

**DUAL FIELD APPROACH TO CORRELATION FUNCTIONS
IN THE HEISENBERG XXZ SPIN CHAIN**

FABIAN H.L. ESSLER

*Physikalisches Institut der Universität Bonn, Nussallee 12
D-53115 Bonn, GERMANY*

and

VLADIMIR E. KOREPIN

*Institute for Theoretical Physics
State University of New York at Stony Brook
Stony Brook, NY 11794-3840, USA*

to appear in the proceedings of SMQFT, Los Angeles 1994

ABSTRACT

We study zero temperature correlation functions of the spin- $\frac{1}{2}$ Heisenberg XXZ model in the critical regime $-1 < \Delta \leq 1$ in a magnetic field by means of the *Dual Field Approach*. We show for one particular example how to derive determinant representations for correlation functions and how to use these to embed the correlation functions in integrable systems of integro-difference equations (IDE). These IDE are associated with a Riemann-Hilbert problem.

1. Introduction

The evaluation of correlation functions in integrable one-dimensional quantum systems is one of the main outstanding problems in Mathematical Physics. Quite recently there has been exciting progress in this direction: the Kyoto group succeeded in deriving integral representations for some correlation functions of the Heisenberg XXZ model in the massive regime $\Delta > 1$ by taking advantage of the infinite quantum affine symmetry of the model on the infinite chain^{4,18}. These integral representations are most powerful for studying the *short distance* behaviour of correlators, whereas it is not obvious how to extract the large distance asymptotics. Also it is not straightforward to extend this approach to the critical regime $-1 < \Delta < 1$ or to include an external magnetic field.

These issues can be very naturally addressed in the framework of the *Dual Field Approach* (DFA) to correlation functions, which was pioneered in [11, 12, 13, 19, 20, 24, 25] for the example of the δ -function Bose gas^{28,29}. A detailed and complete exhibition of this work can be found in the book [22]. By means of the DFA it is possible to derive determinant representations for correlation functions in practically any integrable model, for which the Algebraic Bethe Ansatz can be formulated. Using the determinant representation one can then obtain explicit expressions for the *large distance*

asymptotics of correlation functions (even at finite temperature), and the inclusion of an external magnetic field poses no problem. The DFA thus nicely complements the approach of the Kyoto group. Here we will apply the DFA to the Heisenberg XXZ chain at zero temperature in a magnetic field h , *i.e.* the hamiltonian (with periodic boundary conditions)

$$H = \sum_{j=1}^L \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta (\sigma_j^z \sigma_{j+1}^z - 1) - h \sum_{j=1}^L \sigma_j^z, \quad -1 < \Delta \leq 1, \quad (1)$$

where $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\Delta = \cos(2\eta)$, $\frac{\pi}{2} < \eta \leq \pi$.

Our discussion below follows the logic of the DFA: in section 2 we recall the formulation of the Algebraic Bethe Ansatz (ABA) for the XXZ chain, and then use the ABA to express correlation functions in terms of determinants of Fredholm integral operators. In section 3 we discuss how to embed these determinants in systems of integrable integro-difference equations (IDE) (this is analogous to describing quantum correlation functions by means of differential equations^{1,36}).

2. Determinant Representations

Let us first review the main features of the Algebraic Bethe Ansatz (ABA) for the XXZ Heisenberg magnet^{6,26,31,35,34}. Starting point and central object of the Quantum Inverse Scattering Method (see *e.g.* [27, 34]) is the R-matrix, which is a solution of the Yang-Baxter equation^{40,2}. For the case of the XXZ model it is of the form

$$R(\lambda, \mu) = \begin{pmatrix} f(\mu, \lambda) & 0 & 0 & 0 \\ 0 & g(\mu, \lambda) & 1 & 0 \\ 0 & 1 & g(\mu, \lambda) & 0 \\ 0 & 0 & 0 & f(\mu, \lambda) \end{pmatrix}, \quad (2)$$

where

$$f(\lambda, \mu) = \frac{\sinh(\lambda - \mu + 2i\eta)}{\sinh(\lambda - \mu)}, \quad g(\lambda, \mu) = \frac{i \sin(2\eta)}{\sinh(\lambda - \mu)}. \quad (3)$$

The R-matrix is a linear operator on the tensor product of two two-dimensional linear spaces: $R(\mu) \in \text{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$. From the R-matrix (2) one can construct an L -operator of a "fundamental spin model" (see *e.g.* [22] p.126) by considering $R(\mu)\Pi$, (where Π is the permutation matrix on $\mathbb{C}^2 \otimes \mathbb{C}^2$) as an operator-valued matrix by identifying one of the linear spaces with the two-dimensional Hilbert space \mathcal{H}_n of $SU(2)$ -spins over the n 'th site of a lattice of length L

$$L_n(\mu) = \begin{pmatrix} \sinh(\mu - i\eta\sigma_n^z) & -i \sin(2\eta)\sigma_n^- \\ -i \sin(2\eta)\sigma_n^+ & \sinh(\mu + i\eta\sigma_n^z) \end{pmatrix}. \quad (4)$$

The Yang-Baxter equation for R implies the following relations for the L -operator

$$R(\lambda - \mu) (L_n(\lambda) \otimes L_n(\mu)) = (L_n(\mu) \otimes L_n(\lambda)) R(\lambda - \mu). \quad (5)$$

From the ultralocal L -operator the monodromy matrix is constructed as

$$T(\mu) = L_L(\mu)L_{L-1}(\mu) \dots L_1(\mu) = \begin{pmatrix} A(\mu) & B(\mu) \\ C(\mu) & D(\mu) \end{pmatrix}. \quad (6)$$

Eq. (5) can be lifted to the level of the monodromy matrix

$$R(\lambda - \mu) (T(\lambda) \otimes T(\mu)) = (T(\mu) \otimes T(\lambda)) R(\lambda - \mu). \quad (7)$$

Below we will repeatedly use especially the following matrix elements of (7)

$$\begin{aligned} [B(\lambda), B(\mu)] &= 0 = [C(\lambda), C(\mu)] \\ [B(\lambda), C(\mu)] &= g(\lambda, \mu) (D(\lambda)A(\mu) - D(\mu)A(\lambda)) \\ D(\mu)B(\lambda) &= f(\lambda, \mu)B(\lambda)D(\mu) + g(\mu, \lambda)B(\mu)D(\lambda) \\ A(\mu)B(\lambda) &= f(\mu, \lambda)B(\lambda)A(\mu) + g(\lambda, \mu)B(\mu)A(\lambda). \end{aligned}$$

By tracing (7) over the matrix space one obtains a one-parameter family of commuting *transfer matrices* $\tau(\mu) = \text{tr}(T(\mu)) = A(\mu) + D(\mu)$: $[\tau(\mu), \tau(\nu)] = 0$. The transfer matrix is the generating functional of an infinite number of mutually commuting conserved quantum operators (via expansion in powers of spectral parameter). One of these operators is the hamiltonian

$$H = -2i \sin(2\eta) \frac{\partial}{\partial \mu} \ln(\tau(\mu)) \Big|_{\mu=-i\eta} - 2L \cos(2\eta) - 2hS^z. \quad (8)$$

Below we also make use of some properties of *inhomogenous* XXZ models, which are constructed in the following way: we first note that the intertwining relation for the L -operator (5) still holds, if we shift both spectral parameters λ and μ by an arbitrary amount ν_n , *i.e.*

$$R(\lambda - \mu) (L_n(\lambda - \nu_n) \otimes L_n(\mu - \nu_n)) = (L_n(\mu - \nu_n) \otimes L_n(\lambda - \nu_n)) R(\lambda - \mu). \quad (9)$$

The reason for this is of course that the R -matrix only depends on the difference of spectral parameters. We now can construct a monodromy matrix as

$$T_{inh}(\lambda) = L_L(\lambda - \nu_L)L_{L-1}(\lambda - \nu_{L-1}) \dots L_1(\lambda - \nu_1) = \begin{pmatrix} \mathcal{A}(\mu) & \mathcal{B}(\mu) \\ \mathcal{C}(\mu) & \mathcal{D}(\mu) \end{pmatrix}. \quad (10)$$

The inhomogeneous monodromy matrix (10) obeys the same intertwining relation (7) as (6).

