Available online at www.sciencedirect.com
SCIENCE DIRECT®

Advances in Applied Mathematics 34 (2005) 616–633

ADVANCES IN Applied Mathematics

www.elsevier.com/locate/yaama

Families of matroids induced by classes of graphs

Dillon Mayhew *

Mathematical Institute, University of Oxford, 24-29 St Giles, Oxford OX1 3LB, United Kingdom

Received 19 September 2004; accepted 9 November 2004

Available online 18 January 2005

Abstract

It is easily proved that, if $\mathcal{P}$ is a class of graphs that is closed under induced subgraphs, then the family of matroids whose basis graphs belong to $\mathcal{P}$ is closed under minors. We give simple necessary and sufficient conditions for a minor-closed class of matroids to be induced in this way, and characterise when such a class of matroids contains arbitrarily large connected matroids. We show that five easily-defined families of matroids can be induced by a class of graphs in this manner: binary matroids; regular matroids; the polygon matroids of planar graphs; those matroids for which every connected component is either graphic or cographic; and those matroids for which every connected component is either binary or can be obtained from a binary matroid by a single circuit-hyperplane relaxation. We give an excluded-minor characterisation of the penultimate class, and show that the last of these classes has infinitely many excluded minors.

Keywords: Matroid basis graphs; Hereditary class; Rank-preserving weak map; Excluded-minor characterisation

1. Introduction

Let M be a matroid, and let $\mathcal{B}(M)$ be its set of bases. The *basis graph of* M, denoted by $\mathrm{BG}(M)$, has $\mathcal{B}(M)$ as its set of vertices. Two bases are adjacent in $\mathrm{BG}(M)$ if and only if the size of their symmetric difference is two. The basis graph has been extensively studied in [1,4,5], and others.

* Fax: +44 1865 273583.
E-mail address: mayhew@maths.ox.ac.uk.

0196-8858/$ – see front matter
doi:10.1016/j.aam.2004.11.001

Maurer [5] has proved that, if we let $\mathcal{P}$ be the class of graphs that have no induced subgraph isomorphic to the octahedron, then M is a binary matroid if and only if $\mathrm{BG}(M) \in \mathcal{P}$. We generalise this idea to other classes of graphs that are closed under isomorphism and induced subgraphs. Such a class of graphs will be known as an *hereditary class*.

Suppose that $\mathcal{P}$ is an hereditary class. Let $\mathcal{M}(\mathcal{P})$ be the class of matroids such that $M \in \mathcal{M}(\mathcal{P})$ if and only if $\mathrm{BG}(M) \in \mathcal{P}$. We shall say that $\mathcal{M}(\mathcal{P})$ is *induced by* $\mathcal{P}$. If $\mathcal{M}$ is a class of matroids, and there exists an hereditary class of graphs, $\mathcal{P}$, such that $\mathcal{M} = \mathcal{M}(\mathcal{P})$, then $\mathcal{M}$ is an *induced class*.

A basis graph does not determine a matroid uniquely. For instance, adding a loop or coloop does not change the basis graph of a matroid. However, Holzmann, Norton, and Tobey [1] showed that a basis graph does uniquely determine a loopless and coloopless matroid up to a natural form of equivalence, which we now describe.

Suppose that $M = M_1 \oplus \cdots \oplus M_m$ and $N = N_1 \oplus \cdots \oplus N_n$ are the decompositions of two matroids into their connected components. If $m = n$, and there exists a permutation, $\pi \in S_m$, such that either $M_i \cong N_{\pi(i)}$ or $M_i \cong N^*_{\pi(i)}$ for all $i \in \{1, \ldots, m\}$, then we shall say that M and N are *generalised duals*. In particular, if M is connected and N is a generalised dual of M, then either $N \cong M$ or $N \cong M^*$. It is easy to see that the relation of being generalised duals is an equivalence relation.

Theorem 1.1 [1, Theorem 5.3]. *Two loopless and coloopless matroids have isomorphic basis graphs if and only if they are generalised duals.*

If M and N are two matroids of the same rank, then N is a *rank-preserving weak-map image* of M if N is isomorphic to a matroid N', such that $E(N') = E(M)$, and $\mathcal{B}(N') \subseteq \mathcal{B}(M)$. This relation is denoted by $M \xrightarrow{\text{r.p.}} N$.

It is clear that, if $\mathcal{P}$ is an hereditary class, then $\mathcal{M}(\mathcal{P})$ is closed under generalised duality and the addition of loops and coloops. Furthermore, since, if N is a minor or a rank-preserving weak-map image of M, then $\mathrm{BG}(N)$ is an induced subgraph of $\mathrm{BG}(M)$, it follows that $\mathcal{M}(\mathcal{P})$ is closed under minors and rank-preserving weak maps. These necessary conditions turn out to be sufficient also.

Theorem 1.2. *Let $\mathcal{M}$ be a class of matroids that is closed under isomorphism and minors. Then $\mathcal{M}$ is an induced class if and only if it is closed under generalised duality, the addition of loops and coloops, and rank-preserving weak maps.*

The motivation for studying these classes of matroids came from considering parameters of basis graphs, such as the clique number and the chromatic number. It was natural to look at, for example, the class of matroids with properly k-colourable basis graphs; in other words, the class $\mathcal{M}(\mathcal{P}_k)$, where $\mathcal{P}_k$ is the class of graphs with chromatic number at most k. The characterisation of these classes for small values of k shows that they do not contain large connected matroids [6]. The next result shows exactly when $\mathcal{M}(\mathcal{P})$ does contain large connected matroids.

Theorem 1.3. *Suppose that $\mathcal{P}$ is an hereditary class of graphs. Then $\mathcal{M}(\mathcal{P})$ contains arbitrarily large connected matroids if and only if $\mathcal{P}$ contains arbitrarily large cliques.*

Binary matroids, regular matroids, and the polygon matroids of planar graphs are all induced classes. So too is the set of matroids that are generalised duals of graphic matroids. The excluded-minor characterisations of the first three classes are well known. In Section 5 we provide an excluded-minor characterisation of the last class, which shows that it has 21 non-isomorphic excluded minors. In Section 6 we present an induced class that has an infinite number of excluded minors.

Terminology and notation will follow Oxley [8]. When convenient to do so, we shall make no distinction between the bases of a matroid and the vertices of its basis graph.

2. A characterisation of induced classes

In this section we will prove Theorem 1.2. We require some preliminary results. Suppose that v is a vertex of the graph G. The *closed neighbourhood of* v is the subgraph of G induced by v and all its neighbours. It is denoted by $\widehat{N}_G(v)$, or by $\widehat{N}(v)$ when the context is clear. The *neighbourhood of* v is obtained by deleting v from $\widehat{N}_G(v)$. It is denoted by $N_G(v)$ or $N(v)$. If v' is a vertex in the same connected component of G as v, then $d_G(v, v')$ denotes the length of a shortest path in G that joins v to v'.

The next result is implied by [1, Lemma 3.2] and [4, Lemma 1.4].

Proposition 2.1. *Suppose that v and v' are vertices in a basis graph,* $\mathrm{BG}(M)$, *and that* $d_{\mathrm{BG}(M)}(v, v') = 2$. *There exist two non-adjacent vertices in $V(N(v)) \cap V(N(v'))$.*

Suppose that G is isomorphic to the basis graph of a matroid, N. A *proper labelling of G* is a bijection, $\sigma : V(G) \to \mathcal{B}(M)$, where M is a matroid, and where two vertices are adjacent in G if and only if the symmetric difference of their labels has size two. Note that M and N need not be equal, nor, indeed, isomorphic.

