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Equal Sums of Two kth Powers

T.D. Browning[1]

Mathematical Institute, 24-29, St. Giles, Oxford, OX1 3LB, United Kingdom
E-mail: browning@maths.ox.ac.uk

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Let $k \geqslant 5$ be an integer, and let $x \geqslant 1$ be an arbitrary real number. We derive a bound

$$O_{\varepsilon,k}(x^{2/3k+\varepsilon} + x^{3/k\sqrt{k}+2/k(k-1)+\varepsilon}),$$

for the number of positive integers less than or equal to x which can be represented as a sum of two non-negative coprime kth powers, in essentially more than one way.
Key Words: equal sums; like powers; rational points; upper bound.

1. INTRODUCTION

For integer $k \geqslant 5$, the purpose of this paper is to describe in some quantitative way the paucity of non-trivial rational solutions to the familiar Diophantine equation $x_1^k + x_2^k = x_3^k + x_4^k$. In fact it has long been supposed that there are no non-trivial integer solutions for $k \geqslant 5$. Writing

$$F(\mathbf{x}) = x_1^k + x_2^k - x_3^k - x_4^k, \tag{1}$$

we see that $F = 0$ defines a non-singular projective surface, and by exploiting its intrinsic geometry we shall ultimately glean arithmetic information concerning the number of representations of an integer as a sum of two kth powers. We shall actually consider the slightly more general form

$$F_{\alpha,\beta}(\mathbf{x}) = \alpha x_1^k + \beta x_2^k - \alpha x_3^k - \beta x_4^k, \tag{2}$$

for non-zero $\alpha, \beta \in \mathbb{Z}$ and integer $k \geqslant 5$, so that (1) is just $F = F_{1,1}$. In the equation $F_{\alpha,\beta} = 0$ we may certainly suppose without loss of generality that $\alpha > 0$ and that α, β are both k-free and coprime. Furthermore, when k is

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odd, it will suffice to assume that $\beta > 0$ and that $F_{\alpha,\beta}$ takes the shape

$$F_{\alpha,\beta}(\mathbf{x}) = \alpha x_1^k + \beta x_2^k + \alpha x_3^k + \beta x_4^k. \tag{3}$$

Although there is actually little extra work needed to consider the generalised form $F_{\alpha,\beta}$, the situation as it stands is rather disappointing whenever $F_{\alpha,\beta} \neq F$.

Let $B \geqslant 1$, and suppose that $f \in \mathbb{Z}[x_1, x_2, x_3, x_4]$ is a non-zero form of degree k. In order to be in a position to state our main result, we define the counting function

$$\mathcal{N}(f; B) = \#\{\mathbf{x} \in \mathbb{Z}^4 : \ \mathbf{x} \text{ primitive}, \ |\mathbf{x}| \leqslant B, \ f(\mathbf{x}) = 0\},$$

where $|\mathbf{x}|$ denotes the Euclidean length of the vector $\mathbf{x}$, and such a vector is said to be primitive if the greatest common divisor of its vector components is 1. It is not hard to see that whenever $f = F_{\alpha,\beta}$ is given by (2), the trivial solution $x_1 = x_3, x_2 = x_4$ shows that we have $\mathcal{N}(F_{\alpha,\beta}; B) \gg B^2$. Since it has been recently shown by Heath-Brown [6, Theorem 9] that

$$\mathcal{N}(f; B) \ll_{\varepsilon,k} B^{2+\varepsilon} \tag{4}$$

for any absolutely irreducible form f of degree $k \geqslant 2$ (though this estimate is elementary in the case $f = F_{\alpha,\beta}$), it makes sense to define $\mathcal{N}_1(f; B)$ to be the number of vectors counted by $\mathcal{N}(f; B)$, excluding any that lie on lines in the surface $f = 0$. Thus, $\mathcal{N}_1(f; B)$ can be thought of as counting the non-trivial rational points on the surface. The goal of this paper is to exploit the intrinsic geometry of the surface $F_{\alpha,\beta} = 0$, in order to improve upon the general bound

$$\mathcal{N}_1(f; B) \ll_{\varepsilon,k} B^{1+\varepsilon} + B^{3/\sqrt{k}+2/(k-1)+\varepsilon} \tag{5}$$

due to Heath-Brown [6, Theorem 11], which is valid for any given $\varepsilon > 0$ and non-singular form f of degree $k \geqslant 2$. If $\mathcal{N}_d(f; B)$ denotes the number of vectors counted by $\mathcal{N}(f; B)$, excluding any that lie on curves of degree $\leqslant d$ contained in the surface, then for $\varepsilon > 0$ and non-singular f as above, the same result states that

$$\mathcal{N}_{k-2}(f; B) \ll_{\varepsilon,k} B^{3/\sqrt{k}+2/(k-1)+\varepsilon}. \tag{6}$$

The backbone of our main result—when applied in conjunction with Heath-Brown's bound (6)—is the following description of the rational plane sections of the surface $F_{\alpha,\beta} = 0$.

THEOREM 1. *Let $k \geqslant 5$ be such that $k - 1$ is not divisible by 3. Then there is no rational plane section of the surface $F_{\alpha,\beta} = 0$ which contains an irreducible* *curve of degree* $\delta > 1$, *where* $\delta < k-2$ *for k even and* $\delta < k-3$ *for k odd. In particular, no rational plane section of the surface contains an irreducible quadric curve for* $k \geqslant 6$.

When $\alpha = \beta = 1$, *so that* $F_{\alpha,\beta} = F$, *the previous statements hold for any* $k \geqslant 5$.

The proof of Theorem 1 will be presented in Sections 3 and 4, according to the parity of k, and we implicitly assume throughout the proof of the theorem that $k \geqslant 5$ is such that $k-1$ is not divisible by 3, unless $\alpha = \beta = 1$. Here, and in all that follows, irreducible is taken to mean irreducible over $\mathbb{Q}$ unless otherwise stated.

Our primary result is the following, and will be established in Section 5.

THEOREM 2. *Let* $\alpha, \beta \in \mathbb{Z}$ *be non-zero and fix* $B \geqslant 1$. *Then for any* $\varepsilon > 0$ *and integer* $k \geqslant 5$ *such that* $k-1$ *is not divisible by* 3, *we have*

$$\mathcal{N}_1(F_{\alpha,\beta}; B) \ll_{\varepsilon,k} B^{2/3+\varepsilon} + B^{3/\sqrt{k}+2/(k-1)+\varepsilon}.$$

Moreover, the bound holds for every $k \geqslant 5$ *in the case* $F_{\alpha,\beta} = F$. *In particular, we have*

$$\mathcal{N}_1(F; B) \ll_{\varepsilon,k} B^{2/3+\varepsilon}$$

for every $k \geqslant 27$.

One particular arithmetic consequence of this result—when applied to $F_{\alpha,\beta} = F$—is an estimate for $v_k(x)$, where for arbitrary $x \geqslant 1$ and integer $k \geqslant 3$, we define $v_k(x)$ to be the number of positive integers not exceeding x which are expressible as the sum of two non-negative coprime kth powers, in essentially more than one way. There is a rich body of work dedicated to estimating the quantity $v_k(x)$, and for $k \geqslant 5$ such propositions provide theoretical evidence for the deep conjecture $v_k(x) = 0$ mentioned above. Applying Theorem 2 with $B = x^{1/k}$, it is easy to deduce the following result.

THEOREM 3. *Let* $k \geqslant 5$ *be an integer, and fix* $x \geqslant 1$. *Then for any* $\varepsilon > 0$ *we have*

$$v_k(x) \ll_{\varepsilon,k} x^{2/3k+\varepsilon} + x^{\eta_k+\varepsilon},$$

where $\eta_k = 3/k\sqrt{k} + 2/k(k-1)$. *In particular, we have*

$$v_k(x) \ll_{\varepsilon,k} x^{2/3k+\varepsilon}$$

for every $k \geqslant 27$.

It is interesting to place our result in the context of recent work done in this area. The first estimate to beat the trivial bound $\nu_k(x) \ll_{\varepsilon,k} x^{2/k+\varepsilon}$ arising from (4) for general $k \geqslant 3$, was that of Greaves [4]. Using the two-dimensional large sieve inequality, he developed a flexible argument leading to the bound

$$O_{\varepsilon,k}(x^{11/6k+\varepsilon})$$

for $\nu_k(x)$. Completing earlier work on the subject, Hooley dealt separately with the cases of k having odd and even parity, in an impressive pair of papers [7, 8], respectively. Developing a delicate sieving technique, and a modified method for estimating certain exponential sums, he arrives at the estimate

$$O_{\varepsilon,k}(x^{5/3k+\varepsilon})$$

for $\nu_k(x)$. In an important paper, whose ideas have been successfully extended to further paucity problems in diagonal Diophantine systems, Skinner and Wooley [10] achieve the bound $O_{\varepsilon,k}(x^{3/2k+\delta_k+\varepsilon})$ for $\nu_k(x)$; where

$$\delta_k = \begin{cases} 1/k^2 & \text{if } k = 3, 5, \\ 1/k(k-1) & \text{otherwise.} \end{cases} \tag{7}$$

This latter result so far represented the first means of attacking the general problem without recourse to sieving, but has since been beaten by the second such method—Heath-Brown's bound (5) above—which yields

$$\nu_k(x) \ll_{\varepsilon,k} x^{1/k+\varepsilon} + x^{\eta_k+\varepsilon}, \tag{8}$$

for $k \geqslant 3$ and η_k as in the statement of Theorem 3. Although no longer very important, the author has recently discovered a means by which Skinner and Wooley could have rearranged their existing argument in order to take $\delta_k = 1/k^2$ for all values of $k \geqslant 3$. Estimate (8) is currently the best available for the cases $6 \leqslant k \leqslant 12$, but we see that our Theorem 3 surpasses this bound for $k \geqslant 13$ and becomes the foremost result available for such cases. In the low degree situation $k \leqslant 5$, Hooley's results are surpassed only in the case $k = 3$. Here, a bound $\nu_3(x) \ll_\varepsilon x^{4/9+\varepsilon}$ is provided by Heath-Brown [5], obtained as a corollary to work done on the density of rational points on cubic surfaces containing 3 rational, coplanar lines. The author [2] has provided a conditional treatment of the bound $\nu_4(x) \ll_\varepsilon x^{2/5+\varepsilon}$; this is superior to Hooley's result, but depends upon a certain hypothesis concerning the size of the rank of an elliptic curve.

