Asymptotics of Operator Semigroups and Applications

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Abstract

This thesis is concerned with the quantified asymptotic theory of operator semigroups and its applications. There are two sides of this story that are studied in the two main parts of the thesis respectively. The first side of the story is the asymptotic theory of strongly continuous operator semigroups arising from the study of damped wave equations and the countable spectrum theorem for strong stability. In the first half of the thesis, we apply known theory to two coupled wave-heat systems in order to derive energy decay estimates, before developing new direct integral theory so that one may explore asymptotic ideas there. The second side of the story is the asymptotic theory of discrete operator semigroups arising from the much celebrated Katznelson–Tzafriri and mean ergodic theorems. In the second half of the thesis, we establish new abstract results in this area, proving sharp estimates for the rate of decay between consecutive powers of operators on Hilbert spaces. Additionally, we prove upper and lower bounds for this same decay in the case of multiplication-like operators that improve on previously known theory.

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Since I believe this thesis as a whole is greater than the sum of its parts...

To Mark and Zoe.

Two of my greatest delights at all times as well as two of my sharpest sorrows whenever we must say goodbye.

They are the beloved children of my beautiful sister Joy and her husband Pedro, the coolest man on earth.

Contents

Part I

Preliminaries

Chapter 1

Introduction

In this introductory chapter, we briefly give the background and motivation for the two major aims of this thesis, before outlining its contents. The goal of what follows is to make it evident that these two major aims belong respectively to the two parallel halves of quantified asymptotic theory for operator semigroups – namely the continuous and discrete halves.

Chapter 3 will provide more detailed surveys of the known theory and will have some overlap with this chapter, but the hope is that what follows immediately below will perk the reader's interest with a tantalising taste of what is to come. See Chapter 2 for the relevant definitions.

1.1 Background and motivation

Consider the following *abstract Cauchy problem* for a closed operator A on a Banach space X with initial value $x \in X$:

$$\begin{cases} u'(t) = Au(t), \quad t \geq 0, \\ u(0) = x. \end{cases} \tag{ACP}$$

A well-known and much studied example of such a problem is when A is the Laplacian Δ on the function space $X = L^2(\Omega)$ for some smooth bounded domain $\Omega \subset \mathbb{R}^n$ with $n \geq 1$. In this case, $u : \mathbb{R}_+ \to L^2(\Omega)$ represents the temperature in a material and t denotes time. As in this famous example, the evolution of other quantities in time and (not necessarily physical) space can often be formulated by taking different operators A in (ACP). Hence, many abstract Cauchy problems are motivated by partial differential equations that model various systems.

In order to characterise problems that model a system 'well', a problem is said to be *well-posed in the sense of Hadamard* if 1. a solution exists, 2. the solution is unique, and 3. the solution depends continuously on the initial conditions. In the setting of abstract Cauchy problems for a closed operator A, these three conditions are formalised and satisfied by the requirement that A generates a C_0-semigroup. We motivate and define these concepts as follows.

If A were to be given by multiplication by a scalar a on the space $X = \mathbb{C}$, the solution of (ACP) for any $x \in \mathbb{C}$ would be $u : t \mapsto x \exp(ta)$, an exponential function. In particular, the exponential satisfies $\exp(s)\exp(t) = \exp(s+t)$ for all $s, t \geq 0$. In order to generalise the exponential for non-scalar arguments, consider the following functional equation for a map $T : \mathbb{R}_+ \to \mathcal{B}(X)$:

$$\begin{cases} T(s)T(t) = T(s+t), & s, t \geq 0, \\ T(0) = I. \end{cases} \tag{FE}$$

The first of the above requirements is often referred to as the *semigroup property.* A map $T : \mathbb{R}_+ \to \mathcal{B}(X)$ is called a *strongly continuous semigroup* or C_0-semigroup if T satisfies (FE) and is strongly continuous, that is, $t \mapsto T(t)x$ is continuous for each $x \in X$. In this case, T is also denoted by $(T(t))_{t\geq 0}$ and is commonly referred to by the term *semigroup* alone. Each C_0-semigroup has a unique *generator* A, an operator on X defined by

$$\begin{aligned} D(A) &= \{x \in X : \lim_{t\to 0+} t^{-1}(T(t)x - x) \text{ exists}\}, \\ Ax &= \lim_{t\to 0+} t^{-1}(T(t)x - x), \quad x \in D(A). \end{aligned}$$

In other words, the generator can be thought of as the right-derivative of T at 0 when it exists, and it uniquely determines the C_0-semigroup. This is what one would expect in the scalar case. Along with other properties which we will not state here, A is densely defined, closed, and, importantly, satisfies the conditions that

$$\frac{d}{dt} T(t)x = AT(t)x, \quad x \in D(A), t \geq 0,$$

and

$$T(t)x - x = A \int_0^t T(s)x \, ds, \quad x \in X, t \geq 0.$$

In particular, if A generates a C_0-semigroup T, the orbit $u_x : \mathbb{R}_+ \to X$ given by $u_x(t) = T(t)x$ is continuously differentiable and satisfies (ACP) whenever $x \in D(A)$. In this case, u_x is said to be a *classical solution* of (ACP). More generally, regardless of whether or not x lies in the domain of A, u_x always satisfies the following integral version of (ACP):

$$u(t) = A \int_0^t u(s) \, ds + x.$$

In this case, u_x is called a *mild solution*. Crucially, u_x is the unique solution of the above equation. Furthermore, as T maps into the space of bounded linear operators on X, the orbits depend continuously on the initial conditions.

The problem (ACP) is thus said to be *well-posed* if A generates a C_0-semigroup $T : \mathbb{R}_+ \to \mathcal{B}(X)$. See also [EN00, Chapter 2 Theorem 6.7] and the ensuing discussion there for a lucid exposition of well-posedness in the setting of C_0-semigroups and its relation to Hadamard's criteria.

Having thus motivated C_0-semigroups, a standard question for us to ask for any C_0-semigroup T is what happens as $t \to \infty$. The famous Arendt-Batty-Lyubich-Vũ theorem (also known as the countable spectrum theorem, see [AB88, LVu88] and Theorem 3.1.3) gives sufficient spectral conditions for the generator of a bounded C_0-semigroup T such that

$$\lim_{t\to\infty} \|T(t)x\| \to 0, \quad x \in X. \tag{1.1.1}$$

See Theorem 3.1.3 for the full statement. Whenever a C_0-semigroup T satisfies (1.1.1), it is said to be *strongly stable*. Note that the rate of convergence might depend on the individual $x \in X$. If instead, $\|T(t)\| \to 0$ as $t \to \infty$, that is, if T converges to 0 in operator norm, then T is said to be *uniformly stable*. This convergence is necessarily exponential due to the semigroup property, and uniform stability is much stronger than (ironically) strong stability. See [vN96] for a sense of the intense scrutiny that uniform stability has been subjected to.

The focus of this thesis is a notion of stability that sits in between uniform and merely strong stability. When a C_0-semigroup T is bounded, that is, when there exists $C > 0$ such that $\|T(t)\| \leq C$ for all $t \geq 0$, the spectrum of its generator A, denoted $\sigma(A)$, lies in the closed left half-plane. In this case, $I - A$ is then invertible. The stability of interest is then the following:

$$\lim_{t\to\infty} \|T(t)(I-A)^{-1}\| \to \infty. \tag{1.1.2}$$

This is known in the literature as *semi-uniform stability*. Note that (1.1.2) can be restated as

$$\lim_{t\to\infty} \left(\sup\{\|T(t)x\| : x \in D(A), \|x\|_A = 1\}\right) = 0,$$

where $\|\cdot\|_A$ is the graph norm. Hence, we have stability for classical solutions that can be described with a degree of uniformity.

Batty and Duyckaerts showed that for (1.1.2) to occur, $\sigma(A) \cap i\mathbb{R}$ must be empty. Furthermore, they showed that the rate of decay in (1.1.2) is determined by the rate at which $\|R(is, A)\|$, the norm of the resolvent at $\lambda = is$, grows as $|s| \to \infty$ (see [BD08] and Theorem 3.1.7). Their work was an important catalyst in the explosion of interest in quantified asymptotics for C_0-semigroups (see Section 3.1).

An important class of examples in which the generator A of a bounded C_0-semigroup T satisfies $\sigma(A) \cap i\mathbb{R} = \emptyset$ is that of damped wave equations. Consider the following damped wave equation with Dirichlet boundary conditions and some initial data:

$$\frac{\partial^2 v}{\partial t^2} - \Delta v + a(\xi)\frac{\partial v}{\partial t} = 0, \quad \xi \in \Omega, t > 0. \tag{1.1.3}$$

Taking $\Omega \subset \mathbb{R}^n$ to be a bounded domain for some $n \geq 1$, the function $a : \Omega \to [0, \infty)$ measures the damping within Ω. Assuming that Ω has sufficiently smooth boundary and a is continuous, integration by parts shows that the energy defined by

$$E(t) = \frac{1}{2}\int_\Omega \|\nabla v\|^2 + \left|\frac{\partial v}{\partial t}\right|^2 d\xi, \quad t > 0,$$

is non-increasing. Ignoring degenerate cases, the energy actually decays to zero. In order to quantify the rate at which this energy goes to 0, equation (1.1.3) can be cast as a well-posed abstract Cauchy problem and, without going into the details, the resulting C_0-semigroup T is bounded and has a generator A satisfying $\sigma(A) \cap i\mathbb{R} = \emptyset$. Furthermore, the energy can then be written as

$$E(t) = \frac{1}{2}\|T(t)x\|^2, \quad t > 0,$$

where x encodes the initial data. From this we can see that the theory of quantified asymptotics for C_0-semigroups can demonstrably be used in interesting applications with real-world significance. **The first major aim of this thesis is to obtain energy decay estimates for particular physical systems using quantified asymptotic methods for C_0-semigroups.**

Turning now to the discrete side of the adventure which, in many ways, runs in parallel with the above strongly continuous story, we can take any power-bounded operator T on a Banach space X and produce a bounded discrete operator semigroup by considering $(T^n)_{n\geq 0}$, the powers of T. The central result around which the asymptotic theory of discrete operator semigroups is built is the much celebrated theorem of Katznelson and Tzafriri (see [KT86] and Theorem 3.2.2). In its simplest form, it states that for T with the properties just mentioned, $\|T^n(I - T)\| \to 0$ as $n \to \infty$ if and only if $\sigma(T) \subset \mathbb{T} \cup \{1\}$. As we shall later see in Section 3.2, quantifying this decay is the discrete semigroup analogue of quantifying the decay in (1.1.2).

The main motivation behind quantifying this decay is the mean ergodic theorem (see [Kre85, Section 2.1]). In this discrete setting, it says that if T is power-bounded and X is reflexive, then

$$X = \mathrm{Ker}\,(I - T) \oplus \overline{\mathrm{Ran}\,(I - T)}.$$

Furthermore, if $x \in X$, then

$$\frac{x + Tx + \cdots + T^{n-1}x}{n} \to Px, \quad n \to \infty,$$

where P is the projection onto $\mathrm{Ker}\,(I - T)$, the space of points fixed by T. The average of powers above is often called the *Cesàro average* of T. In particular, if $\sigma(T) \cap \mathbb{T} \subset \{1\}$, Theorem 3.2.2 says that not only do the Cesàro averages converge to the projection, on a dense set the orbits also converge to the projection: For $x = x_1 + x_2 \in \mathrm{Ker}(I - T) \oplus \mathrm{Ran}(I - T)$,

$$T^n x = T^n(x_1 + x_2) = T^n x_1 + T^n(I - T)y_2 \to x_1, \quad n \to \infty.$$

Thus, knowing the rate at which $\|T^n(I - T)\|$ decays tells us how fast these orbits converge to the projection onto the space of fixed points. In applications to dynamical systems, this corresponds to telling us how quickly the orbits would converge to equilibrium.

Seifert was the first to quantify the decay of $\|T^n(I - T)\|$ in terms of resolvent growth near the singularity $1 \in \sigma(T)$ for general power-bounded operators T with $\sigma(T) \cap \mathbb{T} = \{1\}$ (see [Sei15, Sei16] and Theorem 3.2.3). Besides the general setting, he also proved various similar results that yield more precise rates for specific cases (see Section 3.2). Concrete applications of this discrete asymptotic theory range from problems in probability theory to evolution equations and alternating projections (see for example [BS16, BS17, CL16, Dun11, PS18, PS19]). **The second major aim of this thesis is to develop new abstract theorems that fit in and largely complete the quantified asymptotic storyline for discrete operator semigroups.**

1.2 Overview of thesis

This thesis is concerned with the quantified asymptotic theory of operator semigroups, presented in four parts. The main body of this work resides in Parts II and III, where most of the new contributions this thesis makes can be found.

Part I is preliminary, consisting of three chapters. The first is this current chapter, the second is on notation and basic results, and the third provides two surveys. The former survey provides the theoretic machinery that powers the physical applications in Part II. The latter survey gives the background and story so far to which the results in Part III build on and add to.

Part II explores the quantified asymptotic theory of strongly continuous semigroups, with a focus on applications. It consists of the following three chapters. Chapter 4 analyses a coupled wave-heat-wave system in one dimension with Dirichlet boundary conditions. This is modelled after the similar wave-heat system studied by Batty, Paunonen, and Seifert [BPS16] and we follow their approach somewhat closely, though not exactly. In the chapter we prove the well-posedness of the problem via a C_0-semigroup, characterise the spectrum of the generator of the semigroup, obtain resolvent estimates for the generator along the imaginary axis at infinity, and convert these estimates into an optimal rate of decay for the energy of the system using the theorem of Borichev and Tomilov [BT10] first encountered in Chapter 3 (Theorem 3.1.8).

Chapter 5 continues the aim of obtaining optimal decay rates for the energy of coupled systems by looking at a wave-heat system in one dimension with Dirichlet boundary condition for the wave part, but this time with square-integrable heat part on an infinite half-line. This results in a somewhat different analysis than in the previous chapter, despite the similar first steps of showing well-posedness via the existence of a semigroup and characterising the spectrum of the generator. In this case, though we obtain resolvent estimates along the imaginary axis at infinity, we also do so at 0 where, unlike in the previous chapter, there is a singularity. These estimates are then converted into an optimal rate of decay for the energy of a certain class of solutions through the theorem of Batty, Chill, and Tomilov [BCT16] which is again mentioned in Chapter 3 (Theorem 3.1.12). We then turn to the case of where the wave part has Neumann boundary condition on one end, and address this scenario via both the same framework, which leads to an unbounded semigroup, and an alternative framework, which results in a semi-uniformly stable semigroup with the same rate of decay as the Dirichlet case.

Chapter 6 is a bit of an odd ball, in that the majority of its contents are outside the scope of semigroup asymptotics. Nonetheless, we tie it together with the rest of this thesis by its final section on asymptotics. It deals with the Hilbert space direct integrals studied by Dixmier [Dix57] and the corresponding direct integral theory of operators and C_0-semigroups. Some of this is adapted from the Banach direct sum theory of Lachowizc and Moszyński [LM16]. The overall goal of the chapter is to fill the gap in the literature when it comes to the overlap of direct integral theory and C_0-semigroup theory. It begins with the already known basic direct integral and operator-theoretic results needed to develop new theory (though approached slightly differently to what is commonplace in the literature). This is followed by the main two novel results of the chapter. The first states that the direct integral of a family of C_0-semigroups is itself a C_0-semigroup on the direct integral space if and only if the family is uniformly exponentially bounded. The second states that any C_0-semigroup on the direct integral space that is generated by a decomposable operator is itself decomposable. The final few sections deal with special cases, examples, and asymptotic questions that arise out of our central results, including a discussion on a theorem by Maniar and Nafiri [MN16] and how it follows as a simple corollary of our more general quantified asymptotics result.

Part III, comprising of two chapters, takes a turn into the discrete operator semigroup side of quantified asymptotic theory. This time the focus is on new abstract theoretic results and the part as a whole can be viewed as a discrete analogue to the work of Rozendaal, Seifert, and Stahn [RSS19]. Chapter 7 completes the story outlined in the second survey of Chapter 3 by proving that functions of reciprocally positive increase form the largest class for which one gets an optimal quantified Katznelson–Tzafriri theorem on Hilbert spaces. In particular, we first prove that for a power-bounded operator T with $\sigma(T) \cap \mathbb{T} = \{1\}$ that has resolvent estimated by a non-increasing reciprocally positive increase function m, the decay of $\|T^n(I-T)\|$ happens at $O(m^{-1}(n))$, $n \to \infty$. Here, m^{-1} is the so-called minimal right-inverse. We then show that this result is optimal both in rate of decay and in class of function.

Chapter 8 looks more generally at Banach spaces and so-called quasi-multiplicative operators, the twice-removed cousins (in a non-technical sense) of normal operators. There, results are proved that give better rates of decay than the classical '$m_{\log}$ theorem' of Seifert [Sei16] under these special circumstances. Analytic Toeplitz operators are then introduced as a particular example of quasi-multiplication operators in order to provide a specific worked example of where the main theorem of the previous chapter yields a better outcome than all previously known results.

Part IV, appropriately titled 'Final remarks', ends this thesis with a chapter on possible future directions of research connected to or arising out of Parts II and III. Five such potential avenues are briefly explored. In particular, a 'half-result' of sorts (the paltry spoils of an unhappy venture) is provided in the third such avenue.

Several of the chapters in this thesis are based on papers that have been submitted for publication, some of which have already been accepted or published. See the footnotes at the start of each such chapter for the corresponding papers.

Chapter 2

Notation and preliminary results

In this chapter, we provide the basic definitions and results needed in this thesis. The operator theoretic results will be provided without proof in the second and third sections and will be stated as found in standard texts such as [ABHN11, Chapters 1-3, Appendix B] and [EN00, Chapters I-II]. In the fourth section, we use the collection of facts gathered in [RSS19, Section 2] (which is of the same name as our section), referring also to [BGT89] for a comprehensive discussion on functions of regular variation. See [Lax02] for any of the theory of bounded linear operators that might be covered in a typical undergraduate course.

2.1 Basic notation

Throughout this thesis, the (normed) vector spaces that appear will be defined over the complex numbers and will be typically denoted by X, Y, and Z. Norms and inner products will be denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ respectively with subscripts appearing only when necessary for clarity. The set of bounded linear operators from X to Y will be denoted $\mathcal{B}(X,Y)$ for general normed spaces X, Y and $\mathcal{B}(X)$ if $X = Y$. Bounded linear operators will typically be denoted by T and S, though these symbols will often also denote C_0-semigroups in a manner that will be clear. The symbol I will always refer to the identity operator on X. As a matter of convention, for any bounded linear operator T, we set T raised to the power 0 to be I.

We will denote the natural numbers without zero by $\mathbb{N}$, the integers by $\mathbb{Z}$, the non-negative integers by $\mathbb{Z}_+$, the real numbers by $\mathbb{R}$, the non-negative reals by $\mathbb{R}_+$, and the complex numbers by $\mathbb{C}$. We also have the following subsets of the complex plane:

$$\mathbb{C}_- = \{z \in \mathbb{C} : \operatorname{Re} z < 0\}, \quad \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}, \quad \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

For $\lambda \in \mathbb{C}$, we define the square root $\sqrt{\lambda}$ and the logarithm $\log \lambda$ by taking the branch cut along the negative real axis, that is, for $\lambda = re^{i\theta}$ where $r > 0$ and $\theta \in (-\pi, \pi]$, we let $\sqrt{\lambda} = r^{1/2}e^{i\theta/2}$ and $\log \lambda = \log r + i\theta/2$.

We will use standard notation for sequence and function spaces. In particular, for a Banach space X and some $p \in [1, \infty]$, the sequence space $\ell^p(\mathbb{Z}, X)$ denotes the space of all X-valued sequences $x = (x_k)_{k\in\mathbb{Z}}$ with finite (standardly defined) p-norm. Likewise, the function space $L^p(\mathcal{I}, X)$ denotes the space of all X-valued functions on the interval $\mathcal{I}$ with finite (standardly defined) p-norm. If $X = \mathbb{C}$, that is, when the sequences or functions are scalar-valued, we drop the X in the notation.

Given two functions $f, g : (0, \infty) \to \mathbb{R}_+$ and $a \in [0, \infty]$ fixed, we write (say) $f(t) = O(g(t))$, (as) $t \to a$, to indicate that there exists some constant $C > 0$ such that $f(t) \leq Cg(t)$ for all $t > 0$ sufficiently close to a (or sufficiently large in the case where $a = \infty$), the so-called 'big O notation'. We write $f(t) \asymp g(t)$, $t \to a$, when both $f(t) = O(g(t))$ and $g(t) = O(f(t))$ as $t \to a$, that is, when f and g are 'of the same order of growth'. We write $f(t) \sim g(t)$, $t \to a$ to mean $f(t)/g(t) \to 1$ as $t \to a$, that is, when f and g are 'asymptotically equivalent'. If g is strictly positive for all sufficiently large $t > 0$, we write $f(t) = o(g(t))$, $t \to a$, to mean $f(t)/g(t) \to 0$ as $t \to a$, the so-called 'little o notation'. The cases where the functions f and g are defined on $\mathbb{N}$, $\mathbb{R}$ or $\mathbb{R} \setminus \{0\}$ are treated analogously. If p and q are non-negative real-valued quantities, the notation $p \lesssim q$ denotes that $p \leq Cq$ for some constant $C > 0$ that is independent of any varying parameters in a given context.

For $a \geq 0$ and a continuous non-decreasing function $M : (0, a] \to (0, \infty)$, we define the *maximal right-inverse* of M by $M^{-1}(s) = \sup\{r \geq a : M(r) \leq s\}$. This definition implies that $M(M^{-1}(s)) = s$ for $s \geq M(a)$ and $M^{-1}(M(s)) \geq s$ for $s \geq a$. Likewise, for a continuous non-increasing function $m : (0, b] \to (0, \infty)$, we define the *minimal right-inverse* of m by $m^{-1}(t) = \inf\{\theta \in (0, b] : m(\theta) \leq t\}$ for $t \geq m(b)$. This definition implies that $m(m^{-1}(t)) = t$ for $t \geq m(b)$ and $m^{-1}(m(\theta)) \leq \theta$ for $\theta \in (0, b]$.

We will also use 1_G to denote the characteristic function for a set G and the abbreviation 'a.e.' to mean either 'almost every' or 'almost everywhere', depending on the context.

2.2 Closed operators and resolvents

Let X be a (complex) Banach space. An *operator* on X is a linear map

$$A : D(A) \subset X \to X$$

where $D(A)$ is a linear subspace of X called the *domain* of A. General operators will be typically denoted by A, B, and C, particularly if they are unbounded (that is, not elements of $\mathcal{B}(X)$). We write $A \subset B$ to mean that $D(A) \subset D(B)$ and $Ax = Bx$ for all $x \in D(A)$.

The *range* and *kernel* of A are defined respectively by

$$\operatorname{Ran} A = \{Ax : x \in D(A)\} \quad \text{and} \quad \operatorname{Ker} A = \{x \in D(A) : Ax = 0\}.$$

An operator A is densely defined if $D(A)$ is dense in X and closed if its graph $\Gamma(A)$ is closed in $X \times X$ under the product topology, where

$$\Gamma(A) = \{(x, Ax) : x \in D(A)\}.$$

This is equivalent to $D(A)$ being complete under the graph norm $\|\cdot\|_A$ given by

$$\|x\|_A = \|x\|_X + \|Ax\|_X.$$

If X is a Hilbert space, we define the adjoint of a densely defined operator A by the following:

$$y \in D(A^*),\ A^*y = x \iff \langle Az, y\rangle = \langle z, x\rangle, \quad z \in D(A).$$

It is not hard to verify that A^* is closed. On general Banach spaces, the adjoint can be defined similarly on the dual space of X using duality brackets (see for example, [Bre11, Chapter 2]).

For $\lambda \in \mathbb{C}$, it is important to know when the operator $\lambda I - A$ is invertible with bounded inverse. When this operator has an inverse in $\mathcal{B}(X)$, we denote the inverse $(\lambda I - A)^{-1}$ by $R(\lambda, A)$ and call it the *resolvent (operator)*. When $\lambda I - A$ fails to be invertible, it is useful to classify in what way it fails. To this end, we have the following definitions for the *resolvent (set)*, *spectrum*, *point spectrum*, and *approximate point spectrum* of A:

$$\begin{aligned}
\rho(A) &= \{\lambda \in \mathbb{C} : \lambda I - A \text{ has an inverse } R(\lambda, A) \text{ in } \mathcal{B}(X)\};\\
\sigma(A) &= \{\lambda \in \mathbb{C} : \lambda \notin \rho(A)\};\\
\sigma_p(A) &= \{\lambda \in \mathbb{C} : \lambda I - A \text{ has nontrivial kernel}\};\ \text{and}\\
\sigma_{ap}(A) &= \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not bounded below}\}.
\end{aligned}$$

Clearly $\sigma_p(A) \subset \sigma_{ap}(A) \subset \sigma(A)$ and it is not hard to see that if A has nonempty resolvent, then A is closed. Elements of $\sigma_p(A)$ are better known as *eigenvalues* and those of $\sigma_{ap}(A)$ are known as *approximate eigenvalues.* Note that for a densely defined operator A on Hilbert space, $\rho(A^*) = \{\overline{\lambda} : \lambda \in \rho(A)\}$, and $R(\lambda, A)^* = R(\overline{\lambda}, A^*)$.

Resolvents satisfy the equation

$$R(\lambda, A) - R(\mu, A) = (\lambda - \mu)R(\lambda, A)R(\mu, A), \quad \lambda, \mu \in \rho(A)$$

known as the *resolvent identity.* The following proposition is proved using the resolvent identity and so-called Neumann expansion.

Proposition 2.2.1 ([ABHN11, Corollary B.3])**.** *For any operator A, $\rho(A)$ is open and $\sigma(A)$ is closed in $\mathbb{C}$. Morever, if $\mu \in \rho(A)$ and $\lambda \in \mathbb{C}$ with $|\lambda - \mu| < \|R(\mu, A)\|^{-1}$, then $\lambda \in \rho(A)$ and*

$$R(\lambda, A) = \sum_{n=0}^{\infty} (\mu - \lambda)^n R(\mu, A)^{n+1},$$

where the series is norm-convergent. In particular,

$$\|R(\lambda, A)\| \geq \operatorname{dist}(\lambda, \sigma(A))^{-1}, \quad \lambda \in \rho(A).$$

Furthermore, $\lambda \mapsto R(\lambda, A)$ is holomorphic on $\rho(A)$ with values in $\mathcal{B}(X)$ and the derivative of $R(\lambda, A)$ is $-R(\lambda, A)^2$.

Corollary 2.2.2. *Let D be a dense subset of $E \subset \mathbb{C}$. Suppose that for all $\lambda \in D$, $\lambda \in \rho(A)$ and $\|R(\lambda, A)\| \leq M$ for some $M > 0$. Then $E \subset \rho(A)$ and $\|R(\lambda, A)\| \leq M$ for all $\lambda \in E$.*

The spectrum of the resolvent of an operator can be easily related to the spectrum of the original operator by the following spectral mapping theorem.

Theorem 2.2.3 ([EN00, Theorem 1.13])**.** *Let A be an operator with nonempty resolvent set $\rho(A)$. Then*

$$\sigma(R(\lambda, A)) \setminus \{0\} = \{(\lambda - \mu)^{-1} : \mu \in \sigma(A)\}, \quad \lambda \in \rho(A).$$

Having introduced resolvents and spectrum, we define sectoriality.

A closed linear operator A with dense domain $D(A)$ is *sectorial of angle* $\delta \in [0, \frac{\pi}{2})$ if

1. $\Sigma_{\pi/2+\delta} \subset \rho(A)$, where $\Sigma_{\pi/2+\delta} = \left\{\lambda \in \mathbb{C} : |\arg \lambda| < \frac{\pi}{2} + \delta\right\} \setminus \{0\}$; and

2. For each $\varepsilon \in (0, \delta)$, there exists $M_\varepsilon \geq 1$ such that

$$\|R(\lambda, A)\| \leq \frac{M_\varepsilon}{|\lambda|}, \quad 0 \neq \lambda \in \overline{\Sigma}_{\pi/2+\delta-\varepsilon}.$$

Note that it is more common to define sectoriality with the spectrum on the right rather than on the left as we have here. We have chosen this convention as it agrees with [EN00]. In Section 6.4 we refer to the theorems in [EN00, Chapter II §4a] on sectorial generators and holomorphic semigroups rather than state them here.

We finish this section with the notion of dissipativity, which is related to the idea of sectoriality. A linear operator A with domain $D(A)$ on a Banach space X is called *dissipative* if

$$\|(\lambda - A)x\| \geq \|x\|, \quad \lambda > 0, x \in D(A).$$

If X is a Hilbert space, this condition is equivalent to

$$\operatorname{Re}\langle Ax, x\rangle \leq 0, \quad x \in D(A).$$

2.3 C_0-semigroups

As discussed in Chapter 1, C_0-semigroups can be considered as generalised exponentials. Thus they are, perhaps unsurprisingly, exponentially bounded in the following way.

Proposition 2.3.1 ([EN00, Chapter I Proposition 5.5])**.** *For every strongly continuous semigroup T, there exist constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{\omega t}, \quad t \geq 0. \tag{2.3.1}$$

If (2.3.1) holds for some M and ω, we say that T is *exponentially bounded by* ω, and if $\|T(t)\| \leq 1$ for all $t \geq 0$, then T is called a *contraction* semigroup.

The exponential bound of a C_0-semigroup provided by Proposition 2.3.1 both controls the growth of the resolvent of the generator and guarantees the convergence of the Laplace transform of the semigroup in certain cases. As we have already defined generators in Chapter 1, we refer to [ABHN11, Proposition 3.1.9] for more basic properties of the generator and move on to the following result.

Theorem 2.3.2 ([EN00, Chapter II Theorem 1.10])**.** *Let T be a strongly continuous semigroup with generator A on the Banach space X and take constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{\omega t}, \quad t \geq 0.$$

Then $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega\} \subset \rho(A)$,

$$\|R(\lambda, A)\| \leq \frac{M}{\operatorname{Re}\lambda - \omega}, \quad \operatorname{Re}\lambda > \omega,$$

and

$$R(\lambda, A)x = \lim_{t\to\infty} \int_0^t e^{-\lambda s} T(s)x \, ds, \quad x \in X, \operatorname{Re}\lambda > \omega,$$

where the integral converges absolutely.

We will frequently write

$$R(\lambda, A) = \int_0^\infty e^{-\lambda s} T(s) \, ds$$

and refer to it as the Laplace transform representation of $R(\lambda, A)$. We refer to [ABHN11, Chapter 1-2] for a discussion of (scalar- and) vector-valued integration and Laplace transforms in general, and in particular, to [ABHN11, Chapters 3, 5] for the case when $\lambda \in \rho(A)$ but the integral

$$\int_0^\infty e^{-\lambda s} T(s)x \, ds$$

converges only as an improper integral and not absolutely.

The following result is the 1952 generation theorem of Feller–Miyadera–Phillips, the general case of the famous Hille–Yoshida theorem.

Theorem 2.3.3 ([EN00, Chapter II Theorem 3.8]). *Let A be a linear operator on a Banach space X and let $\omega \in \mathbb{R}$ and $M \geq 1$ be constants. Then the following properties are equivalent.*

(i) A generates a strongly continuous semigroup T satisfiying

$$\|T(t)\| \leq M e^{\omega t}, \quad t \geq 0.$$

(ii) A is closed, densely defined, and for every $\lambda > \omega$ one has $\lambda \in \rho(A)$ and

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - \omega)^n}, \quad n \in \mathbb{N}.$$

(iii) A is closed, densely defined, and for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$, one has $\lambda \in \rho(A)$ and

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\operatorname{Re}\lambda - \omega)^n}, \quad n \in \mathbb{N}.$$

Another generation theorem, which is perhaps more commonly used because of the ease with which one can check the sufficient conditions, is the following much celebrated 1961 theorem of Lumer–Phillips.

Theorem 2.3.4 ([EN00, Chapter II Theorem 3.15]). *Let A be a linear operator on a Banach space X. Then A generates a contraction semigroup if and only if A is densely defined, closed, dissipative, and $\lambda_0 I - A$ is surjective for some $\lambda_0 > 0$.*

We refer to [EN00, Chapter I-II] for the definitions, discussions, and characterisations of special kinds of semigroups that appear later in this thesis, such as holomorphic, (eventually) norm-continuous, and eventually differentiable semigroups. We also refer to [ABHN11, Section 3.1] for the discussion on the relationship between C_0-semigroups and Cauchy problems needed in Section 6.5.

2.4 Special classes of functions

In this section, we discuss and define several classes of functions that have specific growth and asymptotic conditions and which play important roles in probability theory, introducing reciprocal versions of previously well-known notions. We will only include the definitions and results that are directly needed in this thesis, though there is a lot more of interest that can be said about some of these classes. For a more comprehensive discussion, see [BCT16, Section 2], [BGT89], and [RSS19, Section 2].

For $a \geq 0$ and $\alpha \in \mathbb{R}$, a measurable function $M : [a, \infty) \to (0, \infty)$ is said to be *regularly varying* or have *regular variation (of index α)* if

$$\lim_{s \to \infty} \frac{M(\lambda s)}{M(s)} = \lambda^{\alpha}, \quad \lambda \geq 1.$$

If $\alpha = 0$ in the above, then M is said to be *slowly varying*. Note that the limit does not always exist, and when it does for all λ in a subset of $[1, \infty)$ with positive measure, the function is automatically regularly varying ([BGT89, Theorem 1.4.3]).

Clearly, polynomials are regularly varying, but not only that, any function of the form $M(s) = Cs^{\alpha}(\log s)^{\beta}$ for $\alpha, \beta \in \mathbb{R}$ and $C > 0$ is regularly varying of index α. Hence, the class of regularly varying functions includes functions of a finer scale of growth than polynomials.

For $b \geq 0$ and $\alpha \in \mathbb{R}$, we now say that a measurable function $m : (0, b] \to (0, \infty)$ is *reciprocally regularly varying* or has *reciprocally regular variation (of index α)* if the function $M : [b^{-1}, \infty) \to (0, \infty)$ given by

$$M(s) = m(s^{-1}), \quad s \in (b^{-1}, \infty),$$

is regularly varying of index α. In other words, m is reciprocally regularly varying if

$$\lim_{\theta \to 0} \frac{m(\mu\theta)}{\mu(\theta)} = \mu^{-\alpha}, \quad 0 < \mu \leq 1.$$

A related notion to regular variation is the following, which we will focus on in Part III. For $a \geq 0$ and $\alpha > 0$, a measurable function $M : [a, \infty) \to (0, \infty)$ is said to have (or be of) *positive increase (with index α)* if there exist constants $c \in (0, 1]$ and $s_0 \geq a$ such that

$$\frac{M(\lambda s)}{M(s)} \geq c\lambda^{\alpha}, \quad \lambda \geq 1, s \geq s_0.$$

Notice that if M has positive increase, then it grows with at least polynomial speed.

Let $\alpha > 0$. As with reciprocally regular variation, for $b \geq 0$ and $\alpha > 0$, we can now say that a measurable function $m : (0, b] \to (0, \infty) \to (0, \infty)$ has (or is of) *reciprocally positive increase (with index α)* if the function $M : [b^{-1}, \infty) \to (0, \infty)$ given by

$$M(s) = m(s^{-1}), \quad s \in (b^{-1}, \infty),$$

has positive increase with index α. In other words, m has reciprocally positive increase (of index α) if there exist $c \in (0, 1]$ and $\theta_0 \in (0, b]$ such that

$$\frac{m(\mu\theta)}{m(\theta)} \geq c\mu^{-\alpha}, \quad 0 < \mu \leq 1, 0 < \theta \leq \theta_0.$$

In a slight abuse of grammar, we will also say 'reciprocally positive increase function(s)' to mean 'function(s) of reciprocally positive increase'.

The following two results ([RSS19, Lemma 2.1 and Proposition 2.2] stated in reciprocal fashion) characterise reciprocally positive increase functions, the first of which does so via a criterion that is easy to verify and hence is extremely useful.

Proposition 2.4.1. *Let $b \geq 0$. If $m : (0, b] \to (0, \infty)$ is non-increasing, then m has reciprocally positive increase if and only if there exists $\mu < 1$ such that*

$$\liminf_{\theta \to 0} \frac{m(\mu\theta)}{m(\theta)} > 1.$$

From the above proposition, it is clear that any function that is reciprocally regularly varying with strictly positive index has reciprocally positive increase.

Proposition 2.4.2. *Let $b \geq 0$ and suppose that $m : (0, \pi] \to (0, \infty)$ is a continuous non-decreasing function such that $m(\theta) \to \infty$ as $\theta \to 0$. If m has reciprocally positive increase, then for all $c > 0$, we have*

$$m^{-1}(t) \asymp m^{-1}(ct), \quad t \to \infty. \tag{2.4.1}$$

Conversely, if (2.4.1) holds for some strictly positive $c \neq 1$, then m has reciprocally positive increase and in particular, (2.4.1) holds for all $c > 0$.

Chapter 3

Surveys

The following two surveys together provide a literature review of the quantified asymptotic theory of operator semigroups. Though we have separated the continuous and discrete cases into two different surveys, it is worth noting that in many ways, they form one (continuous) story.

We refer to [CT07] for a deeper sense of the intertwined nature of the following two semi-parallel streams of research as well as for a comprehensive coverage on the ideas and results from the period up to and including 2007. We also refer to [Lék17] for more indepth coverage of the discrete side of this theory. With these two references in mind, the goal of this chapter is to provide the results within the theory of quantified asymptotics needed in the rest of this thesis and to tie these results together into something of a timeline with a bit of anecdotal history thrown in.

3.1 Quantified rates of decay for C_0-semigroups

We begin this survey with the classical result of Tauber [Tau97], from which this all began and the reason why many of the following theorems are generically referred to as 'quantified Tauberian theorems'.

Theorem 3.1.1 (Tauber 1897). *Let $(a_n)_{n\geq 0}$ be a complex sequence such that $|a_n| = o(n^{-1})$, $n \to \infty$ and suppose that*

$$\lim_{s\nearrow 1} \sum_{n=0}^{\infty} a_n s^n = b.$$

Then $\sum_{n=0}^{\infty} a_n = b$.

Though Hardy and Littlewood generalised Tauber's theorem for functions in the 1920s (see [ABHN11, Chapter 4]), and hence have earned this shout-out, the next real piece of the story is the theorem of Ingham–Karamata [Ing35, Kar34] for Laplace transforms. The following version is less general than their original results.

Theorem 3.1.2 (Ingham, Karamata 1935)**.** *Let f be a bounded measurable function from $\mathbb{R}_+$ to a Banach space X. If $\hat{f}$ extends analytically across the imaginary axis, then*

$$\lim_{t\to\infty}\int_0^t f(s)\,ds = \hat{f}(0).$$

Consider a bounded C_0-semigroup $(T(t))_{t\geq 0}$ generated by an operator A on a Banach space X. Given the Laplace transform representation of the resolvent of the generator for C_0-semigroups (Theorem 2.3.2) and the analyticity of the resolvent, we should somehow expect that if $i\mathbb{R}\subset\rho(A)$, then

$$\lim_{t\to\infty}\int_0^t T(s)x\,ds = -A^{-1}x, \quad x\in X.$$

In particular, since

$$T(t)A^{-1}x = A\int_0^t T(s)A^{-1}x\,ds + A^{-1}x = \int_0^t T(s)x\,ds + A^{-1}x, \quad x\in X,$$

(see [ABHN11, Proposition 3.1.9]), it follows that $\|T(t)A^{-1}x\|\to 0$ as $t\to\infty$ for all $x\in X$ and hence $\|T(t)x\|\to 0$ as $t\to\infty$ for all $x\in X$ since A^{-1} maps onto $D(A)$ which is dense in X and $(T(t))_{t\geq 0}$ is uniformly bounded. This was in fact a key part of Arendt and Batty's proof of the famous countable spectrum theorem [AB88] also proved independently by Lyubich and Vũ [LVu88]. In the rest of this section, $(T(t))_{t\geq 0}$ will be a bounded C_0-semigroup generated by A on either a Banach or a Hilbert space X.

Theorem 3.1.3 (Arendt–Batty, Lyubich–Vũ 1988)**.** *Let X be Banach. Assume that $\sigma(A)\cap i\mathbb{R}$ is countable and $\sigma_p(A^*)\cap i\mathbb{R}=\emptyset$. Then $\lim_{t\to\infty}\|T(t)x\| = 0$ for all $x\in X$.*

Though Lyubich and Vũ's proof avoided transfinite induction, it was Arendt and Batty's proof in the simple case where $\sigma(A)\cap i\mathbb{R}=\emptyset$ which could have been further extended to give an actual rate of decay to zero in operator norm for $T(t)A^{-1}$. This was, however, not of particular interest until Lebeau [Leb96] (with an extra $\log\log$ factor) and his student Burq [Bur98] (in the setting of local energy-decay estimates) produced the following theorem for the Hilbert space case (in the form given by [BD08, Theorem A]).

Theorem 3.1.4 (Burq, Lebeau 1998)**.** *Let X be Hilbert. If $\sigma(A) \cap i\mathbb{R} = \emptyset$ and $M(s) \leq ce^{cs}$ for some constant $c > 0$, then*

$$m(t) \lesssim \frac{1}{\log(2+t)}, \quad t \geq 0.$$

Here and in what follows until otherwise specified,

$$m(t) = \sup_{\tau \geq t} \|T(\tau)(I - A)^{-1}\|, \quad t \geq 0, \tag{3.1.1}$$

and if $\sigma(A) \cap i\mathbb{R} = \emptyset$,

$$M(s) = \sup_{|\tau| \leq s} \|R(i\tau, A)\|, \quad s \geq 0. \tag{3.1.2}$$

This was proved using methods from control theory and arose out of the analysis of damped wave equations. It was also one of the first results to turn resolvent estimates of the generator into a rate of decay for the semigroup. Almost a decade later, Bátkai, Engel, Prüss, and Schnaubelt [BEPS06] (for general Banach spaces) independently from Liu and Rao [LR05] (for the Hilbert space case) focused on polynomial resolvent estimates rather than exponential ones, producing the following theorem (in the form given by [BD08, Theorem B]).

Theorem 3.1.5 (Bátkai–Engel–Prüss–Schnaubelt, Liu–Rao)**.** *Let X be Banach. If $\sigma(A) \cap i\mathbb{R} = \emptyset$ and $M(s) \leq c(1+s)^{\alpha}$ for some constants $c, \alpha > 0$ then for all $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that*

$$m(t) \leq \frac{C_\epsilon}{t^{\frac{1}{\alpha}-\epsilon}}, \quad t \geq 0.$$

Moreover, if X is Hilbert, then there exists $C > 0$ such that

$$m(t) \leq \frac{C \log(2+t)^{\frac{1}{\alpha}+1}}{t^{\frac{1}{\alpha}}}, \quad t \geq 0.$$

Duyckaerts, a student of Burq, continued his ancestral work, collaborating with Batty in the seminal paper [BD08], in which an idea based on a contour integral technique devised first by Newman [New80] and then by Korevaar [Kor82] was used to deduce a rate on general Banach spaces. The proof given in [BD08] is closely related to the original proof by Arendt and Batty of the countable spectrum theorem aforementioned for the simple case where $\sigma(A) \cap i\mathbb{R} = \emptyset$. Anecdotally, this contour integral proof replaced the initial proof based on control theory that Duyckaerts had when he first approached Batty.

The following two results, [BD08, Proposition 1.3] and [BD08, Theorem 1.5] are taken from this paper, the first of which will be used in later chapters.

Proposition 3.1.6 (Batty–Duyckaerts 2008). *Let X be Banach and suppose that $0 \in \rho(A)$. If*

$$\lim_{t\to\infty} m(t) = 0,$$

then $\sigma(A) \cap i\mathbb{R} = \emptyset$ and

$$M(s) \lesssim 1 + m_*^{-1}\left(\frac{1}{2(s+1)}\right), \quad s \geq 0,$$

where m_*^{-1} is a right inverse of the non-increasing function m, mapping $(0, m(0)]$ onto $[0, \infty)$.

Given M as above, we define the following quantity

$$M_{\log}(s) = M(s)[\log(1 + M(s)) + \log(1 + s)], \quad s \geq 0, \tag{3.1.3}$$

which is injective and tends to infinity as t tends to infinity since M is non-decreasing.

Theorem 3.1.7 (Batty–Duyckaerts 2008). *Let X be Banach and suppose that $\sigma(A) \cap i\mathbb{R} = \emptyset$. Then for all $c \in (0, 1/2)$, we have*

$$m(t) \lesssim \frac{1}{M_{\log}^{-1}(ct)}, \quad t \geq 0,$$

where $M_{\log}^{-1}$ is the inverse of $M_{\log}$, mapping $(M_{\log}(0), \infty)$ onto $(0, \infty)$.

Batty and Duyckaerts also formulated the above theorem for Laplace transforms of functions, hearkening back to the original theorem of Ingham and Karamata. Popov and Vodev [PV99] deserve a mention at this point, since Theorem 3.1.7 was already known to them for unitary groups since 1999 in the special case where $M(s) \sim s^\alpha$, $s \to \infty$ for some $\alpha > 0$.

Three questions could then be naturally asked. First, was the above $M_{\log}$ rate optimal and second, if so, could it be improved in specific cases when X is Hilbert? Third, could a similar theorem be proven to allow for the case where the spectrum of A touched the imaginary axis only at 0 (or perhaps at only finitely many other points)?

The first two questions were answered by Borichev and Tomilov in their widely cited paper [BT10]. In it they first proved that the log factor could be gotten rid of in the case of polynomial resolvent estimates on Hilbert spaces, followed by an example in the general Banach space setting showing that the Batty–Duyckaerts result is optimal. Theorem 3.1.8 below will be used in Chapter 4.

Theorem 3.1.8 (Borichev–Tomilov 2010)**.** *Let X be Hilbert and suppose that $\sigma(A)\cap i\mathbb{R}=\emptyset$. Then for any $\alpha>0$, the following are equivalent:*

(i) $\|R(is,A)\|=O(|s|^\alpha)$ *as* $|s|\to\infty$*;*

(ii) $\|T(t)A^{-1}\|=O(t^{-1/\alpha})$ *as* $t\to\infty$*;*

(iii) $\|T(t)x\|=o(t^{-1/\alpha})$ *as* $t\to\infty$ *for all* $x\in D(A)$*.*

Theorem 3.1.9 (Borichev–Tomilov 2010)**.** *Given $\alpha>0$, there exists a bounded C_0-semigroup $(T_\alpha(t))_{t\geq 0}$ on a Banach space X_α with generator A_α such that $\sigma(A_\alpha)\cap i\mathbb{R}=\emptyset$ and $\|R(is,A_\alpha)\|=O(|s|^\alpha)$ as $|s|\to\infty$ but*

$$\limsup_{t\to\infty}\left(\frac{t}{\log t}\right)^{\frac{1}{\alpha}}\|T_\alpha(t)A_\alpha^{-1}\|>0.$$

Martínez answered the third question mentioned above by generalising the Batty–Duyckaerts theorem to allow for the case where $\sigma(A)\cap i\mathbb{R}=\{0\}$ in [Mar11b, Proposition 3.1], which we state below, as well as for the case when the spectrum of A touches the imaginary axis at only finitely many points in [Mar11b, Theorem 3.4]. She achieved by first proving a version for Laplace transforms, continuing the Cauchy integral technique used in [BD08].