Proposition 2.2 [1, Corollary 3.2.1]. *Let $\sigma : V(G) \to \mathcal{B}(M)$ be a proper labelling of a basis graph, G. Suppose that v and v' are vertices of G and that $d_G(v, v') = 2$. Let x and y be two non-adjacent vertices in $V(N(v)) \cap V(N(v'))$. Then*

$$\sigma(v') = \big(\sigma(x) \cap \sigma(y)\big) \cup \big(\sigma(x) - \sigma(v)\big) \cup \big(\sigma(y) - \sigma(v)\big).$$

Proposition 2.3 [1, Lemma 4.1]. *Suppose that v is a vertex in the basis graph of a loopless and coloopless matroid, M. There exist partitions, π and π', of $V(N_{\mathrm{BG}(M)}(v))$ into non-empty sets, such that*:

(i) *If $v_1, v_2 \in V(N_{\mathrm{BG}(M)}(v))$, then v_1 and v_2 are adjacent if and only if a member of π or π' contains both v_1 and v_2.*
(ii) *If $p \in \pi$ and $q \in \pi'$, then $|p \cap q| \leqslant 1$.*

Proof. Suppose that v corresponds to the basis $B = \{x_1, \ldots, x_r\}$ of M. Suppose also that $E(M) - B = \{y_1, \ldots, y_{n-r}\}$, where $r = r(M)$ and $n = |E(M)|$. For $1 \leqslant i \leqslant r$ define p_i to be the set $\{B' \in \mathcal{B}(M) \mid B' \cap B = B - x_i\}$. For $1 \leqslant i \leqslant n - r$ let $q_i = \{B' \in \mathcal{B}(M) \mid$

$B' - B = \{y_i\}\}$. Define $\pi(B, M)$ to be the collection $\{p_1, \ldots, p_r\}$ and $\pi'(B, M)$ to be $\{q_1, \ldots, q_{n-r}\}$. Then $\pi(B, M)$ and $\pi'(B, M)$ are partitions of $V(N(v))$ that satisfy the conditions of the proposition. □

If π and π' are partitions of $V(N(v))$ that satisfy the conditions of Proposition 2.3, then they need not be the same as the natural partitions, $\pi(B, M)$ and $\pi'(B, M)$. However, as we shall see, π and π' must correspond to the natural partitions of some matroid, in fact a generalised dual of M.

Suppose that $M = M_1 \oplus \cdots \oplus M_t$ and $N = N_1 \oplus \cdots \oplus N_t$ are generalised duals. By relabelling we may assume that, for all $i \in \{1, \ldots, t\}$, M_i is isomorphic to either N_i or N_i^*. Therefore there is a bijection, ρ, between $E(M)$ and $E(N)$, such that ρ restricted to $E(M_i)$ is an isomorphism between M_i and one of N_i or N_i^*. Define β to be the bijection between $\mathcal{B}(M)$ and $\mathcal{B}(N)$, so that if B is a basis of M and $i \in \{1, \ldots, t\}$, then $\beta(B) \cap E(N_i) = \rho(B \cap E(M_i))$ when $M_i \cong N_i$, and $\beta(B) \cap E(N_i) = E(N_i) - \rho(B \cap E(M_i))$ when $M_i \cong N_i^*$. It is easy to see that β is an isomorphism between $\mathrm{BG}(M)$ and $\mathrm{BG}(N)$.

In the next result we will use the following, obvious, definitions: if $\mathcal{B}' \subseteq \mathcal{B}(M)$ is a set of bases of M, then $\beta(\mathcal{B}') = \{\beta(B') \mid B' \in \mathcal{B}'\}$; if $\tau = \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$ is a collection of sets of bases, then $\beta(\tau) = \{\beta(\mathcal{B}_1), \ldots, \beta(\mathcal{B}_n)\}$.

Proposition 2.4 [1, Lemma 4.1]. *Suppose that v is a vertex in $\mathrm{BG}(M)$, where M is a loopless and coloopless matroid, and that v corresponds to the basis B. If π and π' are partitions of $V(N(v))$ that satisfy the conditions of Proposition 2.3, then there is a generalised dual, N, of M, such that $\beta(\pi) = \pi(\beta(B), N)$ and $\beta(\pi') = \pi'(\beta(B), N)$, where β is the natural bijection between $\mathcal{B}(M)$ and $\mathcal{B}(N)$.*

The following proposition is easy to verify.

Proposition 2.5. *Let M and N be matroids. If N is a minor or a rank-preserving weak-map image of M, then $\mathrm{BG}(N)$ is an induced subgraph of $\mathrm{BG}(M)$.*

A converse result also holds.

Lemma 2.6. *Suppose that M is a matroid. If G is an induced subgraph of $\mathrm{BG}(M)$, and G is itself isomorphic to a basis graph, then there exist matroids, M_1 and M_2, such that M_1 is a minor of M, and $M_1 \xrightarrow{\text{r.p.}} M_2$, and, furthermore, $\mathrm{BG}(M_2) \cong G$.*

We remark here that, although this result seems not to appear in the literature, it is almost certainly known.

Proof of Lemma 2.6. Let us suppose that $\sigma : V(\mathrm{BG}(M)) \to \mathcal{B}(M)$ is the labelling that maps vertices of $\mathrm{BG}(M)$ to their corresponding bases of M. Suppose also that v_0 is a vertex of G, and that $B = \sigma(v_0) = \{x_1, \ldots, x_r\}$, while $E(M) - B = \{y_1, \ldots, y_{n-r}\}$, where $r = r(M)$ and $n = |E(M)|$.

The partitions, $\pi(B, M) = \{p_1, \ldots, p_r\}$ and $\pi'(B, M) = \{q_1, \ldots, q_{n-r}\}$, of $V(N_{\mathrm{BG}(M)}(v_0))$ were defined in the proof of Proposition 2.3. These partitions naturally induce two partitions on the vertex set of $N_G(v_0)$, although some of the blocks of these induced partitions may be empty. Let us therefore introduce two sets of indices: $I = \{i \mid 1 \leqslant i \leqslant r,\ p_i \cap V(N_G(v_0)) \neq \emptyset\}$ and $J = \{j \mid 1 \leqslant j \leqslant n - r,\ q_j \cap V(N_G(v_0)) \neq \emptyset\}$. We may now define the partitions $\pi = \{p_i \cap V(N_G(v_0)) \mid i \in I\}$ and $\pi' = \{q_j \cap V(N_G(v_0)) \mid j \in J\}$. It is easy to see that π and π' satisfy the conditions of Proposition 2.3. It follows from Proposition 2.4 that there exists a loopless and coloopless matroid, L, such that $\mathrm{BG}(L) \cong G$, and, furthermore, if v_0 corresponds to the basis B' of L, then π and π' correspond to the natural partitions $\pi(B', L)$ and $\pi'(B', L)$.

We now construct a proper labelling, τ, of G. Let $X = \{x_i \mid i \in I\}$ and $Y = \{y_j \mid j \in J\}$. The labelling, τ, will be from $V(G)$ to subsets of $X \cup Y$. Let $\tau(v_0)$ be X. If $v \in N_G(v_0)$, then v is in exactly one member of π and exactly one member of π'. If $v \in p_i \cap q_j$, where $i \in I$ and $j \in J$, then label v with $(X - x_i) \cup y_j$. The rest of the labelling is constructed recursively. Suppose that v' is a vertex of G such that $d_G(v_0, v') = i$ (where $i > 1$) and all the vertices of G that are closer to v_0 than v' have already been labelled. Let P be a path of length i from v_0 to v', and let v be the vertex in P such that $d_G(v, v') = 2$. Suppose that x and y are two non-adjacent vertices in $V(N_G(v)) \cap V(N_G(v'))$. Since v, x, and y have already received labels, we can use Proposition 2.2 to find $\tau(v')$. Proposition 2.2 guarantees that τ is indeed a proper labelling. In fact, if M_2 is the matroid on the ground set $X \cup Y$ that has $\tau(V(G))$ as its set of bases, then $M_2 \cong L$.