An important point in this context is that apart from Theorem 3, each of the preceding estimates for $\nu_k(x)$ remain valid when applied to the quantity

$\nu_k^*(x)$, defined to be the total number of positive integers not exceeding x which are expressible as the sum of two non-negative kth powers in essentially more than one way. It is clear that

$$\nu_k^*(x) \leqslant \sum_{d \leqslant x^{1/k}} \mathcal{N}_1(F; x^{1/k} d^{-1}),$$

so that for $k \geqslant 27$ Theorem 2 implies that $\nu_k^*(x) \ll_{\varepsilon,k} x^{2/(3k)+\varepsilon} \sum_d d^{-2/3}$, and we therefore no longer improve upon the bound afforded by (8), since $\sum_d d^{-2/3} \geqslant x^{1/(3k)}$.

Bennett *et al.* [1] have also studied the surface $F_{\alpha,\beta} = 0$, though their approach is far more arithmetic in nature. Although they achieve the bound $\mathcal{N}_1(F_{\alpha,\beta}; B) \ll_{\varepsilon,k} B^{3/2+k\delta_k+\varepsilon}$, where δ_k is given by (7), their main concern is with an estimate for the total number of integers with absolute value not exceeding $X \geqslant 1$ which are represented by the binary additive form $\alpha x^k + \beta y^k$, in essentially more than one way. In particular this requires a careful analysis of the possible sizes of the components of each integer zero $\mathbf{x}$ of (2) or (3). Although it would be possible to deduce an estimate of this type from Theorem 2 by following the work of Bennett *et al.* [1, Sect. 6], we content ourselves here simply with a sharper bound for $\mathcal{N}_1(F_{\alpha,\beta}; B)$.

2. PRELIMINARY LEMMATA

Our primary tool for counting points on the possible curves $C \subset \mathbb{P}^3$ of varying degree contained in the surface $F_{\alpha,\beta} = 0$, will be the following result due to Heath-Brown [6, Theorem 5].

LEMMA 1. *Let $X \geqslant 1$, and suppose that $C \subset \mathbb{P}^3$ is an irreducible curve of degree d. Then C has $O_{\varepsilon,d}(X^{2/d+\varepsilon})$ primitive points $\mathbf{x} \in \mathbb{Z}^4$, in the cube $|x_i| \leqslant X$.*

The following elementary result will prove valuable in demonstrating the irreducibility of a certain type of curve.

LEMMA 2. *Let $n \geqslant 2$. The ternary form $G(\mathbf{X}) = X_1^n + f(X_2, X_3)$, for some binary form f of degree n, is irreducible over $\bar{\mathbb{Q}}$ unless f is square-full in $\bar{\mathbb{Q}}[X_2, X_3]$.*

Proof. Factorise f as a product of irreducibles in the unique factorisation domain $L = \bar{\mathbb{Q}}[X_2, X_3]$, $f = \prod_{i=1}^r g_i^{e_i}$, and then apply Eisenstein's criterion to the polynomial $X_1^n + f \in L[X_1]$. Thus G is irreducible in $L[X_1]$ unless $e_i \geqslant 2$ for every $1 \leqslant i \leqslant r$. ∎

In order to handle the possibility of reducible plane curves the following four results concerning cyclotomic numbers will prove vital. The fourth is proved using methods originating in an argument produced by Dummigan [10, Lemma 2.4], and it is in Lemma 5 that Theorem 2's restriction that 3 must not divide $k-1$ can be traced back to, for the general case $F_{\alpha,\beta} \neq F$. We begin by stating a result concerning the norm of a certain cyclotomic integer, which ought to be well known. Let K_n denote the cyclotomic field of nth roots of unity.

LEMMA 3. *Let ζ_k be a primitive kth root of unity, and denote by $N = N_{K_k/\mathbb{Q}}$ the norm map. Then*

$$N(1-\zeta_k) = \begin{cases} p, & k = p^e,\ e \geqslant 1, \\ 1 & \textit{otherwise}. \end{cases}$$

Proof. We may assume that k is a prime power, since for $k > 1$ divisible by at least two distinct primes, $1-\zeta_k$ is a unit in K_k [3, Theorem 45]. For any $m \geqslant 1$, we have

$$x^m - 1 = \prod_{d|m} \Phi_d(x),$$

where $\Phi_d(x) = \prod_{(j,d)=1}(x-\zeta_d^j) \in \mathbb{Z}[x]$ is the dth cyclotomic polynomial. Applying Möbius inversion we obtain

$$\Phi_d(x) = \prod_{d|m} (x^d - 1)^{\mu(m/d)},$$

so that in particular

$$\Phi_{p^e}(x) = \frac{x^{p^e}-1}{x^{p^{e-1}}-1} = \sum_{i=0}^{p-1} x^{ip^{e-1}}.$$

But then if $k = p^e$ is a prime power, we have $N(1-\zeta_k) = \Phi_k(1) = p$, as required. ∎

LEMMA 4. *Let $n \geqslant 4$ be an even integer, and let η_1, η_2, η_3 be nth roots of unity such that*

$$1 + \eta_1 + \eta_2 + \eta_3 = 0.$$

Then in fact $1 + \eta_i = 0$, for some $1 \leqslant i \leqslant 3$.

Proof. Viewing the nth roots of unity η_1, η_2, η_3 geometrically, we deduce that $|1+\eta_1| = |\eta_2+\eta_3|$; so that the angle between 1 and η_1 is equal to the angle between η_2 and η_3. But the sums are opposite in direction, and so it follows that each component in the pair $(1, \eta_1)$ has to be opposite a number in the pair (η_2, η_3); unless $1+\eta_1 = \eta_2+\eta_3 = 0$. That is, we must have $1+\eta_i = 0$ for some $1 \leqslant i \leqslant 3$. ∎

LEMMA 5. *Let $n \geqslant 3$ be an integer, and let η_1, η_2 be any two nth roots of ± 1 such that $\eta_i \neq 1$, for $i = 1, 2$. Suppose that*

$$q = \left(\frac{1-\eta_1}{1-\eta_2}\right)^n \in \mathbb{Q}^*.$$

Then either $q = \pm 1$, or

$$\pm q^{\pm 1} \in \{2^n, 2^{n/2}, (2/3)^{n/2}, (4/3)^{n/2}, 3^{n/2}\},$$

and η_1, η_2 belong to the set of twelfth roots of unity.

Specifically, if q is a perfect $(n+1)$th power, or if n is not divisible by 3, then we must have $q = \pm 1$.

Proof. In particular η_1 and η_2 will be $(2n)$th roots of unity, and so we work over K_{2n}, the cyclotomic field of $(2n)$th roots of unity. Furthermore, let r_1 and r_2 denote the exact multiplicative orders of η_1 and η_2, respectively, so that η_i will be a primitive r_ith root of unity for $i = 1, 2$. We recall that no cyclotomic field contains an irrational mth root of an integer for $m > 2$ (see [9, Lemma 2], for example). Thus if $q \in \mathbb{Q}^*$, we must have $q = r^n$ or $q = r^{n/2}$ for some $r \in \mathbb{Q}^*$, where we write $r = a/b$ for coprime $a, b \in \mathbb{Z}$. We first consider the possibility

$$\left(\frac{a}{b}\right)^n = \left(\frac{1-\eta_1}{1-\eta_2}\right)^n, \tag{9}$$

and take the norm $N = N_{K_{2n}/\mathbb{Q}}$ of it

$$\left(\frac{a}{b}\right)^{\phi(2n)} = \frac{N(1-\eta_1)}{N(1-\eta_2)}.$$

Let

$$m_i = N(1-\eta_i) = N_{K_{2n}/K_{r_i}}(N_{K_{r_i}/\mathbb{Q}}(1-\eta_i)) \in \mathbb{Z},$$

for $i = 1, 2$, and observe that since a and b are coprime we have $a^{\phi(2n)} \mid m_1$ and $b^{\phi(2n)} \mid m_2$. By Lemma 3, we deduce that $a = \pm 1$ unless $r_1 = p^e$ for $e \geqslant 1$, say, and $m_1 = p^{\phi(2n)/\phi(r_1)}$. But then $a^{\phi(r_1)} \mid p$, so that we must have $\phi(r_1) = 1$, whence either $a = \pm 1$ or $a = \pm 2$ and $\eta_1 = -1$. One argues similarly for the condition $b^{\phi(2n)} \mid m_2$, and then ultimately concludes that either $q = \pm 1$ as required, or a possibility equivalent to $a/b = \pm 2$ and $\eta_1 = -1$ comes to pass. But then substitution into (9) reveals that $|1 - \eta_2| = 1$, so that η_2 must in fact be a primitive sixth root of unity.

We now turn to the second case and consider the identity

$$\left(\frac{a}{b}\right)^n = \left(\frac{1-\eta_1}{1-\eta_2}\right)^{2n}, \tag{10}$$

which via exactly the same application of Lemma 3 leads to the condition that $a^{\phi(r_1)} \mid p^2$ for $r_1 = p^e$ a prime power, and similarly for b. If $a^{\phi(r_1)} = \pm p$ then we must have $\phi(r_1) = 1$, which can only happen when $r_1 = 2$, since r_1 is assumed to be a prime power. But then $a = \pm 2$ and $\eta_1 = -1$. If $a^{\phi(r_1)} = \pm p^2$ then we must have $\phi(r_1) = 1$ or 2, which can only happen when $r_1 = 2, 3$ or 4, again since r_1 is a prime power. We must therefore have either $a = \pm 4, \pm 3$ or ± 2, respectively, if $a^{\phi(r_1)} = \pm p^2$. Furthermore, the first case corresponds to $\eta_1 = -1$, the second case to η_1 being a primitive cube root of unity, and the third case to η_1 being a primitive fourth root of unity. Proceeding similarly for the condition $b^{\phi(r_2)} \mid p^2$, for $r_2 = p^e$ a further prime power, we use the fact that a and b are coprime to deduce that either $a/b = \pm 1$, or

$$\pm(a/b)^{\pm 1} \in \{\tfrac{2}{3}, \tfrac{4}{3}, 2, 3, 4\}.$$

Moreover, the second possibility can only occur if η_1 and η_2 are both either sixth or twelfth roots of unity. Indeed, integer values of $\pm(a/b)^{\pm 1}$ correspond to a specific value of η_1 or η_2, and then the value of η_2 or η_1 is found by simply substituting the known values of a, b and η_1 or η_2 into (10), and taking the modulus of it.

