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IS THERE A GROWTH-UNEMPLOYMENT TRADE-OFF?

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Abstract

Why might there be a long-run trade-off between growth and unemployment ? In general equilibrium, the returns on the factors of production are interdependent. This paper develops a model where the determination of the wage is central to the evolution of these incentives. The incentive to hire responds little (and in some cases not at all) to changes in the rate of interest. If the wage grows in line with productivity, there is a positive relation between growth and unemployment. If the wage rises as the labour market tightens, the incentive to invest in human and physical capital rises relative to the incentive to hire. There emerges a trade-off between growth and unemployment.

Keywords : endogenous growth; unemployment; human capital. *JEL Classification* : E24; J30; O41.

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1 Introduction

How are growth and unemployment related ? Okun's law suggests that unemployment has a high cost in terms of foregone output. An open question is whether unemployment can have a high cost in terms of long-run growth. Does unemployment reduce the economy's growth potential ? Does productivity growth necessarily generate employment ? The empirical evidence suggests that, if anything, there may be a long-run growth-unemployment trade-off¹. Why might there be a long-run trade-off between growth and unemployment ?

Throughout the post-war era, productivity growth has been higher in Europe than in the United States, while, at least since the early 1980s, unemployment has been higher in Europe. The higher European productivity growth can be partly explained by the 'catching up' that followed the massive destruction of physical capital during the second world war. The rise in European unemployment is partly explained by labour market rigidities in the context of the productivity slowdown and oil price shocks of the 1970s. Wages initially grew ahead of productivity, and subsequently fell, without reversing the rise in unemployment which they had contributed to cause. In the United States, by contrast, wages grew in line with productivity throughout the period.

This paper develops a model where unemployment and growth are endogenously determined. Firms can invest in physical capital, human capital or hire workers. In general equilibrium, the returns on the factors of production are interdependent. This paper develops a model where the determination of the wage is central to the evolution of these incentives. The incentive to hire responds little (and in some cases not at all) to changes in the rate of interest. If the wage grows in line with productivity, there is a positive relation between growth and unemployment. If the wage rises as the labour market tightens, the incentive to invest in human and physical capital rises relative to the incentive to hire. There emerges a trade-off

¹The following discussion draws on the empirical evidence gathered in Gordon (1995).

between growth and unemployment.

The issue of how growth and unemployment interact has recently received some attention, in particular since the work of Pissarides (1990). The present paper develops a model where growth and unemployment are the endogenous outcome of some structural and behavioural relations, following the lead of Bean and Pissarides (1993) and Postel-Vinay (1998). Endogenous growth arises from the complementarity of physical and human capital in the production process, following Uzawa (1963), Lucas (1988), Rebelo (1991) and Bond, Wang and Yip (1996).

Unemployment is modelled as the outcome of a matching process, following Phelps (1968). The focus of the paper is on job creation, so an exogenous process of job destruction is assumed. Following Pissarides (1985), competitive firms bid down the expected rent from advertising a vacancy. The existence of sunk costs implies that a match generates a surplus, which places the match participants in a position of bilateral monopoly. Assuming competitive capital markets, a surplus-sharing rule is necessary to determine wages. The factor distribution of income depends on the surplus-sharing rule prevailing in the economy. This, in turn, may be explained by a whole range of sociological, political, institutional and other factors. The search unemployment framework thus raises a wide range of issues of potential importance for the functioning of the economy. The precise nature of the wage relation determines the nature of the relation between growth and unemployment.

For the purpose of this paper, the crucial assumption is that human capital can be raised by a process of investment. For simplicity, it is assumed that human capital is homogeneous, and that, like output, it is produced within a representative firm. Human capital is a firm-specific, labour-augmenting capital stock similar to the ‘organisation capital’ discussed by Prescott and Visscher (1980). This is the key to the relative simplicity of the model. There is no need to keep track of the history of the workers to determine the stock of human capital of a given firm. Prescott and Visscher (1980) have argued that, to a first approximation, organisational capital is

produced under constant returns to scale. As a result, the production set is a convex cone with a vertex at the origin, and perpetual balanced growth can be sustained endogenously. The two-sector model of endogenous growth developed in this paper abstracts from externalities in production and has physical capital in the education sector².

Section 2 introduces the model. Conditions for existence and uniqueness are discussed in the appendix. Section 3 studies the long-run Growth-Unemployment trade-off exhibited by the model. Section 4 concludes.

2 The Model

The model combines the two-sector theory of endogenous growth with the matching theory of unemployment. There are two reproducible factors of production, physical capital, K , and human capital, H . There is one non-reproducible factor of production, employment, L . All three factors are predetermined in the sense that their initial values are treated as parametric.

2.1 Aggregate Consumption

The approach adopted in this paper is to model unemployment ‘as if’ it was deterministic. Individuals can be in one of two states, either employed or unemployed. If unemployed, individuals participate in the matching process, supplying a constant level of search effort. When employed, they drop out of the matching process and supply 1 unit of labour. Population is constant and normalised to 1 unit –

²A previous version of the paper contains a discussion of the literature on growth and unemployment. Relevant contributions include Pissarides (1990), Aghion and Howitt (1994), Hoon and Phelps (1992, 1997), Hoon (1998), Rowthorn (1999), Caballero and Hammour (1998), Bean and Pissarides (1993), and Postel-Vinay (1998). The following are also relevant to the analysis of this paper : King and Rebelo (1990), Rebelo (1991), Caballé and Santos (1993), Mulligan and Sala-i-Martin (1993), Bond, Wang and Yip (1996), Mino (1996).

no birth, no death. There is a single homogeneous consumption good. Aggregate consumption satisfies the simple Euler equation,

$$\frac{\dot{C}_t}{C_t} = \sigma(r_t - \rho), \quad (1)$$

where r_t denotes the rate of interest in the economy, ρ is the rate of pure time preference, $\sigma > 0$ is the intertemporal elasticity of substitution, and C_t is consumption. A transversality condition is also imposed,

$$\lim_{t \rightarrow \infty} e^{-\rho t} C_t^{-\frac{1}{\sigma}} K_t = 0. \quad (2)$$

The assumptions that workers do not search on the job and that the intensity of search is constant simplify the exposition. One interpretation is that ‘looking for a job is a full-time job’. Abstracting from the choice of search intensity rules out discouraged worker effects. This makes for a sharp contrast with neoclassical theories of unemployment based on the hypothesis of intertemporal substitution of labour supply.

The simple Euler equation assumed here abstracts from wealth effects and unemployment risk. The impact of unemployment on saving, and growth, is likely to be sizeable. Introducing unemployment risk is not a trivial matter. The approach adopted in this paper is to model unemployment ‘as if’ it was deterministic and to leave out wealth effects. This approach has the virtue of simplicity and makes the results of this paper more easily comparable with other studies, in particular with Postel-Vinay (1998) and Bond, Wang and Yip (1996). Similar assumptions have been made, in a related context, by – among others – Pissarides (1990), Aghion and Howitt (1994), Andolfatto (1996), Merz (1995), Postel-Vinay (1998), Shi and Wen (1997).

2.2 Production

There are two kinds of capital, physical capital and human capital. Both types of capital are used in the production of consumption goods, capital goods and human

capital. The technology is putty-putty. Capital goods and consumption goods are produced with the same technology. There are thus two sectors. The goods sector is referred to as sector Y , and the human capital sector is referred to as sector Z . There is a fixed endowment of labour time that may be allocated to the production of goods or to the production of human capital. Goods produced in sector Y may be either consumed or invested. The output of sector Y is the numéraire with price normalised to 1.

The production of goods is written $Y = F(\vartheta_K K, H\vartheta_L L)$, where K is physical capital, H is human capital, L is employment, and ϑ_K and ϑ_L are the allocations of capital and labour between the goods and human capital sectors. The production function has thus two arguments, capital and effective labour. As a result, human capital and employment are substitutes in production. The notation for the production of human capital parallels the notation for the production of output. The production of human capital is written $Z = G[(1 - \vartheta_K)K, H(1 - \vartheta_L)L]$.

Assumption 1 (The Production Functions)

The production functions for goods, $F : \mathfrak{R}_+^2 \rightarrow \mathfrak{R}_+$, and for human capital, $G : \mathfrak{R}_+^2 \rightarrow \mathfrak{R}_+$, are strictly increasing, strictly concave, twice continuously differentiable and homogeneous of degree 1 with respect to the two arguments. There are no externalities.