Theorem 3.1.10 (Martínez 2011)**.** *Let X be Banach and suppose that $\sigma(A)\cap i\mathbb{R}=\{0\}$. Then there exists $c>0$ such that*

$$\|T(t)A(I-A)^{-2}\|\lesssim m_{\log}^{-1}(ct)+\frac{1}{M_{\log}^{-1}(ct)},\quad t\geq c^{-1}\max\{M_{\log}(1),m_{\log}(1)\}.$$

Here

$$M(s)=\sup_{1\leq|\tau|\leq s}\|R(i\tau,A)\|,\quad s\geq 1,$$

with $M_{\log}$ being defined as in (3.1.3) with the adjusted M and where

$$m(s)=\sup_{s\leq|\tau|\leq 1}\|R(i\tau,A)\|,\quad 0<s\leq 1,$$

defines the function

$$m_{\log}(s)=m(s)\log\left(\frac{1+m(s)}{s}\right),\quad 0<s\leq 1.$$

As $M_{\log}$ and $m_{\log}$ are both strictly monotone functions, they have inverses $M_{\log}^{-1}$ and $m_{\log}^{-1}$ respectively defined on their appropriate domains.

Two new natural questions now arose from the previous three answers. The first was whether in the Hilbert space case, there exist classes of functions with finer scales than that of polynomials that permit a better result than Batty–Duyckaerts. Stated in terms of semigroups, do resolvent rates for the generator in between polynomial powers have a corresponding rate of decay of semigroup that falls in between those corresponding polynomial decay rates given in Borichev–Tomilov? The second question was if a Borichev–Tomilov type result for polynomials on Hilbert spaces also existed for the Martínez case for a singularity on the imaginary axis.

These questions (as well as others) were addressed in the rather magisterial work of Batty, Chill, and Tomilov [BCT16], using methods of functional calculus and sectorial operators. The first of the above questions was partially answered in [BCT16, Theorem 5.6] below.

Theorem 3.1.11 (Batty–Chill–Tomilov 2013)**.** *Let X be Hilbert. If $\sigma(A) \cap i\mathbb{R} = \emptyset$. If $\alpha > 0$ and*

$$\|R(is, A)\| = O(|s|^{\alpha}/\ell(|s|)), \quad |s| \to \infty,$$

for some constant $\alpha > 0$ and ℓ increasing and slowly varying, then

$$\|T(t)A^{-1}\| = O((t\ell(t^{1/\alpha}))^{-1/\alpha}), \quad t \to \infty.$$

The second question was answered by [BCT16, Theorem 8.4] stated below (without going into detail about fractional powers) and will be used in Chapter 5.

Theorem 3.1.12 (Batty–Chill–Tomilov 2013)**.** *Let X be Hilbert. Suppose that $\sigma(A) \cap i\mathbb{R} = \{0\}$ and assume that there exist $\alpha \geq 1$ and $\beta > 0$ such that*

$$\|R(is, A)\| = \begin{cases} O(|s|^{-\alpha}), & s \to 0, \\ O(|s|^{\beta}), & |s| \to \infty. \end{cases}$$

Then

$$\|T(t)A^{\alpha}(I - A)^{-(\alpha+\beta)}\| = O(t^{-1}), \quad t \to \infty,$$

and

$$\|T(t)A(I - A)^{2}\| = O(t^{-1/\gamma}), \quad t \to \infty$$

where $\gamma = \max(\alpha, \beta)$.

Up to this point, Cauchy integral techniques were the primary workhorses used to prove these quantified decay rate theorems. However, Chill and Seifert in [CS16] produced a quantified version of the Ingham–Karamata theorem along with several variations, relying on Fourier methods and simple applications of integration by parts. Furthermore, many of the previously stated theorems follow from these results as corollaries (see [CS16, Corollaries 2.6, 2.12, 2.14]). The following is a statement of [CS16, Theorem 2.1(b)] and we refer to [CS16, Section 1] to explain what it means to admit a boundary function.

Theorem 3.1.13 (Chill–Seifert 2015)**.** *Let X be Banach and f be a bounded continuous and Lipschitz function from $\mathbb{R}_+$ to X. Suppose that $\hat{f}$ admits a boundary function F on $i\mathbb{R}$. Then f decays to 0 at infinity. Moreover, if F is smooth and there exist a continuous non-decreasing function $M : \mathbb{R}_+ \to [0, \infty)$ and a constant $C > 0$ such that*

$$\|F^{(j)}(s)\| \leq Cj!M(|s|)^{j+1}, \quad s \in \mathbb{R}, j \geq 0,$$

then for any $c > 0$,

$$\|f(t)\| = O\left(\frac{1}{M_{\log}^{-1}(ct)}\right), \quad t \to \infty,$$

where $M_{\log}^{-1}$ is the inverse of the function $M_{\log}$ defined as in (3.1.3).

This method of proving quantified rates turns out to be especially useful in some of the following theorems.

The question previously asked about whether there are more classes than that of polynomials (and the finer case involving regularly varying functions) in the Hilbert space setting for which a Borichev–Tomilov type result could be found was finally settled by Rozendaal, Seifert, and Stahn in [RSS19].

Theorem 3.1.14 (Rozendaal–Seifert–Stahn 2019)**.** *Let X be Hilbert and suppose that $\sigma(A) \cap i\mathbb{R} = \emptyset$. If M as defined in (3.1.2) has positive increase, then there exist constants $c, C > 0$ such that*

$$\frac{c}{M^{-1}(t)} \leq \|T(t)A^{-1}\| \leq \frac{C}{M^{-1}(t)} \tag{3.1.4}$$

for all sufficiently large $t > 0$. Moreover, if $(T(t))_{t\geq 0}$ is a semigroup of normal operators, then the upper bound above holds if and only if M has positive increase.

This provided the largest possible class of functions for which the log factor could be removed for an optimal Borichev–Tomilov type rate of decay for general bounded semigroups on Hilbert spaces. A version of the statement about semigroups of normal operators in Theorem 3.1.14 was proved in [BCT16], suggesting to Rozendaal, Seifert, and Stahn that functions of positive increase were the correct class of functions to work with. Note that there are no constants appearing in the argument of M^{-1} in (3.1.4). This is due to [RSS19, Proposition 2.2] (the original version of Proposition 2.4.2). The case with a singularity at 0 was also dealt with in [RSS19].

The main part of the story for quantified decay of strongly continuous semigroups might be said to more or less finish here, though we end this survey with three concluding results.

Even as early as [BD08], the formulations for Laplace transforms hinged upon $\hat{f}$ having a holomorphic extension to the region

$$\Omega_M = \left\{ z \in \mathbb{C} : \operatorname{Re} z > -\frac{1}{M(|\operatorname{Im} z|)} \right\} \tag{3.1.5}$$

such that

$$\|\hat{f}\| \leq M(|\operatorname{Im} z|)$$

in that region. This holds for free if $f = T(t)x$ is the semigroup orbit of a unit length vector whenever M is taken to be as in (3.1.2). In [Sta18], Stahn generalised some of the results of [BCT16, BD08] to consider the case when the function estimating $\hat{f}$ (analogously the resolvent) and the function determining the extended domain of holomorphicity were different. A priori, one might know that the Laplace transform is holomorphic on a larger domain than originally dictated by Ω_M. Stahn thus proved an 'M_K-theorem', stated here for C_0-semigroups.

Theorem 3.1.15 (Stahn 2018)**.** *Let X be Banach and $\sigma(A) \cap i\mathbb{R} = \emptyset$. Suppose that $M, K : \mathbb{R}_+ \to (0, \infty)$ are two non-decreasing continuous functions such that for some $\epsilon > 0$, there exists $C_\epsilon > 0$ with*

1. *$K(s) \geq \max\{2, s, M(s)\}$; and*
2. *$K(s) \leq C_\epsilon e^{e^{(sM(s))^{1-\epsilon}}}$ for sufficiently large $s > 0$.*

If $R(\lambda, A)$ extends analytically to Ω_M defined in (3.1.5) and

$$\|R(\lambda, A)\| \leq K(|\operatorname{Im} \lambda|), \quad \lambda \in \Omega_M,$$

then for any $c \in (0, 1)$,

$$\|T(t)A^{-1}\| = O\left(\frac{1}{{M_K}^{-1}(ct)}\right), \quad t \to \infty,$$

where

$$M_K(s) = M(s) \log(K(s)) \tag{3.1.6}$$

is a non-decreasing function and ${M_K}^{-1}$ is the maximal right-inverse of M_K.

It is interesting to note that methods from [BCT16] were used in [Sta18], but that similar results were proven earlier in the unpublished [Sta17] using the methods from [CS16]. It is also worth remarking that Stahn proved a generalised L^p version of Theorem 3.1.13 in [Sta17], following in the steps of Batty, Borichev, and Tomilov in [BBT16] who first proved various L^p-Tauberian theorems in 2016.

The second result worth mentioning in the continuous setting for operator semigroups is the following theorem by Debruyne and Seifert developed first in [DS19a] and then in [DS19b], proving optimality of the 'M_K-theorem' for regularly growing functions in the above setting. We refer to [DS19b] for the definition of regularly growing functions, suffice it to say that these constitute a large class of functions. The proof relied on a clever use of the open mapping theorem, resulting in, even for the simple case where $M = K$, a much shorter and neater proof of the optimality of the '$M_{\log}$-theorem' (Theorem 3.1.7) compared to the one in [BT10].

Theorem 3.1.16 (Debruyne–Seifert 2018)**.** *Let $M, K : \mathbb{R}_+ \to (0, \infty)$ be two regularly growing functions and suppose that $M_K(s) = O(e^{\alpha s})$ as $s \to \infty$ for some $\alpha > 0$. Then there exists a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X with generator A such that for some $\delta > 1$, $\Omega_{\delta M} \subset \rho(A)$ where $\Omega_{\delta M}$ is defined as in (3.1.5) but with δM instead of M and*

$$\|R(\lambda, A)\| \leq CK(|\operatorname{Im} \lambda|), \quad \lambda \in \Omega_{\delta M},$$

for some $C > 0$. Moreover,

$$\limsup_{t\to\infty} \|M_K^{-1}(t)T(t)A^{-1}\| > 0.$$

Here M_K is given by (3.1.6) with M_K^{-1} its inverse.

Finally, we finish with the following result by Maniar and Nafiri who brought the quantified asymptotic theory of C_0-semigroups to the setting of direct sums. In it, they essentially apply the theory of Borichev–Tomilov to collections of C_0-semigroups.

Theorem 3.1.17 ([MN16, Theorem 3.2])**.** *Let $\{T_n(t) : \mathbb{R}_+ \to \mathcal{B}(H_n) : n \in \mathbb{N}\}$ be a uniformly bounded sequence of C_0-semigroups with corresponding generators A_n such that for all $n \in \mathbb{N}, i\mathbb{R} \subset \rho(A_n)$. Further assume that*

$$\sup_{n\in\mathbb{N}} \|R(ir, A_n)\|_{\mathcal{B}(H_n)} < \infty, \quad r \in \mathbb{R}. \tag{3.1.7}$$

Then for a fixed $\alpha > 0$ the following conditions are equivalent:

(i) $\displaystyle\sup_{|r|\geq 1, n\in\mathbb{N}} |r|^{-\alpha}\|R(ir, A_n)\|_{\mathcal{B}(H_n)} < \infty.$

(ii) $\sup_{t\geq 0, n\in\mathbb{N}} \|t^{1/\alpha} T_n(t) A_n^{-1}\|_{\mathcal{B}(H_n)} < \infty.$

We will discuss this result in more depth in Chapter 6 (and Theorem 6.6.2 in particular). There, we use the parameter s for a different purpose, hence the choice of ir to describe elements on the imaginary axis in the above theorem.

3.2 Quantified Katnelson–Tzafriri theorems

We begin this second survey in the chaos waters of probability theory. In 1970 Ornstein and Sucheston proved the following 'zero-two law' ([OS70, Theorem 1.1]) within the context of Markov processes, kickstarting this whole area of study (see [OS70] for the definition of a positive contraction on $L^1(\Omega)$).

Theorem 3.2.1 (Ornstein–Sucheston 1970)**.** *If T is a positive contraction on $L^1(\Omega)$, then either $\|T^n(I-T)\| = 2$ for all $n \geq 0$, or*

$$\lim_{n\to\infty} \|T^n(I-T)\| = 0.$$

In 1976, Foguel [Fog76] proved the above for $L^\infty(\Omega)$ and then in 1986, Zaharopol [Zah86] extended it to include $L^p(\Omega)$ for general $p \in (1,2) \cup (2,\infty)$. Later that year, when trying to understand Zaharopol's proof, Katznelson and Tzafriri [KT86] realised that these results were all just special cases of a general theorem relating the spectrum of a power-bounded operator with the limiting behaviour of the differences in consecutive powers. They were able to prove a general result that has since attracted substantial interest. This result is stated below in its most basic and well-known form.

Theorem 3.2.2 (Esterle, Katznelson–Tzafriri 1986)**.** *Let X be a Banach space and $T \in \mathcal{B}(X)$ be power-bounded, that is,*

$$\sup_{n\in\mathbb{N}} \|T^n\| < \infty.$$

Then

$$\lim_{n\to\infty} \|T^n(I-T)\| = 0 \tag{3.2.1}$$

if and only if $\sigma(T) \cap \mathbb{T} \subset \{1\}$.

In this survey, Esterle is accorded appropriate honour with regards to the above theorem because earlier, in 1983, he studied the operator sequence $T^n(I-T)$, showing, among other things, the if (and more difficult) direction of Theorem 3.2.2 (see [Est83]).

Esterle also proved the strongly continuous analogue of Theorem 3.2.2 in [ESZ92] with Strouse and Zouakia (as did Vũ in [Vũ92]), and used it to provide a new proof of the countable spectrum theorem (Theorem 3.1.3). See also [BV92] for a unification of the continuous and discrete versions of the Esterle–Katznelson–Tzafriri theorem by broadening to the more general framework of representations of locally compact abelian semigroups.

In a similar fashion to Section 3.1, we want to quantify the speed at which the decay in (3.2.1) occurs. In the trivial case where $\sigma(T) \cap \mathbb{T} = \emptyset$, the spectral radius $r(T)$ is strictly less than 1. In particular, for any $\alpha \in (r(T), 1)$, $\|T^n\| \leq \alpha^n$ for all n large enough. Hence $\|T^n(I-T)\| \to 0$ exponentially. Thus the question of quantification is only of interest when the spectrum does touch the unit circle. In this case, the rate of decay cannot happen arbitrarily quickly except in exceptional circumstances. Esterle (again in [Est83]) showed that for any bounded linear operator $T \neq I$ such that $\sigma(T) = \{1\}$, $\liminf_{n\to\infty} n\|T^n(I-T)\|$ was bounded from below by a constant strictly greater than zero. Kalton, Montgomery-Smith, Oleszkiewicz, and Tomilov later determined the exact constant to be $1/e$ in [KMSOT04]. Malinen, Nevalinna, Turunen, and Yuan also found this constant via different methods in [MNTY07], as did Dungey separately in [Dun08].

Furthermore, for any bounded linear operator T (dropping the spectral assumption) on a Banach space X, either

$$\limsup_{n\to\infty} n\|T^n(I-T)\| \geq 1/e,$$

or else X decomposes into the direct sum of two closed invariant subspaces such that the first subspace is fixed by T and on the second, the restriction of T has spectral radius strictly less than 1 (again see [KMSOT04] and also Proposition 5.3.1). Hence for the cases of interest in this thesis, we do not get faster rates of decay in (3.2.1) than $O(n^{-1})$, $n \to \infty$.

An operator is *Ritt* if $\sigma(T) \cap \mathbb{T} = \{1\}$ and there exists a constant $C > 0$ such that

$$\|R(\lambda, T)\| \leq \frac{C}{|1-\lambda|}, \quad |\lambda| > 1. \tag{3.2.2}$$

These operators are relevant for us because an operator is Ritt if and only if it is power-bounded and

$$\|T^n(I-T)\| = O(n^{-1}), \quad n \to \infty.$$

This was proved by independently by Lyubich [Lyu99] and Nagy and Zemanek [NZ99] in 1999. Ritt operators can thus be said to be, in a sense, the class with the extremal rate of decay.

Note that if $1 \in \sigma(T)$, it is impossible to get a better resolvent estimate than (3.2.1), since by Proposition 2.2.1,

$$\|R(\lambda, T)\| \geq \operatorname{dist}(\lambda, \sigma(T))^{-1} \geq |1-\lambda|^{-1}, \quad \lambda \in \rho(T).$$

The study of Ritt operators is often attributed to Nevanlinna, who dealt with them and related topics extensively (see [Nev93, Nev97, Nev01]), but it really first appears in the work of Komatsu [Kom68]. However, Nevanlinna was the first to thoroughly investigate the relationship between resolvent bounds and convergence of iterations.

The other special case for which the rate at which decay takes place in (3.2.1) has been studied is when the decay happens at $O(n^{-1/2})$, $n \to \infty$. In 1973, Foguel and Weiss showed that if T is a contraction in a Banach space, then the convex sum $T_\beta = (1-\beta)I + \beta T$ for $\beta \in (0,1)$ satisfies (3.2.1) with the rate $O(n^{-1/2})$, $n \to \infty$ [FW73, Lemma 2.1]. More recently, in 2008, Dungey showed the converse. In particular, he showed in [Dun08, Theorem 1.2] that if T is a bounded linear operator on a Banach space, then T is power-bounded and $\sup_{n\in\mathbb{N}} n^{1/2}\|T^n(I-T)\| < \infty$ if and only if there exists $\beta \in (0,1)$ and a power-bounded operator S such that $T = (1-\beta)S + \beta S$. Moreover, if either condition holds, then there exists $\beta \in (0,1)$ such that

$$\sigma(T) \subset \{z \in \mathbb{C} : |z-(1-\beta)| \leq \beta \subset \mathbb{D} \cup \{1\}\}.$$

It is worth mentioning that Dungey also considered the aforementioned Ritt operators in [Dun11], relating them to so-called Kreiss operators. We shall return to Dungey's paper in Sections 8.3 and 9.4.

Seifert was the first to prove results for the general case of arbitrary resolvent estimates and decay rates. He did this by obtaining analogous theorems of the type studied in the previous section for discrete operator semigroups.

Heuristically speaking, $T-I$ is the discrete operator semigroup $(T^n)_{n\in\mathbb{N}_0}$ analogue of the generator of a strongly continuous operator semigroup $(T(t))_{t\geq 0}$. We can see this when considering $T-I$ as the 'difference' operator as opposed to A being a 'differentiation' operator:

$$A\int_0^t T(s)x\,ds = T(t)x - x,$$

$$(T-I)\sum_{k=0}^{n-1} T^k x = T^n x - x.$$

To stay consistent with the subsequent literature in discrete operator semigroup asymptotics, we have stated Theorem 3.2.2 as well as our work in Chapter 7 and 8 in terms of $T^n(I-T)$ rather than its negative. Here we notice that, if a spectral mapping theorem held, the condition $\sigma(A) \cap i\mathbb{R} = \{0\}$ would translate to $\sigma(T(t)) \cap \mathbb{T} = \{1\}$. In the discrete case, this is exactly the spectral condition in Theorem 3.2.2.

In [BCT16] and [Mar11b], the goal was to find a rate of decay for $\|T(t)A\|$, that is, for the derivatives of smooth orbits. The extra $(I-A)^{-2}$ terms that appear in Theorems 3.1.10 and 3.1.12 are for regularisation so that the methods of contour integration and functional calculus apply. In the discrete case, where regularisation is not a relevant issue, the analogous decay rate of interest is that of $\|T^n(T-I)\|$.

Thus, the area of so-called quantified Katznelson–Tzafriri theorems began by discretising some of the results of Section 3.1 in order to study the rate at which the decay in (3.2.1) takes place. In the rest of this section, let T be a power-bounded operator on either a Banach or Hilbert space X.

A non-increasing function $m : (0, \pi] \to (0, \infty)$ is called a *dominating function (for the resolvent of T)* if $\|R(e^{i\theta}, T)\| \le m(|\theta|)$ for all θ with $0 < |\theta| \le \pi$. For any such m, we define a further function $m_{\log} : (0, \pi] \to (0, \infty)$ by

$$m_{\log}(\theta) = m(\theta) \log\left(1 + \frac{m(\theta)}{\theta}\right), \quad 0 < \theta \le \pi,$$

a strictly decreasing function defined on the same domain as m with inverse $m_{\log}^{-1}$ defined on the range of $m_{\log}$.

The *minimal dominating function (for the resolvent of T)*, $m_0 : (0, \pi] \to (0, \infty)$, is given by

$$m_0(\theta) = \sup\{\|R(e^{i\varphi}, T)\| : \theta \le |\varphi| \le \pi\}, \quad 0 < \theta \le \pi,$$

and plays an important role in the study of quantified Katznelson–Tzafriri theorems.

In [Sei16], Seifert used some contour integral ideas of Nevanlinna (see the above references to Nevanlinna's work) to prove the following discrete analogue of Theorem 3.1.10 ([Sei16, Theorem 2.11]).

Theorem 3.2.3 (Seifert 2015)**.** *Let X be Banach and suppose that $\sigma(T) \cap \mathbb{T} = \{1\}$ and $m : (0, \pi] \to \infty$ be such that $m \ge m_0$. Then for all $c \in (0, 1)$,*

$$\|T^n(T-I)\| = O(m_{\log}^{-1}(cn))$$

as $n \to \infty$.

However, Seifert later proved a general quantified Tauberian theorem for sequences in [Sei15], published before [Sei16], from which Theorem 3.2.3 followed as an easy corollary (hence the date of Theorem 3.2.3). In that paper, he used the Fourier ideas found in [CS16] as opposed to the contour integral methods employed earlier.

To go with the upper bound in Theorem 3.2.3, the following lower bound was also proved ([Sei16, Corollary 2.6, Remark 2.7]).

Proposition 3.2.4. *Let X be Banach and suppose that $\sigma(T) \cap \{1\}$. Suppose that*

$$\liminf_{\theta \to 0} \max\{\|\theta R(e^{i\theta}, T)\|, \|\theta R(e^{-i\theta}, T)\|\} > \sup\{\|T^n\| : n \geq 0\}.$$

Then there exist constants $c, C > 0$ such that

$$\|T^n(I - T)\| \geq c m_0^{-1}(Cn)$$

for all sufficiently large $n \geq 0$, where m_0^{-1} is the minimal right-inverse of m_0.

In [Sei16], Seifert also proved discrete analogues of the two Borichev–Tomilov results in [BT10] for polynomial rates mentioned in the previous section ([Sei16, Theorem 3.10] and [Sei16, Theorem 3.6] below respectively). The former of these shows that the log term can be dropped when dealing with operators with polynomially estimated resolvents on Hilbert spaces whilst the latter shows that the log factor cannot, in general, be dropped for arbitrary operators on Banach spaces.

Theorem 3.2.5 (Seifert 2016)**.** *Let X be Hilbert and suppose that $\sigma(T) \cap \mathbb{T} = \{1\}$. For any $\alpha \geq 1$, $\|R(e^{i\theta}, T)\| = O(|\theta|^{-\alpha})$ as $\theta \to 0$ if and only if $\|T^n(T - I)\| = O(n^{-1/\alpha})$ as $n \to \infty$.*

Theorem 3.2.6 (Seifert 2016)**.** *Given $\alpha > 2$, there exists a non-trivial Banach space X_α and a power-bounded operator $T_\alpha \in \mathcal{B}(X_\alpha)$ such that $\sigma(T_\alpha) \cap \mathbb{T} = \{1\}$ and $\|R(e^{i\theta}, T_\alpha)\| = O(|\theta|^{-\alpha})$ as $\theta \to 0$ but*

$$\limsup_{n \to \infty} \left(\frac{n}{\log n}\right)^{1/\alpha} \|T_\alpha^n(T_\alpha - I)\| > 0.$$

It is off of the work of Seifert in [Sei15] and [Sei16] that we will later launch Part III. For a more detailed survey on the rates of decay in (3.2.1) up to this point than we have given, see [Lék17]. There, Léka also deals with other related topics (for example, functions of the Volterra operator).

Part II

Continuous operator semigroups

Chapter 4

Optimal energy decay in a one-dimensional coupled wave-heat-wave system

In this chapter[1] and the next, we apply Theorems 3.1.8 (Borichev-Tomilov) and 3.1.12 (Batty-Chill-Tomilov) mentioned in the survey from Section 3.1 to two one-dimensional systems with coupled wave and heat parts. These applications are modelled upon the 2016 paper of Batty, Paunonen, and Seifert [BPS16] where the optimal energy decay in a one-dimensional coupled wave-heat system with finite Neumann wave and Dirichlet heat parts was studied by analysing the following system:

$$
\begin{cases}
u_{tt}(\xi,t) = u_{\xi\xi}(\xi,t), & & \xi \in (-1,0),\ t>0,\\
w_t(\xi,t) = w_{\xi\xi}(\xi,t), & & \xi \in (0,1),\ t>0,\\
u_t(0,t) = w(0,t), & u_\xi(0,t) = w_\xi(0,t), & t>0,\\
u_\xi(-1,t) = 0, & w(1,t) = 0, & t>0,\\
u(\xi,0) = u(\xi), & u_t(\xi,0) = v(\xi), & \xi \in (-1,0),\\
w(\xi,0) = w(\xi), & & \xi \in (0,1),
\end{cases}
\tag{4.0.1}
$$

where the initial data u, v, and w belong to $H^1(-1,0), L^2(-1,0)$ and $L^2(0,1)$ respectively. The energy was then defined, given a vector of initial data $x = (u,v,w)$, as

$$
E_x(t) = \frac{1}{2}\int_{-1}^{1} |u_\xi(\xi,t)|^2 + |u_t(\xi,t)|^2 + |w(\xi,t)|^2 \, d\xi, \quad t \geq 0,
$$

with all the functions being understood to have been extended by zero in ξ to the interval $(-1,1)$. If the solution is sufficiently regular, a routine calculation via inte-

[1]The material in this chapter is based on [Ng20b].

gration by parts shows that

$$E_x'(t) = -\int_0^1 |w_\xi(\xi,t)|^2 \, d\xi, \quad t \geq 0,$$

and, in particular, that the energy of any such solution is non-increasing with respect to time. The main goal of analysing such a model is to quantitatively estimate the rate of energy decay of a given solution. Problems of this type are motivated by the study of so-called fluid-structure models. These arise from the coupling of the nonlinear elasticity equation and the Navier-Stokes equations (see the survey [AT07] for example).

The system (4.0.1) was first studied (with Dirichlet boundary at $\xi = -1$ and a slightly different coupling condition) in [ZZ04], yielding the sharp decay rate $E_x(t) = O(t^{-4})$, $t \to \infty$. The approach in [ZZ04] relied on a rather complicated spectral analysis used in conjunction with the theory of Riesz spectral operators. In contrast to [ZZ04], however, the approach in [BPS16] was based on the semigroup methods of semi-uniform stability pioneered in [BD08], widely popularised in [BT10], and largely completed (perfected) by [RSS19] (see the survey in Section 3.1), greatly simplifying the analysis necessary to obtain the rate of decay. We direct the reader to [BPS16, Section 1] for references to a range of similar problems, including the higher dimensional setting. One important and relevant result that was published after [BPS16] is [ALT16], where Theorem 3.1.8 is applied to a C_0-semigroup that solves a wave-heat fluid-structure system in a 2- or 3-dimensional domain. See also [BPS19] where the same approach as [BPS16] with suitable adjustments is applied to study a wave-heat system on a rectangular domain and [Duy07] where Theorem 3.1.4 (Burq–Lebeau) is applied to another related system from [RZZ05, ZZ07]. It is worth mentioning that this area of research is partially inspired by and connected to another area of partial differential equations - the study of damped wave equations (as mentioned in the survey in Section 3.1).

The goal of this particular chapter, where we add an extra wave component to the system (4.0.1) and take Dirichlet boundary conditions on both ends, is to analyse the following wave-heat-wave-system:

$$\begin{cases} u_{tt}(\xi,t) = u_{\xi\xi}(\xi,t), & & \xi \in (0,1),\ t>0, \\ w_t(\xi,t) = w_{\xi\xi}(\xi,t), & & \xi \in (1,2),\ t>0, \\ \tilde{u}_{tt}(\xi,t) = \tilde{u}_{\xi\xi}(\xi,t), & & \xi \in (2,3),\ t>0, \\ u_t(1,t) = w(1,t), & u_\xi(1,t) = w_\xi(1,t), & t>0, \\ \tilde{u}_t(2,t) = w(2,t), & \tilde{u}_\xi(2,t) = w_\xi(2,t), & t>0, \\ u(0,t) = \tilde{u}(3,t) = 0, & & t>0, \\ u(\xi,0) = u(\xi), & u_t(\xi,0) = v(\xi), & \xi \in (0,1), \\ w(\xi,0) = w(\xi), & & \xi \in (1,2), \\ \tilde{u}(\xi,0) = \tilde{u}(\xi), & \tilde{u}_t(\xi,0) = \tilde{v}(\xi), & \xi \in (2,3). \end{cases} \tag{4.0.2}$$

The initial data is required to satisfy $u = u(\xi,0) \in H^1(0,1), v = u_t(\xi,0) \in L^2(0,1), w = w(\xi,0) \in L^2(1,2), \tilde{u} = \tilde{u}(\xi,0) \in H^1(2,3)$, and $\tilde{v} = \tilde{u}_t(\xi,0) \in L^2(2,3)$.

As in [BPS16], the aim here is to find a quantitative estimate for the rate of energy decay of a given solution. Given a vector of initial data $x = (u, v, w, \tilde{u}, \tilde{v})$ satisfying the conditions above, we similarly define the energy of the corresponding solution as

$$E_x(t) = \frac{1}{2}\int_0^3 |u_\xi(\xi,t)|^2 + |u_t(\xi,t)|^2 + |w(\xi,t)|^2 + |\tilde{u}_\xi(\xi,t)|^2 + |\tilde{u}_t(\xi,t)|^2 \, d\xi, \quad t \geq 0.$$

Again, all functions have been extended by zero in ξ to the interval $(0,3)$. Provided we have sufficient regularity of the solution, a simple calculation via integration by parts shows that

$$E_x'(t) = \operatorname{Re}\left\{\tilde{u}_\xi(3,t)\overline{\tilde{u}_t(3,t)} - u_\xi(0,t)\overline{u_t(0,t)}\right\} - \int_1^2 |w_\xi(\xi,t)|^2 \, d\xi, \quad t \geq 0.$$

Since $u_t(0,t) = \frac{\partial}{\partial t}u(0,t) = \tilde{u}_t(3,t) = \frac{\partial}{\partial t}\tilde{u}(3,t) = 0$ for $t > 0$, the energy of any such solution is non-increasing with respect to time. The rest of the chapter is devoted to obtaining a sharp quantitative estimate for the rate of this decay for classical solutions of (4.0.2).

4.1 Well-posedness – the semigroup and its generator

In this section, we first prove that (4.0.2) is well posed and has solution given by the orbits of a C_0-semigroup of contractions $(T(t))_{t\geq 0}$, before turning to analyse the spectrum of the generator A of $(T(t))_{t\geq 0}$.

4.1.1 Existence of the semigroup

We start by recasting (4.0.2) as an abstract Cauchy problem in order to later apply the methods of semi-uniform stability. Consider the Hilbert space

$$X_0 = H^1(0,1) \times L^2(0,1) \times L^2(1,2) \times H^1(2,3) \times L^2(2,3)$$

and define

$$X = \{(u, v, w, \tilde{u}, \tilde{v}) \in X_0 : u(0) = \tilde{u}(3) = 0\}$$

endowed with the norm (and corresponding inner product) given by

$$\|(u, v, w, \tilde{u}, \tilde{v})\|_X^2 = \|u'\|_{L^2}^2 + \|v\|_{L^2}^2 + \|w\|_{L^2}^2 + \|\tilde{u}'\|_{L^2}^2 + \|\tilde{v}\|_{L^2}^2$$

which is non-degenerate because the fundamental theorem of calculus applied in conjunction with the boundary conditions $u(0) = \tilde{u}(3) = 0$ implies that $\|u\|_{L^2} \lesssim \|u'\|_{L^2}$ and $\|\tilde{u}\|_{L^2} \lesssim \|\tilde{u}'\|_{L^2}$. Here and in the rest of the chapter, the intervals for function spaces appearing as subscripts will often be omitted if they are clear from the context. Let

$$X_1 = X \cap [H^2(0,1) \times H^1(0,1) \times H^2(1,2) \times H^2(2,3) \times H^1(2,3)]$$

and define the operator A on X by $Ax = (v, u'', w'', \tilde{v}, \tilde{u}'')$ for $x = (u, v, w, \tilde{u}, \tilde{v})$ in the domain

$$\begin{aligned} D(A) = \{(u, v, w, \tilde{u}, \tilde{v}) \in X_1 : v(0) = \tilde{v}(3) = 0, u'(1) = w'(1), \\ v(1) = w(1), \tilde{u}'(2) = w'(2), \tilde{v}(2) = w(2)\}. \end{aligned}$$

Lemma 4.1.1. *The following hold:*

(i) A is closed;

(ii) A is densely defined;

(iii) A is dissipative;

(iv) $1 - A$ is surjective.

Proof. (i) Let $x_n = (u_n, v_n, w_n, \tilde{u}_n, \tilde{v}_n) \in D(A)$ be such that

$$x_n \to x = (u, v, w, \tilde{u}, \tilde{v}), \ \ Ax_n = (v_n, u_n'', w_n'', \tilde{v}_n, \tilde{u}_n'') \to y = (f, g, h, \tilde{f}, \tilde{g})$$

in X. Then u_n converges to u in $H^1(0,1)$ and u_n'' converges to g in $L^2(0,1)$. Hence

$$\int u\varphi'' = \lim_{n\to\infty} \int u_n\varphi'' = \lim_{n\to\infty} \int u_n''\varphi = \int g\varphi, \quad \varphi \in C_c^\infty(0,1), \tag{4.1.1}$$

where the integral is taken over $((0,1), d\xi)$ so that $u \in H^2(0,1)$ and $u'' = g$. As v_n converges to both v and f in $L^2(0,1)$, $v = f$. In particular, $v \in H^1(0,1)$. The same argument shows that $\tilde{u} \in H^2(2,3)$ with $\tilde{u}'' = \tilde{g}$ and $\tilde{v} = \tilde{f} \in H^1(2,3)$.

Next, w_n converges to w and w_n'' to h in $L^2(1,2)$. Standard Sobolev theory (see for example [Bre11, Page 217]) ensures the existence of a constant C such that

$$\|\psi'\|_{L^2(1,2)} \leq \|\psi''\|_{L^2(1,2)} + C\|\psi\|_{L^2(1,2)}, \quad \psi \in H^2(1,2).$$

Hence, the sequence w_n' is Cauchy and converges to some H in $L^2(1,2)$. Using similar reasoning to that in (4.1.1), we see that $w \in H^2(1,2)$ with $w' = H$ and $w'' = h$.

To check that the coupling conditions for x to be in the domain $D(A)$ are satisfied, it is enough to pass to a subsequence x_{n_k} that converges pointwise a.e. and note the continuity of $u', v, w', w, \tilde{u}', \tilde{v}$. It follows that $Ax = y$.

(ii) Consider the subspace X_1 equipped with the X norm, which is dense in X. The linear functional $\phi_1 : x = (u, v, w, \tilde{u}, \tilde{v}) \mapsto v(0)$ is unbounded on X_1, and hence

$$X_2 = \operatorname{Ker} \phi_1 = \{(u, v, w, \tilde{u}, \tilde{v}) \in X_1 : v(0) = 0\}$$

is dense in X_1. Similarly,

$$X_3 = \operatorname{Ker} \phi_2 = \{(u, v, w, \tilde{u}, \tilde{v}) \in X_2 : v(1) = w(1)\}$$

is dense in X_2 where ϕ_2 is the unbounded linear functional on X_2 defined by $x \mapsto v(1) - w(1)$. Again, by considering the unbounded linear functional $\phi_3 : x \mapsto u'(1) - w'(1)$ on X_3, we see that

$$X_4 = \operatorname{Ker} \phi_3 = \{(u, v, w, \tilde{u}, \tilde{v}) \in X_3 : u'(1) = w'(1)\}$$

is dense in X_3. The same argument can be repeated for the coupling and boundary conditions for $w, \tilde{u}$, and $\tilde{v}$ to produce a decreasing finite chain of subspaces

$$X \supset X_1 \supset X_2 \supset ... \supset D(A)$$

where each subspace is dense in the preceding one under the X norm. Hence A is densely defined.

(iii) Let $x \in D(A)$. Assuming the appropriate intervals over which to take the L^2 inner products, we have, through integration by parts and the coupling and boundary conditions,

$$\begin{aligned}\langle Ax, x\rangle &= \langle v', u'\rangle_{L^2} + \langle u'', v\rangle_{L^2} + \langle w'', w\rangle_{L^2} + \langle \tilde{v}', \tilde{u}'\rangle_{L^2} + \langle \tilde{u}'', \tilde{v}\rangle_{L^2} \\ &= -\overline{\langle u'', v\rangle_{L^2}} + \langle u'', v\rangle_{L^2} - \langle w', w'\rangle_{L^2} - \overline{\langle \tilde{u}'', \tilde{v}\rangle_{L^2}} + \langle \tilde{u}'', \tilde{v}\rangle_{L^2}.\end{aligned}$$

Hence

$$\operatorname{Re}\langle Ax, x\rangle = -\|w'\|_{L^2}^2 \le 0,$$

showing that A is dissipative.

(iv) Though in the setting of this lemma, we only need to work with $1 - A$, we perform a procedure here with $\lambda - A$ for general $\lambda \neq 0$ in order to avoid repetition that otherwise would be inevitable in later sections. Note that we are closely following the proof of [BPS16, Theorem 3.1].

Let $x = (u, v, w, \tilde{u}, \tilde{v})$ and $y = (f, g, h, \tilde{f}, \tilde{g})$ be in X. Then the equation $(\lambda - A)x = y$ can be rewritten as the following system of boundary value problems:

$$u'' = \lambda^2 u - \lambda f - g, \qquad \xi \in (0,1), \tag{4.1.2a}$$

$$v = \lambda u - f, \qquad \xi \in (0,1), \tag{4.1.2b}$$

$$w'' = \lambda w - h, \qquad \xi \in (1,2), \tag{4.1.2c}$$

$$\tilde{u}'' = \lambda^2 \tilde{u} - \lambda \tilde{f} - \tilde{g}, \qquad \xi \in (2,3), \tag{4.1.2d}$$

$$\tilde{v} = \lambda \tilde{u} - \tilde{f}, \qquad \xi \in (2,3), \tag{4.1.2e}$$

$$u(0) = v(0) = 0, \quad v(1) = w(1), \quad u'(1) = w'(1), \tag{4.1.2f}$$

$$\tilde{u}(3) = \tilde{v}(3) = 0, \quad \tilde{v}(2) = w(2), \quad \tilde{u}'(2) = w'(2). \tag{4.1.2g}$$

Let

$$U_\lambda(\xi) = \frac{1}{\lambda}\int_0^\xi \sinh(\lambda(\xi - r))(\lambda f(r) + g(r))\,dr, \quad \xi \in [0,1],$$

which has derivative

$$U_\lambda'(\xi) = \int_0^\xi \cosh(\lambda(\xi - r))(\lambda f(r) + g(r))\,dr, \quad \xi \in [0,1].$$

The differential equation (4.1.2a) with the boundary condition $u(0) = 0$ has the general solution

$$u(\xi) = a(\lambda)\sinh(\lambda\xi) - U_\lambda(\xi), \quad \xi \in [0,1], \tag{4.1.3}$$

where $a(\lambda) \in \mathbb{C}$ is a parameter free to be varied. In particular,

$$u'(\xi) = \lambda a(\lambda)\cosh(\lambda\xi) - U_\lambda'(\xi), \quad \xi \in [0,1]. \tag{4.1.4}$$

Clearly $u \in H^2(0,1)$ and hence $v \in H^1(0,1)$ with $v(0) = \lambda u(0) - f(0) = 0$.

Similarly, the general solution of (4.1.2d) with boundary condition $\tilde{u}(3) = 0$ can be written as

$$\tilde{u}(\xi) = \tilde{a}(\lambda)\sinh(\lambda(3 - \xi)) + \tilde{U}_\lambda(\xi), \quad \xi \in [2,3], \tag{4.1.5}$$

where $\tilde{a}(\lambda) \in \mathbb{C}$ can be varied freely and

$$\tilde{U}_\lambda(\xi) = \frac{1}{\lambda} \int_\xi^3 \sinh(\lambda(r-\xi))(\lambda \tilde{f}(r) + \tilde{g}(r))\, dr, \quad \xi \in [2,3].$$

Thus

$$\tilde{u}'(\xi) = -\lambda \tilde{a}(\lambda) \cosh(\lambda(3-\xi)) + \tilde{U}'_\lambda(\xi), \quad \xi \in [2,3], \tag{4.1.6}$$

where

$$\tilde{U}'_\lambda(\xi) = -\int_\xi^3 \cosh(\lambda(r-\xi))(\lambda \tilde{f}(r) + \tilde{g}(r))\, dr, \quad \xi \in [2,3].$$

Again, it follows that $\tilde{u} \in H^2(2,3)$ and $\tilde{v} \in H^1(2,3)$ with $\tilde{v}(3) = 0$.

In the same spirit, let

$$W_\lambda(\xi) = \frac{1}{\sqrt{\lambda}} \int_1^\xi \sinh(\sqrt{\lambda}(\xi - r)) h(r)\, dr, \quad \xi \in [1,2],$$

which has derivative

$$W'_\lambda(\xi) = \int_1^\xi \cosh(\sqrt{\lambda}(\xi - r)) h(r)\, dr, \quad \xi \in [1,2].$$

The general solution of (4.1.2c) can then be written as

$$w(\xi) = b(\lambda) \cosh(\sqrt{\lambda}(\xi-1)) + c(\lambda) \sinh(\sqrt{\lambda}(\xi-1)) - W_\lambda(\xi), \quad \xi \in [1,2], \tag{4.1.7}$$

where $b(\lambda), c(\lambda) \in \mathbb{C}$ are free parameters and in particular,

$$w'(\xi) = \sqrt{\lambda} b(\lambda) \sinh(\sqrt{\lambda}(\xi-1)) + \sqrt{\lambda} c(\lambda) \cosh(\sqrt{\lambda}(\xi-1)) - W'_\lambda(\xi), \quad \xi \in [1,2]. \tag{4.1.8}$$

It remains to choose specific constants $a(\lambda), b(\lambda), c(\lambda)$ and $\tilde{a}(\lambda)$ in order to satisfy the coupling conditions. Using (4.1.3) and (4.1.7), the requirement $\lambda u(1) - f(1) = v(1) = w(1)$ holds if and only if

$$\lambda a(\lambda) \sinh(\lambda) - b(\lambda) = \lambda U_\lambda(1) + f(1).$$

Likewise, the conditions $u'(1) = w'(1)$, $\lambda \tilde{u}(2) - \tilde{f}(2) = w(2)$, and $\tilde{u}'(2) = w'(2)$ are equivalent to

$$\lambda a(\lambda) \cosh(\lambda) - \sqrt{\lambda} c(\lambda) = U'_\lambda(1),$$

$$\lambda \tilde{a}(\lambda) \sinh(\lambda) - b(\lambda) \cosh(\sqrt{\lambda}) - c(\lambda) \sinh(\sqrt{\lambda}) = -\lambda \tilde{U}_\lambda(2) + \tilde{f}(2) - W_\lambda(2),$$

and

$$-\lambda \tilde{a}(\lambda) \cosh(\lambda) - \sqrt{\lambda} b(\lambda) \sinh(\sqrt{\lambda}) - \sqrt{\lambda} c(\lambda) \cosh(\sqrt{\lambda}) = -\tilde{U}'_\lambda(2) - W'_\lambda(2)$$

respectively. These four equations can be written in matrix form as

$$M_\lambda \cdot \begin{pmatrix} a(\lambda) \\ b(\lambda) \\ c(\lambda) \\ \tilde{a}(\lambda) \end{pmatrix} = \mathbf{b}, \tag{4.1.9}$$

where

$$M_\lambda = \begin{pmatrix} \lambda\sinh(\lambda) & -1 & 0 & 0 \\ \lambda\cosh(\lambda) & 0 & -\sqrt{\lambda} & 0 \\ 0 & -\cosh(\sqrt{\lambda}) & -\sinh(\sqrt{\lambda}) & \lambda\sinh(\lambda) \\ 0 & \sqrt{\lambda}\sinh(\sqrt{\lambda}) & \sqrt{\lambda}\cosh(\sqrt{\lambda}) & \lambda\cosh(\lambda) \end{pmatrix} \tag{4.1.10}$$

and

$$\mathbf{b} = \begin{pmatrix} \lambda U_\lambda(1) + f(1) \\ U'_\lambda(1) \\ -\lambda\tilde{U}_\lambda(2) + \tilde{f}(2) - W_\lambda(2) \\ \tilde{U}'_\lambda(2) + W'_\lambda(2) \end{pmatrix}. \tag{4.1.11}$$

Thus, (4.1.9) has a solution for any given $y = (f, g, h, \tilde{f}, \tilde{g})$ in X if and only if

$$\det M_\lambda = -\lambda^2[2\sqrt{\lambda}\cosh(\sqrt{\lambda})\cosh(\lambda)\sinh(\lambda) + \sinh(\sqrt{\lambda})(\lambda\sinh^2(\lambda) + \cosh^2(\lambda))]$$

is non-zero. For $\lambda = 1$,

$$\det M_1 = -\sinh(1)[4\cosh^2(1) - 1] \neq 0,$$

proving (iv). □

All the dirty work has now been done (ahead of time). The following theorem follows immediately from Lemma 4.1.1 and Theorem 2.3.4 (Lumer–Phillips).

Theorem 4.1.2. *A generates a contraction C_0-semigroup $T(t)$ on X.*

4.1.2 Spectrum of the generator

From Theorem 4.1.2 and Theorem 2.3.3 (Hille-Yosida), we know that $\sigma(A)$ is contained in the closed left half-plane. However, we can say more about the spectrum.

Theorem 4.1.3. *The spectrum of A consists of isolated eigenvalues and is given by*

$$\sigma(A) = \{\lambda \in \mathbb{C}_- : \det M_\lambda = 0\}.$$

In particular, $\sigma(A) \cap i\mathbb{R} = \emptyset$.

We will need the following lemma in order to prove the theorem above.

Lemma 4.1.4. *If $\lambda \in \rho(A)$, then $R(\lambda, A)$ is a compact operator.*

Proof. Let $\lambda \in \rho(A)$. Then $\lambda - A$ is a bijective bounded (and in particular, closed) linear map from $D(A)$ endowed with the graph norm onto X. Hence the inverse map $R(\lambda, A)$ maps X isomorphically onto $(D(A), \|\cdot\|_A)$. Since

$$\begin{aligned}\|(u, v, w, \tilde{u}, \tilde{v})\|_{D(A)} &= \|(u, v, w, \tilde{u}, \tilde{v})\|_X + \|(v, u'', w'', \tilde{v}, \tilde{u}'')\|_X \\ &\lesssim \|u'\|_{L^2} + \|v\|_{L^2} + \|w\|_{L^2} + \|\tilde{u}'\|_{L^2} + \|\tilde{v}\|_{L^2} \\ &\quad + \|v'\|_{L^2} + \|u''\|_{L^2} + \|w''\|_{L^2} + \|\tilde{v}'\|_{L^2} + \|\tilde{u}''\|_{L^2},\end{aligned}$$

it follows that $(D(A), \|\cdot\|_{D(A)})$ embeds continuously into

$$H^2(0,1) \times H^1(0,1) \times H^2(1,2) \times H^2(2,3) \times H^1(2,3)$$

endowed with its natural norm (see [Bre11, Page 217]). This space in turn embeds compactly into X by the Rellich–Kondrachov theorem of Sobolev theory. Stringing together these embeddings, $R(\lambda, A)$ is a compact operator on X. $\square$

Proof of Theorem 4.1.3. We first show that not only is $\lambda - A$ surjective as shown in Lemma 4.1.1 whenever $\det M_\lambda \neq 0$, it is also injective. Indeed, suppose $(\lambda - A)x = 0$. Then, x is obtained in the same way as in the proof of Lemma 4.1.1(iv) with $\mathbf{b} = 0$ in (4.1.9). As $\det M_\lambda \neq 0$, we get that $x = 0$. Hence $\lambda - A$ is closed and bijective, so has bounded inverse by the closed graph theorem. In particular, $1 \in \rho(A)$ and so the resolvent is non-empty.

