

CODIMENSION ONE AND TWO BIFURCATIONS IN CATTANEO-CHRISTOV HEAT FLUX MODEL

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ABSTRACT. Layek and Pati (Phys. Lett. A, 2017) studied a nonlinear system of five coupled equations, which describe thermal relaxation in Rayleigh-Benard convection of a Boussinesq fluid layer, heated from below. Here we return to that paper and use techniques from dynamical systems theory to analyse the codimension-one Hopf bifurcation and codimension-two double-zero Bogdanov-Takens bifurcation. We determine the stability of the bifurcating limit cycle, and produce an unfolding of the normal form for codimension-two bifurcation, appropriate for the Layek and Pati's model.

1. Introduction. Navier-Stokes and thermodynamic partial differential equations, governing the dynamics of weather and climate are highly nonlinear, so that accurate future predictions are hard to achieve. Although some classical results have been obtained, it is still extremely difficult and costly to make systematic investigations. To this end, lower-dimensional models are derived by Galerkin truncations of the full partial differential equations.

The Lorenz-63 model is the lowest order nonlinear truncated model for force-dissipated hydrodynamic flow [1]. The Lorenz-84 atmospheric circulation model (Lorenz-84 model) is another low order atmospheric circulation [2]. The Lorenz-Stenflo system (Lorenz-96 model) models low-frequency and short-wave acoustic

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gravity disturbances in the atmosphere and can be described by generalized Lorenz equations [3]. In addition, a modified Lorenz-Stenflo system, introduced and analyzed in [4], can show hidden hyperchaos with only one stable equilibrium when parameters vary. When two additional physical ingredients are considered in the governing equations, namely, rotation and density-dependent buoyancy in the fluid, a six-dimensional nonlinear ordinary differential equation system results [5].

Recently, Layek and Pati reported on the influences of thermal time-lag on the onset of convection, and obtained a five-dimensional nonlinear system from a low-order Galerkin expansion of velocity and temperature variables [6]:

$$\begin{cases} \dot{X} = \sigma(Y - X), \\ \dot{Y} = rX - P - XZ, \\ \dot{Z} = XY - W, \\ \dot{P} = -XW - \delta(P - Y), \\ \dot{W} = XP - \delta(W - bZ), \end{cases} \quad (1)$$

where the dot denotes derivative with respect to the normalized time $\tau = \Lambda t$ ($\Lambda = \pi^2 + a^2$, a is the horizontal wavenumber). Here σ is the Prandtl number, r is the normalized Rayleigh number, $b = 4\pi^2/\Lambda$ is the geometrical parameter, $\delta = 1/(2\Lambda C)$ and C is the Cattaneo number. Layek and Pati include some numerical results including bifurcation transition diagrams, plots of the largest Lyapunov exponents, phase-space portraits, and routes to chaos via period-doubling bifurcations for $r > 10$ [6]. In 2019, Daumann and Rech used Lyapunov exponents spectra to characterize the dynamical behavior of system (1), and found numerically the existence of chaos in different parameter regions [7] (see the attractors in the phase-space of system (1) in Fig. 1). Therefore, understanding the local and the global behaviors of hyperchaotic system (1) is of great importance. Moreover, it is important to determine what type of complex dynamics this system is capable of. For a better understanding of complex dynamics, the analysis of bifurcations of vector fields of codimension-one and -two have been widely studied in several papers or books on dynamical systems [8, 9, 10, 12]. As a codimension-one bifurcation, the Hopf bifurcation concerns the relationship between linear stability of an equilibrium state and periodic solutions when the linear system has a pair of pure imaginary eigenvalues. The Bogdanov-Takens bifurcation is relevant for classification and unfolding of these systems with an equilibrium having two zero eigenvalues. This is the main focus of the present report.

The paper is organized as follows. In Section 2, existence and linearised stability of equilibrium solutions is presented. Section 3, summarises the theory behind codimension-one Hopf bifurcation methods, in particular, how to calculate the Lyapunov coefficients, determining the stability of the bifurcating limit cycle. In Section 4, the Bogdanov-Takens bifurcation is discussed, and we obtain the curves of saddle-node bifurcation curve, the Andronov-Hopf bifurcation curve and the homoclinic bifurcation curve by unfolding the Bogdanov-Takens point. Finally, Section 5 contains some concluding remarks.

2. Equilibria and stability. If we set right-hand side of system (1) to zero, we can obtain

$$Y = X, Z = \frac{X^2 - \delta + r\delta}{\delta}, P = \frac{-X^3 + \delta X}{\delta}, W = X^2 \quad (2)$$

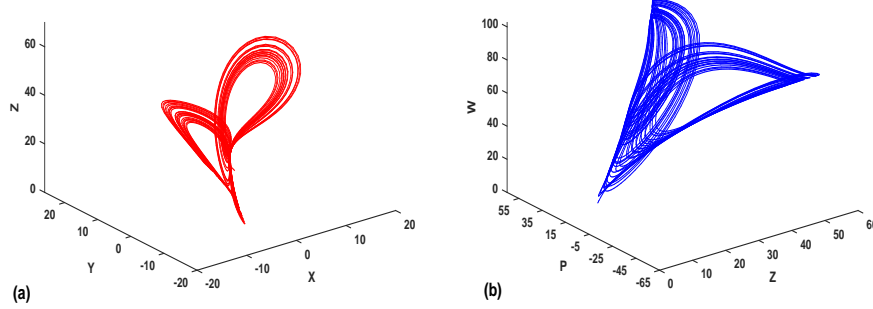


Fig.1. When $\sigma = 10, r = 28, \delta = 15, b = 3$ and initial condition is $(5.1, 6.2, 7.3, 8.4, 9.5)$, chaotic attractors are shown and corresponding Lyapunov exponents are $(0.9263, -0.0000, -11.8503, -14.3371, -14.7389)$: (a) X-Y-Z space; (b) Z-P-W space.