We therefore conclude the proof of Lemma 5, since it is patent that if $q = (a/b)^{n+1}$ for coprime $a, b \in \mathbb{Z}$, then in fact $a/b = \pm 1$, as required. Moreover, K_{2n} can only contain the complete set of sixth or twelfth roots of unity if 6 divides $2n$. ∎

LEMMA 6. *Let $n \geqslant 3$ be an integer, and let η_1, η_2 be any two nth roots of ± 1. Let x, y be non-zero integers such that*

$$y^{n+1}(1 + \eta_1 + \eta_2)^n = \pm x^{n+1}.$$

Then in fact $x = \pm y$.

Proof. With the norm $N = N_{K_{2n}/\mathbb{Q}}$ as in the proof of Lemma 5, we deduce that

$$N(1+\eta_1+\eta_2)^n = \left(\frac{x}{y}\right)^{(n+1)\phi(2n)}.$$

It is clear that $|N(1+\eta_1+\eta_2)| \leqslant 3^{\phi(2n)}$, and that since $N(1+\eta_1+\eta_2) \in \mathbb{Z}$, we can write x/y uniquely as a product of integral prime factors. For any $r(p) > 0$, where $r(p)$ is defined to be the p-adic order of x/y, we may deduce from the fact that $p^{r(p)} \mid N(1+\eta_1+\eta_2)$, that

$$p^{r(p)(n+1)\phi(2n)} \leqslant 3^{n\phi(2n)} < 3^{(n+1)\phi(2n)};$$

whence $p^{r(p)} < 3$. Thus the only possibility is $p = 2$ and $r(2) = 1$, which together with the equation in the statement of Lemma 6, leads to the simplification

$$\{(1+\eta_1+\eta_2)/2\}^n = \pm 2.$$

This implies that K_{2n} contains the nth root of ± 2, which is impossible as stated in the proof of Lemma 5. Thus the statement of Lemma 6 is true for all $n \geqslant 3$. ∎

3. PROOF OF THEOREM 1 FOR THE CASE k EVEN

For $k \geqslant 6$, we begin by rewriting the equation of our surface (2) as

$$F_{\alpha,\beta}(\mathbf{x}) = \alpha x_1^k + \delta\beta x_2^k - \alpha x_3^k - \delta\beta x_4^k, \tag{11}$$

where now $\alpha, \beta > 0$ and δ is 1 or -1 according as our initial β is positive or negative, respectively. Naturally, α, β remain both k-free and coprime. For primitive non-zero $\mathbf{y} \in \mathbb{Z}^4$, let us consider the plane section $\mathbf{x} \cdot \mathbf{y} = 0$, of the surface $F_{\alpha,\beta}(\mathbf{x}) = 0$. In order to get an explicit equation for the curves which are contained in the plane, we shall use the plane equation to substitute for one of the x_i into (2). Indeed, suppose without loss of generality that $y_4 \neq 0$, and let $L(x_1, x_2, x_3) = y_1x_1 + y_2x_2 + y_3x_3$. Then for each integer vector $\mathbf{y}$ we are led to consider the curve

$$\mathscr{C}_{\mathbf{y}}\colon\ G_{\mathbf{y}}(\mathbf{x}) = y_4^k(\alpha x_1^k + \delta\beta x_2^k - \alpha x_3^k) - \delta\beta L(\mathbf{x})^k = 0. \tag{12}$$

In the analysis of these curves it clearly suffices to assume that $y_i \geqslant 0$ for all $1 \leqslant i \leqslant 4$. Furthermore, we shall only be interested in those vectors $\mathbf{y}$ for which $\mathscr{C}_{\mathbf{y}}$ is reducible; and so a singular curve. Thus let $\mathfrak{A}$ denote the set of primitive non-zero vectors $\mathbf{y} \in \mathbb{Z}^4$ for which $\mathscr{C}_{\mathbf{y}}$ is singular. Furthermore, let $\mathfrak{A}'$ denote the subset of $\mathfrak{A}$ consisting of those vectors with each component being non-zero. It should be clear from the identity $G_{\mathbf{y}}(\mathbf{x}) = F_{\alpha,\beta}(y_4 x_1, y_4 x_2, y_4 x_3, -L(\mathbf{x}))$, that points lying on rational lines in the curve $\mathscr{C}_{\mathbf{y}}$ will correspond to points lying on rational lines in the surface $F_{\alpha,\beta} = 0$, and so ultimately contribute nothing to $\mathscr{N}_1(B)$.

In order to get a satisfactory handle on the set of vectors $\mathfrak{A}$, we shall derive an explicit condition for the curve $\mathscr{C}_{\mathbf{y}}$ to be singular. Indeed, $\mathscr{C}_{\mathbf{y}}$ has a singular point at P, say, if and only if $\partial G_{\mathbf{y}}/\partial x_i = 0$ at P, for $i = 1, 2, 3$. But by (12) this is so if and only if there exists a triple (η_1, η_2, η_3) of $(k-1)$th roots of ± 1, where $\eta_1^{k-1} = \delta$, $\eta_2^{k-1} = 1$ and $\eta_3^{k-1} = -\delta$, such that

$$y_4^{k/(k-1)} x_2 = \eta_2 y_2^{1/(k-1)} L(\mathbf{x}) \tag{13}$$

and

$$\alpha^{1/(k-1)} y_4^{k/(k-1)} x_i = \eta_i \beta^{1/(k-1)} y_i^{1/(k-1)} L(\mathbf{x}) \tag{14}$$

at P, for $i = 1$ and 3. Clearly, $L(\mathbf{x}) \neq 0$ at P, since otherwise the fact that $y_4 \neq 0$ would imply that $\mathbf{x} = \mathbf{0}$; which is impossible. Hence we deduce from (13), (14) and the equation for $L(\mathbf{x})$, that

$$1 = \eta_1 \{\beta/\alpha (y_1/y_4)^k\}^{1/(k-1)} + \eta_2 (y_2/y_4)^{k/(k-1)} + \eta_3 \{\beta/\alpha (y_3/y_4)^k\}^{1/(k-1)}.$$

Therefore, $\mathscr{C}_{\mathbf{y}}$ has a singular point at P if and only if there exists a triple of numbers $\Omega = (\omega_2, \omega_3, \omega_4)$, where each ω_i is a $(k-1)$th root of $\delta, -1$ or $-\delta$ according as $i = 2, 3$ or 4, such that $\hat{F}_\Omega(\mathbf{y}) = 0$ where

$$\hat{F}_\Omega(\mathbf{y}) = (\beta y_1^k)^{1/(k-1)} + \omega_2 (\alpha y_2^k)^{1/(k-1)} + \omega_3 (\beta y_3^k)^{1/(k-1)} + \omega_4 (\alpha y_4^k)^{1/(k-1)}, \tag{15}$$

and so if and only if

$$\hat{F}_{\alpha,\beta}(\mathbf{y}) = \prod_{\Omega} \hat{F}_\Omega(\mathbf{y}) = 0,$$

where the product is taken over all of the possible triples Ω. $\hat{F}_{\alpha,\beta}$ is an absolutely irreducible form with rational coefficients and degree at least 2, and is known as the dual of $F_{\alpha,\beta}$. Here it should be specified that we are taking positive $(k-1)$th roots of α, β and y_i^k for $1 \leqslant i \leqslant 4$.

It will be convenient to split our considerations according to precisely how many components of the vector $\mathbf{y}$ are equal to zero. Since $y_4 \neq 0$, it is immediate from (15) that curves corresponding to $y_1 = y_2 = y_3 = 0$ must be absolutely irreducible. Thus the remaining subsections each handle the possibility of precisely two, one or none of the components of $\mathbf{y}$ being equal to zero.

3.1. *Exactly Two Zero Components of* $\mathbf{y}$. We remark that whenever two of the components of the vector $\mathbf{y}$ are zero, a rather trivial consideration of (15) and the primitivity of $\mathbf{y}$ reveals that $\mathscr{C}_{\mathbf{y}}$ is absolutely irreducible unless one of the following three possibilities comes to pass:

1. We have $\alpha = \beta = 1$, $\delta = -1$ and $\mathbf{y}_0 = (0, 0, 1, 1)$.
2. We have $\mathbf{y}_1 = (0, 1, 0, 1)$.
3. We have $\alpha = \beta = 1$, $\delta = +1$ and $\mathbf{y}_2 = (1, 0, 0, 1)$.

Here we have used the fact that $y_4 \neq 0$ in order to conclude that there are precisely three ways in which the vector $\mathbf{y}$ has precisely two zero components. Then in order to deduce Case 1 for example, we consider the possibility $y_1 = y_2 = 0$ and infer from (15) that $\delta\beta y_3^k = -\alpha y_4^k$, whence $\alpha = y_3^k$ and $\beta = -\delta y_4^k$ since $(y_3, y_4) = (\alpha, \beta) = 1$. But then it follows that in fact $\alpha = \beta = 1$ and $\delta = -1$ since α, β are k-free and both strictly positive. Cases 2 and 3 are handled similarly.

Thus we must consider the nature of the curves $\mathscr{C}_{\mathbf{y}}$ for $\mathbf{y} \in \{\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2\}$; which are defined by the forms

$$G_{\mathbf{y}_0} = x_1^k - x_2^k, \qquad G_{\mathbf{y}_1} = \alpha(x_1^k - x_3^k) \qquad \text{and} \qquad G_{\mathbf{y}_2} = x_2^k - x_3^k,$$

respectively. Although the binary form $X^k - Y^k$ will in general have irreducible quadratic factors for arbitrary values of even k, it can easily be seen that points lying on these curves actually lie on lines contained in the surface $F_{\alpha,\beta} = 0$, and so contribute nothing to $\mathscr{N}(F_{\alpha,\beta}; B)$. Indeed, this is plain from the elementary identity

$$(x + y)^k - (x - y)^k = 2xy \sum_{0 \leqslant r < k/2} \binom{k}{2r + 1} x^{k-2r-2} y^{2r} = 2xy S_k(x, y),$$

say, where $S_k(x, y) = 0$ for $x, y \in \mathbb{R}$ if and only if $x = y = 0$.

