expectation is taken over all even characters mod  $q$ .

**Proof.** Given integers  $1 \leq m, n \leq N$ , we have

$$\sum_{\chi \bmod q \text{ even}} \chi(n)\overline{\chi(m)} = \sum_{\chi \bmod q} \left( \frac{\chi(n) + \chi(-n)}{2} \right) \overline{\chi(m)}. \tag{2.2}$$

This equals  $\frac{\varphi(q)}{2}$  if  $n = m$ , and  $(n, q) = 1$ , and 0 otherwise, since we cannot have  $n \equiv -m \pmod q$ . Upon substitution into the right-hand side of Equation (2.1), we get:

$$\mathbb{E}_{\oplus} \left[ \left| \sum_{n=1}^N a_n \chi(n) \right|^2 \right] = \frac{2}{\varphi(q)} \sum_{1 \leq n, m \leq N} a_n \overline{a_m} \sum_{\chi \bmod q \text{ even}} \chi(n)\overline{\chi(m)}. \tag{2.3}$$

Only the diagonal terms survive, and we conclude.  $\square$

We deduce the following splitting theorem:

**Lemma 2.2.** *If  $A$  and  $B$  are two Dirichlet polynomials varying with  $\chi$ , each of length shorter than  $\sqrt{\frac{q}{2}}$  with support on integers with prime factors in disjoint sets, then*

$$\mathbb{E}_{\oplus} \left[ |AB(1/2, \chi)|^2 \right] = \mathbb{E}_{\oplus} [|A(1/2, \chi)|^2] \mathbb{E}_{\oplus} \left[ |B(1/2, \chi)|^2 \right]. \tag{2.4}$$

Note again that the expectation splits exactly due to the exact orthogonality relations, which is a stronger result than in the  $t$ -aspect (cf [AB23], Lemma 13). We also require results on the moments of real parts of Dirichlet polynomials in the  $q$ -aspect. It is convenient to introduce the random variables  $(X(p), p \text{ primes})$  that are independent and uniformly distributed on the unit circle. The random totally multiplicative variable  $X(n)$  is then defined multiplicatively by

$$X(n) = X(p_1)^{a_1} \dots X(p_r)^{a_r}, \tag{2.5}$$

for an integer  $n$  with prime factorisation  $n = p_1^{a_1} \dots p_r^{a_r}$ .

**Lemma 2.3.** *Let  $q$  be large,  $K$  be a polynomial of degree  $d$  with real coefficients,  $N < \frac{q^{\frac{1}{2d}}}{2}$  be a positive integer,  $B = (b_n)_{n=1}^N$  be a sequence of real variables, with  $b_n = 0$  unless  $n$  is a prime or a square of a prime. Then  $\mathbb{E}_{\oplus} \left[ K \left( \Re \left( \sum_{n=1}^N b_n \chi(n) \right) \right)^2 \right] = \mathbb{E} \left[ K \left( \Re \left( \sum_{\substack{n=1 \\ (n,q)=1}}^N b_n X(n) \right) \right)^2 \right]$ .*

**Proof.** Since  $K$  is a polynomial with real coefficients, we may write

$$K \left( \Re \left( \sum_{n=1}^N b_n \chi(n) \right) \right) = \sum_{\substack{j, k \leq N^d \\ p|jk \implies p \leq N}} c_{j, k, B, K} \chi(j)\overline{\chi(k)} \tag{2.6}$$

for certain coefficients  $c_{j,k,B,K}$  depending on the polynomial  $K$  and the coefficients  $B = (b_n)_{n=1}^N$ , satisfying

$$c_{j,k,B,K} = \overline{c_{k,j,B,K}}. \tag{2.7}$$

We may cancel any common factors of  $k$  and  $j$  using the relation:  $\chi(n)\overline{\chi(n)} = 1$  for any character with modulus  $q$  and any integer  $n$  coprime to  $q$ . For coprime integers  $m$  and  $r$  with at most  $N^d$  with prime factors all at most  $N$ , let

$$C_{m,r,B,K} = \sqrt{mr} \sum_{\substack{\frac{j}{(j,k)}=m \\ \frac{k}{(j,k)}=r \\ (jk,q)=1}} c_{j,k,B,K}. \tag{2.8}$$

Note that the sum is zero unless  $m$  and  $r$  are coprime, with all their prime factors at most  $N$  so we may drop these conditions. Using Equation (2.7), we see

$$C_{m,r,B,K} = \overline{C_{r,m,B,K}}. \tag{2.9}$$

Here, the factor  $\sqrt{mr}$  in Equation (2.8) was introduced to give the same scaling convention for coefficients at the point  $s = \frac{1}{2}$  used in other sections of the article, such as for the Dirichlet polynomial from Equation (1.26). From Equation (2.6), we see:

$$K \left( \Re \left( \sum_{n=1}^N b_n \chi(n) \right) \right) = \sum_{m,r \leq N^d} \frac{C_{m,r,B,K}}{\sqrt{mr}} \chi(m)\overline{\chi(r)}. \tag{2.10}$$

Hence,

$$K \left( \Re \left( \sum_{n=1}^N b_n \chi(n) \right) \right)^2 = \sum_{m_1, m_2, r_1, r_2 \leq N^d} \chi(m_1 m_2)\overline{\chi(r_1 r_2)} \frac{C_{m_1, r_1, B, K} C_{m_2, r_2, B, K}}{\sqrt{m_1 r_1 m_2 r_2}}. \tag{2.11}$$

The restriction  $N < \frac{q^{\frac{1}{2d}}}{2}$  ensures that we cannot have  $m_1 m_2 \equiv -r_1 r_2 \pmod q$ , and that

$$m_1 m_2 \equiv r_1 r_2 \pmod q \iff m_1 m_2 = r_1 r_2. \tag{2.12}$$

Moreover, the restrictions on the support of the coefficients  $(m_1, r_1) = (m_2, r_2) = 1$  ensure that the condition in Equation (2.12) is only achieved in the support if

$$m_1 = r_2, m_2 = r_1. \tag{2.13}$$

Using the relation from Equation (2.1) for the Dirichlet polynomial in Equation (2.11), we see:

$$\mathbb{E}_\oplus \left[ K \left( \Re \left( \sum_{n=1}^N b_n \chi(n) \right) \right)^2 \right] = \sum_{m,r \leq N^d} \frac{C_{m,r,B,K} C_{r,m,B,K}}{mr}. \tag{2.14}$$

Finally, using Equation (2.9), we see

$$\mathbb{E}_\oplus \left[ K \left( \Re \left( \sum_{n=1}^N b_n \chi(n) \right) \right)^2 \right] = \sum_{m,r \leq N^d} \frac{|C_{m,r,B,K}|^2}{mr}. \tag{2.15}$$

The corresponding identity with  $\chi(n)$  replaced by  $X(n)$  is proved similarly

$$\mathbb{E} \left[ K \left( \Re \left( \sum_{n=1}^N b_n X(n) \right) \right)^2 \right] = \sum_{m,r \leq N^d} \frac{|C_{m,r,B,K}|^2}{mr}, \tag{2.16}$$

since the orthogonality relation in Equation (2.1) holds by the definition of  $X(n)$ .  $\square$

Using Lemma 2.3, we deduce a splitting relation for the real part of Dirichlet polynomials.

**Lemma 2.4.** *Let  $1 \leq l \leq \mathcal{L}$  and  $F$  be  $l$ -sufficient as defined in Equation (1.30).*

*Then*

$$\mathbb{E}_\oplus [F^2] = \prod_{j=1}^l \mathbb{E}_\oplus \left[ K_j \left( \Re \left( \sum_n b_n^{(j)} \chi(n) \right) \right)^2 \right]. \tag{2.17}$$

**Proof.** Using the notation from the proof of Lemma 2.3, we see that for each  $1 \leq j \leq l$ , the coefficients  $C_{m,r,B^{(j)},K_j}$  are supported on coprime integers with prime factors all lying in  $(q_{j-1}, q_j]$ , and we similarly drop this condition from the sum. Then

$$K_j \left( \Re \left( \sum_n b_n^{(j)} \chi(n) \right) \right)^2 = \sum_{m_1, m_2, r_1, r_2} \chi(m_1 m_2) \overline{\chi(r_1 r_2)} \frac{C_{m_1, r_1, B^{(j)}, K_j} C_{m_2, r_2, B^{(j)}, K_j}}{\sqrt{m_1 m_2 r_1 r_2}}. \tag{2.18}$$

Moreover, for all values of  $r$  and  $m$ , we have

$$C_{m,r,B^{(j)},K_j} = \overline{C_{r,m,B^{(j)},K_j}}, \tag{2.19}$$

and since  $b_n(j)$  is zero unless  $\Omega_j(n) \leq \nu_j$ , we see  $C_{r,m,B^{(j)},K_j}$  is zero unless  $\Omega_j(r), \Omega_j(m) \leq 10(n_j - n_{j-1})^{10^4}$ . By Equation (2.1), we obtain:

$$\mathbb{E}_\oplus \left[ \Re \left( K_j \left( \sum_n b_n^{(j)} \chi(n) \right) \right)^2 \right] = \sum_{m,r} \frac{|C_{m,r,B^{(j)},K_j}|^2}{mr}. \tag{2.20}$$

Given coprime integers  $u$  and  $v$ , each having all their prime factors at most  $q_l$ , we may write

$$u = \prod_{j=1}^l m_j, v = \prod_{j=1}^l r_j, \tag{2.21}$$

where  $m_j$  and  $r_j$  are coprime integers with all their prime factors lying in the interval  $(q_{j-1}, q_j]$ . Then set

$$\tilde{C}_{u,v} = \prod_{j=1}^l C_{m_j, r_j, B^{(j)}, K_j}, \tag{2.22}$$

where we suppress the dependence of  $\tilde{C}_{u,v}$  on the sequences  $B^{(j)}$  and the polynomials  $K_j$  for ease of notation. The relation from Equation (2.19) means that

$$\tilde{C}_{u,v} = \overline{\tilde{C}_{v,u}}. \tag{2.23}$$

We note that  $\tilde{C}_{u,v}$  is supported on coprime integers  $u$  and  $v$  with all their prime factors at most  $q_l$ , so we drop this from the summation criteria to show that for any even primitive character modulo  $q$ ,

$$\prod_{j=1}^l K_j \left( \Re \left( \sum_n b_n^{(j)} \chi(n) \right) \right) = \sum_{u,v} \frac{\tilde{C}_{u,v}}{\sqrt{uv}} \chi(u) \overline{\chi(v)}. \tag{2.24}$$

The restriction on the support of the coefficients  $\tilde{C}_{u,v}$  means  $\tilde{C}_{u,v} = 0$  unless  $u, v \leq \frac{q^{\frac{1}{2}}}{2}$ , so we may apply Equation (2.1) to yield

$$\mathbb{E}_{\oplus} \left[ \prod_{j=1}^l K_j \left( \Re \left( \sum_n b_n^{(j)} \chi(n) \right) \right)^2 \right] = \sum_{u,v} \frac{|\tilde{C}_{u,v}|^2}{uv}. \tag{2.25}$$

Splitting into the components from prime factors in each interval, and using the support of the coefficients  $\tilde{C}_{u,v}$ , we may write the right-hand side of Equation (2.25) as

$$\sum_{\forall 1 \leq j \leq l, p | m_j n_j \implies p \in (q_{j-1}, q_j]} \frac{|\tilde{C}_{\prod_{j=1}^l m_j, \prod_{j=1}^l n_j}|^2}{\prod_{j=1}^l m_j n_j}. \tag{2.26}$$

Using Equation (2.22), we may write this as

$$\prod_{j=1}^l \sum_{p | m_j n_j \implies p \in (q_{j-1}, q_j]} \frac{|C_{m_j, n_j, B^{(j)}, K_j}|^2}{m_j n_j}. \tag{2.27}$$

Finally, using Equation (2.20), we may write the above expression as

$$\prod_{j=1}^l \mathbb{E}_{\oplus} \left[ K_j \left( \Re \left( \sum_n b_n^{(j)} \chi(n) \right) \right)^2 \right]. \tag{2.28}$$

This completes the proof of Lemma 2.4.  $\square$

*2.2. Moments over even primitive characters with a given modulus*

In most of the proofs, we take the expectation of Dirichlet polynomials over all even characters. However, we require a bound on the expectation of certain Dirichlet polynomials where we restrict to just the even primitive characters.

**Lemma 2.5.** *Let  $(a_n)_{n=1}^N$  be a sequence of length  $N \leq q^{\frac{5}{6}}$ . Then we have the following bound on the expectation over even primitive characters with modulus  $q$ :*

$$\mathbb{E}_+ \left[ \left| \sum_{n=1}^N a_n \chi(n) \right|^2 \right] \ll \sum_{n=1}^N |a_n|^2. \tag{2.29}$$

Note that Equation (2.29) is an upper bound for the case where there is no restriction to primitive characters in Equation (2.1), for the restricted length  $N \leq q^{\frac{5}{6}}$ .

**Proof.** Expanding the square out in Equation (2.29) shows that the left-hand side is equal to

$$\frac{1}{\varphi(q)} \sum_{j,k=1}^N a_j \overline{a_k} \left( \sum_{\chi \text{ even, primitive mod } q}^+ \chi(j) \overline{\chi(k)} \right). \tag{2.30}$$

Using the Cauchy-Schwarz inequality, this may be bounded as

$$\ll \frac{1}{\varphi(q)} \sum_{j \leq N} |a_j|^2 \left| \sum_{k \leq N} \sum_{\chi \text{ even, primitive mod } q}^+ \chi(j) \overline{\chi(k)} \right|. \tag{2.31}$$

So it suffices to show that

$$\frac{1}{\varphi(q)} \sum_{k \leq N} \left| \sum_{\chi \text{ even, primitive mod } q}^+ \chi(j) \overline{\chi(k)} \right| \ll 1, \tag{2.32}$$

uniformly for each  $j$  coprime to  $q$ .

Equation 3.2 in [IS99] gives the sum over primitive characters with modulus  $q$  (even and odd), at an individual value. Using Möbius inversion and the orthogonality of characters, they show, with the notation as set out in Section 1.2, that for any  $m$  coprime to  $q$ :

$$\sum^* \chi(m) = \sum_{\substack{vw=q \\ m \equiv 1 \pmod q}} \mu(v)\varphi(w). \tag{2.33}$$

We have

$$\sum^+ \chi(m) = \frac{1}{2} \sum^* (\chi(m) + \chi(-m)). \tag{2.34}$$

Upon substituting the relations from Equations (2.33) and (2.34), we see that the left-hand side in Equation (2.32) is bounded as:

$$\ll \frac{1}{\varphi(q)} \left( \sum_{k \leq N} \left| \sum_{\substack{vw=q \\ j \equiv k \pmod w}} \mu(v)\varphi(w) \right| + \sum_{k \leq N} \left| \sum_{\substack{vw=q \\ j \equiv -k \pmod w}} \mu(v)\varphi(w) \right| \right). \tag{2.35}$$

If we write  $t = (j - k, q)$  and  $u = (j + k, q)$ , then we transform the bound in Equation (2.35) into:

$$\begin{aligned} &\ll \frac{1}{\varphi(q)} \sum_{t,u|q} \left( \#\{k \leq N : (j - k, q) = t\} \left| \sum_{w|t} \mu\left(\frac{q}{w}\right) \varphi(w) \right| \right. \\ &\quad \left. + \#\{k \leq N : (j + k, q) = u\} \sum_{w|u} \left| \mu\left(\frac{q}{w}\right) \varphi(w) \right| \right). \end{aligned} \tag{2.36}$$

But given any factors  $t$  and  $u$  of  $q$ , we have

$$\#\{k \leq N : (j - k, q) = t\} \ll \frac{N}{t} + 1, \quad \#\{k \leq N : (j + k, q) = u\} \ll \frac{N}{u} + 1, \tag{2.37}$$

so that the expression in Equation (2.36) may be bounded as

$$\ll \frac{1}{\varphi(q)} \sum_{u|q} \left( \frac{N}{u} + 1 \right) \left| \sum_{w|u} \mu\left(\frac{q}{w}\right) \varphi(w) \right|. \tag{2.38}$$

We may write the prime factorisations of  $q, u$  and  $w$  as:

$$q = p_1^{a_1} \dots p_n^{a_n}, \quad u = p_1^{b_1} \dots p_n^{b_n}, \quad w = p_1^{c_1} \dots p_n^{c_n}, \tag{2.39}$$

respectively, where the  $p_i$  are distinct,  $a_i \geq 1$  for all  $i$ , and  $0 \leq c_i \leq b_i \leq a_i$ . Examining the term  $\mu\left(\frac{q}{w}\right)$ , we see that the contribution to Equation (2.38) is zero, unless  $b_i \in \{a_i - 1, a_i\}$  for all  $i$ . Then the choices for  $c_i$  giving non-zero contribution to Equation (2.38) are:

$$\begin{cases} c_i = a_i - 1 & b_i = a_i - 1 \\ c_i \in \{a_i - 1, a_i\} & b_i = a_i \end{cases}. \tag{2.40}$$

If we define  $S = \{1 \leq i \leq n : b_i = a_i - 1\}$ , to represent the prime factorisation of  $u$ , then the expression in Equation (2.38) becomes:

$$\frac{1}{\varphi(q)} \sum_{S \subset \{1, \dots, n\}} \left( \frac{N}{q} \prod_{i \in S} p_i + 1 \right) \left| \prod_{i \in S} \mu(p_i) \varphi(p_i^{a_i - 1}) \prod_{i \notin S} (\mu(1) \varphi(p_i^{a_i}) + \mu(p_i) \varphi(p_i^{a_i - 1})) \right|. \tag{2.41}$$

We split the sum up by the factors in the first bracket. We have to bound:

$$\mathcal{B}_1 = \frac{1}{\varphi(q)} \sum_{S \subset \{1, \dots, n\}} \left| \prod_{i \in S} \mu(p_i) \varphi(p_i^{a_i - 1}) \prod_{i \notin S} (\mu(1) \varphi(p_i^{a_i}) + \mu(p_i) \varphi(p_i^{a_i - 1})) \right|, \tag{2.42}$$

and

$$\mathcal{B}_2 = \frac{N}{q\varphi(q)} \sum_{S \subset \{1, \dots, n\}} \prod_{i \in S} p_i \left| \prod_{i \in S} \mu(p_i) \varphi(p_i^{a_i - 1}) \prod_{i \notin S} (\mu(1) \varphi(p_i^{a_i}) + \mu(p_i) \varphi(p_i^{a_i - 1})) \right|. \tag{2.43}$$

We will show that both  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are of order 1, completing the proof of Lemma 2.5.

**Bounding  $\mathcal{B}_1$ .** The expression for  $\mathcal{B}_1$  splits into a product over the prime factors of  $q$ . If the factorisation is as in Equation (2.39), then we see:

$$\mathcal{B}_1 = \prod_{i=1}^r \frac{1}{\varphi(p_i^{a_i})} (|\varphi(p_i^{a_i - 1})| + |\varphi(p_i^{a_i}) - \varphi(p_i^{a_i - 1})|). \tag{2.44}$$

Since  $\varphi(p_i^{a_i - 1}) \leq \varphi(p_i^{a_i})$ , we see that the inner term is identically 1, and so  $\mathcal{B}_1 = 1$ .