There are constant returns to scale with respect to physical capital and effective labour, but increasing returns to scale with respect to physical capital, human capital and raw labour. Homogeneity of degree 1 implies that the production functions may be expressed in intensive form. Let the capital-labour ratio (in effective units) be denoted $k \equiv \frac{K}{HL}$. Let the capital-labour ratio in the goods sector be denoted $k_Y \equiv \frac{\vartheta_K K}{H\vartheta_L L}$. Let the capital-labour ratio employed in the human capital sector be denoted

$k_Z \equiv \frac{(1-\vartheta_K)K}{H(1-\vartheta_L)L}$. The marginal products may be written in intensive form,

$$F_1(\vartheta_K K, H\vartheta_L L) = f'(k_Y) \quad (3a)$$

$$F_2(\vartheta_K K, H\vartheta_L L) = f(k_Y) - k_Y f'(k_Y). \quad (3b)$$

$$G_1[(1-\vartheta_K)K, H(1-\vartheta_L)L] = g'(k_Z) \quad (3c)$$

$$G_2[(1-\vartheta_K)K, H(1-\vartheta_L)L] = g(k_Z) - k_Z g'(k_Z). \quad (3d)$$

Goods may be either consumed or invested. The capital stock at time t evolves according to

$$\dot{K}_t = I_t - \delta_K K_t,$$

where I_t denotes gross investment and $\delta_K \geq 0$ denotes the depreciation rate of physical capital. The effectiveness of hours of work may be raised by investing in the human capital of workers. The stock of human capital evolves according to

$$\dot{H}_t = Z_t - \delta_H H_t,$$

where Z_t is the production of new human capital and $\delta_H \geq 0$ denotes the depreciation rate of human capital. This accumulation equation implies that a change in the level of employment has no immediate effect on the *level* of human capital employed in production. However, other things equal, a reduction in the level of employment reduces the *rate* at which human capital is accumulated.

There is no joint production. Both factors cannot be paid their marginal products. The reason for this is that the reallocation of workers involves sunk costs. It is assumed that the gross rate of return on physical capital is equal to the marginal product of physical capital, whereas the wage rate is strictly less than the marginal product of labour, in order to compensate firms for the sunk costs. There is a large (fixed) number of identical firms. Each firm is sufficiently small that it neglects the effect its actions have on factor prices and marginal products. Hence firms take the rate of interest and the wage as parametric. The representative firm controls the accumulation of human capital. All human capital accumulation is done on the job.

This formulation is adopted to emphasise the existence of a trade-off between the production of goods and the production of human capital.

2.3 Search

There are frictions in the labour market. To hire workers, firms must undertake a costly process of search. Firms are free to post vacancies whenever they wish to hire. The number of job vacancies is one of the firm's control variables, whereas the level of employment is a state variable. Vacancies can be adjusted instantaneously, but employment – and thus unemployment – adjusts only slowly. The number of unemployed workers u_t is also the number of individuals in the search process. Let v_t denote the total number of vacancies posted at time t . The search frictions are captured by the device of a matching function, denoted $M(v_t, u_t)$.

Assumption 2 (The Matching Function)

The matching function, $M : \mathfrak{R}_+^2 \rightarrow \mathfrak{R}_+$, is strictly increasing, strictly concave, twice continuously differentiable, exhibits constant returns to scale, satisfies the Inada conditions, $\lim_{v \rightarrow 0} M_1(v, u) = \lim_{u \rightarrow 0} M_2(v, u) = \infty$, and the boundary conditions $M(v, 0) = M(0, u) = 0$.

The properties of the matching function are analogous to the properties of the production functions. The key difference is that production decisions are co-ordinated, while matching decisions are not. An increase in the number of participants in the search process, from either side of the search, increases the matching rate, but with diminishing marginal returns. The assumption of constant returns to scale is realistic³, and facilitates the characterisation of a balanced growth path.

Since individuals and firms ignore the impact their actions have on the matching rates, there are externalities in the search process. An increase in the number of participants on one side of the search process reduces the probability of a match

³See e.g. Blanchard and Diamond (1989).

on that side but increases it on the other side. The two effects cancel each other out only in a special case. In general, therefore, the decentralised equilibrium is inefficient.

Given constant returns to scale, the matching function may be expressed in intensive form. Let market tightness be defined, $\theta_t \equiv v_t/u_t$. The hiring rate, i.e. the probability, at any time t , that a firm is matched with an applicant is $M(v_t, u_t)/v_t = h(\theta_t)$. Other things equal, an increase in unemployment reduces market tightness, making it easier for firms to hire. The job-finding rate, i.e. the probability that an unemployed worker is matched with a job is $M(v_t, u_t)/u_t = m(\theta_t)$, where $m(\theta_t) = \theta_t h(\theta_t)$. Other things equal, a reduction in unemployment raises market tightness, making it easier for workers to find a job. The hiring rate is decreasing, $h'(\theta_t) < 0$. The job-finding rate is increasing, $m'(\theta_t) > 0$.

A firm wishing to hire must choose how many vacancies to advertise. If posting vacancies were not costly, the firm would advertise as many vacancies as feasible and hire accordingly. However, it is assumed that each vacancy advertised by the firm has an advertising cost of χ_t units of output per unit of time. If v_t vacancies are posted, the total advertising cost is $\chi_t v_t$. This cost is a *flow* cost. The flow cost of advertising one vacancy may be thought of as a per-period average cost of recruiting one worker.

One would expect the hiring activity to be intensive in labour. It may be reasonable to assume that the cost of hiring labour is, to a first approximation, proportional to the cost of employing it. If the ‘hiring sector’ is small compared to the production sector, wages are essentially determined in the production sector. Following Pissarides (1990), let vacancy costs be proportional to wages, $\chi_t = \bar{\chi} w_t H_t$, where $\bar{\chi} \geq 0$ is a constant.

Employment in the representative firm is denoted L_t . Let $\mu \geq 0$ denote the rate at which jobs are destroyed. As the emphasis of this paper is on job creation rather than job destruction, additions to unemployment are exogenous. The jobs destroyed

are randomly selected, so that all employed workers face the same probability of losing their job, irrespective of how long they have been employed. The process of job destruction may be interpreted, for instance, as firm-specific shocks, random obsolescence of machines or ongoing structural changes in the organisation of labour.

As the total workforce is constant and normalised to 1, the unemployment rate is equal to the unemployment level, $u_t = 1 - L_t$. The dynamics of unemployment follows,

$$\dot{u}_t = \mu - [\mu + m(\theta_t)]u_t. \quad (4)$$

Equation (4) is a Beveridge curve, relating the flow out of unemployment with the flow into unemployment.

2.4 Wages

A match between an advertised vacancy and an unemployed worker creates a surplus. This surplus must be shared among the owners of capital and the workers. Institutional features, unemployment compensation, bargaining power, the structure of vacancy costs, and so on, are likely to affect how the surplus is shared.

Let W_t denote the per-effective-unit marginal product of labour. Thus, $W_t H_t$ denotes an employed worker's marginal product; and $W_t H_t L_t$ denotes the value of the marginal product of the employed labour force. When no ambiguity results, the words 'marginal product' is used to refer to the 'per-effective-unit marginal product', W_t . Let w_t denote the wage rate. Thus, $w_t H_t$ denotes the employed worker's wage; and $w_t H_t L_t$ denotes the firm's wage bill.

A simple wage relation is assumed, leaving the surplus-sharing rule implicit. Let $w_t = \Delta_t W_t$. The following condition ensures that hiring and human capital investment are both profitable.

Profitability Condition

The marginal product of labour is greater than the wage,

$$W_t > w_t > 0$$

The Profitability Condition implies a restriction on the set of admissible values of the ‘wedge’ Δ_t ,

$$0 < \Delta_t < 1$$

If a wealth-maximising Nash-bargaining surplus-sharing rule is assumed, it may be shown that the wedge is a function of market tightness alone, $\Delta(\theta_t)$. It will be instructive to make reference to this case. Let $\Delta(\theta_t)$ be continuous and twice differentiable. The sign of $\Delta'(\theta_t)$ indicates whether a tighter labour market is reflected in higher or lower wages. The Nash wage equation typically yields $\Delta'(\theta_t) > 0$ and also $\Delta''(\theta_t) > 0$.

2.5 Decentralised Equilibrium

Consider the problem of the representative firm. The representative firm chooses gross investment, I_t ; chooses the allocation of capital and workers to the sector producing goods, ϑ_{Kt} , and to the sector producing human capital, ϑ_{Lt} ; and chooses the number of vacancies to post, v_t . The stock variables are K_t , H_t , and L_t . The firm takes as given the rate of interest r_t , the wage w_t , market tightness θ_t , the vacancy costs χ_t , and the rate of separation μ . The objective of the firm is to maximise the present discounted value of cash flows π_t ,

$$V(K_0, H_0, L_0) = \max \int_0^\infty e^{-\int_0^t r_s ds} \pi_t dt, \quad (5)$$

where r_t is the rate of interest, and cash flows are : output minus the wage bill, minus vacancy costs, minus gross investment,

$$\pi_t \equiv F(\vartheta_{Kt}K_t, H_t\vartheta_{Lt}L_t) - w_tH_tL_t - \chi_tv_t - I_t. \quad (6)$$

The problem of the representative firm is to maximise the integral in (5) subject to the capital accumulation constraint (7a), the human capital accumulation constraint (7b) and the employment constraint (7c),

$$\dot{K}_t = I_t - \delta_K K_t \quad (7a)$$

$$\dot{H}_t = G[(1 - \vartheta_{Kt})K_t, H_t(1 - \vartheta_{Lt})L_t] - \delta_H H_t \quad (7b)$$

$$\dot{L}_t = h_t v_t - \mu L_t, \quad (7c)$$

subject to the initial conditions $K(0) = K_0 > 0$, $H(0) = H_0 > 0$, $L(0) = L_0 > 0$.