3.2. *Exactly One Zero Component of* $\mathbf{y}$. We now turn to the case in which precisely one of the components of $\mathbf{y}$ is equal to zero. First, consider the case $y_1 = 0$, $y_2 y_3 y_4 \neq 0$, where our necessary condition (15) for the curve $\mathscr{C}_{\mathbf{y}}$ not to be absolutely irreducible becomes

$$(\alpha y_2^k)^{1/(k-1)} + \omega_3 (\beta y_3^k)^{1/(k-1)} + \omega_4 (\alpha y_4^k)^{1/(k-1)} = 0, \tag{16}$$

for ω_3 a $(k - 1)$th root of $-\delta$ and ω_4 a $(k - 1)$th root of -1. The immediate goal of this section is to show that if the curve $\mathscr{C}_{\mathbf{y}}$ is not absolutely irreducible then the vector $\mathbf{y}$ must satisfy a second distinct equation

$$(\alpha y_2^k)^{1/(k-1)} + \omega_3' (\beta y_3^k)^{1/(k-1)} + \omega_4' (\alpha y_4^k)^{1/(k-1)} = 0, \tag{17}$$

where ω_3' and ω_4' are two further $(k-1)$th roots of $-\delta$ and -1, respectively, such that we cannot simultaneously have $\omega_3 = \omega_3'$ and $\omega_4 = \omega_4'$.

We begin by noting that in this situation we may use (12) to write

$$\alpha^{k-1} G_{\mathbf{y}}(\mathbf{x}) = (\alpha y_4 x_1)^k + f(x_2, x_3),$$

where

$$f(x_2, x_3) = \alpha^{k-1}(y_4^k \delta\beta x_2^k - y_4^k \alpha x_3^k - \delta\beta L(\mathbf{x})^k),$$

and $L(\mathbf{x}) = y_2 x_2 + y_3 x_3$. It then follows from an application of Lemma 2 that $G_{\mathbf{y}}$ is absolutely irreducible unless there exists a factorisation into irreducibles $f = \prod_{i=1}^r g_i^{e_i}$ over $\bar{\mathbb{Q}}[x_2, x_3]$, with $e_i \geqslant 2$ for $1 \leqslant i \leqslant r$. But then the binary form f must be highly singular, in that ∇f is zero when evaluated at any given root $\mathbf{x}_0 = (0, x_2, x_3)$ of any given factor g_i of f. Thus we have Eqs. (13) and (14), for $i = 3$, holding at any such $\mathbf{x}_0$; where we must have $x_2 x_3 L(\mathbf{x}) \neq 0$ at $\mathbf{x}_0$, and η_2, η_3 as defined there. But we may then deduce from the equation for $L(\mathbf{x})$, in the same way that (15) was deduced above, that an equation of the form (16) holds, for ω_3, ω_4 as above. We have therefore shown a correspondence between distinct roots $\mathbf{x}_0 = (0, x_2, x_3)$ of any given factor g_i of f, and distinct equations of form (16). Thus whenever $G_{\mathbf{y}}$ is not absolutely irreducible, we conclude that either there are at least two distinct equations of form (16), or f must be a perfect kth power of some linear form $l \in \bar{\mathbb{Q}}[x_2, x_3]$. It remains to eliminate this last possibility in order to conclude the first half of this section. But then if $\mathbf{x}_1$ is the unique zero of $l(x_2, x_3)$, we have

$$\begin{aligned}\frac{\partial^2 f}{\partial x_2 \partial x_3} &= k(k-1)\delta\beta y_2 y_3 L(\mathbf{x})^{k-2} \\ &= k l^{k-2}\left\{(k-1)\frac{\partial l}{\partial x_2}\frac{\partial l}{\partial x_3} + l\frac{\partial^2 l}{\partial x_2 \partial x_3}\right\} \\ &= 0\end{aligned}$$

at $\mathbf{x}_1$, since $k > 2$; whence $L(\mathbf{x}_1) = 0$, which contradicts our previous remarks.

Therefore, if $G_{\mathbf{y}}$ is not absolutely irreducible, we have that $\mathbf{y}$ satisfies at least two distinct Eqs. (16) and (17), which we may subtract in order to deduce that

$$\beta^{1/(k-1)} y_3^{k/(k-1)}(\omega_3 - \omega_3') = \alpha^{1/(k-1)} y_4^{k/(k-1)}(\omega_4' - \omega_4),$$

whence

$$\frac{\beta}{\alpha}\left(\frac{y_3}{y_4}\right)^k = \delta\left(\frac{1-\eta_1}{1-\eta_2}\right)^{k-1}, \tag{18}$$

where η_1 and η_2 are both $(k-1)$th roots of unity, disjoint from 1. A simple application of Lemma 5 reveals that

$$\frac{\beta}{\alpha}\left(\frac{y_3}{y_4}\right)^k = \pm 1,$$

since either $3 \nmid (k-1)$ or $\alpha = \beta = 1$. But then $y_3 = y_4$ and $\alpha = \beta = 1$, since $\alpha, \beta, y_i > 0$ for $i = 2, 3, 4$, and α, β are both k-free and coprime. Substitution into (16) yields

$$y_2^k = \delta y_3^k (1+\eta)^{k-1},$$

for η some $(k-1)$th root of δ. A simplified version of Lemma 6 then enables us to deduce that $y_2 = y_3$, and hence we are left with the case $\mathbf{y} = (0, 1, 1, 1)$, by the primitivity of the vectors $\mathbf{y}$. It follows from (16) that the relation $1 + \omega_3 + \omega_4 = 0$ must hold, where $\omega_3^{k-1} = -\delta$ and $\omega_4^{k-1} = -1$. Since then $1 = |1 + \omega_3| = |1 + \omega_4| = |\omega_3 + \omega_4|$, we easily infer that the set $\{1, \omega_3, \omega_4\}$ is equal to the complete set of cube roots of unity. This is impossible, since if $3 | (k-1)$ then the $(k-1)$th power of each cube root of unity would be 1; and if $3 \nmid (k-1)$ then the set of $2(k-1)$th roots of unity does not contain the cube roots anyway. Thus $\mathscr{C}_{\mathbf{y}}$ must be absolutely irreducible. The cases in which $y_i = 0$ for exactly one of $i = 2$ or $i = 3$ follow similarly, because in either case at least one ω_i will be a $(k-1)$th root of -1 in the equation corresponding to (16).

We conclude therefore that every rational plane section of surface (11), in which precisely one component of the defining vector is equal to zero, produces an absolutely irreducible plane curve of degree k.

3.3. *The General Case* $y_1 y_2 y_3 y_4 \neq 0$. The following simple result will prove useful in dealing with the various possible singular points of $\mathscr{C}_{\mathbf{y}}$, in the general case $y_1 y_2 y_3 y_4 \neq 0$. Recall that $\mathfrak{A}'$ is the set of all such vectors $\mathbf{y}$ which lead to singular forms $G_{\mathbf{y}}$.

LEMMA 7. *Let* $\mathbf{y} \in \mathfrak{A}'$ *and* $k \geqslant 6$ *be even, and suppose that* $\mathscr{C}_{\mathbf{y}}$ *has a singular point at* P. *Then*

$$P = [(\beta y_1)^{1/(k-1)}, \omega_2 (\alpha y_2)^{1/(k-1)}, \omega_3 (\beta y_3)^{1/(k-1)}] \in \mathbb{P}^2(\bar{\mathbb{Q}}), \qquad (19)$$

for some pair (ω_2, ω_3), *where* $\omega_2^{k-1} = \delta$ *and* $\omega_3^{k-1} = -1$. *Furthermore, distinct singular points of* $\mathscr{C}_{\mathbf{y}}$ *correspond to distinct factors* $\hat{F}_{\Omega}$ *of the dual form* $\hat{F}_{\alpha,\beta}$.

Proof. Since no component of $\mathbf{y}$ is equal to zero, we may assume by (13) and (14) that $L(\mathbf{x})x_1x_2x_3 \neq 0$ at P; and so conclude that

$$\left(\frac{x_1}{x_2}\right)^{k-1} = \delta\frac{\beta y_1}{\alpha y_2}, \quad \left(\frac{x_1}{x_3}\right)^{k-1} = -\frac{y_1}{y_3}, \quad \left(\frac{x_2}{x_3}\right)^{k-1} = -\delta\frac{\alpha y_2}{\beta y_3} \tag{20}$$

at P. Writing

$$x_1^{k-1} = \delta\beta y_1 X_1, \quad x_2^{k-1} = \alpha y_2 X_2, \quad x_3^{k-1} = -\delta\beta y_3 X_3,$$

for $X_1, X_2, X_3 \in \mathbb{C}^*$, ratios (20) imply that $X_1 = X_2 = X_3$ at P; and so any singular points P occurring on the curve $\mathscr{C}_\mathbf{y}$ do indeed have the given shape.

It is now clear that distinct singular points of $\mathscr{C}_\mathbf{y}$ correspond bijectively to distinct pairs (ω_2, ω_3). Furthermore, substitution of the point P (as given in the statement of the lemma) into $G_\mathbf{y} = 0$ reveals that the distinct pairs (ω_2, ω_3) each correspond to a distinct factor $\hat{F}_\Omega$ of the dual form $\hat{F}_{\alpha,\beta}$; where $\Omega = (\omega_2, \omega_3, \omega_4)$ is the triple achieved by adjoining ω_4, a $(k-1)$th root of $-\delta$, to the pair (ω_2, ω_3). ∎

In order to classify exactly the possible factorisations of the family of forms $G_\mathbf{y}$ for $\mathbf{y} \in \mathfrak{A}'$, and when each of the various possibilities arises, the following result will prove instrumental.