**Bounding  $\mathcal{B}_2$ .** The term for  $\mathcal{B}_2$  also splits as a product over prime factors of  $q$ .

If the factorisation is as in Equation (2.39), then we may write  $q = q^{\frac{5}{6}} \prod_j p_j^{\frac{a_j}{6}}$ . The contribution of any given  $S \subset \{1, \dots, n\}$  to the right hand side of Equation (2.43) is

$$\frac{N}{q^{\frac{5}{6}}} \cdot \frac{1}{\prod_j p_j^{\frac{a_j}{6}}} \left| \prod_{i \in S} p_i \mu(p_i) \varphi(p_i^{a_i - 1}) \prod_{i \notin S} (\mu(1) \varphi(p_i^{a_i}) + \mu(p_i) \varphi(p_i^{a_i - 1})) \right|. \tag{2.45}$$

Summing over all possible  $S \subset \{1, \dots, n\}$ , we see:

$$\mathcal{B}_2 \leq \frac{N}{q^{\frac{5}{6}}} \prod_{i=1}^n \frac{1}{p_i^{\frac{a_i}{6}} \varphi(p_i^{a_i})} (|p_i \varphi(p_i)^{a_i-1}| + |\varphi(p_i^{a_i}) - \varphi(p_i^{a_i-1})|). \tag{2.46}$$

For  $p_i > 2$ , we bound the term in the product using that

$$\frac{\varphi(p_i^{a_i})}{p_i} \leq \varphi(p_i^{a_i-1}) \leq \frac{\varphi(p_i^{a_i})}{p_i - 2}. \tag{2.47}$$

For  $p_i = 2$  we simply use that

$$|\varphi(p_i^{a_i-1})| \ll |\varphi(p_i^{a_i})|, \tag{2.48}$$

to show that the term associated to  $p_i = 2$  is bounded. Under these bounds we see that Equation (2.46) yields:

$$\mathcal{B}_2 \ll \frac{N}{q^{\frac{5}{6}}} \prod_{p_i > 2} \frac{1}{p_i^{\frac{a_i}{6}}} \left( \frac{p_i}{p_i - 2} + \frac{p_i - 1}{p_i} \right). \tag{2.49}$$

The term in the product:

$$\frac{1}{p_i^{\frac{a_i}{6}}} \left( \frac{p_i}{p_i - 2} + \frac{p_i - 1}{p_i} \right) \tag{2.50}$$

can only be greater than 1 for finitely many choices of  $p_i$  and  $a_i$ , and so we see that  $\mathcal{B}_2 \ll \frac{N}{q^{5/6}}$ . By the bounds on  $N$ , this yields  $\mathcal{B}_2 \ll 1$ .  $\square$

**Remark.** From the proof, one can show that

$$\mathbb{E}_+ \left[ \left| \sum_{n=1}^N a_n \chi(n)^2 \right| \right] \ll \left( 1 + O\left(\frac{N}{q^{\frac{5}{6}}}\right) \right) \sum_{n=1}^N |a_n|^2, \tag{2.51}$$

and hence deduce a version of Lemma 2.2 for splitting expectations over primitive characters. One could then perform all the expectations in this paper over the even primitive characters with modulus  $q$ , without adding in all the non-primitive characters. However, we take the view that, for simplicity, it is easier to just work with all the even characters, to avoid the cumulative error term in Equation (2.51).

Following this proof, we can also get the expectation of polynomials of real parts of Dirichlet characters over primitive even characters, as opposed to over all even characters in Lemma 2.4. We observe that the definition of an  $l$ -sufficient function means that the length is  $O(q^{\frac{1}{100}})$ .

**Lemma 2.6.** *Let  $1 \leq l \leq \mathcal{L}$  and  $F$  be  $l$ -sufficient as defined in (1.30).*

Then the expectation of  $F$  over primitive even characters satisfies

$$\mathbb{E}_+ [F^2] \ll \prod_{j=1}^l \mathbb{E}_\oplus \left[ K_j \left( \Re \left( \sum_n b_n^{(j)} \chi(n) \right) \right)^2 \right]. \tag{2.52}$$

**Lemma 2.7.** *Given  $k$  an integer, let  $\tilde{S}_k$  be as defined in Equation (1.22). If  $m < n$ , and  $2k \leq \frac{\log q}{e^n}$ , then*

$$\mathbb{E}_+ \left[ |\tilde{S}_n - \tilde{S}_m|^{2k} \right] \ll k!(n - m + 1)^k. \tag{2.53}$$

Note this is an analogue of Lemma 3 in [Sou09], but we give more details. We first explain some intuition behind the result. We split the sums  $\tilde{S}_m$  and  $\tilde{S}_n$  into the contributions from the primes and the contribution from the square of the primes. The expectation of the contribution of the primes give the  $n - m$  term. For fixed  $k$ , the contribution of the square of the primes is negligible, and in [Sou09] is neglected. But to take  $k$  increasing, we must consider the contribution of the squares of primes in Equation (2.53). This gives an upper bound of  $(n - m + 1)^k$ , rather than  $(n - m)^k$ , which is what we would expect from just the primes. We first consider the contribution of the primes, without the squares. We require a lemma on the expected value of Dirichlet polynomials supported on primes.

**Lemma 2.8.** *If  $a(p)$  is a sequence of complex numbers,  $x < y$  and  $y$  is a natural number such that  $y^k \leq q$ , then*

$$\sum_\chi^* \left( \sum_{x \leq p \leq y} \left| \frac{a(p)\chi(p)}{p^{\frac{1}{2}}} \right|^{2k} \right) \ll qk! \left( \sum_{x \leq p \leq y} \frac{|a(p)|^2}{p} \right)^k. \tag{2.54}$$

**Proof.** This follows from the proof of Lemma 3 in [Sou09], using the orthogonality relations for Dirichlet characters from Lemma 2.5.  $\square$

**Proof of Lemma 2.7.** The contribution of the primes to the sum  $\tilde{S}_n - \tilde{S}_m$  is:

$$U_1 := \sum_{e^{e^m} \leq p \leq e^{e^n}} \frac{\chi(p)}{p^{\frac{1}{2}}}. \tag{2.55}$$

Hence, applying Lemma 2.8, we see

$$\mathbb{E}_+ \left[ |U_1|^{2k} \right] \ll k! \left( \sum_{e^{e^m} \leq p \leq e^{e^n}} \frac{1}{p} \right)^k \ll k!(n - m)^k, \tag{2.56}$$

where the last inequality follows by Mertens’ estimate.

We need to evaluate the contribution of the terms from the squares of primes. The contribution of the squares of primes to the sum  $\tilde{S}_n - \tilde{S}_m$  is:

$$U_2 := \sum_{e^{\epsilon^m} \leq p \leq e^{\epsilon^n}} \frac{\chi(p^2)}{2p}. \tag{2.57}$$

Given an integer  $N$ , define a variable  $a_{k,m,n}(N)$  as follows: If we can write  $N = \prod_{i=1}^R p_i^{\alpha_i}$ , where the  $p_i$  are distinct primes in the interval  $(e^{\epsilon^m}, e^{\epsilon^n})$ , and  $\sum_{i=1}^R \alpha_i = k$ , then we set

$$a_{k,m,n}(N) = \binom{k}{\alpha_1 \dots \alpha_R} \prod_{i=1}^R a(p_i)^{\alpha_i}, \tag{2.58}$$

and otherwise we set  $a_{k,m,n}(N) = 0$ . We see on expansion that

$$U_2^k = \sum_r \frac{a_{k,m,n}(r)\chi(r^2)}{r}, \tag{2.59}$$

and applying the same arguments as in the proof of Lemma 3 in [Sou09], with the orthogonality relations for Dirichlet characters shows that

$$\mathbb{E}_+ \left[ |U_2|^{2k} \right] \ll k! \left( \sum_{e^{\epsilon^m} \leq p \leq e^{\epsilon^n}} \frac{1}{2p^2} \right)^k \leq k!(1)^k. \tag{2.60}$$

Hence

$$\mathbb{E}_+ \left[ |\tilde{S}_n - \tilde{S}_m|^{2k} \right] = \mathbb{E}_+ \left[ |U_1 + U_2|^{2k} \right] \ll k!(n - m + 1)^k \tag{2.61}$$

by Hölder’s inequality.  $\square$

By setting  $k = \lceil \frac{V^2}{n-m+1} \rceil$  in Equation (2.53), we get by Stirling’s formula and Markov’s inequality:

**Lemma 2.9.** *Let  $q$  be a large integer, with  $q \not\equiv 2 \pmod{4}$ ,  $V > 0$  and  $0 < m < n \leq q\mathcal{L}$ . Then*

$$\frac{1}{\varphi(q)} \#\{ \chi \text{ even primitive mod } q : |\tilde{S}_n - \tilde{S}_m| > V \} \ll \frac{V+1}{(n-m+1)^{\frac{1}{2}}} \exp\left(-\frac{V^2}{n}\right). \tag{2.62}$$

We require a stronger result for the moments of the real parts of the truncated sums.

**Lemma 2.10.** *Let  $m < n < \mathcal{L}$  and  $2k \leq \frac{\log q}{3 \log q_m}$ . Then*

$$\mathbb{E}_+ \left[ |S_{q_n} - S_{q_m}|^{2k} \right] \ll \frac{(2k)!}{2^{2k} k!} (\log \log q_n - \log \log q_m)^k. \tag{2.63}$$

Note that Equation (2.63) is stronger than the analogous Lemma A.2 in [AB23].

**Proof.** Let

$$a_{p,m,n} = \begin{cases} 1 & p \text{ a prime in } (q_m, q_n] \\ 0 & \text{otherwise} \end{cases}. \tag{2.64}$$

Then set  $U_{m,n} = \sum_p \frac{a_{p,m,n}}{p^{1/2}} \chi(p)$ , and  $T_1 = \Re U_{m,n}$  to be the real part of the sum  $U_{m,n}$ . Then  $T_1$  is the contribution of the primes to the difference  $S_n - S_m$ . Then if we set  $T_2 = (S_n - S_m) - T_1$ , then  $T_2$  is the contribution of the square of primes to the difference. Clearly, we have

$$\mathbb{E}_+ [T_1^{2k}] = \frac{1}{2^k} \sum_{j=0}^k \binom{2k}{j} \mathbb{E}_+ [U_{m,n}^j \overline{U_{m,n}}^{2k-j}]. \tag{2.65}$$

Using the same orthogonality relations as in the proof of Lemma 2.7, and observing that the choice of  $k$  means  $(S_n - S_m)^k$  has length  $q_m^{2k} \leq (q_m)^{\frac{\log q}{3 \log q_m}} = q^{\frac{1}{3}}$ , we obtain

$$\mathbb{E}_+ [T_1^{2k}] \ll \frac{1}{2^k} \sum_{j=0}^k \binom{2k}{j} \mathbb{E} \left[ \hat{U}_{m,n}^j \overline{\hat{U}_{m,n}}^{2k-j} \right], \tag{2.66}$$

where

$$\hat{U}_{m,n} = \sum_{q_m \leq p \leq q_n} \frac{a_p}{p^{1/2}} X(p), \tag{2.67}$$

and  $X(r)$  is defined in Equation (2.5). Under this definition, Equation (2.66) becomes:

$$\mathbb{E}(T_1)^{2k} \ll \mathbb{E}[\Re(\hat{U}_{m,n})^{2k}]. \tag{2.68}$$

We need the following proposition, which is a straightforward computation.

**Proposition 2.11.** *For any  $k \in \mathbb{N}$  and choice of complex numbers  $a_p$ , if  $X(p)$  are as defined in Equation (2.5), then we have:*

$$\mathbb{E} \left[ \left( \sum_{x \leq p \leq y} \Re(a_p X(p)) \right)^{2k} \right] \leq \frac{(2k)!}{2^k k!} (s^2)^k, \tag{2.69}$$

where

$$s^2 = \frac{1}{2} \sum_{x \leq p \leq y} |a_p|^2. \tag{2.70}$$

Note that the right-hand side of Equation (2.69) is the moment of a Gaussian random variable with mean 0 and variance  $s^2$ . Substituting the values of  $a_p = a_{p,m,n}$  as defined in Equation (2.64), and setting  $x = q_m, y = q_n$  in Equation (2.69), an application of Mertens’ estimate shows that:

$$s^2 = \sum_{q_m \leq p \leq q_n} \frac{1}{2p} = \frac{(\log \log q_m - \log \log q_n)}{2} + o(1). \tag{2.71}$$

Hence,

$$\mathbb{E} \left[ |T_1|^{2k} \right] \leq \frac{(2k)!}{2^{2k} k!} (\log \log q_n - \log \log q_m)^k. \tag{2.72}$$

We can use a similar method to bound  $\mathbb{E}[|T_2|^{2k}]$ . We take

$$a_p = \begin{cases} \frac{1}{2p^2} & p \text{ a prime in } (q_m, q_n] \\ 0 & \text{otherwise} \end{cases}, \tag{2.73}$$

in Equation (2.69). Since  $T_2^k$  has length  $\leq q^{\frac{1}{3}}$  we may pass to the random variables  $X$ . We obtain:

$$\mathbb{E} \left[ |S_2|^{2k} \right] \leq \frac{(2k)!}{2^{2k} k!} \left( \sum_{p \in (q_m, q_n]} \frac{1}{p^2} \right)^k. \tag{2.74}$$

Another application of Hölder’s inequality shows:

$$\mathbb{E} \left[ |S_n - S_m|^{2k} \right] \leq \frac{(2k)!}{2^{2k} k!} (n - m)^k \left( 1 + O \left( \frac{\sum_{p \in (q_m, q_n]} \frac{1}{p^2}}{\sum_{p \in (q_m, q_n]} \frac{1}{p}} \right) \right)^k. \tag{2.75}$$

We show the error term is negligible. If  $m \geq 1$ , then

$$\frac{\sum_{p \in (q_m, q_n]} \frac{1}{p^2}}{\sum_{p \in (q_m, q_n]} \frac{1}{p}} < \frac{1}{q_m}. \tag{2.76}$$

Since  $k \leq \log q$ , the error term in Equation (2.75) is  $\ll 1 + O \left( \frac{\log q}{q_m} \right) \ll 1$ . If  $m = 0$ , then

$$\sum_{p \in (q_m, q_n]} \frac{1}{p^2} < 1, \text{ whilst}$$

$$\sum_{p \in (q_m, q_n]} \frac{1}{p} \gg \log \log q_n \gg \log \log q. \tag{2.77}$$

Hence, we see  $\left| \frac{S_2}{S_1} \right| \ll \frac{1}{\log \log q}$ . By the choice of  $k$ , we have  $(q_n)^k \leq q$ , so that  $k \leq \log_3 q$ . In either case, we obtain

$$\left( 1 + O \left( \frac{\sum_{p \in (q_m, q_n]} \frac{1}{p^2}}{\sum_{p \in (q_m, q_n]} \frac{1}{p}} \right) \right)^k \ll 1, \tag{2.78}$$

so the sum over squares of primes doesn't change the order of the sum and we conclude.  $\square$

### 3. Twisted second moments in the $q$ -aspect

This section is dedicated to the proof of Theorem 1.6. We first need an expression for the mean value of  $|M_1 \dots M_l F|^2$  given in Proposition 3.1.

Let  $F$  be an  $l$ -sufficient function, as defined in Equation (1.30). We use the same notation as in the proof of Lemma 2.4. By Equation (2.24), we may write  $F$  as

$$F = \sum_{u,v} \frac{\tilde{C}_{u,v}}{\sqrt{uv}} \chi(u) \overline{\chi(v)}. \tag{3.1}$$

By construction of the coefficients, we have

$$u, v \leq \exp \left( \sum_{j=1}^l 10(n_j - n_{j-1})^{10^4} \log q_j \right) \ll q^{10^{-3}}, \tag{3.2}$$

so that we may apply Lemma 2.20 to each  $F_j$  for  $1 \leq j \leq l$ , and Lemma 2.4 to  $F$ .

By Lemma 2.4, the second moment of  $F$  may be expressed as:

$$\mathbb{E}_\oplus [F^2] = \sum_{u,v} \frac{|\tilde{C}_{u,v}|^2}{uv}. \tag{3.3}$$

We need to bound the twisted mollified moment by  $\frac{\log q}{\log q_l} \mathbb{E}_\oplus [F^2]$ . Put  $\tilde{M} = M_1 \dots M_l$ , and let

$$M = \tilde{M} F \tag{3.4}$$

be the mollified twist. Our restrictions on the coefficients of  $M_l$  and  $F$  ensure we may write

$$M(\chi) = \sum_{j,k=1}^{\mathbf{M}} \frac{x_{j,k}}{\sqrt{jk}} \chi(j) \overline{\chi(k)}, \tag{3.5}$$

for some choice of coefficients  $x_{j,k}$  integers  $j$  and  $k$  up to a finite length  $\mathbf{M}$ . We define an array  $X = (x_{j,k})_{j,k}$ .

**Proposition 3.1.** *Let  $M$  and  $X$  be defined as above. Then*

$$\mathbb{E}_+ \left[ |M|^2 \right] \ll \varphi(q) q^{-1} \mathcal{Q}(X) + O \left( q^{-\frac{1}{3}} \max_{u,v} |\tilde{C}_{u,v}|^2 \right), \tag{3.6}$$

where

$$\mathcal{Q}(X) = \sum_{\substack{j_1, j_2, k_1, k_2=1 \\ (j_1, q)=(j_2, q)=(k_1, q)=(k_2, q)=1}}^{\mathbf{M}} x_{j_1, k_1} \overline{x_{j_2, k_2}} \frac{(j_1 k_2, j_2 k_1)}{j_1 j_2 k_1 k_2} \log \left( \frac{R^2 (j_1 k_2, j_2 k_1)^2}{j_1 j_2 k_1 k_2} \right), \tag{3.7}$$

and we use the notation from [IS99]:

$$R = \frac{1}{2} + \frac{1}{2} \psi \left( \frac{1}{4} \right) + \gamma + \eta(q), \tag{3.8}$$

$$\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)},$$

$$\eta(q) = \sum_{p|q} \frac{\log p}{p-1},$$

and  $\tau(q)$  denotes the number of divisors of  $q$ .

**Proof.** In [IS99], (Section 5, Equation 5.5) they produce an estimate for the Fourier transform:

$$B(m_1, m_2) = \sum_{\chi \pmod{q}}^+ \chi(m_1) \overline{\chi(m_2)} \left| L \left( \chi, \frac{1}{2} \right) \right|^2. \tag{3.9}$$

If  $(c, q) = 1$ , then

$$B(cm_1, cm_2) = B(m_1, m_2), \tag{3.10}$$

so in order to understand  $B(m_1, m_2)$ , it suffices to consider the case where  $(m_1, m_2) = 1$ . In [IS99], they bound the terms  $B(m_1, m_2)$  in the following lemma.

**Lemma 3.2** (*Lemma 3.2 in [IS99]*). *Suppose  $(m_1, m_2) = 1$  and let  $B(m_1, m_2)$  be as defined in Equation (3.9). Then*

$$B(m_1, m_2) = \sum_{vw=q} \mu(v)\varphi(w) \sum_{m_1 n_1 \equiv m_2 n_2 \pmod w}^* W(\pi n_1 n_2 / q), \tag{3.11}$$

where

$$W(y) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \Gamma^2\left(\frac{s}{2} + \frac{1}{4}\right) G^2(s) s^{-1} y^{-s} ds, \tag{3.12}$$

and  $G(s)$  is a holomorphic function in the region  $|\Re(s)| < 1$  such that

$$G(s) = G(-s), \quad G\left(\frac{1}{2}\right) = G\left(-\frac{1}{2}\right) = 0, \quad G(0)\Gamma\left(\frac{1}{4}\right) = 1, \tag{3.13}$$

and

$$G(s) \ll |s|^{-3} e^{\frac{\pi|s|}{4}}. \tag{3.14}$$

In Equation (3.11), the  $\star$  restricts the summation to numbers coprime to  $q$ . Moreover, the dominant term is the diagonal contribution to  $B(m_1, m_2)$  in Equation (3.11) when  $m_1 n_1 = m_2 n_2$ , which by their Lemma 4.1 is:

$$B_0(m_1, m_2) = \frac{\varphi^+(q)\varphi(q)}{q\sqrt{m_1 m_2}} \log\left(\frac{R^2}{m_1 m_2}\right) + O(\tau(q)q^{\frac{1}{2}}). \tag{3.15}$$

The error terms from the other terms contribute at most  $\beta(m_1, m_2)$ , where

$$\beta(m_1, m_2) = \sum_{\substack{m_1 n_1 \neq m_2 n_2 \\ (n_1, q) = (n_2, q) = 1}} (m_1 n_1 \pm m_2 n_2, q) (n_1 n_2)^{-\frac{1}{2}} \left| W\left(\frac{\pi n_1 n_2}{q}\right) \right|, \tag{3.16}$$

and the sums for the different signs for  $\pm$  are taken separately. Using Equation (3.16), they obtain:

$$B(m_1, m_2) = \frac{\varphi^+(q)\varphi(q)}{q\sqrt{m_1 m_2}} \log\left(\frac{R^2}{m_1 m_2}\right) + O(\beta(m_1, m_2) + \tau(q)q^{\frac{1}{2}}), \tag{3.17}$$

where  $R$  is as defined in Equation (3.8) and  $\varphi^+(q)$  is the number of even primitive characters modulo  $q$ . In order to get a sufficient error bound when we substitute Equation (3.17) into the terms forming  $M$ , we require a bound on the values of  $j$  and  $k$  which contribute to the sum in Equation (3.5).

**Proposition 3.3.** *If  $\max\{j, k\} \gg q^{\frac{1}{10\sigma}}$ , then  $x_{j,k} = 0$*

**Proof.** Non-zero coefficients in  $M$  come from products of terms of  $M_1 \dots M_l$  and  $F$ . By construction, for each  $1 \leq j \leq l$ ,  $M_j$  has length  $\leq \exp((n_j - n_{j-1})^{10^5} \log q_j)$ , and so by construction of the values  $q_l$ ,  $M_1 \dots M_l$  has length  $\ll \exp((n_{l+1} - n_l)^{10^5}) e^{n_l}$ , which is  $\ll q^{10^{-4}}$ .