The ABA deals with the construction of simultaneous eigenstates of the transfer matrix and the hamiltonian. Starting point is the choice of a *reference state*, which is a trivial eigenstate of $\tau(\mu)$. In our case we make the choice $|0\rangle = \otimes_{n=1}^L |\uparrow\rangle_n$, *i.e.* we choose the completely ferromagnetic state. The action of the L -operator (4) on $|\uparrow\rangle_n$ can be easily computed and implies the following actions of the matrix elements of the monodromy matrix

$$A(\mu)|0\rangle = a(\mu)|0\rangle, \quad a(\mu) = (\sinh(\mu - i\eta))^L,$$

$$\begin{aligned}
D(\mu)|0\rangle &= d(\mu)|0\rangle, & d(\mu) &= (\sinh(\mu + i\eta))^L, \\
C(\mu)|0\rangle &= 0, \\
B(\mu)|0\rangle &\neq 0,
\end{aligned}$$

From (11) it follows that $B(\lambda)$ plays the role of a creation operator, *i.e.* one can construct a set of states of the form

$$\Psi_N(\lambda_1, \dots, \lambda_N) = \prod_{j=1}^N B(\lambda_j)|0\rangle. \quad (11)$$

The requirement that the states (11) ought to be eigenstates of the transfer matrix $\tau(\mu)$ puts constraints on the allowed values of the parameters λ_n : the set $\{\lambda_j\}$ must be a solution of the following system of coupled algebraic equations, called *Bethe equations*^{3,30}

$$\frac{a(\lambda_j)}{d(\lambda_j)} = \prod_{\substack{k=1 \\ k \neq j}}^N \frac{f(\lambda_k, \lambda_j)}{f(\lambda_j, \lambda_k)}, \quad j = 1, \dots, N. \quad (12)$$

These equations are the basis for studying ground state, excitation spectrum and thermodynamics of Bethe Ansatz solvable models. For the case of the XXZ model with $\Delta > -1$ (the case we are interested in here) it was proved by C.N. Yang and C.P. Yang^{38,39} that the ground state is characterized by a set of *real* λ_j subject to the Bethe equations (12). Without an external magnetic field ($h = 0$) their number is $N = L/2$. In the thermodynamic limit the ground state is described by means of an integral equation for the density of spectral parameters $\rho(\lambda)$ ^{10,39}

$$2\pi\rho(\lambda) - \int_{-\Lambda}^{\Lambda} d\mu K(\lambda, \mu) \rho(\mu) = D(\lambda), \quad (13)$$

where the integral kernel K and the driving term D are given by

$$K(\mu, \lambda) = \frac{\sin(4\eta)}{\sinh(\mu - \lambda + 2i\eta) \sinh(\mu - \lambda - 2i\eta)}, \quad D(\lambda) = \frac{-\sin(2\eta)}{\sinh(\lambda - i\eta) \sinh(\lambda + i\eta)}. \quad (14)$$

Here Λ depends on the external magnetic field h . The physical picture of the ground state is that of a filled Fermi sea with boundaries $\pm\Lambda$. The dressed energy of a particle in the sea is given by the solution of the integral equation^{32,33}

$$\epsilon(\lambda) - \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} d\mu K(\lambda, \mu) \epsilon(\mu) = 2h - \frac{2(\sin(2\eta))^2}{\sinh(\lambda - i\eta) \sinh(\lambda + i\eta)}. \quad (15)$$

The requirement of the vanishing of the dressed energy at the Fermi boundary $\epsilon(\pm\Lambda) = 0$ determines the dependence of Λ on h . For small h this relation can be found explicitly by means of a Wiener-Hopf analysis³⁹. For $h \geq h_c = (2 \cos \eta)^2$ the system is in the saturated ferromagnetic state, which corresponds to $\Lambda = 0$.

2.1. Two-Site Generalized Model

For the evaluation of correlation functions the so-called “two-site generalized model” has proven a useful tool. From the mathematical point of view this is simply the application of the co-product associated with the algebra defined by (7). The main idea is to divide the chain of length L into two parts and associate a monodromy matrix with both sub-chains, *i.e.*

$$T(\mu) = T(2, \mu)T(1, \mu) , \quad T(i, \mu) = \begin{pmatrix} A_i(\mu) & B_i(\mu) \\ C_i(\mu) & D_i(\mu) \end{pmatrix} \quad (i = 1, 2) . \quad (16)$$

In terms of L -operators the monodromy matrices are given by

$$\begin{aligned} T(2, \mu) &= L_L(\mu)L_{L-1}(\mu) \dots L_n(\mu) , \\ T(1, \mu) &= L_{n-1}(\mu)L_{n-2}(\mu) \dots L_1(\mu) . \end{aligned} \quad (17)$$

By construction it is clear that both monodromy matrices $T(i, \mu)$ fulfill the same intertwining relation (7) as the complete monodromy matrix $T(\mu)$. Similarly the reference state for the complete chain is decomposed into a direct product of reference states $|0\rangle_i$ for the two sub-chains $|0\rangle = |0\rangle_2 \otimes |0\rangle_1$. The resulting structure can be summarized as

$$\begin{aligned} A_i(\mu)|0\rangle_i &= a_i(\mu)|0\rangle_i , & D_i(\mu)|0\rangle_i &= d_i(\mu)|0\rangle_i , \\ C(\mu)|0\rangle_i &= 0 , & B_i(\mu)|0\rangle_i &\neq 0 , \end{aligned} \quad (18)$$

where the eigenvalues a and d in (11) are given by $a(\mu) = a_2(\mu)a_1(\mu)$ and $d(\mu) = d_2(\mu)d_1(\mu)$. The creation operators $B(\mu)$ for the complete chain are decomposed as $B(\mu) = A_2(\mu) \otimes B_1(\mu) + B_2(\mu) \otimes D_1(\mu)$, which implies that eigenstates of the transfer matrix can be represented as

$$\begin{aligned} \prod_{j=1}^N B(\lambda_j)|0\rangle &= \sum_{I, II} \prod_{j \in I}^{n_1} \prod_{k \in II}^{n_2} a_2(\lambda_j^I) d_1(\lambda_k^{II}) \\ &\quad \times f(\lambda_j^I, \lambda_k^{II}) \left(B_2(\lambda_k^{II})|0\rangle_2 \right) \otimes \left(B_1(\lambda_j^I)|0\rangle_1 \right) , \end{aligned} \quad (19)$$

where the sum is over all partitions $\{\lambda_j^I\} \cup \{\lambda_k^{II}\}$ of the set $\{\lambda_j\}$ with $\text{card}\{\lambda^I\} = n_1$, $\text{card}\{\lambda^{II}\} = n_2 = N - n_1$. A similar equation holds for dual states

$$\begin{aligned} \langle 0| \prod_{j=1}^N C(\lambda_j) &= \sum_{I, II} \prod_{j \in I}^{n_1} \prod_{k \in II}^{n_2} d_2(\lambda_j^I) a_1(\lambda_k^{II}) \\ &\quad \times f(\lambda_k^{II}, \lambda_j^I) \left({}_1\langle 0|C_1(\lambda_j^I) \right) \otimes \left({}_2\langle 0|C_2(\lambda_k^{II}) \right) . \end{aligned} \quad (20)$$

2.2. Reduction of Correlators to Scalar Products

We are now in the position express correlation functions in the framework of the ABA. We will concentrate on the Ferromagnetic String Formation Probability (FSFP) correlation function, which is defined as follows

$$P(m) = \langle GS | \prod_{j=1}^m P_j | GS \rangle, \quad (21)$$

where $|GS\rangle$ is the antiferromagnetic ground state and $P_j = \frac{1}{2}(\sigma_j^z + 1)$ is the projection operator onto the state with spin up at site j . The physical meaning of $P(m)$ is the probability of finding a ferromagnetic string (m adjacent parallel spins up) in the ground state $|GS\rangle$ of the model (1) for a given value of the magnetic field h . From a technical point of view this correlator turns out the simplest one in the DFA²². Let us now rewrite (21) in terms of the ABA. Noting that $\frac{1}{2}(\sigma_j^z + 1) = \exp\left(\frac{\alpha}{2}(1 - \sigma_j^z)\right) \Big|_{\alpha=-\infty}$ we find

$$P(m) = \frac{\langle 0 | \prod_{j=1}^N C(\lambda_j) \exp(\alpha \sum_{l=1}^m \sigma_l^- \sigma_l^+) \prod_{k=1}^N B(\lambda_k) | 0 \rangle}{\langle 0 | \prod_{j=1}^N C(\lambda_j) \prod_{k=1}^N B(\lambda_k) | 0 \rangle} \Big|_{\alpha=-\infty}, \quad (22)$$

where the spectral parameters λ_j are subject the Bethe equations (12) for the ground state. This expression can be rewritten in terms of scalar products by using the two-site generalized model introduced above: we take the first sub-chain to contain sites 1 to m and the second one sites $m + 1$ to L . We note that due to the projection ($\alpha = -\infty$) in (22) the partitions in (19) and (20) must be such that there are no down-spins on the first sub-chain (only the partition where $n_1 = 0$ survives), and thus

$$P(m) = \frac{1}{\sigma_N} {}_2\langle 0 | \prod_{j=1}^N C_2(\lambda_j) \prod_{k=1}^N B_2(\lambda_k) | 0 \rangle_2 \prod_{l=1}^N a_1(\lambda_l) d_1(\lambda_l), \quad (23)$$

where

$$\sigma_N = \langle 0 | \prod_{j=1}^N C(\lambda_j) \prod_{k=1}^N B(\lambda_k) | 0 \rangle. \quad (24)$$