Population is normalised to 1, so that C_t stands for both individual and aggregate consumption. The consumer's problem and the firm's problem are connected by the imposition of a market-clearing condition,

$$C_t + I_t + \chi_t v_t = Y_t. \quad (8)$$

The long-run equilibrium⁴ is characterised by a system of equations in two unknowns p and θ ,

$$R(p) - \delta_K = \frac{h(\theta) [1 - \Delta(\theta)]}{\bar{\chi} \Delta(\theta)} - \mu + \overbrace{\sigma[R(p) - \delta_K - \rho]}^{\text{cost of growth}} \quad \text{KL}$$

$$R(p) - \delta_K = \frac{[1 - \Delta(\theta)] W(p)}{p} \frac{m(\theta)}{\mu + m(\theta)} - \delta_H. \quad \text{KH}$$

The left-hand side of equation KL is the return on the marginal investment in physical capital. The right-hand side is the capitalised return of the marginal vacancy, where the term marked by the braces is the growth-induced rise in vacancy costs implied by the growth in the wage. Thus, equation KL equates the steady-state return earned by one unit of investment in physical capital K with the steady-state return earned by one unit of 'investment' in hiring labour L . Likewise, the left-hand side of equation KH is the return on the marginal investment in physical capital. The right-hand side is the return on the marginal investment in human capital. Thus,

⁴The technical details of the solution and proofs of existence and uniqueness are relegated to the appendix.

equation KH equates the steady-state return earned by one unit of investment in physical capital K with the steady-state return earned by one unit of investment in human capital H . Each equation gives an *implicit* relation between p and θ .

3 The Growth-Unemployment Relation

In the long-run, the return on physical capital must be equated to the return on the hiring activity (equation KL). Likewise, the return on physical capital and the return on human capital must be equated (equation KH). Market tightness θ and the relative price of human capital p solve the system formed by equations KL and KH. Each value of p is associated with a unique value of γ , where $\gamma = \sigma[R(p) - \delta_K - \rho]$. Each value of θ is associated with a unique value of u , where $u = \mu/[\mu + m(\theta)]$. The discussion is centred on the empirically realistic cases $\sigma \leq 1$, and $k_Y > k_Z$, where the latter inequality implies $\gamma'(p) < 0$. The discussion is limited to comparisons of long-run equilibria⁵.

It is instructive to consider the special case where the elasticity of intertemporal substitution is unity, $\sigma = 1$. In that case, equation KL reduces to

$$\frac{h(\theta) [1 - \Delta(\theta)]}{\Delta(\theta)} = (\mu + \rho) \bar{\chi}. \quad \text{KL (bis)}$$

The hiring rate $h(\theta)$ is decreasing in θ . A sufficient condition for uniqueness is that $\Delta(\theta)$ be monotonically increasing. In that case, the left-hand side of equation KL (bis) is monotonically decreasing in θ and there is therefore a unique value of θ that solves equation KL (bis). In the special case where $\sigma = 1$, market tightness, and thus unemployment, is determined solely by the position of the KL line – a vertical line in the (u, γ) plane. This property comes from the intertemporal no-arbitrage condition between physical capital investments and hiring, when $\sigma = 1$. Any change in the capital productivity term implies an *equal* change in the cost of growth term.

⁵The appendix contains a brief discussion of the dynamic forces at work.

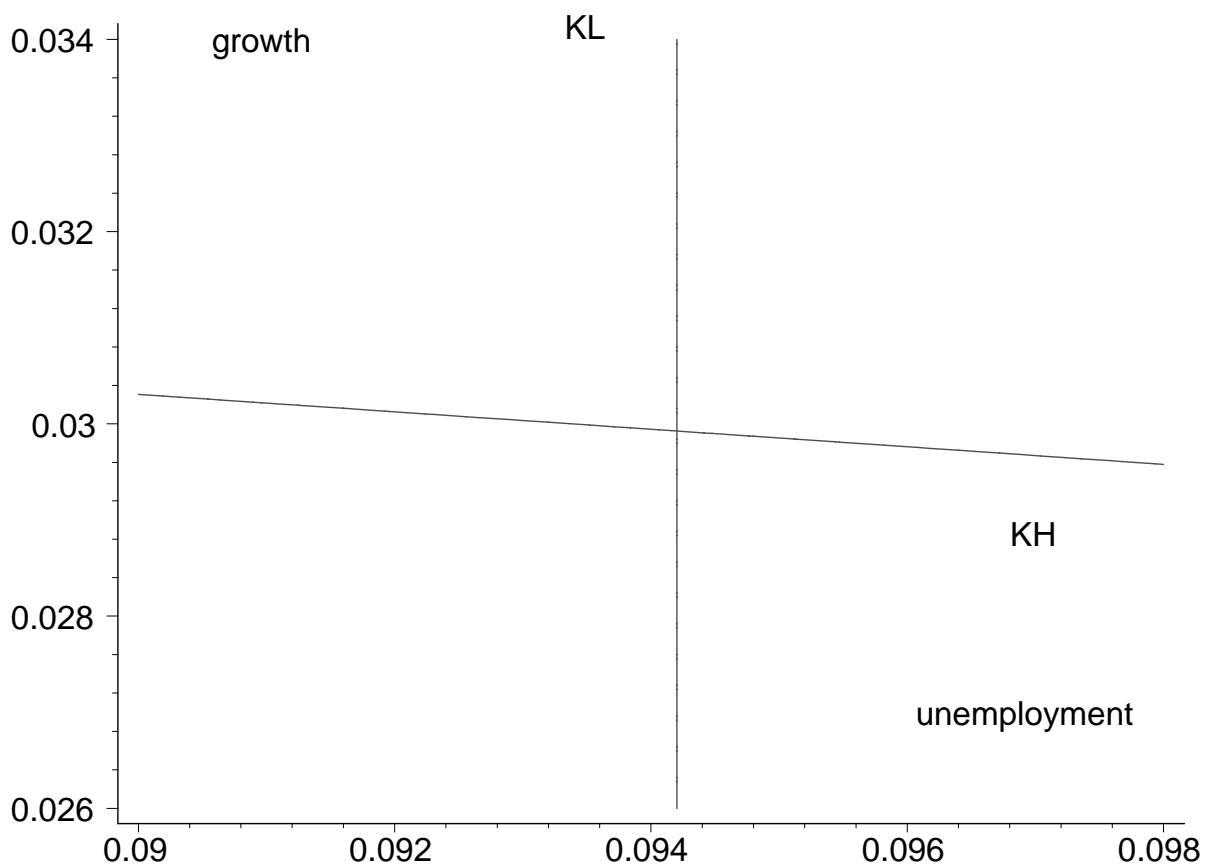


Fig. 1: **A one-sided positive relation**

The change in the cost of growth term is due to the growth-induced rise in vacancy costs.

Consider equation KH. The right-hand side of KH is decreasing if $k_Y > k_Z$, increasing otherwise. Thus, given a value of θ from equation KL (bis), there is a unique value of p given by equation KH. The equilibrium is pinned down in the labour market. Any rate of unemployment implies a unique rate of growth. Equation KL (bis) may be thought of as determining the ‘natural’ rate of unemployment, $u(\theta)$. The ‘natural’ rate of unemployment is determined by the matching technology $h(\theta)$, the wedge $\Delta(\theta)$, vacancy costs $\bar{\chi}$, the job destruction rate μ , and the rate of