The spectral theorem for compact operators used in conjunction with Lemma 4.1.4 implies that the spectrum of $R(1, A)$ consists only of eigenvalues of finite multiplicity with the only possible accumulation point being the origin. By Theorem 2.2.3 (spectral mapping for the resolvent),

$$\sigma(A) = \{1 - \nu^{-1} : \nu \in \sigma(R(1, A)) \setminus \{0\}\}$$

and furthermore, a simple calculation shows that if ν is an eigenvalue of $R(1, A)$, then $1 - \nu^{-1}$ is an eigenvalue of A. Hence $\sigma(A)$ consists only of eigenvalues of finite multiplicity with the only possible accumulation point being at infinity. Thus, $\lambda \in \sigma(A)$ if and only if $\det M_\lambda = 0$.

To show the final statement, suppose that $s \in \mathbb{R}$ with $s \neq 0$ and that $x = (u, v, w, \tilde{u}, \tilde{v}) \in \operatorname{Ker}(is - A)$. From the proof of Lemma 4.1.1(iii), we have

$$0 = \operatorname{Re}\langle (is - A)x, x\rangle = -\operatorname{Re}\langle Ax, x\rangle = \|w'\|_{L^2}. \tag{4.1.12}$$

Thus, from (4.1.2c), $w = (is)^{-1}(w')' = 0$. As in the proof for Lemma 4.1.1(iv), we have

$$u(\xi) = a(is)\sinh(is\xi), \quad v(\xi) = is\ u(\xi), \quad \xi \in [0,1].$$

The coupling conditions imply that $u'(1) = v(1) = 0$. Thus,

$$is\ a(is)\cosh(is) = is\ a(is)\sinh(is) = 0,$$

implying that $a(is) = 0$. Similarly, $\tilde{a}(is) = 0$ so that $x = 0$.

Consider now the case $s = 0$. Rewriting $Ax = 0$ into component differential equations, we get that $u'' = 0$ and $v = 0$ as well as $w' = 0$ as in (4.1.12). As $u'(1) = w'(1)$ and u' is constant, $u' = 0$ and hence $u(0) = 0$ implies that $u = 0$. Similarly, $v(1) = 0$ implies that $w = 0$. The same is true for $\tilde{u}$ and $\tilde{v}$. It follows that $\sigma(A) \cap i\mathbb{R} = \emptyset$. □

4.2 Resolvent estimates

We turn now to obtaining an upper bound on the growth of $\|R(is, A)\|$ as $|s| \to \infty$ which will allow us to deduce a quantitative estimate on the rate of energy decay in the next section.

Theorem 4.2.1. *We have $\|R(is, A)\| = O(|s|^{1/2})$ as $|s| \to \infty$.*

To prove this theorem, we will need explicit forms for the $a(\lambda), b(\lambda), c(\lambda), \tilde{a}(\lambda)$ found in the proof of Lemma 4.1.1(iv) for the case where $\lambda = is$ and to this end, we invert M_λ to get that

$$(\det M_\lambda)^{-1} C^T \mathbf{b} = \begin{pmatrix} a(\lambda) \\ b(\lambda) \\ c(\lambda) \\ \tilde{a}(\lambda) \end{pmatrix}, \tag{4.2.1}$$

where C is the cofactor matrix of M_λ. First, we rewrite $\det M_\lambda$ and define two terms which are ubiquitous in this section:

$$\begin{aligned} \det M_\lambda &= -\lambda^2[2\sqrt{\lambda}\cosh(\sqrt{\lambda})\cosh(\lambda)\sinh(\lambda) + \sinh(\sqrt{\lambda})(\lambda\sinh^2(\lambda) + \cosh^2(\lambda))] \\ &= \frac{\lambda^2}{2}[-e^{\sqrt{\lambda}}(\lambda\sinh^2(\lambda) + 2\sqrt{\lambda}\cosh(\lambda)\sinh(\lambda) + \cosh^2(\lambda)) \\ &\qquad + e^{-\sqrt{\lambda}}(\lambda\sinh^2(\lambda) - 2\sqrt{\lambda}\cosh(\lambda)\sinh(\lambda) + \cosh^2(\lambda))] \\ &= \lambda^2[-e^{\sqrt{\lambda}}T_+^2(\lambda) + e^{-\sqrt{\lambda}}T_-^2(\lambda)], \end{aligned}$$

where

$$T_+(\lambda) = \frac{1}{2}[\cosh(\lambda) + \sqrt{\lambda}\sinh(\lambda)], \quad T_-(\lambda) = \frac{1}{2}[\cosh(\lambda) - \sqrt{\lambda}\sinh(\lambda)].$$

The functions T_+ and T_- are useful because they obey convenient lower bounds on the one hand, and appear many times in the entries of $C = \{c_{ij}\}_{i,j}$ on the other hand. As an example of this, c_{11} is explicitly computed and stated here:

$$\begin{aligned} c_{11} &= -\lambda^{3/2}[\cosh(\sqrt{\lambda})\cosh(\lambda) + \sqrt{\lambda}\sinh(\sqrt{\lambda})\sinh(\lambda)] \\ &= -\lambda^{3/2}[e^{\sqrt{\lambda}}T_+(\lambda) + e^{-\sqrt{\lambda}}T_-(\lambda)]. \end{aligned}$$

The expressions for the other entries can be found in the appendix of this chapter.

We will also need the following two lemmas, the first of which is proved in [BPS16, Lemma 3.3] (over the interval $[-1,0]$ rather than $[0,1]$ or $[2,3]$ as we have here).

Lemma 4.2.2. *There exists a constant $C \geq 0$ such that, for all $f \in H^1(0,1), g \in L^2(0,1), \tilde{f} \in H^1(2,3), \tilde{g} \in L^2(2,3)$, and $\lambda \in i\mathbb{R}$,*

$$\begin{aligned}
\left|\int_0^\xi \sinh(\lambda(\xi - r))(\lambda f(r) + g(r))\,dr\right| &\leq C\|f\|_{H^1} + \|g\|_{L^2}, \quad \xi \in [0,1], \\
\left|\int_0^\xi \cosh(\lambda(\xi - r))(\lambda f(r) + g(r))\,dr\right| &\leq C\|f\|_{H^1} + \|g\|_{L^2}, \quad \xi \in [0,1], \\
\left|\int_\xi^3 \sinh(\lambda(r - \xi))(\lambda \tilde{f}(r) + \tilde{g}(r))\,dr\right| &\leq C\|\tilde{f}\|_{H^1} + \|\tilde{g}\|_{L^2}, \quad \xi \in [2,3], \\
\left|\int_\xi^3 \cosh(\lambda(r - \xi))(\lambda \tilde{f}(r) + \tilde{g}(r))\,dr\right| &\leq C\|\tilde{f}\|_{H^1} + \|\tilde{g}\|_{L^2}, \quad \xi \in [2,3].
\end{aligned}$$

Lemma 4.2.3. *For $\lambda \in i\mathbb{R}$ with $|\lambda| \geq \left(\frac{1}{\sqrt{2}} + 1\right)^2$, we have*

$$1/4 \leq |T_+(\lambda)|, |T_-(\lambda)| \leq \frac{1}{2}(\sqrt{|\lambda|} + 1).$$

Proof. The upper bound is obvious. We prove the lower bound for $T_+(\lambda)$ where $\lambda = is$ with $s \in \mathbb{R}$ and note that $2T_+(is) = \cos(s) + i\sqrt{is}\sin(s)$. Explicit calculation yields

$$\begin{aligned} 4|T_+(\lambda)|^2 &= \left|\frac{1}{\sqrt{is}}(\sqrt{is}\cos(s) - s\sin(s))\right|^2 \\ &= \frac{1}{|s|}\left(\frac{|s|}{2}\cos^2(s) + \left(\sqrt{\frac{|s|}{2}}\cos(s) - s\sin(s)\right)^2\right), \end{aligned}$$

as $\operatorname{Re}\sqrt{\lambda} \geq 0$ for all $\lambda \in \mathbb{C}$ since we have taken the branch cut of the square root along the negative real axis. In the case where $\cos^2(s) \geq 1/2$, it follows that $4|T_+(\lambda)|^2 \geq 1/4$. However, if $\cos^2(s) < 1/2$, then $\sin^2(s) \geq 1/2$, so that

$$2|T_+(\lambda)| \geq |i\sqrt{is}\sin(s)| - |\cos(s)| \geq \sqrt{\frac{|s|}{2}} - \frac{1}{\sqrt{2}} \geq \frac{1}{2}$$

whenever $|\lambda| = |s| \geq \left(\frac{1}{\sqrt{2}} + 1\right)^2$. The case for $T_-(\lambda)$ is similar. $\square$

Proof of Theorem 4.2.1. Let $\lambda = is$ for $s \in \mathbb{R}$ and let $y = (f, g, h, \tilde{f}, \tilde{g}) \in Z$, further defining $x = (u, v, w, \tilde{u}, \tilde{v}) \in D(A)$ by $x = R(\lambda, A)y$. As $v = \lambda u - f$ and $\tilde{v} = \lambda\tilde{u} - \tilde{f}$, we have that

$$\|x\| \lesssim \|\lambda u\|_{L^2} + \|f\|_{L^2} + \|u'\|_{L^2} + \|w\|_{L^2} + \|\lambda\tilde{u}\|_{L^2} + \|\tilde{f}\|_{L^2} + \|\tilde{u}'\|_{L^2}.$$

Thus the result will follow once we have established that each of the summands in the above equation are bounded by $C\sqrt{|\lambda|}\|y\|$ for $|s| \geq N$, where $C, N > 0$ are constants independent of y.

Let u be given by (4.1.3). By Lemma 4.2.2, it is enough to consider $|\lambda a(\lambda)|$ in order to estimate $\|\lambda u\|_{L^2}$ and $\|u'\|_{L^2}$. Now

$$\lambda a(\lambda) = \frac{\lambda}{\det M_\lambda}(c_{11}b_1 + c_{21}b_2 + c_{31}b_3 + c_{41}b_4) \tag{4.2.2}$$

where b_i are the components of the vector $\mathbf{b}$ in (4.1.9). We consider each of these terms separately. Note that by Lemma 4.2.2, the only terms in the components of $\mathbf{b}$ that are not automatically bounded by some constant multiple of $\|y\|$ are $W_\lambda(2)$ and $W_\lambda'(2)$. Looking at the first term in (4.2.2), Lemma 4.2.3 implies that

$$\left|\frac{\lambda}{\det M_\lambda}c_{11}\right| = \sqrt{|\lambda|}\left|\frac{e^{\sqrt{\lambda}}T_+(\lambda) + e^{-\sqrt{\lambda}}T_-(\lambda)}{-e^{\sqrt{\lambda}}T_+(\lambda)^2 + e^{-\sqrt{\lambda}}T_-(\lambda)^2}\right| \lesssim \sqrt{|\lambda|}|T_+(\lambda)|^{-1} \lesssim \sqrt{|\lambda|},$$

since $\operatorname{Re}\sqrt{\lambda} > 0$ for $\lambda \in i\mathbb{R} \setminus \{0\}$ as before, so that $e^{\sqrt{\lambda}}$ dominates $e^{-\sqrt{\lambda}}$. Thus,

$$\left|\frac{\lambda}{\det M_\lambda}c_{11}b_1\right| \lesssim \sqrt{|\lambda|}\|y\|,$$

where the implicit constant is independent of λ and y.

Likewise,

$$\left|\frac{\lambda}{\det M_\lambda}c_{21}\right| = \left|\frac{e^{\sqrt{\lambda}}T_+(\lambda) - e^{-\sqrt{\lambda}}T_-(\lambda)}{-e^{\sqrt{\lambda}}T_+(\lambda)^2 + e^{-\sqrt{\lambda}}T_-(\lambda)^2}\right| \lesssim 1,$$

so that

$$\left|\frac{\lambda}{\det M_\lambda} c_{21} b_2\right| \lesssim \|y\|.$$

Noting that $|\cosh(\lambda)|, |\sinh(\lambda)| \leq 1$, a similar argument shows that the remaining terms in (4.2.2) that do not include $W_\lambda(2)$ and $W'_\lambda(2)$ are bounded by a constant times $\sqrt{|\lambda|}\|y\|$. Consider now

$$\begin{aligned}
\left|\frac{\lambda}{\det M_\lambda} c_{31} W_\lambda(2)\right| &\leq \int_1^2 \left|\frac{\sinh(\sqrt{\lambda}(2-r))h(r)}{-e^{\sqrt{\lambda}}T_+(\lambda)^2 + e^{-\sqrt{\lambda}}T_-(\lambda)^2}\right| dr \\
&\leq \frac{1}{2}\int_1^2 \left|\frac{e^{\sqrt{\lambda}(2-r)} - e^{-\sqrt{\lambda}(2-r)}}{-e^{\sqrt{\lambda}}T_+(\lambda)^2 + e^{-\sqrt{\lambda}}T_-(\lambda)^2}\right| |h(r)|\, dr \\
&\lesssim |T_+(\lambda)|^{-2}\|h\|_{L^2} \lesssim \|h\|_{L^2}
\end{aligned}$$

where the inequality in the final line is justified as before noting that $2 - r \in [0, 1]$. Similarly,

$$\left|\frac{\lambda}{\det M_\lambda} c_{41} W'_\lambda(2)\right| \lesssim \sqrt{|\lambda|}\|h\|_{L^2}.$$

These inequalities combined with (4.2.2) imply that

$$|\lambda a(\lambda)| \lesssim \sqrt{|\lambda|}\|y\|, \quad |\lambda| \geq N,$$

for some constant $N > 0$ independent of y and in particular,

$$\|\lambda u\|_{L^2},\ \|u'\|_{L^2} \lesssim \sqrt{|\lambda|}\|y\|, \quad |\lambda| \geq N.$$

The same arguments show that this also holds for $\|\lambda\tilde{u}\|_{L^2}$ and $\|\tilde{u}'\|_{L^2}$.

We must now estimate w given by (4.1.7), noting that

$$b(\lambda) = \frac{1}{\det M_\lambda}(c_{12}b_1 + c_{22}b_2 + c_{32}b_2 + c_{42}b_4)$$

and

$$c(\lambda) = \frac{1}{\det M_\lambda}(c_{13}b_1 + c_{23}b_2 + c_{33}b_2 + c_{43}b_4).$$

The trick to estimating w is to group the terms together in a specific way. As before, the only terms in the components of $\mathbf{b}$ which are not bounded by $\|y\|$ are $W_\lambda(2)$ and $W'_\lambda(2)$. Hence it is enough to estimate the moduli of

$$\begin{aligned}
w_1(\xi) &= \frac{1}{\det M_\lambda}[c_{12}\cosh(\sqrt{\lambda}(\xi - 1)) + c_{13}\sinh(\sqrt{\lambda}(\xi - 1))], \\
w_2(\xi) &= \frac{1}{\det M_\lambda}[c_{22}\cosh(\sqrt{\lambda}(\xi - 1)) + c_{23}\sinh(\sqrt{\lambda}(\xi - 1))], \\
w_3(\xi) &= \frac{1}{\det M_\lambda}[c_{32}\cosh(\sqrt{\lambda}(\xi - 1)) + c_{33}\sinh(\sqrt{\lambda}(\xi - 1))], \\
\omega_4(\xi) &= \frac{1}{\det M_\lambda}[c_{42}\cosh(\sqrt{\lambda}(\xi - 1)) + c_{43}\sinh(\sqrt{\lambda}(\xi - 1))],
\end{aligned}$$

and

$$w_5(\xi) = \frac{1}{\det M_\lambda}\Big[\big(- c_{32}W_\lambda(2) + c_{42}W'_\lambda(2)\big)\cosh(\sqrt{\lambda}(\xi-1))$$
$$+ \big(- c_{33}W_\lambda(2) + c_{43}W'_\lambda(2)\big)\sinh(\sqrt{\lambda}(\xi-1)) - \det M_\lambda W_\lambda(\xi)\Big],$$

where $\xi \in [1,2]$, since the sum of the w_i is equal to the w after removing the terms of $\mathbf{b}$ that do not include $W_\lambda(2)$ and $W'_\lambda(2)$. These removed terms can be shown to obey the desired estimates using the previous method.

Plugging in the appropriate values gives

$$\begin{aligned} w_1 &= \frac{\lambda^2\cosh(\lambda)}{\det M_\lambda}\Big[e^{\sqrt{\lambda}}T_+(\lambda)\big(\cosh(\sqrt{\lambda}(\xi-1)) - \sinh(\sqrt{\lambda}(\xi-1))\big) \\ &\qquad - e^{-\sqrt{\lambda}}T_-(\lambda)\big(\cosh(\sqrt{\lambda}(\xi-1)) + \sinh(\sqrt{\lambda}(\xi-1))\big)\Big] \\ &= \frac{\lambda^2\cosh(\lambda)}{\det M_\lambda}\left(e^{\sqrt{\lambda}}T_+(\lambda)e^{-\sqrt{\lambda}(\xi-1)} - e^{-\sqrt{\lambda}}T_-(\lambda)e^{\sqrt{\lambda}(\xi-1)}\right) \\ &= \cosh(\lambda)\frac{e^{\sqrt{\lambda}(2-\xi)}T_+(\lambda) - e^{-\sqrt{\lambda}(2-\xi)}T_-(\lambda)}{-e^{\sqrt{\lambda}}T_+(\lambda)^2 + e^{-\sqrt{\lambda}}T_-(\lambda)^2}. \end{aligned}$$

Since $2-\xi \in [0,1]$, as in the case for u,

$$|w_1(\xi)| \lesssim |T_\pm(\lambda)|^{-1} \lesssim 1,$$

where the sign of $\pm$ is determined by that of s and the growth bound is independent of ξ. Likewise, $|w_2(\xi)| \lesssim 1$ with the bound independent of ξ. Next we have that

$$\begin{aligned} w_3 &= \cosh\lambda\frac{\sqrt{\lambda}\sinh(\lambda)\cosh(\sqrt{\lambda}(\xi-1)) + \cosh(\lambda)\sinh(\sqrt{\lambda}(\xi-1))}{-e^{\sqrt{\lambda}}T_+(\lambda)^2 + e^{-\sqrt{\lambda}}T_-(\lambda)^2} \\ &= \cosh(\lambda)\frac{e^{\sqrt{\lambda}(\xi-1)}T_+(\lambda) - e^{-\sqrt{\lambda}(\xi-1)}T_-(\lambda)}{-e^{\sqrt{\lambda}}T_+(\lambda)^2 + e^{-\sqrt{\lambda}}T_-(\lambda)^2}. \end{aligned}$$

Since $\xi - 1 \in [0,1]$, the previous argument again shows that $|w_3(\xi)| \lesssim 1$ with the bound independent of ξ. The same holds for w_4. Thus, it remains to estimate w_5 and after some simple manipulation, we can rewrite this as

$$w_5 = \frac{\lambda^2}{\det M_\lambda}\left[\frac{\cosh^2(\lambda)}{\sqrt{\lambda}}\Omega_1(\xi) + \sqrt{\lambda}\sinh^2(\lambda)\Omega_2(\xi) + \cosh(\lambda)\sinh(\lambda)\Omega_3(\xi)\right], \quad (4.2.3)$$

where

$$\begin{aligned}
\Omega_1(\xi) = &- \int_1^2 \sinh(\sqrt{\lambda}(2-r))\sinh(\sqrt{\lambda}(\xi-1))h(r)\,dr \\
&+ \int_1^\xi \sinh(\sqrt{\lambda}(\xi-1))\sinh(\sqrt{\lambda})h(r)\,dr \\
= &\frac{1}{2}\Big[- \int_\xi^2 \cosh(\sqrt{\lambda}(1+\xi-r))h(r)\,dr \\
&+ \int_1^2 \cosh(\sqrt{\lambda}(3-\xi-r))h(r)\,dr \\
&- \int_1^\xi \cosh(\sqrt{\lambda}(\xi-r-1))h(r)\,dr\Big],
\end{aligned}$$

with the second equality following from the use of identities such as

$$\begin{aligned}
&2\sinh(\sqrt{\lambda}(2-r))\sinh(\sqrt{\lambda}(\xi-1)) \\
&\quad = \cosh(\sqrt{\lambda}(2-r)+\sqrt{\lambda}(\xi-1)) \\
&\quad\quad - \cosh(\sqrt{\lambda}(2-r)-\sqrt{\lambda}(\xi-1)),
\end{aligned}$$

$$\begin{aligned}
\Omega_2(\xi) = &- \int_1^2 \cosh(\sqrt{\lambda}(2-r))\cosh(\sqrt{\lambda}(\xi-1))h(r)\,dr \\
&+ \int_1^\xi \sinh(\sqrt{\lambda}(\xi-r))\sinh(\sqrt{\lambda})h(r)\,dr \\
= &-\frac{1}{2}\Big[\int_\xi^2 \cosh(\sqrt{\lambda}(1+\xi-r))h(r)\,dr \\
&+ \int_1^2 \cosh(\sqrt{\lambda}(3-\xi-r))h(r)\,dr \\
&+ \int_1^\xi \cosh(\sqrt{\lambda}(\xi-r-1))h(r)\,dr\Big],
\end{aligned}$$

and

$$\begin{aligned}
\Omega_3(\xi) = &2\int_1^\xi \sinh(\sqrt{\lambda}(\xi-r))\cosh(\sqrt{\lambda})h(r)\,dr \\
&- \int_1^2 \sinh(\sqrt{\lambda}(2-r))\cosh(\sqrt{\lambda}(\xi-1))h(r)\,dr \\
&- \int_1^2 \cosh(\sqrt{\lambda}(2-r))\sinh(\sqrt{\lambda}(\xi-1))h(r)\,dr \\
= &- \int_1^\xi \sinh(\sqrt{\lambda}(1-\xi+r))h(r)\,dr - \int_\xi^2 \sinh(\sqrt{\lambda}(1+\xi-r))h(r)\,dr.
\end{aligned}$$

Note that for $\xi \in [1,2]$, $|1+\xi-r| \leq 1$ whenever $r \in [\xi,2]$, and $|3-\xi-r| \leq 1$ whenever $r \in [1,2]$, and $|\xi-r-1| \leq 1$ whenever $r \in [1,\xi]$. Thus by pulling the factor of $\big(-e^{\sqrt{\lambda}}T_+(\lambda)^2 + e^{-\sqrt{\lambda}}T_-(\lambda)^2\big)^{-1}$ into the integrand of Ω_1, we see that

$$\left|\frac{\lambda^2}{\det M_\lambda}\frac{\cosh^2(\lambda)}{\sqrt{\lambda}}\Omega_1(\xi)\right| \lesssim \frac{1}{\sqrt{\lambda}}\|h\|_{L^2}.$$

Arguing similarly for Ω_2 and Ω_3, we get from (4.2.3) that

$$|w_5| \lesssim \left(\frac{1}{\sqrt{|\lambda|}} + \sqrt{|\lambda|} + 1\right)\|h\|_{L^2} \lesssim \sqrt{|\lambda|}\|h\|_{L^2},$$

with the implicit constant independent of ξ. It follows that

$$\|w\|_{L^2} \lesssim \sqrt{|\lambda|}\|y\|, \quad |\lambda| \geq N,$$

for some constant $N > 0$ independent of y and, in particular,

$$\|x\| \lesssim \sqrt{|\lambda|}\|y\|,$$

with the implicit constant independent of the specific y and x. □

4.3 Optimal energy decay for classical solutions

Using the abstract but powerful tool Theorem 3.1.8 (Borichev–Tomilov), we can convert the resolvent estimate of Theorem 4.2.1 into a rate of energy decay of classical solutions to (4.0.2), deriving the main result of the chapter. The rate itself will follow easily from Theorem 3.1.8 as we shall soon see, but optimality will require a little bit more work.

Theorem 4.3.1. *If $x \in D(A)$, then $E_x(t) = o(t^{-4})$ as $t \to \infty$. Moreover, this rate is optimal in the sense that, given any positive function r satisfying $r(t) = o(t^{-4})$ as $t \to \infty$, there exists $x \in D(A)$ such that $E_x(t) \neq o(r(t))$ as $t \to \infty$.*

Before we begin the proof, we state the following result needed to show optimality, one that encapsulates the optimality arguments of [BPS16].

Proposition 4.3.2. *Let B be the generator of the C_0-semigroup $S(t)$ on the Hilbert space*

$$Z_* = \{(u,v,w) \in H^1(0,1) \times L^2(0,1) \times L^2(1,3/2) : u(0) = 0\}$$

that solves the following well-posed problem:

$$\begin{cases} u_{tt}(\xi,t) = u_{\xi\xi}(\xi,t), & \xi \in (0,1),\ t>0, \\ w_t(\xi,t) = w_{\xi\xi}(\xi,t), & \xi \in (1,3/2),\ t>0, \\ u_t(1,t) = w(1,t), \quad u_\xi(1,t) = w_\xi(1,t), & t>0, \\ u(0,t) = 0, \quad w(3/2,t) = 0, & t>0, \\ u(\xi,0) = u(\xi), \quad u_t(\xi,0) = v(\xi), & \xi \in (0,1), \\ w(\xi,0) = w(\xi), & \xi \in (1,3/2). \end{cases} \tag{4.3.1}$$

Then

$$\limsup_{|s|\to\infty} |s|^{-1/2} \|R(is,B)\| > 0. \tag{4.3.2}$$

In particular, for any positive function r satisfying $r = o(t^{-4})$ as $t \to \infty$, there exists

$$\begin{aligned} x_* \in D(B) = \{(u,v,w) \in H^2(0,1) \times H^1(0,1) \times H^2(1,3/2) \\ : u(0) = v(0) = w(3/2) = 0, \\ v(1) = w(1),\ u'(1) = w'(1)\} \end{aligned}$$

such that

$$E_{x_*}(t) = \int_0^1 |u'(\xi,t)|^2 + |v(\xi,t)|^2 \, d\xi + \int_1^{3/2} |w(\xi,t)|^2 \, d\xi \neq o(r(t))$$

as $t \to \infty$.

Proof. After a rescaling of the heat component by a factor of 2, this is the same problem as what is studied in [BPS16, Section 5], namely the coupled wave-heat equation that leads to the optimal resolvent bound in [BPS16, Theorem 3.1], but with Dirichlet wave condition. The problem is well-posed, therefore, and the same resolvent estimates hold up to a constant and they remain optimal in the sense of (4.3.2). This is proved in the same exact way as [BPS16, Theorem 3.4] by using the argument found there which is based on Rouché's Theorem (see Proposition 5.2.2(i) where we reproduce the optimality argument in [BPS16, Theorem 3.4] for the problem studied in Chapter 5). Note that in this case, however, $\sigma(B) \cap i\mathbb{R} = \emptyset$.

The final part of the proposition follows from (4.3.2) and is proved along the lines of [BPS16, Remark 4(a)]. We flesh that remark out here. Assume for a contradiction that there exists a positive function r satisfying $r(t) = o(t^{-2})$ as $t \to \infty$ such that for all $x \in D(B)$ $\|S(t)x\| = o(r(t))$. Without loss of generality, r is non-increasing since we can replace r with $r_1(t) = \sup_{t \le \tau} r(\tau)$ which also satisfies $r_1(t) = o(t^{-2})$ and

$\|S(t)x\| = o(r_1(t))$ for all $x \in D(B)$. Then for all $y \in X$, there exists a constant C_y such that

$$r(t)^{-1}\|S(t)R(1,B)y\| \le C_y, \quad t \ge 0.$$

Hence by the uniform boundedness principle, there exists $C > 0$ independent of y such that

$$\|S(t)R(1,B)\| \le Cr(t).$$

In particular, if we adjust the constant C to be slightly larger, $m(t) < Cr(t)$, where $m(t) = \sup_{t\le\tau} \|S(\tau)R(1,B)\|$. By Proposition 3.1.6,

$$\|R(is,B)\| \lesssim 1 + m_*^{-1}\left(\frac{1}{2(|s|+1)}\right), \quad s \in \mathbb{R},$$

where m_*^{-1} is a right-inverse of the function m, mapping $(0, m(0)]$ onto $[0,\infty)$. This contradicts (4.3.2) if $|s|^{-1/2}m_*^{-1}\left(\frac{1}{2(|s|+1)}\right) \to 0$ as $|s| \to \infty$, which we now show.

Notice first that because $t^2Cr(t) \to 0$ as $t \to \infty$ and $(Cr)_*^{-1}(|s|) \to \infty$ as $|s| \to 0$, we have that $(Cr)_*^{-1}(|s|)^2|s| \to 0$ as $|s| \to 0$, where $(Cr)_*^{-1}$ is a right-inverse of the function Cr, mapping $(0, Cr(0)]$ onto $[0,\infty)$.

Hence

$$(Cr)_*^{-1}\left(\frac{1}{2(|s|+1)}\right)^2 \frac{1}{2(|s|+1)} \to 0$$

as $|s| \to \infty$. But since $m < Cr$ and both functions are non-increasing, it follows that $m_*^{-1} \le (Cr)_*^{-1}$ on the interval $(0, m(0)]$ and we are done. □

We finally prove the decay rate in Theorem 4.3.1 using Theorem 3.1.8 as promised and its optimality by effectively showing that the system (4.3.1) is contained in (4.0.2).

Proof of Theorem 4.3.1. By Theorem 3.1.8, we have that

$$E_x(t) = \frac{1}{2}\|T(t)x\|^2 = o(t^{-4}), \quad t \to \infty,$$

for any $x \in D(A)$ since Theorem 4.2.1 gives us the rate $\|R(is,A)\| = O(|s|^{1/2})$ as $s \to \infty$.

To show optimality, assume that there exists a positive function r satisfying $r = o(t^{-4})$ as $t \to \infty$. Proposition 4.3.2 produces an $x_* = (u_*, v_*, w_*) \in D(B) \subset Z_*$ with the classical solution to (4.0.2) given by $x_*(t) = (u_*(t), v_*(t), w_*(t))$ for $t \ge 0$ and for which $E_{x_*}(t) \neq o(r(t))$ as $t \to \infty$. Define $x_1 = (u_1, v_1, w_1, \tilde{u}_1, \tilde{v}_1) \in X$ by

$$\begin{aligned}
u_1(\xi) &= u_*(\xi), \quad \xi \in [0,1], \\
v_1(\xi) &= v_*(\xi), \quad \xi \in [0,1], \\
w_1(\xi) &= \begin{cases} w_*(\xi), & \xi \in [1,3/2], \\ -w_*(3-\xi), & \xi \in [3/2,2], \end{cases} \\
\tilde{u}_1(\xi) &= -u_*(3-\xi), \quad \xi \in [2,3], \\
\tilde{v}_1(\xi) &= -v_*(3-\xi), \quad \xi \in [2,3],
\end{aligned}$$

and note that x_1 satisfies all the conditions necessary to be in $D(A)$, including the $H^2(1,2)$ condition since on a symmetric interval around the only potentially problematic point $\xi = 3/2$, the function w_1 is the negative reflection of an H^2 function around a point at which it is 0. Morever, the classical solution to (4.0.2) for initial data x_1 is given by $x_1(t) = (u_1(t), v_1(t), w_1(t), \tilde{u}_1(t), \tilde{v}_1(t))$ for $t \geq 0$, where

$$\begin{aligned}
u_1(\xi,t) &= u_*(\xi,t), \quad \xi \in [0,1], \\
v_1(\xi,t) &= v_*(\xi,t), \quad \xi \in [0,1], \\
w_1(\xi,t) &= \begin{cases} w_*(\xi,t), & \xi \in [1,3/2], \\ -w_*(3-\xi,t), & \xi \in [3/2,2], \end{cases} \\
\tilde{u}_1(\xi,t) &= -u_*(3-\xi,t), \quad \xi \in [2,3], \\
\tilde{v}_1(\xi,t) &= -v_*(3-\xi,t), \quad \xi \in [2,3].
\end{aligned}$$

It follows that

$$E_{x_1}(t) = 2E_{x_*}(t) \neq o(r(t)), \quad t \to \infty.$$

□

Remark 4.3.3. It is natural to ask what happens when the Dirichlet conditions are replaced by Neumann conditions. In this case, the semigroup is actually unbounded. The function $x(\xi,t) = (at, a, at)$ for any constant $a \neq 0$ solves the variant of (4.0.2) where the sixth line is changed to $u_\xi(0,t) = \tilde{u}_\xi(3,t) = 0$ for the initial condition $(0, a, 0)$, which yields an unbounded orbit of the semigroup in this case. However, an alternative formulation involving a different state space more directly related to the energy can be chosen for the Neumann problem. In that formulation, the same method as for the Dirichlet case can be applied to obtain the same rate of decay (see Section 5.4.2 for this alternative formulation worked out in the setting of Chapter 5).

Appendix – entries for the cofactor matrix C in section 4.2

$$
\begin{aligned}
c_{11} &= -\lambda^{3/2}[e^{\sqrt{\lambda}}T_+(\lambda) + e^{-\sqrt{\lambda}}T_-(\lambda)],\\
c_{12} &= \lambda^2 \cosh(\lambda)(e^{\sqrt{\lambda}}T_+ - e^{-\sqrt{\lambda}}T_-),\\
c_{13} &= -\lambda^2 \cosh(\lambda)[e^{\sqrt{\lambda}}T_+(\lambda) + e^{-\sqrt{\lambda}}T_-(\lambda)],\\
c_{14} &= \lambda^{3/2} \cosh(\lambda),\\
c_{21} &= -\lambda[e^{\sqrt{\lambda}}T_+(\lambda) - e^{-\sqrt{\lambda}}T_-(\lambda)],\\
c_{22} &= -\lambda^2 \sinh(\lambda)[e^{\sqrt{\lambda}}T_+(\lambda) - e^{-\sqrt{\lambda}}T_-(\lambda)],\\
c_{23} &= \lambda^2 \sinh(\lambda)[e^{\sqrt{\lambda}}T_+(\lambda) + e^{-\sqrt{\lambda}}T_-(\lambda)],\\
c_{24} &= -\lambda^{3/2} \sinh(\lambda),\\
c_{31} &= \lambda^{3/2} \cosh(\lambda),\\
c_{32} &= \lambda^{5/2} \sinh(\lambda) \cosh(\lambda),\\
c_{33} &= \lambda^2 \cosh^2(\lambda),\\
c_{34} &= -\lambda^{3/2}[e^{\sqrt{\lambda}}T_+(\lambda)e^{-\sqrt{\lambda}}T_-(\lambda)],\\
c_{41} &= -\lambda^{3/2} \sinh(\lambda),\\
c_{42} &= -\lambda^{5/2} \sinh^2(\lambda),\\
c_{43} &= -\lambda^2 \cosh(\lambda) \sinh(\lambda),\\
c_{44} &= -\lambda[e^{\sqrt{\lambda}}T_+(\lambda) - e^{-\sqrt{\lambda}}T_-(\lambda)].
\end{aligned}
$$

Chapter 5

Optimal energy decay in a one-dimensional coupled wave-heat system with infinite heat part

In this chapter[1], we extend the heat component of (4.0.1) to infinity and take Dirichlet boundary condition for the wave part, analysing the following wave-heat system:

$$\begin{cases} u_{tt}(\xi,t) = u_{\xi\xi}(\xi,t), & & \xi \in (-1,0),\ t>0, \\ w_t(\xi,t) = w_{\xi\xi}(\xi,t), & & \xi \in (0,\infty),\ t>0, \\ u_t(0,t) = w(0,t), & u_\xi(0,t) = w_\xi(0,t), & t>0, \\ u(-1,t) = 0, & & t>0, \\ u(\xi,0) = u(\xi), & u_t(\xi,0) = v(\xi) & \xi \in (-1,0), \\ w(\xi,0) = w(\xi), & & \xi \in (0,\infty). \end{cases} \tag{5.0.1}$$

The initial data is required to satisfy $u \in H^1(-1,0), v \in L^2(-1,0)$, and $w \in L^2(0,\infty)$. Instead of a boundary condition for $w(\xi,t)$ on the uncoupled 'end', we instead stipulate that $w(\xi,t) \in L^2(0,\infty)$ for all $t \geq 0$.

As in the previous chapter (and in [BPS16]), the primary aim here is to obtain a quantitative estimate for the rate of energy decay of a given solution. Due to similarities of both system and aim, the structure of this chapter and many of the phrases used will be the same as the previous chapter. However there are significant differences, which will become apparent as the chapter progresses.

Given a vector $x = (u,v,w)$ of initial data satisfying the conditions above, we again define the energy of the solution corresponding to these initial data by

$$E_x(t) = \frac{1}{2}\int_{-1}^{\infty} |u_\xi(\xi,t)|^2 + |u_t(\xi,t)|^2 + |w(\xi,t)|^2 \, d\xi, \quad t \geq 0, \tag{5.0.2}$$

[1] The material in this chapter is based on [NS20a].

where all functions have been extended by zero in ξ to the interval $(-1,\infty)$. If sufficient regularity of the solution is assumed, we can differentiate the energy and apply integration by parts to see that

$$E_x'(t) = -\int_0^\infty |w_\xi(\xi,t)|^2 \, d\xi, \quad t \geq 0,$$

showing once again, in particular, that the energy of any such solution is non-increasing in time.

The main theorem used in [BPS16] and Chapter 4 to convert resolvent estimates into semigroup decay rates was Theorem 3.1.8 (Borichev-Tomilov). In this chapter, we continue to broadly follow the approach of [BPS16], but appeal instead to the generalised result Theorem 3.1.12 (Batty–Chill–Tomilov). Much of the notation will be the same as in the previous chapter but redefined for appropriate quantities. The reader is advised not to carry any of the previous chapter's definitions into this one.

5.1 Well-posedness – the semigroup and its generator

In this section, we prove that (4.0.2) is solved by the orbits of a contraction semigroup, after which we analyse the spectrum of the semigroup generator A.

5.1.1 Existence of the semigroup

We first translate (5.0.1) into an abstract Cauchy problem. Consider the Hilbert space

$$X = \{(u,v,w) \in H^1(-1,0) \times L^2(-1,0) \times L^2(0,\infty) : u(-1) = 0\}$$

endowed with the norm

$$\|(u,v,w)\|_X^2 = \|u'\|_{L^2}^2 + \|v\|_{L^2}^2 + \|w\|_{L^2}^2$$

and its corresponding inner product. Let

$$X_1 = X \cap [H^2(-1,0) \times H^1(-1,0) \times H^2(0,\infty)]$$

and define the operator A on X by $Ax = (v, u'', w'')$ for $x = (u,v,w)$ in the domain

$$D(A) = \{(u,v,w) \in X_1 : \; v(-1) = 0, \; u'(0) = w'(0), \; v(0) = w(0)\}.$$

We then have the following analogue of Lemma 4.1.1 proved in a mostly analogous way with the key difference being in statement (iv).

Lemma 5.1.1. *The following hold:*

(i) A is closed;

(ii) A is densely defined;

(iii) A is dissipative;

(iv) $I - A$ is surjective.

Proof. (i) Let $x_n = (u_n, v_n, w_n) \in D(A)$ be such that

$$x_n \to x = (u, v, w), \ Ax_n = (v_n, u_n'', w_n'') \to y = (f, g, h)$$

in X. Then u_n converges to u in $H^1(-1, 0)$ and u_n'' converges to g in $L^2(-1, 0)$. Hence

$$\int u\varphi'' = \lim_{n\to\infty} \int u_n\varphi'' = \lim_{n\to\infty} \int u_n''\varphi = \int g\varphi, \quad \varphi \in C_c^\infty(-1, 0), \tag{5.1.1}$$

where the integral is taken over $((-1, 0), d\xi)$ so that $u \in H^2(-1, 0)$ and $u'' = g$. As v_n converges to both v and f in $L^2(-1, 0)$, $v = f$. In particular, $v \in H^1(-1, 0)$.

Next, w_n converges to w and w_n'' to h in $L^2(0, \infty)$. Standard Sobolev theory ensures the existence of a constant C such that

$$\|\psi'\|_{L^2(0,\infty)} \le \|\psi''\|_{L^2(0,\infty)} + C\|\psi\|_{L^2(0,\infty)}, \quad \psi \in H^2(0, \infty),$$

so that (w_n') is Cauchy and converges to some H in $L^2(0, \infty)$. Using similar reasoning to that in (5.1.1), we see that $w \in H^2(0, \infty)$ with $w' = H$ and $w'' = h$.

To check that the coupling conditions for x to be in the domain $D(A)$ are satisfied, it is enough to pass to a subsequence (x_{n_k}) that converges pointwise a.e. (for (u_{n_k}') and (w_{n_k}') as well) and note the continuity of u', v, w', w. It follows that $Ax = y$.

(ii) Consider the subspace X_1 equipped with the X norm, which is dense in X.

The linear functional $\phi_1 : x = (u, v, w) \mapsto v(-1)$ is unbounded on X_1, and hence

$$X_2 = \operatorname{Ker} \phi_1 = \{(u, v, w \in X_1 : v(-1) = 0\}$$

is dense in X_1. Similarly,

$$X_3 = \operatorname{Ker} \phi_2 = \{(u, v, w) \in X_2 : v(0) = w(0)\}$$

is dense in X_2 where ϕ_2 is the unbounded linear functional on X_2 defined by $x \mapsto v(0) - w(0)$. Again, by considering the unbounded linear functional $\phi_3 : x \mapsto u'(0) - w'(0)$ on X_3, we see that

$$X_4 = \operatorname{Ker} \phi_3 = \{(u, v, w) \in X_3 : u'(0) = w'(0)\}$$

is dense in X_3. Thus we have a decreasing finite chain of subspaces

$$X \supset X_1 \supset X_2 \supset X_3 \supset X_4 = D(A)$$

where each subspace is dense in the preceding one under the X norm. Hence A is densely defined.

(iii) Using integration by parts, we have

$$\begin{aligned}\langle Ax, x\rangle &= \langle v', u'\rangle_{L^2} + \langle u'', v\rangle_{L^2} + \langle w'', w\rangle_{L^2} \\ &= v(0)\overline{u'}(0) - w'(0)\overline{w}(0) - \int_{-1}^{0} v\overline{u''} + \int_{-1}^{0} u''\overline{v} - \int_{0}^{\infty} w'\overline{w'}\, d\xi.\end{aligned}$$

Hence

$$\operatorname{Re}\langle Ax, x\rangle = -\|w'\|_{L^2}^2 \leq 0,$$

showing that A is dissipative.

(iv) We perform a procedure here with general $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ that will appear in later sections even though in this setting, we only need to deal with the case where $\lambda = 1$.

Let $x = (u, v, w),\ y = (f, g, h) \in X$. The equation $(\lambda - A)x = y$ can be rewritten as the following system of boundary value problems:

$$u'' = \lambda^2 u - \lambda f - g, \qquad \xi \in (-1, 0), \tag{5.1.2a}$$

$$v = \lambda u - f, \qquad \xi \in (-1, 0), \tag{5.1.2b}$$

$$w'' = \lambda w - h, \qquad \xi \in (0, \infty), \tag{5.1.2c}$$

$$u(-1) = v(-1) = 0, \quad v(0) = w(0), \quad u'(0) = w'(0). \tag{5.1.2d}$$

Let

$$U_\lambda(\xi) = \frac{1}{\lambda}\int_{-1}^{\xi} \sinh(\lambda(\xi - r)(\lambda f(r) + g(r)))\, dr, \quad \xi \in [-1, 0],$$

which has derivative

$$U_\lambda'(\xi) = \int_{-1}^{\xi} \cosh(\lambda(\xi - r))(\lambda f(r) + g(r))\, dr, \quad \xi \in [-1, 0].$$

The differential equation (5.1.2a) with the boundary condition $u(0) = 0$ has the general weak solution

$$u(\xi) = a(\lambda)\sinh(\lambda\xi) - U_\lambda(\xi), \quad \xi \in [-1, 0], \tag{5.1.3}$$

where $a(\lambda) \in \mathbb{C}$ is a parameter free to be varied. In particular, the general solution has derivative

$$u'(\xi) = \lambda a(\lambda)\cosh(\lambda\xi) - U_\lambda'(\xi), \quad \xi \in [-1, 0]. \tag{5.1.4}$$

Clearly $u \in H^2(-1,0)$ and hence $v \in H^1(-1,0)$ with $v(-1) = \lambda u(-1) - f(-1) = 0$.

To find the general solution for (5.1.2c), define the Green's function $G_\lambda \in L^1(\mathbb{R})$ by

$$G_\lambda(\xi) = \frac{1}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}|\xi|} = \frac{1}{2\sqrt{\lambda}}(e^{\sqrt{\lambda}\xi}H(-\xi) + e^{-\sqrt{\lambda}\xi}H(\xi)),[2] \tag{5.1.5}$$

where H is the Heaviside step function. Note that G_λ is integrable because by taking the branch cut of the square root along $(-\infty, 0]$, we ensure that $\operatorname{Re}\sqrt{\lambda} > 0$ for all $\lambda \in \mathbb{C} \setminus (-\infty, 0]$. Considering h as a function in $L^2(\mathbb{R})$ via extension by zero, define

$$w = (h * G_\lambda)(\xi) + b(\lambda)e^{-\sqrt{\lambda}\xi}, \quad \xi \in [0, \infty), \tag{5.1.6}$$

where $h * G_\lambda$ is the convolution

$$(h * G_\lambda)(\xi) = \int_{-\infty}^{\infty} h(r)G_\lambda(\xi - r)\, dr, \quad \xi \in [0, \infty).$$

The following computation shows that $h * G_\lambda$ solves the inhomogeneous equation $w'' = \lambda w - h$ in the weak derivative (locally integrable) sense (see [Bre11, Lemma 8.2]):

$$\begin{aligned}
(h * G_\lambda)(\xi) &= \frac{1}{2\sqrt{\lambda}} e^{\sqrt{\lambda}\xi} \int_\xi^\infty h(r)e^{-\sqrt{\lambda}r} dr + \frac{1}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}\xi} \int_0^\xi h(r)e^{\sqrt{\lambda}r} dr, \\
(h * G_\lambda)'(\xi) &= \frac{1}{2} e^{\sqrt{\lambda}\xi} \int_\xi^\infty h(r)e^{-\sqrt{\lambda}r} dr - \frac{1}{2} e^{-\sqrt{\lambda}\xi} \int_0^\xi h(r)e^{\sqrt{\lambda}r} dr = [h * Q_\lambda](\xi), \\
(h * G_\lambda)''(\xi) &= \frac{\sqrt{\lambda}}{2} e^{\sqrt{\lambda}\xi} \int_\xi^\infty h(r)e^{-\sqrt{\lambda}r} dr + \frac{\sqrt{\lambda}}{2} e^{-\sqrt{\lambda}\xi} \int_0^\xi h(r)e^{\sqrt{\lambda}r} dr - h(\xi) \\
&= \lambda(h * G_\lambda)(\xi) - h(\xi),
\end{aligned}$$

where

$$Q_\lambda(\xi) = \frac{1}{2}(e^{\sqrt{\lambda}\xi}H(-\xi) - e^{-\sqrt{\lambda}\xi}H(\xi)) \in L^1(\mathbb{R}).$$

For later purposes, we specially denote the first term in the top equation:

$$W_\lambda(\xi) = \frac{1}{2\sqrt{\lambda}} e^{\sqrt{\lambda}\xi} \int_\xi^\infty h(r)e^{-\sqrt{\lambda}r} dr. \tag{5.1.7}$$

Thus $w = (h * G_\lambda)(\xi) + b(\lambda)e^{-\sqrt{\lambda}\xi}$ satisfies (5.1.2c) with the constant $b(\lambda)$ being an adjustable parameter for the coupling condition at $\xi = 0$ and furthermore, Young's inequality for convolutions implies that $w \in H^2(0, \infty)$.