2.3. Determinant Representations for Scalar Products

In (23) we have reduced the computation of $P(m)$ to the computation of a scalar product of the form

$$S_N = \langle 0 | \prod_{j=1}^N C(\lambda_j^C) \prod_{k=1}^N B(\lambda_k^B) | 0 \rangle. \quad (25)$$

Following [20, 21] we will show how to represent (25) as a determinant. Here we *do not* assume that the sets of spectral parameters $\{\lambda^B\}$ and $\{\lambda^C\}$ are the same, and we

also *do not* impose the Bethe equations (12). From (8) and (11) it follows that scalar products can be represented as¹⁹

$$S_N = \sum_{A,D} \prod_{j=1}^N a(\lambda_j^A) \prod_{k=1}^N d(\lambda_k^D) K_N \left(\begin{array}{c} \{\lambda^C\} \\ \{\lambda^A\} \end{array} \begin{array}{c} \{\lambda^B\} \\ \{\lambda^D\} \end{array} \right), \quad (26)$$

where the sum is over all partitions of $\{\lambda^C\} \cup \{\lambda^B\}$ into two sets $\{\lambda^A\}$ and $\{\lambda^D\}$. The coefficients K_N are functions of the λ_j and are completely determined by the intertwining relation (7). In particular the K_N 's are identical for the homogeneous model (6) and the *inhomogeneous* model (10), i.e. the K_N 's are independent of the inhomogeneities $\{\nu_n\}$ and also do not depend on the lattice length L as long as $N < L$. The reason for this is that the intertwining relations for the matrix elements $\mathcal{A}(\mu)$, $\mathcal{B}(\mu)$, $\mathcal{C}(\mu)$ and $\mathcal{D}(\mu)$ of (10) are the same as the ones for the matrix elements $A(\mu)$, $B(\mu)$, $C(\mu)$, $D(\mu)$ of (6) (see above). We will exploit this fact by considering special inhomogeneous models for which all terms but one in the sum in (26) vanish, and then represent this term as a determinant. The basic tool for representing scalar products as determinants is a Theorem due to Izergin, Coker and Korepin^{14,15}, which deals with determinant representations for the *partition functions* of inhomogeneous XXZ models constructed according to (10):

Theorem: Consider an inhomogeneous XXZ chain of even length N with inhomogeneities ν_j , $j = 1 \dots N$. Let $|0\rangle$ and $|\bar{0}\rangle$ be the ferromagnetic reference states with all spins up and down respectively. Let $\mathcal{B}(\mu)$ and $\mathcal{C}(\mu)$ be the creation/annihilation operators over the reference state $|0\rangle$. Then the following determinant representation holds

$$\begin{aligned} \langle \bar{0} | \prod_{j=1}^N \mathcal{B}(\lambda_j) | 0 \rangle &= \langle 0 | \prod_{j=1}^N \mathcal{C}(\lambda_j) | \bar{0} \rangle \\ &= (-1)^N \prod_{\alpha=1}^N \prod_{k=1}^N \sinh(\lambda_\alpha - \nu_k - i\eta) \sinh(\lambda_\alpha - \nu_k + i\eta) \\ &\times \left(\prod_{1 \leq \alpha < \beta \leq N} \sinh(\lambda_\alpha - \lambda_\beta) \prod_{1 \leq k < l \leq N} \sinh(\nu_l - \nu_k) \right)^{-1} \det(\mathcal{M}), \end{aligned} \quad (27)$$

where

$$\mathcal{M}_{\alpha k} = \frac{i \sin(2\eta)}{\sinh(\lambda_\alpha - \nu_k - i\eta) \sinh(\lambda_\alpha - \nu_k + i\eta)}. \quad (28)$$

Let us now derive explicit expressions for the coefficients K_N . It will be convenient to work with the following sets of spectral parameters

$$\begin{aligned} \{\lambda^{AC}\} &= \{\lambda^A\} \cap \{\lambda^C\}, \quad \{\lambda^{DC}\} = \{\lambda^D\} \cap \{\lambda^C\}, \\ \{\lambda^{AB}\} &= \{\lambda^A\} \cap \{\lambda^B\}, \quad \{\lambda^{DB}\} = \{\lambda^D\} \cap \{\lambda^B\}, \end{aligned} \quad (29)$$

with cardinalities

$$\begin{aligned} n &= \text{card}\{\lambda^{DC}\} = \text{card}\{\lambda^{AB}\}, \\ N - n &= \text{card}\{\lambda^{AC}\} = \text{card}\{\lambda^{DB}\}. \end{aligned} \quad (30)$$

The partition with $n = 0$ is characterized by $\{\lambda^{AC}\} = \{\lambda^C\}$, $\{\lambda^{DB}\} = \{\lambda^B\}$, $\{\lambda^{AB}\} = \emptyset = \{\lambda^{DC}\}$. The corresponding coefficient $K_N \left(\begin{array}{c} \{\lambda^C\} \\ \{\lambda^C\} \end{array} \begin{array}{c} \{\lambda^B\} \\ \{\lambda^B\} \end{array} \right)$ is called *highest coefficient*. For highest coefficients the following determinant representation holds

$$\begin{aligned} K_N \left(\begin{array}{c} \{\lambda^C\} \\ \{\lambda^C\} \end{array} \begin{array}{c} \{\lambda^B\} \\ \{\lambda^B\} \end{array} \right) &= \left(\prod_{j>k} g(\lambda_j^B, \lambda_k^B) g(\lambda_k^C, \lambda_j^C) \right) \prod_{j,k} h(\lambda_j^C, \lambda_k^B) \det(M_C^B) \\ h(\lambda, \mu) &= \frac{f(\lambda, \mu)}{g(\lambda, \mu)} = \frac{\sinh(\lambda - \mu + 2i\eta)}{i \sin(2\eta)}, \\ (M_C^B)_{jk} = t(\lambda_j^C, \lambda_k^B) &= \frac{g(\lambda_j^C, \lambda_k^B)}{h(\lambda_j^C, \lambda_k^B)} = \frac{(i \sin(2\eta))^2}{\sinh(\lambda_j^C - \lambda_k^B + 2i\eta) \sinh(\lambda_j^C - \lambda_k^B)}. \end{aligned} \quad (31)$$

The proof of (31) is as follows: Consider an inhomogeneous XXZ model on a lattice of length N with inhomogeneities $\nu_j = \lambda_j^C + i\eta$. We have $a(\lambda) = \prod_{j=1}^N \sinh(\lambda - \lambda_j^C - 2i\eta)$ and $d(\lambda) = \prod_{j=1}^N \sinh(\lambda - \lambda_j^C)$. Inspection of (26) yields that in this situation only one term in the sum of the r.h.s of (26) survives, namely the one with $\{\lambda^D\} = \{\lambda^B\}$. Thus for this special scalar product we obtain

$$S_N \Big|_{\nu_j = \lambda_j^C + i\eta} = K_N \left(\begin{array}{c} \{\lambda^C\} \\ \{\lambda^C\} \end{array} \begin{array}{c} \{\lambda^B\} \\ \{\lambda^B\} \end{array} \right) \prod_{j,k} \sinh(\lambda_j^C - \lambda_k^C - 2i\eta) \prod_{m,l} \sinh(\lambda_m^B - \lambda_l^C). \quad (32)$$

On the other hand $B(\lambda)$ flips one spin, and as we have chosen N to be the length of the lattice we find that $\prod_{j=1}^N B(\lambda_j)|0\rangle$ is proportional to the ferromagnetic state with all spins flipped, and thus orthogonal to all states in a basis other than $|\bar{0}\rangle$. Thus

$$S_N \Big|_{\nu_j = \lambda_j^C + i\eta} = \langle 0 | \prod_{j=1}^N C(\lambda_j^C) | \bar{0} \rangle \langle \bar{0} | \prod_{k=1}^N B(\lambda_k^B) | 0 \rangle. \quad (33)$$

By the Theorem both factors can be represented as determinants. By direct computation we find for one of the factors

$$\langle 0 | \prod_{j=1}^N C(\lambda_j^C) | \bar{0} \rangle = \prod_{j,k} \sinh(\lambda_k^C - \lambda_j^C - 2i\eta). \quad (34)$$

Using the determinant representation given by the Theorem on the other factor we arrive at (31).