Lemma 8. *For even $k \geqslant 6$, if the curve $\mathscr{C}_\mathbf{y}$ is not absolutely irreducible, then it has at least $k-1$ distinct singular points.*

Proof. Suppose that $G_\mathbf{y}$ factors non-trivially as $G_\mathbf{y} = G_1G_2$, say. If $\mathscr{C}_1, \mathscr{C}_2$ denote the projective curves defined by $G_1 = 0$ and $G_2 = 0$, respectively, then each intersection point of $\mathscr{C}_1$ and $\mathscr{C}_2$ will produce a singular point in $\mathscr{C}_\mathbf{y}$. However, since $\deg(G_1)\deg(G_2) \geqslant \deg(G_\mathbf{y}) - 1$, Bézout's Theorem implies that $\mathscr{C}_1$ and $\mathscr{C}_2$ intersect in at least $k-1$ points in the projective plane; when counted according to multiplicity. The result will therefore be immediate upon demonstrating that all the intersection points of $\mathscr{C}_1$ and $\mathscr{C}_2$ have intersection multiplicity one.

Suppose that the curves $\mathscr{C}_1$ and $\mathscr{C}_2$ intersect at a point P, say. If P has intersection multiplicity strictly greater than one, then without loss of generality we have

$$\left(\frac{\partial G_1}{\partial x_1}, \frac{\partial G_1}{\partial x_2}, \frac{\partial G_1}{\partial x_3}\right) = \lambda\left(\frac{\partial G_2}{\partial x_1}, \frac{\partial G_2}{\partial x_2}, \frac{\partial G_2}{\partial x_3}\right)$$

at P, for some $\lambda \in \mathbb{C}$. Making use of the fact that $G_1 = G_2 = 0$ at P, we easily establish the relations

$$\frac{\partial^2 G_{\mathbf{y}}}{\partial x_i \partial x_j} = 2\lambda \left(\frac{\partial G_2}{\partial x_i}\right)\left(\frac{\partial G_2}{\partial x_j}\right),$$

at P, for $1 \leqslant i \leqslant j \leqslant 3$; whence

$$\left(\frac{\partial^2 G_{\mathbf{y}}}{\partial x_i \partial x_j}\right)^2 = \left(\frac{\partial^2 G_{\mathbf{y}}}{\partial x_i^2}\right)\left(\frac{\partial^2 G_{\mathbf{y}}}{\partial x_j^2}\right), \tag{21}$$

at P, for $1 \leqslant i \leqslant j \leqslant 3$. But a simple calculation establishes the formulae

$$\frac{\partial^2 G_{\mathbf{y}}}{\partial x_1^2} = k(k-1)\{\alpha y_4^k x_1^{k-2} - \delta\beta y_1^2 L(\mathbf{x})^{k-2}\}, \tag{22}$$

$$\frac{\partial^2 G_{\mathbf{y}}}{\partial x_2^2} = \delta\beta k(k-1)\{y_4^k x_2^{k-2} - y_2^2 L(\mathbf{x})^{k-2}\}, \tag{23}$$

$$\frac{\partial^2 G_{\mathbf{y}}}{\partial x_3^2} = -k(k-1)\{\alpha y_4^k x_3^{k-2} + \delta\beta y_3^2 L(\mathbf{x})^{k-2}\} \tag{24}$$

and

$$\frac{\partial^2 G_{\mathbf{y}}}{\partial x_i \partial x_j} = -k(k-1)\delta\beta y_i y_j L(\mathbf{x})^{k-2}, \tag{25}$$

for $1 \leqslant i < j \leqslant 3$. Hence, taken together with the fact that $\prod_{i=1}^4 y_i \neq 0$, Eqs. (21)–(25) imply that

$$\alpha y_4^k (x_1 x_2)^{k-2} = \{\delta\beta y_1^2 x_2^{k-2} + \alpha y_2^2 x_1^{k-2}\} L(\mathbf{x})^{k-2},$$

$$\alpha y_4^k (x_1 x_3)^{k-2} = \{\delta\beta y_1^2 x_3^{k-2} - \delta\beta y_3^2 x_1^{k-2}\} L(\mathbf{x})^{k-2},$$

$$\alpha y_4^k (x_2 x_3)^{k-2} = \{\alpha y_2^2 x_3^{k-2} - \delta\beta y_3^2 x_2^{k-2}\} L(\mathbf{x})^{k-2},$$

at P. It is clear that as an intersection point of $\mathscr{C}_1$ and $\mathscr{C}_2$, P must also be a singular point of $\mathscr{C}_{\mathbf{y}}$, and so we must have $L(\mathbf{x}) x_1 x_2 x_3 \neq 0$ at P, as noted in the proof of Lemma 7 above. Thus we may take ratios of the previous system of equations in order to conclude that

$$\left(\frac{x_1}{x_2}\right)^{k-2} = \delta \frac{\beta y_1^2}{\alpha y_2^2}, \qquad \left(\frac{x_1}{x_3}\right)^{k-2} = -\frac{y_1^2}{y_3^2}, \qquad \left(\frac{x_2}{x_3}\right)^{k-2} = -\delta \frac{\alpha y_2^2}{\beta y_3^2} \tag{26}$$

at P. For example, the first two equations yield

$$x_2^{k-2}(\delta\beta y_1^2 x_3^{k-2} - \delta\beta y_3^2 x_1^{k-2}) = x_3^{k-2}(\delta\beta y_1^2 x_2^{k-2} + \alpha y_2^2 x_1^{k-2})$$

at P, whence the last equality in (26). It is now straightforward to ascertain a contradiction from (20) and (26); thus we must have the impossible inequality

$$0 < (y_1/y_3)^{k-2} = (x_1/x_3)^{(k-2)(k-1)} = -(y_1/y_3)^{2(k-1)} < 0.$$

This completes the proof of Lemma 8. ∎

The remainder of this section will be spent demonstrating the proof of the following result, which categorises exactly the possible factorisations of the forms $G_{\mathbf{y}}$, and completes the proof of Theorem 1 in the case of even $k \geqslant 6$ such that $3 \nmid (k-1)$ whenever $F_{\alpha,\beta} \neq F$.

Proposition 1. *Let $k \geqslant 6$ be an even integer such that $3 \nmid (k-1)$, whenever $F_{\alpha,\beta} \neq F$. Then for any vector $\mathbf{y} \in \mathfrak{A}'$, the form $G_{\mathbf{y}}$ is absolutely irreducible unless $\mathbf{y} \in \mathfrak{B}_1$, where $\mathfrak{B}_1$ is the set of integer vectors $\mathbf{y} \in \mathfrak{A}'$ which satisfy at least one of the following three conditions*:

1. *We have $y_1 = y_3$ and $y_2 = y_4$.*
2. *We have $y_1 = y_4$, $y_2 = y_3$ and $\alpha = \beta = 1$.*
3. *We have $y_1 = y_2$, $y_3 = y_4$, $\alpha = \beta = 1$ and $\delta = -1$.*

Moreover, for $\mathbf{y} \in \mathfrak{B}_1$ the corresponding form $G_{\mathbf{y}}$ is either the product of a linear factor and an absolutely irreducible factor of degree $k-1$; or if $\alpha = \beta = 1$ and $\mathbf{y} = (1,1,1,1)$, it is the product of two linear factors and an absolutely irreducible factor of degree $k-2$.

Together, Lemmata 7 and 8 show that whenever $\mathscr{C}_{\mathbf{y}}$ is not absolutely irreducible there are at least $k-1$ distinct pairs (ω_2, ω_3) such that Eq. (15) holds for some further number ω_4. We claim that there exist at least two equations $\hat{F}_\Omega = 0$ and $\hat{F}_{\Omega'} = 0$, say, which share the same value of either ω_2, ω_3 or ω_4. Indeed, our claim is only false when the $k-1$ equations all have different values of ω_2. Similarly, all is well unless the equations each have different values of ω_3; and unless there are $k-1$ different values of ω_4. Since $k-1$ is odd, we know in this case that $\delta\omega_2, -\omega_3$ and $-\delta\omega_4$ are all $(k-1)$th roots of unity. Hence we have $k-1$ equations $\hat{F}_\Omega = 0$, no two of which share the same value of $\delta\omega_2, -\omega_3$ or $-\delta\omega_4$. Adding all of these equations together clearly results in the identity $(k-1)y_1^{k/(k-1)} = 0$; contradicting the fact that $y_1 \neq 0$.

We now proceed to show how the claim can be used to deduce that for even k, and $\mathscr{C}_{\mathbf{y}}$ singular but not absolutely irreducible, we must have that one of the following three cases comes to pass:

$$\alpha = \beta = 1 \text{ and } y_3 = y_4, \quad y_2 = y_4, \quad \text{or} \quad \alpha = \beta = 1 \text{ and } y_2 = y_3. \tag{27}$$

Suppose that $\omega_2 = \omega_2'$ in our two equations $\hat{F}_\Omega = 0$ and $\hat{F}_{\Omega'} = 0$, and subtract them from each other to obtain

$$(\omega_3 - \omega_3')\beta^{1/(k-1)}y_3^{k/(k-1)} + (\omega_4 - \omega_4')\alpha^{1/(k-1)}y_4^{k/(k-1)} = 0,$$

where we clearly cannot have $\omega_3 = \omega_3'$ or $\omega_4 = \omega_4'$. Then, since $\omega_i^{k-1} = (\omega_i')^{k-1}$ for $i = 3, 4$, there exist $(k-1)$th roots of unity $\eta_1, \eta_2 \neq 1$ such that

$$\frac{\alpha}{\beta}\left(\frac{y_4}{y_3}\right)^k = \pm\left(\frac{1-\eta_1}{1-\eta_2}\right)^{k-1}.$$

An application of Lemma 5 yields $\alpha/\beta(y_4/y_3)^k = \pm 1$ since either $3 \nmid (k-1)$ or $\alpha = \beta = 1$, whence in fact $\alpha = \beta = 1$ and $y_3 = y_4$, since α, β are k-free, coprime and $\alpha, \beta, y_i > 0$ for $1 \leqslant i \leqslant 4$. The cases in which $\omega_3 = \omega_3'$ or $\omega_4 = \omega_4'$ both proceed in exactly the same manner; leading to the second and third conclusions, respectively, in (27).