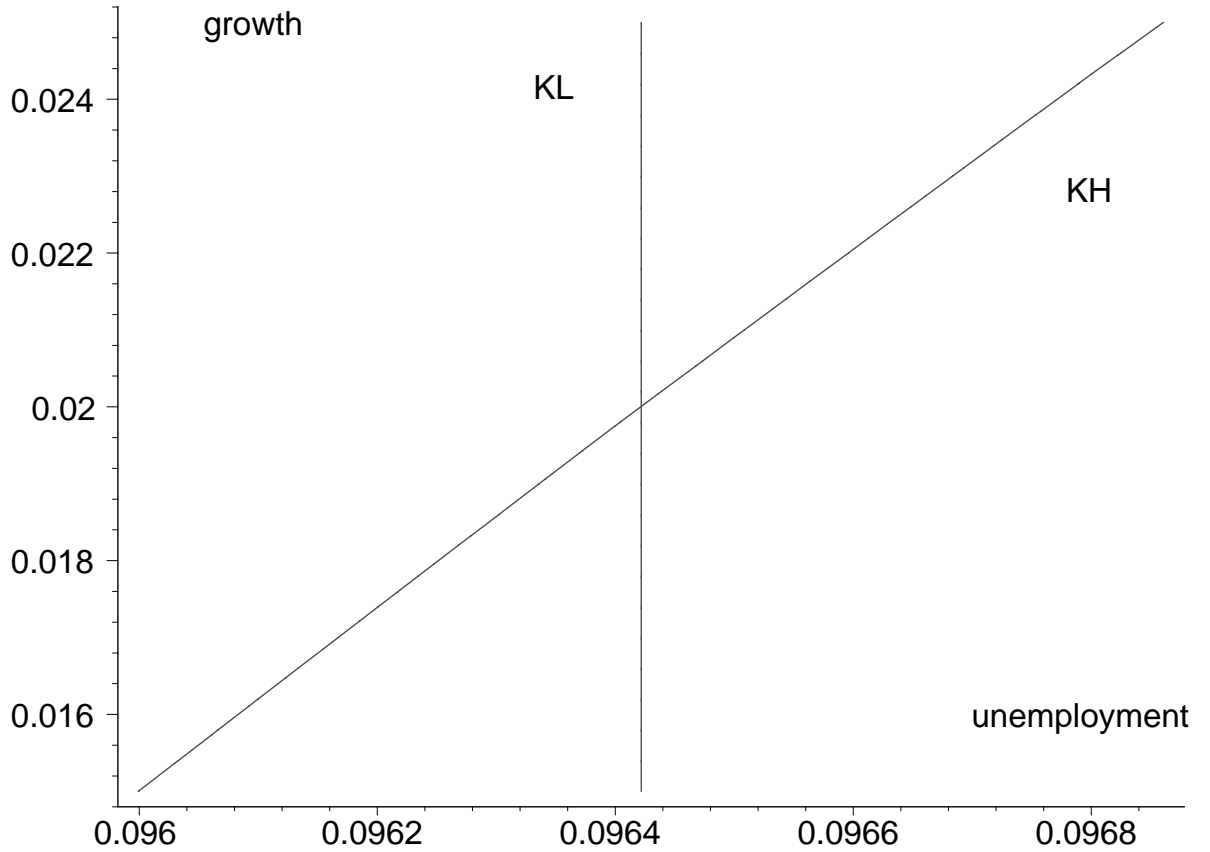


Fig. 2: **A one-sided trade-off**

time preference ρ . Equation KH may be thought of as determining the ‘natural’ rate of growth $\gamma(p)$. The ‘natural’ rate of growth is determined by the production technologies, $R(p)$ and $W(p)$, the depreciation rates δ_K and δ_H , but also by the ‘natural’ rate of unemployment. Thus, conditions in the labour market have repercussions on the long-run growth rate.

In the special case where $\sigma = 1$, the KH line alone determines the nature of the relation between growth and unemployment. Consider first the case where the wedge is constant at all times, which implies $\Delta'(\theta_t) = 0$, for all $\theta_t \geq 0$. The factor intensity ranking determines the nature of the relation between growth and unemployment.

If the goods sector is more capital intensive than the human capital sector, $k_Y > k_Z$, then a low value of θ is associated with a high value of p , and therefore with a low value of $\gamma(p)$. In other words, if the goods sector is more capital intensive than the human capital sector, and if wage claims do not respond to changes in market tightness, a high rate of unemployment is associated with a low rate of growth, while a low rate of unemployment is associated with a high rate of growth.

Figure 1 on Page 14 illustrates the properties of the equilibrium in this case. The KL line is vertical. The KH line is downward-sloping. Shifts in the lines may be interpreted as changes in the structural parameters. A rise in the ‘natural’ rate of unemployment, or a shift of KL to the right, reduces the ‘natural’ rate of growth. The opposite is not true, however : a change in the ‘natural’ rate of growth, or a shift of KH, has no effect on the ‘natural’ rate of unemployment. Thus, if wages grow in line with productivity, a shock that lowers the incentive to hire, such as a rise in the rate of job destruction μ or a fall in the effectiveness of matching M unambiguously reduces growth. Changes in long-run unemployment have a positive effect on changes in long-run growth, but changes in long-run growth have no effect on long-run unemployment. The relation between long-run growth and unemployment is one-sided, going from unemployment to growth.

Consider now the case where the wedge is positively related to market tightness, $\Delta'(\theta) > 0$. In this case, the nature of the relation between growth and unemployment depends on whether $\Delta(\theta)$ increases locally more strongly than $u(\theta)$ decreases, as well as on the factor intensity ranking. If $k_Y > k_Z$, and if $\frac{\Delta'(\theta)}{1-\Delta(\theta)} > \frac{-u'(\theta)}{1-u(\theta)}$, then a low value of θ is associated with a low value of p , and therefore with a high value of $\gamma(p)$. In other words, if the goods sector is more capital intensive than the human capital sector, and if wages fall relative to productivity as unemployment rises, a high rate of unemployment is associated with a high rate of growth. Figure 2 on Page 15 illustrates the properties of the equilibrium in this case. The KL line is vertical. The KH line is locally upward sloping. A rise in the ‘natural’ rate of

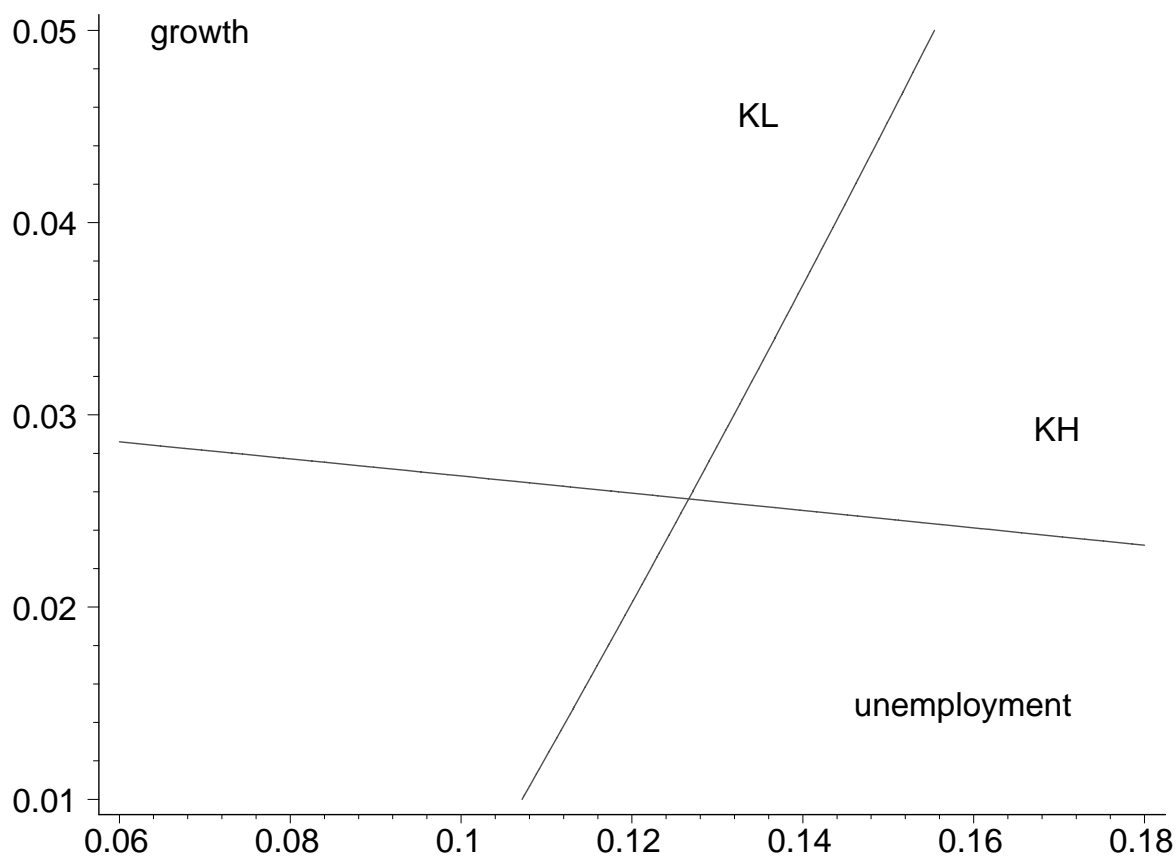


Fig. 3: **A positive relation**

unemployment, or a shift of KL to the right, raises the ‘natural’ rate of growth. The effect of wage claims moderation is to stimulate investment in human capital and physical capital without stimulating hiring. Simulations with a wealth-maximising Nash bargaining surplus-sharing rule show that this condition is typically satisfied.

