A Diagrammatic Approach to Unitary Representations of Rational Cherednik Algebras

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Abstract

This thesis investigates the image of modules of rational Cherednik algebra under the KZ functor and its the compatibility with the unitarity conditions of modules of cyclotomic Hecke algebras. The special case of type A has been well-studied in the literature. Here we aim to generalize the result to rational Cherednik algebras of $G(l,1,n)$ with generic parameters, and postulate that the KZ functor preserves unitarity. The approach is via the isomorphism between the KZ functor and the direct sum of generalized weight space functors, which leads to the introduction of weighted Khovanov-Lauda-Rouquier (KLR) algebras encoding the action of the intertwining operators. The cyclotomic Hecke algebra whose representations are images under the KZ functor is isormorphic to the cyclotomic KLR algebra as a subalgebra of the weighted KLR algebra. We use this subalgebra as a bridge to discuss how the image of the direct sum of generalized weight space functors becomes a representation of the cyclotomic Hecke algebra, and show that $\mathrm{KZ}(M_c(\lambda)) \cong V^{\lambda^{tr}}$ for generic parameters. Then we pass the $*$-operation on the rational Cherednik algebra to the cyclotomic Hecke algebra. We compare the $*$-operation with the star-operation that defines unitarity of cyclotomic Hecke algebras. Furthermore, we prove a stronger statement by equating the asymptotic signature of the Hermitian form $\beta_{c,\lambda}$ of rational Cherednik algebras with the signature of Hermitian forms of cyclotomic Hecke algebras.

Contents

0 Introduction

The KZ functor is a functor sending a module from Category $\mathcal{O}$ of a rational Cherednik algebra to a representation of some cyclotomic Hecke algebra [2]. This thesis studies the images $\mathrm{KZ}(L_c(\tau))$ of simple modules $L_c(\tau)$ under the KZ functor of rational Cherednik algebra of $G(l,1,n)$ and whether the KZ functor is compatible with the unitarity conditions of modules $L_c(\tau)$ and $\mathrm{KZ}(L_c(\tau))$.

The direct computation of the image under the KZ functor is very difficult. We may study a direct sum of generalized weight space functors $\bigoplus_{\mathbf{d}\in D^n} W_{\mathbf{a}_d^{\pm},\mathbf{1}}$ instead, which is isomorphic to the KZ functor [25]. The generalized weight space functors $W_{\mathbf{a},\mathbf{t}}$ send modules of rational Cherednik algebras to their generalized eigenspaces with respect to the generalized eigenvalues $(\mathbf{a},\mathbf{t})$. Hence the question of computing the image under the KZ functor becomes computing certain generalized eigenspaces.

The image under the KZ functor is a representation of the cyclotomic Hecke algebra. However, it is unclear that why the direct sum of some chosen generalized eigenspaces forms a representation of cyclotomic Hecke algebra. According to Webster [25], the direct sum of all generalized eigenspaces of a modules from a rational Cherednk algebra is isomorphic to a module of the weighted KLR algebra, where latter is extended from the concept of the KLR algebra. Moreover, the completion of the affine Hecke algebra is isomorphic to the completion of the KLR algebra [24]. We combine the above two aspects and find a KLR algebra as a subaglebra of the weighted KLR algebra, whose cyclotomic quotient is isomorphic to the cyclotomic Hecke algebra. To simplify the computation, we assume that the parameters are generic for both rational Cherednik algebras and cyclotomic Hecke algebras, It turns out that the image under $\bigoplus_{\mathbf{d}\in D^n} W_{\mathbf{a}_d^{\pm},\mathbf{1}}$ is exactly a representation of this KLR algebra and hence of the cyclotomic Hecke algebra for generic parameters.

Theorem 0.1. Suppose the parameters are generic for both rational Cherednik algebras and cyclotomic Hecke algebras. Then $\mathrm{KZ}(L_c(\tau)) \cong V^{\tau^{tr}}$, where $V^{\tau^{tr}}$ is an irreducible representation of the cyclotomic Hecke algebra.

The unitarity conditions of their representations of type A are already known [9], that is when the parameter lies in an interval plus some isolated rational numbers. The result that the KZ functor preserves the unitarity of $L_c(\tau)$ of type A rational Cherednik algebras for parameters $c \in (-\frac{1}{2}, \frac{1}{2}]$ [19] inspires us to postulate the following conjecture.

Conjecture 0.2. The KZ functor preserves the unitarity if simple modules of rational Cherednik algebra of $G(l,1,n)$, that is $\mathrm{KZ}(L_c(\tau))$ is unitary of $L_c(\tau)$ is.

In order to discuss unitarity of modules of rational Cherednik algebras and cyclotomic Hecke algebras, we need to define star operations on both algebras. Under the assumption that parameters are generic for both algebras, we consider the image under the KZ functor as a subspace of the module of rational Cherednik algebra and pass the star operation of the rational Cherednik algebra (called $*$-operation) to the cyclotomic Hecke algebra via the weighted KLR algebra, and finally compare it with the original star operation on the cyclotomic Hecke algebra. Unfortunately, the $*$-operation passed on from the rational Cherednik algebra is different from the star-operation that defines unitarity in the cyclotomic Hecke algebra. But it still gives us some insight into the KZ functor and the two Hermitian forms involved.

Motivated by our inconclusive attempt to understand the KZ functor and unitarity, there is a need for new directions. In addition, the converse of Conjecture 0.2 does not hold, i.e, for any simple module $L_c(\tau)$ from Category $\mathcal{O}$, the unitarity of the image $\mathrm{KZ}(L_c(\tau))$ does not imply the unitarity of $L_c(\tau)$. To build a bridge that works in both ways, we may weaken the condition of unitarity and consider the concept of quasi-unitarity introduced by Shelley-Abrahamson [18]. It is shown that if $\mathrm{KZ}(L_c(\tau))$ is unitary, then $L_c(\tau)$ is quasi-unitary for type A algebras. Relaxing unitarity to quasi-unitarity may be the modification we need.

In order to define quasi-unitarity, Shelley-Abrahamson invented the concept of asymptotic signature for the modules of rational Cherednik algebras. It measures how large the positive-definite portion is in the module when the degree goes to infinity. A module is quasi-unitary if the asymptotic signature is ± 1, i.e., either positive-definite or negative-definite part of the module dominates the module when the degree goes to infinity. Hence unitarity implies quasi-unitarity. In the same paper [18], a stronger statement is proved that the asymptotic signature of $L_c(\tau)$ is the same as the signature of $\mathrm{KZ}(L_c(\tau))$ up to a sign for real reflection groups. Based on the evidence in the rational Cherednik algebra of type A and cyclic groups, we postulate and prove the main result of the thesis in Section 9.2 for generic parameters

Theorem 0.3. Let $L_c(\tau)$ be a simple module from Category $\mathcal{O}$. Suppose the parameters of the rational Cherednik algebra and the cyclotomic Hecke algebra are generic. The asymptotic signature of $L_c(\tau)$ equals to the signature of $\mathrm{KZ}(L_c(\tau))$ up to a sign.

This theorem implies that for generic parameters the KZ functor preserves unitarity (Conjecture 0.2), and $L_c(\tau)$ is quasi-unitary if and only if $\mathrm{KZ}(L_c(\tau))$ is unitary. However, cases of non-generic parameters still requires further study.

The structure of the thesis is as follows. We introduce the basic concepts of rational Cherednik algebras and their representations in Section 1. We define a star operation on the rational Cherednik algebra H, denoted as $*$-operation. There is a unique Hermitian form on the Verma module M such that $(v \cdot m, m') = (m, v^* \cdot m')$ for $v \in H$ and $m, m' \in M$. The unique simple quotient of the Verma module is said to be unitary if the Hermitian form restricted to this quotient is positive-definite. The section is a review of [9], [8] and [2].

In Section 2, we study closely on the rational Cherednik algebra of $G(l, 1, n)$. In particular, we study the intertwining operators and their action on the basis elements in simple modules. The section is a review of [11], [12] and [13].

Section 3 gives a detailed description of the KZ functor and an example of $H_c(\mathbb{Z}_l, \mathbb{C})$. This section is based on Section 4 of [2].

Section 4 introduces the diagrammatic algebra, i.e., the weighted KLR algebra and the generalized weight space functor. At the end of the section, Theorem 4.20 establishes the direct sum of weight space functors that is isomorphic to the KZ functor. This section is a review of [25], [23], [26].

Section 5 constructs the KLR algebra inside the weighted KLR algebra, whose completion is isomorphic to the cyclotomic Hecke algebra. Section 5.1 is my original work. The first half of Section 5.2 is a review of [24] and the second half is my original work.

In section 6, we compute the cyclotomic Hecke algebra whose representations are images under the KZ functor. We prove Theorem 0.1 that is the images under the KZ functor are isomorphic to irreducible representations of the cyclotomic Hekce algebra for generic parameters as Theorem 6.10. This section is my original work.

In section 7, we compute the unitarity conditions of representations of cyclotomic Hecke algebras. By comparing the unitarity conditions of representations of cyclotomic Hecke algebra and rational Cherednik algebra, we show that in cyclic group and type B the KZ functor preserves the unitarity of the representations of rational Cherednik algebra for generic parameters. This section is my original work.

Section 8 passes the $*$-operation of the rational Cherednik algebra to the cyclotomic Hecke algebra via the weighted KLR algebra. We compare the star operation defining unitarity of representations of the cyclotomic Hecke algebra with the induced $*$-operation. Then we discuss the differences between unitarity conditions for the representation of the rational Cherednik algebra and that of the cyclotomic Hecke algebra. This section is my original work

In Section 9, we introduce the concept of quasi-unitarity and the asymptotic signature of Verma modules. We prove Theorem 0.3 in Section 9.2. Section 9.1 is a review of [18] and Section 9.2 is my original work.

Finally we conclude the paper with a brief discussion of the possible future research directions in Section 10. We introduce the Gaussian inner product and discuss why it fails in the case of $G(l, 1, n)$. Moreover, the Janzten filtration may give us deeper understanding of the connections between the KZ functor and the zeroes of Hermitian forms that is filtered by their orders. This section is a review of [18].

1 Rational Cherednik Algebras

This section introduces the rational Cherednik algebra and its modules. We define unitary representations and give examples at the end of the section. It is a review of [8], [9] and [2].

1.1 Basic Concept

Consider an n-dimensional complex vector space $\mathfrak{h}$ with a positive-definite Hermitian inner product $(\cdot, \cdot)_{\mathfrak{h}}$. Let $\mathfrak{h}^*$ be the dual of $\mathfrak{h}$. We define $\mathrm{T} : \mathfrak{h} \to \mathfrak{h}^*$ to be the anti-linear isomorphism such that $\mathrm{T}(y)(y') = (y', y)_{\mathfrak{h}}$ for all $y, y' \in \mathfrak{h}$. Then we define the inner product on $\mathfrak{h}^*$ to be compatible with T, i.e., $(x, x')_{\mathfrak{h}^*} = (\mathrm{T}^{-1}(x), \mathrm{T}^{-1}(x'))_{\mathfrak{h}}$ for any $x, x' \in \mathfrak{h}^*$ under the inner product in $\mathfrak{h}$. We denote an orthonormal basis of $\mathfrak{h}$ as $\{y_i \mid i = 1, \ldots, n\}$ and the dual basis as $\{x_i \mid i = 1, \ldots, n\}$.

Suppose that W is a finite subgroup of the group of the unitary transformations of $\mathfrak{h}$. An element $s \in W$ is called a reflection if the kernel $\ker(1 - s)$ in the vector space $\mathfrak{h}$ has dimension $n - 1$. Equivalently, s is a reflection if it is conjugate to the matrix $\mathrm{diag}(\lambda, 1, \ldots, 1)$ for some root of unity λ. Let be the set of reflections. We define a W-invariant function $c : S \to \mathbb{C}$ such that for any $g \in W$, and $s \in S$, $c(gsg^{-1}) = c(s)$. For simplicity, we will denote $c(s)$ by c_s .

For any reflection s, we can choose a vector α_s from $\mathrm{im}(1 - s)$ in $\mathfrak{h}^*$. Because the dimension of $\mathrm{im}(1 - s)$ is 1, α_s is unique up to a scalar. Similarly, we can choose a vector $\alpha_s^\vee$ from $\mathrm{im}(1 - s)$ in $\mathfrak{h}$.

Since the reflection s is conjugate to the matrix $\mathrm{diag}(\lambda, 1, \ldots, 1)$ for some non-zero λ, α_s and $\alpha_s^\vee$ are eigenvectors of s with non-trivial eigenvalues in $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively. Moreover, we know that $T(\alpha_s^\vee)$ is an eigenvector of s with some non-zero eigenvalue, which implies α_s is a scalar multiple of $T(\alpha_s^\vee)$. Hence we have $\alpha_s(\alpha_s^\vee) \neq 0$. Let us choose α_s and $\alpha_s^\vee$ such that $\alpha_s(\alpha_s^\vee) = 2$. If we take λ_s to be the non-trivial eigenvalue of s in $\mathfrak{h}^*$, then the non-trivial eigenvalue of s in $\mathfrak{h}$ is λ_s^{-1}.

Since the dimension of $\mathrm{im}(1-s)$ is 1 in $\mathfrak{h}^*$, we have $(1-s)(x) = d_x \alpha_s$ for any $x \in \mathfrak{h}^*$ and some constant $d_x \in \mathbb{C}$. In order to compute the constant d_x, we need to use the equation

$$(s(x), \alpha_s)_{\mathfrak{h}^*} = (x, s^{-1}(\alpha_s))_{\mathfrak{h}^*} = (x, \lambda_s^{-1}\alpha_s)_{\mathfrak{h}^*} = \lambda_s (x, \alpha_s)_{\mathfrak{h}^*} \,. \tag{1.1}$$

It gives

$$d_x(\alpha_s, \alpha_s)_{\mathfrak{h}^*} = ((1-s)(x), \alpha_s)_{\mathfrak{h}^*} = (1-\lambda_s)(x, \alpha_s)_{\mathfrak{h}^*} \,, \tag{1.2}$$

which implies that

$$d_x = \frac{(1-\lambda_s)(x, \alpha_s)_{\mathfrak{h}^*}}{(\alpha_s, \alpha_s)_{\mathfrak{h}^*}} = \frac{1-\lambda_s}{2} x(\alpha_s^\vee) \,. \tag{1.3}$$

Similarly, for any $y \in \mathfrak{h}$, we have

$$(1-s)(y) = \frac{1-\lambda_s^{-1}}{2} \alpha_s(y) \alpha_s^\vee \,. \tag{1.4}$$

We also need the following properties.

Lemma 1.1. If $X_s = \mathbb{C}\alpha_s$ and $Y_s = \mathbb{C}\alpha_s^\vee$, then we have $X_s^\perp = \ker(1-s) \subset \mathfrak{h}^*$ and $Y_s^\perp = \ker(1-s) \subset \mathfrak{h}$. Moreover, we have $\ker(\alpha_s) = Y_s^\perp \subset \mathfrak{h}$ and $\ker(\alpha_s^\vee) = X_s^\perp \subset \mathfrak{h}$.

Proof. For any $x \in \ker(1-s)$, we have $(x, \alpha_s)_{\mathfrak{h}^*} = s(x)$ and $s(\alpha_s) = (\alpha_s, \lambda_s \alpha_s)_{\mathfrak{h}^*}$, which implies that $(1-\lambda_s^{-1})(x, \alpha_s)_{\mathfrak{h}^*} = 0$. Since λ_s is non-trivial, we have $(x, \alpha_s)_{\mathfrak{h}^*} = 0$. As $X_s^\perp$ and $\ker(1-s)$ have the same dimension, they are the same vector space. The same arguments can be applied for Y_s.

To prove the second part, consider $y \in Y_s^\perp$, then

$$\begin{aligned} \alpha_s(y) &= (T^{-1}(\alpha_s), y)_{\mathfrak{h}} \\ &= (\alpha_s, T(y))_{\mathfrak{h}^*} \\ &= \frac{1}{1-\lambda_s^{-1}} ((1-s^{-1})\alpha_s, T(y))_{\mathfrak{h}^*} \end{aligned}$$

$$= \frac{1}{1-\lambda_s^{-1}}(\alpha_s, (1-s)T(y))_{\mathfrak{h}^*}$$

$$= 0\,. \tag{1.5}$$

Since $Y_s^{\perp}$, and $\ker(\alpha_s)$ also have the same dimension, they are the same. This can be applied to $\ker(\alpha_s^{\vee})$ as well. □

Now we are ready to define the rational Cherednik algebra using the above construction.

Definition 1.2. The rational Cherednik algebra $H_c(W, \mathfrak{h})$ is the quotient of the algebra $\mathbb{C}W \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ under the relations

$$[x, x'] = [y, y'] = 0\,, \tag{1.6}$$

$$[y, x] = x(y) - \sum_{s \in S} c_s \alpha_s(y) x(\alpha_s^{\vee}) s, \tag{1.7}$$

for all $x, x' \in \mathfrak{h}^*$ and $y, y' \in \mathfrak{h}$.

We can prove that $sx = s(x)s$ and $sy = s(y)s$ for all $s \in S$, $x \in \mathfrak{h}^*$, and $y \in \mathfrak{h}$, using the product rule of the semi-direct product of two groups.

Let $\mathrm{Sym}(V)$ be the symmetric algebra of the vector space V. It is proved in Proposition 3.5 of [8] the PBW theorem, which is that $H_c(W, \mathfrak{h})$ is isomorphic to $\mathrm{Sym}(\mathfrak{h}^*) \otimes \mathbb{C}W \otimes \mathrm{Sym}(\mathfrak{h})$ as vector space. Hence every element in the rational Cherednik algebra can be expressed as a sum of $x_1^{i_1} \dots x_n^{i_n} y_1^{j_1} \dots y_n^{j_n} g$, where $i_1, \dots, i_n, j_1, \dots, j_n \in \mathbb{N}$, and $g \in W$.

Now let us look at some examples.

Example 1.3. Suppose $\mathbb{Z}_l = \{s^j \mid j = 0, \dots, l-1 \text{ and } s^j = s^0 = 1\}$ is a multiplicative group with identity 1, $\mathfrak{h} = \mathbb{C}\langle y \rangle$, and $\mathfrak{h}^* = \mathbb{C}\langle x \rangle$. Let ζ be an l-th primitive root. The elements of the group W act on $\mathfrak{h}$ by $s^j \cdot v = \zeta^j v$, for $s^j \in W$ and $v \in \mathfrak{h}$. This defines an action of $\mathbb{Z}_l$. For any $s^j \in W$ and $j \neq 0$, s^j is a reflection. We can choose $\alpha_{s^j} = \alpha_j = \sqrt{2}x$ and $\alpha_{s^j}^{\vee} = \alpha_j^{\vee} = \sqrt{2}y$ with eigenvalues ζ^{-j} and ζ^j respectively. Hence the commutation relation becomes

$$[y, x] = 1 - \sum_{j \neq 0} c_j \alpha_j(y) x(\alpha_j^{\vee}) j = 1 - \sum_{j \neq 0} 2 c_j s_j\,. \tag{1.8}$$

Example 1.4. Suppose W is the symmetric group S_n and $\mathfrak{h} = \mathbb{C}^n$. We know that $\mathfrak{h}$ has a standard basis $\{y_1, \ldots, y_n\}$. Then $\mathfrak{h}^* = \mathbb{C}^n$ has a dual basis $\{x_1, \ldots, x_n\}$. The reflections of S_n are $s_{ij} = (i, j)$, where $1 \leq i \neq j \leq n$. Since all the reflections are conjugate, c is a constant map. We can choose $\alpha_{ij} = \alpha_{s_{ij}} = x_i - x_j$ and $\alpha_{ij}^\vee = \alpha_{s_{ij}}^\vee = y_i - y_j$. Then we have $\lambda_{ij} = \lambda_{s_{ij}} = -1$. Therefore the rational Cherednik algebra is generated by $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_n\}$, and W with the relations

$$[x_i, x_j] = [y_i, y_j] = 0\,, \tag{1.9}$$

$$[y_i, x_i] = 1 - c \sum_{i \neq j} s_{ij}\,, \tag{1.10}$$

$$[y_i, x_j] = c s_{ij} \quad \text{for} \quad i \neq j\,. \tag{1.11}$$

1.2 Dunkl Embedding

If we think of the elements in $\mathfrak{h}$ as differential operators on polynomials of $x \in \mathfrak{h}^*$, then we can have an alternative definition of the rational Cherednik algebra in terms of differential operators. It gives an intuitive way of understanding the action of $\mathfrak{h}$, and allows for straightforward computations later.

Let $\mathfrak{h}_{\text{reg}}$ be the set of the regular points of $\mathfrak{h}$, which is the set of the points not fixed by reflections in W. Let $\mathcal{D}(\mathfrak{h}_{\text{reg}})$ be the ring of the differential operators on $\mathfrak{h}_{\text{reg}}$. Then we can define the Dunkl operator as the following.

Definition 1.5. For any $y \in \mathfrak{h}$, we define the Dunkl operator $\mathrm{D}_y \in \mathcal{D}(\mathfrak{h}_{\text{reg}}) \rtimes W$ by

$$\mathrm{D}_y = \partial_y - \sum_{s \in S} \frac{2c_s}{1 - \lambda_s} \frac{\alpha_s(y)}{\alpha_s}(1 - s)\,, \tag{1.12}$$

where ∂_y denotes the partial differentiation with respect to y.

By Proposition 2.15 in [8], we have the following lemma.

Lemma 1.6. For any $x \in \mathfrak{h}^*$ and $y, y' \in \mathfrak{h}$, we have

$$[\mathrm{D}_y, x] = x(y) - \sum_{s \in S} c_s \alpha_s(y) x(\alpha_s^\vee)\,, \tag{1.13}$$

$$[\mathrm{D}_y, D_{y'}] = 0\,. \tag{1.14}$$

Remark. Lemma 1.6 tells us the Dunkl operators and x have the same commutation relation with x and y in the rational Cherednik algebra.

Let A be a filtered algebra over field k, i.e., $k = F^0 A \subset F^1 A \subset \dots$ and $\bigcup_i F^i A = A$. We define the Rees algebra of A to be $\mathrm{Rees}(A) = \bigoplus_{i=0}^{\infty} F^i A$. The semi-product $\mathbb{C}W \ltimes \mathcal{D}(\mathfrak{h}_{\mathrm{reg}})$ has a natural filtration given by setting $\mathbb{C}W \ltimes \mathbb{C}[\mathfrak{h}_{reg}]$ in degree zero and $\mathfrak{h}$ in degree one. We define $\tilde{H}$ to be the subalgebra inside $\mathrm{Rees}(\mathbb{C}W \ltimes \mathcal{D}(\mathfrak{h}_{\mathrm{reg}}))$ generated by $x \in \mathfrak{h}^*$, $g \in W$, and D_y for $y \in \mathfrak{h}$ with respect to this filtration. Then we have the following theorem from Proposition 3.2 in [8].

Theorem 1.7. The subalgebra $\tilde{H}$ is isomorphic to the rational Cherednik algebra $H_c(W, \mathfrak{h})$.

1.3 Modules of Rational Cherednik Algebras

1.3.1 Verma Modules and Category $\mathcal{O}$

Suppose τ is an irreducible representation of W. We can define the Verma module of the rational Cherednik algebra as

$$M_c(\tau) = H_c(W, \mathfrak{h}) \otimes_{\mathbb{C}W \ltimes \mathrm{Sym}(\mathfrak{h})} \tau \,, \tag{1.15}$$

where $\mathfrak{h}$ acts on τ by zero. We may view $M_c(\tau) \cong \mathrm{Sym}(\mathfrak{h}^*) \otimes_{\mathbb{C}} \tau$ as an $\mathrm{Sym}(\mathfrak{h}^*)$-module. As a result, every element in the Verma module can be written as a sum of $x_1^{i_1} \dots x_n^{i_n} \otimes v$ for $x_1, \dots, x_n \in \mathfrak{h}^*$, $i_1, \dots i_n \in \mathbb{N}$ and $v \in \tau$. A quotient of $M_c(\tau)$ is called a lowest weight module with lowest weight τ.

Now we introduce the Euler element $\mathbf{h}$ by

$$\mathbf{h} = \sum_i x_i y_i + \frac{\dim(\mathfrak{h})}{2} - \sum_{s \in S} \frac{2c_s}{1 - \lambda_s} s \,. \tag{1.16}$$

According to Proposition 3.18 of [8], we have the following equations.

$$[\mathbf{h}, x] = x \,, \tag{1.17}$$

$$[\mathbf{h}, y] = -y \,. \tag{1.18}$$

One of the properties of Euler element is in the following proposition.

Proposition 1.8. The Euler element $\mathbf{h}$ acts semi-simply on $M_c(\tau)$, and hence on every lowest weight module M.

Before proving the above proposition, we need to consider the following lemma first.

Lemma 1.9. The sum $\sum_{s\in S} \frac{2c_s}{1-\lambda_s} s$ is in the center of $\mathbb{C}W$.

Proof. It is sufficient to prove the result for $g \in W$,

$$\begin{aligned} g\left(\sum_{s\in S} \frac{2c_s}{1-\lambda_s} s\right) g^{-1} &= \sum_{s\in S} \frac{2c_s}{1-\lambda_s} gsg^{-1} \\ &= \sum_{gsg^{-1}\in S} \frac{2c_s}{1-\lambda_{gsg^{-1}}} gsg^{-1} \\ &= \sum_{s\in S} \frac{2c_s}{1-\lambda_s} s \end{aligned} \tag{1.19}$$

□

Now we are well-equipped to prove Proposition 1.8.

Proof of Proposition 1.8. Let $v \in \tau$. We have

$$\mathbf{h} \cdot v = \left(\sum_i x_i y_i + \frac{\dim(\mathfrak{h})}{2} - \sum_{s\in S} \frac{2c_s}{1-\lambda_s} s\right) \cdot v\,. \tag{1.20}$$

By Schur's Lemma, we know that if $z \in \mathbb{C}W$ is in the centraliser of W, then it acts on any irreducible module τ by a scalar. We denote this scalar by $z\big|_\tau$.

Because y_i acts on τ by zero and Lemma 1.9 is true, we have

$$\mathbf{h} \cdot v = h_c(\tau) v\,, \tag{1.21}$$

where the eigenvalue $h_c(\tau)$ is

$$h_c(\tau) = \left(\frac{\dim(\mathfrak{h})}{2} - \sum_{s\in S} \frac{2c_s}{1-\lambda_s} s\bigg|_\tau\right)\,.$$

Assume that the dimension of $\mathfrak{h}$ is n. Our goal is to use mathematical induction to prove

$$\mathbf{h} \cdot x_1^{i_1} \dots x_n^{i_n} \otimes v = (N + h_c(\tau)) v\,, \tag{1.22}$$

where $\sum_{j=1}^{n} i_j = N$. We already have the base case of $N = 0$. Suppose it is true for $N \geq 0$. Suppose the sum $\sum_{j=1}^{n} i_j$ equals $N+1$. Without loss of generality, we assume that $i_1 > 0$. Then we have

$$\begin{aligned}
\mathbf{h} \cdot x_1^{i_1} \dots x_n^{i_n} \otimes v &= [\mathbf{h}, x_1] x_1^{i_1 - 1} \dots x_n^{i_n} \otimes v + x_1 \mathbf{h} x_1^{i_1 - 1} \dots x_n^{i_n} \otimes v \\
&= x_1 \cdot x_1^{i_1 - 1} \dots x_n^{i_n} \otimes v + x_1 (N + h_c(\tau)) x_1^{i_1 - 1} \dots x_n^{i_n} \otimes v \\
&= (N + 1 + h_c(\tau)) x_1^{i_1} \dots x_n^{i_n} \otimes v \,. \qquad (1.23)
\end{aligned}$$

Therefore, by mathematical induction, (1.22) is true for any $N \in \mathbb{N}$. We conclude that $\mathbf{h}$ acts semi-simply on $M_c(\tau)$. □

Remark. By Proposition 1.8, we may grade the Verma module $M_c(\tau)$ by the following

$$\deg(x) = 1 \qquad \text{for} \quad x \in \mathfrak{h}^* \,, \qquad (1.24)$$

$$\deg(y) = -1 \qquad \text{for} \quad y \in \mathfrak{h} \,, \qquad (1.25)$$

$$\deg(v) = 0 \qquad \text{for} \quad v \in \tau. \qquad (1.26)$$

With respect to the above grading, the defining relations of rational Cherednik algebra are homogeneous. Then $\mathbf{h}$ acts on the elements of degree N by the scalar $N + h_c(\tau)$.

Proposition 1.10. There exists a unique smallest quotient $L_c(\tau)$ of $M_c(\tau)$. Equivalently, there exists a maximal proper submodule of $M_c(\tau)$.

Proof. It suffices to prove that there exists a maximal proper submodule of $M_c(\tau)$. Let N be a proper submodule of $M_c(\tau)$. Since τ is an irreducible representation of W, $\mathbb{C}W \cdot v = \tau$ for any nonzero $v \in \tau$. Hence if any $v \in \tau$ is also belongs to N, then $H_c(W, \mathfrak{h}) \cdot v = M_c(\tau)$ is a submodule of N, which means that $N = M_c(\tau)$. As a result, the degree zero elements are not in any of proper submodules, hence not in their sum. Let J be the sum of all the proper submodules. Then J is the maximal proper submodule and we are done. □

Because of the triangular decomposition by PBW theorem, we can introduce the concept of the Category $\mathcal{O}$ for rational Cherednik algebras, borrowing the idea of Category $\mathcal{O}$ in Lie algebras [2]. We say that a module M is locally finite over $\mathrm{Sym}(\mathfrak{h})$ if $\mathrm{Sym}(\mathfrak{h}) \cdot m$ is finite dimensional for all $m \in M$. A module M is locally nilpotent over $\mathfrak{h}$ if there exists some $N \gg 0$ such that $\mathfrak{h}^N \cdot m = 0$ for each $m \in M$.

Definition 1.11. We then define Category $\mathcal{O}$ to be the subcategory of $H_c(W, \mathfrak{h})$-modules whose objects are finitely generated under the action of $\mathrm{Sym}(\mathfrak{h}^*)$, locally finite under $\mathrm{Sym}(\mathfrak{h})$ and locally nilpotent over $\mathfrak{h}$.

Lemma 1.12. For all τ irreducible representations of W, the modules $M_c(\tau)$ and $L_c(\tau)$ belong to Category $\mathcal{O}$.

Proof. Suppose the dimension of $\mathfrak{h}$ is n. Let $\{y_1, \ldots, y_n\}$ be a basis of $\mathfrak{h}$, and $\{x_1, \ldots, x_n\}$ the dual basis of $\mathfrak{h}^*$. Any element $m \in M_c(\tau)$ can be written as a sum

$$m = \sum_{\substack{i_1, \ldots, i_n \\ v \in \tau}} x_1^{i_1} \ldots x_n^{i_n} \otimes v \,. \tag{1.27}$$

We can pick an element in the sum with the highest degree j. The commutation relations tell us that y_i always lowers the degree by 1. Hence $y_1^{k_1} \ldots y_n^{k_n}$ eliminates m for any $\sum_{s=1}^{n} k_s \gg j$. It shows that $M_c(\tau)$ is locally nilpotent, and so are all its quotients. Since τ is finite dimensional, it is obvious that $M_c(\tau)$ is finitely generated by $\mathrm{Sym}(\mathfrak{h}^*)$. □

Recall that the Euler element $\mathbf{h}$ acts semi-simply on the Verma modules $M_c(\tau)$. We have a similar property for any M in Category $\mathcal{O}$ by Lemma 2.4.3 in [2].

Proposition 1.13. Any object M in Category $\mathcal{O}$ can be decomposed into a direct sum of generalized $\mathbf{h}$-eigenspaces, i.e.,

$$M = \bigoplus_{z \in \mathbb{C}} M_z \,, \tag{1.28}$$

where $M_z = \{m \in M \mid \exists\, N \in \mathbb{N} \text{ such that } (\mathbf{h} - z)^N m = 0.\}$.

By the universality of the Verma modules, we have the following proposition from Lemma 2.6.2 in [2].

Proposition 1.14. The set $\{L_c(\tau) \mid \tau \text{ irreducible representation of } W\}$ is a complete set of non-isomorphic simple modules of Category $\mathcal{O}$.

According to Proposition 1.10, the following proposition can be derived as in Corollary 2.8.7 of [2].

Proposition 1.15. Category $\mathcal{O}$ is semi-simple if and only if $M_c(\tau) = L_c(\tau)$, for all irreducible representation τ of W.

Remark. If the function $c \equiv 0$ is the zero function, then we have $H_c(W, \mathfrak{h}) = \mathcal{D}(\mathfrak{h}) \rtimes W$ whose modules form the precise category of W-equivariant $\mathcal{D}$-modules on $\mathfrak{h}$. From now on, we will assume that $c \neq 0$ for the rest of the paper.

1.3.2 Unitary Representations

Unitary representations are very important in many aspects of mathematics and physics. In unitary representations, each element from the group becomes a unitary operator, which preserves magnitudes of vectors. It is widely applied in quantum physics. Unitary representations of rational Cherednik algebras are previously studied by many mathematicians such as Etingof, Stoica and Griffeth. The unitarity condition of rational Cherednik algebra of type A can be found in [9]. Griffeth described unitarity representations of rational Cherednik algebra of $G(l, 1, n)$ combinatorially in [13]. This section introduces the concept of unitary representations, reviews previous results and presents the problem of determining when a representation is unitary.

A star operation is an anti-linear anti-automorphism on a $\mathbb{C}$-algebra. It is usually denoted by $*$ or $\star$, whose actions are denoted by a^* or $a^\star$ respectively. In order to define unitarity, we need to define some Hermitian form with respect to a star operation on the rational Cherednik algebra. Let $c^\dagger$ be the function that $c_s^\dagger = \overline{c_{s^{-1}}}$. Given by Proposition 2.2 in [9], the following proposition is the key to define the unitary representations of the rational Cherednik algebra.

Proposition 1.16. Suppose the parameter c satisfies $c^\dagger = c$. There exists a unique W-invariant Hermitian form $\beta_{c,\tau}$ on $M_c(\tau)$ such that it coincides with the Hermitian form $(\cdot, \cdot)\big|_\tau$ of the representation τ in degree zero, which satisfies the contravariance condition

$$\beta_{c,\tau}(y \cdot m, m') = \beta_{c,\tau}(m, T(y) \cdot m') \tag{1.29}$$

for all $m, m \in M_c(\tau)$ and $y \in \mathfrak{h}$. The kernel of $\beta_{c,\tau}$ coincides with the maximal proper submodule $J_c(\tau)$ of $M_c(\tau)$. So $\beta_{c,\tau}$ descends to a non-degenerate form $\beta_{c,\tau}$ on the quotient $L_c(\tau) = M_c(\tau)/J_c(\tau)$.

Definition 1.17. We define a star operation on the rational Cherednik algebra by extending

$$y \mapsto T(y),\ x \mapsto T^{-1}(x),\ s \mapsto s^{-1},$$

for any $x \in \mathfrak{h}^*,\ y \in \mathfrak{h}$, and $s \in S$. It is called the $*$-operation.

Remark.

1. The contravariance condition in Proposition 1.16 can be generalized to any element in $H_c(W, \mathfrak{h})$ under the $*$-operation, i.e.,
$$\beta_{c,\tau}(a \cdot m, m') = \beta_{c,\tau}(m, a^* \cdot m') \tag{1.30}$$
for all $m, m' \in M_c(\tau)$ and $a \in H_c(W, \mathfrak{h})$.

2. Given two elements $m, m' \in M_c(\tau))$, $\beta_{c,\tau}(m, m') \neq 0$ implies that m and m' have the same degree. In other words, if m and m' have different degrees, then the form $\beta_{c,\tau}$ must vanish.

Let C be the set of all W-invariant function $c : S \to \mathbb{C}$ such that $c = c^\dagger$.

Definition 1.18. For $c \in C$, the representation $L_c(\tau)$ is said to be unitary if the form $\beta_{c,\tau}$ is positive-definite on $L_c(\tau)$. Moreover, we say $M_c(\lambda)$ is unitary if $L_c(\lambda)$ is.

Definition 1.19. We define $U(\tau)$ to be the subset of C such that $L_c(\tau)$ is unitary, and define $U^*(\tau)$ to be the subset of $U(\tau)$ where c is a constant.

The set $U(\tau)$ is one of our main interests of study in this area. In particular, we want to find the relation between c and τ that makes the simple module $L_c(\tau)$ unitary. Although the general relation is not fully understood, we can demonstrate the relation in some special cases with the examples in the next section.

1.3.3 Examples of Unitary Representations

Example 1.20. Cyclic group of order l: $\mathbb{Z}_l$

Let $W = \mathbb{Z}_l$, $\mathfrak{h} = \mathbb{C}\langle y \rangle$, and $\mathfrak{h}^* = \mathbb{C}\langle x \rangle$ where $T(y) = x$, just like in Example 1.3. Let c_j denote the function value $c(j)$, for all $j = 1, ..., l$. Let τ be the trivial representation of W. We would like to determine the set $U(\tau)$. We know from the last section that as an $\mathrm{Sym}(\mathfrak{h}^*)$-module, the Verma module $M_c(\tau)$ satisfies
$$M_c(\tau) \cong \mathbb{C}[x] \otimes_{\mathrm{Sym}(\mathfrak{h}) \rtimes W} \tau = \mathbb{C}[x]\,. \tag{1.31}$$

Then we normalize $\beta_{c,\tau}$ such that $\beta_{c,\tau}(1, 1) = 1$. If we define a_k to be $\beta_{c,\tau}(x^k, x^k)$, then we can compute a_k inductively using
$$a_k = \beta_{c,\tau}(x^k, x^k) = \beta_{c,\tau}(x^{k-1}, (T^{-1}x)x^k)\,. \tag{1.32}$$

Claim. We have the equation

$$yx^k = \left(k - 2 \sum_{j=1}^{l-1} \frac{1-\zeta^{-jk}}{1-\zeta^{-j}} c_j \right) x^{k-1} \tag{1.33}$$

for $k \in \mathbb{N}$.

Proof. We can prove this claim by induction. In the case of $k = 1$, y and j act on elements of degree zero by 0 and 1 respectively. So by definition we have

$$\begin{aligned} yx &= xy + 1 - \sum_{j \neq 0} c_j \alpha_j(y) x(\alpha_j^\vee) \zeta_j \\ &= 1 - 2 \sum_{j \neq 0} c_j \, . \end{aligned} \tag{1.34}$$

On the other hand, when k equals to one, the right hand side of (1.33) is

$$k - 2 \sum_{j=1}^{l-1} \frac{1-\zeta^{-jk}}{1-\zeta^{-j}} c_j = 1 - 2 \sum_{j \neq 0} c_j \, . \tag{1.35}$$

As (1.34) and (1.35) give the same result, we have shown that (1.33) is true for $k = 1$.

Suppose it is true for the case $k - 1 \geq 0$, we have

$$\begin{aligned} yx^k &= yx \cdot x^{k-1} - xy \cdot x^{k-1} + xy \cdot x^{k-1} \\ &= [y, x] x^{k-1} + x(yx^{k-1}) \\ &= \left(1 - 2 \sum_{j \neq 0} c_j j \right) x^{k-1} + x \left(k - 1 - 2 \sum_{j=1}^{l-1} c_j \frac{1-\zeta^{-j(k-1)}}{1-\zeta^{-j}} \right) x^{k-2} \\ &= x^{k-1} \left(1 - 2 \sum_{j \neq 0} c_j \zeta^{-j(k-1)} \right) + x^{k-1} \left(k - 1 - 2 \sum_{j=1}^{l-1} c_j \frac{1-\zeta^{-j(k-1)}}{1-\zeta^{-j}} \right) \\ &= x^{k-1} \left(k - 2 \sum_{j \neq 0} c_j \left(\frac{1-\zeta^{-j(k-1)}}{1-\zeta^{-j}} + \zeta^{-j(k-1)} \right) \right) \\ &= x^{k-1} \left(k - 2 \sum_{j=1}^{l-1} \frac{1-\zeta^{-jk}}{1-\zeta^{-j}} c_j \right) \end{aligned} \tag{1.36}$$

Hence by mathematical induction the claim holds for any $k \in \mathbb{N}$. □

Applying (1.33) in the above claim, we can complete the computation by

$$\begin{aligned} a_k &= \beta_{c,\tau}(x^{k-1}, (T^{-1}x)x^k) \\ &= \beta_{c,\tau}\left(x^{k-1}, \left(k - 2\sum_{j=1}^{l-1} \frac{1-\zeta^{-jk}}{1-\zeta^{-j}} c_j\right) x^{k-1}\right) \\ &= \left(k - 2\sum_{j=1}^{l-1} \overline{\frac{1-\zeta^{-jk}}{1-\zeta^{-j}} c_j}\right) a_{k-1}\,. \end{aligned} \tag{1.37}$$

Let $b_k = 2\sum_{j=1}^{l-1} \frac{1-\zeta^{-jk}}{1-\zeta^{-j}} c_j$. Since the equation $c_j = \overline{c_{-j}}$ holds, we have

$$\begin{aligned} b_k &= \sum_{j=1}^{l-1} \frac{1-\zeta^{-jk}}{1-\zeta^{-j}} c_j + \sum_{j=1}^{l-1} \frac{1-\zeta^{jk}}{1-\zeta^{j}} c_{-j} \\ &= \sum_{j=1}^{l-1} \left(\frac{1-\zeta^{-jk}}{1-\zeta^{-j}} c_j + \overline{\frac{1-\zeta^{-jk}}{1-\zeta^{-j}} c_j}\right)\,. \end{aligned}$$

Since b_k is the sum of a complex number and its complex conjugate, b_k is a real number. Hence a_k becomes

$$a_k = \prod_{i=1}^{k} (i - b_i)\,. \tag{1.38}$$

Together with the fact that the matrix A, with entries $A_{jk} = \frac{1-\zeta^{-jk}}{1-\zeta^{-j}}$ for $1 \leq j$ and $k \leq l$, is non-degenerate, we conclude that $\{b_1, \ldots, b_{l-1}\}$ forms a real linear coordinate system on C. Hence, the unitarity condition of $M_c(\tau)$ can be described as the following proposition, according to Proposition 3.10 of [9].

Proposition 1.21.

1. The Verma module $M_c(\tau)$ is irreducible if and only if $k - b_k \neq 0$ for all $k \geq 1$. Assuming it is irreducible, it is unitary if and only if $k - b_k > 0$ for all $k = 1, \ldots, l-1$.

2. The set $U(\tau)$ is the set of all vectors $(b_1, \ldots, b_{l-1})$ such that all entries of the vector $(1 - b_1, 2 - b_2, \ldots, l - 1 - b_{l-1})$ are all positive before the first possible zero.

Example 1.22. Symmetric group of n: S_n (Type A)

It can be shown that every irreducible representation of S_n is in one-to-one correspondence to a partition of n [21]. We can label the partition of n by its corresponding irreducible representation

τ of S_n. Each partition τ can be represented by a Young diagram. Let the conjugate partition of some partition τ be the partition obtained by flipping τ along the diagonal and denote it by τ^{tr}. We denote the length of the largest hook of the Young diagram of τ by $l(\tau)$, and denote the multiplicity of the largest part of τ as $m_*(\tau)$. Then we can define

$$N(\tau) = l(\tau) - m_*(\tau) + 1\,. \tag{1.39}$$

Since all the reflections are conjugate to each other, the function c is constant and $C \subset \mathbb{C}$. Moreover, we have $c_s = \overline{c_{s^{-1}}} = \overline{c_s}$. It implies that c is real and $U(\tau) \subset \mathbb{R}$. Then the set $U(\tau)$ can be shown to have the following properties as in Proposition 5.1, Proposition 5.2, and Theorem 5.5 of [9].

Theorem 1.23.

1. If a partition $\tau \neq (1^n)$, then $c \leq \frac{1}{N(\tau)}$ for $c \in U(\tau)$.
2. For each partition τ, we have the interval $[-\frac{1}{l(\tau)}, \frac{1}{l(\tau)}] \subseteq U(\tau)$.
3. If a partition satisfies $\tau \neq (n)$ and $\tau \neq (1^n)$, then the set $U(\tau)$ is the union of the interval $[-\frac{1}{l(\tau)}, \frac{1}{l(\tau)}]$ and the finite set of isolated points $\frac{1}{k}$, where the integer k satisfies $N(\tau) \leq k < l(\tau)$ and $-l(\tau^{tr}) < k \leq -N(\tau^{tr})$.

The computation of unitarity condition on modules of rational Cherednik algebra is generally not a straightforward task. To generalize the above result, we can deploy a tool, called the KZ functor, to translate the problem into more familiar areas. We will discuss it further in later sections.

2 Rational Cherednik Algebras of $G(l, 1, n)$ and Dunkl-Opdam Subalgebra

From now on, we focus on the rational Cherednik algebra of $G(l, 1, n)$. All the result will be about this particular algebra. In this case, Verma modules decompose into a direct sum of generalized eigenspaces with respect to a commutative subalgebra, called the Dunkl-Opdam subalgebra. We also encode the generalized eigenspaces combinatorially by an n-tuple of natural numbers and a Young tableau. The section is a review of [12], [11], [13].

Consider two fixed positive integers $l, n \in \mathbb{N}^+$. First we let $\zeta = \exp(2\pi i/l)$ be an l-th primitive root of unity.

Definition 2.1. The group $G(l,1,n)$ is the subgroup of $\mathrm{GL}(n,\mathbb{C})$ generated by the permutation matrices and the diagonal matrices $\zeta_i = diag(1,\dots,1,\zeta,1,\dots,1)$, where ζ is in the i-th position.

Remark. By definition, we can also write the group $G(l,1,n)$ as a wreath product $S_n \wr \mathbb{Z}_l$.