Arbitrary coefficients K_N are expressed in terms of highest coefficients as follows

$$K_N \left(\begin{array}{c} \{\lambda^C\} \\ \{\lambda^A\} \end{array} \begin{array}{c} \{\lambda^B\} \\ \{\lambda^D\} \end{array} \right) = \left(\prod_{j \in AC} \prod_{k \in DC} f(\lambda_j^{AC}, \lambda_k^{DC}) \right) \left(\prod_{l \in AB} \prod_{m \in DB} f(\lambda_l^{AB}, \lambda_m^{DB}) \right)$$

$$\times K_n \left(\begin{array}{c} \{\lambda^{AB}\} \\ \{\lambda^{AB}\} \end{array} \begin{array}{c} \{\lambda^{DC}\} \\ \{\lambda^{DC}\} \end{array} \right) K_{N-n} \left(\begin{array}{c} \{\lambda^{AC}\} \\ \{\lambda^{AC}\} \end{array} \begin{array}{c} \{\lambda^{DB}\} \\ \{\lambda^{DB}\} \end{array} \right) . \quad (35)$$

To prove (35) we consider an inhomogeneous XXZ model with inhomogeneities $\{\nu_j\} = \{\lambda_j^{AB} + i\eta\} \cup \{\lambda_j^{AC} + i\eta\}$. Now only the term proportional to $K_N \left(\begin{array}{c} \{\lambda^C\} \\ \{\lambda^A\} \end{array} \begin{array}{c} \{\lambda^B\} \\ \{\lambda^D\} \end{array} \right)$ in the sum on the r.h.s. of. (26) survives. Proceeding as above we arrive at (35).

Combining (31) and (35) with (26) we obtain the following expression for general scalar products of XXZ magnets

$$\begin{aligned} S_N &= \prod_{j>k} g(\lambda_j^C, \lambda_k^C) g(\lambda_k^B, \lambda_j^B) \sum \text{sgn}(P_C) \text{sgn}(P_B) \prod_{j,k} h(\lambda_j^{AB}, \lambda_k^{DC}) \prod_{l,m} h(\lambda_l^{AC}, \lambda_m^{DB}) \\ &\times \prod_{l,k} h(\lambda_l^{AC}, \lambda_k^{DC}) \prod_{j,m} h(\lambda_j^{AB}, \lambda_m^{DB}) \det(M_{DC}^{AB}) \det(M_{DB}^{AC}) , \end{aligned} \quad (36)$$

where P_C is the permutation $\{\lambda_1^{AC}, \dots, \lambda_n^{AC}, \lambda_1^{DC}, \dots, \lambda_{N-n}^{DC}\}$ of $\{\lambda_1^C, \dots, \lambda_N^C\}$, P_B is the permutation $\{\lambda_1^{DB}, \dots, \lambda_n^{DB}, \lambda_1^{AB}, \dots, \lambda_{N-n}^{AB}\}$ of $\{\lambda_1^B, \dots, \lambda_N^B\}$, $\text{sgn}(P)$ is the sign of the permutation P , and

$$\left(M_{DC}^{AB} \right)_{jk} = t(\lambda_j^{AB}, \lambda_k^{DC}) d(\lambda_k^{DC}) a(\lambda_j^{AB}), \quad t(\lambda, \mu) = \frac{(g(\lambda, \mu))^2}{f(\lambda, \mu)} . \quad (37)$$

The most important step in the DFA follows next: we introduce *dual quantum fields* in order to simplify (36) and obtain a manageable expression for scalar products. This step was first carried out for the delta-function Bose gas²⁰. The XXZ case can be treated very similarly, so that we will be brief in our discussion. The fundamental observation is that the r.h.s. in(36) looks like the determinant of the *sum* of two matrices, i.e. let A and B be two $N \times N$ matrices over \mathbf{C} . Then the determinant of their sum can be decomposed as follows

$$\det(A + B) = \sum_{P_r, P_c} \text{sgn}(P_r) \text{sgn}(P_c) \det(A_{P_r P_c}) \det(B_{P_r P_c}) . \quad (38)$$

Here P_r and P_c are partitions of the N rows and columns into two subsets \mathcal{R} , $\bar{\mathcal{R}}$ and \mathcal{C} , $\bar{\mathcal{C}}$ of cardinalities n (for \mathcal{R} , \mathcal{C}) and $N - n$ (for $\bar{\mathcal{R}}$, $\bar{\mathcal{C}}$) respectively, $A_{P_r P_c}$ is the $n \times n$ matrix obtained from A by removing all $\bar{\mathcal{R}}$ -rows and $\bar{\mathcal{C}}$ -columns, and $B_{P_r P_c}$ is the $N - n \times N - n$ matrix obtained from B by removing all \mathcal{R} -rows and \mathcal{C} -columns. Finally $\text{sgn}(P_r)$ is the parity of the permutation obtained from $(1, \dots, N)$ by moving all \mathcal{R} -rows to the front. Comparison of (36) with (38) shows that one does not get the $h(\lambda, \mu)$ -factors by simply taking the determinant of the sum of the matrices M_{jk} . This leads to the introduction of the auxiliary variables $\Phi_A(\lambda)$ and $\Phi_D(\lambda)$, which are called *dual quantum fields* and which are represented as sums of “momenta” P_A and “coordinates” Q_A as follows

$$\Phi_A(\lambda) = Q_A(\lambda) + P_D(\lambda), \quad \Phi_D(\lambda) = Q_D(\lambda) + P_A(\lambda),$$

$$[P_D(\lambda), Q_D(\mu)] = \ln(h(\lambda, \mu)), \quad [P_A(\lambda), Q_A(\mu)] = \ln(h(\mu, \lambda)) . \quad (39)$$

All other commutators of P 's and Q 's vanish. A very important property of the fields Φ is that they commute for different values of spectral parameters

$$[\Phi_A(\lambda), \Phi_D(\mu)] = 0 = [\Phi_A(\lambda), \Phi_A(\mu)] = [\Phi_D(\lambda), \Phi_D(\mu)] . \quad (40)$$

The dual quantum fields act on a bosonic Fock space with reference states $|0\rangle$ and $\langle 0|$ defined by

$$P_a(\lambda)|0\rangle = 0, \quad \langle 0|Q_a(\lambda) = 0, \quad a = A, D, \quad \langle 0|0\rangle = 1 . \quad (41)$$

Using the dual fields it is now possible to recast (36) as a determinant of the sum of two matrices in the following way

$$\begin{aligned} S_N &= \prod_{j>k} g(\lambda_j^C, \lambda_k^C) g(\lambda_k^B, \lambda_j^B) \langle 0| \det S |0\rangle, \\ S_{jk} &= t(\lambda_j^C, \lambda_k^B) a(\lambda_j^C) d(\lambda_k^B) \exp(\Phi_A(\lambda_j^C) + \Phi_D(\lambda_k^B)) \\ &\quad + t(\lambda_k^B, \lambda_j^C) d(\lambda_j^C) a(\lambda_k^B) \exp(\Phi_D(\lambda_j^C) + \Phi_A(\lambda_k^B)) . \end{aligned} \quad (42)$$

It is possible to further simplify (42) by eliminating one dual field: we define a new dual vacuum $\langle \tilde{0}|$ according to

$$\langle \tilde{0}| = \langle 0| \exp\left(\sum_{j=1}^N P_D(\lambda_j^C) + P_A(\lambda_j^B)\right), \quad \langle \tilde{0}|0\rangle = 1, \quad (43)$$

and a new dual field

$$\begin{aligned} \varphi(\lambda) &= p(\lambda) + q(\lambda), \\ q(\lambda) &= Q_A(\lambda) - Q_D(\lambda) - \langle \tilde{0}|Q_A(\lambda) - Q_D(\lambda)|0\rangle, \\ p(\lambda) &= P_D(\lambda) - P_A(\lambda), \quad \langle \tilde{0}|q(\lambda) = 0 = p(\lambda)|0\rangle, \\ [p(\lambda), q(\mu)] &= -\ln(h(\lambda, \mu)h(\mu, \lambda)), \\ [p(\lambda), p(\mu)] &= 0 = [q(\lambda), q(\mu)] = [\varphi(\lambda), \varphi(\mu)]. \end{aligned} \quad (44)$$