By using induction on distance from v_0, and again applying Proposition 2.2, it is not difficult to see that the labellings τ and σ are essentially the same.

2.6.1. If $v \in V(G)$, then $\sigma(v) = \tau(v) \cup (B - X)$.

Let M_1 be the matroid $M/(B - X) \setminus (E(M) - (X \cup Y \cup B))$. It follows easily from statement 2.6.1 that $M_1 \xrightarrow{\text{r.p.}} M_2$. This completes the proof of Lemma 2.6. □

We may now prove Theorem 1.2. It will follow immediately from the next result.

Theorem 2.7. *Suppose that $\mathcal{M}$ is a family of matroids that is closed under isomorphism and minors. Let $\mathrm{EX}(\mathcal{M})$ be the set of excluded minors for $\mathcal{M}$. The following conditions are equivalent*:

(i) *The family $\mathcal{M}$ is closed under generalised duality, the addition of loops and coloops, and rank-preserving weak maps.*
(ii) *Every member of $\mathrm{EX}(\mathcal{M})$ is loopless and coloopless, and $\mathrm{EX}(\mathcal{M})$ is closed under generalised duality. Furthermore, if $N \in \mathrm{EX}(\mathcal{M})$, and $N' \xrightarrow{\text{r.p.}} N$, then $N' \notin \mathcal{M}$.*
(iii) *There exists an hereditary class of graphs, $\mathcal{P}$, such that $\mathcal{M} = \mathcal{M}(\mathcal{P})$.*

Proof. It is not difficult to confirm that (iii) implies (i). To show that (i) implies (ii) let us assume that $\mathcal{M}$ is closed under generalised duality, rank-preserving weak maps, and the addition of loops and coloops. It is clear that the excluded minors for $\mathcal{M}$ must be loopless and coloopless, and that $\mathrm{EX}(\mathcal{M})$ must be closed under generalised duality. Suppose that $N \in \mathrm{EX}(\mathcal{M})$, and that $N' \xrightarrow{\text{r.p.}} N$. It cannot be that $N' \in \mathcal{M}$, for then N, too, would be a member of $\mathcal{M}$.

To complete the proof we show that (ii) implies (iii). Suppose that (ii) holds. Let $\mathcal{BG}$ be the set $\{\mathrm{BG}(N) \mid N \in \mathrm{EX}(\mathcal{M})\}$. Define the graph property, $\mathcal{P}$, so that $G \in \mathcal{P}$ if and only if no induced subgraph of G is isomorphic to a member of $\mathcal{BG}$. We wish to show that $\mathcal{M} = \mathcal{M}(\mathcal{P})$. First suppose that $M \notin \mathcal{M}$. Then there must exist a matroid, $N \in \mathrm{EX}(\mathcal{M})$, such that M has an N-minor. Therefore $\mathrm{BG}(M)$ has an induced subgraph isomorphic to $\mathrm{BG}(N)$. Hence $\mathrm{BG}(M) \notin \mathcal{P}$ and $M \notin \mathcal{M}(\mathcal{P})$. From this we conclude that $\mathcal{M}(\mathcal{P}) \subseteq \mathcal{M}$.

Now suppose that $M \notin \mathcal{M}(\mathcal{P})$. Then there must exist a graph, $G \in \mathcal{BG}$, such that $\mathrm{BG}(M)$ contains an induced subgraph isomorphic to G. Let N be a member of $\mathrm{EX}(\mathcal{M})$ such that $\mathrm{BG}(N) \cong G$. Lemma 2.6 implies that there exist matroids, M_1 and M_2, such that M_1 is a minor of M and $M_1 \xrightarrow{\text{r.p.}} M_2$, while $\mathrm{BG}(M_2) \cong \mathrm{BG}(N) \cong G$. We may assume that $E(M_1) = E(M_2)$, and that $\mathcal{B}(M_2) \subseteq \mathcal{B}(M_1)$. Let L be the set of loops of M_2, and L^* the set of coloops. It is not difficult to see that $M_1/L^* \setminus L \xrightarrow{\text{r.p.}} M_2/L^* \setminus L$. Furthermore, $\mathrm{BG}(M_2/L^* \setminus L) \cong \mathrm{BG}(M_2) \cong \mathrm{BG}(N)$. Since both $M_2/L^* \setminus L$ and N are loopless and coloopless, Theorem 1.1 implies that $M_2/L^* \setminus L$ is a generalised dual of N, and must therefore be an excluded minor for $\mathcal{M}$. As $M_1/L^* \setminus L \xrightarrow{\text{r.p.}} M_2/L^* \setminus L$, it follows that $M_1/L^* \setminus L$, and therefore M, is not a member of $\mathcal{M}$. Hence $\mathcal{M} \subseteq \mathcal{M}(\mathcal{P})$, and the proof is complete. □

3. Connected matroids in induced classes

In this section we prove Theorem 1.3, which we restate more formally here.

Theorem 3.1. *Let $\mathcal{P}$ be an hereditary class of graphs. The induced class $\mathcal{M}(\mathcal{P})$ contains a connected matroid of size m, for every positive integer m, if and only if $\mathcal{P}$ contains K_n for every positive integer n.*

Before proving this we will need to establish some preliminary results. Let $\{B_1, \dots, B_t\}$ be a collection of bases of the matroid M. Let $X = \bigcap_{i=1}^{t} B_i$. We shall say that $\{B_1, \dots, B_t\}$ has *property* I *in M* if $|X| = r(M) - 1$, and there exists a set $Y = \{y_1, \dots, y_t\}$ such that $B_i = X \cup y_i$ for all $i \in \{1, \dots, t\}$. We shall say that $\{B_1, \dots, B_t\}$ has *property* II *in M* if $|X| = r(M) - t + 1$, and there exists a set $Y = \{y_1, \dots, y_t\}$ such that $B_i = (X \cup Y) - y_i$ for all $i \in \{1, \dots, t\}$. It is easy to see that a set of bases, $\mathcal{B}$, has property I in M, if and only if the corresponding set of cobases, $\mathcal{B}^* = \{E(M) - B \mid B \in \mathcal{B}\}$, has property II in M^*. Similarly, $\mathcal{B}$ has property II in M if and only if $\mathcal{B}^*$ has property I in M^*.

It is obvious that a set of bases with property I or II forms a clique in the basis graph. The converse also holds.

Lemma 3.2. *Let $\{B_1, \dots, B_t\}$ (where $t \geqslant 2$) be a set of distinct bases of M that forms a clique in* $\mathrm{BG}(M)$*. Then $\{B_1, \dots, B_t\}$ has either property* I *or* II.

Proof. The proof will be by induction on t. If $t = 2$, then $|X| = |B_1 \cap B_2| = r(M) - 1$, since B_1 and B_2 are adjacent in $\mathrm{BG}(M)$. Let y_1 be the single element in $B_1 - B_2$ and y_2 the element in $B_2 - B_1$. It is now clear that $\{B_1, B_2\}$ has property I.

Let us suppose that $t \geqslant 3$, and that the lemma holds for all collections of $t-1$ pairwise adjacent bases. We shall consider the collection $\{B_1, \ldots, B_{t-1}\}$. Suppose that $\{B_1, \ldots, B_{t-1}\}$ has property I. Then $X' = \bigcap_{i=1}^{t-1} B_i$ has cardinality $r(M) - 1$, and there exists a set $Y' = \{y_1', \ldots, y_{t-1}'\}$ such that $B_i = X' \cup y_i'$ for all $i \in \{1, \ldots, t-1\}$. Since B_t is adjacent to B_1, there exist elements, $x \in B_1 - B_t$ and $y \in B_t - B_1$, such that $B_t = (B_1 - x) \cup y$. First assume that $x \in X'$. Then $y_1' \in B_t$, but $y_1' \notin B_2$. Since B_t is adjacent to B_2, it follows that $|B_t - B_2| = 1$, so $B_t - B_2 = \{y_1'\}$. Since $y \in B_t - B_1$ it follows that $y \neq y_1'$. Therefore $y \in B_2$, but $y \notin B_1$. Since $B_2 - B_1 = \{y_2'\}$, this implies that $y = y_2'$. It follows that $t = 3$, for, if $t > 3$, then $B_t \neq B_3$, and since $B_t - B_3$ contains both y_1' and y_2', the bases B_t and B_3 cannot be adjacent. Make the following definitions: $y_1 = y_2'$; $y_2 = y_1'$; and $y_3 = x$. Also, let $Y = \{y_1, y_2, y_3\}$, and let X be $X' - x = \bigcap_{i=1}^{3} B_i$. We may now observe that $\{B_1, B_2, B_3\}$ has property II.