[2] See Acknowledgements to Hansen Chen.

It remains to choose specific constants $a(\lambda)$ and $b(\lambda)$ in order to satisfy the coupling conditions. Using (5.1.3) and (5.1.6), the requirement $\lambda u(0) - f(0) = v(0) = w(0)$ is satisfied if and only if

$$\lambda a(\lambda)\sinh(\lambda) - b(\lambda) = \lambda U_\lambda(0) + f(0) + W_\lambda(0).$$

Likewise, the condition $u'(0) = w'(0)$ is equivalent to

$$\lambda a(\lambda)\cosh(\lambda) + b\sqrt{\lambda} = U_\lambda'(0) + \sqrt{\lambda}W_\lambda(0).$$

These two equations can be written in the form

$$M_\lambda \cdot \begin{pmatrix} a(\lambda) \\ b(\lambda) \end{pmatrix} = \begin{pmatrix} \lambda U_\lambda(0) + f(0) + W_\lambda(0) \\ U_\lambda'(0) + \sqrt{\lambda}W_\lambda(0) \end{pmatrix} \tag{5.1.8}$$

where

$$M_\lambda = \begin{pmatrix} \lambda\sinh(\lambda) & -1 \\ \lambda\cosh(\lambda) & \sqrt{\lambda} \end{pmatrix}.$$

Thus $\lambda - A$ is surjective if and only if (5.1.8) has a solution for any given $y = (f, g, h)$ in X which in turn is equivalent to

$$\det M_\lambda = \lambda(\sqrt{\lambda}\sinh(\lambda) + \cosh(\lambda))$$

being non-zero. For $\lambda = 1$,

$$\det M_1 = \sinh(1) + \cosh(1) = e^1 \neq 0.$$

Hence $1 - A$ is surjective. $\square$

Theorem 2.3.4 (Lumer–Phillips) now kicks in to show that A generates a contraction semigroup.

Theorem 5.1.2. *A generates a contractive C_0-semigroup $(T(t))_{t\geq 0}$ on X.*

5.1.2 Spectrum of the generator

Theorem 5.1.2 implies that $\sigma(A)$ is contained in the closed left half-plane. However, that is not all that can be said about the spectrum.

Theorem 5.1.3. *The spectrum of A is given by the disjoint union*

$$\sigma(A) = (-\infty, 0] \cup \sigma_p(A),$$

where the point spectrum is characterised by

$$\sigma_p(A) = \{\lambda \in \mathbb{C}_- : \cosh(\lambda) + \sqrt{\lambda}\sinh(\lambda) = 0\}.$$

In particular, the spectrum satisfies $\sigma(A) \cap i\mathbb{R} = \{0\}$.

Proof. We first show that $(-\infty, 0]$ is in the spectrum but contains no eigenvalues. Let $\lambda \in (-\infty, 0]$ and define

$$h_n^\lambda(\xi) = \frac{e^{i\sqrt{-\lambda}\xi}}{\sqrt{n}} \Phi\left(\frac{\xi}{n} - 1\right) \quad (\xi \geq 0,\ n = 1, 2, \dots),$$

where Φ is a smooth bump function on $\mathbb{R}$ bounded by 1 with $\Phi = 1$ on $[-1/2, 1/2]$ and $\Phi = 0$ on $(-\infty, -1] \cup [1, \infty)$. Clearly $x_n = (0, 0, h_n^\lambda) \in D(A)$ and moreover,

$$\|x_n\|_X^2 = \frac{1}{n} \int_0^\infty |\Phi\left(\frac{\xi}{n} - 1\right)|^2 \, d\xi \geq \frac{1}{n} \int_{n/2}^{3n/2} d\xi = 1.$$

Note that we have $h_n^\lambda = e^{i\sqrt{-\lambda}\xi} h_0^\lambda$ and the following estimates:

$$\|(h_n^0)'\|_{L^2}^2 \leq \frac{1}{n} \int_0^{2n} \frac{1}{n^2} \|\Phi'\|_\infty^2 = \frac{2}{n^2} \|\Phi'\|_\infty^2, \quad |(h_n^0)''\|_{L^2}^2 \leq \frac{2}{n^4} \|\Phi''\|_\infty^2.$$

Hence

$$\begin{aligned}
\|(\lambda - A)x_n\|_X &= \|2i\sqrt{-\lambda} e^{i\sqrt{-\lambda}\xi} (h_n^0)'(\xi) + e^{i\sqrt{-\lambda}\xi} (h_n^0)''(\xi)\|_{L^2} \\
&\leq 2|\sqrt{-\lambda}| \|(h_n^0)'\|_{L^2} + \|(h_n^0)''\|_{L^2} \to 0
\end{aligned}$$

as $n \to \infty$. Thus λ is an approximate eigenvalue of A.

Assume now that $(\lambda - A)x = 0$ for some non-zero $x = (u, v, w) \in D(A)$. For λ non-zero, this implies that $u'' = \lambda^2 u, v = \lambda u$, and $w'' + |\lambda| w = 0$. Hence $w'' \in H^2(0, \infty)$ so that $w \in H^4(0, \infty) \subset C^3(0, \infty)$. Since w is at least twice classically differentiable, it is forced to be of the form

$$w = a \cos(\sqrt{|\lambda|}\xi) + b \sin(\sqrt{|\lambda|}\xi)$$

for some constants a and b not both zero. But then w is non-zero and periodic and thus cannot be in $L^2(0, \infty)$, a contradiction. Hence $u'(0) = w'(0) = 0$ and $u(0) = \lambda^{-1} v(0) = \lambda^{-1} w(0) = 0$. Since $u = \alpha \cosh(\lambda\xi) + \beta \sinh(\lambda\xi)$ and $u' = \alpha\lambda \sinh(\lambda\xi) + \beta\lambda \cosh(\lambda\xi)$ for some constants α and β, the boundary conditions at 0 force $\alpha = \beta = 0$ so that $u = v = 0$. Likewise, the case for $\lambda = 0$ quickly results in a contradiction. This proves that $(-\infty, 0]$ is in the spectrum of A but contains no eigenvalues.

Following the proof of Lemma 5.1.1(iv) for $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, $(\lambda - A)x = 0$ has a non-zero solution if and only if

$$M_\lambda \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

has a non-zero solution, that is, when $\sqrt{\lambda}\sinh(\lambda)+\cosh(\lambda)=0$. Moreover, the proof of Lemma 5.1.1(iv) shows that $\lambda-A$ is surjective whenever $\sqrt{\lambda}\sinh(\lambda)+\cosh(\lambda)\neq 0$. Thus $\lambda-A$ is invertible whenever $\sqrt{\lambda}\sinh(\lambda)+\cosh(\lambda)\neq 0$.

To prove the final statement of the theorem, note that for $\lambda=is$ where $s\in\mathbb{R}\backslash\{0\}$,

$$\lambda^{-1}\det M_\lambda=\cos(s)+i\sqrt{is}\sin(s).$$

Thus when $\sin(s)=0$, $\mathrm{Re}\,[\det M_\lambda]\neq 0$ and when $\sin(s)\neq 0$, $\mathrm{Im}\,[\det M_\lambda]\neq 0$. Hence $\sqrt{\lambda}\sinh(\lambda)+\cosh(\lambda)$ is non-zero when $\lambda=is$ for $s\in\mathbb{R}\setminus\{0\}$. It follows that $\sigma(A)\cap i\mathbb{R}=\{0\}$. □

5.2 Resolvent estimates

We turn now to the behaviour of the resolvent operator $R(is,A)$ near infinity and zero in order to obtain optimal upper bounds on its growth at these two singularities. The upper bounds will then be converted into estimates on the rate of energy decay in the next section.

5.2.1 Upper bounds

Theorem 5.2.1. *The following hold:*

(i) $\|R(is,A)\|=O(|s|^{1/2})$ *as* $|s|\to\infty$*;*

(ii) $\|R(is,A)\|=O(|s|^{-1})$ *as* $|s|\to 0$.

Proof. Let $\lambda=is$ for $s\in\mathbb{R}\setminus\{0\}$ and let $y=(f,g,h)\in X$, further defining $x=(u,v,w)\in D(A)$ by $x=R(\lambda,A)y$. As $v=\lambda u-f$, we have that

$$\|x\|\lesssim\|\lambda u\|_{L^2}+\|u'\|_{L^2}+\|w\|_{L^2}.$$

Thus the result will follow once we have established estimates for each of the summands in the above equation.

(i) Consider u given by (5.1.3). By Lemma 4.2.2 (with the interval $[0,1]$ changed to $[-1,0]$), it is enough to consider $|\lambda a(\lambda)|$ in order to estimate $\|\lambda u\|_{L^2}$ and $\|u'\|_{L^2}$.

Inverting $\det M_\lambda$ in (5.1.8) we get that

$$\frac{1}{\det M_\lambda}\begin{pmatrix}\sqrt{\lambda} & 1\\ -\lambda\cosh(\lambda) & \lambda\sinh(\lambda)\end{pmatrix}\begin{pmatrix}\lambda U_\lambda(0)+f(0)+W_\lambda(0)\\ U_\lambda'(0)+\sqrt{\lambda}W_\lambda(0)\end{pmatrix}=\begin{pmatrix}a(\lambda)\\ b(\lambda)\end{pmatrix}. \qquad (5.2.1)$$

Note that $\det M_\lambda = \lambda T_+(\lambda)$ from the previous chapter. Hence by Lemmas 4.2.2 and 4.2.3

$$\begin{aligned}|\lambda a(\lambda)| &= |T_+(\lambda)|^{-1}|(\sqrt{\lambda}(\lambda U_\lambda(0) + f(0) + W_\lambda(0)) + U_\lambda'(0) + \sqrt{\lambda}W_\lambda(0))| \\ &\lesssim (1 + |\lambda|^{1/2})(\|f\|_{H^1} + \|g\|_{L^2}) + |\lambda|^{1/2}|W_\lambda(0)| \qquad (5.2.2)\end{aligned}$$

for $\lambda = is$ with large $|s|$. Noticing that

$$\begin{aligned}|W_\lambda(0)| &\leq \frac{1}{2|\lambda|^{1/2}}\|h\|_{L^2}\|e^{-\sqrt{\lambda}r}\|_{L^2} \\ &= \frac{1}{2|\lambda|^{1/2}}\|h\|_{L^2}\left(\int_0^\infty e^{-\sqrt{2|\lambda|}r}\ dr\right)^{1/2} \\ &= \frac{1}{2|\lambda|^{1/2}}\|h\|_{L^2}\left(\frac{1}{\sqrt{2|\lambda|}}\right)^{1/2} \qquad (5.2.3) \\ &\lesssim \|h\|_{L^2}\end{aligned}$$

we see that

$$|\lambda a(\lambda)| \leq |\lambda|^{1/2}\|y\|_X$$

for large $\lambda = is$ with large $|s|$.

We must now estimate w which is given by (5.1.6). Note that

$$\|w\|_{L^2} \leq \|G_\lambda * h\|_{L^2} + \|b(\lambda)e^{-\sqrt{\lambda}r}\|_{L^2} \leq \|G_\lambda\|_{L^1}\|h\|_{L^2} + |b(\lambda)|\left(\frac{1}{\sqrt{2|\lambda|}}\right)^{1/2}, \quad (5.2.4)$$

hence it suffices to estimate $\|G_\lambda\|_{L^1}$ and $|b(\lambda)|$.

As in the case for $a(\lambda)$, estimating each summand in the equation for $b(\lambda)$ given by (5.2.1) and noting that $|\cosh(\lambda)|, |\sinh(\lambda)| \leq 1$ when $\lambda = is$ for s real yields

$$\begin{aligned}|b(\lambda)| \leq |\lambda T_+(\lambda)|^{-1}| &- \lambda\cosh(\lambda)(\lambda U_\lambda(0) + f(0) + W_\lambda(0)) \\ &+ \lambda\sinh(\lambda)(U_\lambda'(0) + \sqrt{\lambda}W_\lambda(0))| \\ &\lesssim \|f\|_{H^1} + \|g\|_{L^2} + |1 + \sqrt{\lambda}||W_\lambda(0)| \qquad (5.2.5) \\ &\lesssim \|f\|_{H^1} + \|g\|_{L^2} + |\lambda|^{1/2}\|h\|_{L^2}\end{aligned}$$

for $\lambda = is$ with large $|s|$. Since

$$\|G_\lambda\|_{L^1} \leq \frac{2}{2|\sqrt{\lambda}|}\int_0^\infty |e^{-\sqrt{\lambda}r}|\ dr \lesssim |\lambda|^{-1} \qquad (5.2.6)$$

for all $\lambda = is$ with $s \in \mathbb{R} \setminus \{0\}$, part (i) of the theorem is proved.

(ii) The estimates given by (5.2.2) and (5.2.3) still apply for $\lambda = is$ with $|s|$ small, so that

$$|\lambda a(\lambda)| \lesssim \|f\|_{H^1} + \|g\|_{L^2} + \left(\frac{1}{\sqrt{|\lambda|}}\right)^{1/2} \|h\|_{L^2} \lesssim |\lambda|^{-1/4}\|y\|_X$$

for $\lambda = is$ with $|s|$ small.

Estimates (5.2.3) (again), (5.2.4), (5.2.5), and (5.2.6) also apply for $\lambda = is$ with $|s|$ small so that

$$\begin{aligned}\|w\|_{L^2} &\lesssim |\lambda|^{-1}\|h\|_{L^2} + \left(\frac{1}{\sqrt{|\lambda|}}\right)^{1/2} \big(\|f\|_{H^1} + \|g\|_{L^2} + (1+|\lambda|^{1/2})|W_\lambda(0)|\big) \\ &\lesssim |\lambda|^{-1}\|h\|_{L^2} + |\lambda|^{-1/4}(\|f\|_{H^1} + \|g\|_{L^2})\end{aligned}$$

for $\lambda = is$ with $|s|$ small, proving part (ii) of the theorem. □

5.2.2 Optimality of the upper bounds

We have, in fact, the following result showing the optimality of both upper bounds in Theorem 5.2.1.

Proposition 5.2.2. *The following hold:*

(i) $\limsup_{|s|\to\infty} |s|^{-1/2}\|R(is, A)\| > 0$*;*

(ii) $\|R(is, A)\| \asymp |s|^{-1}$ *as* $|s| \to 0$*.*

Proof. (i) In our proof we will more or less directly replicate that of [BPS16, Theorem 3.4] with only minor modifications. By Theorem 5.1.3, the roots of the function $F+G$ are eigenvalues, where $F = \sinh(\lambda)$ and $G = \cosh(\lambda)/\lambda$ for $\lambda \in \mathbb{C}_-$. F is πi-periodic with simple zeros at $\lambda = \lambda_n$ for $n \in \mathbb{Z}$ where $\lambda_n = (n+1/2)\pi i$. Since $F'(\lambda) = \cosh(\lambda)$, it follows that $F(\lambda_n) = 1$ and by Taylor expansion, that $|F(\lambda)| > |\lambda - \lambda_n|/2$ provided $|\lambda - \lambda_n|$ is small enough. Let $r_n = |n + 1/2|^{-1/2}$ and $\Omega_n = \{\lambda \in \mathbb{C} : |\lambda - \lambda_n| < 2r_n\}$. Then for $|n|$ sufficiently large, $|F(\lambda)| > r_n$ for $\lambda \in \partial\Omega_n$. If we can show that $|G| \le r_n$ on $\partial\Omega_n$ for large $|n|$, Rouché's theorem implies that $F + G$ has the same number of zeros as F inside Ω_n for large $|n|$, namely one. For any $\epsilon > 0$ small, $|\cosh(\lambda)| < 1+\epsilon$ for λ on $\partial\Omega_n$ since such $|\operatorname{Re}\lambda|$ can be made to be sufficiently small for large enough $|n|$. Hence

$$\sup_{\lambda\in\partial\Omega_n} |G(\lambda)| \le \frac{1+\epsilon}{\sqrt{(n+1/2)\pi - 2r_n}} = \frac{1+\epsilon}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{r_n^{-2} - (2/\pi)r_n}} \le \left(\frac{1+\epsilon}{\sqrt{\pi}} + \epsilon\right) r_n$$

for large enough $|n|$ since $r_n \to 0$. Taking ϵ sufficiently small shows that $|G(\lambda)| < F(\lambda)$ for $\lambda \in \partial\Omega_n$ and $|n|$ large enough. Hence for $|n|$ large, there exist $\mu_n \in \sigma(A)$ such that

$$|\mu_n - is_n| \le 2|n|^{-1/2}$$

where $s_n = (n + 1/2)\pi$. In particular,

$$\operatorname{dist}(is_n, \sigma(A)) \le 2|n|^{-1/2}.$$

By Proposition 2.2.1,

$$|s_n|^{1/2} \lesssim \frac{1}{2}|n|^{1/2} \le \|R(is_n, A)\|,$$

for a sequence $s_n \in \mathbb{R} \setminus \{0\}$ with $|s_n| \to \infty$ as $|n| \to \infty$, establishing (i).

(ii) If $|s| < \|R(is, A)\|^{-1}$ for any $s \in \mathbb{R} \setminus \{0\}$, then $0 \in \rho(A)$ by Proposition 2.2.1, a contradiction. Thus $\|R(is, A)\| \ge |s|^{-1}$ for all $s \in \mathbb{R} \setminus \{0\}$. □

Note that Proposition 5.2.2(i) implies that there cannot be a positive function r with $r(s) = o(|s|^{1/2})$ as $|s| \to \infty$ such that $\|R(is, A)\| \le r(s)$.

5.3 Optimal energy decay for a natural class of solutions

The main machinery that powers this chapter is Theorem 3.1.12 (Batty–Chill–Tomilov). But before we convert the resolvent estimates of Theorem 5.2.1 into a rate of energy decay for a certain class of classical solutions satisfying (5.0.1), we will need the following partial analogue of [BCT16, Theorem 6.9] which is to do with the splitting of X under certain conditions. The proof is described on [BCT16, Page 923] as being almost unchanged from that of [BCT16, Theorem 6.9], so we make the necessary adaptations for what we need and include the proof here.

Proposition 5.3.1. *Let $(T(t))_{t\ge 0}$ be a bounded C_0-semigroup on a Banach space X with generator A. If $0 \in \sigma(A)$ and*

$$\limsup_{t\to\infty} t\|T(t)A(I - A)^{-2}\| = 0,$$

then 0 is an isolated point of $\sigma(A)$.

Proof. Assume for a contradiction that 0 is a limit point of $\sigma(A)$. Then there exists a sequence $(\lambda_n)_{n\ge 0}$ in $\sigma(A) \setminus \{0\}$ such that $\lambda_n \to 0$ as $n \to \infty$. The spectral mapping (inclusion) theorem for the Hille–Phillips functional calculus [HP74, Theorem 16.3.5] tells us that

$$\{e^{\lambda t}\lambda(1-\lambda)^{-2} : \lambda \in \sigma(A)\} \subset \sigma(T(t)A(1-A)^{-2}), \quad t \geq 0,$$

(see the appendix of this chapter for a proof of this claim) and thus, from the spectral radius formula,

$$\|T(t)A(1-A)^{-2}\| \geq \sup\{e^{\operatorname{Re}\lambda t}|\lambda||1-\lambda|^{-2} : \lambda \in \sigma(A)\}, \quad t \geq 0.$$

Letting $t_n = |\operatorname{Re}\lambda_n|^{-1}$ we see that

$$t_n\|T(t_n)A(I-A)^{-2}\| \geq \frac{e^{\operatorname{Re}\lambda_n t_n}|\lambda_n||\operatorname{Re}\lambda_n|^{-1}}{|1-\lambda_n|^2} \geq \frac{e^{-1}}{|1-\lambda_n|^{-2}}$$

for all $n \geq 0$ so that

$$\limsup_{t\to\infty} t\|T(t)A(I-A)^{-2}\| \geq e^{-1} > 0$$

since $\lambda_n \to 0$ as $n \to \infty$, contradicting the premise. □

We can now state and prove the main result of this chapter.

Theorem 5.3.2. *If $x \in D(A) \cap \operatorname{Ran} A$, then $E_x(t) = O(t^{-2})$ as $t \to \infty$. Moreover, this rate is optimal in the sense that, given any positive function r satisfying $r(t) = o(t^{-2})$ as $t \to \infty$, there exists $x \in D(A) \cap \operatorname{Ran} A$ such that $E_x(t) \neq o(r(t))$ as $t \to \infty$.*

Proof. By Theorem 3.1.12, we have that

$$\|T(t)A(I-A)^{-2}\| = O(t^{-1})$$

(in operator norm) as $t \to \infty$. In particular, for any $x \in \operatorname{Ran}(A(I-A)^{-2}) = D(A) \cap \operatorname{Ran} A$, there exists $y \in X$ such that $x = A(I-A)^{-2}y$ so that

$$E_x(t) = \frac{1}{2}\|T(t)x\|_X^2 \lesssim t^{-2}\|y\|^2$$

for large t.

To prove optimality, assume now that there exists a positive function r such that $r(t) = o(t^{-2})$ as $t \to \infty$ and that for all $x \in D(A) \cap \operatorname{Ran} A$, $E_x(t) = O(r(t))$ as $t \to \infty$. Then for each $y \in X$, there exists $C_y > 0$ such that

$$r(t)^{-1/2}\|T(t)A(I-A)^{-2}y\| \leq C_y$$

for all $t > 0$. By the uniform boundedness principle, there exists $C > 0$ such that

$$r(t)^{-1/2}\|T(t)A(I-A)^{-2}\| \le C$$

implying that

$$\limsup_{t\to\infty} t\|T(t)A(I-A)^{-2}\| \le \lim_{t\to\infty} Ctr(t)^{1/2} = 0.$$

Proposition 5.3.1 implies that 0 is an isolated point of $\sigma(A)$, which is a contradiction according to Theorem 5.1.3. □

Remark 5.3.3. From Theorem 3.1.12, we actually have the finer statement that

$$\|T(t)A(I-A)^{-(1+1/2)}\| = O(t^{-1}), \quad t \to \infty,$$

and furthermore, from [BCT16, Proposition 3.10] that

$$\operatorname{Ran}(A(I-A)^{-(1+1/2)}) = D(A^{1/2}) \cap \operatorname{Ran} A.$$

Using the semigroup property, it is easily seen that

$$\|T(t)x\| = O(t^{-2k}), \quad t \to \infty$$

for all $x \in D(A^k) \cap \operatorname{Ran}(A^{2k})$ and all integers $k \ge 1$. Hence, the smoothness of the orbits can contribute to faster decay. However, unlike in [BPS16, Remark 4.3(c)], there is an extra specification that initial values reside in the range of higher powers of A before faster decay can be deduced.

In other words, the energy of smoother orbits can decay faster in a way that depends on the resolvent estimates at 0 and at ∞ (provided the range condition above holds). Thus, when it comes to initial data with higher regularity, the optimality of the resolvent estimate in Theorem 5.2.1(i) as $|s| \to \infty$ noted in Proposition 5.2.2(i) sometimes plays an important role that is not seen for general initial data in $D(A) \cap \operatorname{Ran} A$ where the resolvent estimate for $\|R(is, A)\|$ as $|s| \to 0$ is the only effective determining factor for the rate of energy decay.

We finish this section by providing a characterisation for the range $\operatorname{Ran} A$ of A in order to better understand the space of classical solutions $D(A) \cap \operatorname{Ran} A$ of initial data for which we have the sharp decay estimate.

Proposition 5.3.4. *Let $(f, g, h) \in X$. Then $(f, g, h) \in \operatorname{Ran} A$ if and only if the following hold:*

1. $\xi \mapsto \lim_{a\to\infty} \int_\xi^a h(r)\, dr \in L^2(0, \infty)$;

2. $\xi \mapsto \lim_{b\to\infty} \int_\xi^b \left(\lim_{a\to\infty} \int_t^a h(r)\ dr \right) dt \in L^2(0,\infty)$*; and*

3. $f(0) = \lim_{b\to\infty} \int_0^b \left(\lim_{a\to\infty} \int_t^a h(r)\ dr \right) dt.$

Proof. Suppose that there exists $(u,v,w) \in D(A)$ such that $A(u,v,w) = (f,g,h)$. Then we have

$$u'' = g,\ v = f,\ w'' = h,\ u(-1) = 0,\ u'(0) = w'(0),\ v(0) = w(0),$$

where the derivatives are weak.

In particular, w and w' are continuous and decay to 0 at infinity. Hence the identities

$$w'(a) - w'(\xi) = \int_a^\xi h(r)\ dr, \quad w(b) - w(\xi) = \int_\xi^b w'(t)\ dt$$

imply that the improper integrals of h and w' exist and that

$$w'(\xi) = -\lim_{a\to\infty} \int_t^a h(r)\ dr, \quad w(\xi) = \lim_{b\to\infty} \int_\xi^b \left(\lim_{a\to\infty} \int_t^a h(r)\ dr \right) dt.$$

The coupling condition $v(0) = w(0)$ implies that

$$f(0) = v(0) = w(0) = \lim_{b\to\infty} \int_0^b \left(\lim_{a\to\infty} \int_t^a h(r)\ dr \right) dt,$$

proving one direction.

For the converse, suppose that there exists $(f,g,h) \in X$ such that properties 1., 2., and 3. hold. Let $w \in L^2(0,\infty)$ be given by

$$w(\xi) = \lim_{b\to\infty} \int_\xi^b \left(\lim_{a\to\infty} \int_t^a h(r)\ dr \right) dt, \quad \xi > 0,$$

so that, by the fundamental theorem of calculus,

$$w'(\xi) = -\lim_{a\to\infty} \int_\xi^a h(r)\ dr \in L^2(0,\infty)$$

and $w'' = h \in L^2(0,\infty)$ with $w \in H^2(0,\infty)$. Defining $v = f \in H^1(-1,0)$, we have $v(-1) = f(-1) = 0$ and

$$v(0) = f(0) = \lim_{b\to\infty} \int_0^b \left(\lim_{a\to\infty} \int_t^a h(r)\ dr \right) dt = w(0).$$

Finally let $u \in L^2(-1,0)$ be given by

$$u(\xi) = \int_{-1}^\xi \int_{-1}^t g(r)\ dr dt + \left(w'(0) - \int_{-1}^0 g(r)\ dr \right) (\xi + 1), \quad \xi \in (-1,0).$$

Then $u' \in L^2(-1,0)$ is given by

$$u'(\xi) = \int_{-1}^{\xi} g(r)\ dr + w'(0) - \int_{-1}^{0} g(r)\ dr, \quad \xi \in (-1,0).$$

Moreover, $u'' = g \in L^2(-1,0)$ with $u'(0) = w'(0)$, and $u \in H^2(-1,0)$. Since u is continuous, it is clear from its definition that $u(-1) = 0$. Thus $(u,v,w) \in D(A)$ and $(f,g,h) = (v,u'',w'') = A(u,v,w) \in \operatorname{Ran} A$. □

5.4 The case of Neumann boundary condition

We now consider a one-dimensional coupled wave-heat equation with infinite heat part and Neumann boundary condition for the wave part. In the first subsection, we show that solving the arising system by defininining an analogous generator as before leads to an unbounded semigroup. In the second subsection, we consider an alternative way to define the Neumann semigroup which does, in fact, lead to a semi-uniformly stable semigroup.

5.4.1 Original formulation

Consider the following system:

$$\begin{cases} u_{tt}(\xi,t) = u_{\xi\xi}(\xi,t), & & \xi \in (-1,0),\ t>0, \\ w_t(\xi,t) = w_{\xi\xi}(\xi,t), & & \xi \in (0,\infty),\ t>0, \\ u_t(0,t) = w(0,t), & u_\xi(0,t) = w_\xi(0,t), & t>0, \\ u_\xi(-1,t) = 0, & & t>0, \\ u(\xi,0) = u(\xi), & u_t(\xi,0) = v(\xi) & \xi \in (-1,0), \\ w(\xi,0) = w(\xi), & & \xi \in (0,\infty). \end{cases} \tag{5.4.1}$$

The initial data is again required to satisfy $u \in H^1(-1,0), v \in L^2(-1,0)$, and $w \in L^2(0,\infty)$. The only difference between this system and (5.0.1) is the Dirichlet boundary condition $u(-1,t) = 0$ in (5.0.1) being replaced by the Neumann boundary condition $u_\xi(-1,0) = 0$ for all $t > 0$. In this case, we take the Hilbert spaces $X = H^1(-1,0) \times L^2(-1,0) \times L^2(0,\infty)$ and $Y = H^2(-1,0) \times H^1(-1,0) \times H^2(0,\infty)$ endowed with their natural Hilbert space norms (note that the L^2-norm of the first component must be included here to avoid degeneracy) and define the operator A again by $Ax = (v,u'',w'')$ for $x = (u,v,w)$ in the new domain

$$D(A) = \{(u,v,w) \in Y :\ u'(-1) = 0,\ u'(0) = w'(0),\ v(0) = w(0)\}.$$

Just as in Lemma 5.1.1, it can be shown that A is closed and densely defined, and furthermore, Theorem 2.3.4 (Lumer–Phillips) can be applied to $A - I$ to show that A generates a C_0-semigroup $(T(t))_{t\geq 0}$ (see the proof of [BPS16, Theorem 2.1]).

The proof of the following proposition is almost exactly the same as that of Proposition 5.3.4, so we omit it.

Proposition 5.4.1. *Let $(f, g, h) \in X$. Then $(f, g, h) \in \operatorname{Ran} A$ if and only if the following hold:*

1. $\xi \mapsto \lim_{a\to\infty} \int_\xi^a h(r)\, dr \in L^2(0, \infty)$;

2. $\xi \mapsto \lim_{b\to\infty} \int_\xi^b \left(\lim_{a\to\infty} \int_t^a h(r)\, dr \right) dt \in L^2(0, \infty)$;

3. $f(0) = \lim_{b\to\infty} \int_0^b \left(\lim_{a\to\infty} \int_t^a h(r)\, dr \right) dt$; *and*

4. $\int_{-1}^0 g(r)\, dr = -\lim_{a\to\infty} \int_0^a h(r)\, dr.$

Unlike in the Dirichlet case, however, A here is not injective.

Proposition 5.4.2. *The following hold:*

(i) $\operatorname{Ker} A = \operatorname{span}\{(1, 0, 0)\}$;

(ii) $\overline{\operatorname{Ran} A} = X$.

Proof. (i) If $A(u, v, w) = (v, u'', w'') = 0$ for $(u, v, w) \in D(A)$, then similarly to the Dirichlet case, u and w are both classically twice differentiable. Hence $u = (c, 0, 0)$ for some constant $c \in \mathbb{C}$ and $w = 0$ in order to satisfy integrability conditions, while $v = 0$ trivially.

(ii) Let $(f, g, h) \in X$ and $\epsilon > 0$. Our aim is to construct an element of X that is within a fixed multiple of ϵ from (f, g, h) in X and satisfies the conditions of Proposition 5.4.1. This would show that $\operatorname{Ran} A$ is dense in X. As $h \in L^2(0, \infty)$, there exists $\tilde{h} \in L^2(0, \infty)$ with finite support such that $\|h - \tilde{h}\|_{L^2(0,\infty)} < \epsilon$. Let $\mu > 0$ and $\theta \in [0, 2\pi)$ be such that

$$\mu e^{i\theta} = \int_{-1}^0 g(r)\, dr + \int_0^\infty \tilde{h}(r)\, dr.$$

Since the function $\frac{1}{1+\xi}$ is not integrable over $(0, \infty)$, it follows that there exists $K > 0$ such that

$$\int_0^K \frac{\epsilon}{r+1}\ dr = \mu.$$

Let $F \in L^2(0, \infty)$ be defined by

$$F(\xi) = -\frac{\epsilon e^{i\theta}}{\xi + 1}\chi_{(0,K)}(\xi), \quad \xi > 0,$$

where $\chi_{(0,K)}$ is the indicator function of the interval $(0, K)$. Clearly $\|F\|_{L^2(0,\infty)} < \epsilon$. Now let $\nu > 0$ and $\sigma \in [0, 2\pi i)$ be such that

$$\nu e^{i\sigma} = f(0) - \lim_{b\to\infty} \int_0^b \left(\lim_{a\to\infty} \int_t^a \tilde{h}(r) + F(r)\ dr \right) dt$$

which exists because $\tilde{h} + F$ has finite support.

For $\alpha > 0$, yet to be determined, define the function $H \in L^2(0, \infty)$ by

$$H(\xi) = \frac{\epsilon e^{i\sigma}}{(1+\xi)^2}\chi_{(0,\alpha)}(\xi), \quad \xi > 0,$$

so that

$$\lim_{a\to\infty} \int_\xi^a H(r)\ dr = \frac{\epsilon e^{i\sigma}}{2} \left(\frac{1}{\xi+1} - \frac{1}{\alpha+1} \right), \quad 0 < \xi < \alpha,$$

and

$$\lim_{b\to\infty} \int_0^b \left(\lim_{a\to\infty} \int_t^a H(r)\ dr \right) dt = \frac{\epsilon e^{i\sigma}}{2} \left(\log(\alpha+1) - \frac{\alpha}{\alpha+1} \right).$$

Take $\alpha = e^{2\nu/\epsilon} - 1$ so that

$$\lim_{b\to\infty} \int_0^b \left(\lim_{a\to\infty} \int_t^a H(r)\ dr \right) dt = \frac{\epsilon e^{i\sigma}}{2} \left(\frac{2\nu}{\epsilon} - \frac{\alpha}{\alpha+1} \right) = \nu e^{i\sigma} - \frac{\epsilon e^{i\sigma}}{2} \cdot \frac{\alpha}{\alpha+1}.$$

Note also that $\|H\|_{L^2}^2 \leq \int_0^\infty \frac{\epsilon^2}{(1+r)^4}\ dr \leq \epsilon^2/3$.

Defining

$$\tilde{y} = \left(f - \frac{\epsilon e^{i\sigma}}{2} \cdot \frac{\alpha}{\alpha+1}, g - \frac{\epsilon e^{i\sigma}}{2} + \frac{\epsilon e^{i\sigma}}{2(\alpha+1)}, \tilde{h} + F + H \right) \in X,$$

we quickly check if Proposition 5.4.1 can be applied. Conditions 1. and 2. hold because $\tilde{h} + F + H$ has finite support. Furthermore, by choice of $\mu e^{i\theta}, \nu e^{i\sigma}$ and α, we see that

$$f(0) = \int_0^\infty \int_t^\infty H(r)\ drdt + \frac{\epsilon e^{i\sigma}}{2} \cdot \frac{\alpha}{\alpha+1} + \int_0^\infty \int_t^\infty \tilde{h}(r) + F(r)\ drdt,$$

and

$$\int_{-1}^0 g(r)\ dr + \int_0^\infty \tilde{h}(r)\ dr + \frac{\epsilon e^{i\sigma}}{2} \left(\frac{1}{\xi+1} - \frac{1}{\alpha+1} \right) = -\int_0^\infty F(r)\ dr - \int_0^\infty H(r)\ dr.$$

so that conditions 3. and 4. are satisfied. Hence $\tilde{y} \in \operatorname{Ran} A$. Taking the differences componentwise, we see that

$$\|y - \tilde{y}\|_X^2 \leq \frac{\epsilon^2}{4} + \epsilon^2 + \left(\epsilon + \epsilon + \frac{\epsilon}{\sqrt{3}}\right)^2 < 11\epsilon^2,$$

and we are done. $\square$

We know from ergodic theory (which we will not delve into in this thesis) that if the semigroup $(T(t))_{t\geq 0}$ is bounded (on a reflexive Banach space), then

$$\operatorname{Ker} A \oplus \overline{\operatorname{Ran} A} = X; \tag{5.4.2}$$

see [ABHN11, Section 4.3]. Indeed, because of Theorems 5.1.2 and 5.1.3, (5.4.2) shows that in the Dirichlet setting, the associated operator A there has dense range. However, back in our Neumann setting, (5.4.2) fails, as A here has nontrivial kernel and dense range from the previous proposition.

Corollary 5.4.3. *A generates an unbounded semigroup $(T(t))_{t\geq 0}$.*

Remark 5.4.4. The full strength of the splitting (5.4.2) is not needed to show that A generates an unbounded semigroup. It is enough to show that $\operatorname{Ker} A \cap \overline{\operatorname{Ran} A} \neq \{0\}$ (see again [ABHN11, Section 4.3]), which can be done in a somewhat simpler way by proving that $(1, 0, 0) \in \overline{\operatorname{Ran} A}$. Nonetheless, we have gone for this approach, because (5.4.2) is informative even in the Dirichlet setting as aforementioned.

5.4.2 Alternative formulation

Here, we recast the system given in (5.4.1) with its associated initial data into an alternative abstract Cauchy problem that leads to a semi-uniformly stable semigroup for which the energy defined as in (5.0.2) decays at the same rate modulo a constant as the Dirichlet case. In this case, we take the variables to be the first spatial derivative u_ξ and the first time derivative u_t living in $L^2(-1, 0)$ as opposed to the previous case where the first variable was u in $H^1(-1, 0)$. In this way, we can avoid the L^2-norm of the wave part which is unbounded due to possible upward drift in the initial conditions.

Consider the Hilbert space

$$Y = L^2(-1, 0) \times L^2(-1, 0) \times L^2(0, \infty)$$

endowed with the norm

$$\|(u, v, w)\|_Y^2 = \|u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|w\|_{L^2}^2$$

and its corresponding inner product. Let

$$Y_1 = H^1(-1,0) \times H^1(-1,0) \times H^2(0,\infty)$$

and define the operator B on Y by $Bx = (v', u'', w'')$ for $x = (u, v, w)$ in the domain

$$D(B) = \{(u, v, w) \in Y_1 : u(-1) = 0, u(0) = w'(0), v(0) = w(0)\}.$$

Thus, assuming sufficient regularity, if $z(t) = (u_\xi, u_t, w)$ and $\frac{d}{dt}z(t) = Bz(t)$, then $(u_{\xi t}, u_{tt}, w_t) = (u_{t\xi}, u_{\xi\xi}, w_{\xi\xi})$ as desired. That B is closed, densely defined, and dissipative follows in basically the exact same way as it does for A in Lemma 5.1.1(i)-(iii).

Theorem 5.4.5. *B generates a contractive C_0-semigroup $(S(t))_{t\geq 0}$ on Y.*

Proof. It remains to show that $\lambda - B$ is surjective for some $\lambda > 0$. The approach follows the lines of 5.1.1(iv), dealing with general $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, so we will be terse and leave out much of the explanatory exposition this time around. For $x = (u, v, w) \in D(B)$ and $y = (f, g, h) \in Y$ the equation $(\lambda - B)x = y$ is equivalent to the following system of equations:

$$\lambda u - v' = f, \qquad \xi \in (-1, 0), \tag{5.4.3a}$$

$$\lambda v - u' = g, \qquad \xi \in (-1, 0), \tag{5.4.3b}$$

$$\lambda w - w'' = h, \qquad \xi \in (0, \infty), \tag{5.4.3c}$$

$$u(-1) = 0, \quad v(0) = w(0), \quad u(0) = w'(0). \tag{5.4.3d}$$

Straightforward calculations verify that the following solutions solve the above system:

$$\begin{aligned} u(\xi) &= a(\lambda)\sinh(\lambda(\xi+1)) \\ &\qquad - \int_{-1}^{\xi} \sinh(\lambda(\xi - r)) \left(\lambda \int_{-1}^{r} g(\tau)\, d\tau + f(r)\right) dr - \int_{-1}^{\xi} g(r)\, dr, \end{aligned} \tag{5.4.4a}$$

$$v(\xi) = a(\lambda)\cosh(\lambda(\xi+1)) - \int_{-1}^{\xi} \cosh(\lambda(\xi - r)) \left(\lambda \int_{-1}^{r} g(\tau)\, d\tau + f(r)\right) dr, \tag{5.4.4b}$$

$$w(\xi) = (G_\lambda * h)(\xi) + b(\lambda)e^{-\sqrt{\lambda}\xi}, \tag{5.4.4c}$$

where G_λ is as in (5.1.5).

Writing

$$V_\lambda(\xi) = \frac{1}{\lambda}\int_{-1}^{\xi} \sinh(\lambda(\xi - r)) \left(\lambda \int_{-1}^{r} g(\tau)\, d\tau + f(r)\right) dr$$

and taking W_λ to be defined as in (5.1.7), we see that the coupling conditions (5.4.3d) are satisfied if and only if the following matrix equation holds:

$$\begin{pmatrix} \cosh(\lambda) & -1 \\ \sinh(\lambda) & \sqrt{\lambda} \end{pmatrix} \begin{pmatrix} a(\lambda) \\ b(\lambda) \end{pmatrix} = \begin{pmatrix} V_\lambda'(0) + W_\lambda(0) \\ \lambda V_\lambda(0) + \sqrt{\lambda} W_\lambda(0) + \int_{-1}^{0} g(r)\, dr \end{pmatrix}. \tag{5.4.5}$$

Denoting the leftmost matrix by N_λ we see that the $\lambda - B$ is surjective if and only if

$$\det N_\lambda = \sqrt{\lambda} \cosh(\lambda) + \sinh(\lambda) \neq 0.$$

Since $\det N_1 \neq 0$, Lumer–Phillips kicks in and we are done. □

As in the Dirichlet case, we can characterise the spectrum of B.

Theorem 5.4.6. *The spectrum of B is given by the disjoint union*

$$\sigma(B) = (-\infty, 0] \cup \sigma_p(B),$$

where the point spectrum is characterised by

$$\sigma_p(B) = \{\lambda \in \mathbb{C}_- : \sqrt{\lambda} \cosh(\lambda) + \sinh(\lambda) = 0\}.$$

Proof. The proof that $(\infty, 0]$ lies in the spectrum is the same as the first part in the proof of Theorem 5.1.3. To show that it contains no eigenvalues, assume that $(\lambda - B)x = 0$ for some $x = (u, v, w) \in D(B)$. Again, as in the proof of Theorem 5.1.3, it follows that $w = 0$. If $\lambda < 0$, the equations $\lambda u - v' = 0$ and $\lambda v = u' = 0$ along with the boundary condition $u(-1) = 0$ imply that $\int_{-1}^{\xi} v(r)\, dr = \alpha \cosh(\lambda \xi) + \beta \sinh(\lambda \xi)$ for some constants α and β. The coupling $v(0) = w(0)$ implies that $\beta = 0$. Similarly, $v'(0) = \lambda u(0) = w(0) = 0$ implies that $\alpha = 0$. Hence $u = v = 0$. If $\lambda = 0$, the boundary conditions for u and v force the functions to be 0. The characterisation of the point spectrum of B can be proved in essentially the same way as that of A. □

Remark 5.4.7. The characterisation of the spectrum shows that B generates a strongly stable semigroup, that is, $\|S(t)x\| \to 0$ as $t \to \infty$ for all $x \in Y$, by the countable spectrum theorem since $(S(t))_{t \geq 0}$ is bounded, B has no eigenvalues on the imaginary axis, and $\sigma(B)$ intersects with the imaginary axis only at 0 (see [ABHN11, Theorem 5.5.5]).

The generator B also obeys the same resolvent bounds as A.

Theorem 5.4.8. *The following hold:*

(i) $\|R(is,B)\| = O(|s|^{1/2})$ *as* $|s| \to \infty$;

(ii) $\|R(is,B)\| = O(|s|^{-1})$ *as* $|s| \to 0$.

The proof of the theorem requires two lemmas analogous to Lemmas 4.2.2 and 4.2.3.

Lemma 5.4.9. *There exists a constant* $C \geq 0$ *such that, for all* $f, g \in L^2(-1,0)$ *and* $s \in \mathbb{R}$,

$$\left| \int_{-1}^{\xi} \sinh(\lambda(\xi - r)) \left(\lambda \int_{-1}^{r} g(\tau)\, d\tau + f(r) \right) dr \right| \leq \|f\|_{L^2} + C\|g\|_{L^2}, \quad \xi \in [0,1],$$
$$\left| \int_{-1}^{\xi} \cosh(\lambda(\xi - r)) \left(\lambda \int_{-1}^{r} g(\tau)\, d\tau + f(r) \right) dr \right| \leq \|f\|_{L^2} + C\|g\|_{L^2}, \quad \xi \in [0,1].$$

Proof. This follows from the fact that there exists a constant C such that for all $g \in L^2(-1,0)$,

$$\| \int_{-1}^{\xi} g(r)\, dr \|_{H^1} \leq C\|g\|_{L^2}$$

and the application of Lemma 4.2.2 with f and g there replaced by $\int_{-1}^{\xi} g(r)\, dr$ and f respectively. ☐

Lemma 5.4.10. *For* $s \in \mathbb{R} \setminus \{0\}$ *there exists a constant* c *such that the following hold:*

(i) $|\det N_{is}| \geq c$ *for* $|s|$ *sufficiently large;*

(ii) $|\det N_{is}| \geq c\sqrt{|s|}$ *for* $|s|$ *sufficiently small.*

Proof. (i) Since $\det N_{is} = i\sin(s) + \sqrt{is}\cos(s)$, note that

$$|\operatorname{Re}(\det N_{is})| = \sqrt{\frac{|s|}{2}} \cos(s).$$

Whenever $\sqrt{|s|/2}\cos(s) \geq 1/3$, the desired lower bound holds. If instead we have that $\sqrt{|s|/2}\cos(s) < 1/3$, then whenever $|s| \geq 1$, $\cos(s) < \sqrt{2}/3$ so that $|\sin(s)| \geq \sqrt{7}/3$. Hence

$$|\det N_{is}| \geq |\sin(s)| - \sqrt{|s|}\cos(s) > \frac{\sqrt{7} - \sqrt{2}}{3}.$$

(ii) When $|s|$ is small, $|\sin(s)| \leq |s|$ and $\cos(s) \geq 1/2$. It follows that

$$|\det N_{is}| \geq \sqrt{|s|}\cos(s) - |\sin(s)| \geq \frac{1}{2}\sqrt{|s|} - |s| \geq c\sqrt{|s|}$$

for some constant c and for $|s|$ sufficiently small. ☐

Proof of Theorem 5.4.8. Let $s \in \mathbb{R} \setminus \{0\}$ and suppose that $(is - B)x = y$ where $x = (u, v, w) \in D(B)$ and $y = (f, g, h) \in Y$. In order to estimate the norm of x in terms of the L^2-norms of f, g and h, we solve for u, v, and w explicitly, obtaining the solutions given in (5.4.4a)-(5.4.4c). By Lemma 5.4.9 as well as the estimate (5.2.4) for w which also applies in this setting, it is enough to estimate $|a(is)|$ and $|b(is)|$ together with the previously obtained estimate for $\|G_\lambda\|_{L^1}$.