By Equations (3.1) and (3.2), we may write

$$F = \sum_{u,v \ll q^{10^{-3}}} \frac{\tilde{C}_{u,v}}{\sqrt{uv}} \chi(u) \overline{\chi(v)}. \tag{3.18}$$

We see that we may take

$$\mathbf{M} \ll q^{\frac{1}{100}}. \quad \square \tag{3.19}$$

For the context of  $l$ -sufficient functions, since the coefficients of  $M_l$  have size at most 1, and  $M_1 \dots M_l$  has length  $\ll q^{10^{-4}}$ , the coefficients of  $M$  satisfy:

$$\max_{j,k} \frac{|x_{j,k}|}{\sqrt{jk}} \ll q^{10^{-4}} \max_{u,v} \frac{|\tilde{C}_{u,v}|}{\sqrt{uv}}. \tag{3.20}$$

Using Equation (3.5), we may write

$$|M|^2 = \sum_{\substack{j_1, j_2, k_1, k_2=1 \\ (j_1, q)=(j_2, q)=(k_1, q)=(k_2, q)=1}}^{\mathbf{M}} \frac{\chi(j_1 k_2) \overline{\chi(j_2 k_1)}}{\sqrt{j_1 k_1 j_2 k_2}} x_{j_1, k_1} \overline{x_{j_2, k_2}}. \tag{3.21}$$

The expression in Equation (3.17) for  $B(m_1, m_2)$  is only valid when  $m_1$  and  $m_2$  are coprime. To put all the non-zero terms in the above expression in this form, we rewrite it as:

$$|M|^2 = \sum_{\substack{j_1, j_2, k_1, k_2=1 \\ (j_1, q)=(j_2, q)=(k_1, q)=(k_2, q)=1}}^{\mathbf{M}} \chi\left(\frac{j_1 k_2}{(j_1 k_2, j_2 k_1)}\right) \overline{\chi\left(\frac{j_2 k_1}{(j_1 k_2, j_2 k_1)}\right)} \frac{x_{j_1, k_1} \overline{x_{j_2, k_2}}}{\sqrt{j_1 k_1 j_2 k_2}}. \tag{3.22}$$

Using Equation (3.17), we can write:

$$\begin{aligned} \mathbb{E}_+ \left[ |LM|^2 \right] &\ll \sum_{\substack{j_1, j_2, k_1, k_2=1 \\ (j_1, q)=(j_2, q)=(k_1, q)=(k_2, q)=1}}^{\mathbf{M}} x_{j_1, k_1} \overline{x_{j_2, k_2}} \times \\ &\left\{ \frac{\varphi(q)(j_1 k_2, j_2 k_1)}{q j_1 j_2 k_1 k_2} \log \left( \frac{R^2(j_1 k_2, j_2 k_1)^2}{j_1 j_2 k_1 k_2} \right) + \right. \\ &\left. O \left( \frac{\beta \left( \frac{j_1 k_2}{(j_1 k_2, j_2 k_1)}, \frac{j_2 k_1}{(j_1 k_2, j_2 k_1)} \right) + \tau(q) q^{\frac{1}{2}}}{\varphi(q) \sqrt{j_1 k_1 j_2 k_2}} \right) \right\}. \end{aligned} \tag{3.23}$$

If we put  $X = (x_{j,k})_{j,k}$ , then we may rewrite the right-hand side of Equation (3.23) as:

$$\varphi(q)q^{-1} \mathcal{Q}(X) + O\left(\frac{\max_{j,k} |x_{j,k}|^2}{\varphi(q)} \left(\sum_{j_1, j_2, k_1, k_2} \tau(q)q^{\frac{1}{2}} + \beta\left(\frac{j_1 k_2}{(j_1 k_2, j_2 k_1)}, \frac{j_2 k_1}{(j_1 k_2, j_2 k_1)}\right)\right)\right). \tag{3.24}$$

Using the relation from Equation (3.20), we may re-express the error bound as:

$$\ll O\left(\frac{q^{10^{-3}} \max_{u,v} |\tilde{C}_{u,v}|^2}{\varphi(q)} \left(\sum_{\substack{j_1, j_2, k_1, k_2=1 \\ (j_1, q)=(j_2, q)=(k_1, q)=(k_2, q)=1}}^M \tau(q)q^{\frac{1}{2}} + \beta\left(\frac{j_1 k_2}{(j_1 k_2, j_2 k_1)}, \frac{j_2 k_1}{(j_1 k_2, j_2 k_1)}\right)\right)\right). \tag{3.25}$$

We need to bound

$$\sum_{\substack{j_1, j_2, k_1, k_2=1 \\ (j_1, q)=(j_2, q)=(k_1, q)=(k_2, q)=1}}^M \beta\left(\frac{j_1 k_2}{(j_1 k_2, j_2 k_1)}, \frac{j_2 k_1}{(j_1 k_2, j_2 k_1)}\right). \tag{3.26}$$

We recall from Equation (3.16), if  $(m_1, m_2) = 1$ , then

$$\beta(m_1, m_2) = \sum_{\substack{m_1 n_1 \neq m_2 n_2 \\ (n_1, q)=(n_2, q)=1}} (m_1 n_1 \pm m_2 n_2, q)(n_1 n_2)^{-\frac{1}{2}} \left|W\left(\frac{\pi n_1 n_2}{q}\right)\right|. \tag{3.27}$$

Hence,

$$\begin{aligned} & \sum_{\substack{j_1, j_2, k_1, k_2=1 \\ (j_1, q)=(j_2, q)=(k_1, q)=(k_2, q)=1}}^M \beta\left(\frac{j_1 k_2}{(j_1 k_2, j_2 k_1)}, \frac{j_2 k_1}{(j_1 k_2, j_2 k_1)}\right) \\ &= \sum_{(n_1, q)=(n_2, q)=1} (n_1 n_2)^{-\frac{1}{2}} \left|W\left(\frac{\pi n_1 n_2}{q}\right)\right| \sum_{\substack{j_1, j_2, k_1, k_2=1 \\ n_1 j_1 k_2 \neq n_2 j_2 k_1}}^M \left(\frac{n_1 j_1 k_2 \pm n_2 j_2 k_1}{(j_1 k_2, j_2 k_1)}, q\right). \end{aligned} \tag{3.28}$$

Since  $j_1, j_2, k_1, k_2, n_1$  and  $n_2$  are coprime to  $q$ , the inner sum:

$$\sum_{\substack{j_1, j_2, k_1, k_2=1 \\ n_1 j_1 k_2 \neq n_2 j_2 k_1}}^M \left(\frac{n_1 j_1 k_2 \pm n_2 j_2 k_1}{(j_1 k_2, j_2 k_1)}, q\right) = \sum_{\substack{j_1, j_2, k_1, k_2=1 \\ n_1 j_1 k_2 \neq n_2 j_2 k_1}}^M (n_1 j_1 k_2 \pm n_2 j_2 k_1, q) \tag{3.29}$$

may be bounded as  $\ll \sum_{d|q} \frac{M^4}{d} \ll M^4 \tau(q)$ . For Equation 4.8 in [IS99], the authors use  $W(y) \ll (1+y)^{-1}$ . Hence the sum in Equation (3.28) may be bounded above by

$$\begin{aligned}
 M^4 \tau(q) \sum_{n_1, n_2} (n_1 n_2)^{-\frac{1}{2}} \left(1 + \frac{n_1 n_2}{q}\right)^{-1} &\ll M^4 \left( \tau(q) \sum_{\substack{n_1, n_2 \\ n_1 n_2 < q}} (n_1 n_2)^{-\frac{1}{2} + q} + \sum_{n_1 n_2 > q} (n_1 n_2)^{-\frac{3}{2}} \right) \\
 &\ll M^4 \tau(q) \sqrt{q} \log q.
 \end{aligned}
 \tag{3.30}$$

Using this bound for the error term expressed in (3.25) for the expression in Equation (3.24), the bound in Equation (3.19), and the bounds

$$\frac{q}{\varphi(q)}, \tau(q) \ll_{\epsilon} q^{\epsilon},
 \tag{3.31}$$

we obtain

$$\mathbb{E}_+ \left[ |M|^2 \right] \ll \varphi(q) q^{-1} \mathcal{Q}(X) + O \left( q^{-\frac{1}{3}} \max_{u,v} |\tilde{C}_{u,v}|^2 \right).
 \tag{3.32}$$

This completes the proof of Proposition 3.1.  $\square$

**Proof of Theorem 1.6.** Having proven Proposition 3.1, we proceed with the proof of Theorem 1.6. We want to calculate  $\mathcal{Q}(X)$  for the value of  $X$  determined by the coefficients of  $M$  as defined in Equation (3.4).

We recall from Equation (3.3), the following formula for the expectation of the function  $F$ :

$$\mathbb{E}_{\oplus} [F^2] = \sum_{\substack{p|uv \implies p \leq q \\ (u,v)=1}} \frac{|\tilde{C}_{u,v}|^2}{uv}.
 \tag{3.33}$$

Combining Equations (3.20) and (3.33) with the bounds

$$\frac{q}{\varphi(q)}, \frac{q}{\varphi^+(q)}, \tau(q) \ll_{\epsilon} q^{\epsilon},
 \tag{3.34}$$

for any  $\epsilon > 0$ , we see that the error term in Equation (3.6) is

$$o \left( \mathbb{E}_{\oplus} \left[ |F|^2 \right] \right).
 \tag{3.35}$$

We will show that

$$\mathcal{Q}(X) \ll \frac{\log q}{\log q!} \mathbb{E}_+ \left[ |F|^2 \right],
 \tag{3.36}$$

which upon substitution into Equation (3.6) will complete the proof of Theorem 1.6.

We need to substitute the choice of  $M$  into Equation (3.6). First, we rewrite the expression for  $\mathcal{Q}(X)$  in Equation (3.7) to group terms coming from the same term in  $Q(\chi)$  together. Let  $\tilde{M}$  have coefficients given by:

$$\tilde{M}(\chi) = \sum_f \frac{e_f \chi(f)}{m^{\frac{1}{2}}}. \tag{3.37}$$

Expanding  $M = \tilde{M}F$ , we see

$$x_{j,k} = \sum_{uf=j} \tilde{C}_{u,k} e_f. \tag{3.38}$$

Substituting this relation into Equation (3.7) yields:

$$\mathcal{Q}(X) = \sum_{u_1, k_1, u_2, k_2} \tilde{C}_{u_1, k_1} \overline{\tilde{C}_{u_2, k_2}} \sum_{f_1, f_2} \frac{e_{f_1} e_{f_2}}{[u_1 k_2 f_1, u_2 k_1 f_2]} \log \left( \frac{R^2(u_1 k_2 f_1, u_2 k_1 f_2)^2}{u_1 k_2 f_1 u_2 k_1 f_2} \right). \tag{3.39}$$

For fixed choices of  $u_1, k_1, u_2$  and  $k_2$ , we can separate the log factor in Equation (3.39) into two terms:

$$\log \left( \frac{R^2(u_1 k_1 f_2, u_2 k_2 f_1)^2}{u_1 k_1 f_2 u_2 k_2 f_1} \right) = \log \left( \frac{R^2(u_1 k_2, u_2 k_1)^2}{u_1 k_1 u_2 k_2} \right) - \log \left( \frac{f_1 f_2 (u_1 k_2, u_2 k_1)^2}{(u_1 k_2 f_1, u_2 k_1 f_2)^2} \right). \tag{3.40}$$

If we take this first term on the right hand side, which is constant in  $f_1$  and  $f_2$ , out, then the remaining factor inside the logarithm is doubly multiplicative in  $f_1$  and  $f_2$  for a fixed choice of  $c_1$  and  $c_2$ , which makes it easier to sum over. We write Equation (3.39) as:

$$\mathcal{Q}(X) = P_1 - P_2, \tag{3.41}$$

where

$$P_1 = \sum_{u_1, k_1, u_2, k_2} \tilde{C}_{u_1, k_1} \overline{\tilde{C}_{u_2, k_2}} \log \left( \frac{R^2(u_1 k_2, u_2 k_1)^2}{u_1 k_1 u_2 k_2} \right) \sum_{f_1, f_2} \frac{e_{f_1} e_{f_2}}{[u_1 k_2 f_1, u_2 k_1 f_2]}, \tag{3.42}$$

and

$$P_2 = \sum_{u_1, k_1, u_2, k_2} \tilde{C}_{u_1, k_1} \overline{\tilde{C}_{u_2, k_2}} \sum_{f_1, f_2} \frac{e_{f_1} e_{f_2}}{[u_1 k_2 f_1, u_2 k_1 f_2]} \log \left( \frac{f_1 f_2 (u_1 k_2, u_2 k_1)^2}{(u_1 k_2 f_1, u_2 k_1 f_2)^2} \right). \tag{3.43}$$

We think of  $P_1$  as being the sum where we take an approximate value for the logarithm factor by setting  $f_1 = f_2 = 1$ , and  $P_2$  being the oscillatory term which takes into account

the variation in the logarithmic factor for different values of  $f_1$  and  $f_2$ . We bound both  $P_1$  and  $P_2$ , and show:

$$P_1, P_2 \ll \frac{\log q}{\log ql} \mathbb{E}_\oplus [ |F|^2 ]. \tag{3.44}$$

We first bound  $P_1$  by taking the maximum value of the logarithm factor. From the definition of  $R$ , we see the logarithmic factor be bounded by  $\log q$  and so

$$|P_1| \leq \sum_{u_1, k_1, u_2, k_2} \left| \tilde{C}_{u_1, k_1} \overline{\tilde{C}_{u_2, k_2}} \log \left( \frac{R^2(u_1 k_2, u_2 k_1)^2}{u_1 k_1 u_2 k_2} \right) \right| \left| \sum_{f_1, f_2} \frac{e_{f_1} e_{f_2}}{[u_1 k_2 f_1, u_2 k_1 f_2]} \right| \ll P_3, \tag{3.45}$$

where

$$P_3 = \log q \sum_{u_1, k_1, u_2, k_2} \left| \tilde{C}_{u_1, k_1} \overline{\tilde{C}_{u_2, k_2}} \right| \left| \sum_{f_1, f_2} \frac{e_{f_1} e_{f_2}}{[u_1 k_2 f_1, u_2 k_1 f_2]} \right|. \tag{3.46}$$

We show that  $P_3$  may be bounded by  $\frac{\log q}{\log ql} \mathbb{E}_\oplus [ |Q|^2 ]$  in Section 3.1 below.

**Proposition 3.4.** *Let  $P_3$  be defined as above. Then*

$$P_3 \ll \frac{\log q}{\log ql} \mathbb{E}_\oplus [ F^2 ]. \tag{3.47}$$

For  $P_2$ , we also want to bound the sum absolutely to begin with. We have from Equation (3.43),

$$|P_2| \leq \sum_{u_1, k_1, u_2, k_2} \left| \tilde{C}_{u_1, k_1} \tilde{C}_{u_2, k_2} \right| \left| \sum_{f_1, f_2} \frac{e_{f_1} e_{f_2}}{[u_1 k_2 f_1, u_2 k_1 f_2]} \log \left( \frac{f_1 f_2 (u_1 k_2, u_2 k_1)^2}{(u_1 k_2 f_1, u_2 k_1 f_2)^2} \right) \right|. \tag{3.48}$$

We want to approximate the logarithm by multiplicative functions, to make them easier to sum. Observe that, given  $T > 1$ , as  $\beta \rightarrow 0$  we have by Taylor’s Theorem:

$$\log t = \frac{t^{i\beta} - t^{-i\beta}}{2i\beta} + O(\beta), \tag{3.49}$$

where the error decays uniformly in  $\beta$ , for all  $1 \leq t \leq T$ . This will enable us to bound  $P_2$ .

We substitute  $t$  for

$$\frac{f_1 f_2 (u_1 k_2, u_2 k_1)^2}{(u_1 k_2 f_1, u_2 k_1 f_2)^2}, \tag{3.50}$$

in the definition of  $P_2$  in Equation (3.43). We see that  $t \geq 1$ , whilst the restrictions on prime factorisations ensure that if  $e_{f_1}e_{f_2} \neq 0$ , then  $t$  can be bounded by some  $T$  depending only on  $q$ . Hence,  $P_2$  may be approximated by taking the limit as  $\beta \rightarrow 0$  in Equation (3.49). Indeed, if we set:

$$P_4(\beta) = \frac{1}{2\beta} \sum_{u_1, k_1, u_2, k_2} \left| \tilde{C}_{u_1, k_1} \overline{\tilde{C}_{u_2, k_2}} \right| \tag{3.51}$$

$$\left| \sum_{f_1, f_2} \frac{e_{f_1} e_{f_2}}{[u_1 k_2 f_1, u_2 k_1 f_2]} \left( \left( \frac{f_1 f_2 (u_1 k_2, u_2 k_1)^2}{(u_1 k_2 f_1, u_2 k_1 f_2)^2} \right)^{i\beta} - \left( \frac{f_1 f_2 (u_1 k_2, u_2 k_1)^2}{(u_1 k_2 f_1, u_2 k_1 f_2)^2} \right)^{-i\beta} \right) \right|,$$

then the bound in Equation (3.48) becomes  $|P_2| \leq \lim_{\beta \rightarrow 0^+} P_4(\beta)$ . We show in Section 3.2:

**Proposition 3.5.** *Let  $P_4$  be defined as above. Then*

$$\lim_{\beta \rightarrow 0^+} P_4(\beta) \ll \frac{\log q}{\log q_l} \mathbb{E}_{\oplus} [F^2]. \tag{3.52}$$

Combining Propositions 3.4 and 3.5 completes the proof of Theorem 1.6.  $\square$

3.1. *Proof of Proposition 3.4*

The formula for  $P_3$  is used for estimating the sum  $P_1$  in (3.41). The bound  $P_3 \geq P_1$  was obtained by bounding the absolute value of the logarithmic factor by its maximum, which is smaller than  $\log q$ . The expression for  $P_3$  splits into products over contributions of expectations of the components of the Dirichlet polynomial supported on the integers with primes in the interval  $(q_{i-1}, q_i]$ , for  $1 \leq i \leq l$ . Since  $F$  is  $l$ -sufficient, the contributions of the different intervals are weakly dependent, and the expectation of the  $P_3$  may be bounded by the product of the expectations from each interval. Indeed, we have:

$$P_3 = \log q \prod_i N_i, \tag{3.53}$$

where  $N_i$  is the contribution to  $P_3$  from the  $F_i$  component in the factorisation of  $F = \prod_{i=1}^l F_i$ :

$$N_i = \left| \sum_{\substack{p|f_1, f_2 \implies p \in (q_{i-1}, q_i] \\ \Omega_i(f_1), \Omega_i(f_2) \leq (n_i - n_{i-1})^{10^5}}} \frac{\mu(f_1)\mu(f_2)}{[u_1 k_2 f_1, u_2 k_1 f_2]} \right| \sum_{u_1, k_1, u_2, k_2} |C_{u_1, k_1, B^{(i)}, K_i} C_{u_2, k_2, B^{(i)}, K_i}|. \tag{3.54}$$

In Lemma 3.6 below, we show we may lift the restriction on the number of prime factors of coefficients  $f_1$  and  $f_2$  of the mollifier  $M_l$  with negligible error; in Lemma 3.7 we bound the sum with the restriction removed.

We use Rankin’s trick to show that we can remove the restriction on the number of prime factors of  $f_1$  and  $f_2$ , the terms coming from the mollifier  $\tilde{M}$ , with negligible error. If we were able to remove this restriction completely, then  $\tilde{M}$  would completely mollify the component of the Euler product of the Dirichlet L-function coming from primes in the interval  $(q_{i-1}, q_i]$ ; this next lemma essentially shows that the factors of mollifier  $M_1, \dots, M_l$  are long enough to have this mollification effect. Lemma 3.7 then evaluates the sum with this restriction removed.