We will now assume that $x \notin X'$. It follows that $x = y_1'$. Clearly, $y \notin \{y_1', \ldots, y_{t-1}'\}$. Therefore we may set $y_i = y_i'$ for all $i \in \{1, \ldots, t-1\}$ and $y_t = y$. Then $\{B_1, \ldots, B_t\}$ has property I.

Let us assume that $\{B_1, \ldots, B_{t-1}\}$ has property II. Therefore $\{(E(M) - B_1), \ldots, (E(M) - B_{t-1})\}$ has property I in M^*. We may use the techniques of the last paragraph to show that $\{(E(M) - B_1), \ldots, (E(M) - B_t)\}$ has either property I or II in M^*, and hence $\{B_1, \ldots, B_t\}$ has property I or II in M. □

If B is a basis of M, and $e \notin B$, then $B \cup e$ contains a unique circuit of M, denoted by $C(e, B)$, which contains e. Dually, if $e \in B$, then $(E(M) - B) \cup e$ contains a unique cocircuit, denoted by $C^*(e, E(M) - B)$, which contains e.

Proposition 3.3. *Let M be a matroid on the ground set E, and let $\{B_1, \ldots, B_t\}$ be the vertex set of a maximal clique in* BG(M). *Either there exists a basis, B, and an element $e \in B$, such that $C^*(e, E - B) = \{e_1, \ldots, e_t\}$, and $B_i = (B - e) \cup e_i$ for all $i \in \{1, \ldots, t\}$; or, there exists a basis, B, and an element $e \notin B$, such that $C(e, B) = \{e_1, \ldots, e_t\}$, and $B_i = (B \cup e) - e_i$ for all $i \in \{1, \ldots, t\}$.*

Proof. We will first suppose that $\{B_1, \ldots, B_t\}$ has property I, so that $X = \bigcap_{i=1}^{t} B_i$ has cardinality $r(M) - 1$, and there exists a set $Y = \{y_1, \ldots, y_t\}$ such that $B_i = X \cup y_i$ for all $i \in \{1, \ldots, t\}$. Clearly $\mathrm{cl}(X)$ is a hyperplane, and $Y \subseteq E - \mathrm{cl}(X)$. Assume that Y is not equal to $E - \mathrm{cl}(X)$ and let y be an element in $E - (\mathrm{cl}(X) \cup Y)$. Then $X \cup y$ is a basis, distinct from, and adjacent to, the bases $B_1, \ldots, B_t$. This contradicts the maximality of the clique. Therefore $Y = E - \mathrm{cl}(X)$. If we take an arbitrary element $e \in Y$, then $B = X \cup e$ is the desired basis, and $C^*(e, E - B = E - \mathrm{cl}(X) = Y$ is the desired cocircuit.

The case when $\{B_1, \ldots, B_t\}$ has property II is similar. □

The clique number of a graph, G, is denoted by $\omega(G)$. If M is a matroid, then let $c(M)$ denote the size of the largest circuit of M, and let $c^*(M)$ equal $c(M^*)$. The next result follows easily from Proposition 3.3. Again, this result seems not to be in the literature, although it is presumably already known.

Theorem 3.4. *Let M be a matroid. Then* $\omega(\mathrm{BG}(M)) = \max\{c(M), c^*(M)\}$.

There has been much attention paid to the problem of how large a connected matroid may be if it has upper bounds on $c(M)$ and $c^*(M)$. The best possible result of this sort is due to Lemos and Oxley [2]. The proof of Theorem 3.1 will follow from this result and from Theorem 3.4.

Theorem 3.5 [2, Theorem 1.4]. *Let M be a connected matroid with at least two elements. Then $|E(M)| \leqslant \lfloor c(M)c^*(M)/2 \rfloor$.*

Proof of Theorem 3.1. Suppose that $\mathcal{P}$ contains K_n for all integers $n \geqslant 1$. Then, since $\mathrm{BG}(U_{n-1,n}) \cong K_n$, the induced class $\mathcal{M}(\mathcal{P})$ contains $U_{n-1,n}$ for all $n \geqslant 1$. Hence $\mathcal{M}(\mathcal{P})$ contains a connected matroid of size m for every positive integer m.

We will now assume that $\mathcal{P}$ does not contain every clique. Let t be the greatest integer such that $K_t \in \mathcal{P}$. Let M be a connected member of $\mathcal{M}(\mathcal{P})$. Clearly $\omega(\mathrm{BG}(M)) \leqslant t$. It follows from Theorem 3.4 that $c(M), c^*(M) \leqslant t$, and hence $|E(M)| \leqslant \lfloor t^2/2 \rfloor$. □

4. Well-known induced classes

Certain natural families of matroid are induced classes.

Proposition 4.1. *The families of binary matroids, regular matroids, and the polygon matroids of planar graphs are induced classes.*

Proof. Maurer [5] has noted that the binary matroids are exactly those which have no induced subgraph isomorphic to the octahedron in their basis graphs. In any case, it is easy to see that these three classes are closed under minors, the addition of loops or coloops, and generalised duality. Lucas has proved that they are closed under rank-preserving weak maps [3, Theorem 6.5 and Proposition 6.13]. Hence they are induced classes by Theorem 2.7. □

It is worth remarking here that if $\mathbb{F}$ is a field of size greater than two, then the set of $\mathbb{F}$-representable matroids is not an induced class. If H is a circuit-hyperplane of the matroid M, then the set $\mathcal{B}(M) \cup \{H\}$ is the collection of bases of a matroid on the set $E(M)$. This matroid is said to be produced from M by *relaxing the circuit-hyperplane H*. Let us consider the following matroids: F_7, the Fano plane; F_7^-, which is obtained by relaxing a circuit-hyperplane of F_7; and $F_7^=$, which is obtained by relaxing a circuit-hyperplane of F_7^-. We may also obtain $F_7^=$ by adding a point freely to a 2-point line of $M(K_4)$.

It is an easy exercise to show that, if $\mathbb{F}$ has characteristic two, and is not equal to GF(2), then $F_7^=$ is representable over $\mathbb{F}$, but F_7^- is not. Since $F_7^= \xrightarrow{\text{r.p.}} F_7^-$, it follows that the set of $\mathbb{F}$-representable matroids is not closed under rank-preserving weak maps, and is therefore not an induced class.

Similarly, if the characteristic of $\mathbb{F}$ is not two, then F_7^- is $\mathbb{F}$-representable, but F_7 is not [3]. Since $F_7^- \xrightarrow{\text{r.p.}} F_7$, it again follows that the set of $\mathbb{F}$-representable matroids is not an induced class.

We now consider a lesser-known class of matroids. We shall say that a matroid is *near-graphic* if it is a generalised dual of a graphic matroid. Equivalently, a matroid is near-graphic if and only if every connected component is either graphic or cographic.

Proposition 4.2. *The family of near-graphic matroids is an induced class.*

Proof. Obviously the class of near-graphic matroids is closed under minors and the addition of loops or coloops. It is closed under generalised duality by construction.