In terms of this field we obtain the following determinant representation

$$\begin{aligned} S_N &= \prod_{j>k} g(\lambda_j^C, \lambda_k^C) g(\lambda_k^B, \lambda_j^B) \prod_{j=1}^N a(\lambda_j^C) d(\lambda_j^B) \prod_{j,k} h(\lambda_j^C, \lambda_k^B) \langle \tilde{0}| \det S |0\rangle, \\ S_{jk} &= t(\lambda_j^C, \lambda_k^B) + t(\lambda_k^B, \lambda_j^C) \frac{r(\lambda_k^B)}{r(\lambda_j^C)} \exp(\varphi(\lambda_k^B) - \varphi(\lambda_j^C)) \\ &\quad \times \prod_{m=1}^N \frac{h(\lambda_k^B, \lambda_m^B) h(\lambda_m^C, \lambda_j^C)}{h(\lambda_m^C, \lambda_k^B) h(\lambda_j^C, \lambda_m^B)} \end{aligned} \quad (45)$$

where $r(\lambda) = \frac{a(\lambda)}{d(\lambda)}$.

Norms of Bethe wave functions are special cases of (25). These were first conjectured in [8] (see also [9]). This conjecture was generalized and proved in [19]. The result is

$$\begin{aligned} \langle 0 | \prod_{j=1}^N C(\lambda_j) \prod_{k=1}^N B(\lambda_k) | 0 \rangle &= \prod_{j \neq k} f(\lambda_j, \lambda_k) \prod_{j=1}^N a(\lambda_j) d(\lambda_j) \det \mathcal{N}' , \\ \mathcal{N}'_{jk} &= \sin(2\eta) \left(-K(\lambda_j, \lambda_k) + i \delta_{jk} \frac{\partial}{\partial \lambda_j} \left[\ln(r(\lambda_j)) + \sum_{n=1}^N \ln \left(\frac{h(\lambda_j, \lambda_n)}{h(\lambda_n, \lambda_j)} \right) \right] \right) \\ &= \sin(2\eta) \left(-K(\lambda_j, \lambda_k) + \delta_{jk} \left[i \frac{\partial}{\partial \lambda_j} \ln(r(\lambda_j)) + \sum_{n=1}^N K(\lambda_j, \lambda_n) \right] \right), \end{aligned} \quad (46)$$

where $K(\lambda, \mu)$ and $h(\lambda, \mu)$ are defined in (14). We note that (46) can be obtained directly from (45) by setting all dual fields to zero and then taking the sets of spectral parameters $\{\lambda^C\}$ and $\{\lambda^B\}$ equal and imposing the Bethe equations (12).

2.4. Determinant Representation for FSFP correlation function

Let us now use the machinery built up above express $P(m)$ as a determinant. We will proceed in two steps: we first will analyse (23) *without* using that $\{\lambda^B\} = \{\lambda^C\} = \{\lambda\}$ and *without* imposing the Bethe-equations (12). In the second step we will then impose these two constraints. Using (45) we can represent the scalar product in the two-site generalized models in (23) as a determinant and obtain

$$\begin{aligned} P(m) &= \frac{1}{\sigma_N} \prod_{j>k} g(\lambda_j^C, \lambda_k^C) g(\lambda_k^B, \lambda_j^B) \langle 0 | \det s(\{\lambda^C\}, \{\lambda^B\}) | 0 \rangle , \\ (s(\{\lambda^C\}, \{\lambda^B\}))_{jk} &= a_1(\lambda_j^C) d_1(\lambda_k^B) \left[t(\lambda_j^C, \lambda_k^B) a_2(\lambda_j^C) d_2(\lambda_k^B) \exp(\Phi_A(\lambda_j^C) + \Phi_D(\lambda_k^B)) \right. \\ &\quad \left. + t(\lambda_k^B, \lambda_j^C) d_2(\lambda_j^C) a_2(\lambda_k^B) \exp(\Phi_D(\lambda_j^C) + \Phi_A(\lambda_k^B)) \right]. \end{aligned} \quad (47)$$

Here the dual fields are defined according to

$$\begin{aligned} \Phi_A(\lambda) &= Q_A(\lambda) + P_D(\lambda), & \Phi_D(\lambda) &= Q_D(\lambda) + P_A(\lambda), \\ [P_D(\lambda), Q_D(\mu)] &= \ln(h(\lambda, \mu)), & [P_A(\lambda), Q_A(\mu)] &= \ln(h(\mu, \lambda)). \end{aligned} \quad (48)$$

All other commutators vanish. The reference state $|0\rangle$ and its dual $\langle 0|$ are defined via

$$P_a(\lambda)|0\rangle = 0, \quad \langle 0|Q_a(\lambda) = 0, \quad a = A, D, \quad \langle 0|0\rangle = 1. \quad (49)$$

So far we have not used the fact that we are dealing with expectation values of Bethe states, *i.e.* we have neither used the fact that $\{\lambda^C\} = \{\lambda^B\} = \{\lambda\}$ nor imposed the Bethe equations (12). Let us now impose these constraints. Like for the case of scalar products one of the dual fields can be eliminated

We define a new dual vacuum $(\tilde{0}|$ and a new dual field according to

$$\begin{aligned} (\tilde{0}| &= (0| \exp \left(\sum_{j=1}^N P_D(\lambda_j) + P_A(\lambda_j) \right) , \quad (\tilde{0}|0) = 1 , \\ \phi(\lambda) &= \Phi_A(\lambda) - \Phi_D(\lambda) , \end{aligned} \quad (50)$$

The momenta $\mathring{p}(\lambda)$ and coordinates $\mathring{q}(\lambda)$ of the dual field $\phi(\lambda)$ obey the commutation relations

$$[\mathring{q}(\mu), \mathring{p}(\lambda)] = \ln(h(\lambda, \mu)h(\mu, \lambda)) \quad (51)$$

(all other commutators vanish). By straightforward rewriting of (47) in terms of the new field and the new dual reference state we obtain

$$\begin{aligned} P(m) &= \frac{1}{\sigma_N} \prod_{j>k} f(\lambda_j, \lambda_k) f(\lambda_k, \lambda_j) \prod_{j=1}^N a(\lambda_j) d(\lambda_j) (\tilde{0}| \det \bar{\mathcal{M}} |0) , \\ \bar{\mathcal{M}}_{jk} &= t(\lambda_j, \lambda_k) + t(\lambda_k, \lambda_j) \frac{r_2(\lambda_k)}{r_2(\lambda_j)} \exp(\phi(\lambda_k) - \phi(\lambda_j)) . \end{aligned} \quad (52)$$

Here we have used that

$$(0| \exp \left(\sum_{j=1}^N \Phi_A(\lambda_j) + \Phi_D(\lambda_j) \right) = \prod_{j,k} h(\lambda_j, \lambda_k) (\tilde{0}| . \quad (53)$$

It is found that whereas $\mathring{p}(\lambda)|0) = 0$, the coordinate $\mathring{q}(\lambda)$ of $\phi(\lambda)$ do not annihilate the new dual reference state $(\tilde{0}|$. Therefore we “shift” $\phi(\lambda)$ by subtracting its vacuum expectation value in analogy with (44)

$$\varphi(\lambda) = \phi(\lambda) - (\tilde{0}|\phi(\lambda)|0) = p(\lambda) + q(\lambda) . \quad (54)$$

By construction p and q have the same commutation relations (54) as the momenta/coordinates $\mathring{p}(\lambda)$ and $\mathring{q}(\lambda)$ of $\phi(\lambda)$. Furthermore $p(\lambda)|0) = 0$ and $(\tilde{0}|q(\lambda) = 0$. The shift is equal to

$$\kappa(\lambda) = (\tilde{0}|\phi(\lambda)|0) = \sum_j \ln \left(\frac{h(\lambda, \lambda_j)}{h(\lambda_j, \lambda)} \right) . \quad (55)$$