There are two types of conjugacy classes of reflections in $G(l,1,n)$. Firstly, all the transpositions are conjugate to each other, and $\zeta_i^t s_{ij} \zeta_i^{-t}$ are conjugate to the transpositions. The eigenvectors corresponding to the nontrivial eigenvalue $\lambda_s = -1$ are

$$\alpha_s = \zeta^{-t} x_i - x_j\,, \tag{2.1}$$

$$\alpha_s^\vee = \zeta^t y_i - y_j \tag{2.2}$$

for $s = \zeta_i^t s_{ij} \zeta_i^{-t}$ with $t = 0,\dots,l-1$. We can denote the corresponding c_s as c_0. Secondly, for each fixed $t \in \{0,1,\dots,l-1\}$, we have that ζ_i^t is conjugate to ζ_j^t provided $i,j \le n$. In contrast, ζ_i^t is not conjugate to $\zeta_i^{t'}$ for $t \neq t'$. We denote the function $c_{\zeta_i^t}$ for the conjugacy class of $\{\zeta_i^t\}_{i=1,\dots,n}$ by c_t for some fixed t. For $s = \zeta_i^t$, the eigenvectors α_s and $\alpha_s^\vee$ are given by

$$\alpha_s = x_i\,, \tag{2.3}$$

$$\alpha_s^\vee = 2y_i\,. \tag{2.4}$$

For simplicity, we will also regard the function c as parameters $c_0, \dots, c_{l-1}$. Let the complex vector space $\mathfrak{h}$ be $\mathbb{C}^n$. Then the defining relations 1.7 of the rational Cherednik algebra $H_c(W,\mathfrak{h})$ of $W = G(l,1,n)$ become

$$\begin{aligned}[y_i, x_j] &= (y_i, x_j) - \sum_{s\in S} c_s \alpha_s(y_i) x_j(\alpha_s^\vee) s \\ &= \sum_{t=0}^{l-1} c_0 \zeta^{-t} \zeta_i^t s_{ij} \zeta_i^{-t}\end{aligned} \tag{2.5}$$

for $i \neq j$ and

$$\begin{aligned}[y_i, x_i] &= (y_i, x_i) - \sum_{s\in S} c_s \alpha_s(y_i) x_i(\alpha_s^\vee) s \\ &= 1 - \sum_{t=0}^{l-1} \sum_{k\neq i} c_0 \zeta_i^t s_{ki} \zeta_i^{-t} - \sum_{t=1}^{l-1} 2c_t \zeta_i^t\,.\end{aligned} \tag{2.6}$$

There is an alternative way to parametrize the equations by using $h_0, \ldots, h_{l-1}$ instead of c_s, with the constraint $h_0 = 0$. For any $i = 1, \ldots, n$

$$[y_i, x_i] = 1 - \sum_{t=0}^{l-1} \sum_{k \neq i} c_0 \zeta_i^t s_{ki} \zeta_i^{-t} - \sum_{r=0}^{l-1} \sum_{t=1}^{l-1} (h_r - h_{r-1}) \zeta^{-rt} \zeta_i^t \,, \tag{2.7}$$

where h_{-1} is defined to be h_{l-1}. Comparing the coefficient of ζ_i^t in (2.6) and (2.7), the parameters h_i can be expressed in terms of c_s by the matrix given below

$$\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \zeta^{-1} & \zeta^{-2} & \ldots & \zeta^{-(l-1)} \\ 1 & \zeta^{-2} & \zeta^{-4} & \ldots & \zeta^{-2(l-1)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \zeta^{-(l-1)} & \zeta^{-2(l-1)} & \ldots & \zeta^{-(l-1)^2} \end{pmatrix} \begin{pmatrix} h_0 - h_{-1} \\ h_1 - h_0 \\ h_2 - h_1 \\ \ldots \\ h_{l-1} - h_{l-2} \end{pmatrix} = -2 \begin{pmatrix} 0 \\ c_1 \\ c_2 \\ \ldots \\ c_{l-1} \end{pmatrix} . \tag{2.8}$$

This matrix is invertible and so is the mapping. So there is a bijection between the parametrizations in c_s and the parametrization in h_i. After a computation, we get the relations between c_i and h_j to be

$$h_j = -\frac{2}{l} \sum_{i=1}^{l-1} \frac{1 - \zeta^{ij}}{1 - \zeta^{-i}} c_i. \tag{2.9}$$

From now on we will use the parameters h_i and c_s interchangeably.

We borrow the ideas of the Cartan subalgebra and the Cartan decomposition from Lie algebras, and construct a commutative subalgebra. The generalized eigenspaces of the subalgebra generators in the Verma modules will give us deeper understanding of the modules. We define an element z_i in the rational Cherednik algebra by

$$z_i = y_i x_i + c_0 \phi_i \,, \tag{2.10}$$

where

$$\phi_i = \sum_{1 \leq j < i} \sum_{t=0}^{l-1} \zeta_i^t s_{ij} \zeta_i^{-t} \,. \tag{2.11}$$

Due to Proposition 4.2 and Proposition 4.3 of [12], we have the following lemma.

Lemma 2.2. *The elements $\{z_i\}_{i=1,2,\ldots,n}$ and $\{\zeta_j\}_{j=1,2,\ldots,n}$ in the rational Cherednik algebra generate a commutative subalgebra of $H_c(W, \mathfrak{h})$, which is called the Dunkl-Opdam subalgebra and denoted as*

$\mathfrak{t}$.

The advantage of the Dunkl-Opdam subalgebra is that it is abelian. Hence the generators z_i and ζ_j share the same (generalized) eigenvectors for all $i, j = 1, \ldots, n$. After rearranging its components, it is not hard to see that the euler element $\mathbf{h} = \sum_i z_i - \frac{n}{2} + \sum_i \sum_{t=1}^{l-1} 2c_t \frac{\zeta^t}{1-\zeta^t} \zeta_i^t$ is in the subalgebra. In order to get a finer decomposition of eigenspaces, we may consider the simultaneous eigenspaces of z_i and ζ_i instead of $\mathbf{h}$. We denote the generalized eigenspace of z_i and ζ_i in a module M in Category $\mathcal{O}$ corresponding to a generalized eigenvalue $(\mathbf{a}, \mathbf{t}) = ((a_1, \ldots, a_n), (t_1, \ldots, t_n))$ by

$$M_{\mathbf{a},\mathbf{t}} = \{m \in M \mid (z_i - a_i)^N m = (\zeta^i - t_i)^N m = 0 \text{ for big enough } N\}\,. \tag{2.12}$$

Before we get into the eigenvectors, we need some tools to work with them.

Definition 2.3. For any $i = 1, \ldots, n-1$, we define the intertwining operators σ_i by

$$\sigma_i = s_i + \frac{c_0}{z_i - z_{i+1}} \pi_i\,, \tag{2.13}$$

where

$$\pi_i = \sum_{p=0}^{l-1} \zeta_i^p \zeta_{i+1}^{-p}\,. \tag{2.14}$$

The intertwining operators σ_i are well-defined on the (generalized) eigenspaces of $\mathfrak{t}$, where either $z_i - z_{i+1}$ is invertible or $\pi_i = 0$.

We also define the intertwining operators Φ and Ψ as

$$\Phi = x_n s_{n-1} \ldots s_2 s_1\,, \tag{2.15}$$

$$\Psi = y_1 s_1 s_2 \ldots s_{n-1}\,. \tag{2.16}$$

Let us define an algebra homomorphism $\theta : \mathfrak{t} \to \mathfrak{t}$ by

$$\theta(z_i) = z_{i+1}\,, \tag{2.17}$$

$$\theta(\zeta_i) = \zeta_{i+1} \tag{2.18}$$

for $i = 1, \ldots, n-1$, and

$$\theta(z_n) = z_1 + 1 - \sum_{r=0}^{l-1} \sum_{t=0}^{l-1} (h_r - h_{r-1}) \zeta^{-rt} \zeta_1^t \,, \tag{2.19}$$

$$\theta(\zeta_n) = \zeta^{-1} \zeta_1 \,. \tag{2.20}$$

Then we have the following lemma due to Lemma 5.2 and Lemma 5.3 in [12].

Lemma 2.4.

1. For any $f \in \mathfrak{t}$, we have

$$\sigma_i f = (s_i \cdot f) \sigma_i \,, \tag{2.21}$$

   where s_i permutes z_i with z_{i+1} and permutes ζ_i with ζ_{i+1}.

2. For any $f \in \mathfrak{t}$, we have

$$f \Phi = \Phi \theta(f) \quad \text{and} \quad f \Psi = \Psi \theta^{-1}(f) \,. \tag{2.22}$$

3.

$$\Phi \Psi = z_n - 1 + \sum_{r=0}^{l-1} \sum_{t=0}^{l-1} (h_r - h_{r-1}) \zeta^{-rt} \zeta_1^t \tag{2.23}$$

4.

$$\Psi \Phi = z_1 \tag{2.24}$$

5. For $2 \leq i \leq n-1$, we have

$$\Phi s_i = s_{i-1} \Phi \,. \tag{2.25}$$

6. For $1 \leq i \leq n-2$, we have

$$\Psi s_i = s_{i+1} \Psi \,. \tag{2.26}$$

7.

$$\Phi^2 s_1 = s_{n-1} \Phi^2 \tag{2.27}$$

8.

$$\Psi^2 s_{n-1} = s_1 \Psi^2 \tag{2.28}$$

9.
$$\Psi s_{n-1}\Phi = \Phi s_1 \Psi + \sum_{t=0}^{l-1} \zeta^{-t}\zeta_1^t\zeta_n^{-t} \tag{2.29}$$

Definition 2.5. We define ν to be the bijection $\mathbb{C}^n \to \mathbb{C}^n$ such that

$$\nu((a_1, a_2, \ldots, a_n), (t_1, t_2, \ldots, t_n)) = ((a_n - 1, a_1, \ldots, a_{n-1}), (\zeta t_n, t_1, \ldots, t_{n-1}))\,, \tag{2.30}$$

$$\nu^{-1}((a_1, a_2, \ldots, a_n), (t_1, t_2, \ldots, t_n)) = ((a_2, \ldots, a_n, a_1 + 1), (t_2, \ldots, t_n, \zeta^{-1} t_1))\,. \tag{2.31}$$

Proposition 2.6.

1. The intertwining operators σ_i send a generalized eigenvector in $L_c(\lambda)_{\mathbf{a},\mathbf{t}}$ to a generalized eigenvector in $L_c(\lambda)_{s_i(\mathbf{a},\mathbf{t})}$.
2. The intertwining operator Φ sends a generalized eigenvector in $L_c(\lambda)_{\mathbf{a},\mathbf{t}}$ to a generalized eigenvector in $L_c(\lambda)_{\nu^{-1}(\mathbf{a},\mathbf{t})}$.
3. The intertwining operator Ψ sends a generalized eigenvector in $L_c(\lambda)_{\mathbf{a},\mathbf{t}}$ to a generalized eigenvector in $L_c(\lambda)_{\nu(\mathbf{a},\mathbf{t})}$.

Proof. The proof relies on Lemma 2.4. To prove the statement for σ_i, consider $m \in L_c(\lambda)_{\mathbf{a},\mathbf{t}}$ and apply (2.21) in Lemma 2.4. If $j \neq i, i+1$, we have

$$(z_j - a_j)^N(\sigma_i(m)) = \sigma_i((s_i \cdot z_j) - a_j)^N(m) = 0\,. \tag{2.32}$$

If $j = i$, we have

$$\begin{aligned}(z_i - a_{i+1})^N(\sigma_i(m)) &= \sigma_i((s_i \cdot z_i) - a_{i+1})^N(m)\\ &= \sigma_i(z_{i+1} - a_{i+1})^N(m)\\ &= 0\,.\end{aligned} \tag{2.33}$$

If $j = i+1$, we have

$$\begin{aligned}(z_{i+1} - a_i)^N(\sigma_i(m)) &= \sigma_i((s_i \cdot z_{i+1}) - a_i)^N(m)\\ &= \sigma_i(z_i - a_i)^N(m)\end{aligned}$$

$$= 0\,. \tag{2.34}$$

Hence, we can conclude that $\sigma_i(m) \in L_c(\lambda)_{s_i(\mathbf{a},\mathbf{t})}$. Similaly, the statements on Φ and Ψ can be shown by applying (2.22) in Lemma 2.4. □

More explicitly, Griffeth in [11] shows that there is a basis of $L_c(\lambda)$ containing the common generalized eigenvectors of z_i and ζ_i for $i = 1, \ldots, n$, such that all z_i and ζ_i act in the upper triangular manner. Moreover, the eigenvectors are encoded by a pair (μ, L) of an n-tuple in $\mathbb{Z}^n_{\geq 0}$ and a standard Young tableau of some l-multipartition of n . An l-multipartition of n is an l-tuple of Young diagrams with the total number of boxes to be n. Suppose we have an l-multipartition $\lambda = (\lambda^0, \lambda^1, \ldots, \lambda^{l-1})$ of n. Then we define a function $\beta : \{\text{the boxes in } \lambda\} \to \{0, \ldots, l-1\}$ by

$$\beta(B) = i, \quad \text{if the box } B \text{ lies in } \lambda^i. \tag{2.35}$$

The content of a box B in the i-th row and j-th column in λ^k is defined to be $j - i$, denoted by

$$\mathrm{ct}(B) = j - i. \tag{2.36}$$

A Young tableau is an l-multipartition λ of n with filling $1, \ldots, n$ that each number appears once. Then λ is the called the shape of the Young tableau. A standard Young tableau is a Young tableau such that the fillings are strictly increasing along both rows and columns. Suppose L is a Young tableau. We denote $L(i)$ to be the box containing the number i. Let s_i be the transposition in S_n permuting i and $i + 1$. Then we define $s_i \cdot L$ to be the Young tableau by interchanging the positions of filling i and $i + 1$ in L. Let $\mathrm{SYT}(\lambda)$ be the set of all standard Young tableaux of shape λ. By the Theorem 3.1 in [11], we get that the irreducible representations of $G(l, 1, n)$ are parametrized by l-multipartition λ of n with a basis $\{v_L \mid L \in \mathrm{SYT}(\lambda)\}$.

Let $\mu = (\mu_1, \ldots, \mu_n)$ be an n-tuple in $\mathbb{Z}^n_{\geq 0}$. We denote μ_+ to be the non-descending rearrangement of μ. Assuming that S_n acts on μ by

$$w \cdot \mu = (\mu_{w^{-1}(1)}, \ldots, \mu_{w^{-1}(n)}), \tag{2.37}$$

we define w_μ to be the longest element in S_n such that the order of $w \cdot \mu = \mu_+$. Let $>_d$ be the dominance order. For any two n-tuple $\mu, \nu \in \mathbb{Z}^n_{\geq 0}$, we say $\mu > \nu$ if either $\mu >_d \nu$, or $\mu_+ = \nu_+$ and $w_\mu > w_\nu$ in the Bruhat order. Let us define the partial order of the pair in $\mathbb{Z}^n_{\geq 0} \times \mathrm{STY}(\lambda)$ to be $(\mu, L) \geq (\nu, L')$ exactly if $\mu \geq \nu$. According to Theorem 5.1 in [11], we have the following theorem

Theorem 2.7. Suppose λ is an l-multipartition of n. Let $v^\mu_L = w^{-1}_\mu \cdot v_L$, and $x^\mu = x_1^{\mu_1} \dots x_n^{\mu_n}$. The action of ζ_i and z_i for all $i = 1, \dots, n$ on the Verma module $M_c(\lambda)$ are given by

$$\zeta_i \cdot x^\mu v^\mu_T = \zeta^{\beta(L(w_\mu(i)))-\mu_i} x^\mu v^\mu_T \tag{2.38}$$

$$z_i \cdot x^\mu v^\mu_T = \left((\mu_i+1) - l(h_{\beta(L(w_\mu(i)))} - h_{\beta(L(w_\mu(i)))-\mu_i-1}) - c_0 l \mathrm{ct}(L(w_\mu(i)))\right) f_{\mu,L} + \sum_{(\nu,L')<(\mu,L)} c_{(\nu,L')} x^\nu v^\nu_{L'}, \tag{2.39}$$

for some scalar $c_{(\nu,L')} \in \mathbb{C}$. If we assume that Category $\mathcal{O}$ is semi-simple, then for each pair (μ, L) there is a unique eigenvector $f_{\mu,L}$ of z_i and ζ_i such that $f_{\mu,L} = x^\mu v^\mu_T +$ lower terms .

In the same paper, Lemma 5.3 tells us how the intertwining operators act on the eigenvectors. The result coincides with Proposition 2.6 and gives us a detailed formulae.

Lemma 2.8. Suppose the setting remains the same as above. Then we have (a) If $\mu_i < \mu_{i+1}$ or $\mu_i - \mu_{i+1} \not\equiv \beta(L(w_\mu(i))) - \beta(L(w_\mu(i+1))) \bmod l$, then

$$\sigma_i \cdot f_{\mu,L} = f_{s_i \cdot \mu, L}. \tag{2.40}$$

(b) If $\mu_i > \mu_{i+1}$ and $\mu_i - \mu_{i+1} \equiv \beta(L(w_\mu(i))) - \beta(L(w_\mu(i+1))) \bmod l$, then

$$\sigma_i \cdot f_{\mu,L} = \frac{(\delta - lc_0)(\delta + lc_0)}{\delta^2} f_{s_i \cdot \mu, L}, \tag{2.41}$$

where $\delta = (\mu_i - \mu_{i+1}) - l(h_{\beta(L(w_\mu(i)))} - h_{\beta(L(w_\mu(i+1)))}) - c_0 l(\mathrm{ct}(L(w_\mu(i))) - \mathrm{ct}(L(w_\mu(i+1))))$.
(c) If $\mu_i = \mu_{i+1}$, set $j = w_\mu(i)$, then

$$\sigma_i \cdot f_{\mu,L} = \begin{cases} 0 & \text{if } s_{j-1} \cdot L \text{ is not standard,} \\ f_{\mu, s_{j-1} \cdot L} & \text{if } \zeta^{\beta(L(j))} \neq \zeta^{\beta(L(j-1))}, \\ (1 - (\frac{1}{\mathrm{ct}(L(j-1)) - \mathrm{ct(L(j))}})^2)^{\frac{1}{2}} f_{\mu, s_{j-1} \cdot L} & \text{else.} \end{cases} \tag{2.42}$$

(d)

$$\Phi \cdot f_{\mu,L} = f_{\nu^{-1}\cdot\mu,L}. \tag{2.43}$$

(e)

$$\Psi \cdot f_{\mu,L} = \begin{cases} (\mu_n - l(h_{\beta(L(w_\mu(n)))} - h_{\beta(L(w_\mu(n)))-\mu_n}) - c_0 l\mathrm{ct}(L(w_\mu(n)))) f_{\nu\cdot\mu,L} & \text{if } \mu_n > 0, \\ 0 & \text{if } \mu_n = 0. \end{cases} \tag{2.44}$$

3 KZ Functor

3.1 Construction of KZ functor

The Knizhnik-Zamolodchikov (KZ) functor is a functor from the Category $\mathcal{O}$ to the category of representations of $\mathcal{H}_q(W)$, where $\mathcal{H}_q(W)$ is the cyclotomic Hecke algebra of W with parameter q. The introduction of the KZ functor is to tackle the problem described in Section 1.3.2, by studying the unitarity condition in the cyclotomic Hecke algebra and the compatibility of the KZ functor with the unitarity. The following construction of the KZ functor is based on Section 4 by Bellamy [2].

First, we want to establish an isomorphism between the localized rational Cherednik algebra, and the semi-direct product of the rings of the differential operators $\mathcal{D}(\mathfrak{h}_{\mathrm{reg}})$ and the group W, i.e.,

$$H_c(W, \mathfrak{h})[\delta^{-1}] \xrightarrow{\sim} \mathcal{D}(\mathfrak{h}_{\mathrm{reg}}) \rtimes W\,, \tag{3.1}$$

where $\delta = \prod_s \alpha_s$. Recall the Dunkl operator from Definition 1.5 that

$$D_y = \partial_y - \sum_s \frac{2c_s}{1-\lambda_s}\frac{\alpha_s(y)}{\alpha_s}(1-s)\,.$$

Because α_s is invertible in $\mathfrak{h}_{\mathrm{reg}}$, the Dunkl operator D_y lies in the semi-direct product $\mathcal{D}(\mathfrak{h}_{\mathrm{reg}}) \rtimes W$. If we identify the Dunkl operator D_y with y in $\mathfrak{h}_{\mathrm{reg}}$, then we have

$$H_c(W, \mathfrak{h}) \subseteq \mathcal{D}(\mathfrak{h}_{\mathrm{reg}}) \rtimes W\,. \tag{3.2}$$

Since δ^{-1} is an element of $\mathcal{D}(\mathfrak{h}_{\mathrm{reg}})$, we also get

$$H_c(W, \mathfrak{h})[\delta^{-1}] \subseteq \mathcal{D}(\mathfrak{h}_{\mathrm{reg}}) \rtimes W\,. \tag{3.3}$$

Definition 3.1. An element $f \in \mathbb{C}[\mathfrak{h}]$ is said to be W-semi-invariant if there exists a character function $\chi : W \to \mathbb{C}^\times$ such that $w \cdot f = \chi(w)f$, for all $w \in W$.

Lemma 3.2. The product $\delta = \prod_s \alpha_s$ is W-semi-invariant.

Proof. Let $w \in W$ and $s \in S$. By the definition of the reflections, it is obvious that $wsw^{-1} \in S$. We can prove the lemma by showing that $w(\alpha_s) = \beta\alpha_{wsw^{-1}}$ for some nonzero scalar β. The equation

$$(wsw^{-1})w(\alpha_s) = ws(\alpha_s) = w(\lambda_s\alpha_s) = \lambda_s w(\alpha_s)\,. \tag{3.4}$$

implies that $w(\alpha_s)$ is the nontrivial eigenvector of wsw^{-1} with eigenvalue λ_s. Hence $w(\alpha_s)$ and $\alpha_{wsw^{-1}}$ differ by some scalar $\beta_{s,w}$. Then we apply w to δ to get

$$\begin{aligned} w(\delta) &= w\left(\prod_s \alpha_s\right) \\ &= \prod_s \beta_{s,w}\alpha_{wsw^{-1}} \\ &= \prod_s \beta_{s,w}\delta\,. \end{aligned} \tag{3.5}$$

We define the function $\chi : W \to \mathbb{C}$ by

$$w \mapsto \gamma_w := \prod_s \beta_{s,w}\,. \tag{3.6}$$

Then we have

$$\gamma_{w_1w_2}\delta = w_1w_2(\delta) = \gamma_{w_1}\gamma_{w_2}\delta\,. \tag{3.7}$$

So it is clear that δ is W-semi-invariant. □

Now there exists some r that for any $w \in W$, we have $w^r = 1$. Consequently, we also have $\gamma_w^r = 1$, which implies that $\delta^r \in \mathbb{C}[\mathfrak{h}]^W$. The powers of δ^r form an Ore set in $H_c(W, \mathfrak{h})$ and we may localize $H_c(W, \mathfrak{h})$ at δ^r. Since the sum belongs to the algebra, i.e.,

$$\sum_s \frac{2c_s}{1-\lambda_s}\frac{\alpha_s(y)}{\alpha_s}(1-s) \in H_c(W, \mathfrak{h})[\delta^{-r}]\,, \tag{3.8}$$

by the definition of the Dunkl operators, we conclude that

$$\partial_y = D_y + \sum_s \frac{2c_s}{1-\lambda_s}\frac{\alpha_s(y)}{\alpha_s}(1-s) \in H_c(W,\mathfrak{h})[\delta^{-r}]\,. \tag{3.9}$$

Hence now we have constructed the required isomorphism in (3.1).

By the definition of $\mathfrak{h}_{\mathrm{reg}}$, the group W acts freely on $\mathfrak{h}_{\mathrm{reg}}$, which implies that $\pi : \mathfrak{h}_{\mathrm{reg}} \to \mathfrak{h}_{\mathrm{reg}}/W$ is a finite cover map. In particular, its differential map $\mathrm{d}_x\pi : \mathrm{T}_x(\mathfrak{h}_{\mathrm{reg}}/W) \to \mathrm{T}_{\pi(x)}(\mathfrak{h}_{\mathrm{reg}}/W)$ is an isomorphism for all $x \in \mathfrak{h}$, where T denotes the tangent space.

According to Proposition 4.4.1 of [2], we have the following proposition.

Proposition 3.3. There is a natural isomorphism $\mathcal{D}(\mathfrak{h}_{\mathrm{reg}})^W \cong \mathcal{D}(\mathfrak{h}_{\mathrm{reg}}/W)$.

Then we have a corollary as in Proposition 4.4.3 of [2].

Corollary 3.4. The functor $(\cdot)^W : M \mapsto M^W$ defines an equivalence between the category of the W-equivariant $\mathcal{D}$-modules on $\mathfrak{h}_{\mathrm{reg}}$ (i.e., the category of $\mathcal{D}(\mathfrak{h}_{\mathrm{reg}}) \rtimes W$-modules) and the category of $\mathcal{D}(\mathfrak{h}_{\mathrm{reg}}/W)$-modules.

By the isomorphism defined in (3.1) and Corollary 3.4, we can localize M in Category $\mathcal{O}$ at δ, and apply the $(\cdot)^W$ functor to get a $\mathcal{D}(\mathfrak{h}_{\mathrm{reg}}/W)$-module. Then we can have the solution space to this new module, which is a local system.

Definition 3.5. An integrable connection is a $\mathcal{D}(\mathfrak{h}_{\mathrm{reg}})$-module which is finitely generated as a $\mathbb{C}[\mathfrak{h}_{\mathrm{reg}}]$-module.

Let N be an integrable connection. It is proved that every integrable connection is free as a $\mathbb{C}[\mathfrak{h}_{reg}]$-module [[15], Theorem 1.4.10]. Hence, we have

$$N \cong \bigoplus_{i=1}^{k} \mathbb{C}[\mathfrak{h}_{\mathrm{reg}}]u_i\,, \tag{3.10}$$

for some $u_i \in N$. Recall that $\mathfrak{h}^* = \langle x_1, ..., x_n\rangle$. We have $\mathbb{C}[\mathfrak{h}] = \mathbb{C}[x_1, \ldots, x_n]$. Then we can obtain $n \times k$ differential equations to describe the action of rings of differential operators:

$$\partial_{x_l} u_i = \sum_{j=1}^{k} f_{li}^{j} u_j\,, \tag{3.11}$$

for some polynomial f^j_{li}, and $l = 1, \ldots, n, i = 1, \ldots, k$. One way to study the integrable connections is to examine their space of solutions to the differential equations (3.11). Since very few differential equations have polynomial solutions, this approach only makes sense in analytic topology. Let $\mathfrak{h}^{an}_{\mathrm{reg}}$ be the same space as $\mathfrak{h}_{\mathrm{reg}}$ but equipped with an analytic topology. We write $\mathbb{C}[\mathfrak{h}^{an}_{\mathrm{reg}}]$ as the ring of holomorphic functions on $\mathfrak{h}^{an}_{\mathrm{reg}}$. For instance, we have

$$\mathcal{D}(\mathfrak{h}^{an}_{\mathrm{reg}}) = \mathbb{C}[\mathfrak{h}^{an}_{\mathrm{reg}}] \otimes_{\mathbb{C}[\mathfrak{h}_{\mathrm{reg}}]} \mathcal{D}(\mathfrak{h}_{\mathrm{reg}})\,. \tag{3.12}$$

Then there exists a natural functor:

$$\begin{aligned}(\boldsymbol{\cdot})^{an} :\ \mathcal{D}(\mathfrak{h}_{\mathrm{reg}}) \rtimes W\text{-mod} \to \mathcal{D}(\mathfrak{h}^{an}_{\mathrm{reg}}) \rtimes W\text{-mod}\,,\\ M \mapsto M^{an} := \mathbb{C}[\mathfrak{h}^{an}_{\mathrm{reg}}] \otimes_{\mathbb{C}[\mathfrak{h}_{\mathrm{reg}}]} M\,.\end{aligned} \tag{3.13}$$

Since $\mathbb{C}[\mathfrak{h}^{an}_{\mathrm{reg}}]$ is faithfully flat, then $M^{an} = 0$ implies that $M = 0$.

On any simply connected open subset U of $\mathfrak{h}^{an}_{\mathrm{reg}}$, the dimension of $\mathrm{Hom}_{\mathcal{D}(\mathfrak{h})^{an}_{\mathrm{reg}}}(N^{an}, \mathbb{C}[U])$ is k because it is the solution of a matrix of $k \times k$ first order linear differential equations. These solution spaces glue together to form a local system $\mathrm{Sol}(N)$ on $\mathfrak{h}_{\mathrm{reg}}$.

If $\dim(\mathfrak{h}) = 1$ and $\mathfrak{h}^* = \mathbb{C}\langle x\rangle$, then we have $\mathbb{C}[\mathfrak{h}_{\mathrm{reg}}] = \mathbb{C}[x, x^{-1}]$. We say that N is a regular connection or has regular singularities if (3.11) becomes

$$\partial_x\, u_i = \sum_{j=1}^{k} \frac{a_{ij}}{x} u_j \tag{3.14}$$

for all $a_{ij} \in \mathbb{C}$, with respect to some $\mathbb{C}[\mathfrak{h}^{an}_{\mathrm{reg}}]$-basis of N^{an}. When the dimension of $\mathfrak{h}$ is larger than one, we say that N has regular singularities if the restriction $N\big|_C$ has the same form as (3.14) after the change of basis for any smooth curve $C \subseteq \mathfrak{h}_{\mathrm{reg}}$.

We can denote the category of the integrable connections with regular singularities on $\mathfrak{h}_{\mathrm{reg}}$ by $\mathrm{Conn}^{\mathrm{reg}}(\mathfrak{h}_{\mathrm{reg}})$, and denote the category of the finite dimensional local system on $\mathfrak{h}^{an}_{\mathrm{reg}}$ by $\mathrm{Loc}(\mathfrak{h}_{\mathrm{reg}})$. We define two functors $\mathrm{Conn}^{\mathrm{reg}}(\mathfrak{h}_{\mathrm{reg}}) \to \mathrm{Loc}(\mathfrak{h}_{\mathrm{reg}})$: the solution functor

$$\mathrm{Sol} : N \mapsto \mathrm{Hom}_{\mathcal{D}(\mathfrak{h}^{an}_{\mathrm{reg}})}(N^{an},\ \mathcal{O}^{an}_{\mathfrak{h}_{\mathrm{reg}}})\,, \tag{3.15}$$

and the de Rham functor

$$\mathrm{DR} : N \mapsto \mathrm{Hom}_{\mathcal{D}(\mathfrak{h}_{\mathrm{reg}}^{an})}(\mathcal{O}_{\mathfrak{h}_{\mathrm{reg}}}^{an},\ N^{an})\,. \tag{3.16}$$

There exist two duality functors: $\mathbb{D} : \mathrm{Conn}^{\mathrm{reg}}(\mathfrak{h}_{\mathrm{reg}}) \xrightarrow{\simeq} \mathrm{Conn}^{\mathrm{reg}}(\mathfrak{h}_{\mathrm{reg}})$ and $\mathbb{D}' : \mathrm{Loc}(\mathfrak{h}_{\mathrm{reg}}) \xrightarrow{\simeq} \mathrm{Loc}(\mathfrak{h}_{\mathrm{reg}})$. They are contravariant equivalences whose square is the identity, intertwining Sol and DR such that

$$\mathbb{D}' \circ \mathrm{DR} = \mathrm{DR} \circ \mathbb{D} = \mathrm{Sol}\,, \tag{3.17}$$

$$\mathbb{D}' \circ \mathrm{Sol} = \mathrm{Sol} \circ \mathbb{D} = \mathrm{DR}\,. \tag{3.18}$$

We have the following theorem due to Theorem 4.3.1 of [2].

Theorem 3.6. The de Rham functor and the solution functor are equivalences.

Any function $f \in \mathrm{DR}(N) = \mathrm{Hom}_{\mathcal{D}(\mathfrak{h}_{\mathrm{reg}}^{an})}(\mathcal{O}_{\mathfrak{h}_{\mathrm{reg}}}^{an}, N^{an})$ is completely determined by the value of $f(1)$. Because we have $\partial_{x_i} f(1) = f(\partial_{x_i} 1) = 0$, the image $\mathrm{DR}(N)$ can be identified as

$$\mathrm{DR}(N) = \{n \in N^{an} \mid \partial_{x_i} n = 0\ \forall\ i\} =: N^{\nabla}\,. \tag{3.19}$$

Consider the fundamental group $\pi_1(\mathfrak{h}_{\mathrm{reg}})$. A local system L going along a given loop γ based at a fixed point x_0 gives an invertible endomorphism $\gamma_* : L_{x_0} \to L_{x_0}$. The map γ_* only depends on the homotopy class of $\gamma \in \pi_1(\mathfrak{h}_{\mathrm{reg}})$. Subsequently, the stalk L_{x_0} can be viewed as a representation of $\pi_1(\mathfrak{h}_{\mathrm{reg}})$.

According to Proposition 4.2.1 of [2], we have the following proposition.

Proposition 3.7. If $\mathfrak{h}_{\mathrm{reg}}$ is connected, then the functor $\mathrm{Loc}(\mathfrak{h}_{\mathrm{reg}}) \to \pi_1(\mathfrak{h}_{\mathrm{reg}})$-mod defined by $L \mapsto L_{x_0}$ is an equivalence.

Now we are ready to combine all the ingredients and define the KZ functor. If we take a module M from Category $\mathcal{O}$ and localize it at δ, it becomes the $\mathcal{D}(\mathfrak{h}_{\mathrm{reg}}) \rtimes W$-module, $M[\delta^{-1}]$. Then the functor $(\cdot)^W$ sends it to $(M[\delta^{-1}])^W$, which is a $\mathcal{D}(\mathfrak{h}_{\mathrm{reg}}/W)$-module. It is shown in [10] that $(M[\delta^{-1}])^W$ is a regular connection. Finally, we apply the de Ram functor to get a representation of the fundamental group $\pi_1(\mathfrak{h}_{reg/W})$ given by

$$\mathrm{KZ} :\ \mathcal{O} \to \pi_1(\mathfrak{h}_{\mathrm{reg}}/W)\text{-mod}$$

$$M \mapsto \mathrm{DR}(M[\delta_{-1}]^W) = (((M[\delta^{-1}])^W)^{an})^{\triangledown} . \tag{3.20}$$

Remark. Since all the functors are exact in (3.20), the KZ functor is exact.

3.2 Examples of KZ Functor

We continue with our previous calculations for the case of $W = \mathbb{Z}_l$. Suppose that we have $\mathfrak{h} = \mathbb{C}y$ and $\mathfrak{h}^* = \mathbb{C}x$. We apply he same setup as in Example 1.3 except that here we consider W as a multiplicative group instead of an additive one. Let s be a generator of $\mathbb{Z}_l$ such that $s(y) = \zeta y$ and $s(x) = \zeta^{-1}x$. We know that $\mathbb{Z}_l$ has l different irreducible representations ρ_i for $i = 0, \dots, l-1$ such that s acts on ρ_i by $\zeta^i \mathrm{Id}_{\rho_i}$. Then each representation ρ_i corresponds to an l-partition λ_i of $n = 1$ which is empty everywhere except the box in the i-th coordinate. Recall from Section 1.3.1 that the Verma module $M_c(\rho_i) = \mathbb{C}[x] \otimes \rho_i$ can be regarded as a $\mathbb{C}[x]$-module, and that $y \cdot (1 \otimes \rho_i) = 0$. According to Definition 1.5, the Dunkl operator becomes

$$\begin{aligned} \mathrm{D}_y &= \partial_x - \sum_{j=1}^{l-1} \frac{2c_j}{1-\lambda_{s^j}} \frac{\alpha_{s^j}(y)}{\alpha_{s^j}}(1 - s^j) \\ &= \partial_x - \sum_{j=1}^{l-1} \frac{2c_j}{1-\zeta^{-j}} \frac{1}{x}(1 - s^j) . \end{aligned} \tag{3.21}$$

If we apply D_y to degree zero element in the Verma modules, we obtain

$$\mathrm{D}_y \cdot (1 \otimes \rho_i) = y \cdot (1 \otimes \rho_i) = 0 , \tag{3.22}$$

which implies that

$$\partial_x \, (1 \otimes \rho_i) = \sum_{j=1}^{l-1} \frac{2c_j}{1-\zeta^{-j}} \frac{1}{x}(1 - \zeta^{ij})(1 \otimes \rho_i) . \tag{3.23}$$

If we set

$$a_i = \sum_{j=1}^{l-1} \frac{2c_j}{1-\zeta^{-j}}(1 - \zeta^{ij}) , \tag{3.24}$$

then we have

$$\partial_x \, (1 \otimes \rho_i) = \frac{a_i}{x}(1 \otimes \rho_i) . \tag{3.25}$$

Recall that $\delta = \prod_{i=1}^{l-1} \alpha_{s^i} = 2^{\frac{l-1}{2}} x^{l-1}$ as defined for (3.1). The Verma modules $M_c(\rho_i)$ can be localized at δ to give

$$M_c(\rho_i)[\delta^{-1}] = \mathbb{C}[x, x^{-1}] \otimes \rho_i \,. \tag{3.26}$$

If we let $z = x^l$, then the chain rule gives

$$\partial_x = \frac{\partial z}{\partial x}\,\partial_z = l x^{l-1}\,\partial_z \,, \tag{3.27}$$

which implies that

$$\partial_z = \frac{1}{l x^{l-1}}\,\partial_x \,. \tag{3.28}$$

Applying the functor $(\bullet)^W$, for $W = \mathbb{Z}_l$, on the localized module $M_c(\rho_i)[\delta^{-1}]$ produces

$$M_c(\rho_i)[\delta^{-1}]^{\mathbb{Z}_l} = \mathbb{C}[z, z^{-1}](x^i \otimes \rho_i) \,. \tag{3.29}$$

Since s acts on ρ_i by ζ^i, we can compute (3.14) as

$$\begin{aligned} \partial_z\,(x^i \otimes \rho_i) &= \frac{1}{l x^{l-1}}\,\partial_x\,(x^i \otimes \rho_i) \\ &= \frac{1}{l x^{l-1}} i x^{i-1}(1 \otimes \rho_i) + \frac{1}{l x^{l-1}}\,x^i\,\frac{a_i}{x}(1 \otimes \rho_i) \\ &= \frac{i + a_i}{l z}(x^i \otimes \rho_i) \,. \end{aligned} \tag{3.30}$$

If we denote $x^i \otimes \rho_i$ by m_i, then the result of the above calculation can be written as

$$\partial_z\,m_i = \frac{i + a_i}{l}\,\frac{1}{z} m_i \,, \tag{3.31}$$

which has the solution given by

$$m_i = \mathbb{C}\; z^{\frac{i+a_i}{l}} \,. \tag{3.32}$$

Note that this result is obtained when the solution functor Sol is applied to $M_c(\rho_i)[\delta^{-1}]^{\mathbb{Z}_l}$. The solution for the de Rham functor DR can be computed directly by applying the duality functor $\mathbb{D}'$ to the above result, which sends the local system of $m_i = \mathbb{C}\; z^{\frac{i+a_i}{l}}$ to

$$m_i = \mathbb{C}\; z^{\frac{-(i+a_i)}{l}} \,. \tag{3.33}$$

Now consider the cyclotomic Hecke algebra $\mathcal{H}_{\mathbf{q}}(\mathbb{Z}_l)$ with parameters $\mathbf{Q} = (Q_1, \ldots, Q_l)$, which is generated by only one generator, T_0, with the relation

$$\prod_{i=1}^{l} (T_0 - Q_i) = 0\,. \tag{3.34}$$

This generator T_0 is represented by the loop $t \mapsto \exp\bigl(-2\pi\sqrt{-1}\bigr)$, which acts on the local system as

$$T_0 \cdot z^{\frac{-(i+a_i)}{l}} = \exp\biggl(-2\pi\sqrt{-1}\frac{(i+a_i)}{l}\biggr) z^{\frac{-(i+a_i)}{l}}\,. \tag{3.35}$$

Therefore, this local system is a representation of the corresponding cyclotomic Hecke algebra. If we identify $\exp\Bigl(-2\pi\sqrt{-1}\frac{(i+a_i)}{l}\Bigr)$ with Q_i. Then, the action of T_0 can be written in terms of Q_i as

$$\begin{aligned} T_0 \cdot z^{\frac{-(i+a_i)}{l}} &= \exp\biggl(-2\pi\sqrt{-1}\frac{(i+a_i)}{l}\biggr) z^{\frac{-(i+a_i)}{l}} \\ &= Q_i z^{\frac{-(i+a_i)}{l}} \end{aligned} \tag{3.36}$$

In conclusion, the image of $M_c(\lambda)$ under the KZ functor is a one dimensional vector space on which the cyclotomic Hecke algebra acts by the scalar $Q_i = \exp\Bigl(-2\pi\sqrt{-1}\frac{(i+a_i)}{l}\Bigr)$.

4 Weighted KLR Algebras and Category $\mathcal{O}$ of $G(l,1,n)$

This section introduces weighted KLR algebras and their connection with the weight spaces of modules of rational Cherednik algebras. It can be shown that there is an isomorphism between the KZ functor and a direct sum of weight space functors. It is the key to show the compatibility of the KZ functor and the unitarity condition of rational Cherednik algebras.

4.1 Weighted KLR Diagrams and Weighted KLR Algebras

Even though we can calculate the image of KZ functor by its definition, it is not intuitive and obvious. Fortunately, Webster in his paper [25] showed that KZ functor is isomorphic to a direct sum of weight space functors, which gives us the explicit look of KZ$(M_c(\lambda))$, for each Verma module $M_c(\lambda)$. To acquire this result, we need a detour to weighted KLR algebras and how their modules relate to the modules of rational Cherednik algebras. The section is a review of [23], [26] and [24].First of all, we are going to prepare for the definitions. Let D be a quiver, with vertices and oriented edges. We assume every edge has the same fixed weight $k \in \mathbb{R}$.

Definition 4.1. A weighted Khovanov-Lauda-Rouquier (KLR) diagram attached to the quiver D is a diagram on $\mathbb{R} \times [0, 1]$ with two real lines at $y = 0$ and $y = 1$, between which there are smooth strands whose projections are onto the closed interval $[0, 1]$. These strands satisfy the following:

1. The vertical direction of the strand is unchanged between $y = 0$ and $y = 1$, that is a strand can not go from bottom to top, go back to bottom and then go top.

2. There is no loop in any strand.

3. Each strand has distinct endpoints on both lines at $y = 0, 1$.

4. Each strand is labelled by a vertex of D.

5. Each strand may carry finitely many dots.

6. We add a ghost to each strand, i.e., a dotted strand shifted by k relative to the strand. The shift is to the right if $k > 0$ and to the left otherwise.

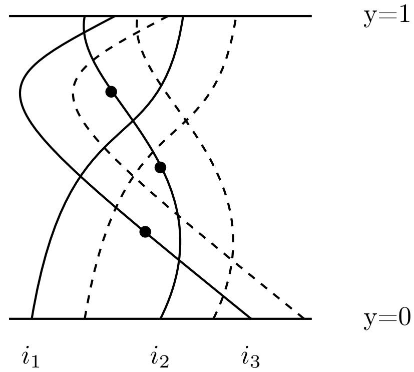
7. There is no tangency between any two strands and no dot on intersection of two strands. All points of intersection of any two strands are distinct from each other. In other words, there are no three strands intersect at the same point.

To make sense of the definition, we give an example of a weighted KLR diagram in Figure 1. There are three strands of positive weight k labelled by the vertices $i_1, i_2, i_3 \in D$. They carry $0, 2, 1$ dots respectively.

The lines at $y = 0$ and $y = 1$ are called the bottom and the top of the diagram respectively. They are omitted in the rest of the paper. Let $\mathbf{i}\mathcal{R}\mathbf{i}'$ denote the collection of all diagrams with bottom labels $\mathbf{i} = \{i_1, \ldots, i_m\}$ and top labels $\mathbf{i}' = \{i'_1, \ldots, i'_m\}$ from left to right. Suppose we have two diagrams A and B. The composition of the two diagram AB is defined by putting A on top of B provided that the bottom of A matches the top of B. Otherwise, the composition is zero.

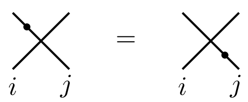
Definition 4.2. We define the loadings as the maps $i : \mathbb{R} \to D \cup 0$ by sending finitely many of points of $\mathbb{R}$ to D and all the other points to 0.

Figure 1. Weighted KLR diagram with three strands of positive weight

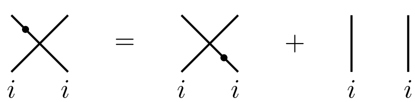


The loadings i encode where the endpoints of weighted KLR diagrams lie in $\mathbb{R}$, and how the endpoints are labelled. We will refer the information of positions and labels of endpoints on top (bottom) of a weighted KLR diagram as top (bottom) loadings.

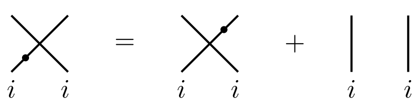
Definition 4.3. A weighted KLR algebra T_D is an algebra over $\mathbb{C}$ generated by the weighted KLR diagrams attached to the quiver D under the local relations



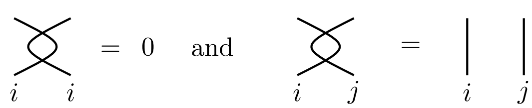
for $i \neq j$, (4.1)



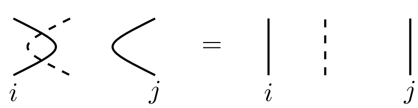
, (4.2)



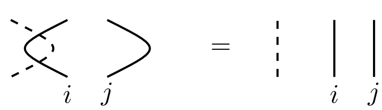
, (4.3)



for $i \neq j$, (4.4)

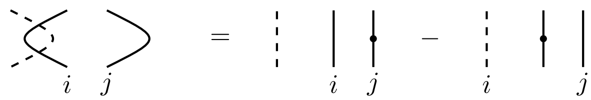


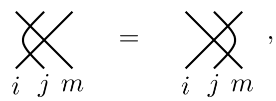
if $i \nrightarrow j$, (4.5)

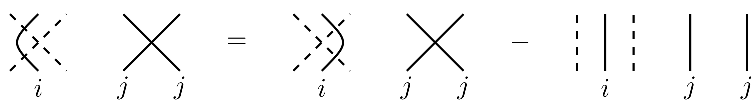


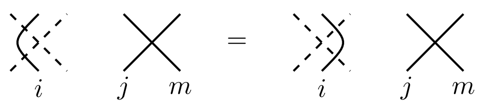
if $i \nrightarrow j$, (4.6)

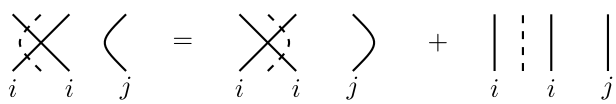
 if $i \to j$, (4.7)

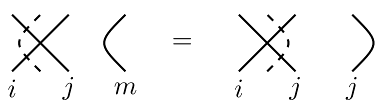
 if $i \to j$, (4.8)

 , (4.9)

 if $i \to j$, (4.10)

 unless $j = m$ and $i \to j$, (4.11)

 if $i \to j$, (4.12)

 unless $i = j$ and $i \to m$. (4.13)

The concept of the weighted KLR algebra is an extension of the original KLR algebra constructed in [16], which has a polynomial representation. Therefore, we expect the weighted KLR algebra to have a polynomial representation as well. According to Proposition 2.7 of [23] and Proposition 3.9 of [24], the polynomial representation is described in the following proposition.

Proposition 4.4. Let use denote $|\mathbf{j}| := m$, for any $\mathbf{j} = \{j_1, \ldots, j_m\}$. The weighted KLR algebra has a polynomial representation

$$P = \bigoplus_{\mathbf{j}} \mathbb{C}[x_{1,\mathbf{j}}, \ldots, x_{|\mathbf{j}|\,,\mathbf{j}}] \,. \tag{4.14}$$

Given a collection of diagrams $\mathbf{i}R\mathbf{i}'$, each diagram in $\mathbf{i}R\mathbf{i}'$ acts on P by mapping the polynomials in the ring $\mathbb{C}[\,x_{1,\mathbf{i}}, \ldots, x_{|\mathbf{i}|,\mathbf{i}}]$ to the polynomials in the other ring $\mathbb{C}[\,x_{1,\mathbf{i}'}, \ldots, x_{|\mathbf{i}'|,\mathbf{i}'}]$, and mapping all the polynomials in other summands to zero. Let $f \in \mathbb{C}[x_{1,\mathbf{j}}, \ldots, x_{|\mathbf{j}|\,,\mathbf{j}}]$, we define the action of the transposition $s_{r,t} = (r, t)$ for $r < t$ by

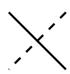
$$f(x_{1,\mathbf{j}}, \ldots, x_{|\mathbf{j}|,\mathbf{j}})^{s_{r,t}} = f(x_{1,s_{r,t}\cdot\mathbf{j}}, \ldots, x_{t,s_{r,t}\cdot\mathbf{j}}, \ldots, x_{r,s_{r,t}\cdot\mathbf{j}} \ldots, x_{|\mathbf{j}|,s_{r,t}\cdot\mathbf{j}})\,. \tag{4.15}$$

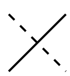
The action of a diagram is defined by the actions of its component dots and crossings. Let the r-th strand in a weighted KLR diagram be the one connected to the r-th endpoint labelled by i_r at the bottom. Recall that the label of each strand is a vertex in quiver D.