It remains to show that the class of near-graphic matroids is closed under rank-preserving weak maps. We first note that the class of graphic matroids is closed under rank-preserving weak maps [3, Proposition 6.13]. It is easy to see that if $M \xrightarrow{\text{r.p.}} N$, then $M^* \xrightarrow{\text{r.p.}} N^*$. It follows that the class of cographic matroids is also closed under rank-preserving weak maps.

Suppose that M is a near-graphic matroid, and that $M \xrightarrow{\text{r.p.}} N$. Let N' be a connected component of N. Then N' is contained in a connected component, M', of M [3, Proposition 5.2]. There exists a minor, M'', of M', such that $M'' \xrightarrow{\text{r.p.}} N'$ [3, Theorem 5.8]. By definition, M', and therefore M'', is either graphic or cographic. Hence N' is either graphic or cographic, and thus N is near-graphic. □

The induced classes that we have discussed in this section all consist of binary matroids. Not all induced classes need be contained in the set of binary matroids, as may be observed by noting that the set of matroids which have no $U_{3,6}$-minors is an induced class.

5. A characterisation of near-graphic matroids

The excluded-minor characterisations of binary matroids, regular matroids and the polygon matroids of planar graphs are classical results of Tutte's [10,11]. However the near-graphic matroids have not been characterised via their excluded minors. We give such a characterisation in this section.

We first require some preliminary material. If M is a matroid, and (X, Y) is a partition of $E(M)$ such that $r(X) + r(Y) \leqslant r(M) + k - 1$, then (X, Y) is a *k-separation of M*. If equality holds then the separation is said to be *exact*. We say that M is n-connected if M has no k-separation where $k < n$.

Let M_1 and M_2 be two matroids such that $E(M_1) \cap E(M_2) = \{p\}$. The *2-sum of M_1 and M_2 along the basepoint p*, denoted by $M_1 \oplus_2 M_2$, is a matroid on the ground set $(E(M_1) \cup E(M_2)) - p$. The collection of circuits of $M_1 \oplus_2 M_2$ is

$$\mathcal{C}(M_1 \setminus p) \cup \mathcal{C}(M_2 \setminus p) \cup \{(C \cup C') - p \mid C \in \mathcal{C}(M_1),\ C' \in \mathcal{C}(M_2),\ p \in C \cap C'\}.$$

It is well known that (X, Y) is a 1-separation of M if and only if $M = (M \mid X) \oplus (M \mid Y)$. Similarly, (X, Y) is an exact 2-separation of M if and only if there exist matroids, M_1 and M_2, on the sets $X \cup p$ and $Y \cup p$ respectively (where $p \notin E(M)$), such that M is equal to the 2-sum of M_1 and M_2 along p [9, (2.6)].

The next fact is well known, and follows easily from [8, Proposition 7.1.15].

Proposition 5.1. *Suppose that M_1 and M_2 are two matroids, and that $E(M_1) \cap E(M_2) = \{p\}$. If $e \in E(M_1) - p$, then $(M_1 \oplus_2 M_2) \setminus e = (M_1 \setminus e) \oplus_2 M_2$, and $(M_1 \oplus_2 M_2)/e = (M_1/e) \oplus_2 M_2$.*

Suppose that M_1 and M_2 are two binary matroids and that $E(M_1) \cap E(M_2) = T$, where $M_1 \mid T = M_2 \mid T \cong U_{2,3}$. Seymour [9] defined the 3-*sum of* M_1 *and* M_2, denoted by $M_1 \oplus_3 M_2$, to be the matroid on $(E(M_1) \cup E(M_2)) - T$, the circuits of which are the minimal non-empty sets that can be expressed as the symmetric difference of a disjoint union of circuits of M_1, and a disjoint union of circuits of M_2.

The next result follows from [8, Proposition 12.4.19].

Proposition 5.2. *Suppose that M_1 and M_2 are binary matroids, and that $E(M_1) \cap E(M_2) = T$. Suppose also that $M_1 \mid T = M_2 \mid T \cong U_{2,3}$. If $e \in E(M_1) - T$, then $(M_1 \oplus_3 M_2) \setminus e = (M_1 \setminus e) \oplus_3 M_2$, and if $e \in E(M_1) - \mathrm{cl}_{M_1}(T)$, then $(M_1 \oplus_3 M_2)/e = (M_1/e) \oplus_3 M_2$.*

The matroids R_{10} and R_{12} are binary self-dual matroids of rank five and six respectively. The matrices in Figs. 1 and 2 represent R_{10} and R_{12} over GF(2).

The matroid R_{12} can also be expressed as the 3-sum of $M^*(K_{3,3})$ and $M(K_5) \setminus e$, where e is any element of $M(K_5)$. Figure 3 shows representations of $M^*(K_{3,3})$ and $M(K_5) \setminus e$, while Fig. 4 shows a representation of their 3-sum, R_{12}. In this diagram, the elements of R_{12} are labelled with the corresponding columns of the matrix in Fig. 2.

The matroids R_{10} and R_{12} play a central role in Seymour's decomposition theorem for regular matroids. The next result will be crucially important for our characterisation of near-graphic matroids.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Fig. 1. A GF(2)-representation of R_{10}.

$$\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Fig. 2. A GF(2)-representation of R_{12}.

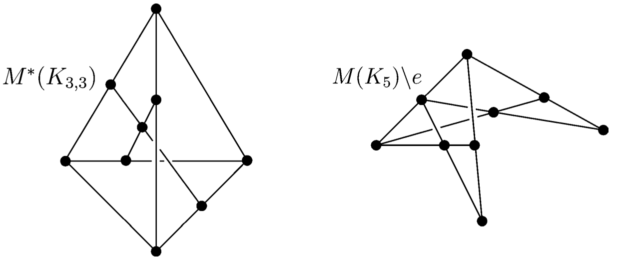

Fig. 3. $M^*(K_{3,3})$ and $M(K_5) \setminus e$.

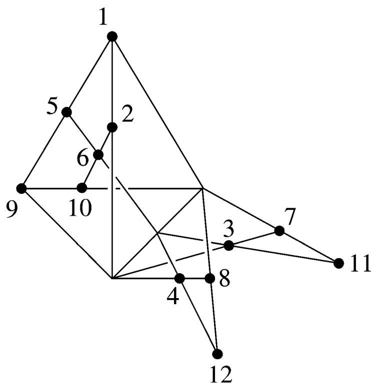

Fig. 4. A representation of R_{12}.

Proposition 5.3 [9, (14.2)]. *Let M be a 3-connected regular matroid. Then M is either graphic or cographic, or has a minor isomorphic to one of R_{10} or R_{12}.*

We can now state and prove our excluded minor characterisation of near-graphic matroids.

Theorem 5.4. *The excluded minors for the class of near-graphic matroids are: $U_{2,4}$; the Fano plane, F_7, and its dual, F_7^*; R_{10}; R_{12}; and those matroids that are formed by taking the 2-sum of the polygon matroid of one of the graphs in Fig. 5 with the bond matroid of one of the same graphs, using the element marked p as the basepoint.*

Proof. Let us first note that the class of near-graphic matroids is closed under direct sums, so all the excluded minors for this class are connected. The excluded minors for graphic matroids are $U_{2,4}$, F_7, F_7^*, $M^*(K_5)$, and $M^*(K_{3,3})$, while the excluded minors for cographic matroids are $U_{2,4}$, F_7, F_7^*, $M(K_5)$, and $M(K_{3,3})$ [11]. Since $U_{2,4}$, F_7, and F_7^* are excluded minors for both graphic matroids and cographic matroids, and are connected, it follows that they are also excluded minors for near-graphic matroids.