To this end, we invert the matrix N_{is} in (5.4.5), obtaining

$$\begin{pmatrix} a(is) \\ b(is) \end{pmatrix} = \frac{1}{\det N_{is}} \begin{pmatrix} \sqrt{is} & 1 \\ -i\sin(s) & \cos(s) \end{pmatrix} = \begin{pmatrix} V'_{is}(0) + W_{is}(0) \\ isV_{is}(0) + \sqrt{is}W_{is}(0) + \int_{-1}^0 g(r)\,dr \end{pmatrix}. \tag{5.4.6}$$

Hence by Lemma 5.4.10

$$\begin{aligned} |a(is)| &= \frac{1}{|\det N_{is}|} \left| \sqrt{is}V'_{is}(0) + 2\sqrt{is}W_{is}(0) + isV_{is}(0) + \int_{-1}^0 g(r)\,dr \right| \\ &\lesssim (1 + |s|^{-1/2})(1 + |s|^{1/2} + |s|^{-1/4})(\|f\|_{L^2} + \|g\|_{L^2} + \|h\|_{L^2}) \\ &\lesssim (1 + |s|^{1/2} + |s|^{-3/4})\|y\|_Y, \end{aligned}$$

where the estimate for $|W_{is}(0)|$ comes from (5.2.3). Likewise,

$$\begin{aligned} |b(is)| &\lesssim \frac{1}{|\det N_{is}|} \Bigg(|V'_{is}(0)| + |isV_{is}(0)| \\ &\qquad\qquad + |i\sin(s) + \sqrt{is}\cos(s)||W_{is}(0)| + \left| \int_{-1}^0 g(r)\,dr \right| \Bigg) \\ &\lesssim (1 + |s|^{-1/2})(1 + |\sin(s)||s|^{-3/4} + |s|^{-1/4})(\|f\|_{L^2} + \|g\|_{L^2} + \|h\|_{L^2}) \\ &\lesssim (1 + |s|^{-1/2})(1 + |s|^{-1/4})\|y\|_Y \\ &\lesssim (1 + |s|^{-3/4})\|y\|_Y. \end{aligned}$$

Thus,

$$\begin{aligned} \|u\|_{L^2} &\lesssim (1 + |s|^{1/2} + |s|^{-3/4})\|y\|_Y, \\ \|v\|_{L^2} &\lesssim (1 + |s|^{1/2} + |s|^{-3/4})\|y\|_Y, \\ \|w\|_{L^2} &\lesssim |s|^{-1}\|h\|_{L^2} + (1 + |s|^{-3/4})|s|^{-1/4}\|y\|_Y \\ &\lesssim |s|^{-1}\|y\|_Y. \end{aligned}$$

Putting this all together, $\|x\|_Y \lesssim (1 + |s|^{1/2} + |s|^{-1})\|y\|_Y$. These estimates are independent of x and y, so that $\|R(is, A)\| \lesssim 1 + |s|^{1/2} + |s|^{-1}$ for $s \in \mathbb{R} \setminus \{0\}$. □

As with the Dirichlet case, the optimality of these estimates can be verified in a similar way to before. Since

$$E_x(t) = \frac{1}{2}\|S(t)\|_Y^2,$$

Theorem 3.1.12 applies directly to give us the rate of energy decay.

Theorem 5.4.11. *If $x \in D(B) \cap \operatorname{Ran} B$, then $E_x(t) = O(t^{-2})$ as $t \to \infty$.*

This energy decay is also optimal in the sense of Theorem 5.3.2.

The last result proved in this section is a characterisation of the range of B, analogous to Propositions 5.3.4 and 5.4.1.

Proposition 5.4.12. *Let $(f, g, h) \in Y$. Then $(f, g, h) \in \operatorname{Ran} B$ if and only if the following hold:*

1. $\xi \mapsto \lim_{a\to\infty} \int_\xi^a h(r)\,dr \in L^2(0,\infty)$;

2. $\xi \mapsto \lim_{b\to\infty} \int_\xi^b \left(\lim_{a\to\infty} \int_t^a h(r)\,dr \right) dt \in L^2(0,\infty)$;

3. $\int_{-1}^0 g(r)\,dr = -\lim_{a\to\infty} \int_0^a h(r)\,dr.$

Proof. The only if direction is almost exactly the same as in the proof of Proposition 5.3.4. Now assume that properties 1., 2., and 3. hold for some $(f, g, h) \in Y$. The if direction follows by defining

$$\begin{aligned}
u &= \int_{-1}^{\xi} g(r)\,dr, \\
v &= \int_{-1}^{\xi} f(r)\,dr - \int_0^1 f(r)\,dr + \lim_{b\to\infty} \int_0^b \left(\lim_{a\to\infty} \int_t^a h(r)\,dr \right) dt, \\
w &= \lim_{b\to\infty} \int_\xi^b \left(\lim_{a\to\infty} \int_t^a h(r)\,dr \right) dt.
\end{aligned}$$

□

At first, it may appear that the same system of equations that produces both an unbounded semigroup $(T(t))_{t\geq 0}$ as defined in the previous section and a strongly stable semigroup $(S(t))_{t\geq 0}$ leads to the following paradox. Given $x = (u, v, w) \in X$ for which $\{T(t)x : t \geq 0\}$ is an unbounded orbit, we know that $\|S(t)y\|_Y \to 0$ as $t \to \infty$ where $y = (u_\xi, v, w) \in Y$. If we consider the first and second components of $S(t)y$ to be respectively the first spatial and first temporal derivatives of a solution to (5.4.1), then the solution $T(t)x$ seems like it should converge to a final state with a flat wave component and a zero heat component, that is to $(c, 0, 0) \in X$ for some constant c. This would imply that the orbit of x under $(T(t))_{t\geq 0}$ is actually bounded.

The resolution to this ‘problem’ lies in the fact that the time derivative of the solution is allowed to go to 0 slow enough so that rather than the solution converging to a constant flat steady state, the wave component becomes more and more flat, but remains ever increasing. The rate of upward increase becomes slower and slower as time progresses, since the time derivative does go to 0, but it is not slow enough to actually force the wave component to be completely constant independent of time. Thus pictorially, the solution ‘converges’ or rather becomes ever closer to a flat heat part coupled with a wave part that looks like an upward floating line decreasing in speed but not being bounded in time. This scenario remains a possibility because even if $y \in D(B) \cap \operatorname{Ran} B$, Theorem 3.1.12 only implies that $\|S(t)y\|$ decays at a $1/t$ rate, allowing for up to log-upward drift.

This hypothetical situation is also intuitive, because the upward floating constant-in-space function $u(\xi, t) = ct$ is flat in space and satisfies the Neumann wave equation, causing an uncoupled Neumann wave system to have unbounded solutions. Hence, in our coupled case, the problematic solution would ‘converge’ to what is essentially a slowed down version of this unbounded function.

Appendix – the spectral inclusion in proof of Proposition 5.3.1

Lemma. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X exponentially bounded by $\omega - 1$ for some $\omega > 0$ and with generator A. Then*

$$\{e^{\lambda t}\lambda(\omega-\lambda)^{-2} : \lambda \in \sigma(A)\} \subset \sigma(T(t)A(\omega - A)^{-2}), \quad t \geq 0.$$

Proof. For $t \geq 0$ and $\omega \in \rho(A)$ define $\mu_t \in L^1(\mathbb{R}_+)$ by

$$\mu_t(s) = \begin{cases} (\omega(s-t)-1)e^{-\omega(s-t)}, & t \leq s \\ 0, & 0 \leq s < t, \end{cases}$$

regarded as an absolutely continuous measure. Then

$$\begin{aligned}\int_0^\infty T(s)\ \mu_t(ds) &= \int_0^\infty T(s)\mu_t(s)\,ds \\ &= \int_t^\infty \omega(s-t)e^{-\omega(s-t)}T(s)\,ds - \int_t^\infty e^{-\omega(s-t)}T(s)\,ds \\ &= \omega T(t)\int_0^\infty e^{-\omega\tau}\tau T(\tau)\,d\tau - T(t)\int_0^\infty e^{-\omega\tau}T(\tau)\,d\tau \\ &= T(t)(\omega(\omega-A)^{-2} - (\omega-A)^{-1}) \\ &= T(t)(\omega(\omega-A)^{-1} - I)(\omega-A)^{-1} \\ &= T(t)A(\omega-A)^{-2}\end{aligned}$$

since $R(\omega, A)$ is the Laplace transform of $T(\tau)$ and so $-R(\omega, A)^2$ is the Laplace transform of $-\tau T(\tau)$ (see Theorem 2.2.1). By the Hille–Phillips functional calculus (see [HP74]) given by,

$$\mathcal{L}\mu(-A) = \int_0^\infty T(s)\ \mu(ds), \quad \mu \text{ measure on } \mathbb{R}_+,$$

where $\mathcal{L}$ denotes the Laplace transform, we get from the spectral inclusion theorem [HP74, Theorem 16.3.5] that

$$\{\mathcal{L}\mu_t(\lambda) : \lambda \in \sigma(-A)\} \subset \sigma(\mathcal{L}\mu_t(-A)).$$

Since

$$\begin{aligned}\mathcal{L}\mu_t(\lambda) &= \int_t^\infty e^{-\lambda s}(\omega(s-t)-1)e^{-\omega(s-t)}\,ds \\ &= \omega e^{-\lambda t}\int_0^\infty e^{-\lambda\tau}\tau e^{-\omega\tau}\,d\tau - e^{-\lambda t}\int_0^\infty e^{-\lambda\tau}e^{-\omega\tau}\,d\tau \\ &= e^{-\lambda t}(\omega(\omega+\lambda)^{-2} - (\omega+\lambda)^{-1}) \\ &= -e^{-\lambda t}\lambda(\omega+\lambda)^{-2},\end{aligned}$$

it follows that

$$\{-e^{-\lambda t}\lambda(\omega+\lambda)^{-2} : \lambda \in \sigma(-A)\} \subset \sigma(T(t)A(\omega-A)^{-2}), \quad t \geq 0,$$

proving the lemma. □

Chapter 6

Hilbert space direct integrals of strongly continuous operator semigroups

In this chapter[1], we combine a recent approach to direct sums of C_0-semigroups, done by Lachowicz and Moszyński [LM16] in 2016, with direct integral theory. Indeed, one can view our results as a generalisation and expansion of those in [LM16] when considering the setting of Hilbert spaces.

Direct integral theory has been around for a long time, originating from the study of operator algebras [Dix57, vN30] in the 1940s-50s. In particular, Nussbaum's generalisation of von Neumann's reduction theory for bounded operators [vN49] to unbounded operators [Nus64] motivated much further work within direct integral theory in areas such as spectral theory [Azo74, Cho70, Len74] and functional calculus [CG71, Gil73] in the 1960s-70s. A book was compiled by Nielsen in 1980 [Nie80], but since then, pure operator theory on direct integrals does not seem to have been developed as frequently. More recent additions to the theory are those of [DNSZ16] on W^*-algebras and [GGST12] on weak measurability.

It is somewhat surprising then, that although the development of C_0-semigroup theory substantially overlapped in time with that of direct integral theory, there appears to be a gap in the literature when it comes to comprehensively combining these two theories. This peculiarity is further compounded by the fact that both theories have already been largely completed for some time. Indeed, it is worth remarking that no comprehensive treatment on direct sums of C_0-semigroups seems to have been produced until 2016 in [LM16] (even though the use of infinite direct sums was already known as a tool to semigroup theorists). The closest that previous

[1]The material in this chapter is based on [Ng20a].

work has come to filling this gap in the literature is the study of semigroups of partial isometries that decompose into direct integrals of truncated shifts (see [EW83, ELW75, Mar11a, PR13, Wal71]). This chapter aims to rectify this situation and properly fill the mostly empty space. Furthermore, we hope to provide an interesting first step towards further developments and applications of direct integral and C_0-semigroup theory in the direction of both Cauchy problems and operator algebras (see Section 9.2).

Note that the first two sections consist of introductory material made of known results, though some reworking of said results is done to fit our particular formulation for direct integrals of closed operators. The remaining sections are, to my knowledge and unless otherwise stated, novel results. It may be the case that some of the results are generalisations of known theory in the literature that I am not currently aware of (outside of [LM16]).

6.1 Direct integrals of Hilbert spaces

Let Ω be a locally compact Hausdorff topological space and μ a σ-finite positive Borel measure on Ω. An important example of this is the counting measure on $\mathbb{N}$ with the discrete topology.

A *Hilbert (space) bundle* over *base space* Ω is a pair $(\mathcal{H}, \pi)$ where $\pi : \mathcal{H} \to \Omega$ is a surjection such that for all $s \in \Omega$, the *fiber* of $\mathcal{H}$ at s, defined by $H_s = \pi^{-1}(s)$, is a Hilbert space with an inner product $\langle \cdot, \cdot \rangle_s$ and induced norm $\| \cdot \|_s$. A *section* x of $\mathcal{H}$ is a function $x : \Omega \to \mathcal{H}$ such that $\pi \circ x(s) = s$ for all $s \in \Omega$. Henceforth, π will be notationally supressed and the Hilbert space bundle will be referred to by $\mathcal{H}$ alone. A *measurable field of Hilbert spaces* is a pair $(\mathcal{H}, \mathcal{F})$ where $\mathcal{H}$ is a Hilbert bundle and $\mathcal{F}$ is a collection of sections satisfying the following conditions:

1. For all $x, y \in \mathcal{F}$, the $\mathbb{C}$-valued function $s \mapsto \langle x(s), y(s) \rangle_s$ is μ-measurable;
2. If z is a section such that for all $x \in \mathcal{F}$, the function $s \mapsto \langle x(s), z(s) \rangle_s$ is μ-measurable, then $z \in \mathcal{F}$; and
3. There exists a sequence $\{\xi_i\}_{i=1}^{\infty} \subset \mathcal{F}$ called a *fundamental sequence* for $\mathcal{H}$ such that for all $s \in \Omega$, the elements $\{\xi_i(s)\}_{i=1}^{\infty}$ are dense in H_s.

$\mathcal{F}$ is called a choice of *measurable sections* of $\mathcal{H}$.

Given a measurable field of Hilbert spaces $(\mathcal{H}, \mathcal{F})$ with associated measure μ, the *direct integral*

$$\int_\Omega^\oplus H_s\, d\mu(s)$$

is the Hilbert space of all measurable sections $x \in \mathcal{F}$ such that

$$\|x\|^2 = \int_\Omega \|x(s)\|_s^2\, d\mu(s) < \infty$$

modulo measurable sections that are 0 μ-a.e. and equipped with the inner product

$$\langle x, y\rangle = \int_\Omega \langle x(s), y(s)\rangle_s\, d\mu(s).$$

We state some simple properties without proof (see [Gar96, Section 3] and [Yet05, Lemma 35]).

Proposition 6.1.1. *Let μ and Ω be as above. The following statements are true.*

(i) If $\mathcal{M} = \{M_k\}_{k=1}^\infty$ is a countable collection of sets with finite measure such that $\Omega = \bigcup_{k=1}^\infty M_k$, the triple sequence given by

$$\xi_{i,j,k}(\cdot) = \xi_i(\cdot)1_{\{s\in\Omega:\|\xi_i(s)\|_s\le j\}}(\cdot)1_{M_k}(\cdot)$$

is also a fundamental sequence after rearrangement into a sequence of one parameter. Furthermore, this fundamental sequence has elements that are pointwise bounded and have finite $\int_\Omega^\oplus H_s\, d\mu(s)$ norm. Hence, from now on we can assume that the fundamental sequence is pointwise bounded and contained in $\int_\Omega^\oplus H_s\, d\mu(s)$.

(ii) There exists a sequence $\{e_i\}_{i=1}^\infty$ of measurable sections such that for every $s \in \Omega$, the vectors $e_i(s)$ form an orthonormal basis for the fiber H_s.

(iii) $\{1_{M_k} e_i : i, k \in \mathbb{N}\}$ is a countable subset such that its span is dense in $\int_\Omega^\oplus H_s\, d\mu(s)$. Hence, $\int_\Omega^\oplus H_s\, d\mu(s)$ is separable.

Given $H = \int_\Omega^\oplus H_s\, d\mu(s)$, if $\Omega' \subset \Omega$ is measurable and $\mu(\Omega') > 0$, then we can form the direct integral $\int_{\Omega'}^\oplus H_s\, d\mu(s)$ using the same definition as before, with Ω' replacing Ω. $\int_{\Omega'}^\oplus H_s\, d\mu(s)$ can be identified with the subspace of H given by

$$\left\{x \in \int_\Omega^\oplus H_s\, d\mu(s) : x(s) = 0 \text{ for a.e. } s \notin \Omega'\right\}.$$

$\int_{\Omega'}^\oplus H_s\, d\mu(s)$ is closed, since if $\{x_n\}_{n=1}^\infty \subset \int_{\Omega'}^\oplus H_s\, d\mu(s)$ and $x_n \to x$ in H, then there exists a subsequence such that $x_{n_k}(s) \to x(s)$ in H_s for a.e. $s \in \Omega$. In particular, $x(s) = 0$ for a.e. $s \notin \Omega'$. Hence $x \in \int_{\Omega'}^\oplus H_s\, d\mu(s)$.

6.2 Direct integrals of operators

In most discussions on direct integrals, for a given measurable field $(\mathcal{H}, \mathcal{F})$, a family of bounded operators $\{T(s) \in \mathcal{B}(H_s) : s \in \Omega\}$ forms a *measurable field of operators* if the function

$$s \mapsto \langle T(s)x(s), y(s)\rangle_s$$

is measurable for all $x, y \in \mathcal{F}$. Some treatments also require that

$$s \mapsto \|T(s)\|_{\mathcal{B}(H_s)}$$

be essentially bounded. When this extra condition is satisfied, we shall call $\{T(s) \in \mathcal{B}(H_s) : s \in \Omega\}$ a *bounded measurable field of operators.* We will not go into a discussion of measurable fields and direct integrals of bounded operators, these can be found in [Dix57, Tak02, Nie80].

The standard generalisation to closed operators, first given by Nussbaum [Nus64], is through the characteristic matrices introduced by Stone [Sto51]. The characteristic matrix $(P_{i,j})$ of a closed operator A on a Hilbert space H is the 2×2 matrix of bounded operators representing the projection P of $H \times H$ onto the closed subspace $\Gamma(A)$, the graph of A. Nussbaum defined a *measurable field of closed operators* to be a family of closed operators $\{A(s) : D(A(s)) \subset H_s \to H_s : s \in \Omega\}$ such that for each i, j, the family $\{P_{i,j}(s) : s \in \Omega\}$ forms a measurable field of bounded operators. This is shown to be consistent with the bounded case in [Nus64, Proposition 6].

We provide an alternative definition which turns out to be equivalent modulo a resolvent condition.

Definition 6.2.1. A *measurable field of closed operators* is a family of closed operators $\{A(s) : D(A(s)) \subset H_s \to H_s : s \in \Omega\}$ such that there exists a fixed $\nu \in \mathbb{C}$ such that for a.e. $s \in \Omega$, $\nu \in \rho(A(s))$ and the family of resolvents $\{R(\nu, A(s)) : s \in \Omega\}$ adjusted on a set of measure zero is a bounded measurable field of operators.

When some $\nu \in \mathbb{C}$ satisfying the stipulations in Definition 6.2.1 exists, our definition and the Nussbaum definition are equivalent. This follows from the simple fact that adding scalar multiples of the identity does not change measurability in either definition and that inverses of Nussbaum measurable fields are measurable when they exist because the characteristic matrix of the inverse is merely a rearrangement of the components (see [Nus64, Proposition 5]). Note that since for any operator B,

$$R(\lambda, B) = (\lambda - \nu + R(\nu, B)^{-1})^{-1}, \quad \lambda, \nu \in \rho(B),$$

$\{R(\nu, A(s)) : s \in \Omega\}$ is Nussbaum measurable if and only if $\{R(\lambda, A(s)) : s \in \Omega\}$ is Nussbaum measurable for all $\lambda, \nu \in \rho(A(s))$ for a.e. $s \in \Omega$.

Using Definition 6.2.1, we can present the following reworked definition.

Definition 6.2.2. The *direct integral of a measurable field of closed operators* $\{A(s) : D(A(s)) \subset H_s \to H_s : s \in \Omega\}$ is the operator A given by

$$D(A) = \Big\{ x \in \int_\Omega^\oplus H_s \, d\mu(s) : x(s) \in D(A(s)) \text{ for a.e. } s \in \Omega, \\ A(\cdot)x(\cdot) \in \int_\Omega^\oplus H_s \, d\mu(s) \Big\},$$

$$Ax = A(\cdot)x(\cdot), \quad x \in D(A),$$

and denoted by $\int_\Omega^\oplus A(s)\, d\mu(s)$.

In accordance with this definition, if we write $A = \int_\Omega^\oplus A(s)\, d\mu(s)$, there is an implicit assumption that there exists at least one $\nu \in \mathbb{C}$ such that for a.e. $s \in \Omega$ $\nu \in \rho(A(s))$ and that $s \mapsto \|R(\nu, A(s))\|_{\mathcal{B}(H_s)}$ is essentially bounded.

Given $A = \int_\Omega^\oplus A(s)\, d\mu(s)$, if $\Omega' \subset \Omega$ is measurable and $\mu(\Omega') > 0$, then we can form the direct integral of closed operators $A_{\Omega'} = \int_{\Omega'}^\oplus A(s)\, d\mu(s)$ on $\int_{\Omega'}^\oplus H_s\, d\mu(s)$ using the same definitions as before, with Ω' replacing Ω. When considering $\int_{\Omega'}^\oplus H_s\, d\mu(s)$ as a closed subspace of $\int_\Omega^\oplus H_s\, d\mu(s)$, $A_{\Omega'}$ coincides with the restriction of A to $\int_{\Omega'}^\oplus H_s\, d\mu(s)$, that is,

$$D(A_{\Omega'}) = \int_{\Omega'}^\oplus H_s\, d\mu(s) \cap D(A) \quad \text{and} \quad A_{\Omega'} = A|_{D(A_{\Omega'})}.$$

Notice that there may exist operators A on a direct integral space $H = \int_\Omega^\oplus H_s\, d\mu(s)$ that are not themselves, the direct integral of a measurable field of closed operators. In the case, however, where $A = \int_\Omega^\oplus A(s)\, d\mu(s)$ for a measurable field of closed operators $\{A(s) : D(A(s)) \subset H_s \to H_s : s \in \Omega\}$, A is said to be *decomposable.*

We now provide some basic properties of direct integrals of operators. First, we deal with boundedness.

Proposition 6.2.3. *Let $A = \int_\Omega^\oplus A(s)\, d\mu(s)$ and $M \geq 0$. The following are equivalent.*

(i) $A(s) \in \mathcal{B}(H_s)$ *for all* $s \in \Omega$ *and* $\operatorname*{ess-sup}_{s\in\Omega} \|A(s)\|_{\mathcal{B}(H_s)} \leq M$.

(ii) $\int_\Omega^\oplus A(s)\, d\mu(s) \in \mathcal{B}\left(\int_\Omega^\oplus H_s\, d\mu(s)\right)$ *and* $\|\int_\Omega^\oplus A(s)\, d\mu(s)\| \leq M$.

Note that $s \mapsto \|A(s)\|_{\mathcal{B}(H_s)}$ *is measurable since* $\int_\Omega^\oplus H_s\, d\mu(s)$ *is separable so that* $\|A(s)\|_{\mathcal{B}(H_s)}$ *is a supremum of countably many measurable functions of the form* $\langle A(s)x(s), A(s)x(s)\rangle_s^{1/2}$.

Proof. Assume (i). Then for any $x \in D(A)$,

$$\|Ax\|^2 = \int_\Omega \|A(s)x(s)\|_s^2 \, d\mu(s) \leq M^2 \int_\Omega \|x(s)\|_s^2 \, d\mu(s) = (M\|x\|)^2,$$

and we are done.

For the converse, we must take care that the function we construct to exceed a given norm bound is measurable. Taking the set $\{e_i\}_{i=1}^\infty$ from Proposition 6.1.1,

$$\left\{ \sum_{i=1}^n a_i e_i : a_i \in \mathbb{Q}, n \in \mathbb{N} \right\}$$

is a countable set which we relabel as $\{f_i\}_{i=1}^\infty$ such that $\{f_i(s)\}_{i=1}^\infty$ is dense in H_s for all $s \in \Omega$. Let

$$K_i = \{s \in \Omega : \|A(s)f_i(s)\| > M\|f_i(s)\|_s\}, \quad i \in \mathbb{N}.$$

These sets are clearly measurable. If for a.e. $s \in \Omega$, $\|A(s)f_i(s)\| \leq M\|f_i(s)\|_s$ for all $i \in \mathbb{N}$, then $\|A(s)\| \leq M$ for a.e. s. Thus, if (i) fails, there exists $j \in \mathbb{N}$ such that $\mu(K_j) > 0$.

Let $x'(\cdot) = f_j(\cdot)1_{K_j}(\cdot)$ and note that it is measurable and in $\int_\Omega^\oplus H_s \, d\mu(s)$. Since

$$\|Ax'\|^2 = \int_{K_j} \|A(s)f_j(s)\|_s^2 \, d\mu(s) > M^2 \int_{K_j} \|f_j(s)\|_s^2 \, d\mu(s) = (M\|x'\|)^2,$$

(ii) also fails. $\square$

Second, we deal with denseness of domain.

Proposition 6.2.4. *Let $A = \int_\Omega^\oplus A(s) \, d\mu(s)$. The following are equivalent.*

(i) $A(s)$ is densely defined for a.e. $s \in \Omega$ and there exists a sequence $\{x_i\}_{i\in\mathbb{N}} \subset D\left(\int_\Omega^\oplus A(s) \, d\mu(s)\right)$ such that $\{x_i(s)\}_{i\in\mathbb{N}}$ is dense in $D(A(s))$ for a.e. $s \in \Omega$;

(ii) A is densely defined.

Proof. Assume (i). Let $\mathcal{M}$ be as in Proposition 6.1.1 and let $x \in \int_\Omega^\oplus H_s \, d\mu(s)$. It is enough to show that for any $G \in \mathcal{M}$, we can approximate $x(\cdot)1_G(\cdot)$ by elements of $D(A)$. Let $\varepsilon > 0$ and $G \in \mathcal{M}$. For $i \in \mathbb{N}$, define the sets G_i as follows:

$$G_1 = \left\{ s \in G : \|x(s) - x_1(s)\|_s^2 < \frac{\varepsilon}{\mu(G)} \right\},$$

$$G_i = \left\{ s \in G : \|x(s) - x_i(s)\|_s^2 < \frac{\varepsilon}{\mu(G)} \right\} \setminus \bigcup_{j=1}^{i-1} G_j, \quad i > 1.$$

Clearly all these sets are measurable. By assumption (i), $G = (\cup_{i=1}^{\infty} G_i) \cup N$ where $N \subset \Omega$ is such that $\mu(N) = 0$.

Now there exists $n \in \mathbb{N}$ such that

$$\int_{G \setminus \cup_{i=1}^{n} G_i} \|x(s)\|_s^2 \, d\mu(s) < \varepsilon.$$

Define the section $\bar{x}$ by $\bar{x}(s) = \sum_{i=1}^{n} x_i(s) 1_{G_i}(s)$ for all $s \in \Omega$. A straightforward checking of the criteria in Definition 6.2.2 shows that for each i, the function $s \mapsto x_i(s) 1_{G_i}(s)$ is both measurable and in $D(A)$. Hence $\bar{x} \in D(A)$ and furthermore,

$$\begin{aligned} \|x(\cdot)1_G(\cdot) - \bar{x}(\cdot)\|^2 &= \sum_{i=1}^{n} \int_{G_i} \|x(s) - x_i(s)\|_s^2 \, d\mu + \int_{G \setminus \cup_{i=1}^{n} G_i} \|x(s)\|_s^2 \, d\mu(s) \\ &< \sum_{j=1}^{n} \mu(G_j) \frac{\varepsilon}{\mu(G)} + \varepsilon \leq 2\varepsilon. \end{aligned}$$

Thus $x(\cdot)1_G(\cdot) \in \overline{D(A)}$ for all $G \in \mathcal{M}$ and, in particular, $x \in \overline{D(A)}$.

To prove the converse, assume (ii). First, there exists a sequence $\{x_i\}_{i=1}^{\infty} \subset D(A)$ that is dense in $\int_{\Omega}^{\oplus} H_s \, d\mu(s)$ since $D(A)$ is dense and $\int_{\Omega}^{\oplus} H_s \, d\mu(s)$ is separable by Proposition 6.1.1. We then claim that $\{x_i(s)\}_{i=1}^{\infty}$ is dense in H_s for a.e. s. Suppose this is not true. Then there exists a measurable subset $G \subset \Omega$ with $0 < \mu(G) < \infty$ such that for all $s \in G$, there exists $n, j_s \in \mathbb{N}$ with

$$\|x_i(s) - \xi_{j_s}(s)\|_s^2 \geq \frac{1}{n}, \quad i \in \mathbb{N}.$$

Here $\{\xi_j\}_{j=1}^{\infty}$ is the fundamental sequence for $\mathcal{H}$. Hence

$$G = \bigcup_{n,j \in \mathbb{N}} \left\{ s \in G : \|x_i(s) - \xi_j(s)\|_s^2 \geq \frac{1}{n} \text{ for all } i \in \mathbb{N} \right\}.$$

Since G has positive measure, one of the sets in the above countable union must have positive measure, say for indices $n = n_0, j = j_0$, and call the set G_{n_0,j_0}. Let

$$x'(s) = \begin{cases} \xi_{j_0}(s) & \text{for } s \in G_{n_0,j_0} \\ 0 & \text{for } s \notin G_{n_0,j_0} \end{cases}$$

and note that $x' \in \mathcal{F}$ by property 2. of measurable fields. By Proposition 6.1.1(i), $x' \in \int_{\Omega}^{\oplus} H_s \, d\mu(s)$. However,

$$\|x_i - x'\|^2 \geq \int_{G_{n_0,j_0}} \|x_i(s) - \xi_{j_0}(s)\|_s^2 \, d\mu(s) \geq \frac{\mu(G_{n_0,j_0})}{n_0}, \quad i \in \mathbb{N},$$

so that $\{x_i\}_{i=1}^{\infty}$ is not dense in $\int_{\Omega}^{\oplus} H_s \, d\mu(s)$. This is a contradiction, so $\{x_i(s)\}_{i=1}^{\infty}$ must be dense in H_s for a.e. $s \in \Omega$. By definition, $x_i \in D(A)$ implies that $x_i(s) \in D(A(s))$ for a.e. $s \in \Omega$. It follows that $D(A(s))$ is dense in H_s for a.e. $s \in \Omega$. $\square$

Third, we state a theorem on adjoints and inverses.

Theorem 6.2.5. *Suppose that $A = \int_\Omega^\oplus A(s)\, d\mu(s)$. Then the following are true.*

(i) A^* exists if and only if $A(s)^*$ exists a.e., in which case

$$A^* = \int_\Omega^\oplus A(s)^*\, d\mu(s).$$

(ii) A^{-1} exists (as a bounded operator) if and only if $A(s)^{-1}$ exists and $s \mapsto A(s)^{-1}$ is essentially bounded, in which case

$$A^{-1} = \int_\Omega^\oplus A(s)^{-1}\, d\mu(s).$$

Proof. This theorem is proved by [Nus64, Theorem 3] or [Cho70, Proposition 3.5] for the Nussbaum case. Since $\rho(B^*) = \{\overline{\lambda} : \lambda \in \rho(B)\}$ and $R(\overline{\lambda}, B^*) = R(\lambda, B)^*$, the resolvent condition in Definition 6.2.1 is satisfied, and so we are in the Nussbaum setting. For (ii), the assumption of boundedness means that once again our setting coincides with the Nussbaum case. □

Some basic spectral properties of direct integrals of operators follow immediately from this.

Corollary 6.2.6. *Suppose that $A = \int_\Omega^\oplus A(s)\, d\mu(s)$. The following statements are true.*

(i) $$\rho(A) = \Big\{\lambda \in \mathbb{C} : \lambda \in \rho(A(s)) \text{ for a.e. } s \in \Omega \text{ and } \operatorname*{ess\text{-}sup}_{s\in\Omega} \|R(\lambda, A(s))\|_{\mathcal{B}(H_s)} < \infty\Big\}$$

and

$$R(\lambda, A) = \int_\Omega^\oplus R(\lambda, A(s))\, d\mu(s), \quad \lambda \in \rho(A).$$

(ii) $$\sigma(A) = \Big\{\lambda \in \mathbb{C} : \text{there exists } G \subset \Omega \text{ such that } \mu(G) > 0 \text{ and } \lambda \in \sigma(A(s)) \text{ for all } s \in G\Big\} \cup \Big\{\lambda \in \mathbb{C} : \lambda \in \rho(A(s)) \text{ for a.e. } s \in \Omega \text{ and } \operatorname*{ess\text{-}sup}_{s\in\Omega} \|R(\lambda, A(s))\|_{\mathcal{B}(H_s)} = \infty\Big\}.$$

The following corollary characterises $\sigma(A)$ under certain conditions as the closure of the first two sets in the union given in part (ii) of Corollary 6.2.6. It generalises [LM16, Corollary 3.7] which applied only to direct sums. We follow the proof of [LM16, Corollary 3.7].

Corollary 6.2.7. *Suppose that $A = \int_\Omega^\oplus A(s)\,d\mu(s)$ and that there exists $\delta > 0$ satisfying the following condition: If $G \subset \Omega$ with $\mu(G) > 0$ and $\lambda \in \rho(A(s))$ for a.e. $s \in G$, then there exist $G_\lambda \subset G$ with $\mu(G_\lambda) > 0$ and $\eta \in \mathbb{C}$ such that $\eta \in \sigma(R(\lambda, A(s)))$ and*

$$|\eta| \geq \delta \|R(\lambda, A(s))\|_{\mathcal{B}(H_s)}$$

for all $s \in G_\lambda$. Then $\sigma(A)$ is equal to the closure of

$$\Big\{\lambda \in \mathbb{C} : \text{there exists } K \subset \Omega \text{ such that } \mu(K) > 0 \\ \text{and } \lambda \in \sigma(A(s)) \text{ for all } s \in K\Big\}.$$

Proof. The inclusion '$\supseteq$' follows from Corollary 6.2.6(ii). For the opposite inclusion let $\lambda \in \rho(A(s))$ for a.e. s and suppose that

$$\operatorname*{ess\text{-}sup}_{s\in\Omega} \|R(\lambda, A(s))\|_{\mathcal{B}(H_s)} = \infty.$$

We are done if we show that λ can be approximated by z such that there exists $K \subset \Omega$ of positive measure with $z \in \sigma(A(s))$ for all $s \in K$.

Let $\varepsilon > 0$. By assumption, there exists $G \subset \Omega$ of positive measure such that $\lambda \in \rho(A(s))$ and $\|R(\lambda, A(s))\|_{\mathcal{B}(H_s)} > \frac{1}{\delta\varepsilon}$ for all $s \in G$. Hence, there exists $G_\lambda \subset G$ of positive measure and $\eta \in \mathbb{C}$ such that $\eta \in \sigma(R(\lambda, A(s))$ for all $s \in G_\lambda$ and $|\eta| > \frac{1}{\varepsilon}$. By the spectral mapping theorem for the resolvent, there exists $z \in \mathbb{C}$ for which $z \in \sigma(A(s))$ for all $s \in G_\lambda$ such that $(z - \lambda)^{-1} = \eta$. In particular, $|z - \lambda| < \varepsilon$ and $\mu(G_\lambda) > 0$. ☐

It is worth mentioning that Azoff [Azo74] and Chow [Cho70] delve much deeper into the spectral theory of direct integrals. In particular, [Azo74, Examples 4.2, 4.4] show the importance of essential boundedness in Corollary 6.2.6 by demonstrating that not much can be said about the spectrum of A given only the spectra of a.e. $A(s)$.

Finally, we turn to compactness. Before we prove the following lemma, recall that a measurable subset $A \subset \Omega$ is called an atom (with respect to μ) if $\mu(A) > 0$ and for all measurable $B \subset A$, either $\mu(B) = 0$ or $\mu(B) = \mu(A)$.

Lemma 6.2.8. *If $\int_\Omega^\oplus A(s)\,d\mu(s)$ is compact and for some $\varepsilon > 0$, $\|A(s)\|_{\mathcal{B}(H_s)} \geq \varepsilon$ for a.e. $s \in \Omega$, then Ω contains an atom.*

Proof. Assume otherwise. Then there exist subsets $\Omega = G_1 \supsetneq G_2 \supsetneq \dots$ such that $\mu(G_i) > \mu(G_{i+1}) > 0$ for all $i \in \mathbb{N}$. Writing $K_i = G_i \setminus G_{i+1}$, we get that $G_1 = \bigcup_{i=1}^\infty K_i$

is a disjoint union and $\mu(K_i) = \mu(G_i) - \mu(G_{i+1}) > 0$ for all $i \in \mathbb{N}$. Using the same construction as in the proof of Proposition 6.2.3, for all $i \in \mathbb{N}$, there exists a measurable subset $K_i' \subset K_i$ with $\mu(K_i') > 0$ and a (normalised) function $y_i \in \int_\Omega^\oplus H_s \, d\mu(s)$ such that $\|A(s)y_i(s)\|_s > \frac{\varepsilon}{2}$ for all $s \in K_i'$, $\|y_i(s)\|_s = 1$ for all $s \in K_i'$, and $y_i(s) = 0$ for all $s \notin K_i'$. Let $x_i = \mu(K_i')^{-1/2} y_i(s)$. Then $x_i \in \int_\Omega^\oplus H_s \, d\mu(s)$ and $\|x_i\| = 1$ for all $i \in \mathbb{N}$. But

$$\begin{aligned}\|Ax_i - Ax_j\|^2 &= \mu(K_i')^{-1} \int_\Omega \|A(s)y_i(s)\|_s^2 \, d\mu(s) + \mu(K_j')^{-1} \int_\Omega \|A(s)y_j(s)\|_s^2 \, d\mu(s) \\ &> \frac{\varepsilon^2}{2}\end{aligned}$$

for all $i \neq j$ as $K_i' \subset K_i$ and $K_j' \subset K_j$ are disjoint. Hence, A cannot be compact. $\square$

Recall that for a direct sum of operators $T = \bigoplus_{i=1}^\infty T_i$ on a direct sum of Hilbert spaces $\bigoplus_{i=1}^\infty H_i$, T is compact if and only if T_i is compact for all $i \in \mathbb{N}$ and $\|T_i\|_{\mathcal{B}(H_i)} \to 0$ (proved via finite-rank approximations). Note also that for Radon measures, if $A \subset \Omega$ is an atom, then A is of the form $\{x\} \cup N$ where $x \in A$ and $\mu(N) = 0$ ([Sim96, Proposition 7.5.1]). In the proof of the following theorem, we incidentally prove[2] (and use) the fact that σ-finite spaces can contain at most countably many atoms modulo sets of measure zero.

Theorem 6.2.9. *Let μ be a Radon measure. Suppose that $A = \int_\Omega^\oplus A(s) \, d\mu(s)$. Then A is compact if and only if $\Omega = \Omega_1 \cup \Omega_2$ where Ω_1 and Ω_2 are disjoint, $\Omega_1 = \{s_1, s_2, \dots\}$ is a countable set of points such that $\mu(\{s_i\}) > 0$ and $A(s_i)$ is compact and non-zero for all $i \in \mathbb{N}$, $\|A(s_i)\|_{\mathcal{B}(H_s)} \to 0$ as $i \to \infty$, and $A(s) = 0$ for a.e. $s \in \Omega_2$.*

Proof. The 'if' direction is simply the case of a direct sum of compact operators tailing to zero, rewriting $A = \bigoplus_{i=1}^\infty A(s_i)$ on $\bigoplus_{i=1}^\infty H_{s_i}$. There the direct sum can clearly be approximated by operators of finite rank.

For the 'only if' direction, let S be the set of points of positive measure and let $\{M_k\}_{k=1}^\infty$ be the collection of sets in Proposition 6.1.1 of finite measure exhausting Ω. Then

$$S = \bigcup_{k=1}^\infty \bigcup_{n=1}^\infty \left\{ s \in S \cap M_k : \mu(\{s\}) \geq \frac{\mu(M_k)}{n} \right\}.$$

Now

$$\left| \left\{ s \in S \cap M_k : \mu(\{s\}) \geq \frac{\mu(M_k)}{n} \right\} \right| \leq n, \quad k, n \in \mathbb{N},$$

[2]The proof of this commonly-known fact is adapted from StackExchange. I have included my adaptation because it is cute and concise. See `https://math.stackexchange.com/questions/41142/questions-on-atoms-of-a-measure` and `https://math.stackexchange.com/questions/20661/the-sum-of-an-uncountable-number-of-positive-numbers`.

so that S is the countable union of finite sets. Writing $S = \{s_1, s_2, \dots\}$, it is clear that the restriction $\int_S^\oplus A(s)\,d\mu(s)$ of A to the subspace $\int_S^\oplus H_s\,d\mu(s)$ is also compact and that we can write

$$\int_S^\oplus H_s\,d\mu(s) = \bigoplus_{i=1}^\infty H_{s_i} \text{ and } \int_S^\oplus A(s)\,d\mu(s) = \bigoplus_{i=1}^\infty A(s_i).$$

It follows from the direct sum case that $A(s_i)$ is compact for all $i \in \mathbb{N}$ and that $\|A(s_i)\|_{\mathcal{B}(H_{s_i})} \to 0$ as $i \to \infty$.

If $A(s) = 0$ for a.e. $s \in \Omega \setminus S$, we are done. If not, there exists $G \subset \Omega \setminus S$ such that $\mu(G) > 0$ and $\|A(s)\|_{\mathcal{B}(H_s)} > 0$ for all $s \in G$. Without loss of generality, we can assume by taking a smaller subset, that $\|A(s)\|_{\mathcal{B}(H_s)} \geq \varepsilon$ for some $\varepsilon > 0$ and a.e. $s \in G$. Applying Lemma 6.2.8 to the subspace $\int_G^\oplus H_s\,d\mu(s)$, we can add another point of positive measure $s_0 \in \Omega \setminus S$ to S. This contradicts that S is the set of all points of positive measure and we are done by taking $\Omega_1 = S$ and $\Omega_2 = \Omega \setminus S$. $\square$

Thus, Theorem 6.2.9 tells us that the only compact direct integrals of operators are, in fact, direct sums.

6.3 Direct integrals of C_0-semigroups

In this section, we prove that we can take direct integrals of C_0-semigroups that have a uniform exponential growth bound and that the generator of the direct integral is the direct integral of the generators.

Theorem 6.3.1. *Let $\left\{T^{(\cdot)}(s) : \mathbb{R}_+ \to \mathcal{B}(H_s) : s \in \Omega\right\}$ be a collection of C_0-semigroups. If for each $t \geq 0$, the direct integral of $\{T^t(s) : s \in \Omega\}$ exists as a bounded operator, then $T : \mathbb{R}_+ \to \mathcal{B}\left(\int_\Omega^\oplus H_s\,d\mu(s)\right)$ defined and denoted by*

$$T^t = \int_\Omega^\oplus T^t(s)\,d\mu(s)$$

is a C_0-semigroup if and only if there exist $M \geq 1, \omega \in \mathbb{R}$ such that

$$\|T^t(s)\|_{\mathcal{B}(H_s)} \leq Me^{\omega t}$$

for a.e. $s \in \Omega$ and all $t > 0$. In this case, $\|T^t\| \leq Me^{\omega t}$ for all $t \geq 0$ and its generator A is given by

$$A = \int_\Omega^\oplus A(s)\,d\mu(s)$$

where $A(s)$ is the generator of $T^t(s)$ for a.e. $s \in \Omega$.

Proof. First, we prove the 'if' direction. $T : \mathbb{R}_+ \to \mathcal{B}(\int_\Omega^\oplus H_s \, d\mu(s))$ clearly satisfies the semigroup property. We now check the strong continuity at 0. Since $T^{(\cdot)}(s)$ is a C_0-semigroup for each $s \in \Omega$, for any $x \in \int_\Omega^\oplus H_s \, d\mu(s)$, $T^t(s)x(s) \to x(s)$ in H_s as $t \to 0$. Hence we have pointwise convergence of $\|T^t(s)x(s) - x(s)\|_s \to 0$. However,

$$\|T^t(s)x(s) - x(s)\|_s^2 \le \big(\|T^t(s)x(s)\|_s + \|x(s)\|_s\big)^2 \le (Me^{\omega t} + 1)^2 \|x(s)\|_s^2$$

for all $t \ge 0$. Since $x \in \int_\Omega^\oplus H_s \, d\mu(s)$, by Lebesgue's dominated convergence theorem,

$$\int_\Omega \|T^t(s)x(s) - x(s)\|_s^2 \, d\mu(s) \to 0.$$

By Proposition 6.2.3, $\|T^t\| \le Me^{\omega t}$ for all $t \ge 0$. Since any C_0-semigroup has an exponential bound, Proposition 6.2.3 also proves the 'only if' direction.

Finally, let B be the generator of T^t. Our goal is to prove that $B = A$, but we must first check that A exists as a direct integral. Let $\lambda = \omega + 1$. Then by the Laplace transform representation for resolvents of generators, $\lambda \in \rho(A(s))$ for a.e. s and

$$\begin{aligned} \langle R(\lambda, A(s))x(s), y(s)\rangle_s &= \left\langle \int_0^\infty e^{-\lambda t} T^t(s)x(s) \, dt, y(s) \right\rangle_s \\ &= \int_0^\infty e^{-\lambda t} \langle T^t(s)x(s), y(s)\rangle_s \, dt \end{aligned}$$

for all $x, y \in \mathcal{F}$. Since the final integrand is measurable by assumption, Fubini's theorem tell us that $\{R(\lambda, A(s)) : s \in \Omega\}$ forms a measurable field of bounded operators. Hence $\{A(s) : s \in \Omega\}$ is a measurable field of closed operators.

We now show that $B = A$. Again, let $\lambda = \omega + 1$. Then $\lambda \in \rho(B)$, $\lambda \in \rho(A(s))$ and for some fixed $C > 0$, $\|R(\lambda, B)\|, \|R(\lambda, A(s))\|_{\mathcal{B}(H_s)} \le C$ for a.e. $s \in \Omega$. By Corollary 6.2.6(i), $\lambda \in \rho(A)$. Once again using the Laplace transform representation for resolvents of generators and Corollary 6.2.6(i), we have

$$\begin{aligned} R(\lambda, B)x &= \int_0^\infty e^{-\lambda t} T^t x \, dt = \int_0^\infty \left(\int_\Omega^\oplus e^{-\lambda t} T^t(s) \, d\mu(s) \right) x \, dt, \qquad (6.3.1) \\ R(\lambda, A)x &= \left(\int_\Omega^\oplus R(\lambda, A(s)) \, d\mu(s) \right) x, \end{aligned}$$

and for a.e. $s \in \Omega$,

$$R(\lambda, A(s))x(s) = \int_0^\infty e^{-\lambda t} T^t(s)x(s) \, dt. \qquad (6.3.2)$$

The outer integral in Equation (6.3.1) is a Bochner integral in $\int_\Omega^\oplus H_s\,d\mu(s)$ and the integral in Equation (6.3.2) is a Bochner integral in H_s. Let $x, y \in \int_\Omega^\oplus H_s\,d\mu(s)$ be arbitrary. Then

$$\begin{aligned}
&\left\langle \int_0^\infty \left(\int_\Omega^\oplus e^{-\lambda t} T^t(s)\,d\mu(s) \right) x\,dt, y \right\rangle \\
&\qquad = \int_0^\infty \int_\Omega \left\langle e^{-\lambda t} T^t(s)x(s), y(s) \right\rangle_s d\mu(s)dt \\
&\qquad = \int_\Omega \int_0^\infty \left\langle e^{-\lambda t} T^t(s)x(s), y(s) \right\rangle_s dtd\mu(s) \qquad (6.3.3) \\
&\qquad = \int_\Omega \left\langle \int_0^\infty e^{-\lambda t} T^t(s)x(s)\,dt, y(s) \right\rangle_s d\mu(s) \\
&\qquad = \int_\Omega \left\langle R(\lambda, A(s))x(s), y(s) \right\rangle_s d\mu(s) \\
&\qquad = \left\langle \left(\int_\Omega^\oplus R(\lambda, A(s))\,d\mu(s) \right) x, y \right\rangle,
\end{aligned}$$

where Equation (6.3.3) holds due to Fubini's theorem. Hence for arbitrary x and y, $\langle R(\lambda, B)x, y\rangle = \langle R(\lambda, A)x, y\rangle$, so that $R(\lambda, B) = R(\lambda, A)$ and in particular, $B = A$. $\square$

A natural question to ask is whether the sufficient conditions for a direct integral of operators to generate a C_0-semigroup are also necessary conditions (see [LM16, Theorem 4.4]). A positive answer for this question is given in the following theorem.