**Lemma 3.6.** *For  $0 \leq i \leq l$ , the expression in Equation (3.54) is*

$$\sum_{u_1, k_1, u_2, k_2} |C_{u_1, k_1, B^{(i)}, K_i} C_{u_2, k_2, B^{(i)}, K_i}| \left| \sum_{\substack{p|f_1, f_2 \implies p \in (q_{i-1}, q_i]}} \frac{\mu(f_1)\mu(f_2)}{[u_1 k_2 f_1, u_2 k_1 f_2]} \right| + O\left(e^{-100(n_i - n_{i-1})} \mathbb{E}_{\oplus} \left[ |F_i|^2 \right] \right). \tag{3.55}$$

**Proof.** The contribution of those values of  $f_1$  where  $\Omega_i(f_1) > (n_i - n_{i-1})^{10^5}$ , which are excluded by the restriction on the length of the mollifier, to the expression in Equation (3.55) may be bounded for any  $\rho > 0$  as:

$$\leq e^{-\rho(n_i - n_{i-1})^{10^5}} \times \sum_{u_1, k_1, u_2, k_2} |C_{u_1, k_1, B^{(i)}, K_i} C_{u_2, k_2, B^{(i)}, K_i}| \sum_{\substack{p|f_1 \implies p \in (q_{i-1}, q_i] \\ PP|f_2 \implies p \in (q_{i-1}, q_i]}} e^{\rho\Omega(f_1)} \frac{\mu^2(f_1)\mu^2(f_2)}{[u_1 k_2 f_1, u_2 k_1 f_2]}. \tag{3.56}$$

We sum over the possible indices of each prime  $p$  in the interval in the prime factorisations of  $f_1$  and  $f_2$ , which both must be squarefree to contribute to the sum. Say the exponent is  $d_{p,i} = v_p(f_i) \in \{0, 1\}$ . Then, for fixed values of  $u_1, u_2, k_1$  and  $k_2$ , the inner sum over  $f_1$  and  $f_2$  in Equation (3.56) is:

$$\prod_{p \in (q_{i-1}, q_i]} \sum_{d_{p,1}, d_{p,2} \in \{0,1\}} \frac{e^{\rho\omega_i(p^{d_{p,1}})}}{p^{\max\{(d_{p,1} + v_p(u_1 k_2)), d_{p,2} + v_p(u_2 k_1)\}}}. \tag{3.57}$$

For a given prime  $p$  in the interval  $(q_{i-1}, q_i]$ , the term in the inner product may be expressed as:

$$\frac{1}{p^{\max\{v_p(u_1 k_2), v_p(u_2 k_1)\}}} \sum_{d_{p,1}, d_{p,2} \in \{0,1\}} \frac{e^{\rho\omega_i(p^{d_{p,1}})}}{p^{\max\{d_{p,1} + v_p(u_1 k_2), d_{p,2} + v_p(u_2 k_1)\} - \max\{v_p(u_1 k_2), v_p(u_2 k_1)\}}}. \tag{3.58}$$

Considering the possible values for the exponent

$$\max\{d_{p,1} + v_p(u_1 k_2), d_{p,2} + v_p(u_2 k_1)\} - \max\{v_p(u_1 k_2), v_p(u_2 k_1)\},$$

this term associated to the prime  $p$  may be bounded by:

$$\frac{1}{p^{\max\{v_p(u_1k_2), v_p(u_2k_1)\}}} \times \begin{cases} 1 + \frac{100e^\rho}{p} & v_p(u_1k_2) = v_p(u_2k_1) \\ 100e^\rho & v_p(u_1k_2) \neq v_p(u_2k_1) \end{cases}. \tag{3.59}$$

Since  $u_1, u_2, k_1$  and  $k_2$  have relatively few prime factors, we expect for typical primes in the interval,  $v_p(u_1k_2) = v_p(u_2k_1) = 0$ , so that Equation (3.59) only yields the larger bound  $100e^\rho$  on a few primes. Note that the contribution of  $p \in (q_{i-1}, q_i]$  with  $v_p(u_1k_2) = v_p(u_2k_1)$  may be bounded by taking the product of the bound in Equation (3.59) over all primes in the interval. Thus, their contribution is at most

$$\prod_{p \in (q_{i-1}, q_i]} 1 + \frac{100e^\rho}{p}. \tag{3.60}$$

An application of the Prime Number Theorem shows that the contribution of these typical primes may be bounded by:

$$\left( \frac{\log q_i}{\log q_{i-1}} \right)^{100e^\rho} = e^{100e^\rho(n_i - n_{i-1})}. \tag{3.61}$$

Hence, if we define a function  $\alpha$  to collate the contribution of all the bounds in Equation (3.59) used for each prime in the interval, then the coefficient for each term in the polynomial, which we are calculating in Equation (3.57) may be bounded by:

$$\frac{\alpha(u_1k_2, u_2k_1)}{[u_1k_2, u_2k_1]} e^{100e^\rho(n_i - n_{i-1})}, \tag{3.62}$$

where  $\alpha$  is a doubly multiplicative function such that:

$$\alpha(p^n, p^n) = 1 + \frac{100e^\rho}{p} \text{ for all } n \geq 1, \text{ and } \alpha(p^n, p^m) = 100e^\rho \text{ for } n, m \geq 0 \text{ with } n \neq m. \tag{3.63}$$

Returning to Equation (3.56), the effect of adding in the extra values of  $f_1$  with many prime factors may be bounded by

$$e^{-\rho(n_i - n_{i-1})^{10^5} + 100e^\rho(n_i - n_{i-1})} \sum_{u_1, k_1, u_2, k_2} |C_{u_1, k_1, B^{(i)}, K_i} C_{u_2, k_2, B^{(i)}, K_i}| \frac{\alpha(u_1k_2, u_2k_1)}{[u_1k_2, u_2k_1]}. \tag{3.64}$$

Using the Cauchy-Schwarz inequality:

$$|C_{u_1, k_1, B^{(i)}, K_i} C_{u_2, k_2, B^{(i)}, K_i}| \leq \frac{1}{2} \left( |C_{u_1, k_1, B^{(i)}, K_i}|^2 + |C_{u_2, k_2, B^{(i)}, K_i}|^2 \right), \tag{3.65}$$

and dropping the restriction on  $\Omega_i(u_1)$  and  $\Omega_i(k_2)$ , we can bound the sum in Equation (3.64) by:

$$\sum_{u_2, k_2} |C_{u_2, k_2, B^{(i)}, K_i}|^2 \sum_{\substack{p|u_1 k_1 \implies p \in (q_{i-1}, q_i] \\ (u_1, k_1) = (u_2, k_2) = 1}} \frac{\alpha(u_1 k_2, u_2 k_1)}{[u_1 k_2, u_2 k_1]}. \tag{3.66}$$

We seek to bound the inner sum,

$$\sum_{\substack{p|u_1 k_2 \implies p \in (q_{i-1}, q_i] \\ (u_1, k_1) = (u_2, k_2) = 1}} \frac{\alpha(u_1 k_2, u_2 k_1)}{[u_1 k_2, u_2 k_1]}. \tag{3.67}$$

The restriction  $(u_1, k_1) = (u_2, k_2) = 1$  ensures that

$$[u_1 k_2, u_2 k_1] = [u_1, u_2][k_1, k_2]. \tag{3.68}$$

Meanwhile, using that for any prime  $p \in (q_{i-1}, q_i]$ , for  $v_p(u_1 k_2) \neq v_p(u_2 k_1)$  we must have either  $v_p(u_1) \neq v_p(u_2)$  or  $v_p(k_1) \neq v_p(k_2)$ , we see that

$$\alpha(u_1 k_2, u_2 k_1) \leq \alpha(u_1, u_2)\alpha(k_2, k_1). \tag{3.69}$$

Hence we may bound the sum in Equation (3.67) for any values of  $u_2$  and  $k_2$  by:

$$\sum_{p|u_1 k_2 \implies p \in (q_{i-1}, q_i]} \frac{\alpha(u_1, u_2)\alpha(k_1, k_2)}{[u_1, u_2][k_1, k_2]} = \left( \sum_{p|u_1 \implies p \in (q_{i-1}, q_i]} \frac{\alpha(u_1, u_2)}{[u_1, u_2]} \right) \times \left( \sum_{p|k_2 \implies p \in (q_{i-1}, q_i]} \frac{\alpha(k_1, k_2)}{[k_1, k_2]} \right). \tag{3.70}$$

We follow [ABR20], Section 8.2, where we change their notation to write  $\alpha$  for their function  $f$ . Using their bounds, we have for any integer  $c_2$  whose prime factors all lie in the interval  $(q_{i-1}, q_i]$ :

$$\sum_{p|c_1 \implies p \in (q_{i-1}, q_i]} \frac{\alpha(c_1, c_2)}{[c_1, c_2]} \leq \frac{1}{c_2} \exp\left(200e^\rho(n_i - n_{i-1}) + 10^4 \rho(n_i - n_{i-1})^{10^4+1}\right). \tag{3.71}$$

Combining Equations (3.62), (3.66) and (3.71) we see that the expression in Equation (3.56) may be bounded above by:

$$e^{-\rho(n_i - n_{i-1})^{10^5} + 500e^\rho(n_i - n_{i-1}) + 2 \cdot 10^4 \rho(n_i - n_{i-1})^{10^4+1}} \sum_{u_2, k_2} \frac{|C_{u_2, k_2, B^{(i)}, K_i}|^2}{u_2 k_2}. \tag{3.72}$$

For  $1 \leq i \leq l$  we have by Equation (1.20),

$$n_i - n_{i-1} = \mathbf{s}(\log_{i+1}(q) - \log_{i+2}(q)), \tag{3.73}$$

and so, by choice of  $\mathbf{s}$ , we have  $n_i - n_{i-1} \geq 10^5$ . Hence, picking  $\rho = 1000$  in Equation (3.72), we may bound the expression in Equation (3.56) by

$$e^{-100(n_i - n_{i-1})} \mathbb{E}_{\oplus} [|F_i|^2].$$

A similar calculation shows we may lift the restriction on  $\Omega_i(f_2)$ , incurring an error of at most order  $e^{-100(n_i - n_{i-1})} \mathbb{E}_{\oplus} [|F_i|^2]$ . This concludes the proof of Lemma 3.6.  $\square$

It remains to estimate the main term in the product for  $P_3$  in (3.55), i.e.,

$$\sum_{u_1, k_1, u_2, k_2} |C_{u_1, k_1, B^{(i)}, K_i} C_{u_2, k_2, B^{(i)}, K_i}| \left| \sum_{p|f_1, f_2 \implies p \in (q_{i-1}, q_i]} \frac{\mu(f_1)\mu(f_2)}{[u_1 k_2 f_1, u_2 k_1 f_2]} \right|. \tag{3.74}$$

**Lemma 3.7.** *The value of the expression in Equation (3.74) is*

$$\mathbb{E}_{\oplus} [|F_i|^2] \times \prod_{p \in (q_{i-1}, q_i]} \left(1 - \frac{1}{p}\right). \tag{3.75}$$

**Proof.** We consider the contribution to the sum for different values of  $u_1, u_2, k_1$  and  $k_2$ . We recall that the coprimality conditions for the support of the summand mean that  $u_1 k_2 = u_2 k_1 \iff u_1 = u_2, k_1 = k_2$ . Suppose  $u_1 k_2 \neq u_2 k_1$ . We show the sum vanishes, so that only the diagonal terms from taking  $u_1 k_2 = u_2 k_1$  remain. Since  $u_1 k_2$  and  $u_2 k_1$  have distinct prime factorisations, there must be some prime which divides them to a different exponent. Hence,

$$\exists p^* \in (q_{i-1}, q_i] : v_{p^*}(u_2 k_1) \neq v_{p^*}(u_1 k_2). \tag{3.76}$$

Without loss of generality, we may assume

$$v_{p^*}(u_2 k_1) > v_{p^*}(u_1 k_2). \tag{3.77}$$

We consider the prime decomposition of  $f_1, f_2, u_2 k_1$  and  $u_1 k_2$  into powers of  $p^*$  and a remainder. Write

$$f_1 = f'_1 (p^*)^{d_1} \quad f_2 = f'_2 (p^*)^{d_2} \quad u_2 k_1 = c_1 (p^*)^{d_3} \quad u_1 k_2 = c_2 (p^*)^{d_4},$$

where  $d_1, d_2 \in \{0, 1\}$  and  $p^* \nmid f'_1, f'_2, c_1, c_2$ . The factor multiplying  $|C_{u_1, k_1, B^{(i)}, K_i} C_{u_2, k_2, B^{(i)}, K_i}|$  in Equation (3.74) is:

$$\sum_{p|f'_1, f'_2 \implies p \in (q_{i-1}, q_i] \setminus p^*} \frac{\mu(f'_1)\mu(f'_2)}{[c_1 f'_1, c_2 f'_2]} \sum_{d_1, d_2=0}^1 \frac{(-1)^{d_1} (-1)^{d_2}}{(p^*)^{\max\{d_1+d_3, d_2+d_4\}}}. \tag{3.78}$$

From Equation (3.77), we see  $d_3 > d_4$ , so that the inner sum over  $(d_1, d_2)$  is:

$$\sum_{d_1=0}^1 \frac{(-1)^{d_1}}{(p^*)^{d_1+d_3}} \sum_{d_2=0}^1 (-1)^{d_2} = 0, \tag{3.79}$$

so the whole sum vanishes in the case  $u_2k_1 \neq u_1k_2$ . We are left with the sum over the diagonal terms  $u_2k_1 = u_1k_2$  in Equation (3.74), which is:

$$\sum_{\substack{p|u_2k_1 \implies p \in (q_{i-1}, q_i] \\ \Omega_i(u_2), \Omega_i(k_1) \leq 10(q_i - q_{i-1})^{10^4}}} \frac{|\gamma(u_2k_1)|^2}{u_2k_1} \left| \sum_{p|f_1, f_2 \implies p \in (q_{i-1}, q_i]} \frac{\mu(f_1)\mu(f_2)}{[f_1, f_2]} \right|. \tag{3.80}$$

Since the functions are multiplicative, the factor coming from the sum over  $f_1$  and  $f_2$  is  $\prod_{p \in (q_{i-1}, q_i]} \left(1 - \frac{1}{p}\right)$ , which multiplies each  $u_2k_1$  term, completing the proof of Lemma 3.7.  $\square$

Combining Lemma 3.6 and Lemma 3.7, we see the contribution to the product in Equation (3.54) is:

$$N_i = \mathbb{E}_{\oplus} \left[ |F_i|^2 \right] \left( \prod_{p \in (q_{i-1}, q_i]} \left(1 - \frac{1}{p}\right) + O(e^{-100(n_i - n_{i-1})}) \right). \tag{3.81}$$

Using the Prime Number Theorem to estimate the product over  $p$ , this yields:

$$N_i = \mathbb{E}_{\oplus} \left[ |F_i|^2 \right] \prod_{p \in (q_{i-1}, q_i]} \left(1 - \frac{1}{p}\right) \left(1 + O(e^{-99(n_i - n_{i-1})})\right). \tag{3.82}$$

Substituting this value of  $N_i$  into Equation (3.53), we take the product to get

$$P_3 \ll \frac{\log q}{\log q_l} \prod_{i=1}^l \mathbb{E}_{\oplus} \left[ |F_i|^2 \right] = \frac{\log q}{\log q_l} \mathbb{E}_{\oplus} \left[ |F|^2 \right], \tag{3.83}$$

where we used Lemma 2.4 to multiply the expectations. This concludes the proof of Proposition 3.4.

### 3.2. Proof of Proposition 3.5

We want to write  $P_4(\beta)$  in terms of the sums of multiplicative functions, which can then be calculated accurately. As in the calculation of  $P_3$ , we first want to remove the restriction on the number of prime factors of  $f_1$  and  $f_2$ , using Rankin’s trick, and then evaluate the sum with these restrictions removed.

Given integers  $u_1, k_1, u_2, k_2$  with prime factors all at most  $q_l$  such that  $(u_1, k_1) = (u_2, k_2) = 1$ , define

$$H(\beta, u_1, k_1, u_2, k_2) = \sum_{p|f_1 f_2 \implies p \in (q_{m-1}, q_m]} \frac{e_{f_1} e_{f_2}}{[u_1 k_2 f_1, u_2 k_1 f_2]} \left( \frac{f_1 f_2 (u_1 k_2, u_2 k_1)^2}{(u_1 k_2 f_1, u_2 k_1 f_2)^2} \right)^{i\beta}, \tag{3.84}$$

so that

$$P_4(\beta) = \frac{1}{2\beta} \sum_{u_1, k_1, u_2, k_2} |\tilde{C}_{u_1, k_1} \tilde{C}_{u_2, k_2}| |H(\beta, u_1, k_1, u_2, k_2) - H(-\beta, u_1, k_1, u_2, k_2)|. \tag{3.85}$$

The expression for  $H(\beta, u_1, k_1, u_2, k_2)$  in Equation (3.84) splits into the products of contribution supported on integers with prime factors all in the intervals  $(q_m, q_{m-1}]$  for each  $1 \leq m \leq l$ . Indeed, we may express Equation (3.84) as:

$$H(\beta, u_1, k_1, u_2, k_2) = \prod_{m=1}^l R(m, \beta, u_1, k_1, u_2, k_2), \tag{3.86}$$

where  $R(m, \beta, u_1, k_1, u_2, k_2)$  is defined as:

$$\sum_{p|f_1 f_2 \implies p \in (q_{m-1}, q_m]} \frac{e_{f_1} e_{f_2}}{[u_1 k_2 f_1, u_2 k_1 f_2]} \left( \frac{f_1 f_2 (u_1 k_2, u_2 k_1)^2}{(u_1 k_2 f_1, u_2 k_1 f_2)^2} \right)^{i\beta}. \tag{3.87}$$

In Lemma 3.8, we show using Rankin’s trick that we may remove the restriction on the number of prime factors of coefficients of the mollifier  $M_1 \dots M_l$  in calculating  $\lim_{\beta \rightarrow 0^+} P_4(\beta)$ , with error bounded by  $O\left(\frac{\log q}{\log q_l} \mathbb{E}_\oplus[|F|^2]\right)$ . Note that this is of the same magnitude as the bound in Theorem 1.6 for the whole twisted moment  $\mathbb{E}_\oplus[|LM_1 \dots M_l F|^2]$ . We want to show  $P_4(\beta)$  may be approximated by  $\Upsilon(\beta)$ , where

$$\Upsilon(\beta) = \frac{1}{2\beta} \sum_{(u_1, k_1) = (u_2, k_2) = 1} |\tilde{C}_{u_1, k_1} \tilde{C}_{u_2, k_2}| |\tilde{H}(\beta, u_1, k_1, u_2, k_2) - \tilde{H}(-\beta, u_1, k_1, u_2, k_2)|, \tag{3.88}$$

and for integers  $u_1, k_1, u_2, k_2$  with prime factors all at most  $q_l$  such that  $(u_1, k_1) = (u_2, k_2) = 1$ ,  $\tilde{H}(\beta, u_1, k_1, u_2, k_2)$  being defined as:

$$\tilde{H}(\beta, u_1, k_1, u_2, k_2) = \sum_{p|f_1 f_2 \implies p \in (q_{m-1}, q_m]} \frac{\mu(f_1)\mu(f_2)}{[u_1 k_2 f_1, u_2 k_1 f_2]} \left( \frac{f_1 f_2 (u_1 k_2, u_2 k_1)^2}{(u_1 k_2 f_1, u_2 k_1 f_2)^2} \right)^{i\beta}. \tag{3.89}$$

In Lemma 3.11, we bound the sum without the restriction on the number of prime factors in the mollifier, and show this may also be bounded as  $O\left(\frac{\log q}{\log q_l} \mathbb{E}_\oplus[|F|^2]\right)$ . This will complete the proof of Lemma 3.5.

**Lemma 3.8.** *Let  $P_4(\beta)$  be defined as in Equation (3.51), and  $\Upsilon(\beta)$  be as defined in Equation (3.88), for  $F$  an  $l$ -sufficient function. Then*

$$P_4(\beta) = \Upsilon(\beta) + O\left(\frac{\log q}{\log q_l} \mathbb{E}_{\oplus} [|F|^2]\right) + o_{\beta \rightarrow 0^+, F}(1). \tag{3.90}$$

In order to prove Lemma 3.8, it is first necessary to understand the terms defining  $\Upsilon(\beta)$  in Equation (3.88) better, so that we may split it into the contribution from primes in different intervals  $(q_{j-1}, q_j]$ , for  $1 \leq j \leq l$ . This follows the proof of the estimate in Lemma 3.6, which allowed the restriction on the coefficients of  $\tilde{M}$  to be lifted in bounding  $P_3$ . Given positive integers  $c_1, c_2, f_1$  and  $f_2$  with all their prime factors at most  $q_l$ , let

$$U(c_1, c_2, f_1, f_2) = \frac{f_1 f_2 (c_1, c_2)^2}{(c_1 f_1, c_2 f_2)^2}. \tag{3.91}$$

Further define

$$\Theta_{\beta}(c_1, c_2) = \sum_{p|f_1, f_2 \implies p \leq q_l} \frac{\mu(f_1)\mu(f_2)}{[c_1 f_1, c_2 f_2]} \frac{U(c_1, c_2, f_1, f_2)^{i\beta} - U(c_1, c_2, f_1, f_2)^{-i\beta}}{2i\beta}. \tag{3.92}$$

Then we see that Equation (3.88) may be rewritten as:

$$\Upsilon(\beta) = \sum_{u_1, k_1, u_2, k_2} |\tilde{C}_{u_1, k_1} \tilde{C}_{u_2, k_2}| |\Theta_{\beta}(u_1 k_2, u_2 k_1)|. \tag{3.93}$$

We will show that most tuples  $(u_1, k_1, u_2, k_2)$  don't contribute to the summand in Equation (3.93). Indeed, we will show only the diagonal terms  $u_1 k_2 = u_2 k_1$ , and those tuples where the prime factorisations of  $u_1 k_2$  and  $u_2 k_1$  differ at a single prime, have non-zero contribution to Equation (3.93), in the limit as  $\beta \rightarrow 0^+$ . We will bound both of these contributions, and show that, summing over  $j$ , both sets of contributions can be bounded by the desired bound  $\ll \mathbb{E}_{\oplus} [|F|^2] \times \frac{\log q}{\log q_l}$ .