It is known [9] that if e is any element of R_{10}, then $R_{10} \setminus e \cong M(K_{3,3})$, and that $R_{10}/e \cong M^*(K_{3,3})$. Therefore R_{10} is neither graphic nor cographic. However, if we remove any element we clearly obtain a matroid that is either graphic or cographic. Since R_{10} is connected, it is therefore an excluded minor for near-graphic matroids.

The next result is slightly more difficult.

5.4.1. *R_{12} is an excluded minor for near-graphic matroids.*

Proof. It is known that R_{12} is neither graphic nor cographic [8, p. 519] (in fact, it is easy to see that R_{12} has both an $M^*(K_{3,3})$-minor, and an $M(K_{3,3})$-minor). Let us assume that R_{12} is labelled as in Fig. 4. Since R_{12} can be expressed as the 3-sum of $M^*(K_{3,3})$ and $M(K_5) \setminus e$, and $M^*(K_{3,3})$ is an excluded minor for graphic matroids, Proposition 5.2 implies that if we remove one of the elements in $\{1, 2, 5, 6, 9, 10\}$ from R_{12}, the resulting matroid can be obtained by taking the 3-sum of two graphic matroids. It follows from [8, Proposition 12.4.19] that the class of graphic matroids is closed under 3-sums. Therefore removing one of these elements from R_{12} produces a graphic matroid.

Let us now consider the matroids produced by removing an element in $\{3, 4, 7, 8, 11, 12\}$ from R_{12}. The class of cographic matroids is not closed under 3-sums, so we must do a more detailed analysis. It is clear that up to isomorphism there are only two matroids that we can obtain. These are shown in Figs. 6 and 7, along with labelled graphs which show that both matroids are cographic.

Since R_{12} is connected and neither graphic nor cographic, but all of its proper minors are graphic or cographic, it is an excluded minor for the class of near-graphic matroids. □

Let $\mathcal{G}$ be the set of graphs shown in Fig. 5.

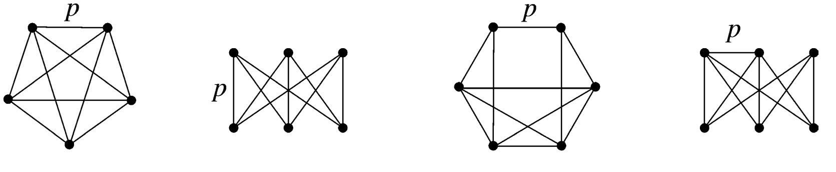

Fig. 5.

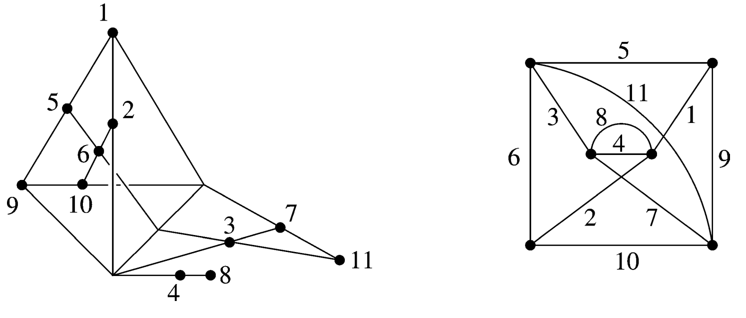

Fig. 6. $R_{12} \setminus \{12\}$ is cographic.

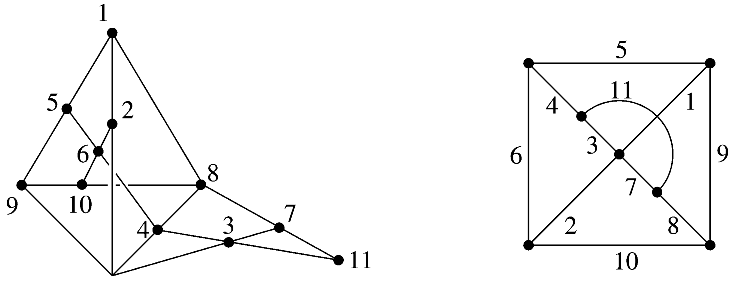

Fig. 7. $R_{12}/\{12\}$ is cographic.

5.4.2. *If G and H are two, not necessarily distinct, graphs from $\mathcal{G}$, then the 2-sum of $M(G)$ and $M^*(H)$ along the basepoint p is an excluded minor for near-graphic matroids.*

Proof. Both $M(G)$ and $M^*(H)$ are minors of $M(G) \oplus_2 M^*(H)$. Therefore $M(G) \oplus_2 M^*(H)$ is neither graphic nor cographic, as $M(G)$ has either an $M(K_5)$-minor, or an $M(K_{3,3})$-minor, and $M^*(H)$ has either an $M^*(K_5)$-minor, or an $M^*(K_{3,3})$-minor.

It is not difficult to show that deleting or contracting an edge other than p from a graph in $\mathcal{G}$ produces a planar graph. Therefore, removing an element other than p from $M(G)$ produces a matroid that is both graphic and cographic. Proposition 5.1 then implies that if $e \in E(M(G)) - p$, both $(M(G) \oplus_2 M^*(H)) \setminus e$ and $(M(G) \oplus_2 M^*(H))/e$ can be expressed as the 2-sum of two cographic matroids. The classes of graphic and cographic matroids are preserved under 2-sums [8, Corollary 7.1.23], so $(M(G) \oplus_2 M^*(H)) \setminus e$ and $(M(G) \oplus_2 M^*(H))/e$ are both cographic. Similarly, if $e \in E(M^*(H)) - p$, then $M^*(H) \setminus e$ and $M^*(H)/e$ are graphic and cographic, so $(M(G) \oplus_2 M^*(H)) \setminus e$ and $(M(G) \oplus_2 M^*(H))/e$ are both graphic. □

We may now complete the proof of Theorem 5.4. Let M be an excluded minor for near-graphic matroids, and suppose that M has no minor isomorphic to $U_{2,4}$, F_7, F_7^*, R_{10}, or R_{12}. This implies that M is regular [10]. It follows immediately from Proposition 5.3 that M is not 3-connected.

Since M is not 3-connected, but is connected, there exist matroids, M_1 and M_2, such that $E(M_1) \cap E(M_2) = \{p\}$ and $M = M_1 \oplus_2 M_2$. Both M_1 and M_2 must be connected, for otherwise M is not connected. Since M_1 and M_2 are proper minors of M, they must be either graphic or cographic. If both were graphic or cographic, then M would be graphic or cographic. Therefore, we will assume that M_1 is graphic, but not cographic, and that M_2 is cographic but not graphic.

It follows that M_1 must have an $M(K_5)$- or an $M(K_{3,3})$-minor. Let N be a minor of M_1 such that N is isomorphic to either $M(K_5)$ or $M(K_{3,3})$. Suppose that $e \in E(M_1) - p$. We wish to show that $M_1 \setminus e$ does not have an N-minor. Suppose that it does. Then $M \setminus e$ has an N-minor, and an M_2-minor, and therefore is neither graphic nor cographic. Since $M \setminus e$ is near-graphic, it follows that $M \setminus e$ is not connected. As $M \setminus e$ is the 2-sum of $M_1 \setminus e$ and M_2, it must be the case that $M_1 \setminus e$ is not connected. Let M' be a connected component of $M_1 \setminus e$ that has an N-minor. It cannot be the case that $e \in \mathrm{cl}_{M_1}(E(M'))$, for in that case M_1 is not connected. Therefore e is a coloop in $M_1 \mid (E(M') \cup e)$, and hence, if we contract e

from $M_1 \mid (E(M') \cup e)$, we obtain a matroid that has an N-minor. Thus M_1/e has an N-minor. Because $M \setminus e$ is not connected, M/e must be connected. However, M_1/e is a minor of M/e, and hence M/e has an N-minor. Since M/e also has an M_2-minor, it follows that M/e is neither graphic nor cographic. This is a contradiction, as M/e is connected. We conclude that $M_1 \setminus e$ does not have an N-minor.