1. If the top and the bottom loading of a diagram $\mathbf{i}R\mathbf{i}$ are the same, and endpoints are connected by straight strands without dots, then it acts by identity on $\mathbb{C}[x_{1,\mathbf{i}}, \ldots, x_{|\mathbf{i}|,\mathbf{i}}]$ and zero elsewhere.

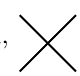
2. If the top and the bottom loading of a diagram $\mathbf{i}R\mathbf{i}$ are the same, and endpoints are connected by straight strands with a dot on the r-th strand, then the diagram acts by multiplying by $x_{r,\mathbf{i}}$ on $\mathbb{C}[x_{1,\mathbf{i}}, \ldots, x_{|\mathbf{i}|,\mathbf{i}}]$ and zero elsewhere.

3. If there is an edge from the label of the r-th strand to label of the t-th strand in quiver D, then the crossing between the r-th strand and the ghost of the t-th strand acts as follows:

   (a) The crossing  acts as the identity if $k < 0$, and acts by multiplying $(x_{t,\mathbf{i}'} - x_{r,\mathbf{i}'})$ if $k > 0$.

   (b) The crossing  acts as the identity if $k > 0$, and acts by multiplying $(x_{t,\mathbf{i}'} - x_{r,\mathbf{i}'})$ if $k < 0$.

4. If there is no edge between the label of the r-th strand and the label of the t-th strand in the quiver D, then both of the crossings in 3 act as the identity.

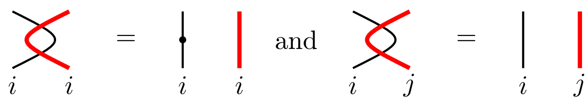
5. The crossing between the label of the r-th strand and the $(r+1)$-th strand,  acts as the permutation s_r if the two strands have different labels, and as the Demazure operator $\frac{s_r - 1}{x_{r+1} - x_r}$ if the two strands have the same labels.

Definition 4.5. A Crawley-Boevey quiver D_∞ is a quiver constructed by adding an extra vertex ∞ to the quiver D and all possible edges from the new vertex ∞ to each original vertex i in D. We assign a weight $\theta_i \in \mathbb{R}$ to the new edge connecting vertices i and ∞.

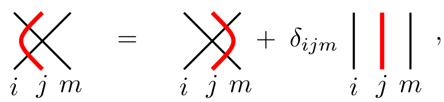
Definition 4.6.

1. The weighted KLR diagrams attached to the Crawley-Boevey quiver D_∞ are the weighted KLR diagrams attached to D with an additional strand labelled by ∞. The ghosts of the additional strand labelled by ∞ are shifted to the right (left) by $|\theta_i|$ if $\theta_i > 0$ ($\theta_i < 0$) and labelled by i. We color these ghosts in red.

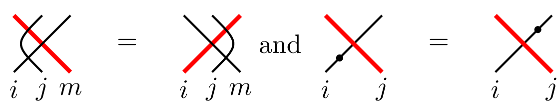
2. The weighted KLR algebra attached to the Crawley-Boevey quiver D_∞ is the algebra generated by the weighted KLR diagram T_{D_∞} under the relations in Definition 4.3, and the following additional relations:



for $i \neq j$, (4.16)



(4.17)



for any i, j, m with their mirror images. (4.18)

3. There is a special case of the weighted KLR algebra attached to the Crawley-Boevey quiver, where the strand labeled by ∞ is a vertical line at $x = 0$ and all the red strands are at $x = \theta_i$. It becomes relevant when considering rational Cherednik algebras.

4.2 Weighted Modules and Weighted KLR Algebras

Recall from Section 2 that the Dunkl-Opdam subalgebra plays an important role in studying simple modules. In this section, we construct an alternative presentation of rational Cherednik algebras in terms of the generators of the corresponding Dunkl-Opdam subalgebra. Based on this presentation, we can give a diagrammatic description of the action of the subalgebra on its modules. This section is a review of [25].

In Section 2, we have parameters c_0 and $h_0, \ldots, h_{l-1}$ of rational Cherednik algebras of $G(l, 1, n)$.

We can define the function $p : \mathbb{C} \to \mathbb{C}$ by

$$p(u) = \sum_{s=0}^{l-1} \sum_{r=0}^{l-1} \zeta^{-rs} h_r u^s \,. \tag{4.19}$$

For any $i = 0, 1, \ldots, l-1$, we denote

$$r_i = -p(\zeta^i) + i = -l h_i + i \,. \tag{4.20}$$

Let us recall the definition of z_i in (2.10). Then we have the modified Dunkl-Opdam operators

$$u_i = z_i - p(\zeta^{-1}\zeta_i) - 1 \,. \tag{4.21}$$

It is obvious that u_i belong to the Dunkl-Opdam subalgebra $\mathfrak{t}$ for all $i = 1, \ldots, n$. Hence the Dunkl-Opdam subalgebra is generated by u_i and ζ_i. The reason why we use u_i instead z_i is that u_i has a simpler commutation relation with other generators. Once we fix these u_i and ζ_i for $i = 1, \ldots, n$, we can define

$$u_i = u_{i-n} + 1 \,, \tag{4.22}$$

$$\zeta_i = \zeta^{-1}\zeta_{i-n} \tag{4.23}$$

for any $n \in \mathbb{N}$. Then the commutation relations can be described in the following lemma.

Lemma 4.7.

1.

$$u_i \Phi = \Phi u_{i+1} \quad \text{and} \quad u_i \Psi = \Psi u_{i-1} \tag{4.24}$$

2.

$$\zeta_i \Phi = \Phi \zeta_{i+1} \quad \text{and} \quad \zeta_i \Psi = \Psi \zeta_{i-1} \tag{4.25}$$

3. If $j \neq i, i+1$, then

$$u_i s_j = s_j u_i \,. \tag{4.26}$$

4.

$$u_i s_i = s_i u_{i+1} - c_0 \pi_i \tag{4.27}$$

Proof. These follow directly from Lemma 2.4. □

We are now prepared to build the alternative presentation of rational Cherednik algebras.

Proposition 4.8. The rational Cherednik algebra $H_c(W, \mathfrak{h})$ is generated by Φ, Ψ, u_i, ζ_i, and s_i for $i = 1, \ldots, n$.

Proof. The generators Φ, Ψ, u_i, ζ_i, and s_i definitely generate a subalgebra of $H_c(W, \mathfrak{h})$, which we denote as $\tilde{H}$. Recall the generators x_i and y_i for $i = 1, \ldots, n$ from Section 1. It suffices to prove that all x_i and y_i belong to $\tilde{H}$.

Because $x_n = \Phi s_1 \ldots s_{n-1}$ by the definition of Φ, we know that x_n lies in $\tilde{H}$. After conjugating x_n with s_{in}, we get $x_i = s_{in} x_n s_{in} \in \tilde{H}$. The same argument applies for y_i and Ψ. □

If we write Proposition 4.8 in terms of abstract algebra, then we get the following theorem due to Theorem 2.3 of [25].

Theorem 4.9. Let A be the free algebra generated by $\tilde{\zeta_i}$, $\tilde{u_i}$, $\tilde{s_i}$, $\tilde{\Phi}$, and $\tilde{\Psi}$ under (2.25)–(2.28) in Lemma 2.4, (4.24)–(4.27) in Lemma 4.7 and the followings:

1.

$$\tilde{\Psi}\tilde{\Phi} = \tilde{u_1} - p(\zeta^{-1}\tilde{\zeta_1}) + 1\,, \tag{4.28}$$

2.

$$\tilde{\Phi}\tilde{\Psi} = \tilde{u}_n - p(\tilde{\zeta}_n)\,, \tag{4.29}$$

3.

$$\tilde{s_i}^2 = 1\,, \tag{4.30}$$

4. For $i \neq (j \pm 1)$, we have

$$\tilde{s}_i \tilde{s}_j = \tilde{s}_j \tilde{s}_i\,, \tag{4.31}$$

5.

$$(\tilde{s}_i \tilde{s}_{i+1})^3 = 1\,, \tag{4.32}$$

6.

$$\tilde{u_i}\tilde{\zeta_j} = \tilde{\zeta_j}\tilde{u_i}\,, \tag{4.33}$$

7.

$$\tilde{u_i}\tilde{u_j} = \tilde{u_j}\tilde{u_i}\,, \tag{4.34}$$

8.

$$\tilde{\zeta_i}\tilde{\zeta_j} = \tilde{\zeta_j}\tilde{\zeta_i}\,, \tag{4.35}$$

9.

$$\tilde{\Phi}\tilde{s}_1\tilde{\Psi} = \tilde{\Psi}\tilde{s}_{n-1}\tilde{\Phi} - c_0(\sum_{p=0}^{l-1}\zeta^p\tilde{\zeta}_n^p\tilde{\zeta}_1^p)\,. \tag{4.36}$$

Then A is isomorphic to the rational Cherednik algebra $H_c(G(l,1,n),\mathbb{C}^n)$.

By Lemma 2.6 in [25], the elements $u_1,\ldots,u_n$ and $\zeta_1,\ldots,\zeta_n$ generate an algebra isomorphic to the quotient $\mathbb{C}[u_1,\ldots,u_n,\zeta_1,\ldots,\zeta_n]/\langle\zeta_1^l-1,\ldots,\zeta_n^l-1\rangle$ of the ring of polynomial . This alternative presentation gives a polynomial representation of the rational Cherednik algebra on this ring. To get ready for next step, we may define $\theta'(u_i) = u_{i+1}$ and $\theta'(\zeta_i) = \zeta_{i+1}$ resembling the original θ defined in Section 2. According to (2.13)–(2.17) in [25], we have the following proposition.

Proposition 4.10. Recall from (2.14), we have $\pi_i = \sum_{p=0}^{l-1}\zeta_i^p\zeta_{i+1}^{-p}$. There is a faithful polynomial representation $\mathbb{C}[U_1,\ldots,U_n,T_1,\ldots,T_n]$ of $H_c(W,\mathfrak{h})$. For any $f\in\mathbb{C}[U_1,\ldots,U_n,T_1,\ldots,T_n]$, the actions are defined as follows:

$$u_i\cdot f = U_if\,, \tag{4.37}$$

$$\zeta_i\cdot f = T_if\,, \tag{4.38}$$

$$s_i\cdot f = f^{s_i} - c_0\frac{f^{s_i}-f}{U_{i+1}-U_i}\pi_i\,, \tag{4.39}$$

$$\Phi\cdot f = \theta'^{-1}(f)\,, \tag{4.40}$$

$$\Psi\cdot f = (U_1 + p(\zeta^{-1}T_1)+1)\theta'(f)\,, \tag{4.41}$$

where f^{s_i} is the polynomial f but interchanging u_i with u_{i+1}, and ζ_i with ζ_{i+1}, and π_i acts on the monomial $\zeta_1^{j_1}\ldots\zeta_n^{j_n}$ by $l\delta_{j_i,j_{i+1}}$.

From now on, we work over the generalized eigenspaces of u_i and ζ_i. The direct sum of these generalized eigenspaces is also a representation of the weighted KLR algebra, which gives us the equivalence between the category of the representations of the rational Cherednik algebra and the category of the representations of the weighted KLR algebra.

Definition 4.11. We define $H_c(W,\mathfrak{h})\text{-mod}_u$ as the subcategory of $H_c(W,\mathfrak{h})\text{-mod}$, where $\mathbb{C}[u_1,\dots,u_n]$ acts locally finite with finite dimensional generalised weight spaces.

Remark. Category $\mathcal{O}$ is a subcategory of $H_c(W,\mathfrak{h})\text{-mod}_u$.

Definition 4.12. Let $\mu_l(\mathbb{C})$ be the set of l-th roots of unity in $\mathbb{C}$. For any $\mathbf{a} = (a_1,\dots,a_n) \in \mathbb{C}^n$ and $\mathbf{t} = (t_1,\dots,t_n) \in \mu_l(\mathbb{C})^n$, we define the exact generalized weight space functor $W_{\mathbf{a},\mathbf{t}} : H_c(W,\mathfrak{h})\text{-mod}_u \to \mathrm{Vect}_{\mathbb{C}}$ as

$$W_{\mathbf{a},\mathbf{t}}(M) = \{m \in M \mid (u_i - a_i)^N m = (\zeta_i - t_i)^N m = 0 \text{ for large enough } N \in \mathbb{N}\}, \tag{4.42}$$

for any $M \in H_c(W,\mathfrak{h})\text{-mod}_u$. The image of the weight space functor is called the weight space and the pair $(\mathbf{a},\mathbf{t})$ is the weight of this weight space.

Remark. In Section 2, we define the generalized eigenspaces of M from Category $\mathcal{O}$ with respect to a generalized eigenvalue $(\mathbf{a},\mathbf{t}) = ((a_1,\dots,a_n),(t_1,\dots,t_n))$ by $M_{\mathbf{a},\mathbf{t}}$. Then we have

$$W_{\mathbf{a},\mathbf{t}}(M) = M_{\mathbf{a},\mathbf{t}}\,,$$

for any $M \in H_c(W,\mathfrak{h})\text{-mod}_u$.

For any pair $(a,t) \in \mathbb{C}\times\mu_l(\mathbb{C})$, there exists an integer m, such that $t = \zeta^m$. We have a well-defined a function $\Sigma : \mathbb{C}\times\mu_l(\mathbb{C}) \to \mathbb{C}/\mathbb{Z}$ by

$$\Sigma(a,t) = \frac{a+m}{l} \bmod \mathbb{Z}\,. \tag{4.43}$$

For some chosen D, the preimage $\Sigma^{-1}(D)$ forms a set of generalized eigenvalues of u_i and ζ_i.

Definition 4.13. We define $H_c(W,\mathfrak{h})\text{-mod}_D$ to be the subcategory of $H_c(W,\mathfrak{h})\text{-mod}_u$ killed by the functors $W_{\mathbf{a},\mathbf{t}}$, where $(a_i,t_i) \notin \Sigma^{-1}(D)$ for some i.

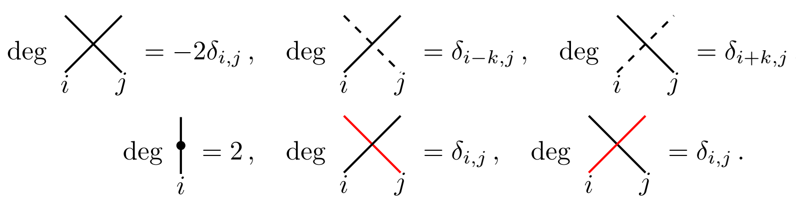
Now we let k be the value $-c_0$, where c_0 is the parameter c_s for transpositions in $H_c(G(l,1,n))$. We turn this subset D into a quiver by adding edges $i \to i+k$ if both vertices are in D. If $k \in \mathbb{Q}$ and $k = \frac{b}{e}$, where b and e are co-prime, then D is a union of e-cycles. Otherwise, D is a union of infinite linear quivers. Recall the definition of r_i from (4.20). Let $\tilde{r_i}$ be $\frac{r_i}{l}$ mod $\mathbb{Z}$. We define the Crawley-Boevey quiver D_∞ attached to the quiver D by adding a new vertex ∞ and new edges connecting $\tilde{r_i}$ and ∞ whenever $\tilde{r_i}$ lies in D.

Let $Y : \mathbb{C} \to \mathbb{R}$ be the $\mathbb{Q}$-linear map, sending x to x if $x \in \mathbb{Q}$ and to 0 otherwise. For all pairs $(\mathbf{a}, \mathbf{t})$ such that $\Sigma(a_i, t_i) \in D$ for all $i = 1, \ldots, n$, we choose a positive real number $\epsilon < \frac{|Y(a_i - a_j)|}{l \cdot n}$ whenever $a_i - a_j \neq 0$. We assign $Y(k)$ as the weights of the old edges in D but not in D_∞, and assign $Y(\frac{p(\zeta^i)}{l}) - i\epsilon$ as the weights of the new edge connecting $\tilde{r_i}$ and ∞.

Let the loading $i_{\mathbf{a},\mathbf{t}} : \mathbb{R} \to D \cup 0$ be the map sending $Y(\frac{a_i}{l}) + i\epsilon$ to $\Sigma(a_i, t_i) \in D$ and all other values to zero.

Definition 4.14. Suppose T_{D_∞} is the weighted KLR algebra attached to the quiver D_∞ as (3) in the Definition 4.6. Then the weighted KLR algebra R_D is defined to be the subalgebra of T_{D_∞} generated by the diagrams, which have the top loading $i_{\mathbf{a},\mathbf{t}}$ and the bottom loadings $i_{\mathbf{a}',\mathbf{t}'}$ satisfying $\Sigma(a_i, t_i), \Sigma(a'_i, t'_i) \in D$ for all $i = 1, \ldots, n$. It is called the weighted KLR algebra associated with the rational Cherednik algebra $H_c(W, \mathfrak{h})$.

We denote $\widehat{R}_D$ as the completion of R_D with respect to the grading



We define the idempotent diagram $e(\mathbf{a}, \mathbf{t})$ such that the loadings of the top and the bottom are both $i_{\mathbf{a},\mathbf{t}}$, and are connected by vertical straight strands. We call the strand corresponding to (a_i, t_i) the i-th stand.

Consider the affine Weyl group $S_n \ltimes \mathbb{Z}^n$. The permutations in S_n act on $(\Sigma^{-1}(D))^n$ in the usual way, and the translation $(m_1, \ldots, m_n) \in \mathbb{Z}^n$ acts by

$$(\mathbf{a}, \mathbf{t}) \mapsto ((a_1 + m_1, \ldots, a_n + m_n), (\zeta^{-m_1} t_1, \ldots, \zeta^{-m_n} t_n)) \,. \tag{4.44}$$

Recall the map ν defined in Definition 2.5. It can be rewritten as an element in $S_n \ltimes \mathbb{Z}^n$ by $(-1, 0, \ldots, 0)s_1, \ldots, s_n$. We can see easily that the affine Weyl group is generated by ν and s_i with $i = 1, \ldots, n-1$. Note that the action of the affine Weyl group on $(\mathbf{a}, \mathbf{t})$ does not change the label $\Sigma(a_i, t_i)$ for all i, i.e., we have $\Sigma(a_i, t_i) = \Sigma(w \cdot (a_i, t_i))$ for $w \in S_n \ltimes \mathbb{Z}^n$.

Consider an element w in the affine Weyl group acting on a pair $(\mathbf{a}, \mathbf{t}) \in \Sigma^{-1}(D)$, i.e., $w \cdot (\mathbf{a}, \mathbf{t}) = (\mathbf{a}', \mathbf{t}')$. We define $\xi(\mathbf{a}, \mathbf{t}, w)$ to be the diagram that has top loading $i_{\mathbf{a},\mathbf{t}}$, bottom loading $i_{\mathbf{a}',\mathbf{t}'}$, and strands connecting $Y(\frac{a_i}{l}) + i\epsilon$ to $Y(\frac{a'_j}{l}) + j\epsilon$, where $a'_j = w \cdot a_i$. All the diagrams in R_D are generated by $\{\xi(\mathbf{a}, \mathbf{t}, w) \mid (w = s_i \text{ with } i = 1, \ldots, n-1) \text{ or } (w = \nu^{\pm})\}$, the idempotent diagrams $e(\mathbf{a}, \mathbf{t})$, and the diagrams obtained from idempotent diagrams by adding a dot on the i-th strand, for $i = 1, \ldots, n$.

There are seven shapes for the diagrams $\xi(\mathbf{a}, \mathbf{t}, s_i)$. If we take $k > 0$ as an example, then the seven shapes can be drawn as in Figure 2, 3, and 4.

Figure 2. Shapes of $\xi(\mathbf{a}, \mathbf{t}, s_i)$ for $a_i > a_{i+1}$

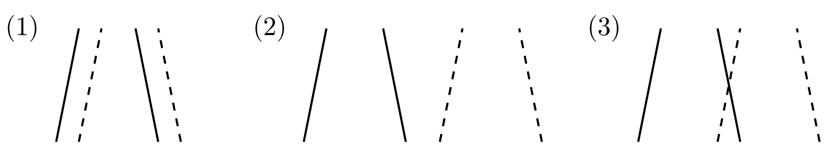


Figure 3. Shapes of $\xi(\mathbf{a}, \mathbf{t}, s_i)$ for $a_i < a_{i+1}$

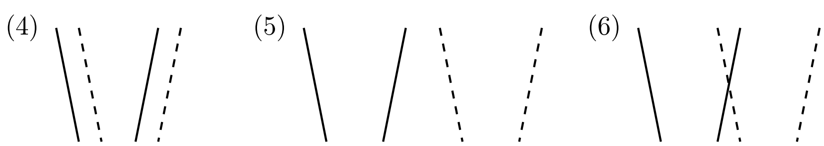
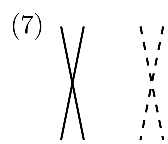


Figure 4. Shapes of $\xi(\mathbf{a}, \mathbf{t}, s_i)$ for $a_i = a_{i+1}$



In the diagram $\xi(\mathbf{a}, \mathbf{t}, \nu)$, all strands except the n-th one are inclined from top left to bottom right, with an off-set of ϵ, while the n-th strand is inclined from bottom left to top right by $\frac{1}{l} + \epsilon(n-1)$. For the inverse diagram $\xi(\mathbf{a}, \mathbf{t}, \nu^{-1})$, all strands except the first one are inclined from bottom left to top right by ϵ, while the first strand is inclined from top left to bottom right by $\frac{1}{l} + \epsilon(n-1)$. Unlike the diagrams $\xi(\mathbf{a},\mathbf{t},s_i)$, The diagrams $\xi(\mathbf{a},\mathbf{t},\nu^{\pm})$ can create crossing between regular strands and red ghosts.

Let $P^N_{\mathbf{a},\mathbf{t}}$ be the quotient

$$P^N_{\mathbf{a},\mathbf{t}} = H_c(W,\mathfrak{h})/\left(\sum (H_c(W,\mathfrak{h})(u_i-a_i)^N + H_c(W,\mathfrak{h})(\zeta_i-t_i)^N)\right) , \tag{4.45}$$

which are modules of H. These modules $P^N_{\mathbf{a},\mathbf{t}}$ form a projective system. We denote the projective limit by

$$P_{\mathbf{a},\mathbf{t}} = \varprojlim P^N_{\mathbf{a},\mathbf{t}} , \tag{4.46}$$

which represents the functor $W_{\mathbf{a},\mathbf{t}}$ shown in Theorem 3.11 in [25].

Finally we are ready for one of our most important results. Due to Lemma 3.9 of [25], the following proposition turns the direct sum of the generalized eigenspaces of H into a representation of the weighted KLR algebra.

Proposition 4.15. There is an isomorphism Ξ between the completion $\widehat{R}_D$ and the endomorphism $\mathrm{End}(\bigoplus_{\mathbf{a},\mathbf{t}} P_{\mathbf{a},\mathbf{t}})$, i.e.,

$$\Xi : \widehat{R}_D \to \mathrm{End}\,(\bigoplus_{\mathbf{a},\mathbf{t}} P_{\mathbf{a},\mathbf{t}})\,. \tag{4.47}$$

Every diagram in $\widehat{R}_D$ with top loadings $i_{\mathbf{a},\mathbf{t}}$ and bottom $i_{\mathbf{a}',\mathbf{t}'}$ maps the elements in $P_{\mathbf{a}',\mathbf{t}'}$ to the elements in $P_{\mathbf{a},\mathbf{t}}$, and the elements outside of $P_{\mathbf{a}',\mathbf{t}'}$ to zero. Recall the actions of s_i, π_i, Ψ, Φ from Proposition 4.10. We can define the action of each diagram on $\bigoplus_{\mathbf{a},\mathbf{t}} P_{\mathbf{a},\mathbf{t}}$ as follows:

1. The idempotent $e(\mathbf{a},\mathbf{t})$ acts as the identity on $P_{\mathbf{a},\mathbf{t}}$ and zero elsewhere.

2. The dot on the strand corresponding to (a_i,t_i) of $e(\mathbf{a},\mathbf{t})$ acts by multiplying by (u_i-a_i).

3. If $t_i \neq t_{i+1}$, then $\Xi(\xi(\mathbf{a},\mathbf{t},s_i))$ acts by $e(\mathbf{a},\mathbf{t})s_i$.

4. If $t_i = t_{i+1}$, then

$$\Xi(\xi(\mathbf{a}, \mathbf{t}, s_i)) = \begin{cases} e(\mathbf{a}, \mathbf{t})\frac{1}{u_i - u_{i+1} - c_0 l}((u_i - u_{i+1})s_i - c_0\pi_i) & \text{for} \quad a_i + c_0 l \neq a_{i+1} \neq a_i \\ e(\mathbf{a}, \mathbf{t})((u_i - u_{i+1})s_i - c_0\pi_i) & \text{for} \quad a_i + c_0 l = a_{i+1} \neq a_i \\ e(\mathbf{a}, \mathbf{t})\frac{1}{u_i - u_{i+1} + c_0 l}(s_i - 1) & \text{for} \quad a_i + c_0 l \neq a_{i+1} = a_i \\ e(\mathbf{a}, \mathbf{t})(1 - s_i) & \text{for} \quad a_i + c_0 l = a_{i+1} = a_i \end{cases} . \tag{4.48}$$

5.

$$\Xi(\xi(\mathbf{a}, \mathbf{t}, \nu)) = \begin{cases} e(\mathbf{a}, \mathbf{t})\Psi & \text{for} \quad a_n = -p(t_n) \\ e(\mathbf{a}, \mathbf{t})\Psi\frac{1}{u_n + p(t_n)} & \text{for} \quad a_n \neq -p(t_n) \end{cases} . \tag{4.49}$$

6.

$$\Xi(\xi(\mathbf{a}, \mathbf{t}, \nu^{-1})) = e(\mathbf{a}, \mathbf{t})\Phi\,. \tag{4.50}$$

Example 4.16. Consider the case where $l = n = 2$. Then we have $\zeta = \exp\left(\frac{2\pi i}{2}\right) = -1$. Let us choose $k = \frac{1}{3}$, $h_0 = 1$, and $h_1 = -1$. By direct computation, we have

$$\begin{aligned} r_0 &= p(\zeta^0) \\ &= \zeta^0 h_0 + \zeta^0 h_0 + \zeta^0 h_1 + \zeta^{-1} h_1 \\ &= 2h_0 \\ &= 2 \end{aligned} \tag{4.51}$$

and

$$\begin{aligned} r_1 &= p(\zeta^{-1}) + 1 \\ &= \zeta^0 h_0 \zeta^0 + \zeta^0 h_0 \zeta^{-1} + \zeta^0 h_1 \zeta^0 + \zeta^{-1} h_1 \zeta^{-1} + 1 \\ &= 2h_1 + 1 \\ &= -1\,. \end{aligned} \tag{4.52}$$

Suppose that the graph is $D = \{[0], [\frac{1}{3}], [\frac{2}{3}]\}$. Since $\tilde{r_0} = 0 \in D$, there is an edge from $\tilde{r_0}$ to ∞, which means that there is a red strand in the diagram at $Y(\frac{p(\zeta^0)}{l}) = 1$. Via calculation we see that

$$\Sigma^{-1}(D) = \{(a, \zeta^m) \mid a + m = 2n \text{ or } a + m = \frac{2}{3} + 2n \text{ or } a + m = \frac{4}{3} + 2n \text{ for some integer } n\}\,. \quad (4.53)$$

We take $(\mathbf{a}, \mathbf{t}) = ((\frac{2}{3}, \frac{1}{3}), (1, -1))$ as an example and choose $\epsilon = \frac{1}{24} < \frac{|Y(\frac{1}{3} - \frac{2}{3})|}{2 \cdot 2} = \frac{1}{12}$. The corresponding loading is then

$$\begin{aligned} i_{\mathbf{a},\mathbf{t}} : \quad \frac{\frac{2}{3}}{2} + \epsilon = \frac{3}{8} &\mapsto \Sigma(\frac{2}{3}, 1) = [\frac{1}{3}]\,, \\ \frac{\frac{1}{3}}{2} + 2\epsilon = \frac{1}{4} &\mapsto \Sigma(\frac{1}{3}, -1) = [\frac{2}{3}]\,. \end{aligned} \quad (4.54)$$

We would like to know the actions of the diagrams $\xi(\mathbf{a}, \mathbf{t}, s_1)$, $\xi(\mathbf{a}, \mathbf{t}, \nu)$, and $\xi(\mathbf{a}, \mathbf{t}, \nu^{-1})$. First we need to compute the loadings of three new pairs of $\mathbf{a}, \mathbf{t}$ as follows:

$$\begin{aligned} i_{s_1(\mathbf{a},\mathbf{t})} : \quad \frac{\frac{1}{3}}{2} + \epsilon = \frac{5}{24} &\mapsto \Sigma(\frac{1}{3}, -1) = [\frac{2}{3}]\,, \\ \frac{\frac{2}{3}}{2} + 2\epsilon = \frac{5}{12} &\mapsto \Sigma(\frac{2}{3}, 1) = [\frac{1}{3}]\,. \end{aligned} \quad (4.55)$$

$$\begin{aligned} i_{\nu(\mathbf{a},\mathbf{t})} : \quad -\frac{\frac{2}{3}}{2} + \epsilon = -\frac{7}{24} &\mapsto \Sigma(-\frac{2}{3}, 1) = [\frac{2}{3}]\,, \\ \frac{\frac{2}{3}}{2} + 2\epsilon = \frac{5}{12} &\mapsto \Sigma(\frac{2}{3}, 1) = [\frac{1}{3}]\,. \end{aligned} \quad (4.56)$$

$$\begin{aligned} i_{\nu^{-1}(\mathbf{a},\mathbf{t})} : \quad \frac{\frac{1}{3}}{2} + \epsilon = \frac{5}{24} &\mapsto \Sigma(\frac{1}{3}, -1) = [\frac{2}{3}]\,, \\ \frac{\frac{5}{3}}{2} + 2\epsilon = \frac{11}{12} &\mapsto \Sigma(\frac{5}{3}, -1) = [\frac{1}{3}]\,. \end{aligned} \quad (4.57)$$

For the above computations in (4.55), (4.56), and (4.57), the corresponding diagrams are shown in Figure 5, 6, and 7.

Since the condition $a_1 + c_0 l \neq a_2 \neq a_1$ holds, by (4.48) in Proposition 4.15, $\Xi(\xi(\mathbf{a}, \mathbf{t}, s_1))$ shown in Figure 5 acts on $P_{\mathbf{a},\mathbf{t}}$ by permuting the first variable and the second variable. For the diagrams $\xi(\mathbf{a}, \mathbf{t}, \nu)$ shown in Figure 6, the condition in (4.49) of Proposition 4.15 is required on a_2 and t_2. We

Figure 5. Generating diagram $\xi(\mathbf{a}, \mathbf{t}, s_1)$

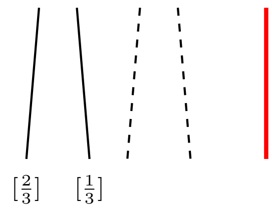


Figure 6. Generating diagram $\xi(\mathbf{a}, \mathbf{t}, \nu)$

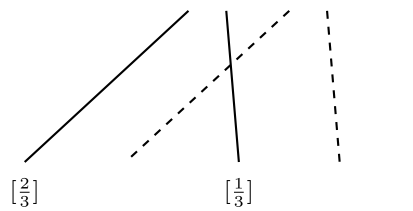
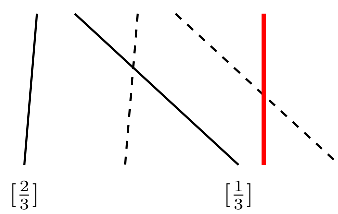


Figure 7. Generating diagram $\xi(\mathbf{a}, \mathbf{t}, \nu^{-1})$



can then compute

$$\begin{aligned} p(t_2) &= \sum_{s=0}^{1} \sum_{r=0}^{1} \zeta^{-rs} h_r t_2^s \\ &= h_0 t_2 - h_1 t_2 \\ &= 2 \\ &\neq -a_2 \,, \end{aligned} \tag{4.58}$$

which means that $\Xi(\xi(\mathbf{a}, \mathbf{t}, \nu))$ acts by $\Psi_{\frac{1}{u_2 + p(t_2)}}$. For the diagram $\xi(\mathbf{a}, \mathbf{t}, \nu^{-1})$ shown in Figure 7, we have $\Xi(\xi(\mathbf{a}, \mathbf{t}, \nu^{-1}))$ acts by Φ due to (4.50) of Proposition 4.15.

Furthermore, Proposition 4.15 leads us to the following theorem, from Theorem 3.13 in [25], which is about the relation between the category of the representations of the rational Cherendik algebra and the category of the representations of the weighted KLR algebra.

Theorem 4.17. Let $\widehat{R}_D$-mod$_{\text{fd}}$ be the category of modules M over the algebra $\widehat{R}_D$ such that $e(\mathbf{a},\mathbf{t})M$ is finite dimensional for all $(\mathbf{a},\mathbf{t})$. The functor W define by

$$\begin{aligned} \mathsf{W} : H\text{-mod}_D &\to \widehat{R}_D\text{-mod}_{\text{fd}} \\ M &\mapsto \bigoplus_{\mathbf{a},\mathbf{t}} W_{\mathbf{a},\mathbf{t}}(M) \end{aligned} \tag{4.59}$$

is an equivalence.

4.3 Weighted KLR Algebra and Category $\mathcal{O}$

Let the graph $D_0 \subset \mathbb{C}/\mathbb{Z}$ be the set

$$\begin{aligned} D_0 : &= \{\tilde{r}_i + mc_0 \mid m \in [-n,n] \text{ and } i = 0,\ldots,l-1\} \\ &= \{-h_i + \frac{i}{l} + mc_0 \mid m \in [-n,n] \text{ and } i = 0,\ldots,l-1\} \end{aligned} \tag{4.60}$$

for the rest of the paper. The preimage $\Sigma^{-1}(D_0)$ of D_0 contains a subset of eigenvalues of u_i and ζ_i for all $i = 1,\ldots,n$ in all Verma modules. The chosen eigenvalues are closely connected to the image under the KZ functor.

Let us recall Category $\mathcal{O}$ from Definition 1.11. A vector $y \in \mathfrak{h}$ acts nilpotently on objects in Category $\mathcal{O}$. Now we denote Category $\mathcal{O}$ by $\mathcal{O}^+$, and define a dual category $\mathcal{O}^-$ such that modules in $\mathcal{O}^-$ are finitely generated and acted by x_i nilpotently. Before we head to our final result, we want to rule out all the 'bad' diagrams in the weighted KLR algebra.

Definition 4.18. Recall that loadings are maps $\mathbb{R} \to D \cup 0$ from Definition 4.2. A loading is said to be positive unsteady if there exist a real number

$$\delta \geq \max_{i=1,..,l} \left\{ Y\left(\frac{p(\zeta^i)}{l}\right)\right\} \tag{4.61}$$

such that a non-empty set of points in the loading have a value $x \in \mathbb{R}$ greater than $\delta + |Y(k)|$ and all the other sets have $x \leq \delta$. Similarly, a loading is said to be negative unsteady if there exist a real number

$$\delta \leq \min_{i=1,\ldots,l} \left\{ Y\left(\frac{p(\zeta^i)}{l}\right)\right\} \tag{4.62}$$

such that a non-empty set of points in the loading have $x < \delta - |Y(k)|$ and all others have $x \geq \delta$. The quotient of R_D by the ideals generated by positive (negative) unsteady loadings are called positive (negative) steady quotient, which is denoted by $R_D(+)$ $(R_D(-))$.

The $\pm$ sign in the above definition gives us a hint of how we can expect the modules of the corresponding algebra are connected.

Due to Theorem 3.12 from [25], we have the following theorem.

Theorem 4.19. The functor W induces an equivalence $\mathcal{O}^{\pm} \cong R_{D_0}(\pm)$-mod.

From Theorem 4.17 and 4.19, we know the connection between the modules in Category $\mathcal{O}$ and the modules of the diagrammatic algebra. We would like to use it to build an isomorphism between the KZ functor and the generalized weight space functor. Suppose that we fix a lift $\eta : D_0 \to \mathbb{C}$ such that $\sum(\eta(d), 1) = d$. We choose a big enough N such that

$$N \gg \max_{\substack{i\in[1,l] \\ d,d'\in D_0}} \left\{|Y(p(\zeta^i))|, |Y(k)|, |Y(\eta(d))|, |Y(\eta(d) - \eta(d'))|\right\} . \tag{4.63}$$

For every $\mathbf{d} = (d_1, \dots, d_n) \in D_0^n$, we define

$$\mathbf{a_d}^{\pm} = ((\eta(d_1) \pm lN), \dots, (\eta(d_n) \pm nlN)) \, , \tag{4.64}$$

$$\mathbf{1} = (1, 1, \dots, 1) \, . \tag{4.65}$$

Then we have the following theorem according to Theorem 3.15 in [25].

Theorem 4.20. The KZ functor on $\mathcal{O}^{\pm}$ is isomorphic to $\bigoplus_{\mathbf{d}\in D_0^n} W_{\mathbf{a_d}^{\pm},\mathbf{1}}$.

Remark. In the proof of Theorem 4.20, Webster used that the cyclotomic Hecke algebra $H_{q,\mathbf{Q}}$ is a quotient of the WF type Hecke algebra $\mathfrak{H}_n^{\mathbf{s}}$, and the category $H_{q,\mathbf{Q}}$-mod is a quotient category of $\mathfrak{H}_n^{\mathbf{s}}$-mod. The equivalence between Category $\mathcal{O}$ and the category $\mathfrak{H}_n^{\mathbf{s}}$-mod intertwines the KZ functor and the quotient functor :$H_{q,\mathbf{Q}}$-mod $\to \mathfrak{H}_n^{\mathbf{s}}$-mod sending $M \mapsto e_{D_{s,n}} M$, where $e_{D_{s,n}}$ is an idempotent. However, Theorem 4.20 doesn't inform us why the direct sum of generalized weight spaces forms a representation of the cyclotomic Hecke algebra remains a mystery and requires us further investigation.

In the next two sections, we will follow a similar path to Webster's and construct the images under the KZ functor as representations of the cyclotomic Hecke algebra. It is proven that the WF type

Hecke algebra is isomorphic to the weighted KLR algebra [26]. We will show that the cyclotomic KLR algebra is a quotient of the weighted KLR algebra and the cyclotomic Hecke algebra is isomorphic to the cyclotomic KLR algebra. As a result, the category $H_{q,\mathbf{Q}}$-mod is a quotient category $\widehat{R}_D$-mod$_{\mathrm{fd}}$ defined in Theorem 4.17. Thus a direct sum of weighted spaces forms a representation of the cyclotomic Hecke algebra under the actions of the weighted KLR algebra, which we will conclude in Theorem 6.10.

5 KLR Algebra and its Isomorphism to Cyclotomic Hecke Algebra

Now we have an idea of how the images of KZ functor look like from Section 4.3. Yet, we still do not know how the image under this direct sum of weight spaces becomes a module of the cyclotomic Hecke algebra. In order to connect the dots, we note the following two facts:

1. By both Brudan and Kleshchev [4], and Webster [24] there is an isomorphism between the (completion of) cyclotomic KLR algebra and the (completion of) cyclotomic Hecke algebra;

2. The weighted KLR algebra is an extension of the KLR algebra. So it is expected to reduce to the original KLR algebra when we impose extra constraints.

Combining the knowledge already known, our goal in Section 5.1 is to find a subalgebra in the weighted KLR algebra that is isomorphic to a KLR algebra, whose representation is $\bigoplus_{\mathbf{d}\in D_0^n} W_{\mathbf{a_d}^+,\mathbf{1}}(M)$, for any module M in Category $\mathcal{O}$. This subsection is my original work. Then we will prove its cyclotomic quotient is isomorphic to the cyclotomic Hecke algebra induced from the KZ functor in Section 5.2. This subsection is partly a review of [24] and [4], and partly my original work.

5.1 KLR Algebra as a Subalgebra in Weighted KLR Algebra

Let us define KLR algebra first as the following.

Definition 5.1. Let D be a quiver. Fix a positive integers n. A KLR algebra is an algebra $R_{D,n}$ generated by the set of elements $\{e(\mathbf{d}) \mid \mathbf{d} = (d_1, \dots, d_n) \in D^n\} \cup \{\mathrm{y}_1, \dots, \mathrm{y}_n\} \cup \{\psi_1, \dots, \psi_{n-1}\}$ over a field K under the relations

$$e(\mathbf{d})e(\mathbf{d}') = \delta_{\mathbf{d},\mathbf{d}'}e(\mathbf{d}), \tag{5.1}$$

$$\sum_{\mathbf{d}} e(\mathbf{d}) = 1, \tag{5.2}$$

$$\mathrm{y}_i e(\mathbf{d}) = e(\mathbf{d})\mathrm{y}_i, \tag{5.3}$$

$$\mathrm{y}_i \mathrm{y}_j = \mathrm{y}_j \mathrm{y}_i, \tag{5.4}$$

$$\psi_i e(\mathbf{d}) = e(s_i \cdot \mathbf{d})\psi_i, \tag{5.5}$$

$$\psi_i \psi_j = \psi_j \psi_i \quad \text{if } |i-j| > 1, \tag{5.6}$$

$$\psi_i \mathrm{y}_j = \mathrm{y}_j \psi_i \quad \text{if } i \neq j, j+1, \tag{5.7}$$

$$\mathrm{y}_i \psi_i e(\mathbf{d}) = \psi_i \mathrm{y}_{i+1} e(\mathbf{d}) \quad \text{unless } d_i = d_{i+1}, \tag{5.8}$$

$$\mathrm{y}_{i+1} \psi_i e(\mathbf{d}) = \psi_i \mathrm{y}_i e(\mathbf{d}) \quad \text{unless } d_i = d_{i+1}, \tag{5.9}$$

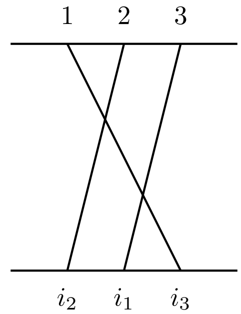
$$\psi_i \mathrm{y}_{i+1} e(\mathbf{d}) - \mathrm{y}_i \psi_i e(\mathbf{d}) = \mathrm{y}_{i+1} \psi_i e(\mathbf{d}) - \psi_i \mathrm{y}_i e(\mathbf{d}) = e(\mathbf{d}) \quad \text{if } d_i = d_{i+1}, \tag{5.10}$$

$$\psi_i^2 e(\mathbf{d}) = \begin{cases} 0 & \text{if } \ d_i = d_{i+1} \\ e(\mathbf{d}) & \text{if } \ d_i \neq d_{i+1},\ d_i \nrightarrow d_{i+1} \text{ and } d_i \nleftarrow d_{i+1} \\ (\mathrm{y}_{i+1} - \mathrm{y}_i)e(\mathbf{d}) & \text{if } \ d_i \to d_{i+1} \text{ and } d_i \nleftarrow d_{i+1} \\ (\mathrm{y}_i - \mathrm{y}_{i+1})e(\mathbf{d}) & \text{if } \ d_i \leftarrow d_{i+1} \text{ and } d_i \nrightarrow d_{i+1} \\ (-\mathrm{y}_i^2 + 2\mathrm{y}_i\mathrm{y}_{i+1} - \mathrm{y}_{i+1}^2)e(\mathbf{d}) & \text{if } \ d_i \rightleftarrows d_{i+1} \end{cases}, \tag{5.11}$$

$$(\psi_i\psi_{i+1}\psi_i - \psi_{i+1}\psi_i\psi_{i+1})e(\mathbf{d}) = \begin{cases} e(\mathbf{d}) & \text{if } \ d_i = d_{i+2} \leftarrow d_{i+1} \text{ and } d_i \nrightarrow d_{i+1} \\ -e(\mathbf{d}) & \text{if } \ d_i = d_{i+2} \to d_{i+1} \text{ and } d_i \nleftarrow d_{i+1} \\ -(\mathrm{y}_i + \mathrm{y}_{i+2})e(\mathbf{d}) & \text{if } \ d_i = d_{i+2} \rightleftarrows d_{i+1} \\ 0 & \text{otherwise} \end{cases}. \tag{5.12}$$

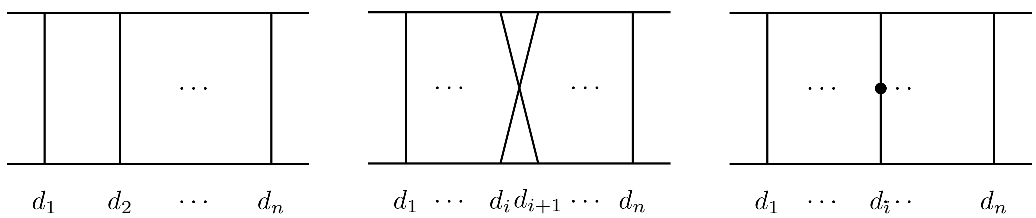
Remark. Similar as the weighted KLR algebra, the KLR algebra can be presented as a diagram on $\mathbb{R} \times [0,1]$, with fixed n points at $x = 1, \ldots, n$ on both the top $y = 1$ and the bottom $y = 0$, connected by n strands. Each strand is labelled by a vertex in the quiver D. For example, if $n = 3$, an element in the KLR algebra would be like Figure 8. Then the generating set of $R_{D,n}$ can be drawn as Figures 9a

Figure 8. An example of a KLR algebra diagram



to 9c. The differences between the weighted KLR algebra and the KLR algebra are

(a) Diagram of $e(\mathbf{d})$ (b) Diagram of ψ_i (c) Diagram of y_i



1. The intersections between a strand and both the top and the bottom of the diagram are fixed in the KLR algebra, but determined by the loadings and can vary vastly in the weighted KLR algebra.

2. There is no ghost in the KLR algebra.

Hence the clever trick that Webster applied in the construction of the weighted space functor $\bigoplus_{\mathbf{d}\in D_0^n} W_{\mathbf{a_d}^+,\mathbf{1}}$ is making ilN much bigger than the lifting $\eta(d_i)$ in the eigenvalues $\eta(d_i)+ilN$, so that we can consider the positions of the strands at the top and the bottom being approximately fixed at $(lN,\ldots,nlN)$. Furthermore, because the distances between strands are huge, comparatively the ghosting shift k is insignificant, meaning it is possible to ignore the ghost in this case. This result is proven in Theorem 4.6 in [24]. Even though it is a theorem about the type W Hecke algebra, it is also a valid statement for the KLR algebra because the completion of the two are isomorphic.

Recall the definition of D_0 in (4.60). According to our goal set at the beginning of the section, we expect to find a subalgebra $R_{D_0,n}$ of weighted KLR algebra associated with quiver D_0, such that the image of a module from Category $\mathcal{O}$ under the weight space functor $\bigoplus_{\mathbf{d}\in D_0^n} W_{\mathbf{a}_d^+,\mathbf{1}}$ is a representation of $R_{D_0,n}$. Before constructing the subalgebra, we note that for any Verma module $M_c(\lambda)$, the direct sum of weight spaces $\bigoplus_{\mathbf{d}\in D_0^n} W_{\mathbf{a_d}^+,\mathbf{1}}(M_c(\lambda))$ is not a representation of the weighted KLR algebra R_{D_0}. For instance, one of the generating elements $\xi(\mathbf{a},\mathbf{t},s_i)$, for some pair $(\mathbf{a},\mathbf{t})$, permutes the i-th and $(i+1)$-th eigenvalues in the eigenspaces. If the sourcing eigenspace has the i-th and $(i+1)$-th eigenvalues to be $\eta(d_i)+ilN$ and $\eta(d_{i+1})+(i+1)lN$ respectively, then the i-th and $(i+1)$-th eigenvalues in the target eigenspace become $\eta(d_{i+1})+(i+1)lN$ and $\eta(d_i)+ilN$ respectively. By comparing the coefficient in front of lN, it is easy to see that the target eigenspace is not a direct summand in $\bigoplus_{\mathbf{d}\in D_0^n} W_{\mathbf{a_d}^+,\mathbf{1}}$.