**Lemma 3.9.** *Let  $c_1$  and  $c_2$  be integers with all their prime factors at most  $q_l$ .*

- If  $c_1 = c_2$ , then

$$\Theta_{\beta}(c_1, c_1) = \frac{1}{c_1} \prod_{p \leq q_l} \left(1 - \frac{1}{p}\right) \sum_{p' \leq q_l} \left(\frac{2 \log p'}{p' - 1}\right) + o_{\beta \rightarrow 0^+, F}(1). \tag{3.94}$$

- If there exists a unique prime  $p^* \leq q_l$  such that  $v_p(c_1) \neq v_p(c_2) \iff p = p^*$ , then

$$\Theta_{\beta}(c_1, c_2) = \frac{(p^* - 1) \log p^*}{(p^*)^{\max\{v_{p^*}(c_1), v_{p^*}(c_2)\} + 1}} \prod_{\substack{p \leq q_l \\ p \neq p^*}} \frac{p - 1}{p^{\max\{v_p(c_1), v_p(c_2)\} + 1}} + o_{\beta \rightarrow 0^+, F}(1). \tag{3.95}$$

- If the prime factorisations of  $c_1$  and  $c_2$  differ in more than one prime, then

$$\lim_{\beta \rightarrow 0^+} \Theta_{\beta}(c_1, c_2) = 0. \tag{3.96}$$

**Proof.** Define

$$\phi_\beta(c_1, c_2) = \sum_{p|f_1, f_2 \implies p \leq q_l} \frac{\mu(f_1)\mu(f_2)}{[c_1 f_1, c_2 f_2]} U(c_1, c_2, f_1, f_2)^{i\beta}. \tag{3.97}$$

Then

$$\Theta_\beta(c_1, c_2) = \frac{\phi_\beta - \phi_{-\beta}}{2i\beta}, \tag{3.98}$$

and we can calculate  $\phi_\beta(c_1, c_2)$  as the sum of functions doubly multiplicative in  $f_1$  and  $f_2$ .

We first calculate the diagonal terms  $\Theta(c_1, c_1)$ . We have

$$\phi_\beta(c_1, c_1) = \frac{1}{c_1} \sum_{p|f_1, f_2 \implies p \leq q_l} \frac{\mu(f_1)\mu(f_2)}{[f_1, f_2]} \left( \frac{f_1 f_2}{(f_1, f_2)^2} \right)^{i\beta}. \tag{3.99}$$

On the diagonal terms,  $\phi_\beta(c_1, c_1)$  factors as an Euler product. If we write  $d_{p,i} = v_p(f_i)$ , then we obtain

$$\phi_\beta(c_1, c_1) = \frac{1}{c_1} \prod_{p \leq q_l} \sum_{d_{p,1}, d_{p,2} \in \{0,1\}} \frac{(-1)^{d_{p,1}+d_{p,2}}}{p^{\max\{d_{p,1}, d_{p,2}\}}} (p^{k_1+k_2-2 \min\{k_1, k_2\}})^{i\beta}. \tag{3.100}$$

We can write this as a product over primes in the interval,

$$\prod_{p \leq q_l} A_{p,\beta}, \quad A_{p,\beta} = 1 + \frac{1}{p} - \frac{2p^{i\beta}}{p}. \tag{3.101}$$

Hence,

$$\Theta_\beta(c_1, c_1) = \frac{1}{2i\beta} (\phi_\beta(c_1, c_1) - \phi_{-\beta}(c_1, c_1)) = \frac{\phi_{-\beta}(c_1, c_1)}{2i\beta} \left( \frac{\phi_\beta(c_1, c_1)}{\phi_{-\beta}(c_1, c_1)} - 1 \right). \tag{3.102}$$

Each term in the Euler product for  $\frac{\phi_\beta}{\phi_{-\beta}}$  is  $1 + O(\beta)$ , hence we may express Equation (3.102) as

$$\Theta_\beta(c_1, c_1) = (\phi_0(c_1, c_1) + O(\beta)) \sum_{p \leq q_l} \frac{1}{2i\beta} \left( \frac{A_{p,\beta}}{A_{p,-\beta}} - 1 \right). \tag{3.103}$$

But setting  $\beta = 0$  in Equation (3.101) shows that:

$$\phi_0(c_1, c_1) = \frac{1}{c_1} \prod_{p \leq q_l} \left( 1 - \frac{1}{p} \right). \tag{3.104}$$

For each prime  $p \leq q_l$ ,

$$A_{p,\beta} = \left(1 - \frac{1}{p}\right) \left(1 - \frac{2i\beta \log p}{p-1}\right) + O(\beta^2), \tag{3.105}$$

hence

$$\frac{A_{p,\beta}}{A_{p,-\beta}} = 1 - \frac{4i\beta \log p}{p-1} + O(\beta^2). \tag{3.106}$$

Hence, as  $\beta \rightarrow 0^+$  in Equation (3.103), and we take the sum over the contribution of all the primes in  $(q_{j-1}, q_j]$  to the sum on the right-hand side, we obtain:

$$\Theta_\beta(c_1, c_1) = \frac{1}{c_1} \prod_{p \leq q_l} \left(1 - \frac{1}{p}\right) \sum_{p' \leq q_l} \left(\frac{2 \log p'}{p' - 1}\right) + o_{\beta \rightarrow 0^+}(F(1)). \tag{3.107}$$

This proves Equation (3.94).

It remains to calculate the off-diagonal terms for  $\Theta_\beta(c_1, c_2)$ . The Euler product, instead of the diagonal term from Equation (3.100), becomes the more complicated formula:

$$\phi_\beta(c_1, c_2) = \prod_{p \in (q_{j-1}, q_j]} \xi_\beta(p, c_1, c_2), \tag{3.108}$$

where

$$\begin{aligned} \xi_\beta(p, c_1, c_2) = & \sum_{d_{p,1}, d_{p,2} \in \{0,1\}} \frac{(-1)^{d_{p,1} + d_{p,2}}}{p^{\max\{v_p(u_1 k_2) + d_{p,1}, v_p(u_2 k_1) + d_{p,2}\}}} \times \\ & \left(\frac{p^{d_{p,1} + d_{p,2} + 2 \min\{v_p(u_1 k_2), v_p(u_2 k_1)\}}}{p^{2 \min\{d_{p,1} + v_p(u_1 k_2), d_{p,2} + v_p(u_2 k_1)\}}}\right)^{i\beta}. \end{aligned} \tag{3.109}$$

This is different to most of the Euler products, since some of the terms may be  $O(\beta)$ , rather than  $1 + O(\beta)$ . We remark that this doesn't occur when splitting  $\phi_\beta(c_1, c_1)$  into the  $A_{p,\beta}$  in Equation (3.103), because the leading term for  $A_{p,\beta}$  is  $A_{p,0} = 1 - \frac{1}{p}$  which is non-zero, and hence  $\frac{A_{p,\beta}}{A_{p,-\beta}} = 1 + O(\beta)$ .

We now evaluate  $\xi_\beta(p, c_1, c_2)$ ; the limiting value will depend on whether or not  $v_p(c_1) = v_p(c_2)$ . If  $v_p(c_1) = v_p(c_2)$ , then

$$\xi_\beta(p, c_1, c_2) = \frac{1 - 2p^{i\beta-1} + p^{-1}}{p^{\max\{v_p(c_1), v_p(c_2)\}}}. \tag{3.110}$$

We expand this to first order terms in  $\beta$  to show:

$$\xi_\beta(p, c_1, c_2) = \frac{p-1}{p^{\max\{v_p(c_1), v_p(c_2)\}+1}} + O(\beta). \tag{3.111}$$

If  $v_p(c_1) \neq v_p(c_2)$ , then

$$\xi_\beta(p, c_1, c_2) = \frac{1 - p^{i\beta-1} - p^{-i\beta} + p^{-1}}{p^{\max\{v_p(c_1), v_p(c_2)\}}}. \tag{3.112}$$

We expand to terms of order  $\beta^2$  to show:

$$\xi_\beta(p, c_1, c_2) = \frac{i(p-1)\beta \log p}{p^{\max\{v_p(c_1), v_p(c_2)\}+1}} \times (1 + O(\beta)). \tag{3.113}$$

From Equation (3.113), we see that if  $v_p(c_1) \neq v_p(c_2)$  for more than one prime  $p \leq q_l$ , then substituting into Equation (3.108), we will have  $\phi_{\pm\beta} = O(\beta^2)$ , since more than one term in the product will be  $O(\beta)$ . Substituting the values for  $\phi_{\pm\beta}$  into Equation (3.98), we see that  $\Theta_\beta(c_1, c_2) = O(\beta)$ , meaning that the coefficient vanishes in the limit as  $\beta$  approaches zero. This proves Equation (3.96).

It remains to consider the case where the prime factorisations differ in exactly one prime. If we have exactly one prime  $p^* \leq q_l$  such that  $v_{p^*}(c_1) \neq v_{p^*}(c_2)$ , then Equation (3.113) gives expressions for  $\xi_\beta(p^*, c_1, c_2)$  and  $\xi_{-\beta}(p^*, c_1, c_2)$ , while Equation (3.111) gives expressions for  $\xi_\beta(p, c_1, c_2)$  and  $\xi_{-\beta}(p, c_1, c_2)$  for all other primes  $p \leq q_l$ . Substituting these expressions into Equation (3.98), we see:

$$\Theta_\beta(c_1, c_2) = \frac{2i(p^* - 1)\beta \log p^*}{(p^*)^{\max\{v_{p^*}(c_1), v_{p^*}(c_2)\}+1}} \prod_{\substack{p \leq q_l \\ p \neq p^*}} \frac{p-1}{p^{\max\{v_p(c_1), v_p(c_2)\}+1}} \times (1 + O(\beta)) \times \frac{1}{2i\beta}. \tag{3.114}$$

This completes the proof of Equation (3.95) in Lemma 3.9.  $\square$

**Proof of Lemma 3.8.** From Lemma 3.9, we have  $\lim_{\beta \rightarrow 0^+} \Theta_\beta(u_1 k_2, u_2 k_1)$  is a non-negative real number for all choices of  $u_1, k_1, u_2$  and  $k_1$ . Hence, we may remove the final modulus signs in Equation (3.93) to write

$$\mathcal{R}(\beta) = \sum_{u_1, k_1, u_2, k_2} |\tilde{C}_{u_1, k_1} \tilde{C}_{u_2, k_2}| \Theta_\beta(u_1 k_2, u_2 k_1) + o_{\beta \rightarrow 0^+} F(1). \tag{3.115}$$

We now split the right-hand side of Equation (3.115) into the contribution from primes in different intervals. Given  $1 \leq j \leq l$ , define

$$\psi(j, \beta) = \sum_{m_1, r_1, m_2, r_2} \left| C_{m_1, r_1, B^{(j)}, K_j} C_{m_2, r_2, B^{(j)}, K_j} \right| \phi_\beta(m_1 r_2, m_2 r_1). \tag{3.116}$$

We may decompose a tuple  $(u_1, k_1, u_2, k_2)$ , such that  $p|u_1 k_1 u_2 k_2 \implies p \in (q_{j-1}, q_j]$  into its factors from each interval. If

$$u_1, u_2, k_1, k_2 = \prod_{j=1}^l m_{j,1}, \prod_{j=1}^l m_{j,2}, \prod_{j=1}^l r_{j,1}, \prod_{j=1}^l r_{j,2}, \tag{3.117}$$

where  $p|m_{j,1} m_{j,2} r_{j,1} r_{j,2} \implies p \in (q_{j-1}, q_j]$ , then we may write Equation (3.115) as:

$$\Upsilon(\beta) = \prod_{j=1}^l \psi(j, \beta) + o_{\beta \rightarrow 0^+, F}(1). \tag{3.118}$$

For any  $1 \leq j \leq l$ , upon setting  $i = j$  we have  $\psi(j, 0)$  is the value of the expression in Equation (3.74), which is evaluated in Lemma 3.7 to be the non-zero quantity,  $\mathbb{E}_{\oplus} \left[ |F_j|^2 \right] \prod_{p \in (q_{j-1}, q_j]} \left( 1 - \frac{1}{p} \right)$ . Hence, using Equation (3.118), we may write

$$\Upsilon(\beta) = \sum_{j=1}^l \left( \frac{\psi(j, \beta) - \psi(j, -\beta)}{2i\beta} \right) \prod_{\lambda=1, \lambda \neq j}^l \mathbb{E}_{\oplus} \left[ |F_{\lambda}|^2 \right] \prod_{p \in (q_{\lambda-1}, q_{\lambda}]} \left( 1 - \frac{1}{p} \right) + o_{\beta \rightarrow 0^+, F}(1). \tag{3.119}$$

We must bound the contribution to Equation (3.119) of the extra terms in  $\tilde{M}$  both with  $\max\{\Omega_j(f_1), \Omega_j(f_2)\} > 10(n_j - n_{j-1})^{10^5}$  for each  $j$ , and the terms with  $\max\{\Omega_{\lambda}(f_1), \Omega_{\lambda}(f_2)\} > 10(n_{\lambda} - n_{\lambda-1})^{10^5}$  for each  $\lambda \neq j$ . In order to bound such terms, we first need to understand the expression related to numbers with prime factors all lying in the interval  $(q_{j-1}, q_j]$ .

**Proposition 3.10.** *Let  $1 \leq j \leq l$  and  $\psi(j, \beta)$  be as above. Then*

$$\left( \frac{\psi(j, \beta) - \psi(j, -\beta)}{2i\beta\psi(j, \beta)} \right) \ll (\log q_j - \log q_{j-1}) \mathbb{E}_{\oplus} \left[ |F_j|^2 \right] \prod_{p \in (q_{j-1}, q_j]} \left( 1 - \frac{1}{p} \right) + o_{\beta \rightarrow 0^+, F}(1). \tag{3.120}$$

**Proof.** We define a restriction of the function  $\Theta_j$  defined in Equation (3.92) to numbers with all their prime factors lying in the interval  $(q_{j-1}, q_j]$ . Given positive integers  $c_1$  and  $c_2$  with all their prime factors lying in the interval  $(q_{j-1}, q_j]$ , define

$$\Theta_{j,\beta}(c_1, c_2) = \sum_{p|f_1, f_2 \implies p \in (q_{j-1}, q_j]} \frac{\mu(f_1)\mu(f_2)}{[c_1 f_1, c_2 f_2]} \frac{U(c_1, c_2, f_1, f_2)^{i\beta} - U(c_1, c_2, f_1, f_2)^{-i\beta}}{2i\beta}, \tag{3.121}$$

where  $U$  is the function defined in Equation (3.91). Then we see that the left-hand side of Equation (3.120) is

$$\sum_{m_{j,1}, r_{j,1}, m_{j,2}, r_{j,2}} \left| C_{m_{j,1}, r_{j,1}, B^{(j)}, K_j} C_{m_{j,2}, r_{j,2}, B^{(j)}, K_j} \right| |\Theta_{j,\beta}(m_{j,1}r_{j,2}, m_{j,2}r_{j,2})|. \tag{3.122}$$

Following the proof of Lemma 3.9, restricted to primes in the interval  $(q_{j-1}, q_j]$ , we see that most tuples  $(m_{j,1}, r_{j,1}, m_{j,2}, r_{j,2})$  don't contribute to the summand in Equation (3.93). We have the same result with the condition  $p \leq q_l$  replaced by  $p \in (q_{j-1}, q_j]$ .

We use this in Equation (3.122), and notice that on the support of the coefficients  $C_{m_{j,1}, r_{j,1}, B^{(j)}, K_j} C_{m_{j,2}, r_{j,2}, B^{(j)}, K_j}$ , we have  $m_{j,1}r_{j,2} = m_{j,2}r_{j,1} \iff m_{j,1} = m_{j,2}, r_{j,1} = r_{j,2}$ . Hence the above equals

$$\prod_{p \in (q_{j-1}, q_j]} \left(1 - \frac{1}{p}\right) \left( \sum_{\substack{m_{j,1}, r_{j,1} \\ v_p(m_{j,1}r_{j,2}) \neq v_p(m_{j,2}r_{j,1}) \iff p=p^*}} \frac{|C_{m_{j,1}, r_{j,1}, B^{(j)}, K_j}|^2}{m_{j,1}r_{j,1}} + \sum_{\substack{m_{j,1}, r_{j,1}, B^{(j)}, K_j \\ m_{j,2}, r_{j,2}, B^{(j)}, K_j \\ v_p(m_{j,1}r_{j,2}) \neq v_p(m_{j,2}r_{j,1}) \iff p=p^*}} \frac{|C_{m_{j,1}, r_{j,1}, B^{(j)}, K_j} C_{m_{j,2}, r_{j,2}, B^{(j)}, K_j}|}{[m_{j,1}r_{j,2}, m_{j,2}r_{j,1}]} \right). \tag{3.123}$$

The diagonal term corresponding to the first sum in Equation (3.122) may be handled directly using Lemma 2.3, to show it contributes:  $\prod_{p \in (q_{j-1}, q_j]} \left(1 - \frac{1}{p}\right) \mathbb{E}_{\oplus} [ |F_j|^2 ]$ . It remains to show the off-diagonal terms from the second sum can be bounded above by:

$$(\log q_j - \log q_{j-1}) \prod_{p \in (q_{j-1}, q_j]} \left(1 - \frac{1}{p}\right) \mathbb{E}_{\oplus} [ |F_j|^2 ] .$$

The contribution of the second sum to Equation (3.122) may be bounded above by:

$$\prod_{p \in (q_{j-1}, q_j]} \left(1 - \frac{1}{p}\right) \left( \sum_{p^* \in (q_{j-1}, q_j]} \log p^* \sum_{\substack{m_{j,1}, r_{j,1}, B^{(j)}, K_j \\ m_{j,2}, r_{j,2}, B^{(j)}, K_j \\ v_p(m_{j,1}r_{j,2}) \neq v_p(m_{j,2}r_{j,1}) \iff p=p^*}} \frac{|C_{m_{j,1}, r_{j,1}, B^{(j)}, K_j} C_{m_{j,2}, r_{j,2}, B^{(j)}, K_j}|}{[m_{j,1}r_{j,2}, m_{j,2}r_{j,1}]} \right). \tag{3.124}$$

Applying Cauchy-Schwarz and using the support of the coefficients  $C_{m_j, r_j, B^{(j)}, K_j}$  shows that Equation (3.124) may be bounded above by:

$$\prod_{p \in (q_{j-1}, q_j]} \left(1 - \frac{1}{p}\right) \left( \sum_{\substack{p | m_{j,1}r_{j,1} \implies p \in (q_{j-1}, q_j] \\ (m_{j,1}, r_{j,1})=1}} |C_{m_{j,1}, r_{j,1}, B^{(j)}, K_j}|^2 \sum_{\substack{p^* \in (q_{j-1}, q_j] \\ (m_{j,2}, r_{j,2})=1 \\ v_p(m_{j,1}r_{j,2}) \neq v_p(m_{j,2}r_{j,1}) \iff p=p^*}} \log p^* \frac{1}{[m_{j,1}r_{j,2}, m_{j,2}r_{j,1}]} \right). \tag{3.125}$$

Considering a fixed prime  $p^*$ , the values of coprime pairs of integers  $m_{j,1}, r_{j,1}$  and  $m_{j,2}, r_{j,2}$  with  $v_p(m_{j,1}r_{j,2}) \neq v_p(m_{j,2}r_{j,1}) \iff p = p^*$  may be expressed as:

$$m_{j,1} = m'(p^*)^{d_1}, \quad r_{j,1} = r'(p^*)^{d_2}, \quad m_{j,2} = m'(p^*)^{d_3}, \quad r_{j,2} = r'(p^*)^{d_4}, \tag{3.126}$$

where  $p^* \nmid m'r'$  and  $d_1, d_2, d_3, d_4$  are non-negative integers. The conditions:

$$(m_{j,1}, r_{j,1}) = (m_{j,2}, r_{j,2}) = 1 \text{ and } v_p(m_{j,1}r_{j,2}) \neq v_p(m_{j,2}r_{j,1}) \iff p = p^* \text{ mean}$$

$$\min\{d_1, d_2\} = \min\{d_3, d_4\} = 0, \tag{3.127}$$

and

$$d_1 + d_4 = d_3 + d_2 + n, \tag{3.128}$$

where  $n$  is a non-zero integer. Hence, using this restriction, the inner sum over  $m_{j,2}$  and  $r_{j,2}$  in Equation (3.125) may be expressed as

$$\sum_{n \neq 0} \frac{\log p^*}{m_{j,1}r_{j,1}} (p^*)^{d_1+d_2-\max\{d_1+d_4, d_2+d_3\}}. \tag{3.129}$$

We handle the terms from positive  $n$  and negative  $n$  separately. We first consider positive  $n$ ; the case of negative  $n$  follows by symmetry.