Using duality, we may also show that, if $e \in E(M_1) - p$, then M_1/e does not have an N-minor. Hence, either M_1 is isomorphic to $M(K_5)$ or $M(K_{3,3})$, or M_1 is isomorphic to a matroid obtained by extending or coextending $M(K_5)$ or $M(K_{3,3})$ by a single element, p. Similarly, M_2 is either isomorphic to $M^*(K_5)$ or $M^*(K_{3,3})$, or can be obtained from one of these matroids by extending or coextending by p.

Let us suppose that M_1 is a single-element extension or coextension of $M(K_5)$ or $M(K_{3,3})$. Recall that M_1 is graphic. Since M_1 is connected, p cannot be a loop or a coloop of M_1. If p is a member of a parallel or series pair of M_1, then it is easy to see that $M_1 \oplus_2 M_2$ has a proper minor isomorphic to either $M(K_5) \oplus_2 M_2$ or $M(K_{3,3}) \oplus_2 M_2$. Neither of these matroids is near-graphic, so we have a contradiction. Given these restrictions, it follows that M_1 must be either the single-element coextension of $M(K_5)$ that is the polygon matroid of the graph (c) in Fig. 5, or the single-element extension of $M(K_{3,3})$ that is the polygon matroid of graph (d). By similar reasoning, M_2 must be the bond matroid of one of the graphs in $\mathcal{G}$. This completes the proof. □

6. An induced class with infinitely many excluded minors

The classes of binary matroids, regular matroids, and the polygon matroids of planar graphs are known to have 1, 3, and 7 non-isomorphic excluded minors respectively. In the previous section we have shown that the class of near-graphic matroids has exactly 21 non-isomorphic excluded minors. It is natural to ask whether there exists an induced class of matroids that has infinitely many excluded minors. In this section we will show that such a class exists. The example we consider was suggested by a referee of this paper.

If H is a circuit-hyperplane of M, and M' is produced by relaxing H in M, then, for every element $e \in E(M) - H$, the set $H \cup e$ is a circuit of M'. If B is any basis of a matroid, and, for every element, $e \notin B$, the set $B \cup e$ is a circuit, then we shall say that B is a *loose basis*. If B is a loose basis of M, then the set $\mathcal{B}(M) - \{B\}$ is the family of bases of a matroid on $E(M)$ [7, (1.5)]. This matroid will be said to be produced from M by *tightening the loose basis B*. Clearly this operation is the inverse of relaxing a circuit-hyperplane.

We now define the class $\mathcal{N}$ of matroids, so that $M \in \mathcal{N}$ if and only if every connected component of M is either binary, or can be obtained from a binary matroid by relaxing a single circuit-hyperplane.

Theorem 6.1. *The class $\mathcal{N}$ is an induced class.*

We defer the proof of Theorem 6.1. Proving that $\mathcal{N}$ is closed under generalised duality and taking minors is relatively simple, but the proof that it is closed under rank-preserving weak maps is more difficult.

We will note here that $\mathcal{N}$ has an infinite number of excluded minors: let J_n denote the $n \times n$ matrix of ones. For $n \geqslant 1$, let M_r be the binary matroid that is represented over GF(2) by $A_r = [I_n \mid J_n - I_n]$. Let the columns of A_r be labelled $a_1 \dots a_r, b_1 \dots b_r$. We will take the ground set of M_r to be the set of column labels. If r is even, then $H_1 = \{a_1, b_2 \dots b_r\}$ and $H_2 = \{a_2 \dots a_r, b_1\}$ are both circuit-hyperplanes of M_r. Let N_r be the matroid obtained by relaxing both of these circuit-hyperplanes.

Proposition 6.2. *If $r \geqslant 4$, and r is even, then N_r is an excluded minor for $\mathcal{N}$.*

Proof. If B is a loose basis of a binary matroid, M, and e and f are two elements of $E(M) - B$, then both $B \cup e$ and $B \cup f$ are circuits of M. Since the symmetric difference of two circuits in a binary matroid is itself a union of circuits, it follows that $\{e, f\}$ must be a circuit of M. Note that H_1 is a loose basis of N_r. Since $r \geqslant 4$, it follows that $\{a_2, b_1\}$ is an independent pair of elements contained in $E(N_r) - H_1$. Therefore N_r cannot be binary. Furthermore, we can show that H_1 and H_2 are the only loose bases of N_r. Let N_r' be the matroid obtained by tightening H_1. Then H_2 is a loose basis in N_r', and a_1 and b_2 are both contained in $E(N_r) - H_2$. Since $r \geqslant 4$, it is not the case that a_1 and b_2 are parallel, so N_r' is not binary. Using the same argument, we may show that tightening H_2 in N_r does not produce a binary matroid. Therefore $N_r \notin \mathcal{N}$.

However, it is easy to see that removing a single element from N_r produces a matroid that has exactly one loose basis, and tightening this basis produces a binary matroid. □

Proposition 6.3. *The class $\mathcal{N}$ is closed under generalised duality.*

Proof. Suppose that M is in $\mathcal{N}$, and that M_1 is a connected component of M. If M_1 is binary, then so is M_1^*, so we need only show that, if M_1 is obtained from a binary matroid by relaxing a circuit-hyperplane, then M_1^* can be obtained from a binary matroid in the same way. This follows immediately from [8, Proposition 2.1.7]. □

The next result follows without difficulty from [8, Proposition 3.3.9].

Proposition 6.4. *The class $\mathcal{N}$ is closed under taking minors.*

Let (M_1, M_2, M_3, M_4) be a sequence of matroids, such that, for $1 \leqslant i \leqslant 4$, the matroid M_i contains at least two elements, and is either uniform of rank one, or uniform of corank one. For $1 \leqslant i \leqslant 4$, let e_i be an element of $E(M_i)$. Let N be isomorphic to $U_{2,4}$, and suppose that the ground set of N is $\{e_1, \dots, e_4\}$. We will use the notation $M(M_1, M_2, M_3, M_4)$ to denote the matroid

$$\big(\big((N \oplus_2 M_1) \oplus_2 M_2\big) \oplus_2 M_3\big) \oplus_2 M_4,$$

where the 2-sum that involves M_i uses the element e_i as its basepoint. Note that $M(M_1, M_2, M_3, M_4)$ is obtained from $U_{2,4}$ by a sequence of up to four parallel or series extensions.

We will need the following result, which can easily be deduced from a theorem of Oxley's.

Theorem 6.5 [7, Theorem 1.2]. *Let M be a non-binary matroid, such that, for every element $e \in E(M)$, either $M \setminus e$ or M/e is binary. Then, either*:

(i) $M \in \mathcal{N}$;
(ii) *the rank or the corank of M is equal to two*; *or,*
(iii) *there exists a sequence, (M_1, M_2, M_3, M_4), such that $M \cong M(M_1, M_2, M_3, M_4)$.*

Lemma 6.6. *The class $\mathcal{N}$ is closed under rank-preserving weak maps.*

Proof. Let us suppose that the lemma does not hold. Then there exists a matroid, M, such that $M_1 \xrightarrow{\text{r.p.}} M$, where $M_1 \in \mathcal{N}$, but $M \notin \mathcal{N}$. Among such counterexamples let M be chosen to be as small as possible, so that if M' is a rank-preserving weak-map image of a matroid in $\mathcal{N}$, and $|E(M')| < |E(M)|$, then M' itself is in $\mathcal{N}$.

6.6.1. *The matroid M is an excluded minor for the class $\mathcal{N}$.*

Proof. Suppose that M' is a proper minor of M. There exists a minor, M_1', of M_1 such that $M_1' \xrightarrow{\text{r.p.}} M'$ [3, Theorem 5.8]. Since $M_1' \in \mathcal{N}$, it follows that M' is in $\mathcal{N}$. □

Since $\mathcal{N}$ is, by construction, closed under direct sums, it follows that M is connected.