Despite this inconvenience, the idea of constructing the generating element of the KLR algebra is straightforward. The only choices of idempotents $e(\mathbf{d})$ are

$$e(\eta(d_1)+lN,\ldots,\eta(d_n)+nlN) \text{ for } (d_1,\ldots,d_n)\in D_0^n.$$

Hence the identity here is

$$\sum_{\mathbf{d}\in D_0^n} e(\eta(d_1)+lN,\ldots,\eta(d_n)+nlN)\,.$$

Let $\mathrm{y}_i e(\mathbf{d})$ be the diagram with a dot on the i-th strand of the diagram $e(\eta(d_1)+lN,\ldots,\eta(d_n)+nlN)$. The last and the most difficult part is to build the elements ψ_i. In a representation of KLR algebra, the elements ψ_i usually send from an eigenspace of y_i to another, by permuting the corresponding the i-th and the $(i+1)$-th eigenvalues. We would want the candidates of generators ψ_i of the KLR algebra to mimic the action of ψ_i, but still to have $\bigoplus_{\mathbf{d}\in D_0^n} W_{\mathbf{a_d}^+,\mathbf{1}}(M)$ for any M in Category $\mathcal{O}$ as their representation, that is to permute $\eta(d_i)$ and $\eta(d_{i+1})$, but leave ilN and $(i+1)lN$ unchanged. In order to achieve this, we

1. interchange the i-th and $(i+1)$-th eigenvalue,
2. put the current i-th eigenvalue to the last coordinate,
3. apply the diagram $\xi(\mathbf{a},\mathbf{t},\nu)$,
4. rotate everything back to its original position,
5. repeat Step 2-4 lN times

Step 2-4 only reduce 1 from the i-th eigenvalue and in order to make the current i-th eigenvalue become of the form $\eta(d_{i+1}) + ilN$ they are required to repeat lN times. Similarly for the current $(i+1)$-th eigenvalue, we rotate necessary eigenvalues, put the current $(i+1)$-th eigenvalue to the first coordinate, apply the diagram $\xi(\mathbf{a}, \mathbf{t}, \nu^{-1})$ and rotate every moved coordinate back to the former position. This procedure is repeated lN times for the same reason as above. Summarizing the construction we have the following definition,

Definition 5.2. We define the potential candidates that generate a KLR algebra inside a weighted KLR algebra by

$$
\begin{aligned}
e(\mathbf{d}) :=& e(\eta(d_1) + lN, \dots, \eta(d_n) + nlN), \quad \text{for any } \mathbf{d} = (d_1, \dots, d_n) \in D_0^n && (5.13)\\
\mathrm{y}_i e(\mathbf{d}) :=& \text{the diagram } e(\eta(d_1) + lN, \dots, \eta(d_n) + nlN) \text{ with a dot on the } i\text{-th strand,}\\
& \text{for any } \mathbf{d} = (d_1, \dots, d_n) \in D_0^n && (5.14)\\
\psi_i e(\mathbf{d}) :=& [\xi(\mathbf{a}, \mathbf{t}, s_{i+1})\xi(\mathbf{a}, \mathbf{t}, s_{i+2}) \dots \xi(\mathbf{a}, \mathbf{t}, s_{n-1})\xi(a, t, s_1)\xi(a, t, s_2) \dots \xi(a, t, s_i)\\
& \xi(\mathbf{a}, \mathbf{t}, \nu^{-1})\xi(\mathbf{a}, \mathbf{t}, s_1)\xi(\mathbf{a}, \mathbf{t}, s_2) \dots \xi\mathbf{a}, \mathbf{t}, s_i)\xi(\mathbf{a}, \mathbf{t}, s_{i-1}) \dots \xi(\mathbf{a}, \mathbf{t}, s_1)]^{lN}\\
& [\xi(\mathbf{a}, \mathbf{t}, s_{n-1})\xi(\mathbf{a}, \mathbf{t}, s_{n-2}) \dots \xi(\mathbf{a}, \mathbf{t}, s_{i+1})\xi(\mathbf{a}, \mathbf{t}, s_{i-1})\xi(\mathbf{a}, \mathbf{t}, s_{i-2}) \dots \xi(\mathbf{a}, \mathbf{t}, s_1)\\
& \xi(\mathbf{a}, \mathbf{t}, \nu)\xi(\mathbf{a}, \mathbf{t}, s_{n-1})\xi(\mathbf{a}, \mathbf{t}, s_{n-2}) \dots \xi(\mathbf{a}, \mathbf{t}, s_i)\xi(\mathbf{a}, \mathbf{t}, s_{i+1}) \dots \xi(\mathbf{a}, \mathbf{t}, s_{n-1})]^{lN}\\
& \xi(\mathbf{a}, \mathbf{t}, s_i), && (5.15)
\end{aligned}
$$

where $(\mathbf{a}, \mathbf{t})$ are the (generalized) eigenvalues of u_i and ζ_i respectively, corresponding to $w \cdot (\mathbf{a}_{\mathbf{d}}, \mathbf{1})$ for some combination of permutations in S_n and $\nu^{\pm 1}$.

With the motivation from above, we are left to show that the proposed candidates of the generating elements indeed form a KLR algebra. The rest of the section is to prove the following theorem.

Theorem 5.3. The candidates $\{e(\mathbf{d}) \mid \mathbf{d} = (d_1, \dots, d_n) \in D^n\} \cup \{\mathrm{y}_1, \dots, \mathrm{y}_n\} \cup \{\psi_1, \dots, \psi_{n-1}\}$ generate a KLR algebra over the complex numbers.

The proof Theorem 5.1 is not straightforward. The idea is to construct an isomorphism between the polynomial representation of the weighted KLR algebra and the polynomial representation of the KLR algebra, and shows that actions of our candidates coincide with the actions of the KLR algebra generators on the polynomial representations of the KLR algebra. By the faithfulness of the representation, our candidates indeed generate a KLR algebra.

Before doing the full proof, we gather necessary ingredients first. Let us recall from Proposition 4.4 that there is a faithful polynomial representation

$$P = \bigoplus_{\mathbf{j}} \mathbb{C}[x_{1,\mathbf{j}}, \dots, x_{|\mathbf{j}|,\mathbf{j}}]$$

of the weighted KLR algebra T_D. Each summand $\mathbb{C}[x_{1,\mathbf{j}}, \dots, x_{|\mathbf{j}|,\mathbf{j}}]$ in P has the same number of variables as the number of labels in $\mathbf{j}$. If the diagram in the weighted KLR algebra does not have the same number of strands as the number of variables in the summand $\mathbb{C}[x_{1,\mathbf{j}}, \dots, x_{|\mathbf{j}|,\mathbf{j}}]$, then it acts on this summand by zero. The diagrams in the weighted KLR algebra R_D associated with the rational Cherendik algebra of $G(l,1,n)$ defined in Definition 4.14 always have n strands. The position and label of the strand uniquely correspond to a pair of eigenvalues of u_i and ζ_i for some $i = 1, \dots, n$. Hence, if we specify the quiver to be D_0, then the polynomial representation of the weighted KLR algebra associated with the rational Cherednik algebra is

$$P = \bigoplus_{\mathbf{d}=(d_1,\dots,d_n)\in D_0^n} \mathbb{C}[x_{1,\mathbf{d}}, \dots, x_{n,\mathbf{d}}]\,.$$

Moreover, by the choice of the lifting η from the vertices in D_0 to $\mathbb{C}$, we can assign each n-tuple $\mathbf{d}$ in D_0^n with eigenvalues of u_i, which is $\mathbf{a_d}$ in (4.64).

In order to determine whether our candidates satisfy the generating relations of the KLR algebra, we need to know their actions on the polynomial representation and compare them with the actions of the generators of the KLR algebra. Recall from Section 4.2 that the label of i-th strand is defined by

$$\Sigma(a_i, t_i) = \Sigma(a_i, \zeta^{m_i}) = \frac{a_i + m_i}{l}. \tag{5.16}$$

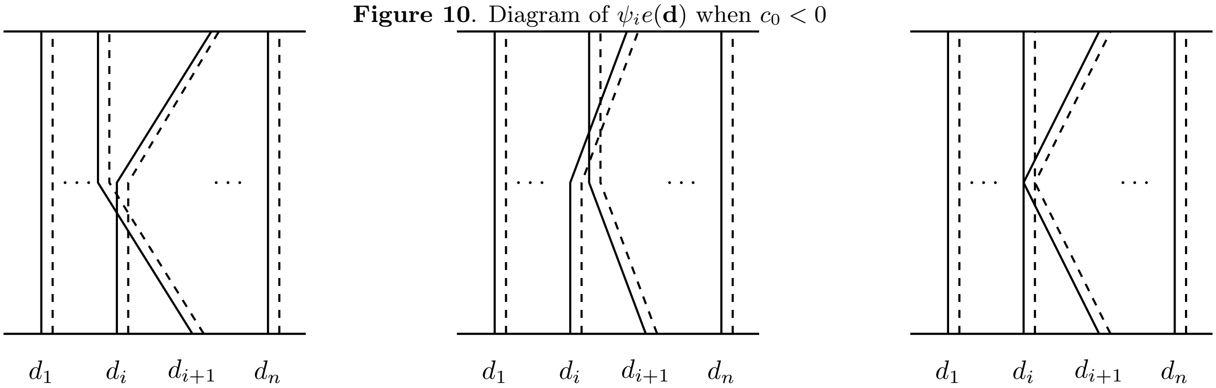
Considering the eigenvalues in the direct summands of the weight space functor, we have

$$\Sigma(\eta(d_i) + ilN, 1) = \Sigma(\eta(d_i) + ilN, \zeta^0) = d_i. \tag{5.17}$$

Because the choice of $\eta(d)$ is fixed for $d \in D_0$, there is a one to one correspondence between the n-tuple $\mathbf{d} = (d_1, \dots, d_n)$ of vertices in D_0 and the n-tuple of eigenvalues $(\eta(d_1) + lN, \dots, \eta(d_n) + nlN)$ via the lifting η and the function Σ. Thus we could use these two notations interchangeably.

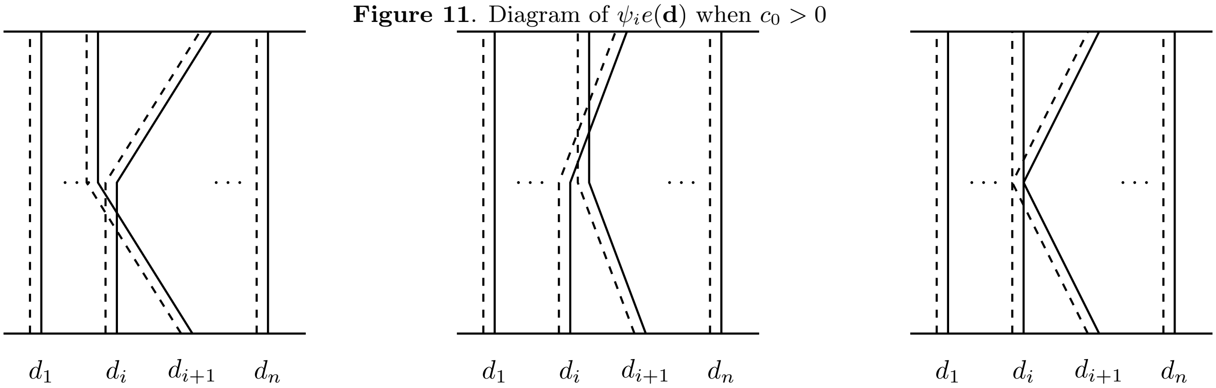
The idempotents $e(\mathbf{d})$ project onto the polynomial ring $\mathbb{C}[x_{1,\mathbf{d}}, \ldots, x_{n,\mathbf{d}}]$ and $\mathrm{y}_i e(\mathbf{d})$ act by multiplying by $x_{i,\mathbf{d}}$ for all i. The tricky part is to compute the action of ψ_i. Firstly, we recall from Section 4.2 the loading of the generating diagrams $\xi(\mathbf{a}, \mathbf{t}, w)$ sends $\frac{a_i}{l} + i\epsilon$ to $\Sigma(a_i, t_i)$, for $w = s_i, \nu^{\pm}$. It is obvious that initially the loadings in the diagrams are far away from each other, hence there is no potential crossing between either two strands or a strand and a ghost for most $\xi(\mathbf{a}, \mathbf{t}, s_i)$ in the product of ψ_i. The action of $\xi(\mathbf{a}, \mathbf{t}, s_i)$ on f would simply be f^{s_i}. However, when we apply $\xi(\mathbf{a}, \mathbf{t}, \nu^{\pm})$, the distance between the i-th and the $(i+1)$-th strands are getting closer by $\frac{1}{l}$. Since the procedure is applied $2lN$ times, and $N > |\eta(d_i) - \eta(d_{i+1})|$, the i-th, the $(i+1)$-th strands and their ghost will eventually cross. In the diagrammatic form, the whole element ψ_i can be presented as three crossings, one between the i-th and $(i+1)$-th strands and the other two between one of the strands and the ghost of the other strand. They can be drawn as in Figure 10 and Figure 11.

Figure 10. Diagram of $\psi_i e(\mathbf{d})$ when $c_0 < 0$



when $\eta(d_i) > \eta(d_{i+1})$ | when $\eta(d_i) < \eta(d_{i+1})$ | when $\eta(d_i) = \eta(d_{i+1})$

Figure 11. Diagram of $\psi_i e(\mathbf{d})$ when $c_0 > 0$



when $\eta(d_i) > \eta(d_{i+1})$ | when $\eta(d_i) < \eta(d_{i+1})$ | when $\eta(d_i) = \eta(d_{i+1})$

Thus, the action of ψ_i is solely determined by these three crossings and the labels of theses two strands. We recall from (4.15) the action of transpositions on the polynomial in the polynomial representation. In the quiver D_0, there is an edge $d \to d'$ from d to d', if $d' = d - c_0$. According to Proposition 4.4, action of ψ_i can be categorized into three cases:

1. If $d_i \nleftarrow d_{i+1}$ and $d_i \neq d_{i+1}$, then the two crossings between a strand and a ghost act by the identity and the crossing between two strands acts by s_i.

2. If $d_i \leftarrow d_{i+1}$, then the one crossing between a strand and a ghost that is below the crossing between two strands, acts by $x_{i,\mathbf{d}} - x_{i+1,\mathbf{d}}$, and the other strand acts by identity. The crossing between strands acts by s_i.

3. If $d_i = d_{i+1}$, then the two crossings between a strand and a ghost act by the identity and the one crossing between two strands acts by $\frac{s_i - 1}{x_{i+1,\mathbf{d}} - x_{i,\mathbf{d}}}$.

Combining both actions, the diagram $\psi_i e(\mathbf{d})$ acts explicitly by the following rule.

$$\psi_i \cdot f = \begin{cases} f^{s_i} & \text{if } d_i \nleftarrow d_{i+1} \text{ and } d_i \neq d_{i+1} \\ (x_{i+1,s_i\cdot\mathbf{d}} - x_{i,s_i\cdot\mathbf{d}})f^{s_i} & \text{if } d_i \leftarrow d_{i+1} \text{ and } d_i \neq d_{i+1} \\ \frac{f^{s_i} - f}{x_{i+1,\mathbf{d}} - x_{i,\mathbf{d}}} & \text{if } d_i \nleftarrow d_{i+1} \text{ and } d_i = d_{i+1} \end{cases}, \tag{5.18}$$

for any $f \in \mathbb{C}[x_{1,\mathbf{d}}, \ldots, x_{n,\mathbf{d}}]$.

Next we construct the polynomial representation of the KLR algebra given by Khovanov and Lauda in Proposition 2.3 and Corollary 2.6 of [16], and also in Equation 3.7 in [24].

Proposition 5.4. Let D be a quiver and $R_{D,n}$ the KLR algebra associated with D. Suppose $\mathcal{P}$ is the free abelian group defined to be direct sum of the polynomial rings

$$\mathcal{P} := \bigoplus_{\mathbf{d}=(d_1,\ldots,d_n)\in D^n} \mathcal{P}_{\mathbf{d}}, \tag{5.19}$$

where

$$\mathcal{P}_{\mathbf{d}} = \mathbb{C}[y_1(\mathbf{d}), \ldots, y_n(\mathbf{d})]. \tag{5.20}$$

The transposition s_i acts on $f \in \mathcal{P}_{\mathbf{d}}$ by

$$f(y_1(\mathbf{d}), \ldots, y_n(\mathbf{d}))^{s_i} = f(y_1(s_i \cdot \mathbf{d}), \ldots, y_{i+1}(s_i \cdot \mathbf{d}), y_i(s_i \cdot \mathbf{d}), \ldots, y_n(s_i \cdot \mathbf{d})). \tag{5.21}$$

Then the abelian group $\mathcal{P}$ is a faithful polynomial representation of the KLR algebra $R_{D,n}$ under the action

$$e(\mathbf{d}') \cdot f = \delta_{\mathbf{d},\mathbf{d}'} f \tag{5.22}$$

$$\mathrm{y}_i \cdot f = y_i(\mathbf{d}) f, \tag{5.23}$$

$$\psi_i \cdot f = \begin{cases} f^{s_i} & \text{if } d_i \not\leftarrow d_{i+1} \text{ and } d_i \neq d_{i+1} \\ (y_{i+1}(s_i \cdot \mathbf{d}) - y_i(s_i \cdot \mathbf{d})) f^{s_i} & \text{if } d_i \leftarrow d_{i+1} \\ \frac{f - f^{s_i}}{y_i(\mathbf{d}) - y_{i+1}(\mathbf{d})} & \text{if } d_i = d_{i+1} \end{cases}, \tag{5.24}$$

where $f \in \mathcal{P}_{\mathbf{d}}$.

There is an isomorphism from the polynomial representation of the weighted KLR algebra and the polynomial representation of the KLR algebra by the following lemma

Lemma 5.5. Let D be a quiver. Let $\mathbf{d} = (d_1, \ldots, d_n)$ be an n-tuple in D. Suppose R_D is the weighted KLR algebra associated with the rational Cherednik algebra of $G(l, 1, n)$, and $R_{D,n}$ is the KLR algebra in Definition 5.1. The weighted KLR algebra R_D has a polynomial representation $P = \bigoplus_{\mathbf{d}=(d_1,\ldots,d_n)\in D^n} \mathbb{C}[x_{1,\mathbf{d}}, \ldots, x_{n,\mathbf{d}}]$ shown in Proposition 4.4. Then there is an isomorphism $J : P \to \mathcal{P}$ that maps $x_{i,\mathbf{d}}$ to $y_i(\mathbf{d})$ and extend it by $J(g_1 g_2) = J(g_1)J(g_2)$ for any $g_1, g_2 \in P$. Via this isomorphism, we can consider the polynomial representation $\mathcal{P}$ as a representation of the weighted KLR algebra by the action

$$r \cdot J(f) := J(r \cdot f), \tag{5.25}$$

for any $r \in R_D, f \in P$.

Proof. We only need to show that the action on $\mathcal{P}$ by the weighted KLR algebra is well-defined. It is obvious that J is injective on $\bigoplus_{\mathbf{d}\in D^n} \mathbb{C}[x_{1,\mathbf{d}}, \ldots, x_{n,\mathbf{d}}]$. So for any non zero $f \in \mathcal{P}$ there is a unique $g \in P$ such that $f = J(g)$. The action of the weighted KLR algebra is compatible with the isomorphism J. For any $r, s \in R_D,\ f \in P$

$$\begin{aligned} r\cdot(s\cdot f) &= r\cdot(s\cdot J(g)) \\ &= r\cdot J(s\cdot g) \\ &= J(rs\cdot g) \\ &= rs\cdot J(g) \\ &= rs\cdot f. \end{aligned} \tag{5.26}$$

□

Now we are fully prepared to prove Theorem 5.3.

proof of Theorem 5.3. Let R be the algebra generated by $\{e(\mathbf{d}) \mid \mathbf{d} = (d_1, \dots, d_n) \in D^n\} \cup \{\mathrm{y}_1, \dots, \mathrm{y}_n\} \cup \{\psi_1, \dots, \psi_{n-1}\}$. Since it is a subalgebra of the weighted KLR algebra, we can view $\mathcal{P}$ as an R-module. For any non zero $f \in \mathcal{P}_{\mathbf{d}}$ with $f = J(g)$ for some $g \in \mathbb{C}[x_{1,\mathbf{d}}, \dots, x_{n,\mathbf{d}}]$, the action of the generators are given by (5.25) and Proposition 4.4 as follows.

$$e(\mathbf{d}')\cdot f = J(e(\mathbf{d}')g) = J(\delta_{\mathbf{d},\mathbf{d}'}g) = \delta_{\mathbf{d},\mathbf{d}'}f, \tag{5.27}$$

$$\mathrm{y}_i\cdot f = J(\mathrm{y}_i\cdot g) = J(x_{i,\mathbf{d}}g) = J(x_{i,\mathbf{d}})J(g) = \mathrm{y}_i(\mathbf{d})f, \tag{5.28}$$

$$\psi_i\cdot f = J(\psi_i\cdot g) \tag{5.29}$$

$$= \begin{cases} J(g^{s_i}) = f^{s_i} & \text{if } d_i \not\leftarrow d_{i+1} \text{ and } d_i \neq d_{i+1} \\ J(x_{i+1,s_i\cdot\mathbf{d}} - x_{i,s_i\cdot\mathbf{d}}g^{s_i}) = (y_{i+1}(s_i\cdot\mathbf{d}) - y_i(s_i\cdot\mathbf{d}))f^{s_i} & \text{if } d_i \leftarrow d_{i+1} \text{ and } d_i \neq d_{i+1}, \\ J(\frac{g-g^{s_i}}{x_{i,\mathbf{d}}-x_{i+1,\mathbf{d}}}) = \frac{f-f^{s_i}}{y_i(\mathbf{d})-y_{i+1}(\mathbf{d})} & \text{if } d_i \not\leftarrow d_{i+1} \text{ and } d_i = d_{i+1}. \end{cases} \tag{5.30}$$

The action is the same as the ones of the generators of the KLR algebra in Proposition 5.4. By the faithfulness of the polynomial representation, the candidates are the generators of the KLR algebra. □

5.2 Isomorphism Between Cyclotomic KLR Algebras and Cyclotomic Hecke Algebras

The next goal is to prove that a cyclotomic quotient of the KLR algebra is isomorphic to a cyclotomic Hecke algebra. In the paper [24], Webster constructed an isomorphism between the completion of the KLR algebra and the completion of the affine Hecke algebra. Based on this theorem, we only need to show that cyclotomic condition in the cyclotomic Hecke algebra is equivalent to the one in the cyclotomic KLR algebra. To get prepared with the theorem, we define the affine Hecke algebra first.

Definition 5.6. Let q be a complex number on the unit circle. Let us assume $q \neq 1$. The affine Hecke algebra $H_q(n)$ is the algebra generated by $\{X_1, \ldots, X_n\} \cup \{T_1, \ldots, T_{n-1}\}$ and their inverses under the relations

$$X_i^{\pm} X_j^{\pm} = X_j^{\pm} X_i^{\pm} \quad \text{for any } i, j = 1, \ldots, n, \tag{5.31}$$

$$(T_i - q)(T_i + 1) = 0 \quad \text{for } i = 1, \ldots, n-1, \tag{5.32}$$

$$T_i X_i T_i = q X_{i+1} \quad \text{for } i = 1, \ldots, n-1, \tag{5.33}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad \text{for } i = 1, \ldots, n-1, \tag{5.34}$$

$$T_i X_j = X_j T_i \quad \text{if } j \neq i, i+1, \tag{5.35}$$

$$T_i T_j = T_j T_i \quad \text{if } i \neq j \pm 1. \tag{5.36}$$

Remark. Equivalently, the affine Hecke algebra can be defined by the algebra generated by T_0, T_1, $\ldots$, T_{n-1} under the relation

$$(T_i - q)(T_i + 1) = 0 \quad \text{for any } i = 1, \ldots, n-1, \tag{5.37}$$

$$T_i T_j = T_j T_i \quad \text{if } i, j = 1, \ldots, n,\ i \neq j \pm 1, \tag{5.38}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad \text{for } i = 1, \ldots, n-1, \tag{5.39}$$

$$T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0. \tag{5.40}$$

Those two definition are equivalent by identifying X_1 with T_0, and $T_i X_i T_i$ with $q X_{i+1}$ inductively.

Definition 5.7. Suppose we fix positive integers l and n. Let $\mathbf{Q} = \{Q_0, \ldots, Q_{l-1}\}$ be a set of complex numbers. The cyclotomic Hecke algebra $H_{q,\mathbf{Q}}(n)$ of $G(l, 1, n)$ is the cyclotomic quotient of affine Hecke algebra under the ideal generated by

$$(X_1 - Q_0)(X_1 - Q_2) \ldots (X_1 - Q_{l-1}) = 0. \tag{5.41}$$

There is a similar definition for the cyclotomic KLR algebra.

Definition 5.8. Suppose $R_{n,D}$ is a KLR algebra associated with the quiver D. Let Λ be a sequence of $|D|$ many non-negative integers $(\Lambda_d)_{d\in D}$ and $\sum_{d\in D}\Lambda_d = l$. The cyclotomic ideal I^Λ of Λ is generated by

$$\{\mathrm{y}_1^{\Lambda_d} e(\mathbf{d}) = 0 \mid \mathbf{d} = (d_1, \dots, d_n) \in D^n \text{ and } d = d_1\}\,.$$

The cyclotomic KLR algebra $R^\Lambda_{n,D}$ is the quotient algebra $R_{n,D}/I^\Lambda$.

Brundan and Kleshchev in their paper [4] constructed a system of orthogonal idempotents

$$\{e(\mathbf{v})\}_{\mathbf{v}=(v_1,\dots,v_n)\in\mathbb{C}^n}$$

such that the idempotent $e(\mathbf{v})$ projects the finite generated module M of $H_{q,\mathbf{Q}}(n)$ to the generalized eigenspaces $e(\mathbf{v})M$ of X_i's with eigenvalues v_i's. We are left to describe the completion of the affine Hecke algebra.

Definition 5.9. Consider a finite subset $V \subset \mathbb{C}\backslash\{0\}$ and add an edge from v to v' if $v' = qv$. Let the ideal $\mathcal{I}$ generated by $\prod_{v\in V}(X_i - v)$, for all $i = 1, \dots, n$. Then we have a nested sequence of ideals in $H_q(n)$ given by $\mathcal{I}_n = H_q(n)\mathcal{I}^n H_q(n)$. The topological algebra $\widehat{H_q(n)}$ is the completion of $H_q(n)$ with respect to the sequence $\mathcal{I}_n$.

With respect to this completion, the identity becomes $1 = \sum_{\mathbf{v}\in V^n} e(\mathbf{v})$. Similarly, we define the completion of the KLR algebra as follows.

Definition 5.10. Let V be the same subset as above. Let J be the ideal in $\mathbb{C}[\mathrm{y}_1, \dots, \mathrm{y}_n]$ generated by y_i for all i in the KLR algebra $R_{V,n}$ associated with V. Consider the nested ideals $J_n = R_{V,n}J^n R_{V,n}$. The completion of $\widehat{R_{V,n}}$ of the KLR algebra $R_{V,n}$ is defined to be the limit respect to the system of ideals J_n.

Now Theorem 5.11 states the isomorphism between the two completions.

Remark. By the generating relations of X_i's and T_i's, the cyclotomic restriction on X_1 implies some cyclotomic restrictions on all X_i. If we apply the cyclotomic quotient of the affine Hecke algebra on its completion, then we get the same cyclotomic Hecke algebra as we directly apply the quotient on the affine Hecke algebra. This is because the cyclotomic quotient gives a limit of degrees on the power series of $X_i - u$, for all $i = 1, \dots, n$. The same argument is true for the cyclotomic KLR algebra and

y_i. However, the construction of completions of both algebras is not in vain, as the isomorphism we will introduce below is between the completions instead of the original algebras.

We define the function $b : \mathbb{C} \to \mathbb{C}$

$$b(y) = \exp\big(2\pi\sqrt{-1}y\big) \tag{5.42}$$

to be the exponential map for the rest of the paper. We have the isomorphism between the completion of the affine Hecke algebra and the completion of the KLR algebra constructed by Theorem 3.10 in [24].

Theorem 5.11. Let q be a complex number on the unit circle. Let V be a quiver in $\mathbb{C}$ where the edges $v \to qv$ are joined by multiplying by $q \in \mathbb{C}$. Let $\widehat{R_{v,n}}$ be the completion of the KLR algebra associated with V and $\widehat{H_q(n)}$ be the completion of the affine Hecke algebra with respect to V. We identify the idempotents in $\widehat{R_{v,n}}$ and $\widehat{H_q(n)}$. Suppose we have $\mathbf{v} = (v_1, \dots, v_n) \in V^n$. Let

$$\Phi_i := T_i + \sum_{\mathbf{v}, v_i \neq v_{i+1}} \frac{1-q}{1 - X_i X_{i+1}^{-1}} e(\mathbf{v}) + \sum_{\mathbf{v}, v_i = v_{i+1}} e(\mathbf{v}). \tag{5.43}$$

There is an isomorphism between the completion of affine Hecke algebra and the completion of the KLR algebra via the map

$$\gamma : \widehat{H_q(n)} \to \widehat{R_{V,n}},$$

such that

$$\gamma(X_i) = \sum_{\mathbf{v}} v_i b(\mathrm{y}_i) e(\mathbf{v}), \tag{5.44}$$

$$\gamma(\Phi_i) = \sum_{\mathbf{v}} \psi_i A_i^{\mathbf{v}} e(\mathbf{v}), \tag{5.45}$$

where the power series $A_i^{\mathbf{v}}$ is defined to be

$$A_i^{\mathbf{v}} = \begin{cases} \frac{b(\mathrm{y}_i) - q b(\mathrm{y}_{i+1})}{b(\mathrm{y}_{i+1}) - b(\mathrm{y}_i)} (\mathrm{y}_{i+1} - \mathrm{y}_i) & \text{if } v_i = v_{i+1} \\ \frac{b(\mathrm{y}_{i+1}) - b(\mathrm{y}_i)}{(q^{-1} b(\mathrm{y}_{i+1}) - b(\mathrm{y}_i))(\mathrm{y}_i - \mathrm{y}_{i+1})} & \text{if } v_i = q v_{i+1} \\ \frac{v_i b(\mathrm{y}_i) - q v_{i+1} b(\mathrm{y}_{i+1})}{v_i b(\mathrm{y}_i) - v_{i+1} b(\mathrm{y}_{i+1})} & \text{otherwise} \end{cases} . \tag{5.46}$$

Remark. The quiver V is different from the quiver D_0. The quiver V has edges $v \to qv$ corresponding to the multiplication by q, while the edges $d \to d - c_0$ of the quiver D_0 correspond to the addition by $-c_0$. We would like to connect this two quiver by the exponential map. Note if we set the vertex set of the quiver $V \subset \mathbb{C}$ to be $\{b(d)\}_{d\in D}$ and edges to be $b(d) \to qb(d)$ where $q = b(-c_0)$, then quiver V is equivalent to the quiver D_0. Hence if we associate the KLR algebra with the quiver $D \subset \mathbb{C}/\mathbb{Z}$ instead of V, then γ can be rewritten as

$$\gamma(X_i) = \sum_{\mathbf{d}\in D_0^n} b(d_i)b(\mathrm{y}_i)e(\mathbf{d}), \tag{5.47}$$

$$\gamma(\Phi_i) = \sum_{\mathbf{d}\in D_0^n} \psi_i A_i^{\mathbf{d}} e(\mathbf{d}), \tag{5.48}$$

where the power series $A_i^{\mathbf{d}}$ are defined accordingly.

Finally we need to show that cyclotomic conditions are the same in two algebras.

Proposition 5.12. Given a quiver D with edges defined by addition and let $R_{D,n}$ and $H_{q,\mathbf{Q}}(n)$ be the KLR algebra and the affine Hecke algebra associated with D. Then the cyclotomic ideal generated by

$$(X_1 - Q_0)(X_1 - Q_1)\dots(X_1 - Q_{l-1}) = 0 \tag{5.49}$$

in the cyclotomic Hecke algebra $H_{q,\mathbf{Q}}(n)$ for $\mathbf{Q} = (Q_0, Q_1, \dots, Q_{l-1}) = (b(\mathfrak{d}_0), b(\mathfrak{d}_1), \dots, b(\mathfrak{d}_{l-1}))$ is isomorphic to the cyclotomic ideal generated by

$$\{\mathrm{y}_1^{\Lambda_{\mathfrak{d}_i}} e(\mathbf{d}) = 0 \mid \mathbf{d} \in D^n \text{ and } d_1 = \mathfrak{d}_i \text{ for some } i = 0, \dots, l-1\}, \tag{5.50}$$

where $Q_i = b(\mathfrak{d}_i)$ appears $\Lambda_{\mathfrak{d}_i}$ many times in (5.49).

Proof. Suppose we have the cyclotomic quotient in the Hecke algebra. If X_1 is substituted by $\gamma(X_1)$, we get the equation

$$\prod_{j=0}^{l-1}\Big(\sum_{\mathbf{d}} b(d_1)b(\mathrm{y}_1)e(\mathbf{d}) - b(\mathfrak{d}_j)\Big) = 0 \tag{5.51}$$

After we multiply the idempotent $e(\mathbf{d})$ for some $\mathbf{d}$, if there is no $i = 0, \dots, l-1$ such that $\mathfrak{d}_i = d_1$, then the term $(b(d_1)b(\mathrm{y}_1) - b(\mathfrak{d}_i))e(\mathbf{d})$ is invertible. The product becomes

$$(b(d_1)b(\mathrm{y}_1) - b(\mathfrak{d}_j))^{\Lambda_{\mathfrak{d}_j}} e(\mathbf{d}) = 0 \quad \text{for some } \mathfrak{d}_j = d_1, \tag{5.52}$$

$$e(\mathbf{d}) = 0 \quad \text{for all } j,\ \mathfrak{d}_j \neq d_1. \tag{5.53}$$

The second equation already gives us one of the cyclotomic condition of KLR algebra. Multiplying by $b(d_1)^{-1}$ and applying Taylor expansion on $b(\mathrm{y}_1)$ rewrites the first equation as

$$(b(\mathrm{y}_1) - 1)^{\Lambda_{\mathfrak{d}_j}} e(\mathbf{d}) = 0 \tag{5.54}$$

$$\Leftrightarrow (2\pi\sqrt{-1}\mathrm{y}_1 - \frac{(2\pi\sqrt{-1}\mathrm{y}_1)^2}{2!} \dots)^{\Lambda_{\mathfrak{d}_j}} e(\mathbf{d}) = 0 \tag{5.55}$$

$$\Leftrightarrow (2\pi\sqrt{-1}\mathrm{y}_1)^{\Lambda_{\mathfrak{d}_j}} (1 - \frac{2\pi\sqrt{-1}\mathrm{y}_1}{2!} \dots)^{\Lambda_{\mathfrak{d}_j}} e(\mathbf{d}) = 0. \tag{5.56}$$

In the completion of KLR algebra, all elements consisting only of y_1 are a product of some power of y_1 and a unit. If this element vanishes, this power of y_1 is zero. Hence it can be deduced that $\mathrm{y}_1^{\Lambda_{\mathfrak{d}_j}} e(\mathbf{d}) = 0$ in the above equation. We are done with one direction.

For the other direction, we assume the cyclotomic condition for the KLR algebra holds. For $\mathbf{d}$ that $e(\mathbf{d}) = 0$, the equation $(X_1 - b(\mathfrak{d}_0))(X_1 - b(\mathfrak{d}_1)) \dots (X_1 - b(\mathfrak{d}_{l-1}))e(\mathbf{d}) = 0$. If the other condition $\{\mathrm{y}_1^{\Lambda_{\mathfrak{d}_i}} e(\mathbf{d}) = 0 \mid d_1 = \mathfrak{d}_i\}$ is met, then the following expression vanishes

$$\begin{aligned} &(X_1 - b(\mathfrak{d}_i))^{\Lambda_{\mathfrak{d}_i}} e(\mathbf{d}) \\ =& b(\mathfrak{d}_i)^{\Lambda_{\mathfrak{d}_i}} (b(\mathrm{y}_1) - 1)^{\Lambda_{\mathfrak{d}_i}} e(\mathbf{d}) \\ =& b(\mathfrak{d}_i)^{\Lambda_{\mathfrak{d}_i}} (1 + 2\pi\sqrt{-1}\mathrm{y}_1 + \frac{(2\pi\sqrt{-1}\mathrm{y}_1)^2}{2!} + \dots - 1)^{\Lambda_{\mathfrak{d}_i}} e(\mathbf{d}) \\ =& b(\mathfrak{d}_i)^{\Lambda_{\mathfrak{d}_i}} \mathrm{y}_1^{\Lambda_{\mathfrak{d}_i}} (\pi\sqrt{-1} + \frac{(2\pi\sqrt{-1})^2\mathrm{y}_1}{2!} + \dots)^{\Lambda_{\mathfrak{d}_i}} e(\mathbf{d}) \\ =& 0. \end{aligned} \tag{5.57}$$

This is true for all $e(\mathbf{d}) \neq 0$. Hence if we add up all the idempotents, then we get the cyclotomic condition on the affine Hecke algebra. □

Corollary 5.13. The cyclotomic KLR algebra $R^{\Lambda}_{n,D}$ is isomorphic to the cyclotomic Hecek algebra $H_{q,\mathbf{Q}}(n)$.

Proof. It is a direct consequence of Theorem 5.11 and Proposition 5.12. □

6 Cyclotomic Hecke Algebra and Image of KZ Functor

6.1 Cyclotomic Hecke Algebra via KZ Functor

In Section 6.1, we want to study the cyclotomic Hecke algebra $H_{q,\mathbf{Q}}(n)$ under the KZ functor and the direct sum of weight space $\bigoplus_{\mathbf{d}\in D_0^n} W_{\mathbf{a_d}^+,\mathbf{1}}$ as a $H_{q,\mathbf{Q}}(n)$-mod. In order to get the cyclotomic Hecke algebra whose representations are images under the KZ functor, we re-parametrize the rational Cherednik algebra first in Section 6.1.1, which is a review of Section 4 of [2]. Then we compute the cyclotomic Hecke algebra $H_{q,\mathbf{Q}}(n)$ directly in Section 6.1.2. This subsection is my original work.

6.1.1 Re-Parametrization and Cyclotomic Hecke Algebra

Recall from Section 1 that in rational Cherednik algebras, we have the function $c : S \to \mathbb{C}$, which is invariant under conjugation. For each $s \in S$, we define a hyperplane $H_s = \ker(\alpha_s) \subset \mathfrak{h}$. Let $\mathcal{A}$ be the set of all hyperplanes H_s for some $s \in S$. We then have the cyclic subgroup of W

$$W_H = \{w \in W \mid w(h) = h \text{ for } h \in H\} \tag{6.1}$$

for each $H \in \mathcal{A}$. Let $W_H^* = W_H \backslash \{1\}$. By Lemma 1.1 and the definition of reflection in Section 1, it is not hard to deduce that

$$S = \bigcup_{H\in\mathcal{A}} W_H^* \,. \tag{6.2}$$

Without loss of generality, we may assume that

$$\alpha_H := \alpha_s = \alpha_{s'} \quad \text{and} \quad \alpha_H^\vee := \alpha_s^\vee = \alpha_{s'}^\vee \qquad \text{for} \quad s, s' \in W_H^* \,. \tag{6.3}$$

Let us denote $n_H = |W_H|$. We obtain the primitive idempotents in $\mathbb{C}W$

$$e_{H,i} = \frac{1}{n_H} \sum_{w\in W_H} (\det(w))^i w \qquad \text{for} \quad 0 \le i \le n_H - 1 \,. \tag{6.4}$$

Based on the above construction, we define the parameter $k_{H,i} \in \mathbb{C}$ by

$$-2 \sum_{s \in W_H^*} c_s s = n_H \sum_{i=0}^{n_H - 1} (k_{H,i+1} - k_{H,i}) e_{H,i} \,, \tag{6.5}$$

$$k_{H,0} = k_{H,n_H} = 0 \,, \tag{6.6}$$

for $H \in \mathcal{A}$. By comparing the coefficients in front of s, we see that

$$c_s = -\frac{1}{2} \sum_{i=0}^{n_H - 1} (\det(s))^i (k_{H,i+1} - k_{H,i}) \,. \tag{6.7}$$

Then the commutation relations (1.7) in the rational Cherednik algebra become

$$\begin{aligned} [y, x] &= x(y) - \sum_{s \in S} c_s \alpha_s(y) x(\alpha_s^\vee) s \\ &= x(y) - \left(\sum_{H \in \mathcal{A}} \alpha_H(y) x(\alpha_H^\vee) \right) \left(\sum_{s \in W_H^*} c_s s \right) \\ &= x(y) + \frac{1}{2} \left(\sum_{H \in \mathcal{A}} \alpha_H(y) x(\alpha_H^\vee) n_H \right) \left(\sum_{i=0}^{n_H - 1} (k_{H,i+1} - k_{H,i}) e_{H,i} \right) \,. \end{aligned} \tag{6.8}$$

Since we have $c_s = c_{gsg^{-1}}$ and $W_{g(H)} = g W_H g^{-1}$, we can get

$$\sum_{i=0}^{n_H - 1} (\det(s))^i (k_{H,i+1} - k_{H,i}) = \sum_{i=0}^{n_g(H) - 1} (\det(gsg^{-1}))^i \left(k_{g(H),i+1} - k_{g(H),i} \right) \,, \tag{6.9}$$

$$\sum_{i=0}^{n_H - 1} (\det(s))^i (k_{H,i+1} - k_{H,i}) = \sum_{i=0}^{n_g(H) - 1} (\det(s))^i \left(k_{g(H),i+1} - k_{g(H),i} \right) \,. \tag{6.10}$$

Because it is true for any reflection, it forces $k_{H,i} = k_{g(H),i}$ for $0 \leq i \leq n_H$. Then we have $H_c(W, \mathfrak{h}) = H_{\mathbf{k}}(W, \mathfrak{h})$, where

$$\mathbf{k} = \{ k_{H,i} \mid H \in \mathcal{A},\, 1 \leq i \leq n_H - 1;\, k_{H,0} = k_{H,n_H} = 0;\, k_{H,i} = k_{w(H),i} \text{ for any } w \in W,\, H, \text{ and } i \} \,.$$

We assume that T_s is the s-generator of the monodromy around $H = \ker(\alpha_s)$. Note that the braid group $\pi_1(\mathfrak{h}_{\text{reg}}/W)$ is generated by $\{T_s \mid s \in S\}$, satisfying the braid relations. Suppose that we have a rational Cherednik algebra with parameter $\mathbf{k}$. We define

$$q_{H,i} = (\det(s_H))^{-i} \exp\bigl(2\pi\sqrt{-1}k_{H,i}\bigr) \tag{6.11}$$

for all $H \in \mathcal{A}$ and $1 \leq i \leq n_H - 1$, where s_H is the generator of W_H. For these fixed parameters $\mathbf{q} = \{q_{H,i\in\mathbb{C}^\times} \mid H \in \mathcal{A}\}$, we have an alternative definition of the cyclotomic Hecke algebra $\mathcal{H}_{\mathbf{q}}(W)$, which is the fundamental group $\pi_1(\mathfrak{h}_{\mathrm{reg}}/W)$ under the relation

$$\prod_{i=0}^{n_H-1} (T_s - q_{H,i}) = 0 \tag{6.12}$$

for all $s \in S$ and $H = \ker(\alpha_s)$.

Based on the above settings, below is one of the most important theorems on the KZ functor, proved by Ginzburg, Guay, Opdam, and Rouquier in Theorem 5.13 of [10].

Theorem 6.1. The KZ functor factors through category of representations of $\mathcal{H}_{\mathbf{q}}(W)$.

6.1.2 Cyclotomic Hecke Algebra of $G(l, 1, n)$ under KZ Functor

We are now prepared to compute the parameters of cyclotomic Hecke algebra. There are two kinds of reflections $\zeta_i^t s_{ij} \zeta_i^{-t}$ and ζ_i^t, for some $0 \leq t \leq l-1$ and $0 < i \neq j \leq n$ with order 2 and l respectively. If we take $s = \zeta_i^t$, it is obvious that $H = \ker(\alpha_s)$ is spanned by $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots y_n$ and $W_H = \{1, \zeta_i, \ldots, \zeta_i^{l-1}\}$, where $\widehat{y_i}$ means omitting the element y_i. Then we have $n_H = l$ and

$$\begin{aligned} e_{H,j} &= \frac{1}{n_H} \sum_{s=0}^{n_H-1} (\det(\zeta_i^s))^j \zeta_i^s \\ &= \frac{1}{l} \sum_{s=0}^{l-1} \zeta^{sj} \zeta_i^s \,. \end{aligned} \tag{6.13}$$

By comparing the coefficient in front of ζ_i^s in (6.5), the expression of c_t in terms of $k_{H,i}$ becomes

$$c_t = -\frac{1}{2} \sum_{j=0}^{n_H-1} \zeta^{tj} (k_{H,j+1} - k_{H,j}) \,, \tag{6.14}$$

for all $t = 1, ..., l-1$. All the equations can be written as matrices, i.e.,

$$\begin{pmatrix} 1 & \zeta^1 & \cdots & \zeta^{(l-1)} \\ 1 & \zeta^2 & \cdots & \zeta^{2(l-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta^{(l-1)} & \cdots & \zeta^{(l-1)(l-1)} \end{pmatrix} \begin{pmatrix} k_{H,1} - k_{H,0} \\ k_{H,2} - k_{H,1} \\ \vdots \\ k_{H,l} - k_{H,l-1} \end{pmatrix} = -2 \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{l-1} \end{pmatrix}. \tag{6.15}$$

We know that $k_{H,0} = k_{H,l} = 0$, we get

$$A \begin{pmatrix} k_{H,1} \\ k_{H,2} \\ \vdots \\ k_{H,l-1} \end{pmatrix} = -2 \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{l-1} \end{pmatrix}, \tag{6.16}$$

where

$$A = \begin{pmatrix} 1-\zeta^1 & \zeta^1 - \zeta^2 & \cdots & \zeta^{(l-2)} - \zeta^{(l-1)} \\ 1-\zeta^2 & \zeta^2 - \zeta^4 & \cdots & \zeta^{2(l-2)} - \zeta^{2(l-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1-\zeta^{(l-1)} & \zeta^{(l-1)} - \zeta^{2(l-1)} & \cdots & \zeta^{(l-2)(l-1)} - \zeta^{(l-1)(l-1)} \end{pmatrix}. \tag{6.17}$$

The inverse of A is given by

$$A^{-1} = \frac{1}{l} \begin{pmatrix} \frac{1-\zeta^{-1}}{1-\zeta^{-1}} & \frac{1-\zeta^{-2}}{1-\zeta^{-2}} & \cdots & \frac{1-\zeta^{-(l-1)}}{1-\zeta^{-(l-1)}} \\ \frac{1-\zeta^{-2}}{1-\zeta^{-1}} & \frac{1-\zeta^{-4}}{1-\zeta^{-2}} & \cdots & \frac{1-\zeta^{-2(l-1)}}{1-\zeta^{-(l-1)}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1-\zeta^{-(l-1)}}{1-\zeta^{-1}} & \frac{1-\zeta^{-2(l-1)}}{1-\zeta^{-2}} & \cdots & \frac{1-\zeta^{-(l-1)(l-1)}}{1-\zeta^{-(l-1)}} \end{pmatrix}. \tag{6.18}$$

Then we can write $k_{H,j}$ in terms of c_t as

$$k_{H,j} = -\frac{2}{l} \sum_{t=1}^{l-1} \frac{1-\zeta^{-tj}}{1-\zeta^{-t}} c_t \,. \tag{6.19}$$

Hence the $q_{H,j}$ in terms of $k_{H,j}$ are

$$q_{H,j} = \det(s_H)^{-j} b(k_{H,j})$$

$$= \zeta^j b(k_{H,j}) \,, \tag{6.20}$$

where we choose the generator s_H of W_H to be ζ_i^{-1} here.