We now note that precisely the same argument can be applied to the $\geqslant k-1$ distinct singular points of the form

$$P = [\eta_1(\beta y_1)^{1/(k-1)}, \eta_2(\alpha y_2)^{1/(k-1)}, (\beta y_3)^{1/(k-1)}] \in \mathbb{P}^2(\bar{\mathbb{Q}}), \tag{28}$$

for pairs (η_1, η_2) where $\eta_1^{k-1} = -1$ and $\eta_2^{k-1} = -\delta$; or equally to the distinct singular points of the form

$$P = [\zeta_1(\beta y_1)^{1/(k-1)}, (\alpha y_2)^{1/(k-1)}, \zeta_3(\beta y_3)^{1/(k-1)}] \in \mathbb{P}^2(\bar{\mathbb{Q}}), \tag{29}$$

for pairs (ζ_1, ζ_3) where $\zeta_1^{k-1} = \delta$ and $\zeta_3^{k-1} = -\delta$. Here, the distinct singular points correspond to equations of type (15), with ω_4 replaced by some further number η_4 or ζ_4, respectively. Thus applying the same argument as above to the singular points of form (28) will yield the possibilities

$$y_2 = y_4, \quad \alpha = \beta = 1 \text{ and } y_1 = y_4 \quad \text{or} \quad \alpha = \beta = 1 \text{ and } y_1 = y_2, \tag{30}$$

and once applied to those singular points of form (29), it yields the possibilities

$$\alpha = \beta = 1 \text{ and } y_3 = y_4, \quad \alpha = \beta = 1 \text{ and } y_1 = y_4 \quad \text{or} \quad y_1 = y_3. \tag{31}$$

Collecting these three conditions (27), (30) and (31) together, we deduce that either $\mathbf{y} \in \mathfrak{B}_1$, or $y_1 = y_2$, $y_3 = y_4$, $\alpha = \beta = 1$ and $\delta = +1$, or 3 of the components of $\mathbf{y}$ are equal and $\alpha = \beta = 1$. In the second case, we easily use (15) to show that for positive coprime integers A, B and $\delta = +1$, the corresponding form $G_{\mathbf{y}}(\mathbf{x}) = B^k(x_1^k + x_2^k - x_3^k) - (Ax_1 + Ax_2 + Bx_3)^k$ is absolutely irreducible unless $\mathbf{y} = (1,1,1,1)$. Indeed, the form $G_{\mathbf{y}}$ will be singular only if $\mathbf{y} = (A, A, B, B)$ satisfies the equation

$$\left(\frac{A}{B}\right)^k = \left(\frac{1-\eta_1}{1-\eta_2}\right)^{k-1},$$

where η_1, η_2 are both $(k-1)$th roots of -1. In particular we have $\eta_i \neq 1$ for $i = 1, 2$ and so an application of Lemma 5 yields $A = B = 1$, by the primitivity of $\mathbf{y}$. In order to complete the proof of the first part of Proposition 1, we note that whenever $\alpha = \beta = 1$ and three of the components of $\mathbf{y}$ are equal, we will obtain an equation of the form handled in Lemma 6. Indeed, suppose that $y_1 = y_2 = y_3$, and substitute into $\hat{F}_{\Omega}(\mathbf{y}) = 0$ to obtain

$$y_1^k(1 + \eta_1 + \eta_2)^{k-1} = y_4^k$$

for some $(k-1)$th roots of ± 1, η_1 and η_2. Since $y_i > 0$ for $1 \leqslant i \leqslant 4$, we deduce from Lemma 6 that $y_1 = y_4$, and so in fact $\mathbf{y} = (1,1,1,1) \in \mathfrak{B}_1$.

It remains to investigate the nature of the curves $\mathscr{C}_{\mathbf{y}}$ for $\mathbf{y} \in \mathfrak{B}_1$, and we begin by considering the case $\mathbf{y} \neq (1,1,1,1)$. Suppose first that $\mathbf{y} = (A, B, A, B)$, for coprime $A, B \in \mathbb{N}$ not both equal to 1. Then $G_{\mathbf{y}}$ has the shape

$$G_{\mathbf{y}}(\mathbf{x}) = B^k(\alpha x_1^k + \delta\beta x_2^k - \alpha x_3^k) - \delta\beta(Ax_1 + Bx_2 + Ax_3)^k. \tag{32}$$

Immediately, we note that $G_{\mathbf{y}}$ has the linear factor $(x_1 + x_3)$, and so there exists a degree $k-1$ form H, say, such that

$$G_{\mathbf{y}}(\mathbf{x}) = (x_1 + x_3)H(\mathbf{x}). \tag{33}$$

We hope to show that H is absolutely irreducible; in fact we shall demonstrate that it is non-singular. Any singular point P of H will be a singular point of $G_{\mathbf{y}}$, so that we may use (15) in order to deduce that

$$(\beta A^k)^{1/(k-1)}(1 + \omega_3) = -(\alpha B^k)^{1/(k-1)}(\omega_2 + \omega_4)$$

for some triple $(\omega_2, \omega_3, \omega_4)$ of $(k-1)$th roots of δ, -1 and $-\delta$, respectively. If $1 + \omega_3$ were not equal to 0, we could go on to derive

$$\frac{\beta}{\alpha}\left(\frac{A}{B}\right)^k = -\delta\left(\frac{1-\eta_1}{1-\eta_2}\right)^{k-1},$$

for $(k-1)$th roots of unity η_1, η_2; so that $\alpha = \beta = A = B = 1$, by the usual application of Lemma 5 and the fact that $(\alpha, \beta) = (A, B) = 1$. Hence we may assume that $1 + \omega_3 = 0$ and Lemma 7 implies that our singular point P satisfies $x_1 + x_3 = 0$ at P. We therefore deduce from (33) that

$$\frac{\partial^2 G_{\mathbf{y}}}{\partial x_1 \partial x_2} = \frac{\partial H}{\partial x_2},$$

at P. By (25), it is not hard to see that for P to be a singular point of H, with $x_1 + x_3 = 0$ at P, then we must have $x_2 = 0$ at P; this contradicts the statement of Lemma 7. Hence H is indeed non-singular.

The case in which $\mathbf{y} = (A, B, B, A)$ and $\alpha = \beta = 1$, for coprime $A, B \in \mathbb{N}$ not both equal to 1, is entirely similar. It suffices to set $\alpha = \beta = 1$ and interchange x_1 and x_2, and also A and B, in the preceding argument. Similarly, in order to treat the remaining case $\mathbf{y} = (A, A, B, B)$, for coprime $A, B \in \mathbb{N}$ both not equal to 1, we set $\alpha = \beta = 1$ and $\delta = -1$, and simply interchange x_2 and x_3 in the argument corresponding to form (32).

We complete the proof of Proposition 1 by considering the special case $\alpha = \beta = 1$ and $\mathbf{y} = (1, 1, 1, 1)$. It is not hard to spot the linear factor $x_1 + x_3$ of $G_{\mathbf{y}}$, but it can be seen that further obvious linear factors will depend upon the sign of δ. Thus if $\delta = +1$, there exists a degree $k-2$ form J_+ such that

$$G_{\mathbf{y}}(\mathbf{x}) = (x_1 + x_3)(x_2 + x_3)J_+(\mathbf{x}). \tag{34}$$

Applying Lemma 4 to the corresponding dual equation $\hat{F}_\Omega(\mathbf{y}) = 1 + \omega_2 + \omega_3 + \omega_4 = 0$, where $\omega_2, \omega_3, \omega_4$ are all $2(k-1)$th roots of unity, we deduce that either $1 + \omega_3 = 0$ or $\omega_2 + \omega_3 = 0$. Indeed, the case $1 + \omega_2 = 0$ cannot come to pass since $\omega_2^{k-1} = \delta = 1$. Thus in order for P to be a singular point of J_+ (and hence of $G_{\mathbf{y}}$), Lemma 7 implies that we must have $x_1 + x_3 = 0$ or $x_2 + x_3 = 0$ at P. Suppose that $x_1 + x_3 = 0$ at P. Then, since $J_+ = \partial J_+/\partial x_i = 0$ at P for $1 \leqslant i \leqslant 3$, we deduce from (34) that $\partial^2 G_{\mathbf{y}}/\partial x_1 \partial x_2 = 0$ at P; whence $L(\mathbf{x}) = 0$ at P by (25), which contradicts the opening line of the proof of Lemma 7. Similarly, if $x_2 + x_3 = 0$ at P we deduce that $\partial^2 G_{\mathbf{y}}/\partial x_1 \partial x_3 = 0$ at P, which again provides the necessary contradiction. Hence J_+ is indeed absolutely irreducible. If $\delta = -1$, Eq. (34) is replaced by the factorisation

$$G_{\mathbf{y}}(\mathbf{x}) = (x_1 + x_2)(x_1 + x_3)J_-(\mathbf{x}),$$

where J_- is a degree $k-2$ form. Furthermore, J_- is easily shown to be absolutely irreducible via an application of Lemma 7, in precisely the same way that J_+ was shown to be so above. This therefore completes the proof of Proposition 1.