If we replace ϕ in (52) by φ we pick up additional factors due to the shifts

$$\begin{aligned} P(m) &= \frac{1}{\sigma_N} \prod_{j \neq k} f(\lambda_j, \lambda_k) \prod_{j=1}^N a(\lambda_j) d(\lambda_j) (\tilde{0}| \det \mathcal{G} |0) , \\ \mathcal{G}_{jk} &= t(\lambda_j, \lambda_k) + t(\lambda_k, \lambda_j) \frac{r_2(\lambda_k)}{r_2(\lambda_j)} e^{\varphi(\lambda_k) - \varphi(\lambda_j)} e^{\kappa(\lambda_k) - \kappa(\lambda_j)} . \end{aligned} \quad (56)$$

The off-diagonal matrix elements of G can be further simplified by simply imposing the Bethe equations. Rewriting the Bethe equations (12) as

$$\begin{aligned} r_2(\lambda_k) \prod_{\substack{j=1 \\ j \neq k}}^N \frac{h(\lambda_k, \lambda_j)}{h(\lambda_j, \lambda_k)} &= \frac{(-1)^{N-1}}{r_1(\lambda_k)}, \\ \frac{1}{r_2(\lambda_k)} \prod_{\substack{j=1 \\ j \neq k}}^N \frac{h(\lambda_j, \lambda_k)}{h(\lambda_k, \lambda_j)} &= (-1)^{N-1} r_1(\lambda_k), \quad k = 1, \dots, N \end{aligned} \quad (57)$$

we find that the additional factors take the form

$$\exp(\kappa(\lambda_k) - \kappa(\lambda_j)) \frac{r_2(\lambda_k)}{r_2(\lambda_j)} = \frac{r_1(\lambda_j)}{r_1(\lambda_k)}. \quad (58)$$

To get the diagonal matrix elements we have to investigate the limit $\lambda_j \rightarrow \lambda_k$ of (56) in detail. In the limit $\lambda_j \rightarrow \lambda_k$ the sum of the first two terms in G_{jk} and the expression in brackets are both of the form “ $\frac{0}{0}$ ”. By using l’Hospital’s rule we find

$$\begin{aligned} \lim_{\lambda_j \rightarrow \lambda_k} \left(t(\lambda_j, \lambda_k) + t(\lambda_k, \lambda_j) \frac{r_2(\lambda_k)}{r_2(\lambda_j)} e^{\varphi(\lambda_k) - \varphi(\lambda_j)} e^{\kappa(\lambda_k) - \kappa(\lambda_j)} \right) &= -2 \cosh(2i\eta) \\ + \sinh(2i\eta) \frac{\partial \varphi(\lambda)}{\partial \lambda} \Big|_{\lambda=\lambda_j} + \sinh(2i\eta) \frac{\partial}{\partial \lambda_j} \left[\ln(r_2(\lambda_j)) + \sum_n \ln \left(\frac{h(\lambda_j, \lambda_n)}{h(\lambda_n, \lambda_j)} \right) \right] & \end{aligned} \quad (59)$$

Putting now everything together we obtain the following representation of the FSFP as a ratio of two determinants

$$\begin{aligned} P(m) &= \frac{(\tilde{0} | \det \mathcal{G} | 0)}{\det \mathcal{N}'}, \\ \mathcal{G}_{jk} &= t(\lambda_j, \lambda_k) + t(\lambda_k, \lambda_j) \frac{r_1(\lambda_j)}{r_1(\lambda_k)} \exp(\varphi(\lambda_k) - \varphi(\lambda_j)) \\ &\quad + \delta_{jk} \sin(2\eta) \left(LD(\lambda_j) + \sum_n K(\lambda_j, \lambda_n) \right), \end{aligned} \quad (60)$$

where φ and \mathcal{N}' are defined in (54) and (46) respectively and where D and K are defined in (14). Eqn (60) is our final result for the FSFP on a *finite* chain of length L . As always in the Bethe Ansatz things simplify essentially in the thermodynamic limit. Let us first investigate the thermodynamic limit for the norm σ_N (46). We write \mathcal{N}' as the product of two matrices:

$$\mathcal{N}'_{jk} = \sin(2\eta) \sum_m I_{jm} J_{mk}, \quad I_{jm} = \delta_{jm} - \frac{K_{jm}}{\theta_m}, \quad J_{jm} = \delta_{jm} \theta_m, \quad (61)$$

where $\theta_m = LD(\lambda_m) + \sum_n K(\lambda_m, \lambda_n)$. The determinant of \mathcal{N}' is the product of the determinants of I and J . Next we use that the set of roots $\{\lambda_j\}$ describes the ground state and the roots thus obey the equations

$$2\pi L\rho(\lambda_j) - \sum_{k=1}^N K(\lambda_j, \lambda_k) = LD(\lambda_j), \quad j = 1 \dots N, \quad (62)$$

which is the discrete version of (13). Here $\rho(\lambda_j) = \frac{1}{L(\lambda_{j+1}-\lambda_j)}$, which becomes $\rho(\lambda)$ defined by (13) in the thermodynamic limit. We thus can rewrite $\theta_m = 2\pi L\rho(\lambda_m)$, which leads to

$$\det J = \prod_{j=1}^N 2\pi L\rho(\lambda_j) . \quad (63)$$

In the thermodynamic limit the matrix I turns into an integral operator $\widehat{I} = id - \frac{1}{2\pi}\widehat{K}$

$$\widehat{I} * f \Big|_{\lambda} = f(\lambda) - \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} d\mu K(\lambda, \mu) f(\mu) , \quad (64)$$

where K is the kernel of \widehat{K} defined by (14). The matrix G_{jk} in (60) is treated in a very similar way. We rewrite it as a product

$$G_{jk} = \sin(2\eta) \sum_m W_{jm} J_{mk} , \quad (65)$$

where $J_{jm} = \delta_{jm} 2\pi L\rho(\lambda_m)$ is the same as above, and

$$W_{jk} = \delta_{jk} + \frac{1}{\sin(2\eta)\theta_k} \left\{ t(\lambda_j, \lambda_k) + t(\lambda_k, \lambda_j) \frac{r_1(\lambda_j)}{r_1(\lambda_k)} \exp(\varphi(\lambda_k) - \varphi(\lambda_j)) \right\} . \quad (66)$$

In the thermodynamic limit the matrix W_{jk} turns into an integral operator $\widehat{W} = id + \frac{1}{2\pi}\widehat{V}$ with kernel

$$V^{(m)}(\lambda, \mu) = \frac{-\sin(2\eta)}{\sinh(\lambda - \mu)} \left\{ \frac{1}{\sinh(\lambda - \mu + 2i\eta)} - \frac{e^{-1}(\lambda)e(\mu)}{\sinh(\mu - \lambda + 2i\eta)} \right\} , \quad (67)$$

$$e(\lambda) = \left(\frac{\sinh(\lambda + i\eta)}{\sinh(\lambda - i\eta)} \right)^m \exp(\varphi(\lambda)) . \quad (68)$$

Thus in the thermodynamic limit the FSFP $P(m)$ for the XXZ magnet can be represented as a ratio of determinants of Fredholm integral operators $(id + \frac{1}{2\pi}\widehat{V})$ and $(id - \frac{1}{2\pi}\widehat{K})$ in the following way

$$P(m) = \frac{(\tilde{0} | \det(id + \frac{1}{2\pi}\widehat{V}) | 0)}{\det(id - \frac{1}{2\pi}\widehat{K})} . \quad (69)$$

Here $(\tilde{0} |$ and $|0)$ are the vacua of the dual bosonic Fock space defined in (50).