The other type of reflections is $s = \zeta_i^t s_{ij} \zeta_i^{-t}$ for some $0 \le t \le l-1$ and $0 < i \ne j \le n$. The non-trivial eigenvector of s is $\alpha_s = \zeta^{-t} x_i - x_j$. Hence the hyperplane H' is $\ker(\alpha_s)$ generated by $\{\zeta^t y_i + y_j, y_k\}_{k \ne i,j}$. The subgroup that stabilizes the hyperplane is $W_{H'} = \{1, s\}$ and $n_{H'} = 2$. Then the idempotents $e_{H',k},\ k = 0, 1$ are

$$e_{H',0} = \frac{1}{2}(1+s) \,, \tag{6.21}$$

$$e_{H',1} = \frac{1}{2}(1-s) \,. \tag{6.22}$$

The relations between c_0 and $k_{H',1}$ is

$$\begin{aligned}
& -2c_0 s = 2[(k_{H',2} - k_{H',1})e_{H',1} + (k_{H',1} - k_{H',0})e_{H',0}] \\
\Leftrightarrow & -c_0 s = (0 - k_{H',1})\frac{1}{2}(1+s) + (k_{H',1} - 0)\frac{1}{2}(1-s) \\
\Leftrightarrow & -c_0 s = -k_{H',1} s \\
\Rightarrow & \, c_0 = k_{H',1} \,.
\end{aligned} \tag{6.23}$$

Hence the $q_{H',j}$ in terms of c_t are

$$q_{H',0} = \det(s_{H'})^0 b(k_{H',0}) = 1 \,, \tag{6.24}$$

$$q_{H',1} = \det(s_{H'})^{-1} b(k_{H',1}) = -b(c_0) \,, \tag{6.25}$$

where the generator of $W_{H'}$ is s with determinant -1. Thus the cyclotomic Hecke algebra $H_{\mathbf{q}}(W)$ is the quotient of braid group generated by $\{T_s \mid s \in S\}$ satisfying the braid relations under the quotient of

$$(T_s - 1)(T_s + b(c_0)) = 0, \text{ for } s = \zeta_i^t s_{ij} \zeta_i^{-t} \tag{6.26}$$

$$\prod_{j=0}^{l-1} (T_s - \zeta^j b(k_{H,j})) = 0, \text{ for } s = \zeta_i^t. \tag{6.27}$$

Since the reflections are generated by $\{\zeta_i\}_{i=1,..,n} \cup \{s_i\}_{i=1,...,n-1}$, we are able to generate the algebra with fewer generators. Let T_i be short for T_{s_i} and T_0 short for T_{ζ_1}. Then the cyclotomic Hecke algebra $H_{c_0,k_{H,j}}(n)$ is the quotient algebra generated by $T_0, T_1, \ldots, T_{n-1}$ under the relations

$$(T_i - 1)(T_i + b(c_0)) = 0 \text{ for any } i = 1, \ldots, n-1\,, \tag{6.28}$$

$$\prod_{j=0}^{l-1}(T_0 - \zeta^j b(k_{H,j})) = 0\,. \tag{6.29}$$

However according to the last section, we would want the cyclotomic Hecke algebra to associate with the quiver D_0, whose vertices are of the form $-h_j + \frac{j}{l} + mc_0$ for some $m \in [-n, n]$. Hence the relations between $k_{H,j}$ and h_j are required. If we examine the (6.19) closely and compare it (2.9), then it is obvious that

$$k_{H,j} = h_{l-j}. \tag{6.30}$$

Then the generating relation of T_0 becomes

$$\prod_{j=0}^{l-1}(T_0 - \zeta^j b(h_{l-j})) = 0, \tag{6.31}$$

After multiplying by $T_0^{-l}\prod_{j=0}^{l-1}\zeta^{-j}\exp\bigl(-2\pi\sqrt{-1}h_{l-j}\bigr)$, we get

$$\prod_{j=0}^{l-1}(T_0^{-1} - \zeta^{-j}\exp\bigl(-2\pi\sqrt{-1}h_{l-j}\bigr) = 0, \tag{6.32}$$

which is the same as

$$\prod_{j=0}^{l-1}(T_0^{-1} - b((-h_{l-j} + \frac{l-j}{l})) = 0. \tag{6.33}$$

Now we get the equation of T_0 in terms of the vertices of D_0. Likewise, we multiply (6.28) by T_i^{-2}, then we get

$$\begin{aligned}&(1 - T_i^{-1})(1 + b(c_0)T_i^{-1}) = 0\\ \Leftrightarrow&((-T_i^{-1}) + 1)((-T_i^{-1}) - b^{-c_0}) = 0\end{aligned} \tag{6.34}$$

We define

$$q = b(-c_0)\,, \tag{6.35}$$

$$Q_j = b((-h_j + \frac{j}{l})) \quad \text{for } j = 0, \ldots, l-1\,. \tag{6.36}$$

Lemma 6.2. The cyclotomic Hecke algebra $H_{\mathbf{q}}(W)$ we get via the KZ functor is isomorphic to $H_{q,\mathbf{Q}}(n)$ by sending T_i to $-T_i^{-1}$ for $i = 1, \ldots, n-1$, and T_0 to T_0^{-1}.

Proof. It is a direct consequence of (6.33), (6.34), and the fact $T_0T_1T_0T_1 = T_1T_0T_1T_0$ is equivalent to $T_0^{-1}T_1^{-1}T_0^{-1}T_1^{-1} = T_1^{-1}T_0^{-1}T_1^{-1}T_0^{-1}$. □

Corollary 6.3. The algebra $H_{\mathbf{q}}(W)$ is isomorphic to the cyclotomic KLR algebra of quiver D_0, quotiented by

$$\{y^{\Lambda_i}e(\mathbf{d}) \mid d_1 = -h_i + \frac{i}{l} \text{ for some } i, \text{ and } \Lambda_i \text{ is the number of times that } Q_i \text{ appears in } \mathbf{Q}.\}\cup$$
$$\{e(\mathbf{d}) = 0 \mid d_1 \neq -h_i + \frac{i}{l} \text{ for any } i.\} \tag{6.37}$$

Proof. It is a direct consequence of Lemma 6.2 and Corollary 5.13 □

6.2 Images of KZ Functor as Modules for Cyclotomic Hecke Algebras

In this section, we will assume that the parameters are generic. In this section we will prove that the image of the Verma module $M_c(\lambda)$ under the KZ functor is isomorphic to the representation $V^{\lambda^{tr}}$ of cyclotomic Hecke algebra, which corresponds to the conjugate of λ. This section is my original work.

Generic parameters have some unique advantages. The first advantage is that the formulae of eigenvalues and eigenvectors in the Verma module of the rational Cherednik algebra are already computed by Griffeth [13] for the generic parameters. The second advantage is that under the same condition, there is a straightforward way to write down the actions of generators on the basis of irreducible modules of the cyclotomic Hecke algebra. So in this section, let us assume that the all parameters are generic.

First of all, we want to clarify what generic parameters mean in this context. We call the cyclotomic Hecke algebra $H_{q,\mathbf{Q}}$ is a semi-simple algebra is all of its modules are projective modules.

Definition 6.4. We say the parameters of a cyclotomic Hecke algbera are generic if $H_{q,\mathbf{Q}}$ is semi-simple.

According to Proposition 2.9 in [1], we have the conditions of generic parameters as follows.

Proposition 6.5. Let $\mathbf{Q} = (Q_0, \ldots, Q_{l-1})$ be an l-tuple in $\mathbb{C}$. Then $H_{q,\mathbf{Q}}$ is semi-simple if and only if $q^i Q_j - Q_k$ $(|i| < n,\ j \neq k)$ and $1 + q + \ldots + q^i$ $(1 \leq i < n)$ are all non zero.

Similarly, we can define the genericity of parameters in rational Cherednik algebra.

Definition 6.6. The parameters in rational Cherednik algebra are generic if Category $\mathcal{O}$ is semi-simple.

As a consequence of generic parameters, the eigenvalues of z_i with respect to $f_{\mu,L}$ in $M_c(\lambda)$ stated in Theorem 2.7 are distinct. Hence z_i acts diagonally on $M_c(\lambda)$ with the right basis.

Corollary 6.7. The parameters of rational Cherednik algebra are generic if and only of $M_c(\lambda) = L_c(\lambda)$ for all irreducible representations λ of $G(l, 1, n)$.

Proof. It is a consequence of Proposition 1.15. □

6.2.1 Image of KZ Functor as a Direct Sum of Weight Spaces

As stated in Theorem 4.20, the KZ functor on Category $\mathcal{O}$ is isomorphic to the direct sum of weight space functor $\bigoplus_{\mathbf{d} \in D_0^n} W_{\mathbf{a_d}^+,\mathbf{1}}$. The weights, or the (generalized) eigenvalues of u_i and ζ_i in the weight spaces are lifted from the vertices in D_0 by η. On the other hand, from Theorem 2.7, the eigenvalues of z_i and ζ_i in $M_c(\lambda)$ can be expressed in terms of a pair $(\mu, L) \in \mathbb{Z}^n_{\geq 0} \times \mathrm{SYT}(\lambda)$. We want to have a closer look at how those two facts are connected, and how certain sum of certain weight spaces of Verma module makes a representation of the cyclotomic Hecke algebra. In the Verma module $M_c(\lambda)$, given a pair $(\mu, L) \in \mathbb{Z}^n_{\geq 0} \times \mathrm{SYT}$, the eigenvalues of z_i and ζ_i with respect to the eigenvector $f_{\mu,L}$ are $(\mu_i + 1) - l(h_{\beta(L(w_\mu(i)))} - h_{\beta(L(w_\mu(i)))-\mu_i-1}) - c_0 l \mathrm{ct}(L(w_\mu(i)))$ and $\zeta^{\beta(L(w_\mu(i)))-\mu_i}$ respectively according to Theorem 2.7. By the definition of u_i in (4.21), u_i acts on the eigenvector $f_{\mu,L}$ by

$$u_i \cdot f_{\mu,L} = [\mu_i - l h_{\beta(w_{mu}(i))} - c_0 l \mathrm{ct}(L(w_\mu(i)))] f_{\mu,L}. \tag{6.38}$$

In order to connect the vertices in D_0, and eigenvalues of u_i and ζ_i, we define the lifting $\eta : D_0 \to \mathbb{C}$ by

$$\tilde{r}_j + mc_0 = -h_j + \frac{j}{l} + mc_0 \mapsto -lh_j + j + lmc_0. \tag{6.39}$$

Because all the parameters are generic, $-h_{j_1} + \frac{j_1}{l} + m_1 c_0 \equiv -h_{j_2} + \frac{j_2}{l} + m_2 c_0 \in D_0 \subset \mathbb{C}/\mathbb{Z}$ cannot be satisfied for either $j_1 \neq j_2$ or $m_1 \neq m_2$. Suppose we have the n-tuple in D_0 to be

$$\mathbf{d} = (-h_{j_1} + \frac{j_1}{l} + m_1 c_0, \ldots, -h_{j_n} + \frac{j_n}{l} + m_n c_0),$$

then the eigenvalues $\mathbf{a_d}$ are

$$\mathbf{a_d} = (-lh_{j_1} + j_1 + lm_1 c_0 + lN, \ldots, -lh_{j_n} + j_n + lm_n c_0 + nlN). \tag{6.40}$$

Since $\bigoplus_{\mathbf{d} \in D_0^n} W_{\mathbf{a_d}^+, \mathbf{1}}$ is a direct sum of the (generalized) eigenspaces of u_i and ζ_i, by (6.38), for every $i = 1, .., n$ we get that the i-th coordinate in $\mathbf{a_d}$ can be written as

$$-lh_{j_1} + j_i + lm_i c_0 + ilN = \mu_i - lh_{\beta(w_\mu(i))} - c_0 l\mathrm{ct}(L(w_\mu(i))), \tag{6.41}$$

for some pair $(\mu, L) \in \mathbb{Z}_{\geq 0}^n \times \mathrm{SYT}(\lambda)$. Since the eigenvalues of $\zeta_i = \zeta^{\beta(L(w_\mu(i))) - \mu_i}$ is 1, we have $\beta(L(w_\mu(i))) \equiv \mu_i \bmod l$. Due to genericity of the parameters, we get

$$j_i = \beta(L(w_\mu(i))), \tag{6.42}$$

$$m_i = -\mathrm{ct}(L(w_\mu(i))), \tag{6.43}$$

$$\mu_i = \beta(L(w_\mu(i))) + ilN. \tag{6.44}$$

It is clear that the eigenvalues are completely determined by the standard Young tableaux L. We will omit μ and denote the eigenvalues of u_i in the direct summand $W_{\mathbf{a_d}^+, \mathbf{1}}(M_c(\lambda))$ by f_L instead of $f_{\mu,L}$. Since the integer N is much bigger than the rest of the numbers, we have the ordering $\mu_1 < \mu_2 < \ldots < \mu_n$. Hence the permutation w_μ is the identity in S_n. Then the first two equations becomes

$$\beta(L(i)) = j_i, \tag{6.45}$$

$$m_i = -\mathrm{ct}(L(i)). \tag{6.46}$$

Using the above data, we can reconstruct the standard Young tableau L by placing the box containing 1 to n one by one, where the box of i is in λ^{j_i} with the content being $-m_i$. The box $L(1)$, no matter which coordinate λ^j of $\lambda = (\lambda^0, \ldots, \lambda^{l-1})$ it is in, is always on the top left. So the content of the box $L(1)$ is zero for any L, implying $m_1 = 0$ if the weight space $W_{\mathbf{a_d},\mathbf{1}}(M_c(\lambda))$ is non empty. If there is no standard Young tableaux of shape λ corresponding to $(-lh_{j_1} + j_1 + lm_1c_0 + lN, \ldots, -lh_{j_n} + j_n + lm_nc_0 + nlN)$, then the weight space $W_{\mathbf{a_d},\mathbf{1}}(M_c(\lambda)) = 0$.

In conclusion, the n-tuple $\mathbf{d} = (d_1, \ldots, d_n) \in D_0^n$ is in one-to-one correspondence with the eigenvalues $\mathbf{a_d}$, where the latter is uniquely associated with a Young tableau. To clarify the relations in the notations, we may use $\mathbf{d}_L = (d_{L1}, \ldots, d_{Ln})$ to represent the n-tuple of vertices in D_0 that corresponds to the standard Young tableau L.

6.2.2 Image of KZ Functor as a Module of Cyclotomic Hecke Algebra

There is still one question remaining, that is why $\bigoplus_{\mathbf{d}\in D_0^n} W_{\mathbf{a_d}^+,\mathbf{1}}(M_c(\lambda))$ is a representation of the cyclotomic Hecke algebra $H_{q,\mathbf{Q}}(n)$, for any Verma module $M_c(\lambda)$. To answer this question, we first need to know how the KLR algebra $R_{D_0,n}$ defined in Section 5 as the subalgebra of the weighted KLR algebra R_{D_0} acts on this direct sum of weight spaces.

Lemma 6.8. Let $M_c(\lambda)$ be the Verma module of $H_c(G(l,1,n), \mathbb{C}^n)$ associated with the irreducible module λ of $G(l,1,n)$. Then the direct sum $\bigoplus_{\mathbf{d}\in D_0^n} W_{\mathbf{a_d}^+,\mathbf{1}}(M_c(\lambda))$ is a representation of the KLR algebra defined in Theorem 5.3 by the actions

1. The idempotents $e(\mathbf{d})$ project onto the weight space $W_{\mathbf{a_d}^+,\mathbf{1}}(M_c(\lambda))$;
2. The element $\mathrm{y}_i e(\mathbf{d})$ acts by zero;
3. The element ψ_i maps from the weight space corresponding to the standard Young tableau L to the weight space corresponding to $s_i \cdot L$, which is the Young tableau after interchanging the boxes containing i and $i+1$ in L if it is standard, and zero otherwise.

Proof. The action of the idempotents is obvious. The element $\mathrm{y}_i e(\mathbf{d})$ acts by $u_i - (\eta(d_i) + ilN)$ according to the statement 2 in Proposition 4.15. Since the parameters are generic, all the generalized weight spaces are eigenspaces. Hence $\mathrm{y}_i e(\mathbf{d})$ acts by zero.

The element ψ_i maps the weight space $W_{\mathbf{a_d},\mathbf{1}}(M_c(\lambda))$ to the weight space $W_{\mathbf{a_{s_i\cdot d}},\mathbf{1}}(M_c(\lambda))$. The eigenvalues in $W_{\mathbf{a_d},\mathbf{1}}(M_c(\lambda))$ and $W_{\mathbf{a_{s_i\cdot d}},\mathbf{1}}(M_c(\lambda))$ can be written as

$$\begin{aligned}\mathbf{a_d} =&(-lh_{j_1}+j_1+lm_1c_0+lN,\dots -lh_{j_i}+j_i+lm_ic_0+ilN, -lh_{j_{i+1}}+j_{i+1}+lm_{i+1}c_0+(i+1)lN,\\ &\dots, -lh_{j_n}+j_n+lm_nc_0+nlN) \text{ and} \qquad (6.47)\\ \mathbf{s_i}\cdot\mathbf{a_d} =&(-lh_{j_1}+j_1+lm_1c_0+lN,\dots -lh_{j_{i+1}}+j_{i+1}+lm_{i+1}c_0+ilN, -lh_{j_i}+j_i+lm_ic_0+(i+1)lN,\\ &\dots, -lh_{j_n}+j_n+lm_nc_0+nlN), \qquad (6.48)\end{aligned}$$

respectively. Suppose the first and second eigenvalues correspond to the standard Young tableau L and L' respectively. By (6.45) and (6.46), we have the relations

$$\beta(L(p)) = j_p = \beta(L'(p)), \quad \text{for } p \neq i, i+1 \qquad (6.49)$$

$$\text{ct}(L(p)) = -m_p = \text{ct}(L'(p)), \quad \text{for } p \neq i, i+1 \qquad (6.50)$$

$$\beta(L(i)) = j_i = \beta(L'(i+1)) \quad \text{and} \quad \beta(L(i+1)) = j_{i+1} = \beta(L'(i)), \qquad (6.51)$$

$$\text{ct}(L(i)) = -m_i = \text{ct}(L'(i+1)) \quad \text{and} \quad \text{ct}(L(i+1)) = -m_{i+1} = \text{ct}(L'(i)). \qquad (6.52)$$

Hence by comparing the positions and the contents of the boxes in L and L', it is obvious that $L' = s_i \cdot L$. If there is no standard Young tableau corresponding to the weight space $W_{\mathbf{a_{s_i\cdot d}},\mathbf{1}}(M_c(\lambda))$, then ψ_i maps to zero. □

Lemma 6.8 and Corollary 5.13 imply $\bigoplus_{\mathbf{d}\in D_0^n} W_{\mathbf{a_d}^+,\mathbf{1}}(M_c(\lambda))$ is a representation of the cyclotomic Hecke algebra. The actions of the generators ψ_i are

$$\psi_i e(\mathbf{d}_L)\cdot f_L = a_L^i f_{s_i\cdot L}, \qquad (6.53)$$

for some scalar $a_L^i \in \mathbb{C}$. Moreover, we would like to check whether this representation can be written in the Young semi-normal form shown in the appendix, where ψ_i acts by

$$\psi_i \cdot v_L = v_{s_i\cdot L}, \qquad (6.54)$$

for some basis $\{v_L \mid L \in \text{SYT}(\lambda)\}$. If $s_i\cdot L$ is not standard, we set $v_{s_i\cdot L} = 0$ by convention. The Young semi-normal form can be achieved by rescaling the basis elements f_L, for some Young tableau L. We fix one f_L and rescale with respect to the pair $(L, s_i \cdot L)$, that is to set $v_L = f_L$ and $v_{s_i \cdot L} = a^i_L f_{s_i \cdot L}$. Then within these two basis elements, ψ_i acts in the desired way. However, there are different paths to scale to get the same v_L. For example, $v_{s_i s_{i+1} s_i \cdot L}$ can be defined along the pairs $(L, s_i \cdot L)$, $(s_i \cdot L, s_{i+1}s_i \cdot L)$, and then $(s_{i+1}s_i \cdot L, s_i s_{i+1} s_i \cdot L)$. On the other hand, we have $v_{s_i s_{i+1} s_i \cdot L} = v_{s_{i+1} s_i s_{i+1} \cdot L}$, where the latter can also be achieved by rescaling along the pairs $(L, s_{i+1} \cdot L)$, $(s_{i+1} \cdot L, s_i s_{i+1} \cdot L)$, and $(s_i s_{i+1} \cdot L, s_{i+1} s_i s_{i+1} s_i \cdot L)$. Thus we need to check if all the paths are compatible. Because of the properties of symmetric group, it is sufficient to show that two paths in the above example are compatible. If there is an edge among the vertices d_{Li}, d_{Li+1} and d_{Li+2}, i.e vertices of the edge differ by c_0, then the boxes corresponding to the involved vertices are in the same tableau λ^k in λ, and are adjacent to each other. Hence they are not interchangeable and $v_{s_i s_{i+1} s_i \cdot L} = 0 = v_{s_{i+1} s_i s_{i+1} \cdot L}$. Otherwise, the boxes corresponding to the vertices d_{Li}, d_{Li+1} and d_{Li+2} are interchangeable. Hence the two rescaling paths are compatible by

$$v_{s_i s_{i+1} s_i \cdot L} = \psi_i \psi_{i+1} \psi_i f_L = \psi_{i+1} \psi_i \psi_{i+1} f_L = v_{s_{i+1} s_i s_{i+1} \cdot L}\,, \tag{6.55}$$

where the second equality holds by (5.12). As the mentioned above, the content of $L(1)$ is zero for all standard Young tableaux L. The only possible $\mathbf{d} = (d_1, \ldots, d_n)$ satisfying $e(\mathbf{d}) \neq 0$ is when $d_1 = -h_j + \frac{j}{l}$ for some $j = 0, ..., l-1$. Under this condition, we have

$$\mathrm{y}_i e(\mathbf{d}) = 0, \text{ for } d_1 = -h_j + \frac{j}{l}\,.$$

By Corollary 5.13, the cyclotomic Hecke algebra, which isomorphic to the cyclotomic KLR algebra, is the cyclotomic quotient of the affine Hecke algebra $H_q(n)$ by the ideal

$$\prod_{j=0}^{l-1} (T_0 - b(-h_j + \frac{j}{l})) = 0. \tag{6.56}$$

It is the same cyclotomic Hecke algebra whose representations are images of modules of $H_c(G(l,1,n), \mathbb{C}^n)$ under the KZ functor. By the isomorphism between the KLR algebra and the affine Hecke algebra in

Theorem 5.11, the generators X_i of the Hecke algebra act on the basis v_L by

$$\begin{aligned} X_i \cdot v_L &= \sum_{\mathbf{d} \in D_0^n} b(d_i) b(\mathrm{y}_i) e(\mathbf{d}) \cdot v_L \\ &= b(d_{L_i}) b(\mathrm{y}_i) e(\mathbf{d}_L) \cdot v_L \qquad (6.57) \\ &= b(d_{L_i}) v_L. \qquad (6.58) \end{aligned}$$

By the correspondence between $\mathbf{d}$ and L, we know that

$$d_{Li} = -lh_{j_i} + j_i + lm_i c_0 = -lh_{\beta(L(i))} + \beta(L(i)) - l\mathrm{ct}(L(i))c_0. \qquad (6.59)$$

Especially we have for X_1,

$$T_0 \cdot v_L = X_1 \cdot v_L = b(-lh_{\beta(L(1))} + \beta(L(1)))v_L = Q_{\beta(L(1))} v_L. \qquad (6.60)$$

Similarly, the action of Φ_i is

$$\begin{aligned} \Phi_i \cdot v_L &= \sum_{\mathbf{d}} \psi_i A_i^{\mathbf{d}} e(\mathbf{d}) \cdot v_L \\ &= \psi_i \frac{b(d_{L_i}) b(\mathrm{y}_i) - q b(d_{L_{i+1}}) b(\mathrm{y}_{i+1})}{b(d_{L_i}) b(\mathrm{y}_{i+1}) - b(d_{L_{i+1}}) b(\mathrm{y}_i)} e(\mathbf{d}_L) \cdot v_L \\ &= \frac{b(d_{L_i}) - q b(d_{L_{i+1}})}{b(d_{L_i}) - b(d_{L_{i+1}})} v_{s_i \cdot L} \\ &= \frac{B_i^L}{C_i^L} v_{s_i \cdot L}, \qquad (6.61) \end{aligned}$$

where

$$\begin{aligned} B_i^L &= b((-lh_{\beta(L(i))} + \beta(L(i)) - l\mathrm{ct}(L(i))c_0)) \\ &\quad - q b((-lh_{\beta(L(i+1))} + \beta(L(i+1)) - l\mathrm{ct}(L(i+1))c_0)) \qquad (6.62) \\ C_i^L &= b((-lh_{\beta(L(i))} + \beta(L(i)) - l\mathrm{ct}(L(i))c_0)) \\ &\quad - b((-lh_{\beta(L(i+1))} + \beta(L(i+1)) - l\mathrm{ct}(L(i+1))c_0)). \qquad (6.63) \end{aligned}$$

By the definition of Φ_i, the action of T_i can be deduced as

$$\begin{aligned} T_i \cdot v_L &= (\Phi_i - \sum_{\mathbf{u}} \frac{1-q}{1-X_iX_{i+1}^{-1}} e(\mathbf{u})) \cdot v_L \\ &= \frac{B_i^L}{C_i^L} v_{s_i \cdot L} + \frac{(1-q)b((-lh_{\beta(L(i+1))} + \beta(L(i+1)) - l\mathrm{ct}(L(i+1))c_0))}{C_i^L} v_L. \end{aligned} \tag{6.64}$$

In order to compare this representation to the well-known representations of cyclotomic Hecke algebra, we define the irreducible representation V^λ from [14] as follows. For any l-multipartition λ of n, we denote the set of standard Young tableaux of shape λ by $\mathrm{SYT}(\lambda)$. Let V^λ be the vector space spanned by the basis $\{v^L \mid L \in \mathrm{SYT}(\lambda)\}$. We define

$$\mathrm{ct}_q(B) = Q_{\beta(B)} q^{\mathrm{ct}(B)}, \text{ for any box } B, \tag{6.65}$$

$$(T_i)_L = \frac{q-1}{1 - \frac{\mathrm{ct}_q L(i)}{\mathrm{ct}_q L(i+1)}}, \text{ for } 1 \le i \le n-1. \tag{6.66}$$

Then the vector space V^λ turns into an irreducible representation of $H_{q,\mathbf{Q}}(n)$ in Equation 2.3 of [14] as follows.

Proposition 6.9. By the above setting, the vector space V^λ is an irreducible representation by the action

$$T_0 \cdot v^L = \mathrm{ct}_q(L(1)) v^L\,, \tag{6.67}$$

$$T_i \cdot v^L = (T_i)_L v^L + (1 + (T_i)_L) v^{s_i \cdot L} \qquad \text{for} \quad 1 \le i \le n-1\,, \tag{6.68}$$

where $s_i \cdot L$ is the standard tableau interchanging the position of i and $i+1$ in L. If $s_i \cdot L$ is not standard, then we set $s_i \cdot L = 0$.

The set $\mathcal{V} = \{V^\lambda \mid \lambda \text{ is an } l\text{-multipartition of } n\}$ forms the complete set of non-isomorphic irreducible representations of $H_{q,\mathbf{Q}}(n)$ [5] [14].

For any Young diagram λ and Young tableau L, we define their conjugate by flipping along the diagonal and denote them λ^{tr} and L^{tr} respectively. The conjugate of any Young diagram remains a Young diagram, and conjugation preserves the standardness of Young tableaux. Suppose we have an l-multipartition $\lambda = (\lambda^0, \dots, \lambda^{n-1})$. Define the conjugate λ^{tr} to be $((\lambda^0)^{tr}, \dots, (\lambda^{l-1})^{tr})$. For a standard Young tableau L of shape λ, we define the conjugate of L as $((L^0)^{tr}, \dots, (L^{l-1})^{tr})$ and denote it by L^{tr}. The conjugate L^{tr} is of shape λ^{tr}. It is easy to see that

$$\beta(L(i)) = \beta(L^{tr}(i)), \tag{6.69}$$

$$\mathrm{ct}(L(i)) = -\mathrm{ct}(L^{tr}(i)), \tag{6.70}$$

for any $i = 1, \ldots, n$. Then we have our crucial result as in Theorem 0.1 as follows.

Theorem 6.10. Let $H_{q,\mathbf{Q}}(n)$ be the cyclotomic Hecke algebra with parameters $q = b(-c_0)$ and $Q_i = b(-h_i + \frac{i}{l})$, $i = 0, \cdots, l-1$. There is an isomorphism $\bigoplus_{\mathbf{d}\in D_0^n} W_{\mathbf{a_d}^+,\mathbf{1}}(M_c(\lambda)) \cong V^{\lambda^{tr}}$ as left modules of $H_{q,\mathbf{Q}}(n)$.

Proof. Based on the rules defined in Proposition 6.9, the cyclotomic Hecke algebra acts on the basis v^L of the irreducible representation V^λ by

$$T_0 \cdot v^L = \mathrm{ct}_q(L(1)) = Q_{\beta(L(1))} q^{\mathrm{ct}(L(1))} = b((-h_{\beta(L(1))} + \frac{\beta(L(1))}{l}))v^L, \tag{6.71}$$

$$\begin{aligned}
&T_i \cdot v^L \\
&= (T_i)_L v^L + (1 + (T_i)_L) v^{s_i\cdot L} \\
&= \frac{q-1}{1-\frac{\mathrm{ct}_q L(i)}{\mathrm{ct}_q L(i+1)}} v^L + (1 + \frac{q-1}{1-\frac{\mathrm{ct}_q L(i)}{\mathrm{ct}_q L(i+1)}}) v^{s_i\cdot L} \\
&= \frac{(q-1)Q_{\beta(L(i+1))}q^{\mathrm{ct}(L(i+1))}}{Q_{\beta(L(i+1))}q^{\mathrm{ct}(L(i+1))} - Q_{\beta(L(i))}q^{\mathrm{ct}(L(i))}} v^L + (1 + \frac{(q-1)Q_{\beta(L(i+1))}q^{\mathrm{ct}(L(i+1))}}{Q_{\beta(L(i+1))}q^{\mathrm{ct}(L(i+1))} - Q_{\beta(L(i))}q^{\mathrm{ct}(L(i))}}) v^{s_i\cdot L} \\
&= \frac{(q-1)b((-h_{\beta(L(i+1))} + \frac{\beta(L(i+1))}{l} + \mathrm{ct}(L(i+1))c_0))}{b((-h_{\beta(L(i+1))} + \frac{\beta(L(i+1))}{l} + \mathrm{ct}(L(i+1))c_0)) - b((-h_{\beta(L(i))} + \frac{\beta(L(i))}{l} + \mathrm{ct}(L(i))c_0))} v^L + \\
&\frac{b((-h_{\beta(L(i))} + \frac{\beta(L(i))}{l} + \mathrm{ct}(L(i))c_0)) - qb((-h_{\beta(L(i+1))} + \frac{\beta(L(i+1))}{l} + \mathrm{ct}(L(i+1))c_0))}{b((-h_{\beta(L(i+1))} + \frac{\beta(L(i+1))}{l} + \mathrm{ct}(L(i+1))c_0)) - b((-h_{\beta(L(i))} + \frac{\beta(L(i))}{l} + \mathrm{ct}(L(i))c_0))} v^{s_i\cdot L} \\
&= \frac{B_i^{L^{tr}}}{C_i^{L^{tr}}} v_{s_i\cdot L} + \frac{(1-q)b((-lh_{\beta(L^{tr}(i+1))} + \beta(L^{tr}(i+1)) - l\mathrm{ct}(L^{tr}(i+1))c_0))}{C_i^{L^{tr}}} v_L.
\end{aligned} \tag{6.72}$$

By comparing the actions of T_i on $\bigoplus_{\mathbf{d}\in D_0^n} W_{\mathbf{a_d}^+,\mathbf{1}}(M_c(\lambda))$ in (6.60) and (6.64), with the actions on $V^{\lambda^{tr}}$ in (6.71) and (6.72), it is clear that there is an isomorphism between the left modules $\bigoplus_{\mathbf{d}\in D_0^n} W_{\mathbf{a_d}^+,\mathbf{1}}(M_c(\lambda))$ and $V^{\lambda^{tr}}$ of $H_{q,\mathbf{Q}}(n)$ by sending $v_L \mapsto v^{L^{tr}}$ and extending it linearly. □

Remark. Corollary 6.11 in [10] states that if c_0 is a positive real number, then $\mathrm{KZ}(M_c(\lambda))$ is isomorphic to the Specht module S^λ corresponding to λ defined in Definition 3.28 in [5]. However the result of Theorem 6.10 does not lead to a contradiction. It is because that the irreducible module V^λ is not necessarily S^λ, assuming the parameters are generic. For example, if we consider the irreducible representation $V^{(n)}$ of the cyclotomic Hecke algebra of type A, then we always have

$$\frac{\mathrm{ct}_q(L(i))}{\mathrm{ct}_q(L(i+1))} = q^{-1}, \text{ for all } i = 1, \dots, n-1. \tag{6.73}$$

Hence the generator T_i acts on the only basis element v^L by

$$T_i \cdot v^L = (T_i)_L v^L = q v^L, \text{ for all } i = 1, \dots, n-1. \tag{6.74}$$

It is not the trivial representation, but the sign representation, which corresponds to the Specht module $S^{(1,\dots,1)}$.

Example 6.11. Let W be the cyclic group $\mathbb{Z}_l$, with irreducible representations ρ_i defined in Section 3.2. Each ρ_i corresponds to the l-multipartition that is an l-tuple of partition with empty partition everywhere except the i-th coordinate. It is one dimensional and generated by $\{v^i_L \mid L \in \mathrm{SYT}(\lambda)\}$. By Theorem 6.10, we have

$$\mathrm{KZ}(M_c(\rho_i)) \cong V^{\rho_i^{tr}} = V^{\rho_i}.$$

By the definition of the representation of V^{ρ_i}, the action of T_0 on v^i_L is

$$T_0 \cdot v^i_L = Q_i v^i_L = b(-h_j + \frac{j}{l}) v^i_L .$$

Recall the definition of a_i in Section 3.2 and we get $a_i = -lh_i$. The action of T_0 becomes

$$T_0 \cdot v^i_L = b(\frac{a_i}{l} + \frac{i}{l}) v^i_L .$$

The cyclotomic Hecke algebra constructed in Section 3.2 is the algebra $H_{q,\mathbf{Q}}(n)$ from Section 6.1.2. Since we obtained the cyclotomic Hecke algebra $H_{q,\mathbf{Q}}(n)$ by sending the generators T_0 of $H_{\mathbf{q}}(W)$ to T_0^{-1} in Lemma 6.2, the generator T_0 in $H_{\mathbf{q}}(W)$ acts by $b(-(\frac{a_i}{l} + \frac{i}{l}))$. The result coincides with Section 3.2.

7 Examples on KZ Functor Preserving Unitarity

In this section, we will warm up and provide some evidence supporting that the KZ functor preserves unitarity of representations of the rational Cherednik algebra. In the appendix of [9], Griffeth discussed the condition for a representation of the rational Cherednik algebra of type A to be unitary. Stoica in [19] proved that when we restrict the value of the function c to be between $(-\frac{1}{2}, \frac{1}{2}]$ for the rational Cherednik algebra of type A, $M_c(\lambda)$ is unitary if and only if $\mathrm{KZ}(M_c(\lambda))$ is unitary. For any $n \geq 2$, the module $M_c(\lambda)$ of $H_c(G(l,1,n), \mathbb{C}^n)$ is not unitary outside the interval $(-\frac{1}{2}, \frac{1}{2}]$ [9]. We will not discuss the type A case further here, but instead focus on other cases, such as the type B or when W is a cyclic group. In order to use the result from Secion 6.2, we assume all parameters are generic. This section is my original work.

7.1 Unitarity Condition of Representations of Cyclotomic Hecke Algebra

We would like to compare the unitary condition on Verma modules $M_c(\lambda)$ of the rational Cherednik algebra and representations $V^{\lambda^{tr}}$ of the cyclotomic Hecke algebra. First let us determine the unitarity condition of the irreducible representation V^λ in this section. Let $H_{q,\mathbf{Q}}(n)$ be the cyclotomic Hecke algebra with generic parameters q, $\mathbf{Q} = (Q_0, \ldots, Q_{l-1})$, and V^λ the irreducible representation defined in Propostion 6.9 with basis $\{v^L \mid L \in \mathrm{SYT}(\lambda)\}$.

Definition 7.1. Let $\star$-operation be the star operation on the cyclotomic Hecke algebra sending X_i to X_i^{-1} and T_i to T_i^{-1}. We say that a representation V of $H_{q,\mathbf{Q}}(n)$ is unitary if it is equipped with a contravariant Hermitian form $\langle \cdot, \cdot \rangle$ with respect to the $\star$-operation, i.e

$$\langle h \cdot v, v' \rangle = \langle v, h^\star \cdot v' \rangle\,, \tag{7.1}$$

for all $v, v' \in V$ and $h \in H_{q,\mathbf{Q}}(n)$, and it is positive definite.

According to to Stoica [20], for any q on the unit circle, the positive-definite contravariant Hermitian form of V^λ is unique up to a rescaling by a positive real number. It suffices to find a positive-definite Hermitian form $\langle \cdot, \cdot \rangle$ such that

$$\langle X_i \cdot v^L, v^{L'} \rangle = \langle v^L, X_i^{-1} \cdot v^{L'} \rangle\,, \tag{7.2}$$

$$\langle T_i \cdot v^L, v^{L'} \rangle = \langle v^L, T_i^{-1} \cdot v^{L'} \rangle\,, \tag{7.3}$$

for all $L, L' \in \mathrm{SYT}(\lambda)$, for all $i = 1, \ldots, n$. The definition of the irreducible V^λ does not specify how X_i acts on the basis for $i > 1$. However from Equation 2.6 and Proposition 2.7 in [14], we know the action of X_i on the basis element v^L is

$$X_i \cdot v^L = \mathrm{ct}_q(L(i)) v^L, \tag{7.4}$$

for any standard Young tableau L of λ, where the function ct_q is defined in (6.65). Moreover, it is easy to see that

$$X_i^{-1} \cdot v^L = \mathrm{ct}_q(L(i))^{-1} v^L. \tag{7.5}$$

Substituting the action of $X_i^{\pm}$ in (7.2), we get

$$\begin{aligned} &\langle \mathrm{ct}_q(L(i)) v^L, v^{L'} \rangle = \langle v^L, \mathrm{ct}_q(L'(i))^{-1} v^{L'} \rangle \\ \Leftrightarrow &\mathrm{ct}_q(L(i)) \langle v^L, v^{L'} \rangle = \mathrm{ct}_q(L'(i)) \langle v^L, v^{L'} \rangle. \end{aligned} \tag{7.6}$$

If the tableaux L and L' are the same, then the equation is automatically satisfied. If not, then that there exists an $i \in \{1, \ldots, n\}$ such that $\mathrm{ct}_q(L(i)) \neq \mathrm{ct}_q(L'(i))$, which implies $\langle v^L, v^{L'} \rangle = 0$. Hence the Hermitian form is orthogonal. We apply the actions of T_i to (7.3) and the get

$$\begin{aligned} &\langle T_i \cdot v^L, v^{L'} \rangle = \langle v^L, T_i^{-1} \cdot v^{L'} \rangle \\ \Leftrightarrow &\langle (T_i)_L v^L + (1 + (T_i)_L) v^{s_i \cdot L}, v^{L'} \rangle = \langle v^L, (\frac{1}{q}(T_i)_{L'} + \frac{1}{q} - 1) v^{L'} + \frac{1}{q}(1 + (T_i)_{L'}) v^{s_i \cdot L'} \rangle. \end{aligned} \tag{7.7}$$

If L and L' are the same tableaux, the equation implies that $(T_i)_L = \overline{(\frac{1}{q}(T_i)_L + \frac{1}{q} - 1)}$, which is trivial by the definition of $(T_i)_L$. If L and L' are different and $s_i \cdot L$ is not standard, then both sides of the equation are zero. If $s_i \cdot L$ is standard, then both sides of the equation are zero unless $L' = s_i \cdot L$. Equation 7.7 becomes

$$\begin{aligned} &\langle (T_i)_L v^L + (1 + (T_i)_L) v^{L'}, v^{L'} \rangle = \langle v^L, (\frac{1}{q}(T_i)_{L'} + \frac{1}{q} - 1) v^{L'} + \frac{1}{q}(1 + (T_i)_{L'}) v^L \rangle \\ \Leftrightarrow &(1 + (T_i)_L) \langle v^{L'}, v^{L'} \rangle = q\overline{(1 + (T_i)_{L'})} \langle v^L, v^L \rangle \\ \Leftrightarrow &(1 + (T_i)_L)(q\overline{(1 + (T_i)_{L'})})^{-1} = \frac{\langle v^L, v^L \rangle}{\langle v^{L'}, v^{L'} \rangle}. \end{aligned} \tag{7.8}$$

If we have $\langle v^L, v^L \rangle = 0$ for some L, then $\langle v^L, v^L \rangle = 0$ holds for all L by (7.8), since any standard Young tableau L' can be obtained by applying series of transpositions on L. Let us assume that $\langle v^L, v^L \rangle \neq 0$. The Hermitian form $\langle \cdot, \cdot \rangle$ is positive-definite if and only if $\langle v^L, v^L \rangle > 0$ for all $L \in \mathrm{SYT}(\lambda)$, or equivalently $(1 + (T_i)_L)(q\overline{(1 + (T_i)_{s_i \cdot L})})^{-1} > 0$ and $\langle v^L, v^L \rangle > 0$ for all $L \in \mathrm{SYT}(\lambda)$ such that $s_i \cdot L \neq 0$.

We want to discover what condition the above inequality $(1 + (T_i)_L)(q\overline{(1 + (T_i)_{s_i \cdot L})})^{-1} > 0$ gives us. Let us define $d_i^L = \mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))$ and $P_i^L = \frac{Q_{\beta(L(i+1))}}{Q_{\beta(L(i))}}$. Then $(T_i)_L$ becomes

$$\frac{q-1}{1 - (P_i^L)^{-1} q^{d_i^L}} \, .$$

Then we have

$$\begin{aligned}
&(1 + (T_i)_L)(q\overline{(1 + (T_i)_{s_i \cdot L})})^{-1} > 0 \\
\Leftrightarrow &\left(1 + \frac{q-1}{1 - (P_i^L)^{-1} q^{d_i^L}}\right) \left(\overline{q\left(1 + \frac{q-1}{1 - (P_i^{s_i \cdot L})^{-1} q^{d_i^{s_i \cdot L}}}\right)}\right)^{-1} > 0 \\
\Leftrightarrow &\left(1 + \frac{q-1}{1 - (P_i^L)^{-1} q^{d_i^L}}\right) \left(\overline{q\left(1 + \frac{q-1}{1 - (P_i^L) q^{-d_i^L}}\right)}\right)^{-1} > 0 \\
\Leftrightarrow &\frac{q - (P_i^L)^{-1} q^{d_i^L}}{1 - (P_i^L)^{-1} q^{d_i^L}} \left(\frac{1 - q(P_i^L)^{-1} q^{d_i^L}}{1 - (P_i^L)^{-1} q^{d_i^L}}\right)^{-1} > 0 \\
\Leftrightarrow &\frac{qP_i^L - q^{d_i^L}}{P_i^L - q^{d_i^L + 1}} > 0 \, .
\end{aligned} \tag{7.9}$$

The fraction $\frac{qP_i^L - q^{d_i^L}}{P_i^L - q^{d_i^L+1}}$ is a positive real number, and so is its complex conjugate. The sum of them is still a positive real number. Hence we have the following inequality.

$$\begin{aligned}
&\frac{qP_i^L - q^{d_i^L}}{P_i^L - q^{d_i^L+1}} + \overline{\left(\frac{qP_i^L - q^{d_i^L}}{P_i^L - q^{d_i^L+1}}\right)} > 0 \\
\Leftrightarrow &\frac{qP_i^L - q^{d_i^L}}{P_i^L - q^{d_i^L+1}} + \frac{q^{-1}(P_i^L)^{-1} - q^{-d_i^L}}{(P_i^L)^{-1} - q^{-(d_i^L+1)}} > 0 \\
\Leftrightarrow &\frac{2q + 2q^{-1} - 2q^{-d_i^L} P_i^L - 2q^{d_i^L} {P_i^L}^{-1}}{2 - q^{-(d_i^L+1)} P_i^L - q^{d_i^L+1} (P_i^L)^{-1}} > 0
\end{aligned} \tag{7.10}$$

We recall that q is defined to be $q = b(-c_0) = \exp\bigl(-2\pi\sqrt{-1}c_0\bigr)$, which implies $q + q^{-1} = \cos(-2\pi c_0) =$

$\cos(2\pi c_0)$. Similarly, by the definition of d_i^L, P_i^L and Q_j, we have

$$\begin{aligned}
&q^{-d_i^L}P_i^L + q^{d_i^L}P_i^{L-1} \\
&= \cos\left(2\pi\left(c_0(\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))) + h_{\beta(L(i))} - \frac{\beta(L(i))}{l} - h_{\beta(L(i+1))} + \frac{\beta(L(i+1))}{l}\right)\right), \quad (7.11)\\
&q^{-(d_i^L+1)}P_i^L + q^{d_i^L+1}(P_i^L)^{-1} \\
&= \cos\left(2\pi\left(c_0(\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1)) + 1) + h_{\beta(L(i))} - \frac{\beta(L(i))}{l} - h_{\beta(L(i+1))} + \frac{\beta(L(i+1))}{l}\right)\right). \quad (7.12)
\end{aligned}$$

Hence the inequality becomes

$$\frac{2\cos(2\pi c_0) - 2\cos\left(2\pi\left(c_0(\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))) + h_{\beta(L(i))} - \frac{\beta(L(i))}{l} - h_{\beta(L(i+1))} + \frac{\beta(L(i+1))}{l}\right)\right)}{2 - 2\cos\left(2\pi\left(c_0(\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1)) + 1) + h_{\beta(L(i))} - \frac{\beta(L(i))}{l} - h_{\beta(L(i+1))} + \frac{\beta(L(i+1))}{l}\right)\right)} > 0 \quad (7.13)$$

The denominator is larger than or equal to zero. However it is never zero, because the inequality (7.13) is equivalent with (7.8) and we assume that $\langle v^L, v^L\rangle \neq 0$. The numerator can be transformed into

$$\begin{aligned}
&-4\sin\left(\pi\left(c_0 + c_0(\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))) + h_{\beta(L(i))} - \frac{\beta(L(i))}{l} - h_{\beta(L(i+1))} + \frac{\beta(L(i+1))}{l}\right)\right)\\
&\cdot\sin\left(\pi\left(c_0 - c_0(\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))) - h_{\beta(L(i))} + \frac{\beta(L(i))}{l} + h_{\beta(L(i+1))} - \frac{\beta(L(i+1))}{l}\right)\right) \quad (7.14)
\end{aligned}$$

It is positive if the two sine factors have opposite signs. Hence we have the following proposition.