For positive  $n$ , we have

$$d_1 + d_2 - \max\{d_1 + d_4, d_2 + d_3\} = d_2 - d_4 - n, \tag{3.130}$$

so that the contribution to the inner sum in Equation (3.125) is a geometric series. In the case  $d_2 > 0$  we have  $d_1 = 0$  by Equation (3.127), so that Equation (3.128) yields that

$$d_4 \geq d_2. \tag{3.131}$$

There are relatively few choices of  $p^*$  for which  $d_2 \neq 0$  since  $r_{j,1}$  is constrained to have few prime factors. Note that for fixed  $n, d_1, d_2$  and  $d_4, d_3$  is determined uniquely by Equation (3.128). The two cases to consider are when  $d_2 = 0$  and  $d_2 > 0$ . In the case  $d_2 = 0$ , we see the contribution to the sum in (3.129) of positive  $n$  is hence

$$\ll \sum_{p^* \in (q_{j-1}, q_j]} \log p^* \sum_{n>0} \frac{1}{m_{j,1}r_{j,1}(p^*)^n} \ll \frac{1}{m_{j,1}r_{j,1}} \sum_{p^* \in (q_{j-1}, q_j]} \frac{\log p^*}{p^*}. \tag{3.132}$$

Using the Prime Number Theorem, the bound above is

$$\ll \frac{\log q_j - \log q_{j-1}}{m_{j,1}r_{j,1}}. \tag{3.133}$$

For the case  $d_2 > 0$ , there are at most  $10(n_j - n_{j-1})^{10^5}$  choices of  $(p^*)$  for which  $d_2 > 0$ . By Equation (3.131), they each have contribution

$$\ll \sum_{n>0} \frac{\log p^*}{m_{j,1} r_{j,1} (p^*)^n} \ll \frac{\log q_{j-1}}{m_{j,1} r_{j,1} q_{j-1}}, \tag{3.134}$$

so that the total contribution to the sum in (3.129) is

$$\ll \frac{(n_j - n_{j-1})^{10^5} \log q_{j-1}}{m_{j,1} r_{j,1} q_{j-1}} \ll \frac{1}{m_{j,1} r_{j,1}}. \tag{3.135}$$

Combining the bounds from (3.133) and (3.135) we see the total contribution to (3.129) for positive  $n$  is

$$\ll \frac{\log q_j - \log q_{j-1}}{m_{j,1} r_{j,1}}. \tag{3.136}$$

By symmetry, the contribution to the sum in (3.129) for negative  $n$  is also

$$\ll \frac{\log q_j - \log q_{j-1}}{m_{j,1} r_{j,1}}. \tag{3.137}$$

Substituting this bound into Equation (3.125), we have that the left-hand side of Equation (3.120) is

$$\ll (\log q_j - \log q_{j-1}) \prod_{p \in (q_{j-1}, q_j]} \left(1 - \frac{1}{p}\right) \left( \sum_{(m_{j,1}, r_{j,1})=1} \frac{|C_{m_{j,1}, r_{j,1}, B^{(j)}, K_j}|^2}{m_{j,1} r_{j,1}} \right). \tag{3.138}$$

Using Lemma 2.3 to handle the sum shows that the above expression is

$$(\log q_j - \log q_{j-1}) \prod_{p \in (q_{j-1}, q_j]} \left(1 - \frac{1}{p}\right) \mathbb{E}_{\oplus} [F_j^2]. \tag{3.139}$$

This completes the proof of Proposition 3.10.  $\square$

We are now in the position to show that we may lift the restriction on the prime factors of coefficients of  $\tilde{M}$  in calculating  $\lim_{\beta \rightarrow 0^+} P_4(\beta)$ . By Rankin’s trick, the contribution to the term  $\left(\frac{\psi(j, \beta) - \psi(j, -\beta)}{2i\beta}\right)$  in the summand in (3.119) of those  $f_1$  with  $\Omega_j(f_1) > 10(n_j - n_{j-1})^{10^5}$  for some  $1 \leq j \leq l$  may be bounded for any  $\rho > 0$  by:

$$e^{-10\rho(n_j - n_{j-1})^{10^5}} \sum_{m_1, r_1, m_2, r_2} \left| C_{m_1, r_1, B^{(j)}, K_j} C_{m_2, r_2, B^{(j)}, K_j} \right| \times \tag{3.140}$$

$$\sum_{p|f_1 f_2 \implies p \in (q_{j-1}, q_j]} e^{\rho \Omega_j(f_1)} \frac{\mu^2(f_1) \mu^2(f_2)}{[m_1 r_2 f_1, m_2 r_1 f_2]} \left| \log \left( \frac{f_1 f_2 (m_1 r_2, m_2 r_1)^2}{(m_1 r_2 f_1, m_2 r_1 f_2)^2} \right) \right| +$$

$$o_{\beta \rightarrow 0^+}, F(1).$$

The restriction on the number of prime factors in the interval  $(q_{j-1}, q_j]$  in the internal sum over  $f_1$  and  $f_2$  in (3.140) is inherited from the support of  $C_{m,r,B^{(j)},K_j}$ . But since

$$1 \leq \frac{f_1 f_2 (m_1 r_2, m_2 r_1)^2}{(m_1 r_2 f_1, m_2 r_1 f_2)^2} \leq f_1 f_2, \tag{3.141}$$

we see:

$$\left| \log \left( \frac{f_1 f_2 (m_1 r_2, m_2 r_1)^2}{(m_1 r_2 f_1, m_2 r_1 f_2)^2} \right) \right| \leq \log(f_1 f_2). \tag{3.142}$$

Hence, the expression in (3.140) may be bounded by:

$$e^{-10\rho(n_j - n_{j-1})^{10^5}} \sum_{m_1, r_1, m_2, r_2} \left| C_{m_1, r_1, B^{(j)}, K_j} C_{m_2, r_2, B^{(j)}, K_j} \right| \times \tag{3.143}$$

$$\sum_{p|f_1 f_2 \Rightarrow p \in (q_{j-1}, q_j]} e^{\rho\Omega_j(f_1)} \frac{\mu^2(f_1)\mu^2(f_2)}{[m_1 r_2 f_1, m_2 r_1 f_2]} \log(f_1 f_2) + o_{\beta \rightarrow 0^+}, F(1).$$

For each tuple  $(m_1, r_1, m_2, r_2)$ , define

$$D_j(m_1, r_1, m_2, r_2) = \sum_{p|f_1 f_2 \Rightarrow p \in (q_{j-1}, q_j]} e^{\rho\Omega_j(f_1)} \frac{\mu^2(f_1)\mu^2(f_2)}{[m_1 r_2 f_1, m_2 r_1 f_2]} \log(f_1 f_2) \tag{3.144}$$

to be the coefficient multiplying  $\left| C_{u_1, k_1, K^{(j)}, B^{(j)}} C_{u_2, k_2, K^{(j)}, B^{(j)}} \right|$  in the internal sum in (3.143), arising from the inner sum over values of  $f_1$  and  $f_2$ .

The term  $\log(f_1 f_2)$  in Equation (3.144) may be written by factorising  $f_1 f_2$  into its prime factors, so that the sum can be rewritten as a sum over logarithms of primes in the interval. Indeed, we have

$$\log(f_1 f_2) = \sum_{p|f_1 f_2} (\log p) v_p(f_1 f_2). \tag{3.145}$$

If we write  $d_{p,i} = v_p(f_i)$ , and observe that the exponents can be at most 1 to contribute to the sum, we see:

$$D_j(m_1, r_1, m_2, r_2) \ll \sum_{p \in (q_{j-1}, q_j]} \log p \sum_{d_{p,1}, d_{p,2} \in \{0,1\}} e^{\rho\Omega_j(p^{d_{p,1}})} \times$$

$$\frac{1}{p^{\max\{d_{p,1} + v_p(m_1 r_2), d_{p,2} + v_p(m_2 r_1)\}}} \times \prod_{\substack{t \leq q_i \\ t \neq p}} h_t, \tag{3.146}$$

where for each prime  $t \in (q_{j-1}, q_j]$ , we define:

$$h_t = \sum_{d_{t,1}, d_{t,2} \in \{0,1\}} \frac{e^{\rho\Omega_j(t^{d_{t,1}+v_t(m_1r_2), d_{t,2}+v_t(m_2r_1)})}}{t^{\max\{d_{t,1}+v_t(m_1r_2), d_{t,2}+v_t(m_2r_1)\}}} \tag{3.147}$$

The sum over  $d_{p,1}$  and  $d_{p,2}$  is at most:

$$\frac{1}{p^{\max\{v_p(m_1r_2), v_p(m_2r_1)\}}} \times \begin{cases} \frac{100e^\rho \log p}{p} & \text{if } v_p(u_1k_2) = v_p(u_2k_1) \\ 100e^\rho \log p & \text{otherwise} \end{cases} \tag{3.148}$$

The sum over  $d_{t,1}$  and  $d_{t,2}$  for  $t$  a prime in the interval not equal to  $p$  in Equation (3.147), which we have denoted by  $h_t$ , is at most:

$$\frac{1}{t^{\max\{v_t(m_1r_2), v_t(m_2r_1)\}}} \times \begin{cases} 1 + \frac{100e^\rho}{t} & \text{if } v_t(m_1r_2) = v_t(m_2r_1) \\ 100e^\rho & \text{otherwise} \end{cases} \tag{3.149}$$

Hence, substituting the bounds from (3.148) and (3.149) into (3.146), we may bound  $D_j(m_1, r_1, m_2, r_2)$  by:

$$D_j(m_1, r_1, m_2, r_2) \leq \frac{1}{[m_1r_2, m_2r_1]} \times \sum_p \begin{cases} \frac{100e^\rho \log p}{p} & \text{if } v_p(m_1r_2) = v_p(m_2r_1) \\ 100e^\rho \log p & \text{otherwise} \end{cases} \tag{3.150}$$

$$\times \prod_{t \neq p} \begin{cases} 1 + \frac{100e^\rho}{t} & \text{if } v_t(m_1r_2) = v_t(m_2r_1) \\ 100e^\rho & \text{otherwise} \end{cases}.$$

We remove the restriction on  $t \neq p$  to obtain that:

$$D_j(m_1, r_1, m_2, r_2) \leq \frac{1}{[m_1r_2, m_2r_1]} \times \sum_p \begin{cases} \frac{100e^{\rho\Omega_j(p)} \log p}{p} & \text{if } v_p(m_1r_2) = v_p(m_2r_1) \\ 100e^{\rho\Omega_j(p)} \log p & \text{otherwise} \end{cases} \tag{3.151}$$

$$\times \prod_t \begin{cases} 1 + \frac{100e^{\rho\Omega_j(t)}}{t} & \text{if } v_t(m_1r_2) = v_t(m_2r_1) \\ 100e^{\rho\Omega_j(t)} & \text{otherwise} \end{cases}.$$

Observe that the above expression factors into a sum over primes  $p$  in the interval, and a product over primes  $t$ . We bound the contribution of both the sum over  $p$  and the product over  $t$ . The contribution of those  $p$  with  $v_p(m_1r_2) = v_p(m_2r_1)$  to the sum in Equation (3.151) may be bounded as:

$$\leq \sum_{p \in (q_{j-1}, q_j]} \frac{100e^\rho \log p}{p} \ll e^\rho e^{(n_j - n_{j-1})}, \tag{3.152}$$

where we used the Prime Number Theorem to get the final bound.

Due to the restriction on the number of prime factors of  $m_1, m_2, r_1$  and  $r_2$ , there are at most  $40(n_j - n_{j-1})^{10^5}$  primes  $p$  with  $v_p(m_1r_2) \neq v_p(m_2r_1)$  in the interval  $(q_{j-1}, q_j]$ . Hence, the contribution of those  $p$  with  $v_p(u_1k_2) \neq v_p(u_2k_1)$ , of which to the sum in Equation (3.151) is

$$\ll (n_j - n_{j-1})^{10^5} (\log q_j) e^{\rho}. \tag{3.153}$$

We now turn to bound the product over  $t$  in Equation (3.151). The contribution of those  $t$  for which  $v_t(m_1r_2) = v_t(m_2r_1) = 0$  to the product in Equation (3.151) is

$$\ll \prod_{t \in (q_{j-1}, q_j]} 1 + \frac{100e^{\rho}}{t} \ll e^{100e^{\rho}(n_j - n_{j-1})}. \tag{3.154}$$

Hence, we may bound the product over  $t$  in Equation (3.151) as

$$\ll e^{100e^{\rho}(n_j - n_{j-1})} \frac{\alpha(m_1r_2, m_2r_1)}{[m_1r_2, m_2r_1]}, \tag{3.155}$$

where  $\alpha$  is the multiplicative function defined as in Equation (3.63). Combining the bound in Equation (3.153) for the sum over  $p$  with the bound in Equation (3.155) for the product over  $t$  in Equation (3.151), we obtain:

$$D_j(m_1, r_1, m_2, r_2) \ll (n_j - n_{j-1})^{10^5} (\log q_j) e^{100e^{\rho}(n_j - n_{j-1})} \frac{\alpha(m_1r_2, m_2r_1)}{[m_1r_2, m_2r_1]}. \tag{3.156}$$

Substituting this bound for the internal sum over  $f_1$  into (3.140), we see the contribution of the extra values of  $f_1$  and  $f_2$  to the term  $\left(\frac{\psi(j, \beta) - \psi(j, -\beta)}{2i\beta}\right)$  is at most:

$$e^{-10\rho(n_j - n_{j-1})^{10^5}} (n_j - n_{j-1})^{10^5} (\log q_j) e^{100e^{\rho}(n_j - n_{j-1})} \times \sum_{m_1, r_1, m_2, r_2} \left| C_{m_1, r_1, K^{(j)}, B^{(j)}} C_{m_2, r_2, K^{(j)}, B^{(j)}} \right| \frac{\alpha(m_1r_2, m_2r_1)}{[m_1r_2, m_2r_1]} + o_{\beta \rightarrow 0^+}, F(1), \tag{3.157}$$

proceed as in the proof of Lemma 3.6. The expression in Equation (3.157) may be bounded by:

$$e^{-10\rho(n_j - n_{j-1})^{10^5} + 100e^{\rho}(n_j - n_{j-1})^{10^5}} (n_j - n_{j-1})^{10^5} (\log q_j) \times \sum_{p|m_2r_2 \implies p \in (q_{j-1}, q_j]} \left| C_{m_2, r_2, K^{(j)}, B^{(j)}} \right|^2 \sum_{\substack{p|m_1r_1 \implies p \in (q_{j-1}, q_j] \\ (m_1, r_1) = (m_2, r_2) = 1}} \frac{\alpha(m_1r_2, m_2r_1)}{[m_1r_2, m_2r_1]} + o_{\beta \rightarrow 0^+}, F(1). \tag{3.158}$$

Using Equations (3.70) and (3.71) to bound the internal sum over  $m_1$  and  $r_1$ , we may bound the expression in (3.158) as

$$\ll e^{-10\rho(n_j-n_{j-1})^{10^5} + 400e^\rho(n_i-n_{i-1}) + 2 \cdot 10^4 \rho(n_j-n_{j-1})^{10^4+1}} (n_j-n_{j-1})^{10^5} (\log q_j) e^{100e^\rho(n_j-n_{j-1})} \tag{3.159}$$

$$\sum_{p|m_2r_2 \implies p \in (q_{j-1}, q_j]} \frac{|C_{m_2, r_2, K^{(j)}, B^{(j)}}|^2}{m_2 r_2} + o_{\beta \rightarrow 0^+, F}(1).$$

Observing that the sum over  $m_2, r_2$  is  $\mathbb{E}_\oplus [ |F_j|^2 ]^2$ , we may bound (3.159) as

$$\ll \log(q) e^{-10\rho(n_j-n_{j-1})^{10^5} + 600e^\rho(n_i-n_{i-1}) + 2 \cdot 10^4 \rho(n_j-n_{j-1})^{10^4+1}} \mathbb{E}_\oplus [ |F_j|^2 ] + o_{\beta \rightarrow 0^+, F}(1). \tag{3.160}$$

Taking  $\rho = 1000$ , the above is  $\ll \log(q) e^{-100(n_j-n_{j-1})} \mathbb{E}_\oplus [ |F_j|^2 ] + o_{\beta \rightarrow 0^+, F}(1)$ . By symmetry, the contribution of those values of  $f_2$  with too many prime factors to the term  $\left( \frac{\psi(j, \beta) - \psi(j, -\beta)}{2i\beta} \right)$  is also  $\ll \log(q) e^{-100(n_j-n_{j-1})} \mathbb{E}_\oplus [ |F_j|^2 ] + o_{\beta \rightarrow 0^+, F}(1)$ .

Substituting this bound into Equation (3.119), the contribution of the extra terms  $f_1$  and  $f_2$  to  $\left( \frac{\psi(j, \beta) - \psi(j, -\beta)}{2i\beta} \right)$ , summed over all  $1 \leq j \leq l$ , is

$$\begin{aligned} &\ll \sum_{j=1}^l \log(q_j) e^{-100(n_j-n_{j-1})} \mathbb{E}_\oplus [ |F_j|^2 ]. \\ &\prod_{\lambda=1, \lambda \neq j}^l \mathbb{E}_\oplus [ |F_\lambda|^2 ] \prod_{p \in (q_{\lambda-1}, q_\lambda]} \left( 1 - \frac{1}{p} \right) + o_{\beta \rightarrow 0^+, F}(1). \end{aligned} \tag{3.161}$$

Using Lemma 2.4, the expression in (3.161) is  $\ll \mathbb{E}_\oplus [ F^2 ]$ .

It remains to show the contribution of the values of  $f_1$  and  $f_2$  with

$\max\{\Omega_\lambda(f_1), \Omega_\lambda(f_2)\} > 10(n_\lambda - n_{\lambda-1})^{10^5}$  to the expression in Equation (3.119) for  $\lambda \neq j$  can also be bounded by  $\mathbb{E}_\oplus [ F^2 ]$ . By Lemma 3.6, we see that the contribution of the extra terms  $f_1$  and  $f_2$  to  $\mathbb{E} [ F_\lambda^2 ]$  is  $O \left( e^{-100(n_\lambda - n_{\lambda-1})} \mathbb{E}_\oplus [ |F_\lambda|^2 ] \right)$ . By Proposition 3.10, the term associated to the interval  $(q_{j-1}, q_j]$  in Equation (3.119) is

$$\left( \frac{\psi(j, \beta) - \psi(j, -\beta)}{2i\beta\psi(j, \beta)} \right) \ll (\log q_j - \log q_{j-1}) \mathbb{E}_\oplus [ |F_j|^2 ] \prod_{p \in (q_{j-1}, q_j]} \left( 1 - \frac{1}{p} \right) + o_{\beta \rightarrow 0^+, F}(1). \tag{3.162}$$

Hence, the contribution of all the extra terms in the intervals with  $\lambda \neq j$  to (3.119) is

$$\ll \sum_{j=1}^l (\log q_j - \log q_{j-1}) \mathbb{E}_{\oplus} \left[ |F_j|^2 \right] \prod_{p \in (q_{j-1}, q_j]} \left( 1 - \frac{1}{p} \right) \sum_{\lambda=1, \lambda \neq j}^l e^{-100(n_{\lambda} - n_{\lambda-1})} \mathbb{E}_{\oplus} \left[ |F_{\lambda}|^2 \right] \tag{3.163}$$

$$\prod_{p \in (q_{\lambda-1}, q_{\lambda}]} \left( 1 - \frac{1}{p} \right) \prod_{\mu=1, \mu \neq \lambda, j}^l \mathbb{E}_{\oplus} \left[ |F_{\mu}|^2 \right] \prod_{p \in (q_{\mu-1}, q_{\mu}]} \left( 1 - \frac{1}{p} \right) + o_{\beta \rightarrow 0^+}, F(1).$$

An application of the Prime Number Theorem and Lemma 2.4 shows the above bound is

$$\ll \frac{\mathbb{E}_{\oplus} [F^2]}{\log q_l} \sum_{j=1}^l (\log q_j - \log q_{j-1}) \sum_{\lambda=1, \lambda \neq j}^l e^{-100(n_{\lambda} - n_{\lambda-1})} + o_{\beta \rightarrow 0^+}, F(1). \tag{3.164}$$

The sum over  $\lambda \neq j$  is bounded uniformly for each  $1 \leq j \leq l$ , hence the contribution of the additional primes to the terms  $\lambda \neq j$  is  $\ll \mathbb{E}_{\oplus} [F^2]$ . This completes the proof of Lemma 3.8.  $\square$

We now bound  $\Upsilon(\beta)$ .