3. Integro-Difference Equations

In this section we embed the numerator of the r.h.s. of (69) in a system of integro-difference equations. The denominator is independent of the distance m and thus merely amounts to an overall normalization of $P(m)$, which is difficult to determine

anyhow and will be dropped in what follows. We start by bringing the kernel of \hat{V} (67) to “standard” form²². This is done in several steps. We first perform the similarity transformation \hat{S} with kernel $S(\lambda) = (e(\lambda))^{\frac{1}{2}}$ on $1 + \frac{1}{2\pi}\hat{V}$, which leaves the determinant unchanged. This “symmetrizes” the kernel

$$V^{(m)}(\lambda, \mu) = \frac{-\sin(2\eta)}{\sinh(\lambda - \mu)} \left\{ \frac{\bar{e}^{-1}(\mu)\bar{e}(\lambda)}{\sinh(\lambda - \mu + 2i\eta)} - \frac{\bar{e}^{-1}(\lambda)\bar{e}(\mu)}{\sinh(\mu - \lambda + 2i\eta)} \right\}, \quad (70)$$

where

$$\bar{e}(\lambda) = \left(\frac{\sinh(\lambda + i\eta)}{\sinh(\lambda - i\eta)} \right)^{\frac{m}{2}} \exp\left(\frac{\varphi(\lambda)}{2}\right). \quad (71)$$

Next we change variables to $\beta = \exp(2\lambda)$, $\alpha = \exp(2\mu)$, $w = \exp(2i\eta)$. Elementary calculations yield

$$\begin{aligned} \frac{1}{2\alpha} V\left(\frac{\ln(\beta)}{2}, \frac{\ln(\alpha)}{2}\right) &= \frac{-2\sin(2\eta)\beta}{\beta - \alpha} \left\{ \left(\frac{\beta w - 1}{\beta - w}\right)^{\frac{m}{2}} \left(\frac{\alpha w - 1}{\alpha - w}\right)^{-\frac{m}{2}} \times \right. \\ &\times \left. \frac{1}{\beta w - \frac{\alpha}{w}} \exp\left(\frac{\varphi\left(\frac{\ln(\beta)}{2}\right)}{2} - \frac{\varphi\left(\frac{\ln(\alpha)}{2}\right)}{2}\right) - \alpha \leftrightarrow \beta \right\}. \end{aligned} \quad (72)$$

Next we use the identity

$$\int_0^\infty ds \exp\left(-is\left(\beta w - \frac{\alpha}{w}\right)\right) = \frac{-i}{\beta w - \frac{\alpha}{w}}, \quad (73)$$

(which holds as $\alpha > 0$, $\beta > 0$ and $\sin(2\eta) < 0$) in order to eliminate the unwanted denominators in (72), and finally perform yet another change of variables to $z(\alpha) = \frac{\alpha w - 1}{\alpha - w}$. This change of variables maps the real axis on the contour $C : \alpha \rightarrow z = \exp i\alpha$ where $-\psi < \alpha < 2\pi + \psi \leq \eta$ ($\psi < 0$ by definition). The endpoints $\xi = e^{i\psi}$ and $\bar{\xi} = e^{-i\psi}$ (we integrate from $\bar{\xi}$ to ξ) of the contour are related to the magnetic field h and the anisotropy Δ . After elementary computations we obtain

$$\frac{d\alpha}{2\alpha} V\left(\frac{\ln(\beta)}{2}, \frac{\ln(\alpha)}{2}\right) = \frac{dz_2}{i} \left(\frac{wz_1 - 1}{wz_2 - 1} \frac{z_1 - w}{z_2 - w}\right) \int_0^\infty ds \frac{e_+^{(m)}(z_1|s)e_-^{(m)}(z_2|s) - z_1 \leftrightarrow z_2}{z_1 - z_2}, \quad (74)$$

where the functions $e_\pm^{(m)}$ are given by

$$\begin{aligned} e_+^{(m)}(z|s) &= \left(2\sin(2\eta) \frac{wz - 1}{z - w} z^{-m} e^{-\phi(z)}\right)^{\frac{1}{2}} \exp\left(i\frac{s}{w} \frac{wz - 1}{z - w}\right) \\ e_-^{(m)}(z|s) &= \left(2\sin(2\eta) \frac{wz - 1}{z - w} z^m e^{\phi(z)}\right)^{\frac{1}{2}} \exp\left(-isw \frac{wz - 1}{z - w}\right). \end{aligned} \quad (75)$$

Note that the ‘‘momenta’’ $p(z)$ and ‘‘coordinates’’ $q(z)$ of the dual field $\phi(z)$ now obey the commutation relations

$$[q(z_2), p(z_1)] = \ln \left[\frac{1}{4 \sin^2(2\eta)} \left(\frac{wz_2 - 1}{wz_1 - 1} \frac{z_1 - w}{z_2 - w} + \frac{wz_1 - 1}{wz_2 - 1} \frac{z_2 - w}{z_1 - w} - w^2 - w^{-2} \right) \right]. \quad (76)$$

Eliminating the factor in brackets before the integral in (74) by a similarity transformation we obtain

$$\begin{aligned} \det\left(1 + \frac{1}{2\pi} \hat{V}\right) &= \det(1 + \hat{V}^{(m)}) \\ V^{(m)}(z_1|z_2) &= \frac{i}{2\pi} \int_0^\infty ds \frac{e_+^{(m)}(z_1|s)e_-^{(m)}(z_2|s) - e_-^{(m)}(z_1|s)e_+^{(m)}(z_2|s)}{z_1 - z_2}. \end{aligned} \quad (77)$$

The integral operator $\hat{V}^{(m)}$ now acts on a function $f(z)$ as

$$(\hat{V}^{(m)} f)(z_1) = \int_C dz_2 V^{(m)}(z_1|z_2) f(z_2), \quad (78)$$

where the integration is to be performed along the contour C . We note that V is symmetric and nonsingular at $z_1 = z_2$. Eqn (77) is the desired standard form for the kernel, and was obtained for the price of introducing the auxiliary s -integration.

The resolvent $\hat{R}^{(m)}$ of $\hat{V}^{(m)}$ is defined by $(1 + \hat{V}^{(m)})(1 - \hat{R}^{(m)}) = 1$ and its kernel $R^{(m)}(z_1|z_2)$ can be written in a form similar to Eq. (77), namely²²

$$R^{(m)}(z_1|z_2) = -\frac{i}{2\pi} \int_0^\infty ds \frac{f_+^{(m)}(z_1|s)f_-^{(m)}(z_2|s) - f_-^{(m)}(z_1|s)f_+^{(m)}(z_2|s)}{z_1 - z_2}. \quad (79)$$

Here the functions $f_\pm^{(m)}$ are solutions of the linear integral equations

$$f_\pm(z_1|s) + \int_C dz_2 V(z_1|z_2) f_\pm(z_2|s) = e_\pm(z_1|s). \quad (80)$$

In terms of these functions we introduce the integral operators $B_{ab}^{(m)}$, $a, b = \pm$ acting as $(B_{ab}^{(m)} f^{(m)})(s) = \int_0^\infty dt B_{ab}^{(m)}(s, t) f(t)$ with the kernel

$$B_{ab}^{(m)}(s, t) = \frac{i}{2\pi} \int_C \frac{dz}{z} f_a^{(m)}(z|s) e_b^{(m)}(z|t), \quad a, b = \pm. \quad (81)$$

The *transpose* B^T acts like $(B^T f)(s) = \int_0^\infty dt B(t, s) f(t)$. We are now able to formulate the embedding of $P(m)$ into a set of integrable integro-difference equations:

- (i) The lattice logarithmic derivative of $\det(1 + \hat{V}^{(m)})$ is given in terms of the integral operator $B_{ab}^{(m)}$ as

$$\frac{\det(1 + \hat{V}^{(m+1)})}{\det(1 + \hat{V}^{(m)})} = \det(1 + B_{-+}^{(m)}) \quad (82)$$

- (ii) The logarithmic derivative of $\det(1 + \widehat{V}^{(m)})$ with respect to the boundaries of the contour C is expressed in terms of the functions $F_{\pm}^{(m)}(s) = f_{\pm}^{(m)}(\xi|s)$ and $G_{\pm}^{(m)}(s) = f_{\pm}^{(m)}(\bar{\xi}|s)$ as follows

$$\begin{aligned} -i\partial_{\psi} \ln \det(1 + \widehat{V}) &= \frac{1}{2\pi} \int_0^{\infty} ds \left\{ F_+(s) \partial_{\psi} F_-(s) - F_-(s) \partial_{\psi} F_+(s) \right. \\ &\quad \left. + G_-(s) \partial_{\psi} G_+(s) - G_+(s) \partial_{\psi} G_-(s) \right\} \\ &\quad + \frac{1}{4\pi^2} \frac{\xi + \bar{\xi}}{\xi - \bar{\xi}} \left(\int_0^{\infty} ds F_+(s) G_-(s) - F_-(s) G_+(s) \right)^2 \end{aligned} \quad (83)$$

- (iii) The following set of completely integrable integro-difference equations for the unknowns F, G, B in (i) and (ii) holds

$$\begin{aligned} \frac{1}{\sqrt{\xi}} F_+^{(m+1)} &= \frac{1}{\xi} \left\{ F_+^{(m)} - (1 - B_{+-}^{(m+1)}) (B_{++}^{(m)})^T F_-^{(m)} \right\} \\ \frac{1}{\sqrt{\bar{\xi}}} F_-^{(m+1)} &= \frac{1}{\xi} \left\{ (\xi + B_{--}^{(m+1)}) (B_{++}^{(m)})^T F_-^{(m)} - B_{--}^{(m+1)} (1 + (B_{+-}^{(m)})^T) F_+^{(m)} \right\} \end{aligned} \quad (84)$$

$$\begin{aligned} \frac{1}{\sqrt{\xi}} G_+^{(m+1)} &= \frac{1}{\bar{\xi}} \left\{ G_+^{(m)} - (1 - B_{+-}^{(m+1)}) (B_{++}^{(m)})^T G_-^{(m)} \right\} \\ \frac{1}{\sqrt{\bar{\xi}}} G_-^{(m+1)} &= \frac{1}{\bar{\xi}} \left\{ (\bar{\xi} + B_{--}^{(m+1)}) (B_{++}^{(m)})^T G_-^{(m)} - B_{--}^{(m+1)} (1 + (B_{+-}^{(m)})^T) G_+^{(m)} \right\} \end{aligned}$$

and

$$\begin{aligned} -i \frac{\partial}{\partial \psi} B_{ab}(s, t) &= \frac{i}{2\pi} \left\{ F_a(s) [F_b(t) + (F_+ B_{-b} - F_- B_{+b})(t)] \right. \\ &\quad \left. + G_a(s) [G_b(t) + (G_+ B_{-b} - G_- B_{+b})(t)] \right\} \end{aligned} \quad (85)$$

where $a, b = \pm$.