In general, if $\sigma < 1$, the KL line is upward-sloping rather than vertical, and the equilibrium is not pinned down solely in the labour market. Conditions in the capital markets have repercussions on long-run unemployment. If wages grow in line with productivity, a positive relation emerges. In this case, a productivity slowdown raises long-run unemployment. On the other hand, if wage claims are moderated by

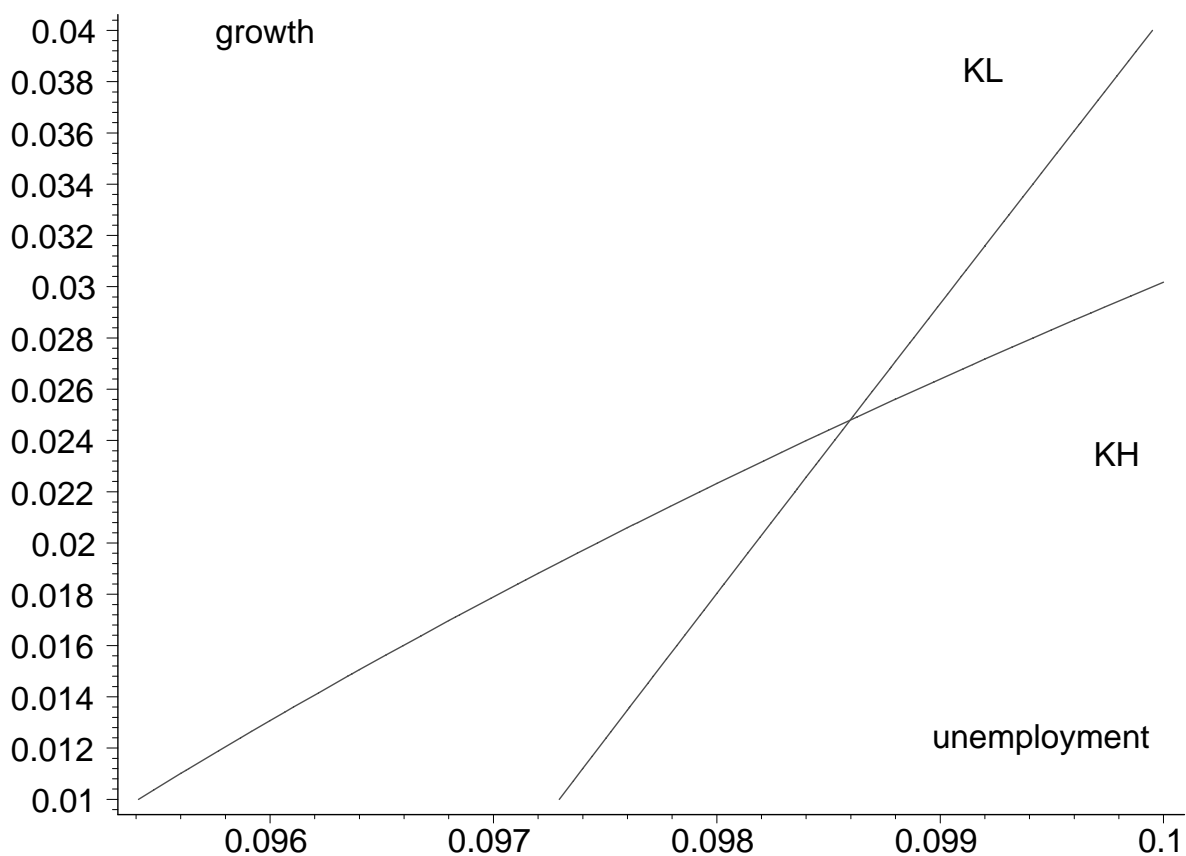


Fig. 4: **The growth-unemployment trade-off**

the incipient rise in unemployment, a negative shock to the productivity of human and physical capital can stimulate employment. Figure 3 and Figure 4 illustrate the more general case where $\sigma < 1$.

Empirical evidence suggests that, in the United States, real wages have grown in line with productivity throughout the period following the productivity slowdown⁶. There is no evidence of dramatic swings in wage claims for the United States, suggesting that the case depicted in Figure 3 may be relevant. The model suggests, in

⁶This is not quite true of those at the very bottom of the distribution, who have suffered a relative decline.

line with evidence, that the productivity slowdown of the 1970s would raise unemployment, while the productivity gains of the 1990s would reduce unemployment. The case of Europe is more subtle. Empirical evidence suggests that wages grew ahead of productivity for about a decade following the productivity slowdown, but have subsequently fallen, suggesting that the case depicted in Figure 3 may be relevant to Europe in the 1970s, while the case depicted in Figure 4 may be relevant to Europe in the 1980s. The model suggests, in line with evidence, that the productivity slowdown of the 1970s would raise unemployment, while the wage claims moderation of the 1980s would raise growth without reducing unemployment.

The timing of events is thus crucial. The model suggests that the moderation of wage claims apparent throughout the 1980s happened too late to contribute to a reduction of unemployment. The model also suggests that if wage claims are sensitive to the degree of tightness in the labour market, the productivity gains of the 1990s need not reduce unemployment, and may even raise unemployment. Broadly speaking, productivity gains make human capital investments more attractive relative to hiring. Firms choose to invest in the quality of their workers rather than raise the size of their workforce.

4 Conclusion

The paper analyses a two-sector model of endogenous growth with unemployment. The incentive to hire responds little (and possibly not at all) to changes in the rate of interest. Two effects are at work. A fall in the rate of interest makes hiring relatively more attractive than investing in physical capital. It also discourages saving, thereby reducing growth and, as the capitalised value of future profits falls, making firms less willing to hire. The incentive to hire responds much less to changes in the rate of interest than the incentive to invest in human capital. If wages grow in line with productivity, there is a positive relation between long-run growth and

unemployment. However, if wage moderation occurs too late, the fall in wages stimulates the accumulation of human capital without much effect on hiring. There emerges an apparent trade-off between growth and unemployment.

5 Appendix

5.1 Decentralised Equilibrium

Let the co-state variables associated with equations (7a), (7b) and (7c) be denoted q_t , p_t , and ν_t respectively. Dropping both the time index and the arguments for clarity, the Hamiltonian conditions necessary for an optimal equilibrium path are :

$$(-1 + q) I = 0 \tag{10a}$$

$$(F_1 - p G_1) K = 0 \tag{10b}$$

$$(F_2 - p G_2) HL = 0 \tag{10c}$$

$$(-\chi + h\nu) v = 0 \tag{10d}$$

$$\dot{q} = rq - \vartheta_K F_1 + \delta_K q - (1 - \vartheta_K) G_1 p \tag{10e}$$

$$\dot{p} = rp + wL - \vartheta_L F_2 L + \delta_H p - (1 - \vartheta_L) G_2 L p \tag{10f}$$

$$\dot{\nu} = r\nu - \vartheta_L F_2 H + wH - (1 - \vartheta_L) G_2 H p + \mu\nu, \tag{10g}$$

where complementary slackness is implicit. In addition, the transversality conditions are $\lim_{t \rightarrow \infty} e^{-rt} q_t K_t = 0$; $\lim_{t \rightarrow \infty} e^{-rt} p_t H_t = 0$; and $\lim_{t \rightarrow \infty} e^{-rt} \nu_t L_t = 0$.

The Hamiltonian and transversality conditions are necessary and sufficient conditions of the firm's problem if Mangasarian's sufficiency condition holds. The concavity assumptions on the production and the matching technologies, together with restrictions on wage costs (to be discussed below), ensure that Mangasarian's sufficiency condition is satisfied.

Homogeneity of degree 1 in the production technologies ensures that the marginal products depend only on the ratio of factor inputs $\frac{K_t}{H_t L_t}$. Likewise, homogeneity in the matching technology implies that the hiring rate and the job-finding rate depend only on the ratio of the search inputs $\theta_t = v_t/u_t$. This implies that the model can be solved in terms of the capital-labour ratio and market tightness, with the actual magnitudes determining the scale of the economy. This six-dimensional system may be reduced to a four-dimensional system with two predetermined variables (k_t, u_t)

and two jump variables (p_t, θ_t) . Adding a demand side to close the model introduces another jump variable, consumption C_t . Let c_t denote scaled consumption, $c_t \equiv C_t/H_t$. After these changes of variables, the transformed system is five-dimensional in $(p_t, \theta_t, u_t, k_t, c_t)$.

The optimality conditions are now used to characterise the dynamic equilibrium. Assuming the existence of an interior optimum, the expressions between brackets in equations (10a)-(10d) are set equal to zero. Conditions for existence are discussed later. Consider first equation (10a). Wherever it is binding, i.e. wherever $I_t > 0$, the co-state variable associated with the capital stock satisfies $q_t = 1$. Note that p_t is the co-state variable associated with human capital, and as such may be interpreted as the shadow price of human capital, in units of the consumption good⁷.

Differentiating equation (10a) with respect to time yields $\dot{q}_t = 0$. This information may be used to obtain an expression for the economy's rate of interest, after substituting equation (10b) into equation (10e),

$$r_t = F_1(\vartheta_{Kt}K_t, H_t\vartheta_{Lt}L_t) - \delta_K. \quad (11)$$

Recall that marginal products may be written in intensive form as in equations (3a), (3b), (3c), and (3d). Equations (10b) and (10c) may be re-written in terms of p , k_Y and k_Z (dropping the time index for clarity),

$$f'(k_Y) = p g'(k_Z) \quad (12a)$$

$$f(k_Y) - k_Y f'(k_Y) = p [g(k_Z) - k_Z g'(k_Z)]. \quad (12b)$$

Equations (12a) and (12b) state that the marginal products of physical capital and effective labour are equalised across sectors at each point in time. Assuming that there is no reversal in factor intensities (i.e. that the sign of $k_{Yt} - k_{Zt}$ does not change) and that both sectors are producing, equations (12a) and (12b) may be used to express the factor intensities and the marginal products in terms of p_t

⁷A flat-rate tax on the firm's profit or a subsidy to physical capital investment introduces a wedge between q_t and 1.

alone, $k_Y(p_t)$, $k_Z(p_t)$, $f'[k_Y(p_t)]$, $f[k_Y(p_t)] - k_Y(p_t)f'[k_Y(p_t)]$, $g'[k_Z(p_t)]$, $g[k_Z(p_t)] - k_Z(p_t)g'[k_Z(p_t)]$. Let $R(p_t) \equiv f'[k_Y(p_t)]$ denote the marginal product of capital and let $W(p_t) \equiv f[k_Y(p_t)] - k_Y(p_t)f'[k_Y(p_t)]$ denote the marginal product of labour, in units of consumption goods. The rate of interest may be written in terms of p_t : $r_t = R(p_t) - \delta_K$. Factor intensities in the Cobb-Douglas case are reported in section 5.5.