4. PROOF OF THEOREM 1 FOR THE CASE k ODD

Let $k \geqslant 5$ be odd, and let $\mathbf{y} \in \mathbb{Z}^4$ be a non-zero primitive vector. Using the form for $F_{\alpha,\beta}(\mathbf{x})$ given by (3), much of the preceding section becomes notationally less encumbered. Thus, for primitive non-zero $\mathbf{y} \in \mathbb{Z}^4$ we consider the plane section $\mathbf{x} \cdot \mathbf{y} = 0$ of the surface $F_{\alpha,\beta}(\mathbf{x}) = 0$, and again assuming without loss of generality that $y_4 \neq 0$ we attain the curve

$$\mathscr{C}_{\mathbf{y}}\colon\ G_{\mathbf{y}}(\mathbf{x}) = y_4^k(\alpha x_1^k + \beta x_2^k + \alpha x_3^k) - \beta L(\mathbf{x})^k = 0$$

where $L(x_1, x_2, x_3) = y_1x_1 + y_2x_2 + y_3x_3$. We can obviously no longer use this equation for $\mathscr{C}_{\mathbf{y}}$ in order to assume that $y_i \geqslant 0$ for $1 \leqslant i \leqslant 4$, though it is clear that points lying on rational lines in $\mathscr{C}_{\mathbf{y}}$ still contribute nothing to $\mathscr{N}_1(B)$. In much the same way as before, it is straightforward to get a satisfactory handle on $\mathfrak{A}$ and $\mathfrak{A}'$, defined to be the same set of vectors $\mathbf{y}$ as defined at the start of the treatment of the even k case. Indeed, $\mathscr{C}_{\mathbf{y}}$ has a singular point at P say, if and only if there exists a triple of $(k-1)$th roots of unity $\Omega = (\omega_2, \omega_3, \omega_4)$ such that $\hat{F}_\Omega(\mathbf{y}) = 0$, where $\hat{F}_\Omega(\mathbf{y})$ is a factor of the dual form $\hat{F}_{\alpha,\beta}$ given by (15). Here it should be specified that we are taking positive $(k-1)$th roots of α, β.

As above, it will be convenient to split our considerations according to how many vector components y_i of $\mathbf{y}$ are equal to zero. However, owing to the similarities between this case and the case of k being even, our investigations will be significantly condensed.

4.1. *The Case* $y_1y_2y_3y_4 = 0$.

The case in which $y_1 = y_2 = y_3 = 0$ obviously produces an absolutely irreducible curve $\mathscr{C}_{\mathbf{y}}$, by Eq. (15) for $\hat{F}_\Omega(\mathbf{y})$ and the fact that $y_4 \neq 0$. Following the previous methods, it is easy to see that whenever precisely two of the components of the vector $\mathbf{y}$ are zero, $\mathscr{C}_{\mathbf{y}}$ must be absolutely irreducible unless one of the following three possibilities comes to pass:

1. We have $\alpha = \beta = 1$ and $\mathbf{y}_0 = (0, 0, 1, 1)$.
2. We have $\mathbf{y}_1^{\pm} = (0, 1, 0, \pm 1)$.
3. We have $\alpha = \beta = 1$ and $\mathbf{y}_2 = (1, 0, 0, 1)$.

Here we have used the fact that $y_4 \neq 0$ in order to conclude that there are precisely three ways in which the vector $\mathbf{y}$ has precisely two zero components. Then in order to deduce Case 1 for example, we consider the possibility $y_1 = y_2 = 0$ and infer from $\hat{F}_\Omega(\mathbf{y}) = 0$ that $\beta y_3^k = \alpha y_4^k$ since $k-1$ is even. Thus $\alpha = \beta = 1$ and $y_3 = y_4 = 1$, because $\alpha, \beta > 0$ are k-free and $(y_3, y_4) = (\alpha, \beta) = 1$. Cases 2 and 3 are handled similarly. We must now consider the nature of the curves $\mathscr{C}_{\mathbf{y}}$ for $\mathbf{y} \in \{\mathbf{y}_0, \mathbf{y}_1^{\pm}, \mathbf{y}_2\}$; these are defined by the forms

$$G_{\mathbf{y}_0} = x_1^k + x_2^k, \qquad G_{\mathbf{y}_1^\pm} = \alpha(\pm x_1^k + x_3^k) \qquad \text{and} \qquad G_{\mathbf{y}_2} = x_2^k + x_3^k,$$

respectively. Although for odd k the binary form $X^k + Y^k$ will in general have irreducible quadratic factors it can easily be seen as in the case of k even, that points lying on these curves actually lie on lines in the surface $F_{\alpha,\beta} = 0$. Indeed, this is plain from the identity

$$(x+y)^k - (x-y)^k = 2y \sum_{0 \leqslant r < k/2} \binom{k}{2r+1} x^{k-2r-1} y^{2r} = 2yT_k(x,y),$$

say, where $T_k(x,y) = 0$ for $x, y \in \mathbb{R}$ if and only if $x = y = 0$.

We now turn to the case in which precisely one of the components of $\mathbf{y}$ is equal to zero; supposing as before that $y_1 = 0$, and $y_2y_3y_4 \neq 0$. Thus the dual factor $\hat{F}_\Omega(\mathbf{y})$ becomes (16) above, but with ω_3 and ω_4 now both $(k-1)$th roots of unity. Following the corresponding case for even k, it is easy to employ Lemma 2 in exactly the same fashion in order to deduce that if the curve $\mathscr{C}_\mathbf{y}$ is not absolutely irreducible then the vector $\mathbf{y}$ must satisfy a second distinct Eq. (17), with ω_3', ω_4' both $(k-1)$th roots of unity such that $\{\omega_3, \omega_4\} \neq \{\omega_3', \omega_4'\}$. It is then straightforward to subtract this from (16) in order to deduce Eq. (18), with δ replaced by 1 and $(k-1)$th roots of unity $\eta_1, \eta_2 \neq 1$. A simple application of Lemma 5 yields

$$\frac{\beta}{\alpha}\left(\frac{y_3}{y_4}\right)^k = \pm 1,$$

since either $3 \nmid (k-1)$ or $\alpha = \beta = 1$, whence $y_3 = \pm y_4$ and $\alpha = \beta = 1$, since $\alpha, \beta > 0$ are k-free and coprime. We substitute this into (16), and then the same simplified application of Lemma 6 that was applied previously, suffices to deduce that in fact $y_3 = \pm y_4 = \pm y_2$. Thus the only possibly reducible curves correspond to $\mathbf{y} = (0, \pm 1, \pm 1, 1)$ when $y_1 = 0$; but it is now perfectly feasible for the roots ω_3 and ω_4 to be cube roots of unity, unlike in the case of k even. Instead, we intersect

$$G_\mathbf{y}(\mathbf{x}) = x_1^k + x_2^k + x_3^k - (\pm x_2 \pm x_3)^k$$

with the line $x_2 - x_3 = 0$, to obtain the binary form $f(x_1, x_2) = x_1^k + ax_2^k$; where $a = 2$ or $2(1 \pm 2^{k-1})$. By Eisenstein's criterion for the prime 2, we deduce that f is irreducible over $\mathbb{Q}$; so that $G_\mathbf{y}$ must also be irreducible over $\mathbb{Q}$. The cases $y_i = 0$ for $i = 2, 3$ are handled identically, and so we may conclude that all such plane sections produce irreducible curves $\mathscr{C}_\mathbf{y}$.

4.2. *The Case* $y_1y_2y_3y_4 \neq 0$. It is clear that a result of the type given by Lemma 7 holds equally well in this setting, with ω_2 and ω_3 now both $(k-1)$th roots of unity. Although practically identical, the following result is sufficiently divergent from the corresponding Lemma 8 to warrant more explicit attention.

LEMMA 9. *For odd* $k \geqslant 5$, *if the curve* $\mathscr{C}_{\mathbf{y}}$ *is not absolutely irreducible then there exist at least* $k-1$ *distinct triples* $\Omega = (\omega_2, \omega_3, \omega_4)$ *such that* $\hat{F}_\Omega(\mathbf{y}) = 0$; *unless* $\alpha = \beta = 1$ *and* $\mathbf{y} = (1, 1, 1, \pm 1)$.

Proof. The proof is entirely similar to the case of even k, except in one important aspect. Essentially, the argument of Lemma 7 leads us to conclude that any singular point P of $\mathscr{C}_{\mathbf{y}}$ will be of the form (19) for $(k-1)$th roots of unity ω_2, ω_3, and we will also have the following odd k version of ratios (20)

$$\left(\frac{x_1}{x_2}\right)^{k-1} = \frac{\beta y_1}{\alpha y_2}, \qquad \left(\frac{x_1}{x_3}\right)^{k-1} = \frac{y_1}{y_3}, \qquad \left(\frac{x_2}{x_3}\right)^{k-1} = \frac{\alpha y_2}{\beta y_3}, \tag{35}$$

holding at P. The proof of Lemma 8 leads us to conclude that

$$\left(\frac{x_1}{x_2}\right)^{k-2} = \frac{\beta y_1^2}{\alpha y_2^2}, \qquad \left(\frac{x_1}{x_3}\right)^{k-2} = \frac{y_1^2}{y_3^2}, \qquad \left(\frac{x_2}{x_3}\right)^{k-2} = \frac{\alpha y_2^2}{\beta y_3^2} \tag{36}$$

at P, corresponding to ratios (26) in the case of k even. We are no longer in a position to use (35) and (36) to get a straightforward contradiction, but they do demand that $\alpha = \beta = 1$ and $y_1 = y_2 = y_3$. To see this, invert ratios (35) and multiply each of them with the corresponding ratio in (36) to deduce

$$\frac{x_2}{x_1} = \frac{y_1}{y_2}, \qquad \frac{x_3}{x_1} = \frac{y_1}{y_3}, \qquad \frac{x_3}{x_2} = \frac{y_2}{y_3},$$

at P. Combining these with the form (19) that P takes is sufficient to establish the claim, since $\alpha, \beta > 0$ are coprime and k-free. But then substitution into $\hat{F}_\Omega(\mathbf{y}) = 0$ yields the familiar equation $y_1^k(1 + \omega_2 + \omega_3)^{k-1} = y_4^k$; to which a straightforward application of Lemma 6 yields $y_4 = \pm y_1$. Thus Lemma 9 fails whenever $\alpha = \beta = 1$ and $\mathbf{y} = (1, 1, 1, \pm 1)$, by the primitivity of the vector $\mathbf{y}$. This completes the proof of the lemma. ∎

We conclude this section by demonstrating the following result, which exactly describes the possible factorisations of the form $G_{\mathbf{y}}$ for odd $k \geqslant 5$ and $\mathbf{y} \in \mathfrak{A}'$. Again, we will take advantage of work already done in the case of even k in order to significantly abridge the proof.