Proposition 7.2. Let $q, \mathbf{Q} = (Q_0, \dots Q_{l-1})$ be the generic parameters of the cyclotomic Hecke algebra $H_{q,\mathbf{Q}}(n)$, and q lie on the unit circle. Then the irreducible representation V^λ is unitary if and only if

$$\begin{aligned}
&\sin\left(\pi\left(c_0 + c_0(\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))) + h_{\beta(L(i))} - \frac{\beta(L(i))}{l} - h_{\beta(L(i+1))} + \frac{\beta(L(i+1))}{l}\right)\right)\\
&\cdot\sin\left(\pi\left(c_0 - c_0(\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))) - h_{\beta(L(i))} + \frac{\beta(L(i))}{l} + h_{\beta(L(i+1))} - \frac{\beta(L(i+1))}{l}\right)\right) < 0, \quad (7.15)
\end{aligned}$$

for all $i = 1, \dots, n-1$ and $L \in \mathrm{SYT}$ such that $s_i \cdot L$ is a standard Young tableau of λ.

7.2 Example of $W = G(l, 1, 1)$

Consider the simplest case where W is the cyclic group $\mathbb{Z}_l$. The irreducible representations of the cyclotomic Hecke algebra of $G(l, 1, 1)$ are all one-dimensional, hence unitary automatically. However, based on Example 1.20, a certain condition needs to be met to have unitary Verma modules of $H_c(G(l, 1, 1), \mathbb{C})$. As a consequence, KZ functor preserves the unitarity, that is if $M_c(\lambda)$ is unitary, then so is $\mathrm{KZ}(M_c(\lambda))$. However the converse is not necessarily true.

7.3 Example of Type B

In the case of type B, the underlying group in the rational Cherednik algebra is $G(2, 1, n)$. Griffith discussed the unitarity condition for a Verma module in Section 8.2 of [13]. For the 2-multipartition $\lambda = (\lambda^0, \lambda^1)$, we define

1. $b_1,\ b_1'$ to be the box of λ^0 and λ^1 with the largest content respectively,
2. $b_4,\ b_4'$ to be the box of λ^0 and λ^1 with the smallest content respectively,

Then the unitarity condition of $M_c(\lambda)$ for generic parameters can be stated in the following proposition, combining the results from Corollary 8.4 and Corollary 8.5 in [13].

Proposition 7.3. Let $H_c(G(2, 1, n), \mathbb{C}^n)$ be the rational Cherednik algebra of type B with generic parameters such that $c_0 > 0$. Recall the construction of $h_i, i = 0, ..., l-1$ in (2.9).

- If $\lambda = ((1^n), \emptyset)$, then $M_c(\lambda)$ is unitary if and only if $h_1 \geq -\frac{1}{2}$.
- If $\lambda = (\lambda^0, \emptyset)$ or $(\emptyset, \lambda^1)$ for some $\lambda^0 \neq (1^n)$, then $M_c(\lambda)$ is unitary if and only if

$$c_0 \leq \frac{1}{\mathrm{ct}(b_1) - \mathrm{ct}(b_4) + 1} \text{ and } -h_1 + \mathrm{ct}(b_1)c_0 \leq \frac{1}{2}, \tag{7.16}$$

or

$$c_0 \leq \frac{1}{\mathrm{ct}(b_1') - \mathrm{ct}(b_4') + 1} \text{ and } -h_1 + \mathrm{ct}(b_1)c_0 \leq \frac{1}{2}, \tag{7.17}$$

respectively.

- If $\lambda = (\lambda^0, \lambda^1)$ where neither λ^0 nor λ^1 is $\emptyset$, then the condition of $M_c(\lambda)$ being unitary is

$$-h_1 + (\mathrm{ct}(b_1) - \mathrm{ct}(b_4') + 1)c_0 \leq \frac{1}{2}, \tag{7.18}$$
$$-h_1 + (\mathrm{ct}(b_4) - \mathrm{ct}(b_1') - 1)c_0 \leq \frac{1}{2}, \tag{7.19}$$
$$h_1 + (\mathrm{ct}(b_1') - \mathrm{ct}(b_4) + 1)c_0 \leq \frac{1}{2}, \tag{7.20}$$
$$h_1 + (\mathrm{ct}(b_4') - \mathrm{ct}(b_1) - 1)c_0 \leq \frac{1}{2}. \tag{7.21}$$

The KZ functor maps $M_c(\lambda)$ to $V^{\lambda^{tr}}$. In the case of type B, we can show that the KZ functor preserves unitarity between these two representations.

Proposition 7.4. Let $H_c(G(2,1,n), \mathbb{C}^n)$ be the rational Cherednik algebra of type B with generic parameters such that $c_0 > 0$. For any 2-multiparition λ of n, $M_c(\lambda)$ is a Verma module of $H_c(G(2,1,n), \mathbb{C}^n)$. Let $H_{q,\mathbf{Q}}(n)$ be the cyclotomic Hecke algebra whose representations V^λ are images of $M_c(\lambda)$ under the KZ functor for all λ. Then $V^{\lambda^{tr}}$ is unitary if $M_c(\lambda)$ is unitary.

Proof. We will prove it by dividing it into three cases.

1. If $\lambda = ((1^n), \emptyset)$, then $V^{\lambda^{tr}}$ is one dimensional, hence always unitary.
2. If $\lambda = (\lambda^0, \emptyset)$ for $\lambda^0 \neq (1^n)$, and $c_0 > 0$, then $\beta(L^{tr}(i)) = \beta(L^{tr}(i+1))$. For any standard Young tableau L of shape λ such that $s_i \cdot L$ is standard, Equation (7.15) becomes

$$\begin{aligned} &\sin\left(\pi\left(c_0 + c_0\left(\mathrm{ct}(L^{tr}(i)) - \mathrm{ct}(L^{tr}(i+1))\right)\right)\right)\sin\left(\pi\left(c_0 - c_0\left(\mathrm{ct}(L^{tr}(i)) - \mathrm{ct}(L^{tr}(i+1))\right)\right)\right) < 0 \\ \Leftrightarrow &\sin\left(\pi\left(c_0 + c_0\left(\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))\right)\right)\right)\sin\left(\pi\left(c_0 - c_0\left(\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))\right)\right)\right) < 0\,. \end{aligned} \tag{7.22}$$

   Assume that $M_c(\lambda)$ satisfies the unitary condition in Proposition 7.3. Because $c_0 > 0$, we have the inequality

$$c_0 + c_0(\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))) \leq c_0(\mathrm{ct}(b_1) - \mathrm{ct}(b_4) + 1) \leq 1. \tag{7.23}$$

It implies the inequality

$$c_0 - c_0(\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))) \le c_0 + c_0(\mathrm{ct}(L(i+1)) - \mathrm{ct}(L(i))) \le c_0(\mathrm{ct}(b_1) - \mathrm{ct}(b_4) + 1) \le 1 \tag{7.24}$$

holds. Moreover, if we relax the restriction on c_0 from $c_0 \le \frac{1}{\mathrm{ct}(b_1)-\mathrm{ct}(b_4)+1}$ to $c_0 \le \frac{1}{\mathrm{ct}(b_1)-\mathrm{ct}(b_4)-1}$, we get the inequalities

$$c_0 + c_0(\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))) \ge c_0(\mathrm{ct}(b_4) - \mathrm{ct}(b_1) + 1) \ge -1\,, \tag{7.25}$$

$$c_0 - c_0(\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))) \ge c_0(\mathrm{ct}(b_4) - \mathrm{ct}(b_1) + 1) \ge -1\,. \tag{7.26}$$

Suppose we have the inequality $\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1)) \ge 0$. It implies that $\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1)) \ge 1$, because $L(i)$ and $L(i+1)$ are in the same tableau and they cannot have the same content. Then we have the inequality

$$c_0 - c_0(\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))) \le 0. \tag{7.27}$$

Multiplying π on inequalities (7.24), (7.25), (7.26) and (7.27) and rearranging terms, we get

$$-\pi \le \pi\left(c_0 + c_0(\mathrm{ct}(L^{tr}(i)) - \mathrm{ct}(L^{tr}(i+1)))\right) \le 0 \tag{7.28}$$

$$0 \le \pi\left(c_0 - c_0(\mathrm{ct}(L^{tr}(i)) - \mathrm{ct}(L^{tr}(i+1)))\right) \le \pi. \tag{7.29}$$

Equalities cannot occur due to the facts that $s_i \cdot L$ is standard implies difference between $\mathrm{ct}(L^{tr}(i))$ and $\mathrm{ct}(L^{tr}(i+1))$ is larger than one. Within this range of c_0, the inequality (7.22) holds, and hence $V^{\lambda^{tr}}$ is unitary.

Likewise, when $\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))$ is less than zero, we have

$$c_0 + c_0(\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))) \le 0. \tag{7.30}$$

After multiplying π, inequalities (7.24), (7.25), (7.26) and (7.27) give us

$$-\pi < \pi\left(c_0 - c_0(\mathrm{ct}(L^{tr}(i)) - \mathrm{ct}(L^{tr}(i+1)))\right) < 0 \tag{7.31}$$

$$0 < \pi \left(c_0 + c_0(\mathrm{ct}(L^{tr}(i)) - \mathrm{ct}(L^{tr}(i+1)))\right) < \pi, \tag{7.32}$$

under which the irreducible representation $V^{\lambda^{tr}}$ is unitary. The same argument applies to $\lambda = (\emptyset, \lambda^1)$.

3. Suppose $\lambda = (\lambda^0, \lambda^1)$ where neither of $\lambda^i,\ i = 0, 1$ is empty. Equations (7.18) to (7.21) imply that

$$-\frac{1}{2} - h_1 \leq (\mathrm{ct}(b_1) - \mathrm{ct}(b_4') + 1)c_0 \leq \frac{1}{2} + h_1 \text{ and} \tag{7.33}$$

$$-\frac{1}{2} + h_1 \leq (\mathrm{ct}(b_1') - \mathrm{ct}(b_4) + 1)c_0 \leq \frac{1}{2} - h_1. \tag{7.34}$$

Adding them together, we get

$$\begin{aligned} & -1 \leq (\mathrm{ct}(b_1) - \mathrm{ct}(b_4') + 1)c_0 - (\mathrm{ct}(b_1') - \mathrm{ct}(b_4) + 1)c_0 \leq 1 \\ \Leftrightarrow & -1 \leq (\mathrm{ct}(b_1) - \mathrm{ct}(b_4) + 1)c_0 + (\mathrm{ct}(b_1') - \mathrm{ct}(b_4') + 1)c_0 \leq 1\,. \end{aligned} \tag{7.35}$$

Since both $\mathrm{ct}(b_1) - \mathrm{ct}(b_4) + 1$ and $\mathrm{ct}(b_1') - \mathrm{ct}(b_4') + 1$ are larger than zero, and $c_0 > 0$, the two summands $\mathrm{ct}(b_1) - \mathrm{ct}(b_4) + 1)c_0$ and $(\mathrm{ct}(b_1') - \mathrm{ct}(b_4') + 1)c_0$ in (7.35) are in the range $(0, 1)$. Hence it guarantees that if $L^{tr}(i)$ and $L^{tr}(i+1)$ are in the same λ^j, then the Hermitian form involving $v_{L^{tr}}$ and $v_{s_i \cdot L^{tr}}$ is positive-definite by the same argument used in the previous case.

Recall that $h_0 = 0$ by definition. If $L(i)$ is in λ^0 and $L(i+1)$ is in λ^1, then the unitarity condition of $V^{\lambda^{tr}}$ (7.15) becomes

$$\begin{aligned} & \sin\left(\pi\left(c_0 + c_0(\mathrm{ct}(L^{tr}(i)) - \mathrm{ct}(L^{tr}(i+1))) - h_1 + \frac{1}{2}\right)\right) \\ & \cdot \sin\left(\pi\left(c_0 - c_0(\mathrm{ct}(L^{tr}(i)) - \mathrm{ct}(L^{tr}(i+1))) + h_1 - \frac{1}{2}\right)\right) < 0 \\ \Leftrightarrow & \sin\left(\pi\left(c_0 - c_0(\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))) - h_1 + \frac{1}{2}\right)\right) \\ & \cdot \sin\left(\pi\left(c_0 + c_0(\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))) + h_1 - \frac{1}{2}\right)\right) < 0\,. \end{aligned} \tag{7.36}$$

By (7.18), we get that

$$\begin{aligned} & c_0 - c_0(\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))) - h_1 + \frac{1}{2} \\ \geq & c_0 - c_0(\mathrm{ct}(b_1') - \mathrm{ct}(b_4)) - h_1 + \frac{1}{2} \geq 0\,. \end{aligned} \tag{7.37}$$

Moreover, (7.20) implies that

$$\begin{aligned} & c_0 - c_0(\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))) - h_1 + \frac{1}{2} \\ & = c_0 + c_0(\mathrm{ct}(L(i+1)) - \mathrm{ct}(L(i))) - h_1 + \frac{1}{2} \\ \leq & c_0 + c_0(\mathrm{ct}(b_1) - \mathrm{ct}(b_4')) - h_1 + \frac{1}{2} \leq 1. \end{aligned} \tag{7.38}$$

Similarly, we have the inequality

$$-1 \leq c_0 - c_0(\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))) + h_1 - \frac{1}{2} \leq 0 \tag{7.39}$$

because of (7.19). Equalities cannot hold, since the boxes $L(i)$ and $l(i+1)$ are interchangeable and their content difference is larger than one. As a result of (7.37), (7.38) and (7.39), the unitarity condition of $V^{\lambda^{tr}}$(7.36) holds. The same argument can be applied for the case where $L(i)$ is in λ^1 and $L(i+1)$ is in λ^0.

In conclusion, under the unitarity condition for $M_c(\lambda)$, the irreducible representation $V^{\lambda^{tr}}$ is unitary.

□

Remark. There is an isomorphism between $H_c(W, \mathfrak{h})$ and $H_{-c}(W, \mathfrak{h})$ that sends elements in $\mathfrak{h}$ and $\mathfrak{h}^*$ to themselves and reflections s to $-s$. The cyclotomic Hecke algebra whose representations are images of modules of $H_{-c}(W, \mathfrak{h})$ under the KZ functor is $H_{q^{-1}, \mathbf{Q}^{-1}}$, which is isomorphic to the original cyclotomic Hecke algebra $H_{q,\mathbf{Q}}$ by mapping the generators to their inverses. Hence Proposition 7.4 is also true for $c_0 < 0$. The result can also be calculated directly from their unitarity conditions.

8 KZ Functor and Unitarity

We are now well-equipped to take on our most important question, which is whether the KZ functor preserves unitarity of the representations of the rational Cherendik algebra of $G(l,1,n)$. We will derive the $*$-operation of the cyclotomic Hecke algebra from the rational Cherednik algebra, and compare it with the $\star$-operation of the cyclotomic Hecke algebra introduced in the previous section. This section is my original work

The rational Cherendik algebra can be parametrized by either c_s or h_j, while the cyclotomic Hecke algebra whose representations are images under the KZ functor is given in terms of h_j. So we need to work with h_j to tackle this problem.

The first step is to derive the condition on h_j equivalent to $c_s = c_s^\dagger$, which is required to define unitarity on the rational Cherendik algebra.

Lemma 8.1. Let us recall that $c_s^\dagger$ is defined to be $\overline{c_{s^{-1}}}$. Then the condition $c_s = c_s^\dagger$ holds for all reflection s if and only if c_0 and h_j are real for all $j = 0, \ldots, l-1$.

Proof. If the equation $c_s = c_s^\dagger$ holds for all $s \in S$, then

$$c_0 = c_{s_{i,j}} = \overline{c_{s_{i,j}^{-1}}} = \overline{c_{s_{i,j}}} = \overline{c_0}, \tag{8.1}$$

which implies that c_0 is real. Moreover, for the other parameters c_j , $j = 1, \ldots, l-1$, we have

$$c_j = c_{\zeta_1^j} = \overline{c_{\zeta_1^{-j}}} = \overline{c_{-j}}. \tag{8.2}$$

By (2.9), h_j can be wrttien as

$$h_j = -\frac{2}{l}\sum_{i=1}^{l-1}\frac{1-\zeta^{ij}}{1-\zeta^{-i}}c_i \tag{8.3}$$

$$= -\frac{1}{l}\sum_{i=1}^{l-1}\frac{1-\zeta^{ij}}{1-\zeta^{-i}}c_i - \frac{1}{l}\sum_{i=1}^{l-1}\frac{1-\zeta^{-ij}}{1-\zeta^{i}}c_{-i} \tag{8.4}$$

$$= -\frac{1}{l}\sum_{i=1}^{l-1}\left(\frac{1-\zeta^{ij}}{1-\zeta^{-i}}c_i + \frac{1-\zeta^{-ij}}{1-\zeta^{i}}c_{-i}\right) \tag{8.5}$$

$$= -\frac{1}{l}\sum_{i=1}^{l-1}\left(\frac{1-\zeta^{ij}}{1-\zeta^{-i}}c_i + \overline{\left(\frac{1-\zeta^{ij}}{1-\zeta^{-i}}c_i\right)}\right). \tag{8.6}$$

The parameter h_j is a sum of a complex number and its conjugate, therefore real.

If h_j are real for all $j = 0, \dots, l-1$, then $k_{H,j}$ are real for all $j = 0, \dots, l-1$ by the fact that $k_{H,j} = h_{l-j}$ in (6.30). The parameters c_i then can be written in terms of $k_{H,j}$ in (6.14)

$$c_i = -\frac{1}{2}\sum_{j=0}^{n_H-1} \zeta^{ij}(k_{H,j+1} - k_{H,j}) \tag{8.7}$$

$$= -\frac{1}{2}\sum_{j=0}^{n_H-1} \overline{\zeta^{-ij}(k_{H,j+1} - k_{H,j})} \tag{8.8}$$

$$= \overline{c_{-i}}. \tag{8.9}$$

Moreover, c_0 is real implies that $c_{s_{ij}} = c^{\dagger}_{s_{ij}}$. That proves the condition $c_s = c^{\dagger}_s$ holds for all reflections s. □

For the rest of the section, we will assume that c_0 and h_j are real. Under this assumption, the eigenvalues of u_i in the Verma module $M_c(\lambda)$, which are $\mu_i - lh_{\beta(w_\mu(i))} - c_0 l\mathrm{ct}(L(w_\mu(i)))$ for some pair $(\mu, L) \in \mathbb{Z}^n_{\geq 0} \times \mathrm{SYT}(\lambda)$, are real. Recall that the $*$-operation on the rational Cherendik algebra in Definition 1.17 sends x_i to y_i, y_i to x_i and s to s^{-1}. If we apply the $*$-operation on u_i, we get

$$\begin{aligned}
u_i^* &= (y_i x_i + c_0 \sum_{1\leq j<i}\sum_{t=0}^{l-1} \zeta_i^t s_{ij}\zeta_i^{-t} - p(\zeta^{-1}\zeta_i) - 1)^* \\
&= x_i^* y_i^* + c_0 \sum_{1\leq j<i}\sum_{t=0}^{l-1} (\zeta_i^{-t})^* s_{ij}^* (\zeta_i^t)^* - \sum_{s=0}^{l-1}\sum_{r=0}^{l-1} (\zeta^{-rs})^* h_r((\zeta^{-1}\zeta_i)^s)^* - 1 \\
&= y_i x_i + c_0 \sum_{1\leq j<i}\sum_{t=0}^{l-1} \zeta_i^t s_{ij}\zeta_i^{-t} - \sum_{s=0}^{l-1}\sum_{r=0}^{l-1} \zeta^{rs} h_r(\zeta\zeta_i^{-1})^s - 1 \\
&= y_i x_i + c_0 \sum_{1\leq j<i}\sum_{t=0}^{l-1} \zeta_i^t s_{ij}\zeta_i^{-t} - \sum_{s=-l+1}^{0}\sum_{r=0}^{l-1} \zeta^{-rs} h_r(\zeta\zeta_i^{-1})^{-s} - 1 \\
&= u_i\,.
\end{aligned} \tag{8.10}$$

Let us recall the projective modules $P_{\mathbf{a},\mathbf{t}}$ defined in (4.46) that represent the weight space functor $W_{\mathbf{a},\mathbf{t}}$. Similar to the construction of idempotents of the cyclotomic Hecke algebra in [4], there is a system $\{e(\mathbf{a},\mathbf{t})\}$ of mutually orthogonal idempotents in the rational Cherednik algebra such that $e(\mathbf{a},\mathbf{t})$ projects on the projective $P_{\mathbf{a},\mathbf{t}}$. We have the following lemma of the $*$-operation acting on the idempotents.

Lemma 8.2. $e(\mathbf{a},\mathbf{t})^* = e(\mathbf{a},\mathbf{t})$.

Proof. Let us consider $H_c(W,\mathfrak{h})$ as the left and right module of the Dunkl-Opdam subalgebra $\mathfrak{t}$ by the left and right multiplication, and denote them by H_l and H_r respectively. We define $P^{\mathbf{a},\mathbf{t}}$ to be the projective limit with respect to the right multiplication, that is

$$P^{\mathbf{a},\mathbf{t}} = \varprojlim_N H_c(W,\mathfrak{h})/((u_i - a_i)^N H_c(W,\mathfrak{h}) + (\zeta_i - t_i)^N H_c(W,\mathfrak{h})), \tag{8.11}$$

for $\mathbf{a} = (a_1,\ldots,a_n) \in \mathbb{R}^n$ and $\mathbf{t} = (t_1,\ldots,t_n) \in \mu_l(\mathbb{C})$. For any $v \in P_{\mathbf{a},\mathbf{t}}$, we have

$$(u_i - a_i)^N \cdot v = 0, \tag{8.12}$$

for large enough N. Since it is an equation of multiplication in the rational Cherednik algebra, we can apply the $*$-operation to get

$$\begin{aligned} &((u_i - a_i)^N \cdot v)^* = 0 \\ \Leftrightarrow &v^*(u_i - a_i)^N = 0. \end{aligned} \tag{8.13}$$

The same argument applies to $(\zeta_i - t_i)^N \cdot v = 0$. It implies that $P^*_{\mathbf{a},\mathbf{t}} \subset P^{\mathbf{a},\mathbf{t}}$. Similarly, we have $P^{\mathbf{a},\mathbf{t}*} \subset P_{\mathbf{a},\mathbf{t}}$. Hence the two subsets $P^*_{\mathbf{a},\mathbf{t}}$ and $P^{\mathbf{a},\mathbf{t}}$ are equal. If we denote the idempotents projecting onto $P^{\mathbf{a},\mathbf{t}}$ in H_r by $\tilde{e}(\mathbf{a},\mathbf{t})$, then we have $H_l e(\mathbf{a},\mathbf{t})^* = H_r \tilde{e}(\mathbf{a},\mathbf{t})$. Hence $e(\mathbf{a},\mathbf{t})^* = \tilde{e}(\mathbf{a},\mathbf{t})$

Claim. There is an anti-involution a of $H_c(W,\mathfrak{h})$ that preserves the Dunkl-Opdam subalgebra by mapping x to $T(x)$, y to $T^{-1}(y)$ and s to s for any $x \in \mathfrak{h}^*,\ y \in \mathfrak{h},\ s \in \{s_1,\ldots,s_{n-1}\} \cup \{\zeta_1,\ldots,\zeta_n\}$.

Proof. It suffices to check that the anti-involution a preserves the generating relations and group actions. It is trivial to show (1.6). It is obvious that (1.7) is equivalent to (2.5) and (2.6). When applying a both (2.5) and (2.6), we get

$$\begin{aligned} a([y_i,x_j]) &= \sum_{t=0}^{l-1} c_0 \zeta^{-t} a(\zeta_i^t s_{ij} \zeta_i^{-t}) \\ \Leftrightarrow [y_j,x_i] &= \sum_{t=0}^{l-1} c_0 \zeta^{-t} \zeta_i^{-t} s_{ij} \zeta_i^t = \sum_{t=0}^{l-1} c_0 \zeta^{-t} \zeta_j^t s_{ij} \zeta_j^{-t} \qquad \text{for } i \neq j; \end{aligned} \tag{8.14}$$

$$a([y_i,x_i]) = 1 - \sum_{t=0}^{l-1}\sum_{k\neq i} c_0 a(\zeta_i^t s_{ki}\zeta_i^{-t}) - \sum_{t=1}^{l-1} 2c_t a(\zeta_i^t)$$
$$\Leftrightarrow [y_i,x_i] = 1 - \sum_{t=0}^{l-1}\sum_{k\neq i} c_0 \zeta_i^{-t} s_{ki}\zeta_i^{t} - \sum_{t=1}^{l-1} 2c_t \zeta_i^t\,. \tag{8.15}$$

We get the same generating relation as before. For the group action, we need to make sure that the anti-involution a preserve the equality of $sy = s(y)s$ and $sx = s(x)s$, for all reflection s, $y \in \mathfrak{h}$ and $x \in \mathfrak{h}^*$. It is enough to check if it is true for the generators s_i for $i = 1, \ldots, n-1$ and ζ_i for $i = 1, \ldots, n$. We apply the anti-involution a, then all the possible situations hold as follows

$$a(s_i y_j) = x_j s_i = s_i x_j = a(y_j s_i), \text{ for } i \neq j, j+1 \tag{8.16}$$
$$a(s_i x_j) = y_j s_i = s_i y_j = a(x_j s_i), \text{ for } i \neq j, j+1 \tag{8.17}$$
$$a(s_i y_i) = x_i s_i = s_i x_{i+1} = a(y_{i+1} s_i) \tag{8.18}$$
$$a(s_i y_i) = x_i s_i = s_i x_{i+1} = a(y_{i+1} s_i) \tag{8.19}$$
$$a(s_i y_{i+1}) = x_{i+1} s_i = s_i x_i = a(y_i s_i) \tag{8.20}$$
$$a(s_i x_{i+1}) = y_{i+1} s_i = s_i y_i = a(x_i s_i) \tag{8.21}$$
$$a(\zeta_i y_j) = x_j \zeta_i = \zeta_i x_j = a(y_j \zeta_i), \text{ for } i \neq j \tag{8.22}$$
$$a(\zeta_i x_j) = y_j \zeta_i = \zeta_i y_j = a(x_j \zeta_i), \text{ for } i \neq j \tag{8.23}$$
$$a(\zeta_i y_i) = x_i \zeta_i = \zeta\zeta_i x_i = a(\zeta y_i \zeta) \tag{8.24}$$
$$a(\zeta_i x_i) = y_i \zeta_i = \zeta^{-1}\zeta_i y_i = a(\zeta^{-1} x_i \zeta). \tag{8.25}$$

When we apply the anti-involution a on the generators of Dunkl-Opdam subalgebra, the equation $a(\zeta_i) = \zeta_i$ holds by the definition of a, and

$$\begin{aligned} a(z_i) &= a\left(y_i x_i + \sum_{1\leq j<i}\sum_{t=0}^{l-1} \zeta_i^t s_{ij}\zeta_i^{-t}\right) \\ &= a(x_i)a(y_i) + \sum_{1\leq j<i}\sum_{t=0}^{l-1} a(\zeta_i^{-t})a(s_{ij})a(\zeta_i^t) \\ &= y_i x_i + \sum_{1\leq j<i}\sum_{t=0}^{l-1} \zeta_i^{-t} s_{ij}\zeta_i^t) \\ &= z_i. \end{aligned} \tag{8.26}$$

Because the Dunkl-Opdam subalgebra is abelian, the anti-involution preserving its generators implies that the algebra itself is also preserved under a. □

Since the Dunkl Opdam subalgebra is commutative, we can consider both H_l and H_r as its bi-modules. Then The involution a gives an isomorphism between these modules by

$$a(t \cdot v) = a(v) \cdot t. \tag{8.27}$$

The idempotents $e(\mathbf{a}, \mathbf{t})$ and $\tilde{e}(\mathbf{a}, \mathbf{t})$ come from the same vector space and they both project onto the generalized eigenspaces with the same eigenvalues. Hence we have $e(\mathbf{a}, \mathbf{t}) = \tilde{e}(\mathbf{a}, \mathbf{t})$. By the fact that $e(\mathbf{a}, \mathbf{t})^* = \tilde{e}(\mathbf{a}, \mathbf{t})$, the equation $e(\mathbf{a}, \mathbf{t})^* = e(\mathbf{a}, \mathbf{t})$ holds. □

8.1 The $*$-Operation on Weighted KLR Algebra

We are left to trace the $*$-operation throughout the process of applying the KZ functor, from the rational Cherednik algebra to the weighted KLR algebra, then to the KLR algebra and, finally to the cyclotomic Hecke algebra. The $*$-operation on the weighted KLR algebra of quiver D_0 associated with the rational Cherednik algebra $H_c(G(l,1,n), \mathbb{C}^n)$ is derived from the $*$-operation on the rational Cherednik algebra by the action on the projective limit $P_{\mathbf{a},\mathbf{t}}$ in Proposition 4.15. The idempotent $e(\mathbf{a}, \mathbf{t})$ in the completed weighted KLR algebra $\widehat{R}_D$ also projects onto $P_{\mathbf{a},\mathbf{t}}$. Hence by the Lemma 8.2, we have

$$e(\mathbf{a}, \mathbf{t}) = e(\mathbf{a}, \mathbf{t})^* \tag{8.28}$$

in $\widehat{R}_D$. The dot on the i-th strand in the diagram $e(\mathbf{a}, \mathbf{t})$ corresponding to the multiplication by $(u_i - a_i)$ is preserved by the $*$-operation because of (8.10). Let us recall that the intertwining operator σ_i is defined in (2.13). If the eigenvalues of ζ_i and ζ_{i+1} are different, then π_i acts by zero. The action of diagram $\xi(\mathbf{a}, \mathbf{t}, s_i)$ in Proposition 4.15 can be expressed in terms of σ_i as follows

$$\Xi(\xi(\mathbf{a}, \mathbf{t}, s_i)) = \begin{cases} e(\mathbf{a}, \mathbf{t})s_i = e(\mathbf{a}, \mathbf{t})\sigma_i & \text{if } t_i \neq t_{i+1} \\ e(\mathbf{a}, \mathbf{t})\frac{u_i - u_{i+1}}{u_i - u_{i+1} - c_0 l}\sigma_i & \text{if } t_i = t_{i+1} \text{ and } a_i + c_0 l \neq a_{i+1} \neq a_i \\ e(\mathbf{a}, \mathbf{t})(u_i - u_{i+1})\sigma_i & \text{if } t_i = t_{i+1} \text{ and } a_i + c_0 l = a_{i+1} \neq a_i \\ e(\mathbf{a}, \mathbf{t})\frac{1}{u_i - u_{i+1} + c_0 l}(s_i - 1) & \text{if } t_i = t_{i+1} \text{ and } a_i + c_0 l \neq a_{i+1} = a_i. \end{cases} \tag{8.29}$$

The fourth case cannot be written in terms of σ_i, since it is not defined on the eigenspace with $a_i = a_{i+1}$ and $t_i = t_{i+1}$. The diagrams $\xi(\mathbf{a}, \mathbf{t}, \nu^{\pm})$ have already been expressed by the intertwining operators Φ and Ψ via the isomorphism Ξ. Hence the actions of the $*$-operation on all three types of diagrams $\xi(\mathbf{a}, \mathbf{t}, s_i)$ and $\xi(\mathbf{a}, \mathbf{t}, \nu^{\pm})$ depend on σ_i^*, Φ^* and Ψ^*, which can be computed by the lemma below.

Lemma 8.3. Recall the $*$-operation defined in Definition 1.17. Then its actions on intertwining operators are

$$\sigma_i^* = \sigma_i \qquad \forall \quad i = 1, \ldots, n-1\,, \tag{8.30}$$

$$\Phi^* = \Psi\,, \tag{8.31}$$

$$\Psi^* = \Phi\,. \tag{8.32}$$

Proof. The result for σ_i follows directly from its definition in (2.13). For Φ, we have

$$\begin{aligned}\Phi^* &= (x_n s_{n-1} \ldots s_1)^* \\ &= (s_{n-1} \ldots s_1 x_1)^* \\ &= x_1^* s_1^* \ldots s_{n-1}^* \\ &= y_1 s_1 \ldots s_{n_1} \\ &= \Psi\,.\end{aligned} \tag{8.33}$$

Moreover, we know that $(\Phi^*)^* = \Phi$ by the definition of the $*$-operation. It is easy to deduce that $\Psi^* = \Phi$. ◻

Applying the $*$-operation on the diagrams, we get

$$\Xi(\xi(\mathbf{a}, \mathbf{t}, s_i))^* = \begin{cases} s_i e(\mathbf{a}, \mathbf{t}) = \sigma_i e(\mathbf{a}, \mathbf{t}) & \text{if } t_i \neq t_{i+1} \\ \sigma_i \frac{u_i - u_{i+1}}{u_i - u_{i+1} - c_0 l} e(\mathbf{a}, \mathbf{t}) & \text{if } t_i = t_{i+1} \text{ and } a_i + c_0 l \neq a_{i+1} \neq a_i \\ \sigma_i (u_i - u_{i+1}) e(\mathbf{a}, \mathbf{t}) & \text{if } t_i = t_{i+1} \text{ and } a_i + c_0 l = a_{i+1} \neq a_i \\ (s_i - 1) \frac{1}{u_i - u_{i+1} + c_0 l} e(\mathbf{a}, \mathbf{t}) & \text{if } t_i = t_{i+1} \text{ and } a_i + c_0 l \neq a_{i+1} = a_i; \end{cases} \tag{8.34}$$

$$\Xi(\xi(\mathbf{a},\mathbf{t},\nu))^* = \begin{cases} \Phi e(\mathbf{a},\mathbf{t}) & \text{if } a_n = -p(t_n) \\ \frac{1}{u_n+p(t_n)}\Phi e(\mathbf{a},\mathbf{t}) & \text{otherwise;} \end{cases} \tag{8.35}$$

$$\Xi(\xi(\mathbf{a},\mathbf{t},\nu^{-1}))^* = \Psi e(\mathbf{a},\mathbf{t}). \tag{8.36}$$

Due to the injectivity of the map Ξ, after comparing the actions of $*$-operation on the generating diagrams with the generating diagrams we already know, it is not hard to deduce that

$$\xi(\mathbf{a},\mathbf{t},s_i)^* = \begin{cases} \xi(s_i\cdot(\mathbf{a},\mathbf{t}),s_i) & \text{if } t_i \neq t_{i+1} \\ \frac{u_{i+1}-u_i+c_0l}{u_{i+1}-u_i-c_0l}\xi(s_i\cdot(\mathbf{a},\mathbf{t}),s_i) & \text{if } t_i = t_{i+1} \text{ and } a_i \pm c_0l \neq a_{i+1} \neq a_i \\ -\frac{1}{u_{i+1}-u_i-c_0l}\xi(s_i\cdot(\mathbf{a},\mathbf{t}),s_i) & \text{if } t_i = t_{i+1} \text{ and } a_i - c_0l = a_{i+1} \neq a_i \\ (u_{i+1}-u_i+c_0l)\xi(s_i\cdot(\mathbf{a},\mathbf{t}),s_i) & \text{if } t_i = t_{i+1} \text{ and } a_i + c_0l = a_{i+1} \neq a_i \\ \frac{u_i-u_{i+1}+c_0l}{u_{i+1}-u_i+c_0l}\xi(\mathbf{a},\mathbf{t},s_i) + 2\frac{u_i-u_{i+1}+c_0\pi_i}{(u_{i+1}-u_i+c_0l)(u_i-u_{i+1}+c_0l)} & \text{if } t_i = t_{i+1} \text{ and } a_i \pm c_0l \neq a_{i+1} = a_i; \end{cases} \tag{8.37}$$

$$\xi(\mathbf{a},\mathbf{t},\nu)^* = \begin{cases} \xi(\nu\cdot(\mathbf{a},\mathbf{t}),\nu^{-1}) & \text{if } a_n = -p(t_n) \\ \frac{1}{u_n+p(t_n)}\xi(\nu\cdot(\mathbf{a},\mathbf{t}),\nu^{-1}) & \text{otherwise;} \end{cases} \tag{8.38}$$

$$\xi(\mathbf{a},\mathbf{t},\nu^{-1})^* = \begin{cases} \xi(\nu^{-1}\cdot(\mathbf{a},\mathbf{t}),\nu) & \text{if } a_n = -p(t_n) \\ \xi(\nu^{-1}\cdot(\mathbf{a},\mathbf{t}),\nu)(u_n+p(t_n)) & \text{otherwise.} \end{cases} \tag{8.39}$$

8.2 The $*$-Operation on KLR Algebra

The $*$-operation of the completion of the KLR algebra $R_{D_0,n}$ is induced from the one of the completion of the weighted KLR algebra R_{D_0}, as the former is a subalgebra of the latter. The same rules apply to the generators $e(\mathbf{d})$ and y_i of the KLR algebra, that is

$$e(\mathbf{d})^* = e(\mathbf{d}) \quad \text{for any } \mathbf{d} \in D_0^n \tag{8.40}$$

$$\mathrm{y}_i^* = \mathrm{y}_i. \tag{8.41}$$

Using the un-braiding generating relations (5.11), the power of ψ_i can be reduced to one or zero in any element. Hence $(\psi_i e(\mathbf{d}))^*$ can be written as

$$(\psi_i e(\mathbf{d}))^* = e(\mathbf{d}) f_{i,\mathbf{d}}(\mathrm{y})\psi_i + e(\mathbf{d}) g_{i,\mathbf{d}}(\mathrm{y}), \tag{8.42}$$

where $f_{i,\mathbf{d}}(\mathrm{y})$ and $g_{i,\mathbf{d}}(\mathrm{y})$ are power series of variables $\mathrm{y}_1, \dots, \mathrm{y}_n$. Since the domains and the images of diagrams $\xi(\mathbf{a}, \mathbf{t}, s_i)$ and $\xi(\mathbf{a}, \mathbf{t}, \nu^\pm)$ are reversed after applying $*$-operation, by definition so are the ones of $\psi_i e(\mathbf{d})$. The element $(\psi_i e(\mathbf{d}))^*$ maps the weight space $W_{s_i \cdot (\mathbf{a_d}, \mathbf{1})}(M)$ to the weight space $W_{(\mathbf{a_d}, \mathbf{1})}(M)$. Therefore, we have

$$(\psi_i e(\mathbf{d}))^* = e(\mathbf{d}) f_{i,\mathbf{d}}(\mathrm{y})\psi_i, \text{ for } d_i \neq d_{i+1}; \tag{8.43}$$

$$(\psi_i e(\mathbf{d}))^* = e(\mathbf{d}) f_{i,\mathbf{d}}(\mathrm{y})\psi_i + e(\mathbf{d}) g_{i,\mathbf{d}}(\mathrm{y}), \text{ otherwise.} \tag{8.44}$$

The generator $\psi_i e(\mathbf{d})$ is a product of many diagrams of $\xi(\mathbf{a}, \mathbf{t}, s_j)$ and $\xi(\mathbf{a}, \mathbf{t}, \nu^\pm)$ as shown in Figures 2, 3 and 4, so $(\psi_i e(\mathbf{d}))^*$ is determined by the $*$-operation on each component.

Flipping the diagram $\xi(\mathbf{a}, \mathbf{t}, s_j)$ along the horizontal line will give us the diagram $\xi(s_j \cdot (\mathbf{a}, \mathbf{t}), s_j)$ for any $\mathbf{a}$, $\mathbf{t}$ and j. When we flip a product of diagrams, the loadings on the top and the bottom of each diagram are switched, and the order of the diagrams in the product is reversed. After flipping the diagram $\psi_i e(\mathbf{d})$, we get the product of the same diagrams in $(\psi_i e(\mathbf{d}))^*$ with the same order, but with an adjustment of the power series $f_{i,\mathbf{d}}(\mathrm{y})$.

If we flip $\psi_i e(\mathbf{d})$ along the horizontal line, we get a digram of two crossings, one between two strands and the other between a strand and a ghost with the bottom loading $s_i \cdot \mathbf{a_d}$ and the top loading $\mathbf{a_d}$. It is exactly the diagram $\psi_i e(s_i \cdot \mathbf{d})$. Hence the power series $f_{i,\mathbf{d}}(\mathrm{y})$ is solely dependent on the difference between the diagrams $\xi(\mathbf{a}, \mathbf{t}, w)^*$ in the product of $(\psi_i e(\mathbf{d}))^*$ and the diagrams $\xi(w \cdot (\mathbf{a}, \mathbf{t}), w)$ for some $w = s_j, \nu^\pm$.

When we apply the $*$-operation on a product of generating diagrams $\xi(\mathbf{a}, \mathbf{t}, s_j)$ and $\xi(\mathbf{a}, \mathbf{t}, \nu)$, we get a power series of $u_1, \dots, u_n$ from the $*$-operation on each generating diagram in (8.37) to (8.39). We need to commute the power series through the generating diagrams after applying the $*$-operation, which rotates the u's according to the rules:

$$\xi(\mathbf{a}, \mathbf{t}, s_j) u_i = u_i \xi(\mathbf{a}, \mathbf{t}, s_j) \quad \text{if } i \neq j, j+1, \tag{8.45}$$

$$\xi(\mathbf{a},\mathbf{t},s_i)u_i = u_{i+1}\xi(\mathbf{a},\mathbf{t},s_i) \quad \text{if } t_i \neq t_{i+1} \text{ or, } t_i = t_{i+1} \text{ and } a_i \neq a_{i+1}, \tag{8.46}$$

$$\xi(\mathbf{a},\mathbf{t},s_i)u_{i+1} = u_i\xi(\mathbf{a},\mathbf{t},s_i) \quad \text{if } t_i \neq t_{i+1} \text{ or, } t_i = t_{i+1} \text{ and } a_i \neq a_{i+1}, \tag{8.47}$$

$$\xi(\mathbf{a},\mathbf{t},s_i)u_i = u_{i+1}\xi(\mathbf{a},\mathbf{t},s_i) - \frac{u_i - u_{i+1} + c_0\pi_i}{u_i - u_{i+1} + c_0 l} \quad \text{if } t_i = t_{i+1} \text{ and } a_i = a_{i+1}, \tag{8.48}$$

$$\xi(\mathbf{a},\mathbf{t},s_i)u_{i+1} = u_i\xi(\mathbf{a},\mathbf{t},s_i) + \frac{u_i - u_{i+1} + c_0\pi_i}{u_i - u_{i+1} + c_0 l} \quad \text{if } t_i = t_{i+1} \text{ and } a_i = a_{i+1}, \tag{8.49}$$

$$\xi(\mathbf{a},\mathbf{t},\nu)u_i = u_{i+1}\xi(\mathbf{a},\mathbf{t},\nu), \tag{8.50}$$

$$\xi(\mathbf{a},\mathbf{t},\nu^{-1})u_i = u_{i-1}\xi(\mathbf{a},\mathbf{t},\nu^{-1}), \tag{8.51}$$

where the last two equations are deduced from (4.24), and for any $i \notin \{1,\ldots,n\}$, u_i is defined in (4.22). For instance, suppose we start with a product of two diagrams $\xi(\mathbf{a},\mathbf{t},s_j)\xi(\mathbf{a},\mathbf{t},s_{j+1})$ with $t_i = t_{i+1} = t_{i+2}$, $a_i \pm c_0 l \neq a_{i+1} \neq a_i$, and $a_{i+1} + c_0 l = a_{i+2} \neq a_{i+1}$. After applying the $*$-operation, the product becomes

$$(\xi(\mathbf{a},\mathbf{t},s_j)\xi(\mathbf{a},\mathbf{t},s_{j+1}))^* \tag{8.52}$$

$$=(u_{i+2} - u_{i+1} + c_0 l)\xi(s_{i+1}\cdot(\mathbf{a},\mathbf{t}),s_{i+1})\frac{u_{i+1} - u_i + c_0 l}{u_{i+1} - u_i - c_0 l}\xi(s_i\cdot(\mathbf{a},\mathbf{t}),s_i) \tag{8.53}$$

$$=(u_{i+2} - u_{i+1} + c_0 l)\frac{u_i - u_{i+1} + c_0 l}{u_i - u_{i+1} - c_0 l}\xi(s_{i+1}\cdot(\mathbf{a},\mathbf{t}),s_{i+1})\xi(s_i\cdot(\mathbf{a},\mathbf{t}),s_i). \tag{8.54}$$

If there are m generating diagrams in the product, then this needs to be done for all u's, resulting a product of power series in the front of diagram after applying the $*$-operation. According to (8.37), (8.38), and (8.39), the product of power series may include factors of $\frac{u_s - u_r + c_0 l}{u_s - u_r - c_0 l}$, $-\frac{1}{u_s - u_r - c_0 l}$, $(u_s - u_r + c_0 l)$, $\frac{u_s - u_r - c_0 l}{u_s - u_r + c_0 l}$, $\frac{1}{u_s + p(t_s)}$, and $(u_s + p(t_s))$ for some r and s from $\{1,\ldots,n\}$, depending on the diagrams and their loadings. Recall that if there exists an edge $d \to d'$ if $d' = d - c_0$. This gives us the power series $f_{i,\mathbf{d}}$ in the following lemma.

Lemma 8.4. Applying $*$-operation on $\psi_i e(\mathbf{d})$ gives us

$$(\psi_i e(\mathbf{d}))^* = e(\mathbf{d}) f_{i,\mathbf{d}}(\mathrm{y})\psi_i, \text{ for } d_i \neq d_{i+1}; \tag{8.55}$$

$$(\psi_i e(\mathbf{d}))^* = e(\mathbf{d}) f_{i,\mathbf{d}}(\mathrm{y})\psi_i + e(\mathbf{d}) g_{i,\mathbf{d}}(\mathrm{y}), \text{ otherwise}, \tag{8.56}$$

where

1. For $d_i \neq d_{i+1}$ and $d_i \not\leftrightarrow d_{i+1}$, all the diagrams $\xi(\mathbf{a},\mathbf{t},s_j)$ for $j = 1,\ldots,n$ in the product $\psi_i e(\mathbf{d})$

act by either

$$e(\mathbf{a},\mathbf{t})s_j = e(\mathbf{a},\mathbf{t})\sigma_j \quad \text{if } t_j \neq t_{j+1} \tag{8.57}$$

$$\text{or } e(\mathbf{a},\mathbf{t})\frac{u_j - u_{j+1}}{u_j - u_{j+1} - c_0 l}\sigma_i \quad \text{if } t_j = t_{j+1} \text{ and } a_j + c_0 l \neq a_{j+1} \neq a_j. \tag{8.58}$$

Let p be the number of times the condition in (8.58) is met. The power series provided by the diagrams $\xi(\mathbf{a},\mathbf{t},\nu^{\pm})$ cancel out each other. The power series $f_{i,\mathbf{d}}$ is a product of p factors of $\frac{u_s - u_r + c_0 l}{u_s - u_r - c_0 l}$, where s and r are different in each term depending on the diagrams and their position.

2. For $d_i \neq d_{i+1}$ and $d_i \leftarrow d_{i+1}$, all the diagrams $\xi(\mathbf{a},\mathbf{t},s_j)$ for $j = 1,\ldots,n$ in the product $\psi_i e(\mathbf{d})$ act by either

$$e(\mathbf{a},\mathbf{t})s_j = e(\mathbf{a},\mathbf{t})\sigma_j \quad \text{if } t_j \neq t_{j+1} \tag{8.59}$$

$$\text{or } e(\mathbf{a},\mathbf{t})\frac{u_j - u_{j+1}}{u_j - u_{j+1} - c_0 l}\sigma_j \quad \text{if } t_j = t_{j+1} \text{ and } a_j + c_0 l \neq a_{j+1} \neq a_j, \tag{8.60}$$

except the one diagram where $a_i - c_0 l = a_{i+1} + m$ and it acts by $e(\mathbf{a},\mathbf{t})(u_i - u_{i+1})\sigma_i$. Let p be the number of times when the condition of the second action $e(\mathbf{a},\mathbf{t})\frac{u_j - u_{j+1}}{u_j - u_{j+1} - c_0 l}\sigma_j$ is met. The power series provided by the diagrams $\xi(\mathbf{a},\mathbf{t},\nu^{\pm})$ cancel out each other. The power series $f_{i,\mathbf{d}}$ is a product of p factors of $\frac{u_s - u_r + c_0 l}{u_s - u_r - c_0 l}$ and one factor of $(u_k - u_j + c_0 l)$ for some $j,k \in \{1,\ldots,n\}$, where s and r are different in each term depending on the diagrams and their position.