Imposing the condition that the firm's factors are fully employed in both sectors, the aggregate factor allocations can be expressed in terms of p_t and the capital-labour ratio k_t ,

$$\vartheta_K(p_t, k_t) = \frac{[k_t - k_Z(p_t)] k_Y(p_t)}{[k_Y(p_t) - k_Z(p_t)] k_t} \quad (13a)$$

$$\vartheta_L(p_t, k_t) = \frac{k_t - k_Z(p_t)}{k_Y(p_t) - k_Z(p_t)}. \quad (13b)$$

Equations (13a)-(13b) determine the allocation of resources across the sectors for given p_t and k_t . The output of each sector is given by

$$z(p_t, k_t) \equiv Z_t/H_t L_t = [1 - \vartheta_L(p_t, k_t)] g[k_Z(p_t)] \quad (14)$$

$$y(p_t, k_t) \equiv Y_t/H_t L_t = \vartheta_L(p_t, k_t) f[k_Y(p_t)]. \quad (15)$$

Equations (10b), (10c) and (11) may be substituted into equations (10f) and (10g). The following is obtained,

$$\dot{p}_t = [R(p_t) - \delta_K + \delta_H] p_t - [W(p_t) - w_t] L_t \quad (16)$$

$$\dot{\nu}_t = [R(p_t) - \delta_K + \mu] \nu_t - [W(p_t) - w_t] H_t. \quad (17)$$

The shadow price of human capital, p_t , has a simple intuitive interpretation. It is the marginal benefit to the firm of a unit investment in human capital. Equation (16) is an intertemporal no-arbitrage condition. It states the equality of factor returns net of capital gains. If the right hand side of the equation is positive the net return on physical capital exceeds that on human capital and there must therefore be a capital gain earned on human capital to offset the difference. The shadow price of

an additional unit of employment, ν_t , also has a simple interpretation. It is the marginal benefit to the firm of posting the marginal vacancy. Like equation (16), equation (17) is an intertemporal no-arbitrage condition. If the right hand side of the equation is positive the net return on physical capital exceeds that on employment and there must therefore be a capital gain earned on additional job creation to offset the difference. The two equations taken together are no-arbitrage conditions linking the accumulation of the three factors of production K_t , H_t , and L_t .

The quantity $R_t - \delta_K$ may be interpreted as the incentive to invest in physical capital; the quantity $(W_t - w_t)L_t/p_t - \delta_H$ may be interpreted as the incentive to invest in human capital; and the quantity $(W_t - w_t)H_t/\nu_t - \mu$ may be interpreted as the incentive to hire. The incentive to invest in human capital therefore depends on the level of employment, while the incentive to hire depends on the level of human capital. As growth is endogenously determined by the accumulation of physical and human capital, the level of employment can have an effect on growth via its effect on the incentive to invest in human capital. A change in the incentive to invest in human capital investment can have an impact on the level of employment. It remains to be seen whether these effects are present in the long-run or whether they are just part of the transitional dynamics.

Consider equation (10d) when it is binding,

$$h(\theta_t) \nu_t = \chi_t. \tag{18}$$

Equation (18) states that vacancies are advertised until the expected rent from job creation is bid down to zero, and implicitly determines market tightness θ_t as a function of the shadow price of employment ν_t . This condition may be used to express θ_t as a function of ν_t . However, it is more convenient to solve the model in terms of θ_t rather than ν_t . Differentiating equation (18) with respect to time, using equation (17), some straightforward manipulations yield a differential equation for

market tightness θ_t ,

$$-\epsilon_{\theta}^h(\theta_t) \frac{\dot{\theta}_t}{\theta_t} = R(p_t) - \delta_K + \mu - h(\theta_t) [W(p_t) - w_t] \frac{H_t}{\chi_t} - \frac{\dot{\chi}_t}{\chi_t}, \quad (19)$$

where $\epsilon_{\theta}^h(\theta_t) \equiv \theta_t h'(\theta_t)/h(\theta_t) < 0$ denotes the elasticity of the hiring rate $h(\theta_t)$ with respect to market tightness θ_t .

As it stands, equation (19) depends on H_t . A balanced growth equilibrium can exist only if χ_t is proportional to H_t in the long-run. Recall that $\chi_t = \bar{\chi} w_t H_t$. As a result, $\dot{\chi}_t/\chi_t$ in (19) depends on \dot{H}_t/H_t , which is given in equation (7b). Equation (7b) may be re-written in terms of p_t , u_t and k_t ,

$$\frac{\dot{H}_t}{H_t} = (1 - u_t) z(p_t, k_t) - \delta_H. \quad (20)$$

To simplify the exposition, let the wage depend on the marginal product of labour and on market tightness, and write⁸ $w_t = \Delta(\theta_t)W(p_t)$. Equation (19) becomes

$$-\left[\epsilon_{\theta}^h(\theta_t) - \epsilon_{\theta}^w(\theta_t)\right] \frac{\dot{\theta}_t}{\theta_t} = \underbrace{R(p_t) - \delta_K}_{[1] (+)} - \underbrace{\frac{h(\theta_t) [1 - \Delta(\theta_t)]}{\bar{\chi} \Delta(\theta_t)}}_{[2] (-)} + \mu - \underbrace{\frac{\dot{H}_t}{H_t}}_{[3] (-)} - \underbrace{\epsilon_p^w(p_t) \frac{\dot{p}_t}{p_t}}_{[4] (+ -)}, \quad (21)$$

where $\epsilon_{\theta}^w(\theta_t)$ denotes the elasticity of the wage with respect to market tightness θ_t , and where $\epsilon_p^w(p_t)$ denotes the elasticity of the wage with respect to the shadow price of human capital p_t . The sign of $\epsilon_p^w(p_t) = p_t W'(p_t)/W(p_t)$ is equal to the sign of $k_Y - k_Z$. The sign of \dot{p}_t/p_t depends on whether the shadow price of new human capital is rising or falling.

The dynamics of market tightness in equation (21) may be compared with equation (24) in Postel-Vinay (1998) from whom the following terminology is borrowed. The first term [1] may be labelled the *capital productivity effect*. This effect works through the complementarity of capital and effective labour. An increase in employment makes capital more productive and thus induces further hiring by firms.

⁸This simple relation holds under a wealth-maximising Nash bargain.

The capital productivity effect thus tends to make employment and tightness move together.

The second term [2] may be labelled the *tightness effect*. An increase in employment reduces unemployment, which, for a given number of vacancies, increases market tightness. An increase in market tightness works through several channels. On the one hand, it increases the average hiring cost, $\chi_t\theta_t$, thus reducing the return to vacancies. On the other hand, if $\Delta'(\theta_t) > 0$, it puts upward pressure on the wage, thus reducing the return to vacancies. The tightness effect thus tends to make employment and tightness move in opposite directions.

The third term [3] may be labelled the *cost of growth effect*. As can be seen from equation (20), an increase in employment stimulates investment in physical and human capital, thus raising vacancy costs and wage costs and reducing the return to vacancies. Like the tightness effect, the cost of growth effect tends to make employment and tightness move in opposite directions.

The fourth term [4] may be labelled the *labour productivity effect*. The direction of the effect depends on the direction of the relative price adjustment process, i.e. on the sign of \dot{p}_t/p_t . Two effects are at work. As can be seen from equation (16), an increase in employment reduces the marginal product of labour and thus lowers the incentive to invest in human capital. Moreover, at the same time the reduction in the marginal product of labour translates into lower vacancy costs. The fall in vacancy costs offsets the fall in the marginal product of labour. This offsetting effect arises because vacancy costs are proportional to the marginal product of labour.

Equation (21) involves several conflicting effects. To assess the relative importance of each effect requires making specific assumptions about wage costs, but also about the production and matching functions.