**Lemma 3.11.** *Let  $\Upsilon(\beta)$  be as defined as in Equation (3.88). Then*

$$\Upsilon(\beta) \ll \frac{\log q}{\log q_l} \mathbb{E}_{\oplus} \left[ |F|^2 \right] + o_{\beta \rightarrow 0^+}, F(1). \tag{3.165}$$

**Proof.** Recall from Equation (3.119):

$$\Upsilon(\beta) = \sum_{j=1}^l \left( \frac{\psi(j, \beta) - \psi(j, -\beta)}{2i\beta} \right) \prod_{\lambda=1, \lambda \neq j}^l \mathbb{E}_{\oplus} \left[ |F_{\lambda}|^2 \right] \prod_{p \in (q_{\lambda-1}, q_{\lambda}]} \left( 1 - \frac{1}{p} \right) + o_{\beta \rightarrow 0^+}, F(1). \tag{3.166}$$

By Proposition 3.10, we may rewrite Equation (3.166) as:

$$\begin{aligned} \Upsilon(\beta) &\ll \sum_{j=1}^l (\log q_j - \log q_{j-1}) \mathbb{E}_{\oplus} \left[ |F_j|^2 \right] \prod_{p \in (q_{j-1}, q_j]} \prod_{\lambda=1, \lambda \neq j}^l \mathbb{E}_{\oplus} \left[ |F_{\lambda}|^2 \right] \prod_{p \in (q_{\lambda-1}, q_{\lambda}]} \left( 1 - \frac{1}{p} \right) \\ &+ o_{\beta \rightarrow 0^+}, F(1). \end{aligned} \tag{3.167}$$

Using Lemma 2.4 and the Prime Number Theorem yields

$$\begin{aligned} \Upsilon(\beta) &\ll \mathbb{E}_{\oplus} [F^2] \sum_{j=1}^l (\log q_j - \log q_{j-1}) \prod_{p \leq q_l} \left( 1 - \frac{1}{p} \right) + o_{\beta \rightarrow 0^+}, F(1) \\ &\ll \mathbb{E}_{\oplus} [F^2] + o_{\beta \rightarrow 0^+}, F(1). \quad \square \end{aligned} \tag{3.168}$$

This completes the proof of Lemma 3.11. Using Lemma 3.8 and Lemma 3.11 we conclude the proof of Proposition 3.5, and hence the proof of Theorem 1.6.

**Proof of Theorem 1.5.** This essentially follows from writing  $|Q_j|^2 = \Re(Q_j^2) + \Re(iQ_j)^2$ . Taking  $K_j(x) = x$  to be the identity polynomial and for each subset  $S \subset \{1, \dots, l\}$  taking  $F_j$  to be  $Q_j i^{1(j \in S)}$  in Theorem 1.6, we obtain

$$\sum_{S \subset \{1, \dots, l\}} \mathbb{E}_{\oplus} \left[ |LM|^2 \prod_{j=1}^l \Re(Q_j i^{1(j \in S)})^2 \right] \ll \frac{\log q}{\log ql} \sum_{S \subset \{1, \dots, l\}} \mathbb{E}_{\oplus} \left[ \prod_{j=1}^l \Re(Q_j i^{1(j \in S)})^2 \right], \tag{3.169}$$

or

$$\mathbb{E}_{\oplus} \left[ |LMQ|^2 \right] \ll \frac{\log q}{\log ql} \mathbb{E}_{\oplus} \left[ |Q|^2 \right]. \quad \square \tag{3.170}$$

#### 4. Large deviations

In this section, we prove Theorems 1.1 and 1.4. In order to prove the large deviations results, we deploy a barrier method to show the random walk model for  $\log \left( L \left( \mathfrak{X}, \frac{1}{2} \right) \right)$  stays within a controllable region for most characters,  $\mathfrak{X}$ .

##### 4.1. Method of proof

The orthogonality relations in the previous section are very similar to the orthogonality relations used in the  $t$ -aspect in [AB23], and Theorem 1.5 provides the analogue for the twisted second moment of zeta. Using these points of reference, the method of proof for the large deviations in the  $q$ -aspect uses a recursive scheme in the same spirit as [AB23].

We consider the event

$$H = \left\{ \log \left( L \left( \mathfrak{X}, \frac{1}{2} \right) > V \right) \right\}, \tag{4.1}$$

where  $\mathfrak{X}$  is drawn uniformly at random from the even primitive characters with modulus  $q$ . Since the number of even, primitive characters is asymptotic to  $\frac{\varphi(q)}{2}$ , we have

$$\frac{1}{\varphi(q)} \#\left\{ \chi \text{ even, primitive mod } q : \left| L \left( \chi, \frac{1}{2} \right) \right| > V \right\} \asymp \mathbf{P}(H). \tag{4.2}$$

We partition the event  $H$  on *good events* pertaining to the partial sums  $S_{n_l}$  defined in Equation (1.23) (and Equation (1.22) for the complex ones). More precisely, we consider the restrictions  $\{S_{n_l} \in [L_l, U_l]\}$  where

$$L_l = \kappa n_l - \mathfrak{s}(\log_{l+2} q), \quad U_l = \kappa n_l + \mathfrak{s}(\log_{l+2} q), \tag{4.3}$$

and  $\kappa$  is the *slope*

$$\kappa = \frac{V}{\log \log q}. \tag{4.4}$$

We recall that  $V \sim \alpha \log \log q$  for some  $0 < \alpha < 1$ , so that  $\kappa \rightarrow \alpha$ . The idea is that if that the “full sum”  $\log(L(\mathfrak{X}, \frac{1}{2}))$  is close to  $\kappa \log \log q$ , then the most likely values for  $S_{n_l}$  should be  $\kappa n_l$ . The terms  $\pm \mathfrak{s}(\log_{l+2} q)$  handle the fluctuations around the linear interpolation. To properly mollify  $\log(L(\mathfrak{X}, \frac{1}{2}))$ , more restrictions are needed. We would expect the slope of the random sub-walk  $\tilde{S}_{n_l} - \tilde{S}_{n_{l-1}}$  to be not too much larger than the total slope for most characters, i.e. bounded by  $\kappa$ , and so we condition on the event that the slope is smaller than some large parameter,  $\mathcal{A}$ . For this slope to obey a typical behaviour, and not be influenced too much by large local fluctuations, the interval  $[n_{l-1}, n_l]$  must be sufficiently large, and we see from Equation (1.19) that  $n_l - n_{l-1} = \mathfrak{s}(\log_{l+2}(q) - \log_{l+1}(q))$ , where  $\mathfrak{s}$  is a large parameter. The exact values of these parameters in the recursive scheme are informed by the necessary inequalities which must be satisfied to prove the Gaussian bound, and are selected in Section 4.2. We recursively define the following events, where the dependence on  $\frac{1}{2}$  and  $\mathfrak{X}$  is suppressed for simplicity:

$$\begin{aligned} G_l &= A_l \cap B_l \cap C_l \cap D_l \\ A_l &= A_{l-1} \cap \{|\tilde{S}_{n_l} - \tilde{S}_{n_{l-1}}| \leq \mathcal{A}(n_{l-1} - n_l)\} \\ B_l &= B_{l-1} \cap \{S_{n_l} \leq U_l\} \\ C_l &= C_{l-1} \cap \{S_{n_l} \geq L_l\} \\ D_l &= D_{l-1} \cap \left\{ \left| L e^{-S_{n_l}} \right|^2 \leq c_l |LM_1 \dots M_l|^2 + e^{-\mathcal{D}(\log \log q - n_{l-1})} \right\}, \end{aligned} \tag{4.5}$$

where  $c_l = \prod_{j=1}^l (1 + e^{-n_{j-1}})$ , and  $A_0, B_0, C_0, D_0 = \{\text{primitive even characters mod } q\}$  is the full sample space. Here, the event  $D_l$  ensures the truncation of the logarithm of the mollifier  $M_1 \dots M_l$  into  $S_{n_l}$  is a sufficiently good approximation, and  $\mathcal{D}$  is a large parameter which ensures that the approximation has a small error. The following proposition gathers the necessary estimate to bound the probability of  $H$  on the good events.

**Proposition 4.1.** *Let  $V = \kappa \log \log q$  with  $\frac{1}{\sqrt{\log \log q}} \ll \kappa < 1$ . Then for some  $\delta \gg \kappa$  dependent on  $\kappa$ , we have*

- (i)  $\mathbf{P}(H \cap G_1^c) \ll \frac{e^{-\frac{V^2}{\log \log q}}}{\sqrt{\log \log q}} (\log \log q)^{-\delta};$
- (ii) for  $1 \leq l \leq \mathcal{L} - 1$ ,  $\mathbf{P}(H \cap G_l \cap G_{l+1}^c) \ll \frac{e^{-\frac{V^2}{\log \log q}}}{\sqrt{\log \log q}} (\log_{l+2} q)^{-\delta};$
- (iii)  $\mathbf{P}(H \cap G_{\mathcal{L}}) \ll \frac{e^{-\frac{V^2}{\log \log q}}}{\sqrt{\log \log q}}.$

**Proof of Theorem 1.1.** The proof of Theorem 1.1 is completed by writing

$$\mathbf{P}(H) \leq \mathbf{P}(H \cap G_1^c) + \sum_{l=1}^{\mathcal{L}-1} \mathbf{P}(H \cap G_l \cap G_{l+1}^c) + \mathbf{P}(H \cap G_{\mathcal{L}}) \tag{4.6}$$

and observing that  $\sum_{l=1}^{\mathcal{L}} (\log_{l+2}(q))^{-\delta} = O(1)$  as  $q \rightarrow \infty$ .  $\square$

**Proof of Theorem 1.4.** We employ the same method to prove Theorem 1.4, but since we may take  $V = o(\log \log q)$ , we cannot obtain as sharp bounds as in Theorem 1.1. If  $V = o(\log \log q)$  and  $N$  is a fixed positive integer, then by the construction of  $\mathcal{L}$  we have  $\log_{\mathcal{L}-n}(q) \gg 1$  as  $q \rightarrow \infty$  for each  $1 \leq n \leq N$ . Since  $\kappa = o(1)$ , we can only obtain  $\delta = o(1)$  from these bounds. Hence the contribution from  $\mathbf{P}(H \cap G_{\mathcal{L}-n} \cap G_{\mathcal{L}+1-n}^c)$  in Proposition 4.1 may be  $O(1) \cdot \frac{e^{-\frac{V^2}{\log \log q}}}{\sqrt{\log \log q}}$ , and we obtain the weaker bound:

$$\mathbf{P}(H) \ll \mathcal{L} \times \frac{e^{-\frac{V^2}{\log \log q}}}{\sqrt{\log \log q}}. \quad \square \tag{4.7}$$

*4.2. Choice of parameters*

In order to ensure the bounds on  $\mathbf{P}(H)$  are sharp enough to prove Proposition 4.1, we require certain bounds on the parameters to ensure certain inequalities below are satisfied. For Equation (4.38), we will require:

$$1 + \mathbf{s}(\kappa^2 - \mathcal{A}^2 + 2\kappa) < 0, \tag{4.8}$$

whilst for Equation (4.50) we will also require  $\mathbf{s}$  to satisfy:

$$\frac{1}{2} + \kappa^2 + 2(\kappa - 1)\mathbf{s} < 0. \tag{4.9}$$

Equation (4.9) forces us to take  $\mathbf{s}$  proportional to  $\frac{1}{1-\kappa}$ . We take

$$\mathbf{s} = \frac{10^5}{1 - \kappa}, \tag{4.10}$$

which ensures Equation (4.9) holds. The factor  $10^5$  is to ensure Equation (1.21) holds. Taking  $\mathcal{A} = 10^3$  ensures Equation (4.8) holds. Finally, we take  $\mathcal{D} = 10^4$  as in [ABR20], which gives sufficiently good bounds on  $\mathbf{P}(A_l \cap D_l^c)$ , to apply Lemma 4.2.

With these choice of parameters, we see that given  $\lambda < 1$ , there exists some  $\epsilon < 0$  depending on  $\lambda$  such that the difference of left-hand side and the right-hand side of the inequalities in Equations (4.8) and (4.9) are smaller than  $\epsilon\kappa$  for all  $0 < \kappa \leq \lambda$ . This enables us to consider the range with  $\kappa \rightarrow 0$  in Theorem 1.4, i.e.,  $V = o(\log \log q)$ . In

contrast, the inequalities with the choice of parameters in [AB23] do not hold with this uniformity, which prevents them from extending their range of  $V$ .

4.3. Proof of Proposition 4.1(i)

We split the event  $H \cap G_1^c$  as

$$H \cap G_1^c \subset A_1^c \cup B_1^c \cup (H \cap C_1^c \cap A_1 \cap D_1) \cup (D_1^c \cap A_1). \tag{4.11}$$

We bound each of the probabilities of each of the four event on the right-hand side, very similarly as in the proof of Proposition 2.1 in [AB23]. We give the details for  $A_1^c$  and  $D_1^c \cap A_1$ . The estimates involving  $B_1$  and  $C_1$  are similar to those done in the next section, but simpler.

Clearly,  $\mathbf{P}(A_1^c) = \mathbf{P}(|\tilde{S}_{n_1}| > \mathcal{A}n_1)$ . Thus, using Theorem 2.9 with  $k = \lceil 2\mathcal{A}^2n_1 \rceil$ , we obtain

$$\mathbf{P}(A_1^c) \ll \sqrt{n_1} \exp(-\mathcal{A}^2n_1) \ll \frac{e^{-\kappa^2 \log \log q}}{\sqrt{\log \log q}} (\log \log q)^{-\delta}, \tag{4.12}$$

for some  $\delta > 0$ .

For  $\mathbf{P}(D_1^c \cap A_1)$ , we first require a lemma, which allows the restriction  $A_1$  to force  $L$  to be large unless  $D_1$  is satisfied.

**Lemma 4.2.** *Let  $l \geq 1$  and suppose  $|\tilde{S}_{n_l} - \tilde{S}_{n_{l-1}}| \leq 10^3(n_l - n_{l-1})$ . Then*

$$e^{-(S_{n_l} - S_{n_{l-1}})} \leq (1 + e^{-n_{l-1}}) |M_l| + e^{-10^5(n_l - n_{l-1})} \tag{4.13}$$

**Proof.** This follows from the proof of Lemma 23 in [ABR20] using the identity:

$$\prod_{p \in (q_{l-1}, q_l]} (1 - \chi(p)p^{-s}) = e^{-(\tilde{S}_{n_l} - \tilde{S}_{n_{l-1}}) - R},$$

where  $R = \sum_{k \geq 3} \sum_{p \in (q_{l-1}, q_l]} \frac{1}{k} p^{-ks}$ .  $\square$

The event  $A_1 \cap \{|L| \leq (\log q)^2\}$  is contained in  $A_1 \cap D_1$ . This is because, on  $A_1$ , one can apply Lemma 4.2 to get

$$|Le^{-S_{n_1}}| \leq 2|LM_1| + |L|e^{-10^5n_1} \leq 2|LM_1| + (\log q)^2e^{-10^5n_1}.$$

Hence,  $\mathbf{P}(D_1^c \cap A_1) \leq \mathbf{P}(L > (\log q)^2)$ . By Markov’s inequality this probability is

$$\leq (\log q)^{-4} \mathbb{E}_+ [|L|^2] \ll \frac{e^{-\kappa^2 \log q}}{\sqrt{\log \log q}} e^{-\log \log q}. \tag{4.14}$$

Here we used that

$$\mathbb{E}_+ [|L|^2] \sim \log q, \tag{4.15}$$

which is a standard bound and may be seen by taking  $M = 1$  in Equation (3.7).

4.4. Proof of Proposition 4.1(ii)

As for the proof of Proposition 4.1(i), we use a union bound on the event  $H \cap G_l \cap G_{l+1}^c$  and bound each probability in the union bound individually.

We have the containment:

$$H \cap G_l \cap G_{l+1}^c \subset (A_{l+1}^c \cap G_l) \cup (B_{l+1}^c \cap G_l) \cup (H \cap C_{l+1}^c \cap A_{l+1} \cap D_{l+1} \cap G_l) \cup (D_{l+1}^c \cap A_{l+1} \cap G_l). \tag{4.16}$$

In order to bound the probabilities of each event, we need two important lemmas that express the probabilities of the events involving the partial sums in terms the random model where the characters  $\chi$  are replaced by the random phases  $X$  defined in Equation (2.5). We first prove an analogue of Lemma 2.6 in [AB23], which allows us to bound the twisted moment with the truncated sums bounded above and below by conditioning on  $B_l \cap C_l$ .

**Lemma 4.3.** *Let  $1 \leq l \leq \mathcal{L}$ , and  $Q_l = Q_l(\chi)$  be a Dirichlet polynomial of length  $N \leq q^{\frac{1}{100}}$  supported on integers all of whose prime factors are greater than  $q_l$ . Then for  $w \in [L_l, U_l]$  we have*

$$\mathbb{E}_+ \left[ |Q_l(1/2, \chi)|^2 \mathbf{1}(B_l \cap C_l \cap \{S_{n_l} \in (w, w + 1]\}) \right] \ll \mathbb{E}_\oplus \left[ |Q_l(1/2, \chi)|^2 \right] \frac{e^{-\frac{w^2}{n_l}}}{\sqrt{n_l}}. \tag{4.17}$$

**Proof.** The proof is by approximating the indicator functions in terms of a short Dirichlet polynomial, as in [AB23] and [ABR20]. We give the details for completeness. We define  $\mathfrak{J}$  to be the set of  $l$ -tuples  $\mathbf{u} = (u_1, \dots, u_l)$  satisfying:

$$\sum_{i=1}^j u_i \in [L_j - 1, U_j + 1], \forall j < l, \quad \sum_{i=1}^l u_i \in [w - 1, w + 2]. \tag{4.18}$$

Since  $\kappa < 1$ , we see for  $\mathbf{u}$  to satisfy these bounds on the partial sums  $S_{n_j}$  for  $1 \leq j \leq l$ , we require for all  $j > 1$ ,

$$|u_j| \leq 2(2\Delta_j + 1). \tag{4.19}$$

Moreover, we must have the inclusion

$$B_l \cap C_l \cap \{S_{n_l} \in (w, w + 1]\} \subset \cup_{\mathbf{u} \in \mathfrak{J}} \{Y_j \in [u_j, u_j + \Delta_j^{-1}]\}. \tag{4.20}$$

Hence, we obtain:

$$1(B_l \cap C_l \cap \{S_{n_l} \in (w, w + 1]\}) \leq \sum_{\mathbf{u} \in \mathfrak{J}} \prod_{1 \leq j \leq l} 1\{Y_j \in [u_j, u_j + \Delta_j^{-1}]\}. \tag{4.21}$$

We then proceed to bound each indicator function with a short Dirichlet polynomial.

Let  $A \geq 10$ ,  $j \leq l$  and  $\Delta_j = (n_j - n_{j-1})$ . We take  $D_{\Delta_j, A}(x)$  to be the polynomial of Lemma 2.8 in [AB23]. Specifically,  $D_{\Delta_j, A}(x)$  is a polynomial of degree at most  $\Delta_j^{10A}$  such that

$$1(x \in [u_j + \Delta_j^{-1}]) \leq |D_{\Delta_j, A}(x - u_j)|^2 \left(1 + ce^{-\Delta_j^{A-1}}\right), \tag{4.22}$$

where  $c$  is an absolute constant. If we let  $X = Y_j$  then, by construction of the support of the coefficients of  $Y_j$ , we see that  $D_{\Delta_j, A}(Y_j - u_j)$  is a Dirichlet polynomial on integers with all prime factors in the interval  $(q_{j-1}, q_j]$  of length at most  $\exp(2e^{n_j} \Delta_j^{10A})$ .