Theorem 6.3.2. *Suppose $A = \int_\Omega^\oplus A(s)\,d\mu(s)$ generates a C_0-semigroup $T(\cdot)$ on $\int_\Omega^\oplus H_s\,d\mu(s)$ with constants $M \geq 1, \omega \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\omega t}$ for all $t > 0$. Then for a.e. $s \in \Omega$, $A(s)$ generates a C_0-semigroup denoted $T^{(\cdot)}(s)$ on H_s such that $\|T^t(s)\|_{\mathcal{B}(H_s)} \leq Me^{\omega t}$ for all $t > 0$. Furthermore,*

$$T(t) = \int_\Omega^\oplus T^t(s)\,d\mu(s).$$

Proof. If A generates a C_0-semigroup on $\int_\Omega^\oplus H_s\,d\mu(s)$, then there exist $M \geq 1, \omega \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ and for every $\lambda > \omega$, one has $\lambda \in \rho(A)$ and

$$\|[(\lambda - \omega)R(\lambda, A)]^n\| \leq M$$

for all $n \in \mathbb{N}$. By Corollary 6.2.6(ii) and 6.2.3, for each $\lambda > \omega$ and $n \in \mathbb{N}$, there exists a measurable $U_{\lambda,n} \subset \Omega$ with $\mu(\Omega \setminus U_{\lambda,n}) = 0$ such that $\lambda \in \rho(A(s))$ and

$$\|[(\lambda - \omega)R(\lambda, A(s))]^n\|_{\mathcal{B}(H_s)} \leq M, \quad s \in \Omega \setminus U_{\lambda,n}.$$

Let $G = \bigcup_{\lambda \in \mathbb{Q}_{\omega+}, n \in \mathbb{N}} U_{\lambda,n}$ where $\mathbb{Q}_{\omega+} = \{\mu \in \mathbb{Q} : \mu > \omega\}$. Then $\mu(\Omega \setminus G) = 0$ and for all $s \in G$, we have $\mathbb{Q}_{\omega+} \subset \rho(A(s))$ and

$$\|[(\lambda - \omega)R(\lambda, A(s))]^n\|_{\mathcal{B}(H_s)} \leq M, \quad \lambda \in \mathbb{Q}_{\omega+}, n \in \mathbb{N}. \tag{6.3.4}$$

Clearly $\mathbb{Q}_{\omega+}$ is dense in $\{\mu \in \mathbb{R} : \mu > \omega\}$ so by considering bounded subsets of $\{\mu \in \mathbb{R} : \mu > \omega\}$ and applying Proposition 2.2.2 to the relation (6.3.4), we get that for all $\lambda \in \mathbb{R}$ with $\lambda > \omega$, we have $\lambda \in \rho(A(s))$ and by continuity, (6.3.4) holds for all $s \in G$. It follows that for all $s \in G$ and hence for a.e. $s \in \Omega$, $A(s)$ generates a C_0-semigroup $T^t(s)$ such that

$$\|T^t(s)\|_{\mathcal{B}(H_s)} \leq Me^{\omega t}, \quad t > 0.$$

Let $\operatorname{Re}\lambda > \omega$ so that $\lambda \in \rho(A(s))$ for a.e. $s \in \Omega$. Since $\{A(s) : s \in \Omega\}$ is a measurable field of closed operators, $\{R(\lambda, A(s)) : s \in \Omega\}$ forms a measurable field of bounded operators. Hence for any $x, y \in \mathcal{F}$, the Laplace transform representation for resolvents of generators gives us that

$$\begin{aligned} s \mapsto \left\langle R(\lambda, A(s))x(s), y(s) \right\rangle_s &= \left\langle \int_0^\infty e^{-\lambda t} T^t(s)x(s)\, dt, y(s) \right\rangle_s \\ &= \int_0^\infty e^{-\lambda t} \left\langle T^t(s)x(s), y(s) \right\rangle_s dt \end{aligned}$$

is measurable. Let

$$F(\lambda, s) = \int_0^\infty e^{-\lambda t} \left\langle T^t(s)x(s) - x(s), y(s) \right\rangle_s dt,$$

which is measurable by Fubini's theorem. Notice that for fixed s, $F(\lambda, s)$ is the Laplace transform with respect to t of $g(t,s) = \langle T^t(s)x(s) - x(s), y(s) \rangle_s$ which satisfies $g(0,s) = 0$, is continuous in t by strong continuity of $T^t(s)$, and

$$\begin{aligned} |g(t,s)| &= \left| \left\langle T^t(s)x(s) - x(s), y(s) \right\rangle_s \right| \\ &\leq (ce^{\omega t} + 1)\|x(s)\|_s \|y(s)\|_s \leq (c+1)\|x(s)\|_s \|y(s)\|_s e^{\omega' t} \end{aligned}$$

where $\omega' = \max\{\omega, 0\}$ so that g is exponentially bounded with respect to t. Applying Fubini's theorem two more times, we get that

$$s \mapsto \frac{1}{u} \int_0^u \frac{1}{2\pi} \int_{\omega'+1-ir}^{\omega'+1+ir} e^{\lambda t} F(\lambda, s)\, d\lambda dr$$

is also measurable for each $t \geq 0$. By [ABHN11, Theorem 4.2.21(b)],

$$g(t,s) = \lim_{u \to \infty} \frac{1}{u} \int_0^u \frac{1}{2\pi} \int_{\omega'+1-ir}^{\omega'+1+ir} e^{\lambda t} F(\lambda, s)\, d\lambda dr$$

for a.e. $s \in \Omega$. Hence $s \mapsto \langle T^t(s)x(s) - x(s), y(s)\rangle_s$ is the pointwise a.e. limit of measurable functions for each $t \geq 0$. This together with the uniform exponential bounds $\|T^t(s)\|_{\mathcal{B}(H_s)} \leq Me^{\omega t}$ implies that $\{T^t(s) : s \in \Omega\}$ form a measurable field of bounded operators for each $t \geq 0$.

Theorem 6.3.1 then implies that A generates $\int_\Omega^\oplus T^t(s)\, d\mu(s)$, hence

$$T(t) = \int_\Omega^\oplus T^t(s)\, d\mu(s).$$

□

Theorem 6.3.2 says that decomposable generators generate C_0-semigroups that are decomposable into individual C_0-semigroups.

Remark 6.3.3. If A is a decomposable operator that generates a C_0-semigroup $T(\cdot)$, the following argument is an alternative way via the characterisation by Nussbaum to see that for each $t \geq 0$, $T(t)$ is a decomposable operator. Since A is decomposable, Corollary 6.2.6(i) implies that $R(\lambda, A)$ is decomposable for all $\lambda \in \rho(A)$. In particular, $R(\lambda, A)$ commutes with all bounded diagonalisable operators [Nus64, Corollary 4] and hence by [ABHN11, Proposition 3.1.5], $T(t)$ commutes with all bounded diagonalisable operators for any $t \geq 0$. The characteristaion [Nus64, Corollary 4] again implies that $T(t)$ is decomposable for all $t \geq 0$. However, unlike Theorem 6.3.2 this argument does not guarantee that $T(\cdot)$ decomposes into a family of C_0-semigroups, since all we would know from this is that $T(t)$ is decomposable for each $t \geq 0$. Difficulties arise when trying to show the existence of a semigroup on H_s for any particular s.

We also have the following corollary which generalises [LM16, Theorem 4.4] in the Hilbert space case in a less satisfying way than Theorem 6.3.2, as it requires that the subsets we are 'projecting' down on have positive measure.

Corollary 6.3.4. *Suppose that $A = \int_\Omega^\oplus A(s)\, d\mu(s)$ generates a C_0-semigroup on $\int_\Omega^\oplus H_s\, d\mu(s)$. Then for every measurable subset $\Omega' \subset \Omega$ with positive measure, $\int_{\Omega'}^\oplus A(s)\, d\mu(s)$ generates a C_0-semigroup on the closed subspace $\int_{\Omega'}^\oplus H_s\, d\mu(s)$.*

Proof. Theorem 6.3.2 gives the pointwise and uniform a.e. conditions for $\{A(s) : s \in \Omega'\}$ necessary to apply Theorem 6.3.1 with Ω' replacing Ω. □

Remark 6.3.5. A proof of this corollary can be obtained in an analogous way as the proof for [LM16, Theorem 4.4]. The key step was to show that $\int_{\Omega'}^\oplus H_s\, d\mu(s)$ is invariant under T, the C_0-semigroup generated by A. Let $x \in \int_{\Omega'}^{\oplus} H_s \, d\mu(s)$. By the Post-Widder inversion formula [EN00, Chapter III Corollary 5.5],

$$T(t)x = \lim_{n\to\infty} \left[\frac{n}{t} R\Big(\frac{n}{t}, A\Big)\right]^n x = \lim_{n\to\infty} \left[\int_{\Omega}^{\oplus} \frac{n}{t} R\Big(\frac{n}{t}, A(s)\Big) \, d\mu(s)\right]^n x$$

for any $t \geq 0$ where the resolvents in the integrand of the direct integral exist for a.e. $s \in \Omega$. By definition of the direct integral of operators,

$$\left(\left[\int_{\Omega}^{\oplus} \frac{n}{t} R\Big(\frac{n}{t}, A(s)\Big) \, d\mu(s)\right]^n x\right)(s) = 0$$

for a.e. $s \notin \Omega'$. Hence $T(t)x$ is the limit in $\int_{\Omega}^{\oplus} H_s \, d\mu(s)$ of elements in $\int_{\Omega'}^{\oplus} H_s \, d\mu(s)$ and by closedness, is in $\int_{\Omega'}^{\oplus} H_s \, d\mu(s)$. We can apply [EN00, Chapter II Proposition 2.3] to complete the proof.

However, this method could not be directly used to prove Theorem 6.3.2 since H_s does not embed into $\int_{\Omega}^{\oplus} H_s \, d\mu(s)$ in any meaningful way and hence, like in Remark 6.3.3, difficulties potentially arise as to the existence of a semigroup for all $t \geq 0$ on H_s for any particular s.

6.4 Direct integrals of special classes of semigroups

A natural question to ask is whether there are similar results to Theorems 6.3.1 and 6.3.2 for special classes of semigroups (see [EN00, Chapter II Section 4]). These classes are not looked at in [LM16][2]. Ahead of the following discussion, we note that these classes can be classified through spectral conditions in such a way that their direct integral theory once again, like in the general case, reduces to a.e. uniform conditions on the individual fibres.

6.4.1 Bounded analytic semigroups

First, we provide an easy consequence of Corollary 6.2.6 that deals with sectoriality.

Proposition 6.4.1. *Suppose that $A = \int_{\Omega}^{\oplus} A(s) \, d\mu(s)$. Then A is sectorial of angle $\delta \in [0, \frac{\pi}{2})$ if and only if $\Sigma_{\pi/2+\delta} \setminus \{0\} \subset \rho(A(s))$ and for all $\varepsilon \in (0, \delta)$ there exists $M_\varepsilon \geq 1$ such that*

$$\|R(\lambda, A(s))\|_{\mathcal{B}(H_s)} \leq \frac{M_\varepsilon}{|\lambda|}, \quad \lambda \in \overline{\Sigma}_{\pi/2+\delta-\varepsilon} \setminus \{0\},$$

for a.e. $s \in \Omega$.

[2]It would be unsurprising if direct sums of semigroups from these special classes have already been previously looked at elsewhere for particular cases, but I have been unable to find any references in the literature.

Proof. The 'if' direction follows immediately from Corollary 6.2.6(ii) and Proposition 6.2.3 as the uniform resolvent bound for a.e. s given $\lambda \in \Sigma_{\pi/2+\delta}$ guarantees that such $\lambda \in \rho(A)$.

Now let $Q = \{a + ib \in \mathbb{C} : a, b \in \mathbb{Q}\}$. The 'only if' direction is then proved by showing, in the same way as the proof for Theorem 6.3.2, that $\Sigma_{\pi/2+\delta} \cap Q \setminus \{0\} \subset \rho(A(s))$ and that for each $\varepsilon \in (0, \delta)$, the same constant M_ε for bounding $\|\lambda R(\lambda, A)\|$ also bounds $\|\lambda R(\lambda, A(s))\|_{\mathcal{B}(H_s)}$ for all $\lambda \in \overline{\Sigma}_{\pi/2+\delta-\varepsilon} \cap Q \setminus \{0\}$ for a.e. $s \in \Omega$. Again, as in the proof of Theorem 6.3.2, we are done by Proposition 2.2.2 and continuity of the resolvent applied to compact subsets of the sectors. $\square$

In other words, A is sectorial of angle $\delta \in [0, \frac{\pi}{2})$ if and only if $A(s)$ is uniformly a.e. sectorial of angle δ for a.e. s. We restate this in terms of bounded analytic semigroups without proof (see [EN00, Chapter II Section 4a.] and [IK02, Chapter 3]).

Theorem 6.4.2. *Suppose that $A = \int_\Omega^\oplus A(s)\, d\mu(s)$. Then A generates a bounded analytic semigroup T of angle $\delta \in [0, \frac{\pi}{2})$ if and only if $A(s)$ generates an analytic semigroup $T^{(\cdot)}(s)$ of angle δ such that for every $0 < \delta' < \delta$, there exists $M_{\delta'} \geq 1$ with*

$$\|T^z(s)\|_{\mathcal{B}(H_s)} \leq M_{\delta'}, \quad z \in \Sigma_{\delta'},$$

uniformly for a.e. $s \in \Omega$.

These results can also be extended for general unbounded semigroups (see [IK02, Propositions 3.16, 3.18, Theorem 3.19]).

6.4.2 Eventually differentiable semigroups

In light of the characterisation of differentiable semigroups found in [EN00, Chapter II Theorem 4.14], a similar result to Theorem 6.4.2 can be given for eventually differentiable semigroups.

Theorem 6.4.3. *Suppose that $A = \int_\Omega^\oplus A(s)\, d\mu(s)$. Then A generates an eventually differentiable semigroup exponentially bounded by $Me^{\omega t}$ for $\omega \in \mathbb{R}$ if and only if $A(s)$ generates a C_0-semigroup also exponentially bounded by $Me^{\omega t}$ and there exist constants $a, b, C > 0$ with*

$$\Theta = \left\{\lambda \in \mathbb{C} : ae^{-b \operatorname{Re} \lambda} \leq |\operatorname{Im} \lambda|\right\} \subset \rho(A(s))$$

and

$$\|R(\lambda, A(s))\|_{\mathcal{B}(H_s)} \leq C|\operatorname{Im} \lambda|$$

for all $\lambda \in \Theta$ with $\operatorname{Re} \lambda \leq \omega$ uniformly for a.e. $s \in \Omega$.

Proof. The proof is almost exactly the same as that of Proposition 6.4.1. However, for the 'if' direction, we must also check that

$$\operatorname*{ess-sup}_{s\in\Omega} \|R(\lambda, A(s))\|_{\mathcal{B}(H_s)} < \infty$$

holds for all $\lambda \in \Theta$. For $\operatorname{Re}\lambda \leq \omega$, the uniform a.e. resolvent bound is given by $C|\operatorname{Im}\lambda|$. For $\operatorname{Re}\lambda > \omega$,

$$\|R(\lambda, A(s))\|_{\mathcal{B}(H_s)} \leq \frac{M}{\operatorname{Re}\lambda - \omega}$$

for a.e. $s \in \Omega$ by virtue of standard semigroup generator properties. $\square$

In other words, the direct integral operator generates an eventually differentiable semigroup if individual $A(s)$ are eventually differentiable in a uniform a.e. way.

6.4.3 Immediately norm-continuous semigroups

We introduce the natural notion of norm-continuous in t uniformly a.e. on Ω in order to formulate an unsurprising result concerning immediately norm-continuous semigroups. We say that $\{T^t(s) : s \in \Omega\}$ is *norm-continuous for $t > t_0$ uniformly a.e. on* Ω if for every $t > t_0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|T^t(s) - T^{t_1}(s)\|_{\mathcal{B}(H_s)} < \varepsilon, \quad |t - t_1| < \delta,$$

for a.e. $s \in \Omega$. Note that δ is independent of s.

Proposition 6.4.4. *Suppose that $A = \int_\Omega^\oplus A(s)\, d\mu(s)$. Assume that $A(s)$ generates a semigroup $T^{(\cdot)}(s)$ for a.e. $s \in \Omega$ that are uniformly a.e. exponentially bounded and that $\{T^t(s) : s \in \Omega\}$ is norm-continuous for $t > t_0$ uniformly a.e. on Ω. Then A generates an eventually norm-continuous semigroup that is norm-continuous for $t > t_0$.*

Proof. The direct integral semigroup $T^{(\cdot)}$ exists by Theorem 6.3.1. Let $t > t_0, \varepsilon > 0$, and $x \in \int_\Omega^\oplus H_s\, d\mu(s)$. Then by norm-continuity for $t > t_0$ a.e. on Ω, there exists $\delta > 0$ such that

$$\begin{aligned}
\|(T^t - T^{t_1})x\|^2 &= \int_\Omega \|(T^t(s) - T^{t_1}(s))x(s)\|_s^2\, d\mu(s) \\
&\leq \int_\Omega \|T^t(s) - T^{t_1}(s)\|_{\mathcal{B}(H_s)}^2 \|x(s)\|_s^2\, d\mu(s) \\
&< \varepsilon^2 \|x\|^2
\end{aligned}$$

for all $|t - t_1| < \delta$. Hence $\|T^t - T^{t_1}\| < \varepsilon$ for all $|t - t_1| < \delta$, proving that T^t is norm-continuous for $t > t_0$. $\square$

We get a useful characterisation of immediately norm-continuous semigroups on Hilbert spaces from [EN00, Chapter II Theorem 4.20], enabling us to prove the following result.

Theorem 6.4.5. *Suppose that $A = \int_\Omega^\oplus A(s)\,d\mu(s)$. Then A generates an immediately norm-continuous semigroup bounded by $Me^{-\varepsilon t}$ for some $\varepsilon > 0$ if and only if $A(s)$ generates a C_0-semigroup $T^{(\cdot)}(s)$ bounded by $Me^{-\varepsilon t}$ for a.e. $s \in \Omega$ and*

$$\lim_{r\to\pm\infty} \operatorname*{ess\text{-}sup}_{s\in\Omega} \|R(ir, A(s))\|_{\mathcal{B}(H_s)} = 0. \tag{6.4.1}$$

Proof. For the 'if' direction, Theorem 6.3.1 ensures that A generates a C_0-semigroup that is bounded by $Me^{-\varepsilon t}$. By Proposition 6.2.3, Corollary 6.2.6(i), and (6.4.1), we have that

$$\lim_{r\to\pm\infty} \|R(ir, A)\| = 0.$$

Thus by [EN00, Chapter II Theorem 4.20], A generates an immediately norm-continuous semigroup. For the converse, it is enough to see that as in the proof of Theorem 6.3.2, there is a set $G \subset \Omega$ with $\mu(\Omega \setminus G) = 0$ such that for all $s \in G$, $i\mathbb{Q} \subset \rho(A(s))$ and $\|R(ir, A(s))\|_{\mathcal{B}(H_s)} \leq \|R(ir, A)\|$ for all $r \in \mathbb{Q}$. Continuity of the resolvent and density of $i\mathbb{Q}$ in $i\mathbb{R}$ completes the proof. ☐

6.4.4 Immediately compact semigroups

We know from [EN00, Chapter II Theorem 4.29] that a C_0-semigroup T is immediately compact if and only if T is immediately norm-continuous and its generator A has compact resolvent. For an operator B, the resolvent $R(\lambda, B)$ for any $\lambda \in \rho(B)$ cannot be zero. Hence, in light of Theorem 6.2.9, the only possible setting for a direct integral of semigroups to be immediately compact when μ is a Radon measure is the direct sum case. Combining [EN00, Chapter II Theorem 4.29] with Theorem 6.2.9 and Theorem 6.4.5, we get the following result with trivial proof.

Theorem 6.4.6. *Let μ be the counting measure on $\mathbb{N}$ and suppose that $A = \int_\mathbb{N}^\oplus A(n)\,d\mu(n)$. Then A generates an immediately compact semigroup if and only if $\lim_{r\to\pm\infty} \|R(ir, A(n))\|_{\mathcal{B}(H_n)} = 0$ uniformly for all $n \in \mathbb{N}$ and there exists $\lambda \in \mathbb{C}$ such that $\lambda \in \rho(A(n))$ and $R(\lambda, A(n))$ is compact for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|R(\lambda, A(n))\|_{\mathcal{B}(H_n)} \to 0$. In particular, if A generates an immediately compact semigroup, then so does $A(n)$ for all $n \in \mathbb{N}$.*

6.5 Examples and applications

In this section, we discuss two examples where Theorem 6.3.1 can be applied. Regrettably, these are somewhat special cases, where extra structure allows us to work more easily with the relevant spaces. The point, however, is that examples exist.

The first example is as follows. Let $\Omega = (0,1)$ and take μ to be the Lebesgue measure on $(0,1)$. It is worth informing the reader here that there is some ambiguity in the literature about what measures and measure spaces are used in direct integral theory. Most papers describe μ as a Borel measure on a locally compact topological space, as we have done so in Section 6.1. This, however, does not mean we are only restricted to working with the Borel σ-algebra, as the known results hold on measure spaces that contain the Borel σ-algebra (see for example [AC80, GGST12, Yet05]). Now define $\mathcal{H}$ by fixing H_s to be the constant Hilbert space $L^2(\mathbb{R})$, the space of square integrable complex-valued functions on the real line, for all $s \in (0,1)$. For the choice of measurable sections, we take $\mathcal{F}$ to be all sections $x : (0,1) \to \mathcal{H}$ such that

$$s \mapsto \langle x_s, g\rangle_{L^2}$$

is measurable for all $g \in L^2(\mathbb{R})$. In this case, $\int_\Omega^\oplus H_s \, d\mu(s) = L^2((0,1); L^2(\mathbb{R})) = L^2((0,1)\times\mathbb{R})$, the space of square integrable complex-valued functions on $(0,1)\times\mathbb{R}$. This is also known as the Hilbert space tensor product $L^2(0,1) \otimes L^2(\mathbb{R})$.

Define the operator A on $L^2(\mathbb{R})$ by

$$\begin{aligned} D(A) &= \{f \in L^2(\mathbb{R}) : f \text{ absolutely continuous}, f' \in L^2(\mathbb{R})\}, \\ Af &= f', \quad f \in D(A). \end{aligned}$$

This is widely known to generate the left-shift semigroup $T^{(\cdot)}$ given by $(T^t f)(r) = f(r+t)$ for $f \in L^2(\mathbb{R})$ and $r \in \mathbb{R}$ (see for example [EN00, Chapter II Section 2]). Now for each $s \in (0,1)$, define the operator A_s by $D(A_s) = D(A)$ and $A_s f = (A - sI)f$ for $f \in D(A_s)$. Elementary differential equation theory shows that $\{A_s : D(A_s) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R}) : s \in (0,1)\}$ is a measurable field of closed operators according to Definition 6.2.1 (since $\sigma(A) = i\mathbb{R}$ so that $\sigma(A_s) = -s + i\mathbb{R}$, allowing for a common element of the resolvent sets to exist). Now each A_s generates the re-scaled semigroup $T_s^{(\cdot)} = e^{-s\cdot}T^{(\cdot)}$ on $L^2(\mathbb{R})$ and the family $\{T_s^{(\cdot)} : \mathbb{R}_+ \to \mathcal{B}(L^2(\mathbb{R})) : s \in (0,1)\}$ is uniformly bounded by 1 as $T^{(\cdot)}$ is bounded by 1. Futhermore for every $t > 0$,

$$s \mapsto \langle e^{-st}T^t x_s, y_s\rangle_s = e^{-st}\int_{\mathbb{R}} x_s(r+t)\overline{y_s(r)}\, dr$$

is measurable for all $x, y \in \mathcal{F}$.

Thus by Theorem 6.3.1, the direct integral $B = \int_\Omega^\oplus A_s \, d\mu(s)$ generates a C_0-semigroup $S^{(\cdot)}$ given by $S^t = \int_\Omega^\oplus T_s^t \, d\mu(s)$ for all $t \geq 0$. Of course, one can also follow the same computation that is done in general terms in the proof of Theorem 6.3.1 to show this directly. In this simple example, B can be thought of as encoding the range of a continuous pertubation of the differentiation operator. Note that it is essential that Ω be bounded below to ensure the re-scaled semigroups are uniformly exponentially bounded.

The second example is the case where the family of closed operators $\{A_s : D(A_s) \subset H_s \to H_s : s \in \Omega\}$ is a bounded measurable field of operators. In this case, exponentiation yields a family of uniformly bounded norm-continuous C_0-semigroups $\{\exp(\cdot A_s) : \mathbb{R}_+ \to \mathcal{B}(H_s) : s \in \Omega\}$ (in fact, C_0-groups). From this point, we can follow two separate routes to obtain a decomposable C_0-semigroup on the direct integral space. The first is, of course, Theorem 6.3.1. The second, more interesting method, is to apply functional calculus. We define the direct integral $A = \int_\Omega^\oplus A_s \, d\mu(s)$ which is also a bounded operator. From here, we can apply the Riesz-Dunford functional calculus to the exponential function $\exp(t\cdot)$ for each $t \geq 0$. This is clearly holomorphic on an open neighbourhood containing $\sigma(A_s)$ for a.e. $s \in \Omega$. Using Gilfeather's functional calculus result [Gil73, Theorem 1], we see that

$$\exp(tA) = \int_\Omega^\oplus \exp(tA_s) \, d\mu(s).$$

We end this section by mentioning the setting where μ is the counting measure. Here, the theory of direct integrals reduces to that of direct sums which is dealt with in [LM16], where the theory is developed for Banach spaces more generally and not restricted to Hilbert spaces. In particular, an example of a stochastic particle system [LM16, Section 5] is used to show that the direct sum equivalent of Theorem 6.3.1, [LM16, Theorem 4.3], automatically upgrades 'separately weak' solutions to 'strong solutions', resulting in an extra degree of differentiability for certian initial value problems.

It remains an interesting avenue of exploration to see if there are physical applications of Theorem 6.3.1 that require more than the scenario where the individual fibres of the Hilbert space bundle are equal to a fixed Hilbert space. It may be that more applications can be found, perhaps in the flavour of the stochastic example in [LM16] if our theory can be extended to the Banach space setting. In particular, the motivating link between C_0-semigroups and Cauchy problems lends itself to the hope that useful applications remain waiting to be found. Note that direct integrals of Banach spaces have been studied under the rubric of randomly normed spaces in [HLR91].

6.6 Asymptotics of direct integral semigroups

We motivate this final section by recalling Theorem 3.1.17 (Maniar–Nafiri) concerning the stability of a uniformly bounded countable sequence of C_0-semigroups on Hilbert spaces. The assumption (3.1.7) is missing from the statement of [MN16, Theorem 3.2] as found in [MN16]. It is likely that the authors of [MN16] implicitly assume this condition in [MN16, Theorem 3.2] without explicitly stating it. The introduction, preparatory work in Section 2, and application in Theorem 4.4 of their paper indicate that they are, in fact, at least assuming uniform boundedness of the resolvents at 0. However, their statement of [MN16, Proposition 3.6] seems to indicate otherwise. In any case, we both believe that the authors of [MN16] merely forgot to add (3.1.7) as an assumption in [MN16, Theorem 3.2] and use the following example[3] to show that the theorem fails if it is not assumed. We drop the subscripts for the norms as the context provides enough clarity (all operators involved are bounded multiplication operators).

Example 6.6.1. Let $r_0 \in (-1, 1)$ and take $A_n = ir_0 - 1/n$ to be the multiplication operator acting on $H_n = \mathbb{C}$ for all $n \in \mathbb{N}$. Then A_n generates the multiplication semigroup $T_n(t) = e^{(ir_0 - 1/n)t}$ on H_n for all $n \in \mathbb{N}$ which is uniformly bounded by 1. Moreover, $ir - A_n = i(r - r_0) + 1/n$ has bounded inverse given by multiplication by $(i(r - r_0) + 1/n)^{-1}$ for all $r \in \mathbb{R}, n \in \mathbb{N}$ so that $i\mathbb{R} \subset \rho(A_n)$ for all $n \in \mathbb{N}$. Fix $\alpha > 0$. Condition (i) of Theorem 3.1.17 is satisfied since

$$|r|^{-\alpha}\|R(ir, A_n)\| = |r|^{-\alpha}|i(r - r_0) + 1/n|^{-1} \leq \frac{|r|^{-\alpha}}{|r - r_0|}, \quad n \in \mathbb{N}, |r| \geq 1.$$

However condition (ii) of Theorem 3.1.17 fails since for $t = n$,

$$\|n^{1/\alpha}T_n(n)A_n^{-1}\|^2 = |n^{1/\alpha}e^{ir_0n-1}(ir_0 - 1/n)^{-1}|^2 = \frac{n^{2/\alpha}e^{-2}}{r_0^2 + 1/n^2} \to \infty, \quad n \to \infty,$$

so that $\sup_{t\geq 0, n\in\mathbb{N}} \|t^{1/\alpha}T_n(t)A_n^{-1}\| = \infty$.

As

$$\|R(ir_0, A_n)\| = n \to \infty, \quad n \to \infty,$$

[3] Suggested to me by my examiners during my viva voce. See Acknowledgements to Lassi Paunonen.

(3.1.7) is violated at $r = r_0$. Since this construction can be done for any $r_0 \in (-1, 1)$, we see that (3.1.7) cannot be omitted.

We now prove the following theorem which generalises Theorem 3.1.17 using a much simpler argument than the one found in [MN16], while still using Theorem 3.1.8 (Borichev–Tomilov) as in [MN16].

Theorem 6.6.2. *Let $\{T^{(\cdot)}(s) : \mathbb{R}_+ \to \mathcal{B}(H_s) : s \in \Omega\}$ be a collection of C_0-semigroups with generators $A(s)$ such that for each $t \geq 0$, the direct integral of $\{T^t(s) : s \in \Omega\}$ exists as a bounded operator on $H = \int_\Omega^\oplus H_s \, d\mu(s)$ and*

$$\operatorname*{ess-sup}_{s\in\Omega} \sup_{t\geq 0} \|T^t(s)\|_{\mathcal{B}(H_s)} < \infty.$$

Further assume that $i\mathbb{R} \subset \rho(A(s))$ for a.e. $s \in \Omega$ and

$$\operatorname*{ess-sup}_{s\in\Omega} \|R(ir, A(s))\|_{\mathcal{B}(H_s)} < \infty, \quad r \in \mathbb{R}.$$

Then the following are equivalent:

(i) $\operatorname*{ess-sup}_{s\in\Omega} \sup_{|r|\geq 1} |r|^{-\alpha} \|R(ir, A(s))\|_{\mathcal{B}(H_s)} < \infty.$

(ii) $\operatorname*{ess-sup}_{s\in\Omega} \sup_{t\geq 0} \|t^{1/\alpha} T^t(s) A(s)^{-1}\|_{\mathcal{B}(H_s)} < \infty.$

Proof. There exists $M > 0$ such that $\|T^t(s)\|_{\mathcal{B}(H_s)} \leq M$ for all $t \geq 0$ and a.e. $s \in \Omega$. Thus,

$$T(t) = \int_\Omega^\oplus T^t(s) \, d\mu(s)$$

is a C_0-semigroup on H bounded by M with generator

$$A = \int_\Omega^\oplus A(s) \, d\mu(s)$$

by Theorem 6.3.1. Moreover, $i\mathbb{R} \subset \rho(A)$ by Corollary 6.2.6(i).

Assuming (i), we see by Proposition 6.2.3 (and again by Corollary 6.2.6(i)) that

$$\sup_{|r|\geq 1} |r|^{-\alpha} \|R(ir, A)\| < \infty. \tag{6.6.1}$$

Thus by Theorem 3.1.8,

$$\sup_{t\geq 0} \|t^{1/\alpha} T(t) A^{-1}\| < \infty. \tag{6.6.2}$$

Since

$$T(t)A^{-1} = \int_\Omega^\oplus T^t(s) A(s)^{-1} \, d\mu(s),$$

Proposition 6.2.3 then implies (ii).

Now assuming (ii), Proposition 6.2.3 similarly implies (6.6.2), which in turn implies (6.6.1) by Theorem 3.1.8 once again. Since

$$R(ir, A) = \int_{\Omega}^{\oplus} R(ir, A(s))\, d\mu(s), \quad r \in \mathbb{R},$$

by Corollary 6.2.6(i), (i) follows once again from Proposition 6.2.3. □

Thus, Theorem 3.1.17 follows by taking the measure space underlying the direct integral space to be $\mathbb{N}$ endowed with the counting measure. In fact, one only has to take direct sums in order to prove Theorem 3.1.17 without needing to use direct integral theory.

The method of passing to a direct integral (or sum) in order to obtain the relation between uniform resolvent bounds and 'uniformly strong' stability can be applied to other quantified Tauberian and Katznelson–Tzafriri theorems such as those found in Chapter 3.

Part III

Discrete operator semigroups

Chapter 7

Optimal rates of decay for discrete operator semigroups on Hilbert spaces

The goal of this chapter[1] is to do for discrete operator semigroup asymptotics what Theorem 3.1.14 (Rozendaal–Seifert–Stahn) did for its continuous counterpart, namely, give a fitting end to the discrete quantified asymptotics story.

7.1 The m^{-1}-theorem

In this section we adapt and combine arguments from [RSS19] and [Sei15] in order to prove a so-called 'm^{-1}-type' Katznelson–Tzafriri theorem for power bounded operators on Hilbert spaces. The following result is a discrete analogue of Theorem 3.1.14 and of the same flavour as Theorem 3.2.3 (Seifert).

Theorem 7.1.1. *Let X be a (complex) Hilbert space and let $T \in \mathcal{B}(X)$ be a power-bounded operator such that $\sigma(T) \cap \mathbb{T} = \{1\}$. Suppose that $m : (0, \pi] \to (0, \infty)$ is a continuous non-increasing function of reciprocally positive increase such that*

$$\|R(e^{i\theta}, T)\| \leq m(|\theta|), \quad 0 < |\theta| \leq \pi.$$

Then

$$\|T^n(I - T)\| = O\left(m^{-1}(n)\right), \quad t \to \infty,$$

where m^{-1} is the minimal right-inverse of m.

Before we prove the above theorem, we need the following three technical lemmas. The first two are fundamentally just applications of summation and integration by

[1] The material of this chapter and the next is based on [NS20b].

parts respectively while the third is a simple calculation using the language of the first two lemmas.

Lemma 7.1.2. *Let $\phi = (\phi_k)_{k\in\mathbb{Z}}$ be a sequence in $\mathbb{R}_+$ and $(\eta_k)_{k\in\mathbb{Z}}$ be a sequence in X such that $\eta_k = 0$ for $k \le 0$. Assuming all the following series converge absolutely, define $\Phi_l : \mathbb{Z} \to \mathbb{C}$ for $l \in \mathbb{N}$ inductively by*

$$\Phi_1(k) = \phi_k, \quad \Phi_{l+1}(k) = \begin{cases} \displaystyle\sum_{p=-\infty}^{k} \Phi_l(p), & k < 0, \\ \displaystyle -\sum_{p=k+1}^{\infty} \Phi_l(p), & k \ge 0. \end{cases}$$

If we write $\langle \Phi_l \rangle = \sum_{-\infty}^{\infty} \Phi_l(p)$ for all $l \in \mathbb{N}$, then

$$(\eta * \Phi_l)(j) = \sum_{k\ge 0} [\eta_k - \eta_{k-1}]\Phi_{l+1}(j-k) + \eta_j \langle \Phi_l \rangle, \quad l \in \mathbb{N}, j \in \mathbb{Z}.$$

Proof. First note that $\Phi_{l+1}(k) - \Phi_{l+1}(k-1) = \Phi_l(k)$ when $k \ne 0$ and

$$\Phi_{l+1}(0) - \Phi_{l+1}(-1) = -\sum_{p=1}^{\infty} \Phi_l(p) - \sum_{p=-\infty}^{-1} \Phi_l(p) = \Phi_l(0) - \langle \Phi_l \rangle.$$

Thus,

$$\begin{aligned}
(\eta * \Phi_l)(j) &= \sum_{k\ge 0} \eta_k \Phi_l(j-k) \\
&= \sum_{k\ge 0, k\ne j} \eta_k [\Phi_{l+1}(j-k) - \Phi_{l+1}(j-k-1)] + \eta_j \Phi_l(0) \\
&= \sum_{k\ge 1, k\ne j} \eta_k \Phi_{l+1}(j-k) - \sum_{k\ge 1, k\ne j+1} \eta_{k-1} \Phi_{l+1}(j-k) + \eta_j \Phi_l(0) \\
&= \sum_{k\ge 0} [\eta_k - \eta_{k-1}]\Phi_{l+1}(j-k) + \eta_j \Phi_l(0) - \eta_j \Phi_{l+1}(0) + \eta_j \Phi_{l+1}(-1) \\
&= \sum_{k\ge 0} [\eta_k - \eta_{k-1}]\Phi_{l+1}(j-k) + \eta_j \left(\Phi_l(0) + \sum_{p=1}^{\infty} \Phi_l(0) + \sum_{p=-\infty}^{-1} \Phi_l(p) \right) \\
&= \sum_{k\ge 0} [\eta_k - \eta_{k-1}]\Phi_{l+1}(j-k) + \eta_j \langle \Phi_l \rangle.
\end{aligned}$$

□

Lemma 7.1.3. *Let $\varphi : \mathbb{R} \to \mathbb{R}_+$ be a smooth function such that $\|\varphi\|_\infty \le 1$ and $\varphi(\theta) = 0$ for $|\theta| \ge 2$. Then for each $\alpha \in \mathbb{Z}_+$, there exists a constant C_α (only depending on α) such that*

$$\left| \int_{-\pi}^{\pi} e^{ik\theta}(1 - e^{-i\theta})\varphi(\theta/r)\, d\theta \right| \le C_\alpha \frac{r^2}{(r|k|)^\alpha}, \quad k \in \mathbb{Z} \setminus \{0\}, 0 < r \le 1.$$

Proof. For all $k \in \mathbb{Z} \setminus \{0\}$, an application of integration by parts repeated α times leads to

$$\int_{-\pi}^{\pi} e^{ik\theta}(1-e^{-i\theta})\varphi(\theta/r)\,d\theta = \left(\frac{-1}{ik}\right)^{\alpha} \int_{-\pi}^{\pi} e^{ik\theta} \frac{d^\alpha}{d\theta^\alpha}\left[(1-e^{-i\theta})\varphi(\theta/r)\right]\,d\theta$$

as the boundary terms disappear due to the fact that $\varphi^{(\alpha)}(\pm\pi/r) = 0$ for all $0 < r \leq 1$. Furthermore,

$$\frac{d^\alpha}{d\theta^\alpha}\left[(1-e^{-i\theta})\varphi(\theta/r)\right] = -\sum_{p=0}^{\alpha-1}\binom{\alpha}{p}(-i)^{\alpha-p}e^{-i\theta}\frac{1}{r^p}\varphi^{(p)}(\theta/r) + (1-e^{-i\theta})\frac{1}{r^\alpha}\varphi^{(\alpha)}(\theta/r)$$

so that

$$\begin{aligned}\left|\int_{-\pi}^{\pi} e^{ik\theta}(1-e^{-i\theta})\varphi(\theta/r)\,d\theta\right| &\leq \frac{1}{|k|^\alpha}\int_{-\pi}^{\pi}\left(\sum_{p=0}^{\alpha-1}\binom{\alpha}{p}\frac{1}{r^p}|\varphi^{(p)}(\theta/r)|\right.\\ &\qquad\left. + |1-e^{-i\theta}|\frac{1}{r^\alpha}|\varphi^{(\alpha)}(\theta/r)|\right)d\theta\\ &\leq \frac{C}{|k|^\alpha}\int_{-2r}^{2r}\left(\sum_{p=0}^{\alpha-1}\binom{\alpha}{p}\frac{1}{r^p} + \frac{|\theta|}{r^\alpha}\right)d\theta\\ &\leq \frac{C_\alpha}{|k|^\alpha}\frac{r^2}{r^\alpha}\end{aligned}$$

for C_α chosen big enough to absorb the binomial coefficients dependent on α. The second inequality immediately above holds because $\varphi^{(\alpha)}(\theta/r) = 0$ for all $|\theta| \geq 2r$ and $|1-e^{-i\theta}| = 2|\sin(\theta/2)| \leq |\theta|$ for $|\theta| \leq \pi$. □

Lemma 7.1.4. *Let C_α be positive constants for each $\alpha \in \mathbb{Z}_+$. There exist constants B_l for each $l \in \mathbb{N}$ (depending only on the C_α and l) such that the following holds: If $(\phi_k)_{k\in\mathbb{Z}}$ is a sequence in $\mathbb{R}_+$ such that there exists $r \in (0,1]$ with $0 \leq \phi_0 \leq C_0 r^2$ and*

$$0 \leq \phi_k \leq C_\alpha \frac{r^2}{(r|k|)^\alpha}, \quad k \in \mathbb{Z}\setminus\{0\}, \alpha \in \mathbb{Z}_+,$$

and for each $l \in \mathbb{N}$, $\Phi_l : \mathbb{Z} \to \mathbb{C}$ is as in Lemma 7.1.2, then

$$\|\Phi_l\|_{\ell^1(\mathbb{Z})} \leq B_l r^{2-l}, \quad l \in \mathbb{N}.$$

Proof. We prove this by induction. Consider the case when $l = 1$. Then there exists constants C_0 and C_2 such that

$$|\Phi_1(k)| = \phi_k \leq \min\left(C_0 r^2, C_2 \frac{1}{|k|^2}\right).$$

Let $Q_1 = \{k \in \mathbb{Z} : |k| \le 1/r\}$ and $Q_2 = \{k \in \mathbb{Z} : |k| > 1/r\}$. Then

$$\|\Phi_1\|_{\ell^1(\mathbb{Z})} \le C_0 \sum_{k \in Q_1} r^2 + C_2 \sum_{k \in Q_2} \frac{1}{|k|^2} \le B_1 r, \quad k \in \mathbb{Z} \setminus \{0\},$$

by the integral test. Now suppose that $\|\Phi_l\|_{\ell^1(\mathbb{Z})} \le B_l r^{2-l}$ for some constant B_l. In particular, $|\Phi_{l+1}(k)| \le B_l r^{2-l}$ for all $k \in \mathbb{Z}$. Note that there also exists a constant C_{l+2} such that

$$|\Phi_1(k)| \le C_{(l+1)+1} r^{1-(l+1)} \frac{1}{|k|^{(l+1)+1}}, \quad k \in \mathbb{Z} \setminus \{0\}.$$

Hence by l applications of the integral test,

$$|\Phi_{l+1}(k)| \le C'_{l+2} r^{-l} \frac{1}{|k|^2}, \quad k \in \mathbb{Z} \setminus \{0\},$$

for some adjusted constant C'_{l+2}. Taking Q_1 and Q_2 as before,

$$\|\Phi_{l+1}\|_{\ell^1(\mathbb{Z})} \le B_l r^{-l} \sum_{k \in Q_1} r^2 + C'_{l+2} r^{-l} \sum_{k \in Q_2} \frac{1}{|k|^2} \le B_{l+1} r^{1-l} = B_{l+1} r^{2-(l+1)},$$

completing the proof. □

Proof of Theorem 7.1.1. We begin by fixing $x \in X$ and $n \in \mathbb{N}$ and construct the following sequence of functions $P_w : \mathbb{Z} \to \mathbb{Z}_+$ for $w \in \mathbb{Z}_+$ inductively:

$$P_0(j) = \chi_{\{1,2,\dots,n\}}(j), \quad j \in \mathbb{Z}, \quad P_{w+1}(j) = \begin{cases} \sum_{k=0}^{j} P_w(k), & j \ge 0, \\ 0, & j < 0. \end{cases}$$

If instead we had chosen $F_0 = \chi_{\mathbb{N}}$, and inductively defined F_w as above, then $F_w(j)$ would be the well-known multiset coefficients or figurate numbers. In fact, for all $w \in \mathbb{N}$ and $1 \le j \le n$,

$$P_w(j) = F_w(j) = \binom{w+j-1}{w}.$$

The first identification follows from the fact that the definition of $F_w(j)$ depends only on the values of $F_l(k)$ for $0 \le l \le w$ and $0 \le k \le j$, which coincide with $P_l(k)$ over the same range since $F_0(k) = P_0(k)$ for $0 \le k \le n$. The second equality is the famous closed form for figurate numbers, the higher order generalisations of triangular and tetrahedral numbers (see for example [Dic66, Chapter 1]). We will use the explicit form of these numbers later.

By definition of P_w and the fact that $P_w(n) \ge 1$ for all $n, w \in \mathbb{N}$, we can then rewrite $T^n(T-I)x$ in an analogous way to [RSS19, Equation 3.5] as follows:

$$T^n(T-I)x = \frac{1}{P_{w+1}(n)} \sum_{j=0}^{n} T^{n-j} P_w(j) T^j (T-I)x = \frac{1}{P_{w+1}(n)} \sum_{j=1}^{n} T^{n-j} h_w(j) \quad (7.1.1)$$

where the map $h_w : \mathbb{Z} \to X$ is given by $h_w(j) = P_w(j)T^j(T-I)x$ for $j > 0$ and 0 for $j \leq 0$. Note that we are allowed to start the index in the second summation from $j = 1$ because $P_w(0) = 0$.

Following now in the footsteps taken to prove [Sei15, Theorem 2.1], let $\psi : [-\pi, \pi] \to \mathbb{R}$ be a smooth function such that $\psi(\theta) = 0$ for $|\theta| \leq 1, 0 \leq \psi(\theta) \leq 1$ for $1 \leq |\theta| \leq 2$ and $\psi(\theta) = 1$ for $2 \leq |\theta| \leq \pi$ and for $r \in (0, 1]$, let $\psi_r : [-\pi, \pi] \to \mathbb{R}$ be given by $\psi_r(\theta) = \psi(\theta/r)$ for $\theta \in [-\pi, \pi]$.

Let the discrete time Fourier transform $\mathcal{F} : \ell^2(\mathbb{Z}, X) \to L^2([-\pi, \pi], X)$ be defined by

$$\mathcal{F}y = \sum_{k=-\infty}^{\infty} y_k e_{-k}, \quad y = (y_j)_{j\in\mathbb{Z}} \in \ell^2(\mathbb{Z}, X), \tag{7.1.2}$$

where for each $k \in \mathbb{Z}$, $e_k : [-\pi, \pi] \to \mathbb{C}$ is given by $e_k(\theta) = e^{ik\theta}$ for $\theta \in [-\pi, \pi]$, and the sum converges in $L^2([-\pi, \pi], X)$. Let its inverse $\mathcal{F}^{-1} : L^2([-\pi, \pi], X) \to \ell^2(\mathbb{Z}, X)$ be given by $\mathcal{F}^{-1}f = (\mathcal{F}^{-1}f(j))_{j\in\mathbb{Z}}$ for $f \in L^2([-\pi, \pi], X)$ where

$$(\mathcal{F}^{-1}f)(j) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e_j(\theta)f(\theta)\, d\theta, \quad j \in \mathbb{Z}.$$

Note that the sum in (7.1.2) also converges pointwise for every $\theta \in [-\pi, \pi]$ if the sequence is in $\ell^1(\mathbb{Z}, X)$.