Consider the dynamics of the shadow price of human capital p_t . Substituting $w(p_t, \theta_t) = \Delta(\theta_t)W(p_t)$ and $u_t = 1 - L_t$ in (16) yields

$$\dot{p}_t = [R(p_t) - \delta_K + \delta_H] p_t - [1 - \Delta(\theta_t)] W(p_t) (1 - u_t). \quad (22)$$

If unemployment is ‘removed’ from the model, the dynamics of p_t are independent of θ_t and u_t ,

$$\dot{p}_t = [R(p_t) - \delta_K + \delta_H] p_t - W(p_t), \quad (23)$$

Equation (23) has the property that it is always stable if $k_Y > k_Z$ – the more relevant case empirically. If $k_Y < k_Z$, on the other hand, it is unstable, in which case p_t jumps immediately to its (unique) steady-state value. In the model with unemployment, the dynamics of p_t depend also on market tightness θ_t and unemployment, u_t . A complete characterisation of the dynamics is beyond the scope of this paper. To get a feel for the dynamics, suppose Δ is constant for all t , and suppose that the system is saddle-path stable. In this case, if $k_0 > k$, then p_t increases during the transition to the steady state. It follows that the labour productivity effect [4] is negative. On the other hand, if $k < k_0$, then p_t decreases during the transition to the steady state, and the labour productivity effect [4] is positive.

The demand side of the model is captured by a simple Euler equation for aggregate consumption, equation (1), which may be used together with equation (20) to obtain an equation of motion for scaled consumption, $c_t \equiv C_t/H_t$,

$$\frac{\dot{c}_t}{c_t} = \sigma[R(p_t) - \delta_K - \rho] - (1 - u_t) z(p_t, k_t) + \delta_H. \quad (24)$$

Recall that government consumption and taxes are left out of the model. The market equilibrium condition (8) may be combined with the equation for the accumulation of physical capital (7a),

$$\frac{\dot{K}_t}{K_t} = \frac{y(p_t, k_t)}{k_t} - \delta_K - \frac{c_t}{k_t(1 - u_t)} - \frac{\bar{\chi} \theta_t \Delta(\theta_t) W(p_t) u_t}{k_t(1 - u_t)}. \quad (25)$$

The dynamic equation for the evolution of the capital-labour ratio k_t may be obtained from $\dot{k}_t/k_t = \dot{K}_t/K_t - \dot{H}_t/H_t + \dot{u}_t/u_t$ together with equations (25), (20) and (4),

$$\begin{aligned} \dot{k}_t = & y(p_t, k_t) - \delta_K k_t - \frac{c_t}{1 - u_t} - \frac{\bar{\chi} \theta_t \Delta(\theta_t) W(p_t) u_t}{1 - u_t} - \frac{m(\theta_t) u_t k_t}{1 - u_t} + \mu k_t \\ & - (1 - u_t) z(p_t, k_t) k_t + \delta_H k_t. \end{aligned} \quad (26)$$

Definition 1 (Decentralised Equilibrium)

A decentralised equilibrium is a set of sequences for consumption, C_t , for gross investment, I_t , for vacancies, v_t (and thus for market tightness, θ_t), for the allocation of capital, ϑ_{Kt} , for the allocation of labour, ϑ_{Lt} , for the capital stock, K_t , for human capital, H_t , for total employment, L_t (and thus for unemployment, u_t), for the rate of interest, r_t (and thus for the marginal product of capital R_t), for the marginal product of labour W_t , and for the wage rate, w_t , that satisfy the dynamic constraint for physical capital accumulation (7a), the dynamic constraint for human capital accumulation (7b), the dynamic constraint for employment (7c), the Hamiltonian conditions for the firm's problem (10a)-(10g), the Euler equation (1), the market equilibrium condition (8), the transversality conditions, and the initial conditions.

An analytical study of the dynamics of the model is difficult given the high dimensionality of the system. Simulations for a calibrated version of the model show that saddle path stability typically obtains. However, convergence to the steady state need not be monotonic. If $\Delta(\theta_t)$ is strongly increasing, an increase in θ_t may induce a rise in p_t in equation (22). A rise in p_t , in turn, may induce a fall in θ_t in equation (21). A fall in θ_t , in turn, may induce a fall in p_t in equation (22). The resulting dynamics could be oscillating.

5.2 Steady State : Existence and Uniqueness**Definition 2 (Balanced Growth Path)**

A balanced growth path (or steady-state) is a decentralised equilibrium such that C_t , K_t , H_t are growing at the same constant rate and p_t , u_t , θ_t , R_t , W_t and w_t are constant.

Attention is restricted to interior steady states. To find a steady state where aggregate consumption C_t , physical capital K_t , and human capital H_t grow at the same constant rate, one looks for a stationary state of the transformed system where

$c_t = C_t/H_t$ and $k_t = K_t/H_tL_t$ are constant for all t . The stationary-state values of the system are denoted by simply omitting the time index. The stationary state of the system, denoted (p, θ, u, c, k) , is such that $\dot{p}_t = \dot{\theta}_t = \dot{u}_t = \dot{c}_t = \dot{k}_t = 0$. Let γ denote the common growth rate of the economy.

Maximal Growth Condition

The maximal attainable rate of growth γ_{max} satisfies $\rho > (1 - \frac{1}{\sigma})\gamma_{max}$.

The Maximal Growth Condition imposes an upper bound on the maximal growth rate and ensures that the transversality condition (2) holds.

Positive Growth Condition

The production technologies satisfy $R(p) - \delta_K > \rho$, where p is evaluated at the steady state.

The Positive Growth Condition is imposed to ensure that production technologies are sufficiently productive to yield positive growth at the shadow price consistent with balanced growth.

Setting $\dot{u}_t = 0$ in equation (4) yields the steady-state value of u as a function of θ ,

$$u = \frac{\mu}{\mu + m(\theta)}. \quad (27)$$

Equation (27) establishes that, for a given value of market tightness, there is a unique rate of unemployment associated with it. Note that if $\mu = 0$, the only level of employment consistent with balanced growth is full employment, $u = 0$. Let $u(\theta)$ denote the function defined in equation (27). It is easily established that if $\mu > 0$, then $u'(\theta) < 0$, $u''(\theta) > 0$, $\lim_{\theta \rightarrow 0} u(\theta) = 1$, and $\lim_{\theta \rightarrow \infty} u(\theta) = 0$. Equation (27) may be used to eliminate u from the system.

In equation (24), setting $\dot{c}_t = 0$ and imposing $c \neq 0$, then c drops out of the equation,

$$\sigma[R(p) - \delta_K - \rho] = (1 - u) z(p, k) - \delta_H. \quad (28)$$

Equation (28) equates the rate of growth of consumption with the rate of growth of human capital. Substituting equation (28) into equation (21), with $\dot{\theta}_t = 0$, yields equation KL in the text, where k has been eliminated. Substituting equation (27) into equation (16), with $\dot{p}_t = 0$, yields equation KH in the text.

In the remaining part of this section, sufficient conditions for existence and uniqueness of decentralised equilibria are derived. Introducing new notation will clarify the exposition. Let,

$$\begin{aligned}\tilde{R}(p) &\equiv [(1 - \sigma)[R(p) - \delta_K] + \mu + \sigma \rho] \bar{\chi} \\ \Omega(p) &\equiv \frac{R(p) - \delta_K + \delta_H}{W(p)/p}.\end{aligned}$$

Equations KL, and KH may be re-written,

$$h(\theta) [1 - \Delta(\theta)]/\Delta(\theta) = \tilde{R}(p) \tag{29}$$

$$[1 - \Delta(\theta)] [1 - u(\theta)] = \Omega(p) \tag{30}$$

Consider the right-hand side of each equation. The expressions on the right-hand side are monotonic in p . Monotonicity is a consequence of the properties of the ‘factor marginal returns frontier’ – a more accurate name for the ‘factor price frontier’ since the price of labour is not equal to its marginal product. It is well-known (as the Rybczynski theorem) that, with well-behaved production functions, the marginal product of physical capital and the marginal product of labour are monotonic and inversely related. The precise direction depends on the factor-intensity ranking. It is straightforward to establish the monotonicity of $\tilde{R}(p)$ and $\Omega(p)$, using the fact that they are monotonically related to $R(p)$. Some useful results are now summarised.

Lemma 1

$$\text{sign} [\tilde{R}'(p)] = \text{sign} (1 - \sigma) \text{sign}[R'(p)] = -\text{sign} (1 - \sigma)(k_Y - k_Z).$$

$$\text{sign} [W'(p)] = -\text{sign} [R'(p)] = \text{sign} (k_Y - k_Z).$$

$\Omega(p)$ is monotonic, irrespective of factor intensity ranking.

Proof. The proof is standard. See section 5.3. ■

It follows from Lemma 1 that the right-hand side of each of equations (29) and (30) is monotonic, irrespective of factor intensity ranking, and irrespective of whether the elasticity of intertemporal substitution σ is greater or smaller than 1.

Consider the left-hand side of each equation. It has been assumed that the hiring rate $h(\theta)$ is decreasing. It has been shown that $u(\theta)$ is non-increasing. The curvature of $\Delta(\theta)$ has been left unspecified. The following condition on $\Delta(\theta)$ guarantees that the equilibrium is unique.