PROPOSITION 2. *Let $k \geqslant 5$ be an odd integer such that $3 \nmid (k-1)$, whenever $F_{\alpha,\beta} \neq F$. Then for any vector $\mathbf{y} \in \mathfrak{A}'$, the form $G_{\mathbf{y}}$ is irreducible unless $\mathbf{y} \in \mathfrak{B}_2$, where $\mathfrak{B}_2$ is the set of vectors satisfying any one of the three conditions defining the set $\mathfrak{B}_1$ of Proposition 1, with all references to δ suppressed.*

Moreover, for $\mathbf{y} \in \mathfrak{B}_2$ the corresponding form $G_{\mathbf{y}}$ is either the product of a linear factor and an absolutely irreducible factor of degree $k-1$; or if $\alpha = \beta = 1$ and $\mathbf{y} = (1,1,1,1)$, it is the product of three linear factors and an absolutely irreducible factor of degree $k-3$.

By Lemma 9, we know that whenever $\mathscr{C}_{\mathbf{y}}$ is not absolutely irreducible and $\mathbf{y} \neq (1,1,1,\pm 1)$, there are at least $k-1$ distinct triples of $(k-1)$th roots of unity $\Omega = (\omega_2, \omega_3, \omega_4)$ such that $\hat{F}_{\Omega}(\mathbf{y}) = 0$. The claim made at the beginning of the proof of Proposition 1 equally holds in this framework, and so we may assume that there exist at least two distinct equations $\hat{F}_{\Omega} = 0$ and $\hat{F}_{\Omega'} = 0$, say, which share the same value of either ω_2, ω_3 or ω_4. Since we are no longer in a position to assume that $y_i \geqslant 0$ for $1 \leqslant i \leqslant 4$, applying Lemma 5 and the techniques employed in the proof of Proposition 1 will lead to the conclusion that $G_{\mathbf{y}}$ is absolutely irreducible unless

$$y_1 = \pm y_3 \qquad \text{and} \qquad y_2 = \pm y_4, \tag{37}$$

or $\alpha = \beta = 1$ and

$$\begin{aligned} y_1 &= \pm y_2 \qquad \text{and} \qquad y_3 = \pm y_4 \qquad \text{or} \\ y_1 &= \pm y_4 \qquad \text{and} \qquad y_2 = \pm y_3, \end{aligned} \tag{38}$$

where the case in which the modulus of three of the components of $\mathbf{y}$ are equal is handled exactly as in the case of k even, and shown to be included in possibilities (37) and (38).

We defer the treatment of the curves $\mathscr{C}_{\mathbf{y}}$ arising from those $\mathbf{y}$ whose components all have modulus 1 until later; concentrating instead on the nature of the curves $\mathscr{C}_{\mathbf{y}}$ for $\mathbf{y}$ satisfying $|y_i| > 1$ for some $1 \leqslant i \leqslant 4$, and (37) or (38). This laborious task will be made easier by noting that it suffices to consider those $G_{\mathbf{y}}$ for which $\mathbf{y} = (A, B, \pm A, \pm B)$; where $A, B \in \mathbb{Z}$ are coprime and not both with modulus equal to 1. Indeed, the arguments for the alternative cases in which $\alpha = \beta = 1$ and $\mathbf{y} = (A, B, \pm B, \pm A)$ or $\mathbf{y} = (A, \pm A, B, \pm B)$, will follow in precisely the same way. Indeed for both cases one sets $\alpha = \beta = 1$ and interchanges the roles of both x_1 and x_2, and those of A and B in the first alternative; and just the roles of x_2 and x_3 in the second.

There are four basic permutations of $\mathbf{y}$ to consider; and we use the equation $\hat{F}_{\Omega}(\mathbf{y}) = 0$ to deduce that

$$(\beta A^k)^{1/(k-1)} + \omega_2 (\alpha B^k)^{1/(k-1)} + \omega_3 (\pm \beta A^k)^{1/(k-1)} + \omega_4 (\pm \alpha B^k)^{1/(k-1)} = 0,$$

for $(k-1)$th roots of unity $\omega_2, \omega_3, \omega_4$. Thus there exists a pair (η_1, η_2) of $(k-1)$th roots of ± 1, such that

$$\beta A^k(1-(-\omega_3\eta_1))^{k-1} = \alpha B^k(1-(-\omega_4\eta_2/\omega_2))^{k-1}.$$

If we put $\zeta_1 = -\omega_3\eta_1$ and $\zeta_2 = -\omega_4\eta_2/\omega_2$, we see that if one of ζ_1 or ζ_2 is equal to 1, then so is the other, since $\alpha\beta AB \neq 0$. But it is clear that $\zeta_1 = \zeta_2 = 1$ if and only if η_1 and η_2 are both $(k-1)$th roots of $+1$; which can only occur when $\mathbf{y} = (A, B, A, B)$. Alternatively, we have $\zeta_i \neq 1$ for $i = 1, 2$, and we can apply Lemma 5 in addition to the fact that $3 \nmid (k-1)$ or $\alpha = \beta = 1$, in order to deduce that $\alpha = \beta = 1$ and $\mathbf{y} = (\pm 1, 1, \pm 1, \pm 1)$, since $(\alpha, \beta) = (A, B) = 1$. Before analysing this special case, we return to the case $\mathbf{y} = (A, B, A, B)$, and immediately notice a linear factor $(x_1 + x_3)$ appearing in the corresponding form $G_{\mathbf{y}}$. The argument previously applied to form (33), once translated to the setting of odd k, will suffice to show that $G_{\mathbf{y}}$ is the product of a rational linear form and an absolutely irreducible form of degree $k-1$.

Now let $\alpha = \beta = 1$ and $\mathbf{y} = (\pm 1, \pm 1, \pm 1, 1)$. We observe that the equation for $\hat{F}_\Omega(\mathbf{y}) = 0$ demands that the corresponding curve $\mathscr{C}_{\mathbf{y}}$ has a singular point if and only if there exists a triple $(\omega_2, \omega_3, \omega_4)$ of $(k-1)$th roots of unity such that

$$(\pm 1)^{1/(k-1)} + \omega_2(\pm 1)^{1/(k-1)} + \omega_3(\pm 1)^{1/(k-1)} + \omega_4 = 0.$$

This can clearly be rewritten as

$$\eta_1 + \eta_2 + \eta_3 + 1 = 0, \tag{39}$$

where η_1, η_2 and η_3 are $(k-1)$th roots of ± 1. Further, for $1 \leqslant i \leqslant 3$, each η_i is clearly a $(k-1)$th roots of $+1$ if and only if the corresponding component y_i of $\mathbf{y}$ is equal to $+1$. We hope to show that the only possibility is that at least two components of $\mathbf{y}$ can be taken to be $+1$, and the remaining two share the same sign. Indeed, writing $n = 2(k-1)$ and working over the cyclotomic field of nth roots of unity, we deduce from Lemma 4 that there can only be trivial solutions to (39), whence

$$\mathbf{y} \in \{(1,1,1,1), (1,-1,-1,1), (-1,1,-1,1), (-1,-1,1,1)\}.$$

Consider first those $\mathbf{y}$ containing components of opposite sign. Thus for the vector $\mathbf{y} = (-1, 1, -1, 1)$, we have $G_{\mathbf{y}}(\mathbf{x}) = x_1^k + x_2^k + x_3^k + (x_1 - x_2 + x_3)^k$, and we easily notice that $G_{\mathbf{y}}$ can be written in form (33), for some degree $k-1$ form H. By our application of Lemma 4 to (39) and since $\mathbf{y} = (-1, 1, -1, 1)$, we use exactly the same sort of argument that was used to show that the form J_+ in (34) was non-singular, in order to deduce that

$x_1 + x_3 = 0$ at any singular point of H. Following the usual argument, we are led to the fact that H itself must be non-singular and hence absolutely irreducible. One proceeds in a similar manner for the vectors $\mathbf{y} = (1, -1, -1, 1)$ and $\mathbf{y} = (-1, -1, 1, 1)$, with corresponding linear factors $(x_2 + x_3)$ and $(x_1 + x_2)$ of $G_{\mathbf{y}}$.

Turning to the final special case $\mathbf{y} = (1, 1, 1, 1)$, we have the factorisation

$$\begin{aligned} G_{\mathbf{y}}(\mathbf{x}) &= x_1^k + x_2^k + x_3^k - (x_1 + x_2 + x_3)^k \\ &= (x_1 + x_2)(x_1 + x_3)(x_2 + x_3)J(\mathbf{x}), \end{aligned}$$

where J is some form of degree $k - 3$. Again, by our previous application of Lemma 4, and through basically the same sort of argument that was used above, we deduce that J itself must be absolutely irreducible; which completes the proof of Proposition 2.

5. PROOF OF THEOREM 2

In light of Heath-Brown's bound (6), all that remains is to estimate the contribution to $\mathcal{N}_1(F_{\alpha,\beta}; B)$ from those primitive points $\mathbf{x} \in \mathbb{Z}^4$ for which $|\mathbf{x}| \leqslant B$, and $\mathbf{x}$ lies on some irreducible curve of degree $\leqslant k - 2$ which is contained in the surface $F_{\alpha,\beta} = 0$. When $F_{\alpha,\beta} \neq F$ we assume that $k - 1$ is not divisible by 3, with no such restriction when $F_{\alpha,\beta} = F$.

By a useful theorem of Colliot-Thélène [6, Appendix], any degree k non-singular surface in $\mathbb{P}^3$ contains just $O_k(1)$ irreducible curves of degree $\leqslant k - 2$. Moreover, the case in which the curve is a line is to be omitted by definition of $\mathcal{N}_1(F_{\alpha,\beta}; B)$. Whenever the irreducible curve has degree d, for $3 \leqslant d \leqslant k - 2$, an application of Lemma 1 provides the bound $O_{\varepsilon,k}(B^{2/3+\varepsilon})$ for each of the curves. Thus it remains to handle the case of $F_{\alpha,\beta} = 0$ containing irreducible quadric curves, which will necessarily be planar, and so an application of Theorem 1 neatly eliminates this possibility whenever $k \geqslant 6$. This therefore completes the proof of Theorem 2, since for $k = 5$ any quadric curves in the surface will each produce a contribution of $O_\varepsilon(B^{1+\varepsilon})$ to $\mathcal{N}_1(F_{\alpha,\beta}; B)$ by Lemma 1, which is satisfactory.

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