Define the sequences $y^r, z^r : \mathbb{Z} \to \mathbb{C}$ by $y^r = \mathcal{F}^{-1}\psi_r$ and $z^r = \mathcal{F}^{-1}(1 - \psi_r)$. Explicitly then,

$$y^r(j) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ij\theta}\psi_r(\theta)\, d\theta, \quad z^r(j) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ij\theta}(1 - \psi_r(\theta))\, d\theta, \quad j \in \mathbb{Z}.$$

Since ψ_r is smooth, repeated applications of integration by parts, in the same way as in the proof of Lemma 7.1.3, show that y^r and z^r are rapidly decaying. In particular, $(|j|^\alpha y^r(j))_{j\in\mathbb{Z}}$ and $(|j|^\alpha z^r(j))_{j\in\mathbb{Z}}$ are in $\ell^1(\mathbb{Z})$ for any $\alpha \geq 1$. Hence, we can take well-defined convolutions with h_w since $\|h_w(j)\| \lesssim P_w(j) \leq F_w(j) \lesssim (j + w)^w$ for all $j \geq 0$. Since $y^r(0) = 1 - z^r(0)$ and $y^r(k) = -z^r(k)$ for $k \neq 0$,

$$h_w = (h_w - h_w * y^r) + h_w * y^r = h_w * z^r + h_w * y^r$$

and substituting this into (7.1.1), our goal is to independently estimate

$$\frac{1}{P_{w+1}(n)}\sum_{j=1}^{n} T^{n-j}(h_w * z^r)(j) \quad \text{and} \quad \frac{1}{P_{w+1}(n)}\sum_{j=1}^{n} T^{n-j}(h_w * y^r)(j).$$

This method of splitting into so-called high-frequency and low-frequency components was previously harnessed in [BCT16], [CS16], and [RSS19] in the continuous case and [Sei15] in the discrete case.

In order to deal with $h_w * z^r$, let $H_w : \mathbb{Z} \to X$ be given by

$$H_w(j) = \begin{cases} \sum_{k=0}^{j} h_w(k), & j \geq 1, \\ 0, & j \leq 0, \end{cases}$$

so that by partial summation,

$$\begin{aligned} H_w(j) &= \sum_{k=0}^{j} P_w(k)(T^{k+1}x - T^k x) \\ &= P_w(j)T^{j+1}x - P_w(0)T^0 x - \sum_{k=1}^{j}(P_w(k) - P_w(k-1))T^k x \\ &= P_w(j)T^{j+1}x - \sum_{k=1}^{j} P_{w-1}(k)T^k x. \end{aligned}$$

In particular,

$$\|H_w(j)\| \leq KP_w(j)\|x\| + K\sum_{k=1}^{j} P_{w-1}(k)\|x\| = 2KP_w(j)\|x\|$$

where $K = \sup\{\|T^k\| : k \in \mathbb{Z}_+\} < \infty$. The rapidly decaying nature of z^r guarantees that the subsequent series converge. As

$$(h_w * z^r)(j) = \sum_{k\geq 0} H_w(k)(z^r(j-k) - z^r(j-k-1)), \quad j \in \mathbb{Z},$$

it follows that

$$\|(h_w * z^r)(j)\| \leq 2K\|x\| \sum_{k\geq 0} P_w(k)\phi_{j-k}, \quad j \in \mathbb{Z},$$

where $(\phi_k)_{k\in\mathbb{Z}}$ is the non-negative sequence given by $\phi_k = |z^r(k) - z^r(k-1)|$ for $k \in \mathbb{Z}$. Note that $(\phi_k)_{k\in\mathbb{Z}}$ depends on r even though the constants in the estimates we later develop will be independent of r. Applying Lemma 7.1.2 to $(\eta_k)_{k\in\mathbb{Z}} = (P_w(k))_{k\in\mathbb{Z}}$ and $\phi = (\phi_k)_{k\in\mathbb{Z}}$ with $\Phi_l : \mathbb{Z} \to \mathbb{C}$ for $l \in \mathbb{N}$ defined as in the lemma, we have that

$$\begin{aligned} \|(h_w * z^r)(j)\| &\lesssim \|x\||(P_w * \Phi_1)(j)| \\ &= \|x\| \left| \sum_{k\geq 0} [P_w(k) - P_w(k-1)]\Phi_2(j-k) + P_w(j)\langle \Phi_1 \rangle \right| \\ &= \|x\| \left| (P_{w-1} * \Phi_2)(j) + P_w(j)\langle \Phi_1 \rangle \right|, \end{aligned}$$

and hence by recursively applying the argument behind the two equalities above,

$$\begin{aligned}\|(h_w * z^r)(j)\| &\lesssim \|x\| \left| \sum_{k\geq 0} P_0(k)\Phi_{w+1}(j-k) + \sum_{l=1}^{w} P_l(j)\langle \Phi_{w+1-l}\rangle \right| \\ &\leq \|x\| \left(\|\Phi_{w+1}\|_{\ell^1(\mathbb{Z})} + \left| \sum_{l=1}^{w} P_l(j)\langle \Phi_{w+1-l}\rangle \right| \right) \\ &\leq \|x\| \sum_{l=0}^{w} P_l(j) \|\Phi_{w+1-l}\|_{\ell^1(\mathbb{Z})}\end{aligned}$$

for all $j \geq 1$. Thus,

$$\begin{aligned}&\left\| \frac{1}{P_{w+1}(n)} \sum_{j=1}^{n} T^{n-j}(h_w * z^r)(j) \right\| \\ &\qquad \lesssim \frac{1}{P_{w+1}(n)} \|x\| \sum_{j=1}^{n} \sum_{l=0}^{w} P_l(j) \|\Phi_{w+1-l}\|_{\ell^1(\mathbb{Z})} \\ &\qquad = \frac{1}{P_{w+1}(n)} \|x\| \sum_{l=0}^{w} P_{l+1}(n) \|\Phi_{w+1-l}\|_{\ell^1(\mathbb{Z})} \\ &\qquad = \|x\| \sum_{l=0}^{w} \frac{(w+1)!}{n(n+1)\cdots(n+w)} \frac{n(n+1)\cdots(n+l)}{(l+1)!} \|\Phi_{w+1-l}\|_{\ell^1(\mathbb{Z})} \\ &\qquad \leq \|x\|(w+1)! \sum_{l=0}^{w} n^{l-w} \|\Phi_{w+1-l}\|_{\ell^1(\mathbb{Z})}.\end{aligned}$$

Applying Lemma 7.1.3 to $\varphi = 1 - \psi$ we get that for each $\alpha \in \mathbb{Z}_+$, there exists a constant C_α depending only on α such that

$$\phi_k = |z^r(k) - z^r(k-1)| = \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} e^{ik\theta}(1-e^{-i\theta})\varphi(\theta/r)\, d\theta \right| \leq C_\alpha \frac{r^2}{(r|k|)^\alpha}, \quad k \in \mathbb{Z}\backslash\{0\}.$$

By Lemma 7.1.4, for each $l \in \mathbb{N}$, we have that $\|\Phi_l\|_{\ell^1(\mathbb{Z})} \leq B_l r^{2-l}$ for constants $B_l > 0$ independent of r. It follows that for all $m \in \mathbb{N}$, there exists a constant $D_w > 0$ depending only on w, and in particular independent of n, r, and x, such that

$$\left\| \frac{1}{P_{w+1}(n)} \sum_{j=1}^{n} T^{n-j}(h_w * z^r)(j) \right\| \leq D_w r \|x\| \sum_{l=0}^{w} (nr)^{l-w}. \tag{7.1.3}$$

We now deal with the second convolution term in the frequency splitting. By Hölder's inequality,

$$\left\| \frac{1}{P_{w+1}(n)} \sum_{j=1}^{n} T^{n-j}(h_w * y^r)(j) \right\| \leq \frac{K n^{1/2}}{P_{w+1}(n)} \|h_w * y^r\|_{\ell^2(\mathbb{Z},X)} \tag{7.1.4}$$

so our goal is to estimate the ℓ^2-norm of $h_w * y^r$. The idea is to use the Fourier transform in order to recover ψ_r, introduced earlier in the proof. Thinking of ψ_r as a function on $\mathbb{T}$, we want to end up multiplying it with the resolvent operator in order to 'cut out' the spectrum at 1.

To this end, for $\rho > 1$, define $T_\rho : \mathbb{Z} \to \mathcal{B}(X)$ and $h_{w,\rho} : \mathbb{Z} \to X$ by $h_{w,\rho}(j) = \rho^{-j} h_w(j)$ for $j \in \mathbb{Z}$ and $T_\rho(j) = \rho^{-j} T^j$ for $j \geq 0$ and $T_\rho(j) = 0$ for $j < 0$. Observe that since

$$\begin{aligned} h_{w+1,\rho}(j) &= \rho^{-j} P_{w+1}(j) T^j (T-I)x \\ &= \rho^{-j} \sum_{k=0}^{j} P_w(k) T^j (T-I)x \\ &= \sum_{k=0}^{j} \rho^{-(j-k)} T^{j-k} \rho^{-k} P_w(k) T^k (T-I)x \\ &= \sum_{k=0}^{j} T_\rho(j-k) h_{w,\rho}(k) = (T_\rho * h_{w,\rho})(j) \end{aligned}$$

for all $j \in \mathbb{Z}_+$ and $w \in \mathbb{Z}_+$, we have $h_{w,\rho} = T_\rho^{*w} * h_{0,\rho}$ as $h_{w,\rho}$ and T_ρ are both zero on the negative integers. By the convolution theorem,

$$\mathcal{F}(h_{w,\rho})(\theta) = F_\rho(\theta)^w \mathcal{F}(h_{0,\rho})(\theta), \quad \theta \in [-\pi, \pi],$$

where $F_\rho : [-\pi, \pi] \to \mathcal{B}(X)$, the 'operator-valued Fourier transform of T_ρ', is given by $F_\rho(\theta) = \sum_{k=0}^{\infty} (\rho e^{i\theta})^{-k} T^k$. Notice that since the sequence of powers of the operator $(\rho e^{i\theta})^{-1} T$ decays exponentially, the corresponding Neumann series converges, so that

$$F_\rho(\theta) = (I - (\rho e^{i\theta})^{-1} T)^{-1} = \rho e^{i\theta} R(\rho e^{i\theta}, T), \quad \theta \in [-\pi, \pi].$$

Note also that

$$\begin{aligned} \mathcal{F}(h_{0,\rho})(\theta) &= \sum_{k=1}^{n} (\rho e^{i\theta})^{-k} (T-I) P_0(k) T^k x \\ &= (T-I) \sum_{k=1}^{n} (\rho e^{i\theta})^{-k} P_0(k) T^k x = (T-I) \mathcal{F}(g_\rho)(\theta) \end{aligned}$$

where $g_\rho : \mathbb{Z} \to X$ is given by $g_\rho(k) = \rho^{-k} P_0(k) T^k x$ for $k \in \mathbb{Z}$. Thus,

$$\mathcal{F}(h_{w,\rho}) = \mu_{w,\rho} \mathcal{F}(g_\rho)$$

where $\mu_{w,\rho} : [-\pi, \pi] \to \mathcal{B}(X)$ is given by $\mu_{w,\rho}(\theta) = (\rho e^{i\theta})^w R(\rho e^{i\theta}, T)^w (T - I)$.

Now $h_{w,\rho}(k) \to h_w(k)$ for each $k \in \mathbb{Z}$ as $\rho \to 1+$ straight from definition. Furthermore, if $h : \mathbb{Z} \to X$ is given by $h(k) = P_0(k)T^k x = \chi_{\{1,2,\ldots,n\}}(k)T^k x$ for $k \in \mathbb{Z}$, $\mathcal{F}(g_\rho)$ and $\mathcal{F}(h)$ are finite sums so that $\mathcal{F}(g_\rho)(\theta) \to \mathcal{F}(h)(\theta)$ as $\rho \to 1+$ for each $\theta \in [-\pi, \pi]$. By continuity we also have that $\mu_{w,\rho}(\theta) \to \mu_w(\theta)$ for each $\theta \in [-\pi, \pi]$ as $\rho \to 1+$ where $\mu_w : [-\pi, \pi] \setminus \{0\} \to \mathcal{B}(X)$ is given by $\mu_w(\theta) = e^{i\theta w} R(e^{i\theta}, T)^w (T - I)$ for $0 < |\theta| \leq \pi$. Thus, we can use the dominated convergence theorem twice (once in the series setting, once in the integral setting) as $\rho^{-k} \leq 1$ for all $\rho > 1$ and $k \in \mathbb{Z}_+$ along with the fact that $\psi_r(\theta) = 0$ for $|\theta| \leq r$ and $R(\rho e^{i\theta}, T)$ is uniformly bounded in ρ and θ for $r \leq |\theta| \leq \pi$ to see that for all $j \in \mathbb{Z}$,

$$\begin{aligned}
(h_w * y^r)(j) = \sum_{k\geq 0} h_w(k) y^r(j-k) &= \lim_{\rho\to 1+} \sum_{k \geq 0} h_{w,\rho}(k) y^r(j-k) \\
&= \lim_{\rho\to 1+} (h_{w,\rho} * y^r)(j) \\
&= \lim_{\rho\to 1+} \mathcal{F}^{-1}[\mathcal{F}(h_{w,\rho})\mathcal{F}(y^r)](j) \\
&= \lim_{\rho\to 1+} \mathcal{F}^{-1}[\mu_{w,\rho}\mathcal{F}(g_\rho)\psi_r](j) \\
&= \lim_{\rho\to 1+} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ij\theta} \mu_{w,\rho}(\theta) \mathcal{F}(g_\rho)(\theta)\psi_r(\theta)\, d\theta \\
&= \lim_{\rho\to 1+} \frac{1}{2\pi} \int_{r\leq|\theta|\leq\pi} e^{ij\theta} \mu_{w,\rho}(\theta) \mathcal{F}(g_\rho)(\theta)\psi_r(\theta)\, d\theta \\
&= \frac{1}{2\pi} \int_{r\leq|\theta|\leq\pi} e^{ij\theta} \mu_w(\theta)\mathcal{F}(h)(\theta)\psi_r(\theta)\, d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ij\theta} \mu_w(\theta)\mathcal{F}(h)(\theta)\psi_r(\theta)\, d\theta \\
&= \mathcal{F}^{-1}[\mu_w \psi_r \mathcal{F}(h)](j).
\end{aligned}$$

Hence

$$h_w * y^r = \mathcal{F}^{-1}[\mu_w \psi_r \mathcal{F}(h)]$$

and so by Parseval's theorem,

$$\|h_w * y^r\|_{\ell^2(\mathbb{Z},X)} \leq \|\mu_w \psi_r\|_{L^\infty([-\pi,\pi],\mathcal{B}(X))} \|h\|_{\ell^2(\mathbb{Z},X)}. \tag{7.1.5}$$

Clearly $\|h\|_{\ell^2(\mathbb{Z},X)} \leq K n^{1/2}\|x\|$ so it remains to estimate $\|\mu_w\psi_r\|_{L^\infty([-\pi,\pi],\mathcal{B}(X))}$.

For $0 < |\theta| \leq \pi$,

$$\begin{aligned}
\mu_w(\theta) &= -e^{i\theta w}[(1 - e^{i\theta}) + (e^{i\theta} - T)]R(e^{i\theta}, T)^w \\
&= -e^{i\theta w}[(1 - e^{i\theta})R(e^{i\theta}, T)^w + R(e^{i\theta}, T)^{w-1}]
\end{aligned}$$

so that

$$\begin{aligned}\|\mu_w(\theta)\| &\leq |1-e^{i\theta}|\|R(e^{i\theta},T)\|^w + \|R(e^{i\theta},T)\|^{w-1} \\ &\leq 2|1-e^{i\theta}|\|R(e^{i\theta},T)\|^w \leq 2|\theta| m(|\theta|)^w \end{aligned} \tag{7.1.6}$$

as $\|R(e^{i\theta},T)\| \geq \operatorname{dist}(e^{i\theta},\sigma(T))^{-1} \geq |1-e^{i\theta}|^{-1}$ for $e^{i\theta} \in \rho(T)$. Since m has reciprocally positive increase, there exists $\beta > 0, c \in (0,1]$, and $\theta_0 \leq \pi$ such that

$$\frac{m(\nu\theta)}{m(\theta)} \geq c\nu^{-\beta}, \quad 0 < \nu \leq 1, 0 < \theta \leq \theta_0. \tag{7.1.7}$$

In particular, picking $w = \lceil \beta^{-1} \rceil$ so that $w^{-1} \leq \beta$ and letting $\nu = r/|\theta|$ in (7.1.7) for $r \leq |\theta| \leq \theta_0$, we get from (7.1.6) that

$$\|\mu_w(\theta)\| \leq 2r\left(\frac{m(r)}{c}\right)^w, \quad r \leq |\theta| \leq \theta_0.$$

Now μ_w is bounded on the interval $[\theta_0,\pi]$ and ψ_r is always bounded by 1. Moreover, $\psi_r(\theta) = 0$ for $|\theta| \leq r$. Hence

$$\|\mu_w\psi_r\|_{L^\infty([-\pi,\pi],\mathcal{B}(X))} \lesssim \sup_{r\leq|\theta|\leq\theta_0} \|\mu_w(\theta)\| \lesssim r\left(\frac{m(r)}{c}\right)^w.$$

Substituting this into (7.1.5) and then into (7.1.4) gives

$$\begin{aligned}\left\|\frac{1}{P_{w+1}(n)}\sum_{j=1}^{n} T^{n-j}(h_w * y^r)(j)\right\| &\lesssim \frac{n^{1/2}n^{1/2}}{P_{w+1}(n)}\|x\| r\left(\frac{m(r)}{c}\right)^w \\ &\lesssim r\|x\|\left(\frac{m(r)}{cn}\right)^w,\end{aligned}$$

where the implicit constant is independent of n, r, and x as

$$P_{w+1}(n) = \binom{w+n}{w+1} = \frac{n(n+1)\cdots(n+w)}{(w+1)!} \geq \frac{n^{w+1}}{(w+1)!}.$$

Combining this with (7.1.3) in (7.1.1) gives

$$\|T^n(I-T)x\| \lesssim r\|x\|\left(\sum_{l=0}^{w}(nr)^{l-w} + \left(\frac{m(r)}{cn}\right)^w\right)$$

for our particular choice of w, where again, the implicit constant is independent of n, r, and x. In particular, if we choose $r = m^{-1}(n)$, then for n large enough so that $r^{-1} \geq \pi^{-1}$,

$$nr = rm(r) \geq r\|R(e^{ir},T)\| \geq 1$$

as before and $m(r)/(cn) = c^{-1}$. Since w is fixed and $\sum_{l=0}^{w} t^{l-w}$ is non-increasing in t,

$$\|T^n(I-T)x\| \lesssim m^{-1}(n)\|x\|,$$

for n sufficient large. The theorem follows. □

Remark 7.1.5. This proof can be greatly simplified for the case where the reciprocally positive increase function of interest has index greater or equal to 1, since then we can pick $w = 1$ in our proof and the multiple inductive arguments used would be unnecessary. Whether or not there exist examples of operators with resolvents that are closely estimated by reciprocally positive increase functions with index strictly less than 1 remains unclear.

It is natural to ask if Theorem 7.1.1 is practically useful or not, and whether or not there are operators with resolvents that are more sharply bounded by functions of reciprocally positive increase than by polynomials. We will explore a non-trivial worked example in Section 8.3. For now, suffice it to say that given a specific rate, one can often cook up a multiplication operator for which the range of the multiplier approaches the unit circle at such a rate as to yield an example of a normal operator with the desired resolvent estimate.

7.2 Optimality of the m^{-1}-theorem

In this short section, we prove two results. The first is a discrete analogue of [CPS+19, Proposition 5.4]. It says that provided the resolvent grows near the singularity at a fast enough rate, m^{-1} is the best possible rate. The second is a discrete analogue of [RSS19, Theorems 3.4, 3.7]. It shows that the class of reciprocally positive increase functions is the largest class of functions for which one can reasonably expect an m^{-1}-theorem. These two results combine to show that Theorem 7.1.1 is optimal in both rate of decay and class of function.

Proposition 7.2.1. *Let X be a (complex) Banach space, let $T \in \mathcal{B}(X)$ be a power-bounded operator such that $\sigma(T) \cap \mathbb{T} = \{1\}$, and suppose that*

$$\liminf_{\theta \to 0} \max\{\|\theta R(e^{i\theta}, T)\|, \|\theta R(e^{-i\theta}, T)\|\} > \sup\{\|T^n\| : n \geq 0\}.$$

If $m : (0, \pi] \to (0, \infty)$ is any continuous non-increasing function with $m(\theta) \to \infty$ as $\theta \to 0$ such that

$$\limsup_{|\theta| \to 0} \frac{\|R(e^{i\theta}, T)\|}{m(|\theta|)} > 0,$$

then there exists $c > 0$ such that

$$\limsup_{n \to \infty} \frac{\|T^n(I - T)\|}{m^{-1}(cn)} > 0,$$

where m^{-1} is the minimal right-inverse of m.

Proof. First let m_0 be the minimal dominating function for the resolvent of T as defined in Section 3.2 and let m_0^{-1} be its minimal right-inverse. Then there exist a constant $d > 0$ and a sequence θ_n going to 0 such that

$$m_0(|\theta_n|) > dm(|\theta_n|) = m_0(m_0^{-1} dm(|\theta_n|)), \quad n \geq 0.$$

In particular, as m_0 is non-increasing and m^{-1} is minimal,

$$m^{-1}(m(|\theta_n|)) \leq |\theta_n| < m_0^{-1}(dm(|\theta_n|)).$$

By Proposition 3.2.4, there exist constants $c_0, C > 0$ and $n_0 \geq 0$ such that

$$\|T^n(I-T)\| \geq c_0 m_0^{-1}(C'n), \quad n \geq n_0, C' \geq C.$$

Since $|\theta_n| \to 0$ as $n \to \infty$, we have that $m(|\theta_n|) \to \infty$ as $n \to \infty$, so that for n large enough, there exists $p_n \in \mathbb{N}$ such that

$$dm(|\theta_n|)) \in [Cp_n, C(p_n+1)).$$

In particular, there exists $C_n \in [C, 2C]$ such that $dm(|\theta_n|)) = C_n p_n$. Note that as C_n is bounded from above by $2C$, $p_n \to \infty$ as $n \to \infty$. Thus for n large enough so that $p_n \geq n_0$,

$$\begin{aligned}
\|T^{p_n}(I-T)\| \geq c_0 m_0^{-1}(C_n p_n) &= c_0 m_0^{-1}(dm(|\theta_n|)) \\
&> c_0 m^{-1}(m(|\theta_n|)) \\
&= c_0 m^{-1}\left(\frac{C_n}{d} p_n\right) \geq c_0 m^{-1}\left(\frac{2C}{d} p_n\right).
\end{aligned}$$

Taking $c = (2C)/d$ proves the proposition. □

Theorem 7.2.2. *Let X be a Banach space and let $T \in \mathcal{B}(X)$ be such that $\sigma(T) \cap \mathbb{T} = \{1\}$. Suppose that $m : (0, \pi] \to (0, \infty)$ is a continuous non-increasing function such that*

$$\delta m(\theta) \leq \sup_{\theta \leq |\varphi| \leq \pi} \frac{1}{\operatorname{dist}(e^{i\varphi}, \sigma(T))} \leq \sup_{\theta \leq |\varphi| \leq \pi} \|R(e^{i\varphi}, T)\| \leq m(\theta), \quad 0 < \theta \leq \pi, \quad (7.2.1)$$

for some $\delta \in (0, 1]$. If m^{-1} is the minimal right-inverse of m and there exist constants $C, c > 0$ such that

$$\|T^n(I-T)\| \leq Cm^{-1}(cn), \quad n \in \mathbb{N}, \quad (7.2.2)$$

then m has reciprocally positive increase.

In the following proof, we follow closely the ideas found in [Sei16, Proposition 2.8], the discrete result analogous to [BCT16, Proposition 6.13].

Proof. Note first, because m^{-1} is non-increasing, (7.2.2) holds for c replaced with d and the same constant C for any positive $d < c$. In particular, we will use this fact for $d \in (c/2, c]$. From (7.2.1), the spectral mapping theorem for polynomials, and the fact that the spectral radius of a bounded linear operator is always at most the norm of the operator, we see that

$$|\lambda^n(1-\lambda)| \le \|T^n(I-T)\| \le Cm^{-1}(dn),$$

so that,

$$n \log \frac{1}{|\lambda|} \ge \log \frac{|1-\lambda|}{Cm^{-1}(dn)}$$

for all $\lambda \in \sigma(T) \setminus \{0, 1\}, d \in (c/2, c]$, and $n \in \mathbb{N}$. The function $g : (0, 1) \to \mathbb{R}$ defined by $g(s) = (s-1)/\log s$ is a continuous increasing function satisfying $g(s) \to 1$ as $s \to 1$. Thus, there exists $s_0 \in (0, 1)$ such that

$$dg(r) \ge \frac{c}{2} g(s_0) > b =: \frac{c}{4}$$

whenever $r \in (s_0, 1)$ and $d \in (c/2, c]$.

Let θ be such that $m(\theta) = dn$ for some $n \in \mathbb{N}$ and $d \in (c/2, c]$. It follows that

$$m(\theta) \ge \frac{d}{\log \frac{1}{|\lambda|}} \log \frac{|1-\lambda|}{Cm^{-1}(dn)}, \quad \lambda \in \sigma(T) \setminus \{0, 1\}.$$

Suppose now that $\gamma \in (0, r_0)$ where $r_0 \le \delta(1-s_0)$ is small enough so that $|1-e^{i\varphi}| \ge \frac{5}{6}\varphi$ for all $\varphi \in (0, r_0)$. By (7.2.1), there exist $\lambda \in \sigma(T)$ and $\varphi \in [-\pi, \pi]$ such that $\gamma \le |\varphi|$ and

$$\delta m(\gamma) \le |\lambda - e^{i\varphi}|^{-1}.$$

Let $r = |\lambda|$. Since $m(\gamma) \ge \gamma^{-1}$ and $1-r \le |\lambda - e^{i\varphi}|$ by the reverse triangle inequality, it follows that

$$1 - r \le (\delta m(\gamma))^{-1} \le \delta^{-1}\gamma < \delta^{-1}\delta(1-s_0) = 1 - s_0,$$

in particular, $r > s_0 > 0$, so that $dg(r) > b$. In particular, for γ small, $\lambda \neq 0$. Thus, if $|1-\lambda| \ge \gamma/2$, in which case $\lambda \neq 1$,

$$\begin{aligned} \frac{m(\theta)}{m(\gamma)} \ge \frac{d\delta|\lambda - e^{i\varphi}|}{\log \frac{1}{|\lambda|}} \log \frac{|1-\lambda|}{Cm^{-1}(dn)} &\ge \frac{d\delta(1-r)}{\log(1/r)} \log \frac{\gamma}{2C\theta} \\ &= \delta d g(r) \log \frac{\gamma}{2C\theta} > \delta b \log \frac{\gamma}{2C\theta}. \end{aligned}$$

On the other hand, if $|1-\lambda| < \gamma/2$, then $|\lambda - e^{i\varphi}| \geq \gamma/3$ as

$$|\lambda - e^{i\varphi}| \geq |1-e^{i\varphi}| - |1-\lambda| > |1-e^{i\varphi}| - \frac{\gamma}{2} \geq \frac{5|\varphi|}{6} - \frac{\gamma}{2} \geq \frac{\gamma}{3}.$$

Hence,

$$\frac{m(\theta)}{m(\gamma)} \geq m(\theta)\delta|\lambda - e^{i\varphi}| \geq \frac{\delta m(\theta)\gamma}{3} \geq \frac{\delta\gamma}{3\theta} \geq \delta b \log \frac{\gamma}{3b\theta}.$$

Thus, if we take $\mu = \max\{1, 3b, 2C\} \exp\left(\frac{2}{\delta b}\right)$, it follows that

$$\frac{m(\theta)}{m(\mu\theta)} \geq \delta b \log\left(\exp\left(\frac{2}{\delta b}\right)\right) = 2$$

for all θ for which $m(\theta) = dn$ for some $d \in (c/2, c]$ and $n \in \mathbb{N}$.

In particular, if θ is small enough, $m(\theta) \in (c(p-1), cp]$ for some integer $p > 1$. Let $d = m(\theta)/p$ so that

$$\frac{c}{2} \leq \frac{c(p-1)}{p} < d \leq c$$

and $m(\theta) = pd$. Hence

$$\frac{m(\theta)}{m(\mu\theta)} \geq 2$$

for all θ small enough, so that

$$\liminf_{\theta\to 0} \frac{m(\nu\theta)}{m(\theta)} \geq 2 > 1,$$

where $\nu = \mu^{-1}$. Since $\nu < 1$, Proposition 2.4.1 implies that m has reciprocally positive increase. $\square$

Remark 7.2.3. If T is a normal operator on a Hilbert space, or indeed any quasi-multiplication operator on a Banach space (see Chapter 8), then m_0, the minimal dominating function for the resolvent of T automatically satisfies (7.2.1) for $\delta = 1$. Thus, for such operators, an m_0^{-1}-rate of decay for $\|T^n(I-T)\|$ implies that m_0 has reciprocally positive increase, showing that the class of reciprocally positive increase functions is the optimal class.

Chapter 8

Rates of decay for quasi-multiplication operators

This chapter aims to provide analogous results to [RSS19, Section 4] to obtain rates for operators without reciprocally positive increase resolvent bounds but which nonetheless have other special properties. We then turn to a class of operators that have these properties before finishing with a worked example where Theorem 7.1.1 (and indeed Theorem 8.1.1 below) yield better results than Theorem 3.2.3.

8.1 The $m_{\max}$-theorem

In this section, we introduce the new concept of quasi-multiplication operators, for which we can prove additional results related to the m^{-1}-theorem. These are the discrete analogues of the so-called quasi-multiplicative semigroups (see [BCT16, Section 5]).

An operator T on a Banach space X is a *quasi-multiplication operator* or *quasi-multiplicative* if

$$\|f(T)\| = \sup_{\lambda\in\sigma(T)} |f(\lambda)|$$

for all rational functions $f = p/q$ with p, q being polynomials and q having no roots in $\sigma(T)$. Since q has no roots in $\sigma(T)$, we can factorise it into $q = (\mu_1 - z)\cdots(\mu_p - z)$ with $\mu_1, \ldots, \mu_p \in \rho(T)$ so that $q(T)^{-1}$ can be defined as the product of the corresponding resolvents.

Note that all normal operators on Hilbert spaces are quasi-multiplication operators. Hence the following discrete analogue of [RSS19, Theorem 4.4] is true, in particular, for normal operators on Hilbert spaces.

Theorem 8.1.1. *Let X be a Banach space and $T \in \mathcal{B}(X)$ be quasi-multiplicative such that $1 \in \sigma(T) \subset \mathbb{D} \cup \{1\}$. Define the functions $m, m_{\max} : (0, \pi] \to (0, \infty)$ by*

$$m(\theta) = \sup_{\theta \le |\varphi| \le \pi} \frac{1}{\operatorname{dist}(e^{i\varphi}, \sigma(T))}, \quad 0 < \theta \le \pi,$$

and

$$m_{\max}(\theta) = \max_{\theta \le \varphi \le \pi} m(\varphi) \log\left(\frac{\varphi}{\theta}\right), \quad 0 < \theta \le \pi,$$

respectively. Let $m_{\max}^{-1}$ be the minimal right inverse of $m_{\max}$.

(i) Suppose that there exists $\varepsilon \in (0, 1]$ and $\theta_0 \in (0, \pi]$ such that

$$m(\theta) \ge \frac{1}{\varepsilon\theta}, \quad \theta \le \theta_0. \tag{8.1.1}$$

Then there exists $n_0 \in \mathbb{N}$ such that

$$\|T^n(I - T)\| \le (1 + \varepsilon) m_{\max}^{-1}(n), \quad n \ge n_0.$$

(ii) Suppose that (8.1.1) holds for some $\varepsilon \in (0, 1)$ and $\theta_0 \in (0, \pi]$. Then for any constants $b \in (\varepsilon, 1)$ and $C > 1$, there exists $n_0 \in \mathbb{N}$ such that

$$\|T^n(I - T)\| \ge (b - \varepsilon) m_{\max}^{-1}(Cn), \quad n \ge n_0.$$

Proof. (i) We have that

$$\|T^n(I - T)\| = \sup_{\lambda \in \sigma(T)} |p_n(\lambda)|, \quad n \ge 0,$$

where $p_n(\lambda) = \lambda^n(1 - \lambda)$. Define the domains

$$\Theta = \left\{re^{i\varphi} : 0 \le r \le 1, |\varphi| \le \pi, r \ge \max\{0, 1 - \varepsilon|\varphi|\}\right\},$$
$$\Omega = \left\{re^{i\varphi} : 0 \le r \le 1, |\varphi| \le \pi, r \le \max\{0, 1 - \varepsilon|\varphi|\}\right\}.$$

Note that $\Theta \cup \Omega = \overline{\mathbb{D}}$.

For $re^{i\varphi} \in \sigma(T) \setminus \{1\}$,

$$m(|\varphi|) \ge \frac{1}{|e^{i\varphi} - re^{i\varphi}|} = \frac{1}{1 - r},$$

so that

$$|p_n(re^{i\varphi})| \le |1 - re^{i\varphi}| \left(1 - \frac{1}{m(|\varphi|)}\right)^n \le (1 - r + |\varphi|) \exp\left(-\frac{n}{m(|\varphi|)}\right).$$

Since $1-r \le \varepsilon|\varphi|$ for $re^{i\varphi} \in \sigma(T) \cap \Theta \setminus \{0\}$, we get for such $re^{i\varphi}$ that

$$|p_n(re^{i\varphi})| \le (1+\varepsilon)|\varphi| \exp\left(-\frac{n}{m(|\varphi|)}\right).$$

Let $n \ge 1$ and $\theta = m_{\max}^{-1}(n)$. If $|\varphi| \le \theta$, then

$$|p_n(re^{i\varphi})| \le (1+\varepsilon)\theta = (1+\varepsilon)m_{\max}^{-1}(n).$$

If instead, $\theta \le |\varphi| \le \pi$, then

$$n = m_{\max}(\theta) \ge m(|\varphi|) \log\left(\frac{|\varphi|}{\theta}\right),$$

and hence

$$|\varphi| \exp\left(-\frac{n}{m(|\varphi|)}\right) \le \theta.$$

So once again, $|p_n(re^{i\varphi})| \le (1+\varepsilon)m_{\max}^{-1}(n)$. As $p_n(\lambda) = 0$ for the case $\lambda = 0$,

$$\sup_{\lambda \in \sigma(T) \cap \Theta} |p_n(\lambda)| \le (1+\varepsilon)m_{\max}^{-1}(n), \quad n \ge 1.$$

Moving on, the maximum modulus principle implies that

$$\sup_{\lambda \in \sigma(T) \cap \Omega} |p_n(\lambda)| \le \sup_{\lambda \in \partial\Omega} |p_n(\lambda)|.$$

Now $r = 1 - \varepsilon|\varphi|$ for $re^{i\varphi} \in \partial\Omega \setminus \{0\}$, so once again, as $p_n(\lambda) = 0$ in the case where $\lambda = 0$,

$$\sup_{\lambda \in \sigma(T) \cap \Omega} |p_n(\lambda)| \le \sup_{0 \le \varphi \le \pi} (1+\varepsilon)\varphi(1-\varepsilon\varphi)^n = \sup_{0 \le \varphi \le \pi} (1+\varepsilon^{-1})\varepsilon\varphi(1-\varepsilon\varphi)^n.$$

The function $\varphi \mapsto \varphi(1-\varphi)^n$ has its maximum at $\varphi = (n+1)^{-1}$, so

$$\begin{aligned}\sup_{\lambda \in \sigma(T) \cap \Omega} |p_n(\lambda)| &\le (1+\varepsilon^{-1})\frac{1}{n+1}\left(1 - \frac{1}{n+1}\right)^n \\ &= (1+\varepsilon^{-1})\frac{1}{n+1}\left(1 - \frac{1}{n+1}\right)^{n+1}\frac{n+1}{n} \le \frac{1+\varepsilon^{-1}}{en}.\end{aligned}$$

Now

$$m_{\max}(\varphi) \ge m(e\varphi) \ge \frac{1}{\varepsilon e\varphi}, \quad 0 < \varphi \le \frac{\theta_0}{e},$$

so for $n \ge n_0$ where $n_0 \ge m_{\max}(\pi/e)$ and $m_{\max}^{-1}(n_0) \le \theta_0$,

$$\begin{aligned}m_{\max}^{-1}(n) &= \inf\{\varphi \in (0, \pi/e] : m_{\max}(\varphi) \le n\} \\ &\ge \inf\{\varphi \in (0, \pi/e] : (\varepsilon e\varphi)^{-1} \le n\} = \frac{1}{\varepsilon en}.\end{aligned}$$

Thus,

$$\sup_{\lambda\in\sigma(T)\cap\Omega} |p_n(\lambda)| \le \varepsilon(1+\varepsilon^{-1})m_{\max}^{-1}(n), \quad n \ge n_0.$$

The result follows.

(ii) Define $\omega : \mathbb{Z}_+ \to \mathbb{R}_+$ by $\omega(n) = \|T^n(I-T)\|$, $n \in \mathbb{Z}_+$. Let ω_* be given by

$$\omega_*(s) = \min\{p \in \mathbb{Z}_+ : \omega(p) \le s\}, \quad s > 0,$$

so that $\omega(\omega_*(s)) \le s$ and let ω^* be given by

$$\omega^*(s) = \max\{p \in \mathbb{Z}_+ : \omega(p) \ge s\}, \quad 0 < s \le \omega(0),$$

so that $\omega(\omega^*(s)) \ge s$. Note that $\omega(\omega_*(s)-1)) > s$ for $\omega_*(s) \ge 1$ so that $\omega_*(s) - 1 \le \omega^*(s)$.

Given $\varphi \in (0,\pi]$, let $\lambda \in \sigma(T)$ and ψ with $|\psi| \ge \varphi$ be such that

$$m(\varphi) = |e^{i\psi} - \lambda|^{-1}.$$

If $r = |\lambda|$, then $m(\varphi)^{-1} \ge 1 - r$. Since $m(\varphi) \ge \varphi^{-1}$, we see that $r \to 1-$ as $\varphi \to 0+$. In particular, for any $b \in (\varepsilon, 1)$ and $c \in (0,1)$, there exists $\theta_0 > 0$ such that whenever $0 < \varphi \le \theta_0$, we have $\lambda \ne 0$, $1 - r \ge c\log r^{-1}$, and $|e^{i\varphi} - 1| \ge b\varphi$. Furthermore our assumption about (8.1.1) implies that θ_0 can be chosen such that

$$|\lambda - e^{i\psi}| = m(\varphi)^{-1} \le \varepsilon\varphi \tag{8.1.2}$$

so that

$$|1-\lambda| \ge |e^{i\psi} - 1| - |\lambda - e^{i\psi}| \ge |e^{i\varphi} - 1| - \varepsilon\varphi \ge (b-\varepsilon)\varphi$$

whenever $\varphi \le \theta_0$. Note that (8.1.2) also implies that we can choose θ_0 small enough to force $\lambda \ne 1$ since $\varepsilon < 1$ is fixed and $|1 - e^{i\psi}| \ge |1 - e^{i\varphi}|$. As T is quasi-multiplicative,

$$\omega(n) \ge |\lambda^n(1-\lambda)|, \quad \lambda \in \sigma(T), n \ge 0,$$

so that

$$n \log\frac{1}{|\lambda|} \ge \log\left(\frac{|1-\lambda|}{\omega(n)}\right), \quad \lambda \in \sigma(T) \setminus \{0,1\}, n \ge 0.$$

Thus

$$m(\varphi)^{-1} \ge 1 - r \ge c\log r^{-1} \ge \frac{c}{n}\log\left(\frac{|1-\lambda|}{\omega(n)}\right) \ge \frac{c}{n}\log\left(\frac{(b-\varepsilon)\varphi}{\omega(n)}\right), \quad \varphi \le \theta_0.$$

Now fix $\theta \le \theta_0$ and take $n = \omega_*((b-\varepsilon)\theta)$. Then,

$$m(\varphi)^{-1} \ge \frac{c}{\omega_*((b-\varepsilon)\theta)}\log\left(\frac{\varphi}{\theta}\right), \quad \theta \le \varphi \le \theta_0,$$

so that

$$c^{-1}(\omega^*((b-\varepsilon)\theta)+1) \geq c^{-1}\omega_*((b-\varepsilon)\theta) \geq m(\varphi)\log\left(\frac{\varphi}{\theta}\right), \quad \theta \leq \varphi \leq \theta_0.$$

Hence, as $w^*(s) \to \infty$ as $s \to \infty$, for any $d < c$, we can go back and pick θ_0 even smaller such that

$$d^{-1}\omega^*((b-\varepsilon)\theta) \geq c^{-1}(\omega^*((b-\varepsilon)\theta)+1), \quad \theta \leq \theta_0.$$

In particular, for any $d < 1$, we can pick retrospectively pick $c \in (d, 1)$ to obtain this θ_0. Thus,

$$\omega^*((b-\varepsilon)\theta) \geq dm(\varphi)\log\left(\frac{\varphi}{\theta}\right), \quad \theta \leq \varphi \leq \theta_0,$$

so that for θ small,

$$\omega^*((b-\varepsilon)\theta) \geq d \max_{\theta \leq \varphi \leq \theta_0} m(\varphi)\log\left(\frac{\varphi}{\theta}\right) = dm_{\max}(\theta). \tag{8.1.3}$$

To see why this last inequality holds assume $\theta_0 \leq \psi \leq \pi$ and note that $m(\theta) \to \infty$ as $\theta \to 0$. Then if θ is small enough such that $\sqrt{\theta} \leq \theta_0$, $m(\sqrt{\theta}) \geq 4m(\theta_0)$, and $\log(1/\theta) \geq \log \pi$, we have that

$$\begin{aligned} m(\psi)\log\left(\frac{\psi}{\theta}\right) &\leq m(\theta_0)\log\left(\frac{\pi}{\theta}\right) \leq \frac{1}{2}m(\sqrt{\theta})\log\left(\frac{1}{\theta}\right) \\ &= m(\sqrt{\theta})\log\left(\frac{\sqrt{\theta}}{\theta}\right) \leq \max_{\theta \leq \varphi \leq \theta_0} m(\varphi)\log\left(\frac{\varphi}{\theta}\right). \end{aligned}$$

Now choose n_0 large enough so that for all $n \geq n_0$, $\theta = m_{\max}^{-1}(n/d)$ is small enough that (8.1.3) applies. Thus,

$$\omega^*((b-\varepsilon)\theta) \geq dm_{\max}(\theta) = n, \quad n \geq n_0. \tag{8.1.4}$$

Notice that ω is decreasing, since if $\lambda_0 \in \sigma(T)$ is such that

$$\omega(n+1) = \|T^{n+1}(I-T)\| = \sup_{\lambda \in \sigma(T)} |\lambda^{n+1}(1-\lambda)| = |\lambda_0^{n+1}(1-\lambda_0)|,$$

that is, $\lambda = \lambda_0$ is the point where $|\lambda^{n+1}(1-\lambda)|$ attains its maximum on the compact set $\sigma(T)$, then

$$\omega(n) \geq |\lambda_0^n(1-\lambda_0)| \geq |\lambda_0^{n+1}(1-\lambda_0)| = \omega(n+1),$$

as $|\lambda_0| \leq 1$. Thus applying ω to both sides of (8.1.4), we get that

$$(b-\varepsilon)m_{\max}^{-1}(n/d) = (b-\varepsilon)\theta \leq \omega(\omega^*(b-\varepsilon)\theta) \leq \omega(n) = \|T^n(I-T)\|, \quad n \geq n_0.$$

For any $C > 1$, take $d = C^{-1}$ and we are done. $\square$

Remark 8.1.2. Note that (8.1.1) always holds for $\varepsilon = 1$ and often holds for other $\varepsilon > 0$. As expected, the more information in the form of (8.1.1) we have about m, the more precise we can be with the estimates on $\|T^n(I-T)\|$. In particular, if (8.1.1) holds for all $\varepsilon > 0$ (for example, if m grows quadratically), then

$$\frac{\|T^n(I-T)\|}{m_{\max}^{-1}(n)} = 1 + o(1), \quad n \to \infty.$$

We turn now to a comparison of the three different rates for m, $m_{\log}$, and $m_{\max}$.

Proposition 8.1.3. *Let $m : (0,\pi] \to (0,\infty)$ be non-increasing. Then for $\theta \in (0,\pi]$*

(i) $m_{\log}(\theta) \geq m_{\max}(\theta)$ for θ small so that $m_{\log}^{-1}(s) \geq m_{\max}^{-1}(s)$ for s large;

(ii) $m_{\max}(\theta) \geq c^{-1}m(e^{1/c}\theta)$ for θ small so that $m_{\max}^{-1}(s) \geq e^{-1/c}m^{-1}(cs)$ for s large and $c > 0$; and

(iii) $m_{\max}(\theta) \leq (ce\alpha)^{-1}m(\theta)$ for θ small so that $m_{\max}^{-1}(s) \geq m^{-1}(ce\alpha s)$ for s large if m has reciprocally positive increase with index $\alpha > 0$ and constant $c > 0$.

Proof. (i) Clearly

$$m_{\max}(\theta) \leq m(\theta)\log\left(\frac{\pi}{\theta}\right) \leq m(\theta)\log\left(1 + \frac{m(\theta)}{\theta}\right) = m_{\log}(\theta)$$

for θ such that $m(\theta) \geq \pi$.
(ii) Taking $\psi = e^c\theta$ for θ small in the definition of $m_{\max}$, we see that

$$m_{\max}(\theta) \geq m(e^{1/c}\theta)\log(e^{1/c}) = c^{-1}m(e^{1/c}\theta).$$

(iii) By assumption, there exists $\theta_0 > 0$ such that

$$\frac{m(\mu\varphi)}{m(\varphi)} \geq c\nu^{-\alpha}, \quad 0 < \nu \leq 1, 0 < \varphi \leq \theta_0.$$

In particular, whenever $\theta \leq \varphi \leq \theta_0$, we can take $\mu = \theta/\varphi$ and use the basic inequality $t \geq e\log t$ for $t > 0$ to see that

$$m(\theta) \geq c\left(\frac{\theta}{\varphi}\right)^{-\alpha} m(\varphi) \geq ce\log\left(\frac{\theta}{\varphi}\right)^{-\alpha} m(\varphi) = (ce\alpha)m(\varphi)\log\left(\frac{\varphi}{\theta}\right).$$

As in the proof of Theorem 8.1.1(ii), for θ small,

$$m_{\max}(\theta) = \max_{\theta\leq\varphi\leq\theta_0} m(\varphi)\log\left(\frac{\varphi}{\theta}\right).$$

Hence for θ small, $m(\theta) \geq (ce\alpha)m_{\max}(\theta)$. ☐

We have shown that $m_{\max}$ is always 'slower' than m, except that when m has reciprocally positive increase, the rates are essentially the same. However, the $m_{\max}^{-1}$ rate is always at least as quickly decaying as the $m_{\log}^{-1}$ rate. Furthermore, Theorem 8.1.1 provides an upper bound for $\|T^n(I-T)\|$ with no constant C inside the argument of $m_{\max}^{-1}$. This is reminiscent of reciprocally positive increase functions. Whether we can remove the inner constant C for the lower bound is unclear, though the singularity of the resolvent at $\lambda = 1$ may mean that having a constant on the inside of the lower bound is unavoidable.

8.2 Analytic Toeplitz operators

In this section, we turn to a practical class of quasi-multiplication operators – analytic Toeplitz operators. This is followed by a particular example where we have a reciprocally positive increase (in fact, reciprocally regularly varying) estimate on the resolvent, providing an application of Theorem 7.1.1 that is not covered by Theorem 3.2.5 and where the operator is non-normal.