Wage Claims Condition

The wedge Δ_t satisfies

- (i) either $\Delta_t = \Delta$, for all t ; or
- (ii) $\Delta'(\theta_t) > 0$; and
- (iii) $\Delta''(\theta_t) \geq 0$; and
- (iv)
$$\left(\frac{\frac{u'(\theta)}{1-u(\theta)} + \frac{\Delta'(\theta)}{1-\Delta(\theta)}}{\frac{-h'(\theta)}{h(\theta)} + \frac{\Delta'(\theta)}{\Delta(\theta)[1-\Delta(\theta)]}} \right) \left(\frac{\theta \Delta(\theta)}{\mu + m(\theta)} \right) < \frac{\Omega'(p)}{(1-\sigma)R'(p)},$$

for all $\theta_t \in [0, \psi)$, where ψ is defined implicitly as $\Delta(\psi) = 1$.

Intuitively, the Wage Claims Condition states that the wedge should not be too increasing (in a special sense) so that equations (29) and (30) have only one solution. The following set of results is immediate.

Lemma 2

If either (i) or (ii) in the Wage Claims Condition holds, equation (29) implicitly defines a monotonic relation between p and θ , irrespective of the factor intensity ranking.

If either (i) or (iii) in the Wage Claims Condition holds, Mangasarian's sufficiency condition holds.

If (iv) in the Wage Claims Condition holds, equation (30) implicitly defines a relation

between p and θ that is not generally monotonic. The relation is monotonic if, in addition, $\frac{\Delta'(\theta)}{1-\Delta(\theta)} < \frac{-u'(\theta)}{1-u(\theta)}$, for all $\theta \in (0, \psi)$, which holds if $\Delta'(\theta) = 0$, for all $\theta \in (0, \psi)$.

Proof. The proof follows from Lemma 1. It is straightforward to check that if $\Delta'(\theta_t) \geq 0$ and $\Delta''(\theta_t) \geq 0$, Mangasarian's sufficiency condition holds. Condition (iv) is obtained by restricting the determinant of the Jacobian matrix associated with equations (29) and (30) to strictly negative values. ■

The following proposition summarises the existence and uniqueness results.

Proposition 1 (Existence and Uniqueness)

If the conditions on Wage Claims, Profitability, Maximal Growth, and Positive Growth hold, then there exists a unique balanced growth equilibrium.

Proof. See section 5.4. ■

5.3 Proof of Lemma 1

Total differentiation of equations (12a) and (12b) yields

$$k'_Y(p) = \frac{Rk_Z + W}{(k_Z - k_Y)f''} \tag{31a}$$

$$k'_Z(p) = \frac{Rk_Y + W}{(k_Z - k_Y)pg''} \tag{31b}$$

$$R'(p) = f''k'_Y(p) \tag{31c}$$

$$W'(p) = -k_Y(p)g''k'_Y(p). \tag{31d}$$

Equations (31a)-(31d) can be used to sign $R'(p_t)$ and $W'(p_t)$, depending on the factor intensity ranking $k_Z(p_t) - k_Y(p_t)$.

5.4 Proof of Proposition 1

The Wage Claims Condition guarantees that there exists a unique pair (θ, p) that solves equations (29) and (30). This may be checked by totally differentiating the

equations and applying Lemma 1 and Lemma 2. Next, a unique u may be found from (27). The rest of the proof consists in showing existence and uniqueness of c and k , given p , θ and u , and checking that the transversality conditions hold. Checking the transversality conditions is trivial. The condition on Maximal Growth ensures that they are satisfied.

Once the steady-state values of p , θ and u have been found, the steady-state values of c and k follow. From equation (24), the capital-labour ratio k may be found,

$$z(p, k) = \frac{\sigma[R(p) - \delta_K - \rho]}{1 - u} + \frac{\delta_H}{1 - u} = (\gamma + \delta_H)/(1 - u) \quad (32)$$

Equation (32) may be inverted as follows. Substituting equation (14) into equation (13b) yields

$$z(p, k) = \frac{[k_Y(p) - k]g[k_Z(p)]}{k_Y(p) - k_Z(p)} \quad (33)$$

Combining equation (32) with (33) yields

$$k = k_Y(p) + \frac{(\gamma + \delta_H)[k_Z(p) - k_Y(p)]}{(1 - u) g[k_Z(p)]} \quad (34)$$

Equation (34) gives the capital-labour ratio k as a function of p and u . Thus, for given values of p and u , it is unique.

Lemma 3

If the Maximal Growth Condition holds, then for given values of p , θ , and u , there exists a unique value of k such that $\dot{k}_t = 0$.

Proof. The solution given in (34) is feasible if $\min\{k_Y, k_Z\} < k < \max\{k_Y, k_Z\}$, the condition that both sectors are active. Using equation (34) feasibility requires $0 < \gamma + \delta_H < (1 - u) g[k_Z(p)]$. However, $p g[k_Z(p)] = R(p) k_Z(p) + W(p)$ implies $g[k_Z(p)] > W(p)/p > [W(p) - w(p, \theta)]/p$. A sufficient condition for $(1 - u) g[k_Z(p)] - \delta_H > \gamma$ is $[W(p) - w(p, \theta)](1 - u)/p - \delta_H > \gamma$. Using equation (16) in the steady state, $R(p) - \delta_K + \delta_H = [W(p) - w(p, \theta)](1 - u)/p$, the sufficient condition may

be written $R(p) - \delta_K > \gamma$. Using equation (1), $R(p) - \delta_K - \rho = \gamma/\sigma$, implies $\rho > (1 - 1/\sigma)\gamma$. This feasibility condition is always satisfied for $\sigma < 1$. Imposing the Maximal Growth Condition implies $\rho > (1 - 1/\sigma)\gamma_{\max} > (1 - 1/\sigma)\gamma$. ■

Lemma 4

If the Maximal Growth Condition holds, then for given values of p , θ , u , and k , there exists a unique value of c such that $\dot{c}_t = 0$.

Proof. From equation (25), scaled consumption c may be obtained,

$$c = (1 - u) y(p, k) - (1 - u) (\delta_K + \gamma)k. \quad (35)$$

It follows that the solution, if feasible, is unique. Feasibility is now checked. Using $y + pz = R(p)k + W(p)$ and equation (32), equation (35) may be re-written $c/(1 - u) = [R(p) - \delta_K - \gamma]k + W(p) - p(\gamma + \delta_H)/(1 - u)$. Recall that it has been shown that, under the Maximal Growth Condition, $R(p) - \delta_K - \gamma > 0$. The second term on the right-hand side of the equality is positive if $(1 - u)W(p)/p - \delta_H > \gamma$. However, $(1 - u)W(p)/p > (1 - u)[W(p) - w(p, \theta)] = R(p) - \delta_K + \delta_H$, where the latter equality follows from equation (16). Next, equation (1) implies $R(p) - \delta_K + \delta_H = \gamma/\sigma + \rho + \delta_H$, and thus $W(p) - p(\gamma + \delta_H)/(1 - u) > 0$ if $\rho > (1 - 1/\sigma)\gamma$. The Maximal Growth Condition is sufficient for the latter inequality to hold. This ends the proof of existence and uniqueness of a decentralised equilibrium. ■

5.5 Cobb-Douglas Production Functions

$$Y(\vartheta_K K, H\vartheta_L L) = A(\vartheta_K K)^\alpha (H\vartheta_L L)^{1-\alpha}, \text{ where } \alpha \in (0, 1).$$

$$G[(1 - \vartheta_K)K, H(1 - \vartheta_L)L] = B[(1 - \vartheta_K)K]^\varphi [H(1 - \vartheta_L)L]^{1-\varphi}, \text{ where } \varphi \in (0, 1).$$

With Cobb-Douglas technologies, factor intensities are :

$$K_Y(p) = \left[\frac{B}{A}\right]^{1/(\alpha-\varphi)} \left[\frac{\varphi}{\alpha}\right]^{\varphi/(\alpha-\varphi)} \left[\frac{1-\varphi}{1-\alpha}\right]^{(1-\varphi)/(\alpha-\varphi)} p^{1/(\alpha-\varphi)}$$

$$K_Z(p) = \left[\frac{B}{A}\right]^{1/(\alpha-\varphi)} \left[\frac{\varphi}{\alpha}\right]^{\alpha/(\alpha-\varphi)} \left[\frac{1-\varphi}{1-\alpha}\right]^{(1-\alpha)/(\alpha-\varphi)} p^{1/(\alpha-\varphi)}$$

The marginal products are :

$$R(p) = A\alpha \left[\frac{A}{B} \right]^{(1-\alpha)/(\alpha-\varphi)} \left[\frac{\alpha}{\varphi} \right]^{\varphi(1-\alpha)/(\alpha-\varphi)} \left[\frac{1-\alpha}{1-\varphi} \right]^{(1-\varphi)(1-\alpha)/(\alpha-\varphi)} (1/p)^{(1-\alpha)/(\alpha-\varphi)}$$

$$W(p) = A(1-\alpha) \left[\frac{B}{A} \right]^{\alpha/(\alpha-\varphi)} \left[\frac{\varphi}{\alpha} \right]^{\alpha\varphi/(\alpha-\varphi)} \left[\frac{1-\varphi}{1-\alpha} \right]^{\alpha(1-\varphi)/(\alpha-\varphi)} p^{\alpha/(\alpha-\varphi)}$$

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