6.6.2. *The matroid M_1 is not binary*.

Proof. It has already been noted, and is easy to prove directly, that a rank-preserving weak-map image of a binary matroid is itself binary. Hence, if M_1 were binary, then M would be binary, and would therefore belong to $\mathcal{N}$. □

We conclude that M_1 can be obtained from a binary matroid, M_2, by relaxing a circuit-hyperplane, H.

6.6.3. $r(M) > 2$.

Proof. Suppose otherwise. Since every matroid of rank at most one belongs to $\mathcal{N}$, the rank of M, and therefore M_1, must be two. Since M_1 is not binary, but can be obtained from a binary matroid by relaxing a circuit-hyperplane, it follows that M_1 must contain exactly four parallel classes. Any connected rank-preserving weak-map image of M_1 that is not isomorphic to M_1 contains at most three parallel classes, and is therefore binary. From this contradiction we conclude that $r(M) > 2$. □

Since $\mathcal{N}$ is closed under duality, M^* is also a minimal counterexample to Lemma 6.6. By applying the arguments above, we may conclude that $r(M^*) > 2$.

6.6.4. *If $e \in H$, then $M \setminus e$ is binary, and if $e \notin H$, then M/e is binary*.

Proof. Let e be an element of $E(M)$. It is not the case that e is a coloop of M_1 or M, for M is connected, and if e were a coloop of M_1 it would be a coloop of M. Therefore $M_1 \setminus e \xrightarrow{\text{r.p.}} M \setminus e$. Suppose that $e \in H$. Then [8, Proposition 3.3.9] implies that $M_1 \setminus e = M_2 \setminus e$. Therefore $M_1 \setminus e$ is binary. Since $M \setminus e$ is the rank-preserving weak-map image of a binary matroid, it is also binary.

Similarly, e is not a loop in M_1 or in M. Therefore $M_1/e \xrightarrow{\text{r.p.}} M/e$. Also, if $e \notin H$, then $M_1/e = M_2/e$. It follows that M/e is binary. □

From Theorem 6.5, and our assumption on the rank and corank of M, we conclude that there exists a sequence, (M_1, M_2, M_3, M_4), such that $M \cong M(M_1, M_2, M_3, M_4)$.

If, for some $i \in \{1, \dots, 4\}$, the matroid M_i is isomorphic to $U_{1,2}$, we will say that M_i is *trivial*. Note that $M \oplus_2 U_{1,2} \cong M$, for any matroid M.

6.6.5. *No more than two of the non-trivial matroids in* (M_1, M_2, M_3, M_4) *are cocircuits, and no more than two are circuits.*

Proof. It is easy to see that

$$\big(M(M_1, M_2, M_3, M_4)\big)^* = M\big(M_1^*, M_2^*, M_3^*, M_4^*\big).$$

Furthermore, the order of the matroids in (M_1, M_2, M_3, M_4) is insignificant. Therefore, if the claim in 6.6.5 is false, then, by duality and relabelling, we may assume that $M_i \cong U_{1,n_i}$, where $n_i > 2$, for all $i \in \{1, 2, 3\}$. It follows that M_4 is a circuit of size at least three, for otherwise $r(M) = 2$. By deleting all but two elements from $E(M_i) - e_i$, for $1 \leqslant i \leqslant 3$, and contracting all but one element from $E(M_4) - e_4$, we see that M has a minor isomorphic to $M' = M(U_{1,3}, U_{1,3}, U_{1,3}, U_{1,2})$. This is the rank-2 matroid that has three parallel classes of size two, and one parallel class of size one. It is not difficult to show that M' is an excluded minor for $\mathcal{N}$, so M must be isomorphic to M'. But this contradicts our assumption that $r(M) > 2$. □

6.6.6. *There are no trivial matroids in* (M_1, M_2, M_3, M_4).

Proof. If the claim is false, then we may assume that $M_1 \cong U_{1,2}$. It cannot be the case that every matroid in (M_1, M_2, M_3, M_4) is trivial, for then M would have rank two. By referring to 6.6.5, and using duality and relabelling, we may assume that M_2 is a circuit of size at least three, and that M_3 and M_4 are cocircuits. Then the rank of M is $|E(M_2)|$, and it is easy to see that $(E(M_1) \cup E(M_2)) - \{e_1, e_2\}$ is a loose basis of M.

The only non-spanning circuits in the matroid obtained by tightening this basis are: $(E(M_1) \cup E(M_2)) - \{e_1, e_2\}$; any pair of elements in $E(M_3) - e_3$; any pair of elements in $E(M_4) - e_4$; and, any triple of elements containing the single element of $E(M_1) - e_1$, an element from $E(M_3) - e_3$, and an element from $E(M_4) - e_4$. It is easy to see that this matroid is isomorphic to the 2-sum of two binary matroids, and is therefore binary. This implies that M is a member of $\mathcal{N}$. □

We may now assume that $|E(M_i)| > 2$ for all $i \in \{1, \ldots, 4\}$. Furthermore, from 6.6.5, and by relabelling if necessary, we will assume that M_1 and M_2 are cocircuits, while M_3 and M_4 are circuits. It is easily demonstrated that the rank of M is $|E(M_3)| + |E(M_4)| - 2$, and that $(E(M_3) \cup E(M_4)) - \{e_3, e_4\}$ is a loose basis of M. The only non-spanning circuits in the matroid obtained by tightening this basis are: $(E(M_3) \cup E(M_4)) - \{e_3, e_4\}$; any pair of elements in $E(M_1) - e_1$; any pair of elements in $E(M_2) - e_2$; the union of $E(M_3) - e_3$ with an element from $E(M_1) - e_1$ and an element from $E(M_2) - e_2$; and, the union of $E(M_4) - e_4$ with an element from $E(M_1) - e_1$ and an element from $E(M_2) - e_2$. It is not difficult to see that this matroid is binary, and that M is therefore in $\mathcal{N}$. This contradiction completes the proof of the lemma. □

Proof of Theorem 6.1. The proof follows from Theorem 2.7, Proposition 6.3, Proposition 6.4, Lemma 6.6, and the obvious observation that $\mathcal{N}$ is closed under the addition of loops and coloops. □

Acknowledgments

I thank my supervisor, Professor Dominic Welsh, for his valuable guidance, and the referees, for their very useful suggestions.

References

[1] C.A. Holzmann, P.G. Norton, M.D. Tobey, A graphical representation of matroids, SIAM J. Appl. Math. 25 (1973) 618–627.
[2] M. Lemos, J. Oxley, A sharp bound on the size of a connected matroid, Trans. Amer. Math. Soc. 353 (2001) 4039–4056.
[3] D. Lucas, Weak maps of combinatorial geometries, Trans. Amer. Math. Soc. 206 (1975) 247–279.
[4] S. Maurer, Matroid basis graphs I, J. Combin. Theory Ser. B 14 (1973) 216–240.
[5] S. Maurer, Matroid basis graphs II, J. Combin. Theory Ser. B 15 (1973) 121–145.
[6] D. Mayhew, DPhil Thesis, University of Oxford, 2005.
[7] J. Oxley, A characterization of a class of non-binary matroids, J. Combin. Theory Ser. B 49 (1990) 181–189.
[8] J. Oxley, Matroid Theory, Oxford Univ. Press, Oxford, 1992.
[9] P.D. Seymour, Decomposition of regular matroids, J. Combin. Theory Ser. B 28 (1980) 305–359.
[10] W.T. Tutte, A homotopy theorem for matroids I, II, Trans. Amer. Math. Soc. 88 (1958) 144–174.
[11] W.T. Tutte, Matroids and graphs, Trans. Amer. Math. Soc. 90 (1959) 527–552.