The proof of (i)-(iii) is analogous to the one for the XXX-case ⁷ so that we only sketch the main steps. (i) is a direct consequence of the *shift-equation*

$$z_1^{-\frac{1}{2}} V^{(m+1)}(z_1|z_2) z_2^{\frac{1}{2}} = V^{(m)}(z_1|z_2) + \frac{i}{2\pi} \int_0^{\infty} ds \frac{e_+^{(m)}(z_1|s) e_-^{(m)}(z_2|s)}{z_1}. \quad (86)$$

which follows directly from (67) and (76), whereas (ii) and (iii) are consequences of the Lax-representation

$$\begin{aligned} \frac{1}{\sqrt{z}} f_-^{(m+1)}(z|s) &= f_-^{(m)}(z|s) - \frac{1}{z} B_{--}^{(m+1)} \left(\left[1 + (B_{-+}^{(m)})^T \right] f_+^{(m)} - (B_{++}^{(m)})^T f_-^{(m)} \right) (z|s) \\ \frac{1}{\sqrt{z}} f_+^{(m+1)}(z|s) &= \frac{1}{z} f_+^{(m)}(z|s) - \left(\left[1 - B_{+-}^{(m+1)} \right] (B_{++}^{(m)})^T f_-^{(m)} \right) (z|s), \end{aligned} \quad (87)$$

and

$$\begin{aligned} \frac{\partial}{\partial \psi} f_{\pm}(z|s) = & \frac{1}{2\pi} \left\{ \frac{\xi}{z-\xi} f_{\pm}(\xi|s) \int_0^{\infty} dt (f_+(\xi|t)f_-(z|t) - f_-(\xi|t)f_+(z|t)) \right. \\ & \left. + \frac{\bar{\xi}}{z-\bar{\xi}} f_{\pm}(\bar{\xi}|s) \int_0^{\infty} dt (f_+(\bar{\xi}|t)f_-(z|t) - f_-(\bar{\xi}|t)f_+(z|t)) \right\}. \end{aligned} \quad (88)$$

This concludes our derivation of the embedding of $P(m)$ in a system of integrable (due to the Lax-representation (87)-(88)) integro-difference equations. We note that there is a integral-operator-valued Riemann-Hilbert problem associated with these IDE⁵, and that the asymptotics of $P(m)$ for large m can be extracted from the solution of the Riemann-Hilbert problem. So far this has been done only for some limiting cases⁵, but we hope to report the full solution in the future.

4. Acknowledgements

V.E.K. thanks the organizers of the conference for support.

5. References

1. E. Barouch, B.M. McCoy, T.T. Wu, *Phys. Rev. Lett.* **31** (1973) 1409.
2. R.J. Baxter, *Ann. Phys. (N.Y.)* **70** (1972) 323.
3. H. Bethe, *Z.Phys.* **71** (1931) 205.
4. B. Davies, O. Foda, M. Jimbo, T. Miwa, A. Nakayashiki, *Comm. Math. Phys.* **151** (1993) 89.
5. F.H.L. Eßler, H. Frahm, A.R. Its, V.E. Korepin, *preprint*.
6. L.D. Faddeev, L.A. Takhtajan, *Sov. Sci. Rev. Math.* **C1** (1981) 107.
7. H. Frahm, A.R. Its, V.E. Korepin, *Nucl. Phys.* **B428** (1994) 694.
8. M. Gaudin, internal report *Centre d'Études Nucléaires de Saclay*, CEA-N-1559, (1), (1972).
9. M. Gaudin, B.M. McCoy, T.T. Wu, *Phys. Rev.* **D23** (1981) 417.
10. L. Hulthen, *Arkiv Mat. Astron. Fysik* **26A** (1938).
11. A.R. Its, A.G. Izergin, V.E. Korepin, *Comm. Math. Phys.* **130** (1990) 471, *Physica* **D53** (1991) 187, *Phys. Lett.* **141A** (1989) 121.
12. A.R. Its, A.G. Izergin, V.E. Korepin, N. Slavnov, *Int. J. Mod. Phys.* **B4** (1990) 1003.
13. A.R. Its, A.G. Izergin, V.E. Korepin, G.G. Varzugin, *Physica* **D54** (1992) 351.
14. A.G. Izergin, *Sov. Phys. Dokl.* **32** (1987) 878.
15. A.G. Izergin, D. Coker, V.E. Korepin, *J. Physics* **A25** (1992) 4315.
16. A.G. Izergin, V.E. Korepin, *Comm. Math. Phys.* **113** (1987) 117.
17. M. Jimbo, T. Miwa, Y. Mori, M. Sato, *Physica* **1D** (1980) 80.
18. M. Jimbo, K. Miki, T. Miwa, A. Nakayashiki, *Phys. Lett.* **168A** (1992) 256.

19. V.E. Korepin, *Comm. Math. Phys.* **86** (1982) 391.
20. V.E. Korepin, *Comm. Math. Phys.* **113** (1987) 177.
21. V.E. Korepin, *Func. Analysis Appl.* **23** (1989) 15.
22. V.E. Korepin, G. Izergin and N.M. Bogoliubov, *Quantum Inverse Scattering Method, Correlation Functions and Algebraic Bethe Ansatz*, Cambridge University Press, 1993
23. V.E. Korepin, A.G. Izergin, F.H.L. Eßler, D. Uglov, *Phys. Lett.* **190A** (1994) 182.
24. V.E. Korepin, N. Slavnov, *Comm. Math. Phys.* **129** (1990) 103.
25. V.E. Korepin, N. Slavnov, *Comm. Math. Phys.* **136** (1991) 633.
26. P.P. Kulish, N. Yu. Reshetikhin, E.K. Sklyanin, *Letters in Math. Phys.* **5** (1981) 393.
27. P.P. Kulish, E.K. Sklyanin, *J. Soviet Math.* **19** (1982) 1596.
28. E.H. Lieb, W. Liniger, *Phys. Rev.* **130** (1963) 1605.
29. E.H. Lieb, *Phys. Rev.* **130** (1963) 1616.
30. R. Orbach, *Phys. Rev.* **112** (1958) 309.
31. E.K. Sklyanin, L.A. Takhtajan, L.D. Faddeev, *Theor. Mat. Phys.* **40** (1979) 688.
32. M. Takahashi, *Prog. Theor. Phys.* **46** (1971) 401.
33. M. Takahashi, M. Suzuki, *Prog. Theor. Phys.* **48** (1972) 2187.
34. L.A. Takhtajan, *Exactly Solvable Problems in Condensed Matter and Relativistic field Theory*, B.S. Shastry, S.S. Jha, V. Singh (eds.) Lecture Notes in Physics, v.242, Berlin: Springer Verlag, (1985), p.175.
35. L.A. Takhtajan, L.D. Faddeev, *Russ. Math. Survey* **34** (1979) 11.
36. C. Tracy, B.M. McCoy, *Phys. Rev. Lett.* **31** (1973) 1500.
37. H.G. Vaidya, C. Tracy, *Phys. Lett.* **68A** (1978) 378.
38. C.N. Yang, C.P. Yang, *Phys. Rev.* **150** (1966) 321.
39. C.N. Yang, C.P. Yang, *Phys. Rev.* **150** (1966) 327.
40. C.N. Yang, *Phys. Rev. Lett.* **19** (1967) 1312.