The *Hardy space* $H_+^2(\mathbb{T})$ is defined by

$$H_+^2(\mathbb{T}) = \{f \in L^2(\mathbb{T}) : \hat{f}(n) = 0,\ n < 0\},$$

where $\{\hat{f}(n)\}_{n\in\mathbb{Z}}$ is the (two-sided) sequence of Fourier coefficients of f given by

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta}\, d\theta.$$

The Hardy space inherits the Hilbert space structure from $L^2(\mathbb{T})$.

It is well known that every function $f \in L^2(\mathbb{T})$ can be represented in $L^2(\mathbb{T})$ as

$$f(z) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e_n(z), \quad z \in \mathbb{T},$$

where $e_n : \mathbb{T} \to \mathbb{T}$ is given by $e_n(z) = z^n$ for $n \in \mathbb{Z}$. Hence we can define the (orthogonal) projection

$$P_+ : L^2(\mathbb{T}) \to H_+^2(\mathbb{T}), \quad \sum_{n\in\mathbb{Z}} \hat{f}(n)e_n \mapsto \sum_{n\in\mathbb{Z}_+} \hat{f}(n)e_n.$$

For the *symbol* (function) $a \in L^\infty(\mathbb{T})$, the *Toeplitz operator* $T(a) : H_+^2(\mathbb{T}) \to H_+^2(\mathbb{T})$ is defined by $T(a)f = P_+(af)$. It is worth remarking that the Fourier transform maps $T(a)$ onto the so-called class of matrix Toeplitz operators on $\ell_2(\mathbb{Z}_+)$, the space of square-summable sequences with index in $\mathbb{Z}_+$. These are the infinite matrices of the type $\{a_{n-m}\}_{n,m\geq 0}$.

When it comes to the spectrum and resolvent of $T(a)$, things get real messy real quick if a is only in $L^\infty(\mathbb{T})$. However, things are rather neat when a is in the space of bounded analytic functions on the unit disk, denoted by $H^\infty(\mathbb{T})$. This is the case we will focus on – the so-called class of *analytic Toeplitz operators* (Toeplitz operators with analytic symbol). Though a function $a \in H^\infty(\mathbb{T})$ is not necessarily continuous on the closed unit disk, it is nonetheless true that $\|a\|_{L^\infty(\mathbb{D})} = \|a\|_{L^\infty(\mathbb{T})}$ (see for example [Par88, Theorem 2.6]).

The following proposition states some basic properties of analytic Toeplitz operators, the first two of which are also true for general L^∞ symbols (see [MAR07, Theorem 3.2.11, Theorem 3.3.8]).

Proposition 8.2.1. *Let $a, b \in H^\infty(\mathbb{T})$. Then the following hold:*

(i) $T(a)^* = T(\overline{a})$;

(ii) $\|T(a)\| = \|a\|_{L^\infty(\mathbb{T})}$;

(iii) $T(a)T(b) = T(ab)$;

(iv) $\sigma(T(a)) = \overline{a(\mathbb{D})}$.

Corollary 8.2.2. *Analytic Toeplitz operators are quasi-multiplicative.*

Proof. Let T be an analytic Toeplitz operator. From Proposition 8.2.1, we see that $T(f(a)) = f(T(a))$ for all f that are polynomials or functions of the form $f(z) = (\mu - z)^{-1}$ for $\mu \in \rho(T(a))$. In particular, for such functions f, we have $f(a)$ remaining in $H^\infty(\mathbb{T})$ so that

$$\|f(T(a))\| = \|f(a)\|_{L^\infty} = \sup_{z\in\mathbb{T}} |f(a(z))| = \sup_{z\in\mathbb{D}} |f(a(z))| = \sup_{\lambda\in\sigma(T)} |f(\lambda)|$$

where the third equality follows from the property of functions in $H^\infty(\mathbb{T})$ discussed immediately above the proposition and the final equality follows from continuity of f. $\square$

For more that might be said regarding specific classes of analytic Toeplitz operators in relation to the so-called subordinated semigroups and the work of Dungey, see Section 9.4.

8.3 A worked example

We now turn to a particular analytic Toeplitz operator, whose Fourier coefficients appear as a discrete probability distribution in [Dun11, Example 5.2]. The point of the following example is to find an operator T whose resolvent can be estimated sharply by a reciprocally regularly varying function. The theory in Chapter 7 then allows us to deduce a sharp rate of decay for $\|T^n(I-T)\|$.

Example 8.3.1. Let $\phi : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ be given by

$$\phi(z) = \sum_{k\geq 2} \frac{1}{k(k-1)} z^k, \quad z \in \overline{\mathbb{D}}.$$

As in Dungey's example, we can use the expansion $\log(1-z) = -\sum_{k\geq 1} k^{-1} z^k$ and some quick calculus to check that

$$1 - \phi(z) = 1 - \sum_{k\geq 2} \frac{1}{k(k-1)} z^k = (1-z)[1 - \log(1-z)], \quad z \in \overline{\mathbb{D}} \setminus \{1\},$$

where log is the logarithm with branch cut on the negative real axis. Letting $z \to 1$, we see that $\phi(1) = 1$, so that ϕ is bounded on the unit disk and hence is in $H^\infty(\mathbb{T})$ (Dungey used this to show that the Fourier coefficients of ϕ form a positive probability). Thus $T = T(\phi)$ is a quasi-multiplication operator and

$$\|R(\mu, T)\| = \sup_{z\in\mathbb{T}} |(\mu - \phi(z))^{-1}|, \quad \mu \in \rho(T).$$

In particular,

$$\|R(e^{i\varphi}, T)\| = \sup_{0\leq|\theta|\leq\pi} |e^{i\varphi} - \gamma(\theta)|^{-1}, \quad 0 < |\varphi| \leq \pi,$$

where $\gamma : [-\pi, \pi] \to \mathbb{C}$ is given by $\gamma(\theta) = e^{i\theta} + (1 - e^{i\theta}) \log(1 - e^{i\theta})$ for $\theta \neq 0$ and $\gamma(0) = 1$. If this were a physics paper, we would say that by Taylor expansions, we see that $\|R(e^{i\varphi}, T)\| \sim (2/\pi)|\log(|\varphi|)/\varphi|$. Unfortunately, this is a mathematics doctoral thesis, which means that any examiner or supervisor or mathematically trained stranger who stumbles across a copy should be able to rigorously understand it whilst sitting in an armchair by the fire sipping a cup of tea. We spell out the details below.

We begin by fixing φ small and (without loss of generality) positive and noting that $|e^{i\varphi} - \gamma(\theta)|$ achieves its minimum for some θ such that $|1 - \gamma(\theta)| \leq 4\varphi$. This can be seen by realising that

$$1 - e^{i\varphi} = 2\sin\left(\frac{\varphi}{2}\right) e^{i\left(\frac{\theta}{2} - \frac{\pi}{2}\right)},$$

so that $|1-e^{i\varphi}| \le \varphi$. Thus the point 1 lies in the circle centred at $e^{i\varphi}$ of radius 2φ. As $\gamma(0)=1$, the closest point on the curve γ cannot lie outside the aforementioned circle. Hence if θ_φ is the value at which the minimum occurs, the distance from 1 to $\gamma(\theta_\varphi)$ must be less than twice the radius of the circle, namely 4φ. Since

$$|1-\gamma(\theta_\gamma)| = \left|2\sin\left(\frac{\theta_\gamma}{2}\right)\right| |1-\log(1-e^{i\theta_\gamma})| \ge \left|\frac{\theta_\gamma}{2}\right| \left|1-\log\left(2\sin\frac{|\theta_\gamma|}{2}\right)\right|,$$

there exists a constant $\alpha \ge 1$ independent of φ such that $|\theta_\varphi| \le \alpha\varphi$. In fact, this means that if φ is small and positive, $e^{i\theta}$ is the term that dominates in $\gamma(\theta)$ so that θ_φ must be non-negative.

Next, Taylor expansion yields

$$\begin{aligned} e^{i\varphi}-\gamma(\theta) &= \cos\varphi + i\sin\varphi - \cos\theta - i\sin\theta \\ &\qquad + 2i\sin\left(\frac{\theta}{2}\right)e^{i\theta/2}\left[\log\left(2\sin\left(\frac{\theta}{2}\right)\right) + i\left(\frac{\theta}{2}-\frac{\pi}{2}\right)\right] \\ &= i\varphi + \frac{\pi}{2}\theta - i\theta(1-\log\theta) + \theta^2\beta_1(\varphi) + \varphi^2\beta_2(\theta), \end{aligned}$$

where β_1, β_2 is bounded for small φ, θ respectively, as

$$\lim_{\theta\to 0+} \log\left(\frac{\theta}{2\sin(\theta/2)}\right) = 1.$$

Thus,

$$\begin{aligned} \inf_{0\le|\theta|\le\pi} |e^{i\varphi}-\gamma(\theta)| &\ge \inf_{0\le\theta\le\alpha\varphi} \left|\frac{\pi}{2}\theta + i(\varphi - f(\theta))\right| - \sup_{0\le\theta\le\alpha\varphi} \left(\theta^2|\beta_1(\varphi)| + \varphi^2|\beta_2(\theta)|\right) \\ &\ge \inf_{0\le|\theta|\le\pi} \left(\frac{\pi^2}{4}\theta^2 + (\varphi - f(\theta))^2\right)^{1/2} - \alpha'\varphi^2 \end{aligned}$$

for some adjusted constant α' and where $f(\theta) = \theta(1-\log\theta)$. Using some basic calculus, $\frac{\pi^2}{4}\theta^2 + (\varphi - f(\theta))^2$ achieves its minimum at $\theta = \theta_0$ where θ_0 satisfies

$$\varphi = \frac{\pi^2}{4}\frac{\theta_0}{\log(1/\theta_0)} + f(\theta_0). \tag{8.3.1}$$

Let $F(\theta)$ be the function defined by the right hand side of the above equation. Then for any $\delta > 1$, we have that $\delta h(\theta) \ge F(\theta)$ for θ small enough, where $h(\theta) = \theta\log(1/\theta)$. Since both h and F are strictly increasing functions for small θ, it follows that both functions have right-inverses close to 0 and that on this domain of definition, $(\delta h)^{-1}(\varphi) \le F^{-1}(\varphi)$. Thus,

$$\inf_{0\le|\theta|\le\pi} |e^{i\varphi}-\gamma(\theta)| \ge \left(\frac{\pi^2}{4}(F^{-1}(\varphi))^2 + \left(\frac{\pi^2}{4}\frac{\theta_0}{\log(1/\theta_0)}\right)^2\right)^{1/2} \ge \frac{\pi}{2}(\delta h)^{-1}(\varphi).$$

Now if $\varphi = \delta\theta \log(1/\theta)$, then

$$\theta = \delta^{-1} \frac{\varphi}{\log(1/\theta)} = \delta^{-1} \frac{\varphi}{\log(1/\varphi) + \log(\log(1/\theta))}.$$

Note that $\varphi = \theta \log(1/\theta) < \sqrt{\theta}$ for small θ. So

$$\log(\log(1/\theta)) = \log(2\log(1/\sqrt{\theta})) < \log 2 + \log(\log(1/\varphi)),$$

and in particular,

$$\frac{\log(\log(1/\theta))}{\log(1/\varphi))} \to 0$$

as $\varphi \to 0$. Hence

$$\frac{\varphi}{\log(1/\varphi)} \sim \delta(\delta h)^{-1}(\varphi), \quad \varphi \to 0.$$

Putting it all together, we see that

$$\limsup_{\varphi \to 0} \frac{\|R(e^{i\varphi}, T)\|}{\varphi^{-1} \log(1/\varphi)} \leq \frac{2}{\pi}\delta. \tag{8.3.2}$$

Similarly,

$$\inf_{0 \leq |\theta| \leq \pi} |e^{i\varphi} - \gamma(\theta)| \leq \inf_{0 \leq \theta \leq \alpha\varphi} \left| \frac{\pi}{2}\theta + i(\varphi - f(\theta)) \right| + \sup_{0 \leq \theta \leq \alpha\varphi} \left(\theta^2 |\beta_1(\varphi)| + \varphi^2 |\beta_2(\theta)| \right).$$

Note that $\theta_0 \leq f(\theta_0) \leq \varphi \leq \alpha\varphi$ for θ_0 satisfying (8.3.1). Hence the minimum of $|(\pi/2)\theta + i(\varphi - f(\theta))|$ in $[0, \pi]$ is achieved in $[0, \alpha\varphi]$ so that for some adjusted constant $\alpha' \geq 1$ and any $\varepsilon > 0$,

$$\begin{aligned} \inf_{0 \leq |\theta| \leq \pi} |e^{i\varphi} - \gamma(\theta)| &\leq \left(\frac{\pi^2}{4}\theta_0^2 + \left(\frac{\pi^2}{4} \frac{\theta_0}{\log(1/\theta_0)} \right)^2 \right)^{1/2} + \alpha'\varphi \\ &\leq (1+\varepsilon)\frac{\pi}{2}\theta_0 + \alpha'\varphi \\ &\leq (1+\varepsilon)\frac{\pi}{2}F^{-1}(\varphi) + \alpha'\varphi \end{aligned}$$

for φ small. Since $F \geq h$ so that, for small φ, $F^{-1}(\varphi) \leq h^{-1}(\varphi)$, and arguing as before, we see that

$$\inf_{0 \leq |\theta| \leq \pi} |e^{i\varphi} - \gamma(\theta)| \leq (1 + 2\varepsilon)\frac{\pi}{2} \frac{\varphi}{\log(1/\varphi)}$$

for small enough φ. Thus,

$$\liminf_{\varphi \to 0} \frac{\|R(e^{i\varphi}, T)\|}{\varphi^{-1} \log(1/\varphi)} \geq \frac{2}{\pi}(1 + 2\varepsilon)^{-1}. \tag{8.3.3}$$

Since (8.3.2) holds for all $\delta > 1$ and (8.3.3) holds for all $\varepsilon > 0$, it follows that

$$\|R(e^{i\varphi}, T)\| \sim \frac{2}{\pi}\frac{|\log(1/\varphi)|}{|\varphi|}, \quad |\varphi| \to 0. \tag{8.3.4}$$

In [Dun11, Example 5.2], Dungey shows that T is not a Ritt operator via a clever geometric argument, but (8.3.4) shows that $\log(1/\varphi)/\varphi$ is the exact rate describing the resolvent growth, up to a constant.

Defining $m : (0, \pi] \to (0, \infty)$ by

$$m(\varphi) = \frac{2}{\pi}\frac{\log(1/\varphi)}{\varphi}, \quad 0 < \varphi \leq \pi,$$

we see that m is reciprocally regularly varying with strictly positive index, and so has reciprocally positive increase. From (8.3.4), there are constants $c, C > 0$ such that

$$cm(|\theta|) \leq m_0(|\theta|) \leq Cm(|\theta|), \quad 0 < |\theta| \leq \pi,$$

where m_0 is the minimal dominating function for the resolvent of T. Applying Theorem 7.1.1, we get that

$$\|T^n(I - T)\| = O(m^{-1}(n)), \quad n \to \infty.$$

Now Proposition 3.2.4 implies that there exist $d, D > 0$ such that

$$Dm_0^{-1}(dn) \leq \|T^n(I - T)\|$$

for n large enough. Since m is strictly decreasing, m^{-1} is a proper inverse. Hence $m_0^{-1}(s) \geq m^{-1}(s/c)$, implying by Proposition 2.4.2, that

$$\|T^n(I - T)\| \asymp m^{-1}(n), \quad n \to \infty.$$

As argued previously, if $s = \log(1/\varphi)/\varphi$, then

$$\varphi = \frac{\log(1/\varphi)}{s} = \frac{\log s - \log(\log(1/\varphi))}{s}.$$

Similar to before,

$$\log\left(\frac{1}{\log(1/\varphi)}\right) = \log(1/2) + \log\left(\frac{1}{\log(1/\sqrt{\varphi})}\right)$$

and $s \leq \varphi^{-1/2}$ so that

$$-\frac{\log\left(\log(1/\varphi)\right)}{s} \to 0$$

as $s \to \infty$. Hence if φ is given by $s = \log(1/\varphi)/\varphi$, then $\varphi \sim (\log s)/s,\ s \to \infty$. In particular, we get that

$$\|T^n(I-T)\| \asymp \frac{\log n}{n}, \quad n \to \infty,$$

the fruit of our long labours. Note that there is a convenient way of handling inverses of reciprocally regularly varying functions using the so-called de Bruijn conjugates (see [BCT16, Subsection 2.2] and [BGT89, Section 1.5.7]), but we have gone for the more explicit route in this simple case.

Now for $m = \log(1/\varphi)/\varphi$,

$$m_{\log}(\varphi) = \frac{\log(1/\varphi)}{\varphi} \log\left(1 + \frac{\log(1/\varphi)}{\varphi^2}\right) \geq \frac{\log(1/\varphi)^2}{\varphi}$$

for small φ. Applying the same argument for inverses as before, we see that there exists $c > 0$ such that

$$m_{\log}^{-1}(s) \geq c\frac{(\log s)^2}{s}$$

for large s, so that the rate given by Theorem 3.2.3 is one factor of $\log n$ slower than that given by Theorem 7.1.1.

Hence the analytic Toeplitz operator for our symbol ϕ falls in the setting where Theorem 7.1.1 gives a better result than Theorem 3.2.3 and Theorem 3.2.5 (in fact, the best result). Note also that we could have used Theorem 8.1.1 together with Proposition 8.1.3 since our operator is quasi-multiplicative.

Remark 8.3.2. Theorem 9.4.1 ([Dun11, Theorem 2.4]), discussed in Section 9.4 automatically shows, without going through resolvent estimates, that for the operator $T = T(\phi)$ in the above example, $\|T^n(I-T)\| = O(n^{-1/2})$ as $n \to \infty$. Nonetheless, the rate we get by Theorem 7.1.1 is finer and indeed the optimal one.

Part IV

Final remarks

Chapter 9

Future directions

In this final chapter, we comment on a few possible avenues of research coming out of the work in Parts II and III. Some starts were made, but no substantial results were established as of yet. Various observations and connections to other areas of mathematics unexplored in this thesis are also mentioned.

9.1 Coupled systems and homogenisation theory

Out of Chapter 4, we pose and comment on a few questions about possible future directions arising out of systems similar to that described by (4.0.2). The last of these questions could potentially be very interesting and not easily tractable.

Firstly, *does the rate of energy decay remain the same up to a constant when extra wave and heat parts are added, for example, in a wave-heat-wave-heat or a wave-heat-wave-heat-wave system?*

Secondly, *if the energy remains optimally bounded by Ct^{-4} for C dependent on the particular system, can we find an explicit N-formula for the multiplicative constant C_N bounding the energy of the system composed of N wave-heat pairs all coupled together?*

Our first reaction at the thought of answering this question is one of horror, as the methods used in Chapter 4 involved inverting a 4×4 matrix, and a system composed of N wave-heat pairs would require the inversion of a $(4N-2) \times (4N-2)$ matrix. Though we believe that the answer to the first question is yes, answering the second question would require some clever matrix tricks to avoid total carnage. It is notable, however, that the matrices involved would have 0 entries everywhere except on a diagonal band of width at most four. So perhaps it is doable.

The idea of, perhaps inductively, obtaining a formula for C_N as above leads to the question of homogenisation, that is, the computation of a limit equation. For $N \in \mathbb{N}$

and a given smooth function f, consider the following system of mixed hyperbolic and elliptic type that was studied in [Wau16]:

$$\begin{cases} \partial_t^2 u_N(\xi,t) - \partial_\xi^2 u_N(\xi,t) = \partial_t f(\xi,t), & \xi \in \bigcup\limits_{j\in\{1,\dots,N\}} \left(\dfrac{j-1}{N}, \dfrac{2j-1}{2N}\right), \ t \in \mathbb{R}, \\ u_N(\xi,t) - \partial_\xi^2 u_N(\xi,t) = f(\xi,t), & \xi \in \bigcup\limits_{j\in\{1,\dots,N\}} \left(\dfrac{2j-1}{2N}, \dfrac{j}{N}\right), \ t \in \mathbb{R}, \\ \partial_\xi u_N(0,t) = \partial_\xi u_N(1,t), & t \in \mathbb{R}, \end{cases} \tag{9.1.1}$$

subject to zero initial conditions and the requirement that the u_N and their derivatives are continuous. Here we have used the notation ∂ for derivatives as in [Wau16] to avoid creating a mess. Note that requiring conditions of continuity at the junction points results in the coupling considered in [ZZ04] rather than that of [BPS16] and (4.0.2).

Waurick showed in [Wau16] that as $N \to \infty$, the sequence of solutions $(u_N)_{N\in\mathbb{N}}$ converges weakly in $L^2_{loc}([0,1]\times\mathbb{R})$ to u, the solution to the limit equation

$$\frac{1}{2}\partial_t^2 u(\xi,t) + \partial_t u(\xi,t) + \frac{1}{2}u(\xi,t) - 2\partial_\xi^2 u(\xi,t) = f(\xi,t) + \partial_t f(\xi,t), \quad t \in \mathbb{R},$$

subject to zero initial conditions and Neumann boundary on both ends. Furthermore, he showed that this limit admitted exponentially stable solutions in the sense of [Tro13]. However, when the elliptic part, $u_N(\xi,t) - \partial_\xi^2 u_N(\xi,t) = f(\xi,t)$, is replaced with the corresponding parabolic part, $\partial_t u_N(\xi,t) - \partial_\xi^2 u_N(\xi,t) = f(\xi,t)$, the limit equation becomes

$$\frac{1}{2}\partial_t^2 u(\xi,t) + \partial_t u(\xi,t) - 2\partial_\xi^2 u(\xi,t) = f(\xi,t) + \partial_t f(\xi,t), \quad t \in \mathbb{R},$$

subject again to zero initial conditions and Neumann boundary. Crucially, this limit equation is not exponentially stable, raising the following question.

Third and finally, *for the homogenised limit equation with mixed hyperbolic and parabolic parts as above, can the limit solution be posed as an abstract Cauchy problem and solved by a semi-uniformly stable semigroup?*

In [FW18], resolvent estimates of some kind are calculated in a way that depends on N using the Gelfand transform, before a numerical analysis is conducted. How this might somehow be converted to a resolvent estimate for the limit problem itself remains an interesting unanswered question.

We end this section with a comment on a rather unexpected link between homogenisation and direct integral theory. The 'homogenisation of periodic structures via Bloch decomposition' was studied in [CV97], applying the Bloch–Floquet transform into the field of homogenisation. Resolvent convergence estimates for homogenisation of periodic fractional elasticity problems were then proved by Cherednichenko and Waurick in [CW18] via these Bloch decompositions into direct integrals (referred to in [CW18] as fibres) that arose from the Fourier–Laplace transform in time and the Gelfand transform in space applied to the original fractional elasticity problem. Building on this, Waurick then investigated so called fibre homogenisation with Cooper in [CW19]. Down a slightly different track, the topic of 'second order periodic differential operators ... and homogenisation' was studied much earlier by Birman and Suslina in [BS03], prominently featuring direct integral theory and mentioning Floquet–Bloch decompositions. Finally, the topological Bloch–Floquet transform was introduced by De Nittis and Panati in [DNP12] and shown to 'induce a direct integral decomposition of the algebra of observables'. This brief survey on homogenisation and Bloch–Floquet theory shows that direct integrals (or fibres), studied in the context of C_0-semigroups in Chapter 6 of this thesis, may become an area of renewed interest.

9.2 Quantum Markov semigroups

From a C_0-semigroup theoretic perspective, the results of Chapter 6 are already deeply interesting, answering natural fundamental questions and laying the groundwork for the aforementioned applications to specific Cauchy problems, quantified asymptotic theory, and rates of decay. However, our results may lead further to a rich area of application, yet to be explored.

Since every von Neumann algebra on a separable Hilbert space is a direct integral of factors [vN30], we hope that further progress can be made in the theory of quantum Markov semigroups (QMSs) on von Neumann algebras by building on our results for C_0-semigroups on direct integral spaces. QMSs, a natural generalisation of classical Markov semigroups that have interested mathematical physicists for almost fifty years, are a certain type of one-parameter operator semigroup on von Neumann algebras, originally arising out of the study of open quantum systems. Recently, there has been an interest in deriving results concerning QMSs and their generators inspired from classical strongly continuous semigroup theory (see [AZ15, AZ16] for example). The key difference between QMSs and C_0-semigroups is that QMSs act on what are essentially spaces of operators (subspaces of $\mathcal{B}(H)$, the space of bounded linear operators on a Hilbert space H) whereas C_0-semigroups act directly on a Banach space X, leading to the use of different topologies. Nonetheless, ideas from C_0-semigroup theory evidently can be useful in developing an understanding of QMSs. On the theory of QMSs and their generators, see [Dav79, Kra71, Lin76, Sti55] to name but a few milestone papers.

Thus, it seems reasonable to ask the following rather general and open-ended question. *Can the work in Chapter 6 be extended and applied to the study of QMSs?*

9.3 m_k-type theorems

Originally, it had been hoped that there would be an additional chapter to Part III, namely, to address the question of whether or not there is a discrete operator semigroup analogue of Theorem 3.1.15 (Stahn). In other words, *is it possible to introduce a separate function that dominates the resolvent from the function that determines the domain of holomorphicity around $\mathbb{T}\setminus\{1\}$ in order to have more control on the rate of decay of $\|T^n(T-I)\|$ in the form of an 'm_k-theorem'?*

This was approached via two methods. First, the contour integral method which was introduced into the area of quantified Tauberian theorems by Batty and Duyckaerts in [BD08] and used in [Sei16], and second, the Fourier method pioneered by Chill and Seifert in [CS16] and used in [Sei15].

The first of these methods procured the following half-result, unsatisfying because of its unoriginality and restrictiveness. The proof below is almost an exact replica of the proof of [Sei16, Theorem 2.11] with only minimal modifications. In many places, we have only slightly reworded the arguments made by Seifert, with no originality claimed for those parts, and completely adopted the original notation. The first half, in particular, remains completely unchanged.

Theorem 9.3.1. *Let X be a (complex) Banach space, and let $T \in \mathcal{B}(X)$ be a power-bounded operator such that $\sigma(T) \cap \mathbb{T} = \{1\}$. Furthermore, let $m, k : (0, \pi] \to [1, \infty)$ be non-increasing functions such that*

$$\frac{k(\theta)}{\theta \log\left(\frac{k(\theta)}{\theta} + 1\right)} \lesssim m(\theta)^2. \tag{9.3.1}$$

Suppose that $\sigma(T) \subset \Omega_m$ and

$$\|R(re^{i\theta}, T)\| \le k(|\theta|), \quad re^{i\theta} \in \mathbb{D} \setminus \Omega_m, \tag{9.3.2}$$

where

$$\Omega_m = \left\{ re^{i\theta} \in \mathbb{C} : 0 \le r < 1 - \frac{1}{m(|\theta|)}, 0 < |\theta| \le \pi \right\}.$$

Then for all $c \in (0,1)$,

$$\|T^n(I-T)\| = O(m_k^{-1}(cn)), \quad n \to \infty,$$

where m_k^{-1} is the inverse of the strictly decreasing function $m_k : (0,\pi] \to (0,\infty)$ defined by

$$m_k(\theta) = m(\theta)\log\left(1 + \frac{k(\theta)}{\theta}\right).$$

Proof. Fixing a constant $c \in (0,1)$, denote by Ω^c the closure of the set

$$\left\{re^{i\theta} \in \mathbb{C} : 0 \le r \le 1 - \frac{c}{m(|\theta|)}, 0 < |\theta| \le \pi\right\}.$$

Since $\sigma(T) \subset \Omega_m \subset \Omega^c$, we can define the holomorphic function $F_n : \mathbb{C} \setminus \Omega^c \to \mathcal{B}(X)$ by

$$F_n(\lambda) = T^n(2-T)(I-(\lambda-1)R(\lambda,T)) = T^n(2-T)^2R(\lambda,T)(I-R(2,T)).$$

The second equality above follows from the resolvent identity. Since $F_n(2) = T^n(I-T)$, it follows by Cauchy's integral formula that

$$T^n(I-T) = \frac{1}{2\pi}\oint_\Gamma \frac{h(\lambda)}{\lambda-2}F_n(\lambda)\,d\lambda, \tag{9.3.3}$$

where Γ is any contour outside Ω^c around the point 2 and h is a function to be exactly determined later that is holomorphic on a suitable region and satisfies $h(2) = 1$. What we now do is to take $\Gamma = \Gamma_{\text{out}} \cup \Gamma_{\text{in}}$, where Γ_{out} is an outer contour enclosing both the point 2 and the set Ω^c and Γ_{in} is an inner contour which lies in the interior of Γ_{out} and encloses Ω but not the point 2. Γ can be considered as a closed contour by inserting a cut from any point on Γ_{in} to any point on Γ_{out} and cancelling the contributions to the contour integral that occur from the opposite directions along this cut.

Let φ be the Cayley transform given by $\varphi(\lambda) = (1-\lambda)/(1+\lambda)$ and, for $r \in (0,1)$ and $R > 0$, define

$$\gamma_r = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1+r^2}{1-r^2}\right| = \frac{2r}{1-r^2}\right\}$$

and $\Gamma_R = \{\lambda \in \mathbb{C} : |\lambda - 1| = R\}$.

In case the reader is wondering where φ came from, it was chosen in such a way that it maps γ_r onto $r\mathbb{T}$, the real line onto itself, and the unit circle onto the imaginary axis. Moving on, suppose that $r \in (0,1/3)$ and $R > 2$, and let $\Gamma_{\text{out}} = \Gamma_R$ and $\Gamma_{\text{in}} = C_r \cup \gamma_r^+$, where γ_r^+ is the part of γ_r that lies outside the unit disc and C_r is a path in $\mathbb{D} \setminus \Omega^c$ connecting the end points of γ_r^+ to be suitably chosen later. Before determining C_r, we first choose h in (9.3.3) as the map h_r given by

$$h_r(\lambda) = \frac{1}{1+9r^2}\left(1 + \frac{r^2}{\varphi(\lambda)^2}\right).$$

Thus, h_r is holomorphic away from 1. Let $M = \sup\{\|T^n\| : n \geq 0\}$. Since

$$R(\lambda, T) = \lambda^{-1}(1-\lambda^{-1}T)^{-1} = \lambda^{-1}\sum_{k=0}^{\infty}(\lambda^{-1}T)^k, \quad |\lambda| > 1,$$

we have that

$$\|R(\lambda, T)\| \leq \frac{M}{|\lambda|-1}, \quad |\lambda| > 1, \tag{9.3.4}$$

and hence by the power-boundedness of T, that $\|F_n(\lambda)\| \lesssim (|\lambda|-1)^{-1}$. Clearly h_r is bounded above in modulus independently of r and R along Γ_R, so it follows that

$$\left\|\oint_{\Gamma_R} \frac{h_r(\lambda)}{\lambda - 2} F_n(\lambda)\, d\lambda\right\| \lesssim \frac{1}{R}, \quad R > 2.$$

Letting $R \to \infty$ shows that the contribution from $\Gamma_{\text{out}} = \Gamma_R$ can neglected by Cauchy's theorem.

Turning back to Γ_{in}, note that

$$|1 + r^2 z^{-2}| = 2\frac{|\operatorname{Re} z|}{r}, \quad z \in r\mathbb{T}$$

and also that

$$1 - |\lambda|^2 = |1+\lambda|^2 \operatorname{Re}\varphi(\lambda) = \frac{4\operatorname{Re}\varphi(\lambda)}{|1+\varphi(\lambda)|^2}.$$

Now if $\lambda \in \gamma_r$, then $\varphi(\lambda) \in r\mathbb{T}$ and in particular, $|1+\varphi(\lambda)|^2 = 1 + 2\operatorname{Re}\varphi(\lambda) + r^2$. Thus,

$$|h_r(\lambda)| \lesssim \frac{|\operatorname{Re}\varphi(\lambda)|}{r} \leq c\frac{||\lambda|-1|}{r}, \quad \lambda \in \gamma_r. \tag{9.3.5}$$

Note further, that for such λ,

$$|1-\lambda| \leq \left|1 - \frac{1+r^2}{1-r^2}\right| + \left|\lambda - \frac{1+r^2}{1-r^2}\right| = \frac{2r^2+2r}{1-r^2} \lesssim r. \tag{9.3.6}$$

From the definition of F_n and (9.3.4), we see that

$$\|F_n(\lambda)\| \lesssim \|I - (\lambda-1)R(\lambda,T)\| \leq |\lambda - 1|\left(\frac{1}{|1-\lambda|} + \frac{M}{|\lambda|-1}\right) \lesssim \frac{r}{|\lambda|-1}, \quad \lambda \in \gamma_r^+.$$

Hence

$$\left\|\int_{\gamma_r^+} \frac{h_r(\lambda)}{\lambda-2} F_n(\lambda)\, d\lambda\right\| \lesssim r,$$

where the implicit constant is independent of $n \geq 0$ and r. What remains is to estimate the contribution along C_r.

To this end, let $\theta_r \in (0, \pi/2)$ denote the argument of the point in the upper half-plane at which γ_r intersects the unit circle. Define the curve C_r° which lives on the boundary of Ω^c by

$$C_r^\circ(\theta) = \left(1 - \frac{c}{m(|\theta|)}\right) e^{i\theta}, \quad \theta_r \leq |\theta| \leq \pi.$$

Let $C_r^\pm$ denote the rays defined by

$$C_r^\pm(s) = s e^{\pm i\theta_r}, \quad 1 - cm(\theta_r)^{-1} \leq s \leq 1,$$

and let $C_r = C_r^\circ \cup C_r^+ \cup C_r^-$. With this, our contour Γ is completely set. Finally, define the function p_n by

$$p_n(\lambda) = \sum_{k=0}^{n-1} \lambda^{n-k-1} T^k, \quad \lambda \in \mathbb{C} \setminus \Omega^c.$$

Using the fact that $(\lambda - T)p_n(\lambda) = (\lambda^n - T^n)$ so that $p_n(\lambda) = (\lambda^n - T^n)R(\lambda, T)$, the resolvent identity, and some elemntary manipulations, we get that

$$F_n(\lambda) = \frac{1}{\lambda - 2}(2 - T)^2 \left((\lambda - 1)(\lambda^n R(\lambda, T) - p_n(\lambda)) - T^n R(2, T)\right), \quad 2 \neq \lambda \in \mathbb{C} \setminus \Omega^c.$$

By Cauchy's theorem,

$$\begin{aligned} R(2,T)^2 \int_{C_r} \frac{h_r(\lambda)}{\lambda - 2} F_n(\lambda)\, d\lambda &= \int_{C_r} \frac{h_r(\lambda)(\lambda - 1)\lambda^n}{(\lambda - 2)^2} R(\lambda, T)\, d\lambda \\ &\quad - \int_{\gamma_r^-} \frac{h_r(\lambda)}{(\lambda - 2)^2}((\lambda - 1)p_n(\lambda) + T^n R(2, T))\, d\lambda, \end{aligned}$$

where $\gamma_r^- = \gamma_r \cap \mathbb{D}$.

Note that outside of the circle γ_r around the point 1, $|1 - \lambda|$ is bounded from below by $(1 - r)^2/(2r)$. Hence h_r is uniformly bounded independently of r along C_r. Instead of applying the Neumann series argument as in the original proof of [Sei16, Theorem 2.11], we use (9.3.2) to see that

$$\left\| \int_{C_r^\circ} \frac{h_r(\lambda)(\lambda - 1)\lambda^n}{(\lambda - 2)^2} R(\lambda, T)\, d\lambda \right\| \lesssim k(\theta_r) \left(1 - \frac{c}{m(\theta_r)}\right)^n. \tag{9.3.7}$$

Notice now that for $|z| \geq r$, we have $|1 + r^2/z^2| \leq 2|\operatorname{Re} z|/|z|$. This is because for $z = a + ib$, $a^2 + b^2 \geq r^2$ implies that $a^2 - b^2 + r^2 \leq 2a^2$ so that

$$|z^2 + r^2|^2 = (a^2 - b^2 + r^2)^2 + 4a^2b^2 \leq 4a^2(a^2 + b^2) = 4|\operatorname{Re} z|^2|z|^2.$$

Using that φ is a Möbius transform mapping the line segments $C_r^{\pm}$ to arcs of circles that pass through $z = \pm ir$ and $z = -1$, we see that $|\varphi(\lambda)| \geq r$ for $\lambda \in C_r^{\pm}$. Thus, as $|1+\lambda|$ is bounded away from 0 for on the rays $C_r^{\pm}$,

$$|h_r(\lambda)| \leq \frac{2|\operatorname{Re}\varphi(z)|}{|\varphi(z)|} \lesssim \frac{|\lambda|^2 - 1}{|\lambda - 1|} \lesssim \frac{1 - |\lambda|}{|\lambda - 1|}, \quad \lambda \in C_r^{\pm},$$

so that

$$\begin{aligned} \left\| \int_{C_r^{\pm}} \frac{h_r(\lambda)(\lambda-1)\lambda^n}{(\lambda-2)^2} R(\lambda, T)\, d\lambda \right\| &\lesssim \int_{C_r^{\pm}} (1 - |\lambda|)|\lambda|^n \|R(\lambda, T)\| d|\lambda| \\ &\lesssim \int_{1 - cm(\theta_r)^{-1}}^{1} \frac{k(\theta_r)}{m(\theta_r)} s^n \, ds \leq \frac{k(\theta_r)}{m(\theta_r)} \frac{1}{n}. \end{aligned} \tag{9.3.8}$$

For sufficiently large $n \geq 0$, choose $\theta_r = m_k^{-1}(cn)$ so that

$$cn = m(\theta_r) \log\left(\frac{k(\theta_r)}{\theta_r} + 1\right).$$

By (9.3.1), we know that

$$\frac{1}{nm(\theta_r)} \lesssim \frac{1}{m(\theta_r)^2 \log\left(\frac{k(\theta_r)}{\theta_r} + 1\right)} \lesssim \frac{\theta_r}{k(\theta_r)} + 1 = \exp\left(-\frac{cn}{m(\theta_r)}\right).$$

Since the function $f(a) = a + \log(1 - a)$ is less than or equal to 0 for small positive a, $(1 - a)^n \leq \exp(-an)$. In particular,

$$\left(1 - \frac{c}{m(\theta_r)}\right)^n \leq \exp\left(-\frac{cn}{m(\theta_r)}\right).$$

Hence both the norms of the left integrals in (9.3.7) and (9.3.8) are bounded by some constant times

$$k(\theta_r) \exp\left(-\frac{cn}{m(\theta_r)}\right) \leq \theta_r.$$

By (9.3.6),

$$\|(\lambda - 1)p_n(\lambda)\| \leq M|\lambda - 1| \sum_{k=0}^{n-1} |\lambda|^{n-k-1} \lesssim \frac{r}{1 - |\lambda|}, \quad \lambda \in \gamma_r.$$

Combining this with (9.3.5), both $h_r(\lambda)$ and $h_r(\lambda)(\lambda-1)p_n(\lambda)$ are uniformly bounded, independently of r and n, for λ along γ_r^-. Hence,

$$\left\| \int_{\gamma_r^-} \frac{h_r(\lambda)}{(\lambda - 2)^2} ((\lambda - 1)p_n(\lambda) + T^n R(2, T))\, d\lambda \right\| \lesssim r.$$

Since γ_r meets $\mathbb{T}$ at right angles by the preservation of angles through the conformal mapping φ, $\tan(\theta_r) = \frac{2r}{1-r^2}$, hence for r small, $\theta_r \geq r$. Since all the contributions of our contour integral are bounded by $C\theta_r$ for some constant C independent of n and r, we see that $\|T^n(I - T)\| = O(m_k^{-1}(cn))$, $n \to \infty$. $\square$

When $k = m$, it is clear that we get the classical $m_{\log}$ theorem (Theorem 3.2.3) back, since $1 \leq \theta m(\theta)$ so that $m(\theta) \leq \theta m(\theta)^2$, satisfying condition (9.3.1).

Regarding the second of our two approaches, it is worth remarking that the proof of [Sei15, Theorem 2.1], which used Fourier methods, can be adapted to the m_k setting with the condition (9.3.1) being allowed to have an extra power on the log term. Nonetheless, the way to convert the quantified Tauberian theorem for sequences into the Katznelson–Tzafriri m_k-theorem is not as simple as the $m_{\log}$ case of [Sei15, Theorem 2.5] might suggest (see [Sta17, Lemma 3.1]). Previously, we had hoped to prove a much less restrictive theorem by following the approach of [Sta17] by using the Denjoy–Carleman theorem. However it became apparent fairly quickly that there were major obstacles to this approach when used in the setting with singularities (in our case, the spectrum at the point $\lambda = 1$ was the culprit).

9.4 Subordinated discrete semigroups

In [Dun11], Dungey explores so-called subordinated discrete semigroups of operators, these being discrete analogues of C_0-semigroups that are subordinated to uniformly bounded C_0-semigroups via convolution semigroups (see [CK91] and [Dav80, Section 2.4]). He does so as follows.

Let F be a probability on Z_+, that is, $F : \mathbb{Z}_+ \to [0, 1]$ is such that $\sum_{k\geq 0} F(k) = 1$. For a power-bounded operator T on a Banach space X, define

$$\Psi(F; T) = \sum_{k\geq 0} F(k)T^k \in \mathcal{B}(X).$$

The discrete operator semigroup $(\Psi(F; T)^n)_{n\geq 0}$ is said to be subordinated to the bounded discrete operator semgroup $(T^n)_{n\geq 0}$.

For a probability F, define its so-called generating function $\phi_F : \overline{\mathbb{D}} \to \mathbb{C}$ by

$$\phi_F(z) = \sum_{k\geq 0} F(k)w^k, \quad z \in \overline{\mathbb{D}}.$$

This is continuous on $\overline{\mathbb{D}}$ and analytic in $\mathbb{D}$.

Dungey uses the notion of aperiodicity, which we will not go into (see the discussion found in [Dun11, Page 1725]), to prove the following result on rates. Suffice it to say that F is aperiodic whenever the support of F contains consecutive integers.

Theorem 9.4.1 ([Dun11, Theorem 2.4]). *Let $T \in \mathcal{B}(X)$ be power-bounded and let F be an aperiodic probability. Then*

$$\sup_{n\in\mathbb{N}} n^{1/2} \|\Psi(F; T)^n - \Psi(F; T)^{n+1}\| < \infty$$

and there exists a $\beta \in (0,1)$ *such that*

$$\sigma(\Psi(F;T)) \subset \{z \in \mathbb{C} : |z-(1-\beta)| \leq \beta\} \subset \mathbb{D} \cup \{1\}.$$

This theorem is reminiscent of Dungey's older work in [Dun08] discussed in Section 3.2 of this thesis.

Analytic Toeplitz operators come in at this point, for if T_R is the right shift (also known as the unilateral shift) on $\ell^2(\mathbb{Z}_+)$ defined by

$$T_R(x_0, x_1, x_2, \dots) = (0, x_0, x_1, \dots), \quad x = (x_0, x_1, x_2, \dots) \in \ell^2(Z_+),$$

then $\Psi(F;T_R)$ is the (infinite) Toeplitz matrix that corresponds to the analytic Toeplitz operator with symbol ϕ_F. Thus, operators of the form $\Psi(F;T)$ for a probability F and a power-bounded operator T can also be considered as generalisations of analytic Toeplitz operators that have non-negative real-valued symbols. This is why Remark 8.3.2 is true.

Dungey then characterises those probabilities F for which $\Psi(F;T)$ are always Ritt whenever T is power-bounded. For the sake of our discussion, we do not need the full characterisation of [Dun11, Theorem 3.1], but only the following necessary condition.

Theorem 9.4.2. *Let F be a probability. If for some Banach space X and power-bounded operator $T \in \mathcal{B}(X)$, the operator $\Psi(F;T) \in \mathcal{B}(X)$ is Ritt then there is a $\theta \in (0, \pi/2)$ such that*

$$\phi_F(w) \in \mathbb{D} \cup \{1\}, \quad 1 - \phi_F(w) \in \{z \in \mathbb{C} : |\arg z| \leq \theta\} \cup \{0\}, \quad w \in \overline{\mathbb{D}}. \tag{9.4.1}$$

Furthermore, if F is not the point mass at 0, that is, the probability with $F(0) = 1$ and $F(k) = 0$ for $k \geq 1$, then

$$\sum_{k \geq 0} kF(k) = \infty. \tag{9.4.2}$$

Hence F must have infinite support.

Note that if F is the point mass, $\Psi(F;T)$ is just the identity, yielding trivial 'decay'. Our example in Example 8.3.1 is not Ritt because it fails (9.4.1), but interestingly, it still satisfies (9.4.2). See [Dun11, Example 5.2] for a proof of this.

It can be shown (by working with $\phi_F(\overline{\mathbb{D}})$ and using Taylor series in a similar way to Example 8.3.1) that when $T = T_R$ so that we are in the case of analytic Toeplitz operators, if F has finite support (in other words, we have a polynomial symbol) or is of the form $F(k) = C\alpha^k$, $k \in \mathbb{Z}_+$, for some $\alpha \in (0,1)$ and normalising constant C, then $\|\Psi(F;T)^n - \Psi(F;T)^{n+1}\| \asymp n^{-1/2}$, $n \to \infty$.

Putting this and Theorem 9.4.2 together, we see that in certain cases, operators of the form $\Psi(F;T)$ sit on the decay scale of $\|\Psi(F;T)^n - \Psi(F;T)^{n+1}\|$ between $O(n^{-1})$ and $O(n^{-1/2})$, $n \to \infty$, in such a way that we may heuristically say that the fastest decay occurs when F decays slowly and the slowest decay occurs when F decays quickly. Example 8.3.1 gives a probability that sits in between the extremal decay rates of $O(n^{-1})$ and $O(n^{-1/2})$, $n \to \infty$. See also [Sei16, Example 3.2] for an example of the slowest decay rate occuring. There, T is the left shift rather than the right shift, resulting in $\Psi(F;T)$ being a Toeplitz operator that is not analytic.

This leads us to ask the following question. *Can we say more about this relationship between decay rate of F and decay rate of $\|\Psi(F;T)^n - \Psi(F;T)^{n+1}\|$ in both the general case and the right shift case?* In the specific case of where T is the right shift, one wonders if some results could be established for conditions such as if

$$\sum_{k\geq 0} k^\gamma F(k) < \infty$$

for some or all $\gamma > 0$. For the general case, the following spectral characterisation gives rise to some hope that the relationship between the decay of probabilities F and the decay of $\|\Psi(F;T)^n - \Psi(F;T)^{n+1}\|$ for operators T other than the right shift might be characterised in the future.

Theorem 9.4.3 ([Dun11, Theorem 2.1])**.** *Let $T \in \mathcal{B}(X)$ be power-bounded. Then for any probability F,*

$$\sigma(\Psi(F;T)) = \phi_F(\sigma(T)) \subset \phi_F(\overline{\mathbb{D}}).$$

At the very least, these subordinated discrete semigroups are proving to be an interesting class containing potential applications for the theory of discrete quantified asymptotics, both because they have various desirable properties to work with and because of their inherent relation to probability theory.

9.5 An alternative optimality approach

We end this chapter with the following observation and subsequent question.

The optimality of the $M_{\log}$ rate given in Theorem 3.1.7 was first proved in [BT10] via a rather tedious construction involving function spaces with carefully concocted measures. As mentioned in Section 3.1, Debruyne and Seifert managed to prove Theorem 3.1.16 via a much simpler method using the open mapping theorem. In the discrete world, Seifert had discretised the rather tedious construction of [BT10] in order to prove Theorem 3.2.6. That proof was also rather tedious.

Thus, a fitting way to end this thesis is to ask the following natural question. *Is there a discrete operator semigroup analogue to Theorem 3.1.16 that can be proved by a discrete version of the proof found in [DS19a]?* The answer, if ever shown to be affirmative, would find an open-armed welcome from Part III of this thesis.

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