3. For $d_i \neq d_{i+1}$ and $d_i \rightarrow d_{i+1}$, all the diagrams $\xi(\mathbf{a},\mathbf{t},s_j)$ for $j = 1,\ldots,n$ in the product $\psi_i e(\mathbf{d})$ act by either

$$e(\mathbf{a},\mathbf{t})s_j = e(\mathbf{a},\mathbf{t})\sigma_j \quad \text{if } t_j \neq t_{j+1} \tag{8.61}$$

$$\text{or } e(\mathbf{a},\mathbf{t})\frac{u_j - u_{j+1}}{u_j - u_{j+1} - c_0 l}\sigma_i \quad \text{if } t_j = t_{j+1} \text{ and } a_j + c_0 l \neq a_{j+1} \neq a_j. \tag{8.62}$$

Let p be the number of times the second action applies. The power series provided by the diagrams $\xi(\mathbf{a},\mathbf{t},\nu^{\pm})$ cancel out each other. All the diagrams behave the same as in case 1 except $\xi(\mathbf{a},\mathbf{t},s_i)$, where $a_i - c_0 l = a_{i+1}$. The $*$-operation on this diagram gives an extra power series $-\frac{1}{u_{i+1} - u_i - c_0 l}$. The power series $f_{i,\mathbf{d}}$ is a product of p factors of $\frac{u_s - u_r + c_0 l}{u_s - u_r - c_0 l}$ and one of $-\frac{1}{u_k - u_j - c_0 l}$

for some $j, k \in \{1, \ldots, n\}$, where s and r are different in each term depending on the diagrams and their position.

4. For $d_i = d_{i+1}$ and $d_i \not\leftrightarrow d_{i+1}$, all the diagrams $\xi(\mathbf{a}, \mathbf{t}, s_j)$ for $j = 1, \ldots, n$ in the product $\psi_i e(\mathbf{d})$ act by either

$$e(\mathbf{a}, \mathbf{t}) s_j = e(\mathbf{a}, \mathbf{t}) \sigma_j \quad \text{if } t_j \neq t_{j+1} \tag{8.63}$$

$$\text{or } e(\mathbf{a}, \mathbf{t}) \frac{u_j - u_{j+1}}{u_j - u_{j+1} - c_0 l} \sigma_i \quad \text{if } t_j = t_{j+1} \text{ and } a_j + c_0 l \neq a_{j+1} \neq a_j, \tag{8.64}$$

except the diagram $\xi(\mathbf{a}, \mathbf{t}, s_i)$ for $a_i = a_{i+1}$ and $t_i = t_{i+1} = 1$. It gives an extra power series $\frac{u_i - u_{i+1} + c_0 l}{u_{i+1} - u_i + c_0 l}$. Let p be the number of times when the condition of the action $e(\mathbf{a}, \mathbf{t}) \frac{u_j - u_{j+1}}{u_j - u_{j+1} - c_0 l} \sigma_j$ is met. The power series provided by the diagrams $\xi(\mathbf{a}, \mathbf{t}, \nu^{\pm})$ cancel out each other. The power series $f_{i,\mathbf{d}}$ is a product of p factors of $\frac{u_s - u_r + c_0 l}{u_s - u_r - c_0 l}$ and one of $\frac{u_j - u_k + c_0 l}{u_k - u_j + c_0 l}$ for some $j, k \in \{1, \ldots, n\}$, where s and r are different in each term depending on the diagrams.

Moreover, we have $g_{i,\mathbf{d}}(\mathrm{y}) e(\mathbf{d}) = \frac{f_{i,\mathbf{d}}(\mathrm{y}) - 1}{\mathrm{y}_i - \mathrm{y}_{i+1}} e(\mathbf{d})$.

Proof. The computation of $f_{i,\mathbf{d}}$ is straightforward by applying (8.37) to (8.39) to the product $\psi_i e(\mathbf{d})$ and moving all the power series given by the $*$-operation to the front of the terms by (8.45) to (8.51).

When $d_i = d_{i+1}$, we can compute $g_{i,\mathbf{d}}$ in term of $f_{i,\mathbf{d}}$ by (5.10). Applying the $*$-operation on $(\mathrm{y}_{i+1}\psi_i - \psi_i \mathrm{y}_i) e(\mathbf{d}) = e(\mathbf{d})$, we get

$$\begin{aligned}
& e(\mathbf{d})(\psi_i^* \mathrm{y}_{i+1} - \mathrm{y}_i \psi_i^*) = e(\mathbf{d}) \\
\Leftrightarrow & e(\mathbf{d}) \left((f_{i,\mathbf{d}}(\mathrm{y}) \psi_i + g_{i,\mathbf{d}}(\mathrm{y})) \mathrm{y}_{i+1} - \mathrm{y}_i (f_{i,\mathbf{d}}(\mathrm{y}) \psi_i + g_{i,\mathbf{d}}(\mathrm{y})) \right) = e(\mathbf{d}) \\
\Leftrightarrow & e(\mathbf{d}) \left(f_{i,\mathbf{d}}(\mathrm{y})(\mathrm{y}_i \psi_i + 1) - g_{i,\mathbf{d}}(\mathrm{y}) \mathrm{y}_{i+1} - \mathrm{y}_i f_{i,\mathbf{d}}(\mathrm{y}) \psi_i - \mathrm{y}_i g_{i,\mathbf{d}}(\mathrm{y}) \right) = e(\mathbf{d}) \\
\Leftrightarrow & e(\mathbf{d}) g_{i,\mathbf{d}}(\mathrm{y})(\mathrm{y}_i - \mathrm{y}_{i+1}) = e(\mathbf{d})(f_{i,\mathbf{d}}(\mathrm{y}) - 1)(\mathrm{y}) \\
\Leftrightarrow & g_{i,\mathbf{d}}(\mathrm{y}) e(\mathbf{d}) = \frac{f_{i,\mathbf{d}}(\mathrm{y}) - 1}{\mathrm{y}_i - \mathrm{y}_{i+1}} e(\mathbf{d}).
\end{aligned} \tag{8.65}$$

□

8.3 The $*$-Operation on Cyclotomic Hecke Algebra

Let $H_{q,\mathbf{Q}}$ be the cyclotomic Hecke algebra with parameters $q = b(-c_0)$ and $\mathbf{Q} = (Q_0, \ldots, Q_{l-1}) = (b(-h_0 + \frac{0}{l}), \ldots, b(-h_{l-1} + \frac{l-1}{l}))$. Finally we can pass the $*$-operation onto the cyclotomic Hecke algebra $H_{q,\mathbf{Q}}$ via the isomorphism to the cyclotomic KLR algebra in Theorem 5.11. Furthermore, we want to compare this $*$-operation with the $\star$-operation that defines unitarity on the cyclotomic Hecke algebra's modules in Definition 7.1. Let us recall that γ is the isomorphism between the completion of affine Hecke algebra and the completion of KLR algebra from Theorem 5.11 If we apply the $*$-operation on $\gamma(X_i)$ for any i, then we get

$$\begin{aligned}\gamma(X_i)^* &= \left(\sum_{\mathbf{d}\in D_0^n} b(d_i)b(\mathrm{y}_i)e(\mathbf{d})\right)^* \\ &= \sum_{\mathbf{d}} b(-d_i)b(-\mathrm{y}_i)e(\mathbf{d}) \\ &= \gamma(X_i^{-1}). \end{aligned} \tag{8.66}$$

Similarly, applying the $*$-operation on $\gamma(\Phi_i)$ leads to

$$\begin{aligned}\gamma(\Phi_i)^* &= \left(\sum_{\mathbf{d}\in D_0^n} \psi_i A_i^{\mathbf{d}} e(\mathbf{d})\right)^* \\ &= \left(\sum_{\mathbf{d}\in D_0^n} \psi_i e(\mathbf{d}) A_i^{\mathbf{d}}\right)^* \\ &= \sum_{\mathbf{d}} (A_i^{\mathbf{d}})^* (\psi_i e(\mathbf{d}))^*. \end{aligned} \tag{8.67}$$

The summands in $\gamma(\Phi_i)^*$ can be divided into trichotomy.

1. For $d_i \neq d_{i+1}, d_{i+1} - c_0$, the summand $(A_i^{\mathbf{d}})^*(\psi_i e(\mathbf{d}))^*$ becomes

$$\begin{aligned}(A_i^{\mathbf{d}})^*(\psi_i e(\mathbf{d}))^* &= \left(\frac{b(d_i)b(\mathrm{y}_i) - qb(d_{i+1})b(\mathrm{y}_{i+1})}{b(d_i)b(\mathrm{y}_i) - b(d_{i+1})b(\mathrm{y}_{i+1})}\right)^* f_{i,\mathbf{d}}\psi_i e(s_i\cdot\mathbf{d}) \\ &= \frac{b(-d_i)b(-\mathrm{y}_i) - qb(-d_{i+1})b(-\mathrm{y}_{i+1})}{b(-d_i)b(-\mathrm{y}_i) - b(-d_{i+1})b(-\mathrm{y}_{i+1})} f_{i,\mathbf{d}}\psi_i e(s_i\cdot\mathbf{d}) \\ &= \frac{b(d_{i+1})b(\mathrm{y}_{i+1}) - q^{-1}b(d_i)b(\mathrm{y}_i)}{b(d_{i+1})b(\mathrm{y}_{i+1}) - b(d_i)b(\mathrm{y}_i)} f_{i,\mathbf{d}}\psi_i e(s_i\cdot\mathbf{d}) \\ &= \frac{1}{q}\frac{b(d_i)b(\mathrm{y}_i) - qb(d_{i+1})b(\mathrm{y}_{i+1})}{b(d_i)b(\mathrm{y}_i) - b(d_{i+1})b(\mathrm{y}_{i+1})} f_{i,\mathbf{d}}\psi_i e(s_i\cdot\mathbf{d}) \end{aligned} \tag{8.68}$$

2. For $d_i = d_{i+1} - c_0$, the summand $(A_i^{\mathbf{d}})^*(\psi_i e(\mathbf{d}))^*$ is

$$\begin{aligned}
\left(A_i^{\mathbf{d}}\right)^*\left(\psi_i e(\mathbf{d})\right)^* &= \left(\frac{b(\mathrm{y}_{i+1}) - b(\mathrm{y}_i)}{(q^{-1}b(\mathrm{y}_{i+1} - b(\mathrm{y}_i)))(\mathrm{y}_i - \mathrm{y}_{i+1})}\right)^* f_{i,\mathbf{d}}\psi_i e(s_i \cdot \mathbf{d}) \\
&= \frac{b(-\mathrm{y}_{i+1}) - b(-\mathrm{y}_i)}{(qb(-\mathrm{y}_{i+1}) - b(-\mathrm{y}_i))(\mathrm{y}_i - \mathrm{y}_{i+1})} f_{i,\mathbf{d}}\psi_i e(s_i \cdot \mathbf{d}) \\
&= \frac{b(\mathrm{y}_i) - b(\mathrm{y}_{i+1})}{(qb(\mathrm{y}_i) - b(\mathrm{y}_{i+1}))(\mathrm{y}_i - \mathrm{y}_{i+1})} f_{i,\mathbf{d}}\psi_i e(s_i \cdot \mathbf{d}) \\
&= f_{i,\mathbf{d}}\psi_i \frac{b(\mathrm{y}_{i+1}) - b(\mathrm{y}_i)}{(qb(\mathrm{y}_{i+1}) - b(\mathrm{y}_i))(\mathrm{y}_{i+1} - \mathrm{y}_i)} e(s_i \cdot \mathbf{d}) \\
&= f_{i,\mathbf{d}}\psi_i A_i^{s_i \cdot \mathbf{d}} \frac{b(\mathrm{y}_{i+1}) - b(\mathrm{y}_i)}{q^2 b(\mathrm{y}_{i+1}) - b(\mathrm{y}_i)} (\mathrm{y}_{i+1} - \mathrm{y}_i) e(s_i \cdot \mathbf{d}).
\end{aligned} \tag{8.69}$$

3. For $d_i = d_{i+1}$, we have the summand

$$\begin{aligned}
\left(A_i^{\mathbf{d}}\right)^*\left(\psi_i e(\mathbf{d})\right)^* &= \left(\frac{b(\mathrm{y}_i) - qb(\mathrm{y}_{i+1})}{b(\mathrm{y}_{i+1}) - b(\mathrm{y}_i)}(\mathrm{y}_{i+1} - \mathrm{y}_i)\right)^* (f_{i,\mathbf{d}}\psi_i + g_{i,\mathbf{d}})e(\mathbf{d}) \\
&= \frac{b(-\mathrm{y}_i) - q^{-1}b(-\mathrm{y}_{i+1})}{b(-\mathrm{y}_{i+1}) - b(-\mathrm{y}_i)}(\mathrm{y}_{i+1} - \mathrm{y}_i)(f_{i,\mathbf{d}}\psi_i + g_{i,\mathbf{d}})e(\mathbf{d}) \\
&= \frac{b(\mathrm{y}_{i+1}) - q^{-1}b(\mathrm{y}_i)}{b(\mathrm{y}_i) - b(\mathrm{y}_{i+1})}(\mathrm{y}_{i+1} - \mathrm{y}_i)(f_{i,\mathbf{d}}\psi_i + g_{i,\mathbf{d}})e(\mathbf{d})\,.
\end{aligned} \tag{8.70}$$

Now the $*$-operation in the rational Cherednik algebra $H_c(G(l,1,n),\mathbb{C}^n)$ has already been passed onto the cyclotomic Hecke algebra $H_{q,\mathbf{Q}}$. If the Verma module $M_c(\lambda)$ of the rational Cherednik algebra $H_c(G(l,1,n),\mathbb{C}^n)$ is unitary, then contravariant Hermitian form of the representation $V^{\lambda^{tr}}$ with respect to $*$-operation on the cyclotomic Hecke algebra is positive-definite.

Let us recall the definition of $\star$-operation and unitary representations of cyclotomic Hecke algebra in Definition 7.1. The star operation $\star$ gives us explicit information about its actions onto T_i instead of Φ_i and $e(\mathbf{d})$. In order to compare the two star operations, we need to know $\Phi_i^\star$ and $e(\mathbf{d})^\star$, which motivates the following lemma.

Lemma 8.5. Suppose $H_{q,\mathbf{Q}}(n)$ is a cyclotomic Hecke algebra with q and Q_i on the unit cirle for all $i = 0, \dots, n-1$. Let $V = \{\mathbf{v} = (v_1, \dots, v_n)\}$ be the set of possible eigenvalues of $X_i's$. Then we have

$$e(\mathbf{v})^\star = e(\mathbf{v}), \tag{8.71}$$

$$\Phi_i^\star = \frac{1}{q}\Phi_i. \tag{8.72}$$

Proof.

1. Proof of $e(\mathbf{v})^\star = e(\mathbf{v})$.

   The proof of $e(\mathbf{v})^\star = e(\mathbf{v})$ is similar to the proof of Lemma 8.2. Consider the cyclotomic Hecke algebra $H_{q,\mathbf{Q}}(n)$ as a left (right) module of itself by the left (right) multiplication, and denote it as H_l (H_r). Let $\widehat{e}(\mathbf{d})$ be the idempotents in the right module of the cyclotomic Hecke algebra. Since H_l and H_r are both finitely generated $H_{q,\mathbf{Q}}(n)$-modules, they can be written as a direct sum of generalized eigenspaces of $X_i's$, which are $\bigoplus_{\mathbf{v}\in V} H_l e(\mathbf{v})$ and $\bigoplus_{\mathbf{v}\in V} \widehat{e}(\mathbf{v})H_r$. Let m be a generalized eigenvector in $H_l e(\mathbf{v})$. Then we have

   $$(X_i - v_i)^N m = 0, \tag{8.73}$$

   for any $i = 1,..,n$ and $N \in \mathbb{N}$ large enough. Since the left hand side of the equation is an element in the cyclotomic Hecke algebra, we can apply the $\star$-operation on the equation,

   $$\big((X_i - v_i)^N v\big)^\star = 0 \tag{8.74}$$

   $$\Leftrightarrow m^\star(X_i^{-1} - v_i^{-1}) = 0 \tag{8.75}$$

   $$\Leftrightarrow m^\star(X_i - v_i) = 0, \tag{8.76}$$

   for any i. It implies that $(H_l e(\mathbf{v}))^\star \subset \widehat{e}(\mathbf{v})H_r$. The same argument applies for generalized eigenvalues in $\widehat{e}(\mathbf{v})H_r$ and $H_l e(\mathbf{v}) = \widehat{e}(\mathbf{v})H_r$. Let $\widehat{a}$ be an anti-involution on $H_{q,\mathbf{Q}}(n)$ that maps X_i to X_i and T_i to T_i. It preserves the commutative subalgebra generated by $\{X_i\}_{i=1,\dots,n}$. If we consider H_l and H_r as representations of the abelian subalgebra generated by X_i, then both of them become bi-modules of this abelian subalgebra. The involution $\widehat{a}$ gives an isomorphism between H_l and H_r by

   $$h \cdot v = \widehat{a}(v) \cdot h, \tag{8.77}$$

   for any $h, v \in H_{q,D}(n)$. Hence the idempotents $e(\mathbf{d})$ and $\widehat{e}(\mathbf{d})$ coincide.

2. Proof of $\Phi_i^\star = \frac{1}{q}\Phi_i$.

Let us compute $\Phi_i^\star$ directly from definition.

$$
\begin{aligned}
\Phi_i^\star &= \left(T_i + \sum_{\mathbf{v}, v_i \neq v_{i+1}} \frac{1-q}{1-X_iX_{i+1}^{-1}} e(\mathbf{v}) + \sum_{\mathbf{v}, v_i = v_{i+1}} e(\mathbf{v})\right)^\star \\
&= T_i^{-1} + \sum_{\mathbf{v}, v_i \neq v_{i+1}} \frac{1-q^{-1}}{1-X_i^{-1}X_{i+1}} e(\mathbf{v}) + \sum_{\mathbf{v}, v_i = v_{i+1}} e(\mathbf{v}) \\
&= \frac{1}{q}T_i - \left(1 - \frac{1}{q}\right) + \sum_{\mathbf{v}, v_i \neq v_{i+1}} \left(\frac{1}{q}\frac{q-1}{X_iX_{i+1}^{-1}-1}\frac{1}{X_i^{-1}X_{i+1}}\right) e(\mathbf{v}) + \sum_{\mathbf{v}, v_i = v_{i+1}} e(\mathbf{v}) \\
&= \frac{1}{q}T_i + \frac{1}{q} + \sum_{\mathbf{v}, v_i \neq v_{i+1}} \left(\frac{1}{q}\frac{q-1}{X_iX_{i+1}^{-1}-1}\frac{1}{X_i^{-1}X_{i+1}} - 1\right) e(\mathbf{v}) + \sum_{\mathbf{v}, v_i = v_{i+1}} (1-1)e(\mathbf{v}) \\
&= \frac{1}{q}T_i + \sum_{\mathbf{v}, v_i = v_{i+1}} \frac{1}{q}e(\mathbf{v}) + \sum_{\mathbf{v}, v_i \neq v_{i+1}} \left(\frac{X_i^{-1}X_{i+1} - q^{-1}}{1-X_i^{-1}X_{i+1}} + \frac{1}{q}\right) e(\mathbf{v}) \\
&= \frac{1}{q}T_i + \sum_{\mathbf{v}, v_i = v_{i+1}} \frac{1}{q}e(\mathbf{v}) + \sum_{\mathbf{v}, v_i \neq v_{i+1}} \frac{1}{q}\frac{q-1}{X_iX_{i+1}^{-1}-1} e(\mathbf{v}) \\
&= \frac{1}{q}\Phi_i.
\end{aligned} \tag{8.78}
$$

□

Proposition 8.6. The $*$-operation is different from the $\star$-operation.

Proof. Suppose the $*$-operation is the same as the $\star$-operation. Then we have $\Phi_i^* = \frac{1}{q}\Phi_i$. As a result, when $d_i \neq d_{i+1},\ d_{i+1} - c_0$, Equation (8.68) becomes

$$
\frac{b(d_i)b(\mathrm{y}_i) - qb(d_{i+1})b(\mathrm{y}_{i+1})}{b(d_i)b(\mathrm{y}_i) - b(d_{i+1})b(\mathrm{y}_{i+1})} f_{i,\mathbf{d}} = 1 \tag{8.79}
$$

by (8.68). Recall that $\mathrm{y}_i e(\mathbf{d}) = (u_i - a_i)e(\mathbf{d})$. The power series $f_{i,\mathbf{d}}$ contains variables $\mathrm{y}_1, \ldots, \mathrm{y}_n$ given in Lemma 8.4, it can never be the inverse of the power series $\frac{b(d_i)b(\mathrm{y}_i) - qb(d_{i+1})b(\mathrm{y}_{i+1})}{b(d_i)b(\mathrm{y}_i) - b(d_{i+1})b(\mathrm{y}_{i+1})}$ that only has variables y_i and y_{i+1}. Hence the two operations are different. □

Unfortunately the $*$-operation on the cyclotomic Hecke algebra is different from the $\star$-operation required to define unitarity. Hence this method cannot give a conclusive result on whether the KZ functor preserves unitarity. But we will shift our focus to quasi-untarity. We will compute asymptotic signature of Verma modules and discover its connection with representations of cyclotomic Hecke algebra.

9 Quasi-Unitarity, Asymptotic Signature and KZ Functor

The example in Section 7.2 tells us that the KZ functor automatically sends a unitary representation in Category $\mathcal{O}$ of $H_c(\mathbb{Z}_l, \mathbb{C})$ to a unitary representation of $\mathcal{H}_{\mathbf{q}}(\mathbb{Z}_l)$. However, it also reveals that the converse of the statement is not true. Because it is shown in Example 1.20 that $L_c(\tau)$ has to satisfy certain conditions to be unitary, for trivial representation τ, while the image $V^{\tau^{tr}}$ of $L_c(\tau)$ under KZ functor is automatically unitary.

In order to have a statement that holds in both ways, we need to loosen the condition on unitarity. There is a weaker version of unitarity, called quasi-unitarity introduced by Shelley-Abrahamson in [18]. It is proved that if W is a Coxeter group, then for any irreducible representation λ of W, the unitarity of the image $\mathrm{KZ}(L_c(\lambda))$ under the KZ functor implies the quasi-unitarity of $L_c(\lambda)$. It is worth extending this result to the more general group $G(l, 1, n)$.

9.1 Definitions and Examples

This section introduces the concepts of asymptotic signature and quasi-unitarity. It is a review of [18]. I will give an original example at the end of the section.

Suppose we have the rational Cherednik algebra $H_c(W, \mathfrak{h})$, for some group $G(l, 1, n)$. By Proposition 1.13, we know any $M \in$ Category $\mathcal{O}$ can be written as a direct sum of generalized eigenspaces of the Euler element $\mathbf{h}$, i.e.,

$$M = \bigoplus_{z \in \mathbb{C}} M_z \, . \tag{9.1}$$

Recall that Verma modules are objects in Category $\mathcal{O}$. Then we can define the character of Verma module $M_c(\lambda)$ based on this decomposition.

Definition 9.1. Let $M_c(\lambda)$ be a Verma module for some irreducible representation λ of $G(l, 1, n)$.

1. The character $\mathrm{ch}(M_c(\lambda))$ of $M_c(\lambda)$ is the formal series

$$\mathrm{ch}(M_c(\lambda))(w, t) := \sum_{z \in \mathbb{C}} t^z \, \mathrm{Tr}_{M_c(\lambda)_z}(w) \tag{9.2}$$

   for $w \in W$.

2. The shifted character $\mathrm{ch}_0(M_c(\lambda))$ is defined by

$$\mathrm{ch}_0(M_c(\lambda))(w,t) := t^{-h_c(\lambda)}\,\mathrm{ch}(M_c(\lambda))(w,t)\,, \tag{9.3}$$

where $h_c(\lambda)$ is defined in (1.21).

We still need other two ingredients to define quasi-unitary.

Definition 9.2. Let β be a Hermitian form on a finite dimensional complex vector space. We define the signature $\mathrm{sig}(\beta)$ of β to be

$$\mathrm{sig}(\beta) = p - q\,, \tag{9.4}$$

where p and q are the dimension of any maximal positive-definite and negative definite subspace with respect to β respectively.

Definition 9.3. Let $M_c(\lambda)$ be a Verma module for some irreducible representation λ of $G(l,1,n)$ with the Hermitian form $\beta_{c,\lambda}$.

1. The signature character $\mathrm{sch}(M_c(\lambda),\beta_{c,\lambda})$ is defined by

$$\mathrm{sch}(M_c(\lambda),\beta_{c,\lambda})(t) := \sum_{z\in\mathbb{C}} t^z\ \mathrm{sig}((\beta_{c,\lambda})_z)\,, \tag{9.5}$$

where $(\beta_{c,\lambda})_z$ is the restriction of the form $\beta_{c,\lambda}$ to the weight space $M_c(\lambda)_z$.

2. Similarly, the shifted signature character $\mathrm{sch}_0(M_c(\lambda),\beta_{c,\lambda})$ is defined to be

$$\mathrm{sch}_0(M_c(\lambda),\beta_{c,\lambda})(t) := t^{-h_c(\lambda)}\ \mathrm{sch}(M_c(\lambda),\beta_{c,\lambda})(t)\,. \tag{9.6}$$

In practice, we usually omit $\beta_{c,\lambda}$ in $\mathrm{sch}_0(M_c(\lambda),\beta_{c,\lambda})$ and $\mathrm{sch}(M_c(\lambda),\beta_{c,\lambda})$ when it is clear from the context.

Recall that the set C is $\{c\,|\,c^\dagger = c\}$. The next lemma defines the asymptotic signature of $\beta_{c,\lambda}$, according to Lemma 3.5.2 in [18].

Lemma 9.4. Let c be in C, and let λ be an irreducible representation of W. Let $\beta_{c,\lambda}^{\leq n}$ be the restriction of $\beta_{c,\lambda}$ to the space $L_c(\lambda)^{\leq n} :=$ {elements in $L_c(\lambda)$ that have degree less than or equal to n}, for any

$n \in \mathbb{N}$. Then the limits

$$\lim_{t\to 1^-} \frac{\mathrm{sch}_0(L_c(\lambda))(t)}{\mathrm{ch}_0(L_c(\lambda))(1,t)}\,, \lim_{n\to\infty} \frac{\mathrm{sig}(\beta_{c,\lambda}^{\le n})}{\mathrm{dim} L_c(\lambda)^{\le n}} \tag{9.7}$$

exist in the interval $[-1,1]$ and equal to the same rational number denoted by $a_{c,\lambda}$. We call this rational number the asymptotic signature of $L_c(\lambda)$.

Proof. The argument is the same as in the proof of Lemma 3.5.2 in [18] for the real reflection group. □

Definition 9.5. The module $L_c(\lambda)$ is said to be quasi-unitary if $a_{c,\lambda} = \pm 1$.

Remark. If a Verma module $M_c(\lambda)$ is unitary, then its asymptotic signature $a_{c,\lambda}$ equals to one. By definition, $M_c(\lambda)$ is quasi-unitary. However, quasi-unitarity doesn't ensure unitarity, which will be shown in the next example.

Example 9.6. Now we move on to compute if $L_c(\lambda)$ is quasi-unitary for any irreducible representation of $\mathbb{Z}_l$. According to Proposition 1.8 and (1.22) in its proof, we know that for $W = \mathbb{Z}_l$, the Verma module can be decomposed as

$$M_c(\lambda) = \bigoplus_{k\in\mathbb{N}} M_c(\lambda)_{k+h_c(\lambda)}\,, \tag{9.8}$$

where $M_c(\lambda)_{k+h_c(\lambda)} = \mathbb{C}x^k \otimes \lambda$. Recall the definition of ζ in Example 1.3. We choose s to be a generator of $\mathbb{Z}_l$. For any $i = 0,1,\ldots,l-1$, we denote ρ_i as the irreducible representation of $\mathbb{Z}_l$, where the generator s acts by ζ^i. After normalization, we have the Hermitian form $(1,1) = 1$ on ρ_i for each i. Duplicating the construction of the numbers a_k and b_k in the representation ρ_0 from Example 1.20, we have similar numbers for each ρ_i, given by

$$a_k^i = \beta_{c,\rho_i}(x^k, x^k) \qquad \text{for} \quad k \in \mathbb{N} \quad \text{and} \quad i = 0,..,l-1\,. \tag{9.9}$$

Inductively we can build b_k^i such that

$$\begin{aligned} a_k^i &= \beta_{c,\rho_i}(x^k, x^k) \\ &= \beta_{c,\rho_i}(yx^k, x^{k-1}) \\ &= \beta_{c,\rho_i}(\mathrm{D}_y\, x^k, x^{k-1}) \\ &= \beta_{c,\rho_i}((\partial_y - \sum_{j=1}^{l-1} \frac{2c_j}{1-\lambda_{s^j}} \frac{\alpha_{s_j}(y)}{\alpha_{s_j}}(1-s_j))x^k, x^{k-1}) \end{aligned}$$

$$= (k - b^i_k) a^i_{k-1} \,, \tag{9.10}$$

where D_y is the Dunkl operator with respect to y.

$$b^i_k = \sum_{j=1}^{l-1} c_j \frac{1-\zeta^{-jk}}{1-\zeta^{-j}} \zeta^{-ji} \qquad \text{for} \quad k > 0 \,. \tag{9.11}$$

For any $i \in 0, \ldots, l-1$, we have $\overline{b^i_k} = b^i_k$. Hence b^i_k are real numbers.

On the one hand, if there is no $k \in \mathbb{N}^+$ such that $b^i_k = k$, then $\ker(\beta_{c,\rho_i}) = 0$, which implies $L_c(\rho_i) = M_c(\rho_i)$. By Corollary 6.7, the parameters are generic in this case. By From (9.10), the signatures $\operatorname{sig}(\beta_{k+h_c(\rho_i)}) = 1$ if $(k - b^i_k)$ and a^i_{k-1} have the same sign, and $\operatorname{sig}(\beta_{k+h_c(\rho_i)}) = -1$ otherwise. Since we have $b^i_k = b^i_{k+l}$ for all $k \in \mathbb{N}^+$, then there exists some large $N \in \mathbb{N}^+$ such that $N > \max_{k=1,\ldots,l} \{b^i_k\}$. It implies that $k - b^i_k > 0$ for all $k \geq N$. Hence for $k \geq N$ the signatures $\operatorname{sig}(\beta_{k+h_c(\rho_i)})$ are either all 1 or all -1. By definition, we have

$$\operatorname{sch}_0(L_c(\rho_i))(t) = \sum_{k \in \mathbb{N}^+} t^k \operatorname{sig}(\beta_{k+h_c(\rho_i)}) \,, \tag{9.12}$$

and

$$\operatorname{ch}_0(L_c(\rho_i))(1,t) = \sum_{k \in \mathbb{N}^+} t^k \,. \tag{9.13}$$

Hence we can compute the numbers a_{c,ρ_i} by

$$a_{c,\rho_i} = \lim_{t \to 1^-} \frac{\sum_{k \in \mathbb{N}^+} t^k \operatorname{sig}(\beta_{k+h_c(\rho_i)})}{\sum_{k \in \mathbb{N}^+} t^k} = \pm 1 \,. \tag{9.14}$$

Suppose there exists some $i \in \{0, \ldots, l-1\}$ the parameters satisfy $1 < b^i_1$, $2 < b^i_2$ and $k > b^i_k$ for all $k > 2$. Then $M_c(\rho_i)$ is not unitary, as $sig(\beta_{1+h_c(\rho_i)}) < 0$. However it is quasi-unitary as shown above.

On the other hand, if there exists $k \in \mathbb{N}^+$ such that $b^i_k = k$ in $M_c(\rho_i)$. Then $\ker(\beta_{c,\rho_i}) \neq \{0\}$ and the parameters are non-generic. We can pick the smallest of such numbers as

$$B_i = \min_{k \in \mathbb{N}^+} \{b^i_k \mid b^i_k = k\} \,. \tag{9.15}$$

Then the kernels $\ker(\beta_{c,\rho_i})$ become

$$\ker(\beta_{c,\rho_i}) = \bigoplus_{k \geq B_i} M_c(\rho_i)_{k+h_c(\rho_i)} \,. \tag{9.16}$$

Since the simple modules $L_c(\rho_i)$ are the quotient of $M_c(\rho_i)$ by the kernels of the Hermitian forms β_{c,ρ_i}, we get

$$L_c(\rho_i) \cong \bigoplus_{0 \leq k \leq B_i - 1} M_c(\rho_i)_{k+h_c(\rho_i)} \,. \tag{9.17}$$

Then we have the signature characters as

$$\mathrm{sch}_0(L_c(\rho_i))(t) = \sum_{0 \leq k \leq B_i - 1} t^k \ \mathrm{sig}(\beta_{k+h_c(\rho_i)}) \,, \tag{9.18}$$

and a_{c,ρ_i} become

$$a_{c,\rho_i} = \lim_{t \to 1^-} \frac{\sum_{0 \leq k \leq B_i - 1} t^k \, \mathrm{sig}(\beta_{k+h_c(\rho_i)})}{\sum_{0 \leq k \leq B_i - 1} t^k} \,. \tag{9.19}$$

Hence we have $a_{c,\rho_i} = 1$ if β_{c,ρ_i} are positive-definite on $L_c(\rho_i)$, and a_{c,ρ_i} can never be -1 as they are defined to be positive-definite on degree 0. Therefore, $L_c(\rho_i)$ are quasi-unitary if and only they are unitary.

It is obvious to see the difference in a_{c,ρ_i} between the two situations of $L_c(\rho_i) = M_c(\rho_i)$ and $L_c(\rho_i) \neq M_c(\rho_i)$. When $L_c(\rho_i) = M_c(\rho_i)$, the signatures $\mathrm{sig}(\beta_{c,\rho_i})$ oscillate between 1 and -1 in the generalized eigenspaces of lower degrees, and stabilize at 1 or -1 in higher degrees. Hence if we add up the $\mathrm{sig}(\beta_{c,\rho_i})$ of all degrees, the contributions from higher degrees dominate, thus giving limit values $a_{c,\rho_i} = \pm 1$, as written in (9.14). In contrast, if $L_c(\rho_i) \neq M_c(\rho_i)$, then the Hermitian forms β_{c,ρ_i} are killed at higher degrees. So the signatures $\mathrm{sig}(\beta_{c,\rho_i})$ do not have the chance to stabilize. As a result, the numbers a_{c,ρ_i} only depend on the first B_i degrees where β_{c,ρ_i} are nonzero, as stated in (9.19).

Example 9.7. Shelley-Abrahamson has proved the result in Coxeter groups that $M_c(\lambda)$ is quasi-unitary if and only if $\mathrm{KZ}(M_c(\lambda))$ is unitary in [18].

Remark. If $L_c(\lambda)$ is of finite dimensional, then $\mathrm{KZ}(L_c(\lambda)) = 0$. The irreducible modules $L_c(\lambda)$ are infinite dimensional in most cases. For example, in type A case, the only possible finite dimensional representation of $H_c(S_n, \mathbb{C}^n)$ are the trivial and sign representation with certain parameters [3]. This result indicates that it is still possible for the Hermitian form $\beta_{c,\lambda}$ to stabilize at higher degrees.

9.2 Asymptotic Signature and KZ Functor

For the rest of the section, we would like to explore the connection between the notion of quasi-unitarity and the KZ functor, and answer the question whether $M_c(\lambda)$ is quasi-unitary if and only $\mathrm{KZ}(M_c(\lambda))$ is unitary for the groups $G(l,1,n)$. Furthermore, if so, does it indicate any relation between the KZ functor and the asymptotic signature, as the value of the latter is a strict implication of quasi-unitarity? Since we only have the unitarity condition of representations of the cyclotomic Hecke algebra with generic parameters, and for the sake of simplicity in computations, we will assume that the parameters are generic for both the rational Cherednik algebra and the cyclotomic Hecke algebra for the rest of the section. Hence we have that the simple quotient $L_c(\lambda)$ is the Verma module $M_c(\lambda)$. Inspired by Theorem 5.0.1 in [18] that the asymptotic signature $a_{c,\lambda}$ of the simple module $L_c(\lambda)$ equals to $\frac{\mathrm{sig}(\langle\cdot,\cdot\rangle)}{\dim(\mathrm{KZ}(L_c(\lambda)))}$ up to a sign, where $\langle\cdot,\cdot\rangle$ is the contravariant Hermitian form with respect to the $\star$-operation for the cyclotomic Hecke algebra of the finite Coxeter group, we propose and prove the most important theorem here. This subsection is my original work.

Theorem 9.8. Suppose W is the group $G(l,1,n)$, and λ is an irreducible representation of W. Let the parameters be generic for both the rational Cherednik algebra $H_c(W,\mathfrak{h})$ and the cyclotomic Hecke algebra $H_{q,\mathbf{Q}}(n)$, whose irreducible representation $V^{\lambda^{tr}}$ is the image of $M_c(\lambda)$ under the KZ functor. Let $\langle\cdot,\cdot\rangle$ be a $\star$-operation invariant Hermitian form of $V^{\lambda^{tr}}$. The asymptotic signature $a_{c,\lambda}$ of $M_c(\lambda)$ is given by the formula

$$a_{c,\lambda} = \frac{\mathrm{sig}(\langle\cdot,\cdot\rangle)}{\dim(V^{\lambda^{tr}})} \tag{9.20}$$

up to a sign.

By the definition of the asymptotic signature in Lemma 9.4, it can be presented as a limit of the fraction $\lim_{n\to\infty}\frac{\mathrm{sig}(\beta_{c,\lambda}^{\leq n})}{\dim(M_c(\lambda)^{\leq n})}$. We will tackle the problem by directly computing the signature $\mathrm{sig}(\beta_{c,\lambda}^{\leq n})$ and comparing it with $\dim(M_c(\lambda)^{\leq n})$. The common eigenvectors $f_{\mu,L}$ of z_i and ζ_i in Verma module $M_c(\lambda)$ are labelled by pairs $(\mu,L)\in\mathbb{Z}_{\geq 0}^n\times\mathrm{STY}(\lambda)$ and they generate the entire Verma module. For any $m\in M_c(\lambda)$, let us call $\beta_{c,\lambda}(m,m)$ by the norm squre of m. In order to check the signature of $\beta_{c,\lambda}$, it is sufficient to consider the norm square of the eigenvectors $f_{\mu,L}$ for all $(\mu,L)\in\mathbb{Z}_{\geq 0}^n\times\mathrm{STY}(\lambda)$, of which Griffeth gave a formula in Theorem 6.1 [11] as follows.

Proposition 9.9. For convenience, we define

$$a_i^{\mu,L} = \mathrm{ct}(L(w_\mu(i))) \quad \text{and} \quad b_i^{\mu,L} = \beta(L(w_\mu(i)))\,. \tag{9.21}$$

Then the norm square of eigenvectors $f_{\mu,L}$ equals to

$$\begin{aligned}\beta_{c,\lambda}(f_{\mu,L}, f_{\mu,L}) = &\prod_{i=1}^{n}\prod_{k=1}^{\mu_i}(k - l(h_{b_i^{\mu,L}} - h_{b_{i-k}^{\mu,L}}) - c_0 a_i^{\mu,L}) \\ &\times \prod_{\substack{1\le i<j\le n,\\ \mu_i>\mu_j}} \prod_{\substack{1\le k\le \mu_i-\mu_j,\\ k\equiv b_i^{\mu,L}-b_j^{\mu,L} \bmod l}} \frac{(k - l(h_{b_i^{\mu,L}} - h_{b_j^{\mu,L}}) - c_0 l(a_i^{\mu,L} - a_j))^2 - (c_0 l)^2}{(k - l(h_{b_i^{\mu,L}} - h_{b_j^{\mu,L}}) - c_0 l(a_i^{\mu,L} - a_j))^2} \\ &\times \prod_{\substack{1\le i<j\le n,\\ \mu_i<\mu_j-1}} \prod_{\substack{1\le k\le \mu_j-\mu_i-1,\\ k\equiv b_j^{\mu,L}-b_i^{\mu,L} \bmod l}} \frac{(k - l(h_{b_j^{\mu,L}} - h_{b_i^{\mu,L}}) - c_0 l(a_j - a_i^{\mu,L}))^2 - (c_0 l)^2}{(k - l(h_{b_j^{\mu,L}} - h_{b_i^{\mu,L}}) - c_0 l(a_j - a_i^{\mu,L}))^2},\end{aligned} \tag{9.22}$$

Proof of Theorem 9.8. : The general plan of the proof is as follows.

1. We consider the the norm square of eigenvectors $f_{\mu,L}$ in the Verma module and rearrange the product in (9.22).

2. We introduce the concept of stabilization point of n-tuple μ and show that we can only consider about μ above the stabilization point.

3. We show that the order of coordinates of μ does not matter for μ above the stabilization point.

4. We compute the condition needed to have consistent signs for the norm square of eigenvectors for non-descending μ above the stabilization point.

5. We compare this condition with the unitarity condition of the representation $V^{\lambda^{tr}}$ of cyclotomic Hecke algbera, and compute the asymptotic signature in terms of the sign of the Hermitian form of $V^{\lambda^{tr}}$.

We rewrite (9.22) as

$$\beta_{c,\lambda}(f_{\mu,L}, f_{\mu,L}) = A_{\mu,L} B_{\mu,L} C_{\mu,L}, \tag{9.23}$$

where

$$A_{\mu,L} = \prod_{i=1}^{n}\prod_{k=1}^{\mu_i}(k - l(h_{b_i^{\mu,L}} - h_{b_{i-k}^{\mu,L}}) - c_0 a_i^{\mu,L}) \tag{9.24}$$

$$B_{\mu,L} = \prod_{\substack{1\le i<j\le n,\\ \mu_i>\mu_j}} \prod_{\substack{1\le k\le \mu_i-\mu_j,\\ k\equiv b_i^{\mu,L}-b_j^{\mu,L} \bmod l}} \frac{(k - l(h_{b_i^{\mu,L}} - h_{b_j^{\mu,L}}) - c_0 l(a_i^{\mu,L} - a_j^{\mu,L}))^2 - (c_0 l)^2}{(k - l(h_{b_i^{\mu,L}} - h_{b_j^{\mu,L}}) - c_0 l(a_i^{\mu,L} - a_j^{\mu,L}))^2} \tag{9.25}$$

$$C_{\mu,L} = \prod_{\substack{1\le i<j\le n,\\ \mu_i<\mu_j-1}} \prod_{\substack{1\le k\le \mu_j-\mu_i-1,\\ k\equiv b_j^{\mu,L}-b_i^{\mu,L} \bmod l}} \frac{(k - l(h_{b_j^{\mu,L}} - h_{b_i^{\mu,L}}) - c_0 l(a_j^{\mu,L} - a_i^{\mu,L}))^2 - (c_0 l)^2}{(k - l(h_{b_j^{\mu,L}} - h_{b_i^{\mu,L}}) - c_0 l(a_j^{\mu,L} - a_i^{\mu,L}))^2}. \tag{9.26}$$

The sign of norm square of $f_{\mu,L}$ is determined by $A_{\mu,L}$, $B_{\mu,L}$ and $C_{\mu,L}$, while the signature of the Hermitian form $\beta_{c,\lambda}$ is determined by its signs on large degree eigenvectors. As long as μ_i are large enough for all $i = 1, \ldots, n$, the equation $A_{\mu,L}$ produces the same sign. For each standard Young tableau L, the sign of the product $B_{\mu,L}C_{\mu,L}$ oscillates when the value of μ_i for some $i = 1, \ldots, n$ are small and stabilizes when $|\mu_i - \mu_j| \gg 0$ for all pair $i \neq j$. That is, although the absolute value of $\beta_{c,\lambda}(f_{\mu,L}, f_{\mu,L})$ increases as the values of μ_i increase, the sign will not change when the differences $|\mu_i - \mu_j|$ is larger than $l(h_{b_i^{\mu,L}} - h_{b_j^{\mu,L}}) + c_0 l(a_i^{\mu,L} - a_j^{\mu,L}) + |c_0 l|$ for all $i \neq j$. We say such μ is "above the stabilization point", and others are "below the stabilization point". Let us denote the number of standard Young tableaux of shape λ by $|\mathrm{STY}(\lambda)|$. Then the dimension of elements of degree N in $M_c(\lambda)$ is $(N+1)^{n-1}|\mathrm{STY}(\lambda)|$. Let $k_{i,j}^{\mu,L}$ be

$$k_{i,j}^{\mu,L} = l(h_{b_i^{\mu,L}} - h_{b_j^{\mu,L}}) + c_0 l(a_i^{\mu,L} - a_j^{\mu,L})\,. \tag{9.27}$$

For any standard Young tableau L, the number of n-tuples $\mu = (\mu_1, \ldots, \mu_n)$ with some μ_i, μ_j such that $|\mu_i - \mu_j| < l(h_{b_i^{\mu,L}} - h_{b_j^{\mu,L}}) + c_0 l(a_i^{\mu,L} - a_j^{\mu,L}) + |c_0 l|$ is

$$K_L^N = \sum_{i\neq j}\sum_{k=0}^{\lfloor k_{i,j}^{\mu,L}\rfloor}\sum_{x=0}^{\lfloor \frac{N-k_{i,j}^{\mu,L}}{2}\rfloor}(N-2x-k)^{n-3}\,, \tag{9.28}$$

which is a polynomial of degree at most $(n-2)$. Thus the number of eigenvectors above the stabilization point is a polynomial of degree $(n-1)$. It is much more significant compared to the the number of eigenvectors below the stabilization point when $N \to \infty$. As a consequence, from now on we only consider the eigenvectors with μ that are above the stabilization point.