Hence we obtain

$$1(Y_j \in [u_j, u_j + \Delta_j^{-1}]) \leq |D_{\Delta_j, A}(Y_j - u_j)|^2 (1 + ce^{-\Delta_j^{A-1}}), \tag{4.23}$$

where  $Y_j = S_{n_j} - S_{n_{j-1}}$  for  $1 \leq j \leq l$ . Putting this back in the expectation, we get

$$\begin{aligned} & \mathbb{E}_+ \left[ |Q_l(1/2, \chi)|^2 1(B_l \cap C_l \cap \{S_{n_l} \in (w, w + 1]\}) \right] \\ & \ll \sum_{\mathbf{u} \in \mathfrak{J}} \mathbb{E}_+ \left[ |Q_l(1/2, \chi)|^2 \prod_j (1 + ce^{-\Delta_j^{A-1}}) |D_{\Delta_j, A}(Y_j - u_j)|^2 \right]. \end{aligned} \tag{4.24}$$

Following the decomposition of  $|Q_l|^2$  into the contribution from its real and imaginary parts as in the proof of Theorem 1.5, we may apply Theorem 1.6 and Lemma 2.4 to this combination of products of expectations of Dirichlet polynomials,  $Q_j$ , and real parts of Dirichlet polynomials,  $D_{\Delta_j, A}(Y_j - u_j)$ . Using Lemmas 2.4 and 2.6 to remove the restriction on the characters being primitive in the expectation in Equation (4.24), and bounding the error terms  $1 + ce^{-\Delta_j^{A-1}}$  trivially, we see that the expectation is

$$\ll \mathbb{E}_\oplus \left[ |Q_l(1/2, \chi)|^2 \prod_j |D_{\Delta_j, A}(Y_j - u_j)|^2 \right]. \tag{4.25}$$

Using Lemma 2.4, the expectation above splits, so the right-hand side is simply:

$$\mathbb{E}_\oplus \left[ |Q_l(1/2, \chi)|^2 \right] \prod_j \mathbb{E}_\oplus \left[ |D_{\Delta_j, A}(Y_j - u_j)|^2 \right]. \tag{4.26}$$

The orthogonality relation for the real part of even Dirichlet characters, Lemma 2.3, implies that for each  $1 \leq j \leq l$ ,

$$\mathbb{E}_\oplus \left[ |D_{\Delta_j, A}(Y_j - u_j)|^2 \right] = \mathbb{E} \left[ |D_{\Delta_j, A}(\mathcal{Y}_j - u_j)|^2 \right], \tag{4.27}$$

where  $(\mathcal{Y}_j, j \leq l)$  are independent random variables of the form

$$\mathcal{Y}_j = \sum_{q_{j-1} \leq p \leq q_j} \frac{\operatorname{Re} X(p)}{p^{\frac{1}{2}}}. \tag{4.28}$$

These are the exact analogue of the random variables defined in Equation 54 in [AB23]. Their distribution doesn't depend on being defined for the  $t$ -aspect or the  $q$ -aspect. These variables can be compared to Gaussian explicitly as in [AB23] starting at Equation 54. Proceeding as such, we obtain the result.  $\square$

We also require a lemma which is similar to Lemma 4.3, but includes the mollifier  $LM_1 \dots M_l$  in the moment. This is analogous to Lemma 2.7 in [AB23].

**Lemma 4.4.** *Let  $1 \leq l \leq \mathcal{L}$ . Let  $Q_l$  be a Dirichlet polynomial as defined in Lemma 4.3, and  $w \in [L_l, U_l]$ . Then*

$$\mathbb{E}_+ \left[ |LM_1 \dots M_l Q_l|^2 1(B_l \cap C_l, S_{n_l} \in (w, w + 1]) \right] \ll \mathbb{E}_\oplus \left[ |Q_l|^2 \right] \times \frac{\log q}{\log q_l} \frac{e^{-\frac{w^2}{n_l}}}{\sqrt{n_l}}. \tag{4.29}$$

**Proof.** The proof follows from the proof of Lemma 4.3. We see using Equations (4.22) and (4.23), that the left-hand side of Equation (4.29) becomes

$$\ll (1 + ce^{-\Delta_j^{A-1}}) \sum_{u \in \mathfrak{J}} \mathbb{E}_\oplus \left[ |LM_1 \dots M_l|^2 |Q_l|^2 \prod_j |D_{\Delta_j, A}(Y_j - u_j)|^2 \right]. \tag{4.30}$$

Setting

$$Q = Q_l \prod_j D_{\Delta_j, A}(Y_j - u_j), \tag{4.31}$$

we see that  $Q$  is a well-factorable Dirichlet polynomial. Applying Theorem 1.5 shows that the expression in (4.30) is

$$\ll (1 + ce^{-\Delta_j^{A-1}}) \frac{\log q}{\log q_l} \times \sum_{u \in \mathfrak{J}} \mathbb{E}_\oplus \left[ |Q_l|^2 \prod_j |D_{\Delta_j, A}(Y_j - u_j)|^2 \right]. \tag{4.32}$$

Proceeding as from Equation (4.25), we obtain Equation (4.30).  $\square$

Returning to the events in Equation (4.16), we begin by showing that for  $l \geq 1$ ,

$$\mathbf{P}(A_{l+1}^c \cap G_l) \ll \frac{e^{-\frac{V^2}{\log \log q}}}{\sqrt{\log \log q}} (\log_{l+1}(q))^{-\delta}. \tag{4.33}$$

For any  $k > 1$ , we have,  $\mathbf{P}(A_{l+1}^c \cap G_l)$  is bounded by:

$$\sum_{u \in [L_l, U_l]} \mathbb{E}_+ \left[ \frac{|\tilde{S}_{n_{l+1}} - \tilde{S}_{n_l}|^{2k}}{(\mathcal{A}(n_{l+1} - n_l))^{2k}} 1(B_l \cap C_l \cap \{S_{n_l} \in (u, u + 1]\}) \right]. \tag{4.34}$$

We let  $k = \lceil \mathcal{A}^2(n_{l+1} - n_l) \rceil$ . The Dirichlet polynomial  $(\tilde{S}_{n_{l+1}} - \tilde{S}_{n_l})^k$  satisfies the conditions of Lemma 4.3 for each  $u \in [L_l, U_l]$ , so we may bound the expression from (4.34) as:

$$\ll \frac{\mathbb{E}_\oplus [|\tilde{S}_{n_{l+1}} - \tilde{S}_{n_l}|^{2k}]}{(\mathcal{A}(n_{l+1} - n_l))^{2k}} \sum_{u \in [L_l, U_l]} \frac{e^{-\frac{u^2}{n_l}}}{\sqrt{n_l}}. \tag{4.35}$$

We can then use Lemma 2.10 to bound the moment, giving the above expression as:

$$\ll \frac{k!(n_{l+1} - n_l + 1)^k}{(\mathcal{A}(n_{l+1} - n_l))^{2k}} \sum_{u \in [L_l, U_l]} \frac{e^{-\frac{u^2}{n_l}}}{\sqrt{n_l}}. \tag{4.36}$$

We apply Stirling’s formula to bound the above expression as:

$$\begin{aligned} &\ll \frac{\sqrt{k} \left( \frac{\mathcal{A}^2(n_{l+1} - n_l + 1)^2}{e} \right)^k}{(\mathcal{A}(n_{l+1} - n_l))^{2k}} \sum_{u \in [L_l, U_l]} \frac{e^{-\frac{u^2}{n_l}}}{\sqrt{n_l}} \\ &\ll e^{-\mathcal{A}^2(n_{l+1} - n_l)} \sqrt{n_{l+1} - n_l} \sum_{r \geq -\mathbf{s} \log_{l+2} q} \frac{e^{-\frac{(\kappa n_l + r)^2}{n_l}}}{\sqrt{n_l}}. \end{aligned} \tag{4.37}$$

Since  $L_l = \kappa n_l - \mathbf{s} \log_{l+2} q$ , the above expression is

$$\ll \frac{e^{-\kappa^2 n_l}}{\sqrt{n_l}} (\log_{l+1} q)^{-\mathcal{A}^2 \mathbf{s} + 2\kappa \mathbf{s}} (\log_{l+2} q)^{\frac{1}{2} + \mathcal{A}^2 \mathbf{s}} \ll \frac{e^{-\kappa^2 \log \log q}}{\sqrt{\log \log q}} (\log_{l+1} q)^{\kappa^2 \mathbf{s} - \mathcal{A}^2 \mathbf{s} + 2\kappa \mathbf{s} + 1}. \tag{4.38}$$

The choice of parameters for Equation (4.8) ensures the exponent is negative.

Next, we bound  $\mathbf{P}(B_{l+1}^c \cap G_l)$ . Here again we have a sharper bound than the analogous result in [AB23], which allows us to take  $V = o(\log \log q)$ . We have:

$$\mathbf{P}(B_{l+1}^c \cap G_l) \ll \sum_{u \geq U_{l+1}} \mathbf{P}(\{S_{n_{l+1}} \in [u, u + 1]\} \cap G_l). \tag{4.39}$$

Using Lemma 4.3 with  $Q_l = 1$ , we see this is:

$$\ll \sum_{u \geq U_{l+1}} \frac{e^{-\frac{u^2}{n_{l+1}}}}{\sqrt{\log \log q_l}} \ll \frac{1}{\sqrt{\log \log q}} e^{-\frac{U_{l+1}^2}{n_{l+1}}} \sum_{k \geq 0} e^{\frac{U_{l+1}^2 - (U_{l+1} - k)^2}{n_{l+1}}}, \tag{4.40}$$

by writing  $k = \lfloor U_{l+1} - u \rfloor$ . The sum over  $k$  is  $\ll \sum_{k \geq 0} e^{-\frac{2kU_{l+1}}{n_{l+1}}} \ll 1$ . So we see that the right-hand side in Equation (4.40) is

$$\ll \frac{e^{-\frac{U_{l+1}^2}{n_{l+1}}}}{\sqrt{\log \log q}} \ll \frac{e^{-\kappa^2 \log \log q}}{\sqrt{\log \log q}} e^{\kappa^2 s \log_{l+3} q - 2\kappa s \log_{l+3} q} \ll \frac{e^{-\kappa^2 \log \log q}}{\sqrt{\log \log q}} (\log_{l+2} q)^{\kappa(\kappa-2)s}. \tag{4.41}$$

The choice of parameters ensures the exponent is negative.

We now turn to bound  $\mathbf{P}(H \cap C_{l+1}^c \cap A_{l+1} \cap D_{l+1} \cap G_l)$ . By using Lemma 4.2 and partitioning on the values  $S_{n_l} = u$  according to the range allowed by the event  $G_l$ , as well as the values  $v = S_{n_{l+1}} - S_{n_l}$  allowed by the event  $A_{l+1} \cap C_{l+1}^c$ , we see that the probability may be bounded above by:

$$\sum_{\substack{u \in [L_l, U_l] \\ u+v \leq L_{l+1} \\ |v| \leq \mathcal{A}(n_{l+1} - n_l)}} \mathbf{P}(\{S_{n_l} \in (u, u + 1], S_{n_{l+1}} - S_{n_l} \in (v, v + 1], \\ |LM_1 \dots M_{l+1}| > \frac{1}{100} e^{V-(u+v)}\} \cap B_l \cap C_l). \tag{4.42}$$

We improve on the analogous bound in [AB23] by using Lemma 2.10, which is a better upper bound on the moment of the truncated sums  $S_{n_{l+1}} - S_{n_l}$  than was achieved in Equation (79) in [AB23]. This allows us to attain an implicit constant in Theorem 1.1 which is bounded as  $\alpha \rightarrow 0$ , and gives us the bounds to prove Theorem 1.4 for values of  $V$  where we may have  $V = o(\log \log q)$ . The probability in (4.42) may be bounded by:

$$\sum_{\substack{u \in [L_l, U_l] \\ u+v \leq L_{l+1} \\ |v| \leq \mathcal{A}(n_{l+1} - n_l)}} e^{2(u+v-V)} \mathbb{E}_+ [ |LM_1 \dots M_{l+1}|^2 \mathbf{1}(\{S_{n_l} \in (u, u + 1], \\ S_{n_{l+1}} - S_{n_l} \in (v, v + 1]\} \cap B_l \cap C_l)]. \tag{4.43}$$

By Markov’s inequality, if we set  $r = \left\lceil \frac{v^2}{(n_{l+1} - n_l)} \right\rceil$ , then this is:

$$\ll \sum_{\substack{u \in [L_l, U_l] \\ u+v \leq L_{l+1} \\ |v| \leq \mathcal{A}(n_{l+1} - n_l)}} e^{-2V+2(u+v)} \times \\ \mathbb{E} \left[ \frac{|S_{n_{l+1}} - S_{n_l}|^{2r}}{v^{2r}} |LM_1 \dots M_{l+1}|^2 \mathbf{1}(\{S_{n_l} \in (u, u + 1]\} \cap B_l \cap C_l) \right]. \tag{4.44}$$

Using Lemma 4.4 with  $Q_l = (S_{n_{l+1}} - S_{n_l})^r$  and Equation (2.63) yields that this is

$$\ll \sum_{u+v \leq L_{l+1}} e^{-2V+2(u+v)} e^{2(\log \log q - n_{l+1})} \frac{(2r)!}{(2v)^{2r} r!} (n_{l+1} - n_l)^r \times \frac{e^{-\frac{u^2}{n_l}}}{\sqrt{n_l}}. \tag{4.45}$$

We can use Stirling’s formula to bound this as

$$\ll \frac{1}{\sqrt{\log \log q}} \sum_{u+v \leq L_{l+1}} e^{-2V+2(u+v)} e^{2(\log \log q - n_{l+1})} \left( \frac{r(n_{l+1} - n_l)}{ev^2} \right)^r e^{-\frac{u^2}{n_l}}. \tag{4.46}$$

Substituting the value for  $r$ , this is:

$$\ll \frac{1}{\sqrt{\log \log q}} \sum_{u+v \leq L_{l+1}} e^{-2V+2(u+v)} e^{2(\log \log q - n_{l+1})} e^{-\frac{v^2}{n_{l+1} - n_l}} e^{-\frac{u^2}{n_l}}. \tag{4.47}$$

Making the change of variables  $a = u - \kappa n_l, b = v - \kappa(n_{l+1} - n_l)$  shows this is:

$$\ll \sum_{a+b \leq L_{l+1} - \kappa n_{l+1}} \frac{e^{-\kappa^2 n_{l+1}}}{\sqrt{\log \log q}} e^{2(\log \log q - n_{l+1})} e^{(2-2\kappa)(a+b)} e^{-\frac{b^2}{n_{l+1} - n_l}}. \tag{4.48}$$

Performing the sum over  $a$  gives that the above expression is

$$\ll \sum_b \frac{e^{-\kappa^2 n_{l+1}}}{\sqrt{\log \log q}} e^{2(\log \log q - n_{l+1})} e^{-\frac{b^2}{n_{l+1} - n_l} - (2-2\kappa)s \log_{l+2} q}. \tag{4.49}$$

Using a Gaussian bound, the above expression is:

$$\begin{aligned} &\ll \frac{e^{-\kappa^2 n_{l+1}}}{\sqrt{\log \log q}} e^{2(\log \log q - n_{l+1})} \sqrt{n_{l+1} - n_l} e^{-(2-2\kappa)s \log_{l+2} q} \\ &\ll \frac{e^{-\kappa^2 \log \log q}}{\sqrt{\log \log q}} (\log_{l+1} q)^{\frac{1}{2} + \kappa^2 - (2-2\kappa)s}. \end{aligned} \tag{4.50}$$

By construction of  $s$ , the exponent of  $\log_{l+1} q$  is negative. We remark that  $\log_{l+1} q \gg \log_{l+2} q$ , so that this bound is even sharper than the total bound for Proposition 4.1(ii).

It remains to bound  $\mathbf{P}(D_{l+1}^c \cap A_{l+1} \cap G_l)$ . The event  $D_{l+1}^c \cap A_{l+1} \cap G_l$  is contained in

$$\{|LM_1 \dots M_l| > e^{\mathcal{A}(\log \log q - n_l)}\} \cap G_l. \tag{4.51}$$

This is because  $\{|LM_1 \dots M_l| \leq e^{\mathcal{A}(\log \log q - n_l)}\} \cap D_l$  is in  $A_{l+1} \cap D_{l+1}$  by using Lemma 4.2. By Markov’s inequality, we have

$$\mathbf{P}(\{|LM_1 \dots M_l| > e^{\mathcal{A}(\log \log q - n_l)}\} \cap G_l) \ll e^{-2\mathcal{A}(\log \log q - n_l)} \mathbb{E}_{\oplus} \left[ |LM_1 \dots M_l|^2 \mathbf{1}(G_l) \right]. \tag{4.52}$$

Using Lemma 4.4, we can bound the right-hand side as

$$\ll e^{-2(\mathscr{A}-1)(\log \log q - n_l)} \times \frac{e^{-\frac{L_l^2}{n_l}}}{\sqrt{n_l}} \ll \frac{e^{-\kappa^2 \log \log q}}{\sqrt{\log \log q}} (\log_{l+1} q)^{2s(\kappa+1-\mathscr{A})}. \tag{4.53}$$

The choice of  $\mathscr{A}$  ensures the exponent is negative.

4.5. Proof of Proposition 4.1(iii)

We have the obvious bound  $\mathbf{P}(H \cap G_{\mathcal{L}}) \leq \mathbf{P}(G_{\mathcal{L}})$ . It then suffices to apply Lemma 4.4. Note that the length of the interval  $[L_{\mathcal{L}}, U_{\mathcal{L}}]$  is of order one. Therefore,

$$\mathbf{P}(G_{\mathcal{L}}) \ll \sup_{v \in [L_{\mathcal{L}}, U_{\mathcal{L}}]} \frac{1}{\sqrt{n_{\mathcal{L}}}} e^{-v^2/n_{\mathcal{L}}} \ll \frac{e^{-\kappa^2 \log \log q}}{\sqrt{\log \log q}}. \tag{4.54}$$

5. Proof of Corollary 1.2

Corollary 1.2 follows from Theorem 1.1 in a similar way to the proof of Corollary 1.2 in [AB23].

**Proof.** Consider the CDF of the random variable  $\log L(\mathfrak{X}, \frac{1}{2})$  with  $\mathfrak{X}$  drawn uniformly from the even primitive characters mod  $q$ , i.e.,  $F(V) = \mathbf{P}(\log |L(\mathfrak{X}, \frac{1}{2})| \leq V)$ . Then if we set

$$\mathcal{M}_{\beta} = \mathbb{E}_+ \left[ \left| L\left(\mathfrak{X}, \frac{1}{2}\right) \right|^{2\beta} \right], \tag{5.1}$$

for  $0 < \beta < 1$ , the moments can be written as:

$$\mathcal{M}_{\beta} = \int_{-\infty}^{+\infty} e^{2\beta V} dF(V) = [e^{2\beta V} S(V)]_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} 2\beta e^{2\beta V} S(V) dV. \tag{5.2}$$

Since  $S(V) \leq 1$ , the boundary term at  $-\infty$  is zero. Since there are only finitely many characters, the boundary term at  $+\infty$  is also zero. The contribution of negative values of  $V$  in Equation (5.2) is negligible, since

$$\int_{-\infty}^0 2\beta e^{2\beta V} S(V) dV \leq \int_{-\infty}^0 2\beta e^{2\beta V} dV = 1. \tag{5.3}$$

In order to estimate  $\int_0^{\infty} 2\beta e^{2\beta V} S(V)$ , consider  $\beta_-$  and  $\beta_+$  with

$$0 < \beta_- < \beta < \beta_+ < 1. \tag{5.4}$$

We want to show the dominant contribution of to the integral on the right-hand side of Equation (5.2) comes from the interval  $[\beta_- \log \log q, \beta_+ \log \log q]$ .

We take  $\beta_- = \frac{\beta}{4}$ ,  $\beta_+ = \frac{3+\beta}{4}$ . Using Theorem 1.1 to bound  $S(V)$ , we have

$$\begin{aligned} \int_{\beta_- \log \log q}^{\beta_+ \log \log q} \beta e^{2\beta V} S(V) dV &\ll \int_{\beta_- \log \log q}^{\beta_+ \log \log q} e^{2\beta V} \frac{e^{-\frac{V^2}{\log \log q}}}{\sqrt{\log \log q}} dV \\ &= \frac{e^{\beta^2 \log \log q}}{\sqrt{\log \log q}} \int_{\beta_- \log \log q}^{\beta_+ \log \log q} e^{-(\beta \log \log q - V)^2 / \log \log q} dV \\ &\ll e^{\beta^2 \log \log q}. \end{aligned} \tag{5.5}$$

The contribution of the interval  $[0, \beta_- \log \log q]$  to the integral in Equation (5.2) is of a lower order since it is bounded by:

$$\int_0^{\beta_- \log \log q} 2\beta e^{2\beta V} dV \leq e^{\frac{\beta^2 \log \log q}{2}}. \tag{5.6}$$

For the interval  $[\beta_+ \log \log q, \infty)$ , we require a bound on  $S(V)$  in this range. But we have

$$\mathbb{E}_+ \left[ \left| L \left( \chi, \frac{1}{2} \right) \right|^2 \right] \ll \log q, \tag{5.7}$$

and so Markov’s inequality yields  $S(V) \ll e^{-2V} \log q$ . Hence the contribution in the range  $[\beta_+ \log \log q, \infty)$  to the integral in Equation (5.2) is

$$\ll \log q \int_{\beta_+ \log \log q}^{\infty} 2\beta e^{2(\beta-1)V} dV = \frac{\beta}{1-\beta} e^{(2(\beta-1)\beta_++1) \log \log q}. \tag{5.8}$$

This is  $o(e^{\beta^2 \log \log q})$  by the choice of  $\beta_+$ .  $\square$

Since we chose the parameters in Section 4.2 such that the inequalities held uniformly for  $0 < \kappa < 1$ , we see that the implicit constant in Equation 1.2 is bounded as  $\beta \rightarrow 0$ . As previously stated, this contrasts with the proof in [AB23], where in Equation (1.17), their bound had  $C_\beta \rightarrow \infty$  as  $\beta \rightarrow 0$ .

**Data availability**

No data was used for the research described in the article.

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