Let $\mu^+ = w_\mu(\mu)$ be the non-decreasing arrangement of μ, where w_μ is the longest element in S_n that makes μ non-decreasing. Suppose the inequality $\mu_i > \mu_j$ holds, for some $i \neq j \in \{1, \ldots, n\}$. Let a and b be integers such that $i = w_\mu^{-1}(a)$ and $j = w_\mu^{-1}(b)$. By the definition of w_μ, we have $a > b$. It implies $w_\mu(i) > w_\mu(j)$. Hence in both $B_{\mu,L}$ and $C_{\mu,L}$, no matter the value of i and j, the factors $k - l(h_{b_i^{\mu,L}} - h_{b_j^{\mu,L}}) - c_0 l(a_i^{\mu,L} - a_j^{\mu,L})$ in the products are always $k - l(h_{b_i^{\mu,L}} - h_{b_j^{\mu,L}}) - c_0 l(a_i^{\mu,L} - a_j^{\mu,L})$ for $w_\mu(i) > w_\mu(j)$. Hence the product $B_{\mu,L}C_{\mu,L}$ can be rewritten as

$$
\begin{aligned}
B_{\mu,L}C_{\mu,L} = &\prod_{\substack{w_\mu(i)>w_\mu(j),\\ |\mu_i-\mu_j|>1}} \prod_{\substack{1\le k\le \mu_i-\mu_j,\\ k\equiv b_i^{\mu,L}-b_j^{\mu,L} \bmod l}} \\
&\frac{(k - l(h_{b_i^{\mu,L}} - h_{b_j^{\mu,L}}) - c_0 l(a_i^{\mu,L} - a_j^{\mu,L}))^2 - (c_0 l)^2}{(k - l(h_{b_i^{\mu,L}} - h_{b_j^{\mu,L}}) - c_0 l(a_i^{\mu,L} - a_j^{\mu,L}))^2} \\
\times &\prod_{\substack{1\le i<j\le n,\\ \mu_i=\mu_j+1}} \prod_{\substack{1\le k\le \mu_i-\mu_j,\\ k\equiv b_i^{\mu,L}-b_j^{\mu,L} \bmod l}} \\
&\frac{(k - l(h_{b_i^{\mu,L}} - h_{b_j^{\mu,L}}) - c_0 l(a_i^{\mu,L} - a_j^{\mu,L}))^2 - (c_0 l)^2}{(k - l(h_{b_i^{\mu,L}} - h_{b_j^{\mu,L}}) - c_0 l(a_i^{\mu,L} - a_j^{\mu,L}))^2}\,.
\end{aligned} \tag{9.29}
$$

Since the n-tuple μ is above the stabilization point, we have $\mu_i \neq \mu_j$ for all $i \neq j$. Hence w_{μ^+} is the identity element in S_n. With some straightforward computations, we have

$$a_i^{\mu^+,L} = \mathrm{ct}(L(i)) = \mathrm{ct}\left(L(w_\mu\left(w_\mu^{-1}(i)\right))\right) = a^{\mu,L}_{w_\mu^{-1}(i)}\,, \tag{9.30}$$

$$b_i^{\mu^+,L} = \beta(L(i)) = \beta\left(L(w_\mu\left(w_\mu^{-1}(i)\right))\right) = b^{\mu,L}_{w_\mu^{-1}(i)}\,. \tag{9.31}$$

Hence the product $B_{\mu^+,L}C_{\mu^+,L}$ is

$$B_{\mu^+,L}C_{\mu^+,L} = \prod_{i>j} \prod_{\substack{1\le k\le \mu_i-\mu_j,\\ k\equiv b_i^{\mu^+,L}-b_j^{\mu^+,L} \bmod l}}$$

$$\frac{(k-l(h_{b_i^{\mu^+,L}}-h_{b_j^{\mu^+,L}})-c_0l(a_i^{\mu^+,L}-a_j^{\mu^+,L}))^2-(c_0l)^2}{(k-l(h_{b_i^{\mu^+,L}}-h_{b_j^{\mu^+,L}})-c_0l(a_i^{\mu^+,L}-a_j^{\mu^+,L}))^2}$$

$$=\prod_{w_\mu(w_\mu^1(i))-w_\mu(w_\mu^1(j))}\prod_{\substack{1\le k\le\mu_i-\mu_j,\\ k\equiv b^{\mu,L}_{w_\mu^{-1}(i)}-b^{\mu,L}_{w_\mu^{-1}(j)} \bmod l}}$$

$$\frac{(k-l(h_{b^{\mu,L}_{w_\mu^{-1}(i)}}-h_{b^{\mu,L}_{w_\mu^{-1}(j)}})-c_0l(a^{\mu,L}_{w_\mu^{-1}(i)}-a^{\mu,L}_{w_\mu^{-1}(j)}))^2-(c_0l)^2}{(k-l(h_{b^{\mu,L}_{w_\mu^{-1}(i)}}-h_{b^{\mu,L}_{w_\mu^{-1}(j)}})-c_0l(a^{\mu,L}_{w_\mu^{-1}(i)}-a^{\mu,L}_{w_\mu^{-1}(j)}))^2}$$

$$=B_{\mu,L}C_{\mu,L}\,, \tag{9.32}$$

which is the same as the product of index (μ, L). Hence the order of μ does not play a role in the result of the product. From now on, we assume that μ is non-decreasing.

Let L be a standard Young tableau of shape λ. The boxes $L(g)$ and $L(g+1)$ are interchangeable for some $g \in \{1,\dots,n-1\}$, i.e., $L' = s_g \cdot L$ is also a standard Young tableau. We have the relations between the terms $a_i^{\mu,L}, a_i^{\mu,L'}, b_i^{\mu,L}$, and $b_i^{\mu,L'}$ associated with (μ, L) and (μ, L') to be

$$a_i^{\mu,L}=\mathrm{ct}(L(i))=\mathrm{ct}(L'(i))=a_i^{\mu,L'}\qquad\text{for } i\neq g,g+1, \tag{9.33a}$$

$$a_g^{\mu,L}=\mathrm{ct}(L(g))=\mathrm{ct}(L'(g+1))=a_{g+1}^{\mu,L'}, \tag{9.33b}$$

$$a_{g+1}^{\mu,L}=\mathrm{ct}(L(g+1))=\mathrm{ct}(L'(g))=a_g^{\mu,L'}, \tag{9.33c}$$

$$b_i^{\mu,L}=\beta(L(i))=\beta(L'(i))=b_i^{\mu,L'}\qquad\text{for } i\neq g,g+1, \tag{9.33d}$$

$$b_g^{\mu,L}=\beta(L(g))=\beta(L'(g+1))=b_{g+1}^{\mu,L'}, \tag{9.33e}$$

$$b_{g+1}^{\mu,L}=\beta(L(g+1))=\beta(L'(g))=b_g^{\mu,L'}. \tag{9.33f}$$

Compare $B_{\mu,L}C_{\mu,L}$ and $B_{\mu,L'}C_{\mu,L'}$ by the above relations, we get

$$B_{\mu,L'}C_{\mu,L'}$$

$$=\prod_{\substack{1\le k\le\mu_{g+1}-\mu_g,\\ k\equiv b_g^{\mu,L}-b_{g+1}^{\mu,L} \bmod l}}$$

$$\frac{(k-(d_{b_g^{\mu,L}}-d_{b_{g+1}^{\mu,L}})-c_0l(a_g^{\mu,L}-a_{g+1}^{\mu,L})+c_0l)(k-(d_{b_g^{\mu,L}}-d_{b_{g+1}^{\mu,L}})-c_0l(a_g^{\mu,L}-a_{g+1}^{\mu,L})-c_0l)}{(k-(d_{b_g^{\mu,L}}-d_{b_{g+1}^{\mu,L}})-c_0l(a_g^{\mu,L}-a_{g+1}^{\mu,L}))^2}$$

$$\times \prod_{k\equiv b^{\mu,L}_{g+1}-b^{\mu,L}_{g} \mod l} \left(\frac{(k-(d_{b^{\mu,L}_{g+1}}-d_{b^{\mu,L}_{g}})-c_0l(a^{\mu,L}_{g+1}-a^{\mu,L}_{g})+c_0l)(k-(d_{b^{\mu,L}_{g+1}}-d_{b^{\mu,L}_{g}})-c_0l(a^{\mu,L}_{g+1}-a^{\mu,L}_{g})-c_0l)}{(k-(d_{b^{\mu,L}_{g+1}}-d_{b^{\mu,L}_{g}})-c_0l(a^{\mu,L}_{g+1}-a^{\mu,L}_{g}))^2}\right)^{-1}$$

$$\times B_{\mu,L}C_{\mu,L}$$

$$= \prod_{\substack{1\le k\le \mu_{g+1}-\mu_g,\\ k\equiv b^{\mu,L}_{g}-b^{\mu,L}_{g+1} \mod l}} \frac{(k-k^{\mu,L}_{g,g+1})^2-(c_0l)^2}{(k-k^{\mu,L}_{g,g+1})^2} \prod_{\substack{1\le k\le \mu_{g+1}-\mu_g,\\ k\equiv b^{\mu,L}_{g+1}-b^{\mu,L}_{g} \mod l}} (\frac{(k-k^{\mu,L}_{g+1,g})^2-(c_0l)^2}{(k-k^{\mu,L}_{g+1,g})^2})^{-1} B_{\mu,L}C_{\mu,L}$$

$$= \prod_{\substack{1\le k\le \mu_{g+1}-\mu_g,\\ k\equiv b^{\mu,L}_{g}-b^{\mu,L}_{g+1} \mod l}} \frac{(k-k^{\mu,L}_{g,g+1}+c_0l)(k-k^{\mu,L}_{g,g+1}-c_0l)}{(k-k^{\mu,L}_{g,g+1})^2}$$

$$\times \prod_{\substack{1\le k\le \mu_{g+1}-\mu_g,\\ k\equiv b^{\mu,L}_{g+1}-b^{\mu,L}_{g} \mod l}} \left(\frac{(k-k^{\mu,L}_{g+1,g}+c_0l)(k-k^{\mu,L}_{g+1,g}-c_0l)}{(k-k^{\mu,L}_{g+1,g})^2}\right)^{-1} B_{\mu,L}C_{\mu,L}. \tag{9.34}$$

For simplicity, we assign

$$\kappa_1 = k^{\mu,L}_{g,g+1} - c_0 l\,, \tag{9.35}$$

$$\kappa_2 = k^{\mu,L}_{g,g+1} + c_0 l\,. \tag{9.36}$$

Note that $k^{\mu,L}_{g,g+1} = -k^{\mu,L}_{g+1,g}$. Then $B_{\mu,L}C_{\mu,L}$ and $B_{\mu,L'}C_{\mu,L'}$ have the same sign if and only if

$$\prod_{\substack{1\le k\le \mu_{g+1}-\mu_g,\\ k\equiv b^{\mu,L}_{g}-b^{\mu,L}_{g+1} \mod l}} (k-\kappa_1)(k-\kappa_2) \prod_{\substack{1\le k\le \mu_{g+1}-\mu_g,\\ k\equiv b^{\mu,L}_{g+1}-b^{\mu,L}_{g} \mod l}} ((k+\kappa_1)(k+\kappa_2))^{-1} > 0 \tag{9.37}$$

The sign of product in (9.37) oscillates when $\mu_{g+1}-\mu_g$ is less than both $|\kappa_1|$ and $|\kappa_2|$, and remains the same for any μ_g and μ_{g+1} when $\mu_{g+1}-\mu_g > |\kappa_1|\,,|\kappa_1|$. There are four cases in terms of the signs of κ_1 and κ_2 after the sign stabilizes. Before the discussion of the cases, we have an observation as follows.

Observation For any positive number k, any integer a and any positive integer l,

- there are even number of integers in the interval $[0,k]$ that are the same as a modulo l if and only if $2ml \le k-a < (2m+1)l$ for some integer m.

- there are odd number of integers in the interval $[0,k]$ that is the same as a modulo l if and only if $(2m-1)l \le k-a < 2ml$ for some integer m.

Case I Suppose $\kappa_1 > 0$ and $\kappa_2 > 0$. Then the sign of (9.37) is determined by $\prod_{k\equiv b_g^{\mu,L}-b_{g+1}^{\mu,L} \bmod l}(k-\kappa_1)(k-\kappa_2)$. It is positive if and only if $\prod_{\substack{1\le k\le \mu_{g+1}-\mu_g, \\ k\equiv b_g^{\mu,L}-b_{g+1}^{\mu,L} \bmod l}}(k-\kappa_1)$ and $\prod_{\substack{1\le k\le \mu_{g+1}-\mu_g, \\ k\equiv b_g^{\mu,L}-b_{g+1}^{\mu,L} \bmod l}}(k-\kappa_2)$ have the same sign, which is the same as that either there are even number of integers that are less than or equal to both κ_1 and κ_2 that are $b_g^{\mu,L}-b_{g+1}^{\mu,L}$ modulo l, or there are odd number of integers that are less than or equal to both κ_1 and κ_2 that are $b_g^{\mu,L}-b_{g+1}^{\mu,L}$ modulo l. By the above observation, this condition is equivalent to

$$2ml \le \kappa_1 - (b_g^{\mu,L}-b_{g+1}^{\mu,L}) < (2m+1)l$$

and

$$2m'l \le \kappa_2 - (b_g^{\mu,L}-b_{g+1}^{\mu,L}) < (2m'+1)l\,, \tag{9.38}$$

or

$$(2m-1)l \le \kappa_1 - (b_g^{\mu,L}-b_{g+1}^{\mu,L}) < 2ml$$

and

$$(2m'-1)l \le \kappa_2 - (b_g^{\mu,L}-b_{g+1}^{\mu,L}) < 2m'l \tag{9.39}$$

for some integers m and m'. Substituting in the definitions of κ_1 and κ_2, we get

$$2ml \le l(h_{b_g^{\mu,L}} - h_{b_{g+1}^{\mu,L}}) + c_0 l(a_g^{\mu,L}-a_{g+1}^{\mu,L}) - c_0 l - (b_g^{\mu,L}-b_{g+1}^{\mu,L}) < (2m+1)l \tag{9.40}$$

and

$$2m'l \le l(h_{b_g^{\mu,L}} - h_{b_{g+1}^{\mu,L}}) + c_0 l(a_g^{\mu,L}-a_{g+1}^{\mu,L}) + c_0 l - (b_g^{\mu,L}-b_{g+1}^{\mu,L}) < (2m'+1)l\,, \tag{9.41}$$

or

$$(2m-1)l \le l(h_{b_g^{\mu,L}} - h_{b_{g+1}^{\mu,L}}) + c_0 l(a_g^{\mu,L}-a_{g+1}^{\mu,L}) - c_0 l - (b_g^{\mu,L}-b_{g+1}^{\mu,L}) < 2ml, \tag{9.42}$$

and

$$(2m'-1)l \le l(h_{b_g^{\mu,L}} - h_{b_{g+1}^{\mu,L}}) + c_0 l(a_g^{\mu,L}-a_{g+1}^{\mu,L}) + c_0 l - (b_g^{\mu,L}-b_{g+1}^{\mu,L}) < 2m'l\,. \tag{9.43}$$

Case II Suppose $\kappa_1 < 0$ and $\kappa_2 < 0$. Then the sign of (9.37) is determined by

$$\prod_{\substack{1 \le k \le \mu_{g+1} - \mu_g, \\ k \equiv b^{\mu,L}_{g+1} - b^{\mu,L}_g \bmod l}} (k + \kappa_1)(k + \kappa_2)\,.$$

By similar arguments, it is positive if and only if

$$2ml \le -l(h_{b^{\mu,L}_g} - h_{b^{\mu,L}_{g+1}}) + c_0 l(a^{\mu,L}_g - a^{\mu,L}_{g+1}) + c_0 l - (b^{\mu,L}_{g+1} - b^{\mu,L}_g) < (2m+1)l$$

and

$$2m'l \le -l(h_{b^{\mu,L}_g} - h_{b^{\mu,L}_{g+1}}) + c_0 l(a^{\mu,L}_g - a^{\mu,L}_{g+1}) - c_0 l - (b^{\mu,L}_{g+1} - b^{\mu,L}_g) < (2m'+1)l\,, \qquad (9.44)$$

or

$$(2m-1)l \le -l(h_{b^{\mu,L}_g} - h_{b^{\mu,L}_{g+1}}) + c_0 l(a^{\mu,L}_g - a^{\mu,L}_{g+1}) + c_0 l - (b^{\mu,L}_{g+1} - b^{\mu,L}_g) < 2ml$$

and

$$(2m'-1)l \le -l(h_{b^{\mu,L}_g} - h_{b^{\mu,L}_{g+1}}) + c_0 l(a^{\mu,L}_g - a^{\mu,L}_{g+1}) - c_0 l - (b^{\mu,L}_{g+1} - b^{\mu,L}_g) < 2m'l \qquad (9.45)$$

for some integers m and m'. If we multiply by -1 on the inequalities, we get the same condition as in *Case I*.

Case III Suppose $\kappa_1 > 0$ and $\kappa_2 < 0$. Then the sign of (9.37) is determined by

$$\prod_{\substack{1 \le k \le \mu_{g+1} - \mu_g, \\ k \equiv b^{\mu,L}_g - b^{\mu,L}_{g+1} \bmod l}} (k - \kappa_1) \prod_{\substack{1 \le k \le \mu_{g+1} - \mu_g, \\ k \equiv b^{\mu,L}_{g+1} - b^{\mu,L}_g \bmod l}} (k + \kappa_2)\,.$$

By similar arguments, the product is positive if and only if

$$2ml \le l(h_{b^{\mu,L}_g} - h_{b^{\mu,L}_{g+1}}) + c_0 l(a^{\mu,L}_g - a^{\mu,L}_{g+1}) - c_0 l - (b^{\mu,L}_g - b^{\mu,L}_{g+1}) < (2m+1)l$$

and

$$2m'l \le -l(h_{b^{\mu,L}_g} - h_{b^{\mu,L}_{g+1}}) + c_0 l(a^{\mu,L}_g - a^{\mu,L}_{g+1}) - c_0 l - (b^{\mu,L}_{g+1} - b^{\mu,L}_g) < (2m'+1)l\,, \qquad (9.46)$$

or

$$(2m-1)l \le l(h_{b_g^{\mu,L}} - h_{b_{g+1}^{\mu,L}}) + c_0 l(a_g^{\mu,L} - a_{g+1}^{\mu,L}) - c_0 l - (b_g^{\mu,L} - b_{g+1}^{\mu,L}) < 2ml$$

and

$$(2m'-1)l \le -l(h_{b_g^{\mu,L}} - h_{b_{g+1}^{\mu,L}}) + c_0 l(a_g^{\mu,L} - a_{g+1}^{\mu,L}) - c_0 l - (b_{g+1}^{\mu,L} - b_g^{\mu,L}) < 2m'l \tag{9.47}$$

for some integers m and m'. If we multiply by -1 on the second and fourth inequalities, we get the same condition as in *Case I*.

Case IV Suppose $\kappa_1 < 0$ and $\kappa_2 > 0$. Then the sign of (9.37) is determined by

$$\prod_{\substack{1\le k\le \mu_{g+1}-\mu_g, \\ k\equiv b_g^{\mu,L}-b_{g+1}^{\mu,L} \bmod l}} (k-\kappa_2) \prod_{\substack{1\le k\le \mu_{g+1}-\mu_g, \\ k\equiv b_{g+1}^{\mu,L}-b_g^{\mu,L} \bmod l}} (k+\kappa_1)\,.$$

By similar arguments, the product is positive if and only if

$$2ml \le -l(h_{b_g^{\mu,L}} - h_{b_{g+1}^{\mu,L}}) + c_0 l(a_g^{\mu,L} - a_{g+1}^{\mu,L}) + c_0 l - (b_{g+1}^{\mu,L} - b_g^{\mu,L}) < (2m+1)l$$

and

$$2m'l \le l(h_{b_g^{\mu,L}} - h_{b_{g+1}^{\mu,L}}) + c_0 l(a_g^{\mu,L} - a_{g+1}^{\mu,L}) + c_0 l - (b_g^{\mu,L} - b_{g+1}^{\mu,L}) < (2m'+1)l\,, \tag{9.48}$$

or

$$(2m-1)l \le -l(h_{b_g^{\mu,L}} - h_{b_{g+1}^{\mu,L}}) + c_0 l(a_g^{\mu,L} - a_{g+1}^{\mu,L}) + c_0 l - (b_{g+1}^{\mu,L} - b_g^{\mu,L}) < 2ml$$

and

$$(2m'-1)l \le l(h_{b_g^{\mu,L}} - h_{b_{g+1}^{\mu,L}}) + c_0 l(a_g^{\mu,L} - a_{g+1}^{\mu,L}) + c_0 l - (b_g^{\mu,L} - b_{g+1}^{\mu,L}) < 2m'l \tag{9.49}$$

for some integers m and m'. If we multiply by -1 on the first and third inequalities, we get the same condition as in *Case I*.

Recall that the unitarity condition of the irreducible representation V^λ of cyclotomic Hecke algebra with generic parameters is

$$\sin\left(\pi\left(c_0 + c_0(\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))) + h_{\beta(L(i))} - \frac{\beta(L(i))}{l} - h_{\beta(L(i+1))} + \frac{\beta(L(i+1))}{l}\right)\right)$$

$$\cdot \sin\left(\pi\left(c_0 - c_0(\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))) - h_{\beta(L(i))} + \frac{\beta(L(i))}{l} + h_{\beta(L(i+1))} - \frac{\beta(L(i+1))}{l}\right)\right) < 0, \tag{9.50}$$

for all $i = 1, \ldots, n-1$ and $L \in \mathrm{SYT}(\lambda)$ such that $s_i \cdot L$ is a standard Young tableau of λ.

For any $i = 1, \ldots, n$ and any interchangeable boxes $L(i)$ and $L(i+1)$ in the standard Young tableau L of shape λ, the unitarity condition can be rewritten as

$$2m < c_0 + c_0(\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))) + h_{\beta(L(i))} - \frac{\beta(L(i))}{l} - h_{\beta(L(i+1))} + \frac{\beta(L(i+1))}{l} < 2m+1$$

and

$$2m' - 1 < c_0 - c_0(\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))) - h_{\beta(L(i))} + \frac{\beta(L(i))}{l} + h_{\beta(L(i+1))} - \frac{\beta(L(i+1))}{l} < 2m', \tag{9.51}$$

or

$$2m - 1 < c_0 + c_0(\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))) + h_{\beta(L(i))} - \frac{\beta(L(i))}{l} - h_{\beta(L(i+1))} + \frac{\beta(L(i+1))}{l} < 2m$$

and

$$2m' < c_0 - c_0(\mathrm{ct}(L(i)) - \mathrm{ct}(L(i+1))) - h_{\beta(L(i))} + \frac{\beta(L(i))}{l} + h_{\beta(L(i+1))} - \frac{\beta(L(i+1))}{l} < 2m'+1 \tag{9.52}$$

for some integers m and m'. By comparing the condition with the inequalities (9.40) to (9.43), the unitarity condition of $V^{\lambda^{tr}}$ is the same as the condition for $B_{\mu,L}C_{\mu,L}$ to have the same sign for all standard Young tableau L, and μ is non-descending and above the stabilization point. For any pair of standard Young tableau L and $L' = s_g \cdot L$, it implies that $B_{\mu,L}C_{\mu,L}$ and $B_{\mu,L'}C_{\mu,L'}$ have the same sign for μ above the stabilization point if and only if they satisfy the conditions (9.40) to (9.43), or equivalently the square norms of basis element v^L and $v^{s_g \cdot L}$ in $V^{\lambda^{tr}}$ have the same sign.

Any standard tableau J can be obtained from applying a series of transpositions to a standard tableau L on the interchangeable boxes. Whether or not the sign of $B_{\mu,J}C_{\mu,J}$ is the same as $B_{\mu,L}C_{\mu,L}$ is determined by the number of pairs of standard Young tableaux satisfying the condition (9.40) to (9.43) in the process of applying the transpositions. They have the same sign if there are even number of them, and opposite sign otherwise. The same argument applies to the square norms of basis elements

v^L and v^J in the representation $V^{\lambda^{tr}}$. As a result, the sign of $B_{\mu,J}C_{\mu,J}$ is the same as $B_{\mu,L}C_{\mu,L}$ if and only if the square norm of v^L and v^J are of the same sign. If initially the square norms of v^L has the same sign as $A_{\mu,L}B_{\mu,L}C_{\mu,L}$, then $\mathrm{sig}\langle\cdot,\cdot\rangle = p-q$, where p and q are respectively the numbers of the standard tableaux L such that $A_{\mu,L}$ and $B_{\mu,L}C_{\mu,L}$ have the same sign and opposite sign. Otherwise $\mathrm{sig}\langle\cdot,\cdot\rangle = q-p$. Inheriting the notation of p and q, finally let us compute the asymptotic signature as follows.

$$\lim_{M\to\infty}\sum_{N=1}^{M}\frac{\left(\left((N+1)^{n-1}-K_L^N\right)(p-q)-K_L^N\right)}{(N+1)^{n-1}} \leq a_{c,\lambda} \leq \lim_{M\to\infty}\sum_{N=1}^{M}\frac{\left(\left((N+1)^{n-1}-K_L^N\right)(p-q)+K_L^N\right)}{(N+1)^{n-1}}. \tag{9.53}$$

It implies that $a_{c,\lambda} = p-q$, which is the signature of $\langle\cdot,\cdot\rangle$ of $V^{\lambda^{tr}}$, up to a sign. ☐

Corollary 9.10. The Verma module $M_c(\lambda)$ of the rational Cherednik algebra $H_c(G(l,1,n),\mathbb{C}^n)$ is quasi-unitary if and only if the irreducible module $V^{\lambda^{tr}}$ of the cyclotomic Hecke algebra $H_{q,\mathbf{Q}}(n)$ is unitary.

Proof. It is a direct consequence of Theorem 9.8. ☐

10 Exploration of Further Study

10.1 Gaussian Inner Product

As discussed in Section 8, the connection between the β-Hermitian form and the Hermitian form $\langle\cdot,\cdot\rangle$ of representations of cyclotomic Hecke algebra via the KZ functor is not what we expected. In this section, we will explore other Hermitian forms that may have stronger links with the Hermitian form $\langle\cdot,\cdot\rangle$. Etingof defined a Gaussian inner product for real reflection groups based on the β-Hermitian form [9]. Since the Gaussian inner product admits a star operation that sends x to x, it has an integral representation. Shelley-Abrahamson showed that the "kernel" of this integral representation corresponds to the Hermitian form $\langle\cdot,\cdot\rangle$ [18]. We would like to extended this idea to the groups $G(l,1,n)$.

Let us review the definition and properties of the Gaussian inner product for real reflection groups first. Suppose W is a real reflection group and $\mathfrak{h}_{\mathbb{R}}$ is the n-dimensional real reflection representation of W. Let $\mathfrak{h} = \mathbb{C}\otimes_{\mathbb{R}}\mathfrak{h}_{\mathbb{R}}$ be the complexified reflection representation with basis $\{y_1,...,y_n\}$. Let $\mathfrak{h}^*$

be the dual of $\mathfrak{h}$ with dual basis $\{x_1, ..., x_n\}$. Let $H_c(W, \mathfrak{h})$ be the rational Cherednik algebra of W. In real reflection groups, any non trivial eigenvalues of reflections are -1. Recall the Euler element $\mathbf{h} = \sum_{i=1}^n x_i y_i + \frac{n}{2} - \sum_{s\in S} \frac{2c_s}{1-\lambda_s} s = \sum_{i=1}^n x_i y_i + \frac{n}{2} - \sum_{s\in S} c_s s$. We define the operators F, E, H to be

$$F := \frac{1}{2}\sum_{i=1}^n y_i^2, \quad E := -\frac{1}{2}\sum_{i=1}^n y_i^2.$$

The set $\{E, F, \mathbf{h}\}$ forms a sl_2-triple.

Definition 10.1. Recall that $c_s^\dagger = \overline{c_{s^{-1}}}$. Let the parameter c satisfy $c = c^\dagger$. Let $M_c(\lambda)$ be a Verma module acquired by tensoring the irreducible representation λ of W. We define the Gaussian inner product of $M_c(\lambda)$ with respect to the parameter c to be

$$\gamma_{c,\lambda}(m, m') = \beta_{c,\lambda}(\exp(F) \cdot m, \exp(F) \cdot m') \,. \tag{10.1}$$

This bilinear form is also called γ-Hermitian form.

Remark. The Gaussian inner product is well-defined, since the operator F acts nilpotently on the module $M_c(\lambda)$.

The following proposition introduces some nice properties of the Gaussian inner product, based on Proposition 4.2.3 in [18]

Proposition 10.2. The Gaussian inner product satisfies

$$\gamma_{c,\lambda}(xm, m') = \gamma_{c,\lambda}(m, xm'), \quad \text{for any } x \in \mathfrak{h}^*{}_{\mathbb{R}},\ m, m' \in M_c(\lambda)\,, \tag{10.2}$$

$$\gamma_{c,\lambda}(ym, m') = \gamma_{c,\lambda}(m, (-y + T(y))m'), \quad \text{for any } y \in \mathfrak{h}_{\mathbb{R}},\ m, m' \in M_c(\lambda)\,. \tag{10.3}$$

We can extend the map sending x to x and y to $-y+T(y)$, to a star operation on $H_c(W, \mathfrak{h})$, which we call the $\bullet$-operation .

Lemma 10.3. The $\bullet$-operation is conjugate to the $*$-operation by $\exp(-F)\exp(E)$, i.e.,

$$a^\bullet = \exp(-F)\exp(E)a^*\exp(-E)\exp(F), \quad \text{for any } a \in H_c(W, \mathfrak{h})\,. \tag{10.4}$$

Proof. By the definition of the Gaussian inner product, for any $m, m' \in M_c(\lambda)$ and $a \in H_c(W, \mathfrak{h})$ we have

$$\begin{aligned}
\gamma_{c,\lambda}(a \cdot m, m') &= \beta_{c,\lambda}(\exp(F)(a \cdot m), \exp(F)m') \\
&= \beta_{c,\lambda}(m, \exp(-E)a^* \exp(F)m') \\
&= \beta_{c,\lambda}(\exp(-F)\exp(F)m, a^* \exp(-E)\exp(F)m') \\
&= \beta_{c,\lambda}(\exp(F)m, \exp(E)a^* \exp(-E)\exp(F)m') \\
&= \beta_{c,\lambda}(\exp(F)m, \exp(F)\exp(-F)\exp(E)a^* \exp(-E)\exp(F)m') \\
&= \gamma_{c,\lambda}(m, \exp(-F)\exp(E)a^* \exp(-E)\exp(F)m'). \qquad (10.5)
\end{aligned}$$

□

Without assuming $c = c^\dagger$, we can extend the idea of the Hermitian form $\beta_{c,\lambda}$ on the Verma module $M_c(\lambda)$ to a pairing

$$\beta_{c,\lambda} : M_c(\lambda) \times M_{c^\dagger}(\lambda) \to \mathbb{C}$$

given in Definition 4.1.1 [18] that coincides with the Hermitian form of λ on degree zero and contravariant with respect to the $*$-operation. It descends to the Hermitian form $\beta_{c,\lambda}$ on $M_c(\lambda)$, once the restriction $c = c^\dagger$ is added back. The Gaussian inner product $\gamma_{c,\lambda}$ on $M_c(\lambda) \times M_{c^\dagger}(\lambda)$ can be defined accordingly. The integral representation of the Gaussian inner product is constructed by Shelley-Abrahamson in Theorem 4.3.1 [18] as follows.

Theorem 10.4. For any finite Coxeter group W and irreducible representation λ of W, there is a unique family $K_{c,\lambda}$ holomorphic in c invariant under conjugation, of $\mathrm{End}_{\mathbb{C}}(\lambda)$-valued tempered distributions on $\mathfrak{h}_{\mathbb{R}}$, such that the following integral representation of the Gaussian inner product $\gamma_{c,\lambda}$ holds for all c invariant under conjugation:

$$\gamma_{c,\lambda}(m, m') = \int_{\mathfrak{h}_{\mathbb{R}}} m'(x)^\dagger K_{c,\lambda}(x) m(x) \exp\left(\frac{-|x|^2}{2}\right) \mathrm{d}x \,, \qquad (10.6)$$

for all $m, m' \in \mathbb{C}[\mathfrak{h}] \otimes \lambda$, where we make the standard identification of $M_c(\lambda)$ and $M_{c^\dagger}(\lambda)$ with $\mathbb{C}[\mathfrak{h}] \otimes \lambda$. Furthermore, $K_{c,\lambda}$ satisfies the additional property: For any $x \in \mathfrak{h}_{\mathbb{R},reg}$, the operator $K_{c,\lambda}(x) \in \mathrm{End}_{\mathbb{C}}(\lambda)$ determines a pairing that is compatible with the $\star$-operation in the cyclotomic

Hecke algebra

$$\langle \cdot, \cdot \rangle : \mathrm{KZ}(M_c(\lambda)) \times \mathrm{KZ}(M_{c^\dagger}(\lambda)) \to \mathbb{C}$$

by the formula $(v_1, v_2) = v_2^\dagger K_{c,\lambda} v_1$. When we have $c = c^\dagger$, the pairing reduces to the Hermitian form $\langle \cdot, \cdot \rangle$ of the cyclotomic Hechke algebra up to a positive scalar mentioned in the previous sections.

There is a direct link between the Hermitian form $\beta_{c,\lambda}$ and $\langle \cdot, \cdot \rangle$ of cyclotomic Hecke algbera via the operator $K_{c,\lambda}$ for real reflection groups. The question is whether we can extend it to $G(l, 1, n)$. The definition of the Gaussian Hermitian form remains valid for the Verma modules of $H_c(W, \mathfrak{h})$ for $G(l, 1, n)$. It is still compatible with the $\bullet$-operation. However, we do not always have $x^\bullet = x$ for any group $G(l, 1, n)$. Hence, there is no obvious integral representation for the Gaussian inner product of $G(l, 1, n)$. One method to tackle the problem is to compute the operator $K_{c,\lambda}$ directly, like in [7] and [6]. Another method is to find other connections between the Gaussian inner product and $K_{c,\lambda}$, which will finally leads to the Hermitian form on representations of cyclotomic Hecke algebra.

In the construction of the KZ functor, one needs to compute the local system and derive the representation of the fundamental group. Losev in [17] showed that it is possible to reverse this process and create a module of the rational Cherednik algebra out of a representation of the cyclotomic Hecke algebra. It is worth exploring how the reversed process is related to the Hermitian forms and star operations of both algebras.

10.2 Janzten Filtration

In the search of the solution to the unitarity conditions of modules of rational Cherednik algebras for the non-generic parameters, we encounter the Janzten filtration. Instead of viewing the Hermitian form $\beta_{c,\lambda}$ in terms of a binary operation, we could view it as an analytic function parametrized by the parameters c. It filters the Verma modules of rational Cherednik algebra by the order of the zeros of its $\beta_{c,\lambda}$ Hermitian form.

Let us define the Janzten filtration in a general setting [22].

Definition 10.5. Let E be a finite dimensional complex vector space, and $(\cdot, \cdot)_t$ an analytic family of Hermitian forms on E defined for small real t. Assume that $(\cdot, \cdot)_t$ is non-degenerate for sufficiently small non-zero t. The Jantzen filtration of E is the sequence of subspaces

$$E = E_0 \supset E_1 \supset E_2 \supset \dots E_N = \{0\},$$

defined as follows. Fix $k \in \mathbb{N}$. Then an element $e \in E$ belongs to E_k if and only if for some $\epsilon > 0$ there is an analytic function

$$f_e : (-\epsilon, \epsilon) \to E,$$

with the following properties:

1. $f_e(0) = e$,

2. $(f_e(t), e')_t$ vanishes at least to order k at $t = 0$ for any $e' \in E$.

Suppose e, e' are in E_k; choose $f_e, f_{e'}$ accordingly. Set

$$(e, e')^k = \lim_{t \to 0} \frac{1}{t^k} (f_e(t), f_{e'}(t))_t \,.$$

It is independent of the choices of f_e and $f_{e'}$.

Jantzen also showed the connection between $(\cdot, \cdot)^k$ and the submodule E_k and E_{k-1}. It is quoted by Vogan in Theorem 3.2 of [22].

Theorem 10.6. In the setting of Definition 10.5, $(\cdot, \cdot)^k$ is a Hermitian form on E_k, with radical exactly equal to E_{k-1}. In particular,

1. $\mathrm{Rad}(\cdot, \cdot)^0 = E_1$; and

2. $(\cdot, \cdot)^k$ is a non-degenerate Hermitian form on E_k / E_{k-1}.

Suppose in rational Cherednik algebra $H_c(W, \lambda)$ of $G(l, 1, n)$, the parameter c satisfies $c = c^\dagger$. For all $i = 0, \ldots, l-1$, let $c_i(t) = c_i + tc_i'$, for some $c_i \in \mathbb{C}$ such that $c(t) = c(t)^\dagger$. Assume that $\beta_{c(t),\lambda}$ is non-degenerate for $t \in (-\delta, \delta) \setminus \{0\}$. Recall that $M_c(\lambda)$ is isomorphic to $S[\mathfrak{h}^*] \otimes_{\mathbb{C}} \lambda$, which we will treat as E in Definition 10.5. Let $M_c(\lambda)[d]$ be the finite dimensional subspace of $M_c(\lambda)$ containing elements of degree d, and $\beta_{c(t),\lambda}[d]$ is the Hermitian form $\beta_{c(t),\lambda}$ restricted on $M_c(\lambda)[d]$. Then the subspace $M_c(\lambda)[d]$ along with $\beta_{c(t),\lambda}[d]$ forms a filtration with variables t,

$$M_c(\lambda)[d] = M_c(\lambda)[d]^{\geq 0} \supset M_c(\lambda)[d]^{\geq 1} \supset \ldots \supset M_c(\lambda)[d]^{\geq N} = \{0\}.$$

Let the Hermitian form $\beta_{c,\lambda}[d]^{\geq k}$ be the Hermitian form $(\cdot, \cdot)^k$ in the definition, that is

$$\beta_{c,\lambda}[d]^{\geq k}(m, m') = \lim_{t \to 0} \frac{1}{t^k} \beta_{c(t),\lambda}[d](f_m(t), f_{m'}(t)),$$

for some analytic functions $f_m(t), f_{m'}(t)$ defined in Definition 10.5. By Theorem 10.6, the radical of the Hermitian form $\beta_{c,\lambda}[d]^{\geq k}$ is exactly $M_c(\lambda)[d]^{\geq k}$, and $\beta_{c,\lambda}[d]^{\geq k}$ is non-degenerate on $M_c(\lambda)[d]^{\geq k}/M_c(\lambda)[d]^{\geq k-1}$. Let us define $M_c(\lambda)^{\geq k} := \bigoplus_{d\geq 0} M_c(\lambda)[d]^{\geq k}$. Then there is filtration

$$M_c(\lambda)^{\geq 0} = M_c(\lambda) \supset M_c(\lambda)^{\geq 1} \supset \dots .$$

Lemma 10.7. The filtration of $M_c(\lambda)$ by the subspaces $M_c(\lambda)^{\geq k}$ is a filtration by $H_c(W, \mathfrak{h})$-submodules. We have $M_c(\lambda)^{\geq k} = 0$ for sufficiently large k.

Proof. The argument is the same as the proof of Lemma 3.1.3 in [18]. □

The filtration in the above lemma is referred to as Janzten filtration of the Verma module $M_c(\lambda)$. If we denote $M_c(\lambda)^k$ as the quotient $M_c(\lambda)^{\geq k}/M_c(\lambda)^{\geq k-1}$, then we have an exact sequence

$$0 \to M_c(\lambda)^{\geq k-1} \to M_c(\lambda)^{\geq k} \to M_c(\lambda)^k \to 0 .$$

In particular the second submodule $M_c(\lambda)^{\geq 1}$ in the filtration is the kernel of $\beta_{c,\lambda}$, i.e the largest submodule inside $M_c(\lambda)$. Hence the quotient $M_c(\lambda)^1$ is the irreducible module $L_c(\lambda)$ by definition. Since the KZ functor is exact, it preserves the Janzten filtration by preserving these exact sequences.

Example 10.8. In this example, we will compute the Janzten filtration of the Verma module $M_c(\lambda)$ of $H_c(\mathbb{Z}_l, \mathbb{C})$. Recall that the rational Cherednik algebra $H_c(\mathbb{Z}_l, \mathbb{C})$ has parameters $c_1, \dots, c_{l-1}$, where c_i corresponds to ζ_1^i for each i. Let $c_i(t) = c_i + tc_i'$ be the parameters defined above. Let $M_c(\lambda)$ be the Verma module, where λ is the trivial representation of $\mathbb{Z}_l$. In the Example 1.20, we compute the value $\beta_{c,\lambda}(x^d, x^d)$, which equals to

$$\beta_{c,\lambda}(x^d, x^d) = \prod_{i=1}^{d}(i - b_i) ,$$

where $b_i = 2\sum_{j=1}^{l-1} \frac{1-\zeta^{-ij}}{1-\zeta^{-j}} c_j$. The subspace $M_c(\lambda)[d]$ is one-dimensional and is generated by x^d. Suppose there are k many zeros of $\beta_{c,\lambda}(x^d, x^d)$ at degree d, that is there are integers $0 < i_1 < \dots, < i_k \leq d$ such that $i_j = b_{i_j}$ for all $j = 1, \dots, k$. Then the Janzten filtration looks like

$$M_c(\lambda)[d] \supset M_c(\lambda)[d]^{\geq 1} \supset \dots \supset M_c(\lambda)[d]^{\geq k} \supset M_c(\lambda)[d]^{\geq k+1} = \{0\} .$$

Let $b_i(t)$ denote $2\sum_{j=1}^{l-1} \frac{1-\zeta^{-ij}}{1-\zeta^{-j}} c_j(t)$. Then we have

$$b_i(t) = b_i + 2t \sum_{j=1}^{l-1} \frac{1-\zeta^{-ij}}{1-\zeta^{-j}} c'_j \,.$$

Let the analytic function f_{x^d} be the identity map. The Hermitian form $\beta^k_{c,\lambda}$ becomes

$$\begin{aligned}
\beta_{c,\lambda}[d]^{\geq k}(x^d, x^d) &= \lim_{t\to 0} \frac{1}{t^k} \prod_{i=1}^{d} (i - b_i(t)) \\
&= \lim_{t\to 0} \frac{1}{t^k} \prod_{\substack{i=1,\\ i\neq i_1,\ldots,i_k}}^{d} (i - b_i(t)) \prod_{j=1}^{k} (j - b_j - 2t \sum_{j=1}^{l-1} \frac{1-\zeta^{-ij}}{1-\zeta^{-j}} c'_j) \\
&= \lim_{t\to 0} \frac{1}{t^k} \prod_{\substack{i=1,\\ i\neq i_1,\ldots,i_k}}^{d} (i - b_i(t)) \prod_{j=1}^{k} (-2t \sum_{j=1}^{l-1} \frac{1-\zeta^{-ij}}{1-\zeta^{-j}} c'_j) \\
&= \prod_{\substack{i=1,\\ i\neq i_1,\ldots,i_k}}^{d} (i - b_i) \prod_{j=1}^{k} (-2 \sum_{j=1}^{l-1} \frac{1-\zeta^{-ij}}{1-\zeta^{-j}} c'_j) \,.
\end{aligned} \tag{10.7}$$

The direction of the deformation differs, for different parameters c'_i. There are at most $l-1$ many zeros in $\beta_{c,\lambda}(x^d, x^d)$, since there are at most $l-1$ many distinct b_i. Hence $M_c(\lambda)^k = \{0\}$, if $k \geq l$.

Appendix

A Young's Semi-Normal Form of Cyclotomic Hecke Algebra of Complex Reflection Group

Let $q \in \mathbb{C}^{\times}$. We define e_q to be the smallest positive integer e such that $1 + q + \ldots q^{e-1} = 0$. If there is no such positive integer e, we set $e_q := 0$. Brudan and Kleshchev constructed Young's semi-normal form of the integral KLR algebra of symmetric groups when $e_q = 0$ in Section 5 of [4]. In this appendix, we want to extend the result and show that Young semi-normal forms exist for all KLR algebras of $G(l, 1, n)$.

Let us fix a complex number c and a positive integer l. Let D be a finite subset of $\mathbb{C}/\mathbb{Z}$. We make D a quiver by adding edges $d \to d + c$ joined by adding c, whenever both d and $d + c$ are in D. Let $\Lambda = (\Lambda_d)_{d \in D}$ be a sequence of integers with $\sum_{d \in D} \Lambda_d = l$. Then $R^{\Lambda}_{D,n}$ is the quotient of the KLR algebra associated with the quiver D under the ideal I^{Λ} as in Definition 5.8. Let $s(\Lambda)$ be some l-tuple $(g_1, \ldots, g_l)$, where g_i are vertices of D and g_i appear Λ_{g_i} times in $s(\Lambda)$. Consider an l-multipartition of n. For any standard Young tableau L of λ, we define the residue sequence

$$\mathbf{r}^{\Lambda}_L = (g_{\beta(L(1))} + c\mathrm{ct}(L(1)), \ldots, g_{\beta(L(n))} + c\mathrm{ct}(L(n))). \tag{A.1}$$

We then construct the Young's semi-normal form as follows.

Theorem A.1. Assume D is a union of the line segments $g_i - (n-1)c \to \ldots g_i \to g_i + c \ldots \to g_i + (n-1)c$ for all i. These line segments don't interact with each other, and there is no loop in D. Then for any l-multiparititon λ of n, there is a representation W^{λ} of $R^{\Lambda}_{D,n}$, generated by the basis $\{v_L \mid L \in \mathrm{SYT}(\lambda)\}$. The generators of $R^{\Lambda}_{D,n}$ acts on the basis element v_L by

$$e(\mathbf{d})v_L = \delta_{\mathbf{d},\mathbf{r}_L} v_L, \quad \text{for } \mathbf{d} \in D^n \tag{A.2}$$

$$\mathrm{y}_i \cdot v_L = 0, \quad \text{for } i = 1, \ldots, n \tag{A.3}$$

$$\psi_i \cdot v_L = v_{s_i \cdot L}, \quad \text{for } i = 1, \ldots, n-1 \tag{A.4}$$

where $s_i \cdot L$ is the tableau L after switching the positions of i and $i+1$ if $s_i \cdot L$ is standard, and $v_{s_i \cdot L}$ is zero otherwise.

Proof. It is sufficient to check that the actions satisfy (5.1) to (5.12). The equations (5.1) to (5.4) are trivial. If we switch the position of i and $i+1$ in the standard Young tableau L, then the boxes containing i and $i+1$ exchange the contents and the positions in λ^j. Hence we have

$$\mathbf{r}^{\Lambda}_{s_i \cdot L} = s_i \cdot \mathbf{r}^{\Lambda}_L, \tag{A.5}$$

which implies the action satisfies (5.5). For (5.6), since the distance between i and j is larger than 1, we have

$$(\psi_j \psi_i) \cdot v_L = \psi_j \cdot v_{s_i \cdot L} = v_{s_j \cdot (s_i \cdot L)} = v_(s_i \cdot (s_j \cdot L)) = \psi_i \cdot v_{s_j \cdot L} = (\psi_i \psi_j) \cdot v_L. \tag{A.6}$$

Both the left hand side and the right hand side of (5.7) to (5.9) are zero when acting on the basis element v_L. Hence those equations are satisfied.

In (5.10) and (5.11), if $d_i = d_{i+1}$ in $\mathbf{d} \in D^n$, then the corresponding $\mathbf{r}^{\Lambda}_L$ satisfies

$$g_{\beta(L(i))} + c\mathrm{ct}(L(i)) = g_{\beta(L(i+1))} + c\mathrm{ct}(L(i+1))\,.$$

Because of the shape of D, the boxes containing i and $i+1$ are in the exact same position, which is impossible. Hence there is no standard Young tableau L corresponding to $\mathbf{d}$ and the equation acts on the basis element by zero.

There are five cases in (5.11), where the last case doesn't apply here. We have covered the first case $d_i = d_{i+1}$ in the previous argument. If $d_i \to d_{i+1}$ or $d_i \leftarrow d_{i+1}$, then the boxes containing i and $i+1$ in the corresponding standard Young tableau L are in the same coordinate λ^k in λ, and switching the position of the two boxes gives us a non-standard tableaux. Hence ψ_i acts by zero, which is the same as $\pm(\mathrm{y}_{i+1} - \mathrm{y}_i)$. If none of the constrains apply, then $L(i)$ and $L(i+1)$ are interchangeable. We have

$$\psi_i^2 \cdot v_L = \psi_i \cdot v_{s_i \cdot L} = v_L = e(\mathbf{d}) \cdot v_L. \tag{A.7}$$

In the first three cases of (5.12), there is no standard Young tableau L corresponding to $\mathbf{d}$. Thus both sides of the equation are zero. In the last case, on one hand, if one of the pairs $(L(i), L(i+1))$, $(L(i+1), L(i+2))$, and $(L(i), L(i+2))$ is not interchangeable, then both sides of the equation act by zero. On the other hand, if they are all interchangeable, then both sides of the equation also act by zero because of $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$. $\square$

Index

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