and

$$X^4 + \delta(\delta - 1 - b)X^2 - b(r - 1)\delta^2 = 0. \quad (3)$$

We introduce the following notation

$$\begin{aligned} r^* &= \frac{-1 + 2b - b^2 + 2\delta + 2b\delta - \delta^2}{4b}, X_0 = \frac{\sqrt{\delta + b\delta - \delta^2}}{\sqrt{2}}, \\ X_1 &= \frac{\sqrt{\delta + b\delta - \delta^2 + \delta\sqrt{1 - 2b + b^2 + 4br - 2\delta - 2b\delta + \delta^2}}}{\sqrt{2}}, \\ X_2 &= \frac{\sqrt{\delta + b\delta - \delta^2 - \delta\sqrt{1 - 2b + b^2 + 4br - 2\delta - 2b\delta + \delta^2}}}{\sqrt{2}}. \end{aligned}$$

Hence, we obtain the following results:

(A) if $r < r^*$, system (1) has only one trivial equilibrium: $O(0, 0, 0, 0, 0)$;

(B) if $r = r^*$ and $1 + b - \delta > 0$, system (1) has three equilibria:

$$\begin{aligned} &O(0, 0, 0, 0, 0), O_1 \left(X_0, X_0, \frac{X_0^2 - \delta + r\delta}{\delta}, \frac{-X_0^3 + \delta X_0}{\delta}, X_0^2 \right), \\ &O_2 \left(-X_0, -X_0, \frac{X_0^2 - \delta + r\delta}{\delta}, \frac{X_0^3 - \delta X_0}{\delta}, X_0^2 \right), \end{aligned}$$

otherwise system (1) has only one trivial equilibrium: $O(0, 0, 0, 0, 0)$;

(C) if $r > r^*$ and $1 + b - \delta < -\sqrt{(b-1)^2 + 4br - 2\delta - 2b\delta + \delta^2}$, system (1) has only one trivial equilibrium point origin $O(0, 0, 0, 0, 0)$;

(D) if $r > r^*$ and

$$-\sqrt{(b-1)^2 + 4br - 2\delta - 2b\delta + \delta^2} < 1 + b - \delta < \sqrt{(b-1)^2 + 4br - 2\delta - 2b\delta + \delta^2},$$

system (1) has three equilibria:

$$\begin{aligned} &O(0, 0, 0, 0, 0), E_1 \left(X_1, X_1, \frac{X_1^2 - \delta + r\delta}{\delta}, \frac{-X_1^3 + \delta X_1}{\delta}, X_1^2 \right), \\ &E_2 \left(-X_1, -X_1, \frac{X_1^2 - \delta + r\delta}{\delta}, \frac{X_1^3 - \delta X_1}{\delta}, X_1^2 \right); \end{aligned}$$

(E) if $r > r^*$ and $1 + b - \delta > \sqrt{(b-1)^2 + 4br - 2\delta - 2b\delta + \delta^2}$, system (1) has five equilibria:

$$\begin{aligned} &O(0, 0, 0, 0, 0), E_1 \left(X_1, X_1, \frac{X_1^2 - \delta + r\delta}{\delta}, \frac{-X_1^3 + \delta X_1}{\delta}, X_1^2 \right), \\ &E_2 \left(-X_1, -X_1, \frac{X_1^2 - \delta + r\delta}{\delta}, \frac{X_1^3 - \delta X_1}{\delta}, X_1^2 \right), \\ &E_3 \left(X_2, X_2, \frac{X_2^2 - \delta + r\delta}{\delta}, \frac{-X_2^3 + \delta X_2}{\delta}, X_2^2 \right), \\ &E_4 \left(-X_2, -X_2, \frac{X_2^2 - \delta + r\delta}{\delta}, \frac{X_2^3 - \delta X_2}{\delta}, X_2^2 \right). \end{aligned}$$

The linear stability of equilibrium O leads to the characteristic equation:

$$(\lambda^3 + (\delta + \sigma)\lambda^2 + (\sigma\delta - \sigma r + \delta)\lambda + \sigma\delta - r\sigma\delta)(\lambda^2 + \delta\lambda + b\delta) = 0. \quad (4)$$

Routh-Hurwitz criterion shows that the trivial equilibrium state O is asymptotically stable when

$$r < \min \left\{ 1, \frac{\delta^2 + \delta^2\sigma + \delta\sigma^2}{\sigma^2} \right\}. \quad (5)$$

Linearizing (1) about the equilibrium

$$E_i \left(X_i, X_i, \frac{X_i^2 - \delta + r\delta}{\delta}, \frac{-X_i^3 + \delta X_i}{\delta}, X_i^2 \right) \quad (i = 0, 1, 2)$$

yields the characteristic equation:

$$\begin{aligned} &\lambda^5 + (2\delta + \sigma)\lambda^4 + \frac{2X_i^2\delta + \delta^2 + b\delta^2 + \delta^3 + X_i^2\sigma - \delta\sigma + 2\delta^2\sigma}{\delta}\lambda^3 \\ &+ (2X_i^2\delta + \delta^2 + b\delta^2 + 4X_i^2\sigma - \delta\sigma + b\delta\sigma + \delta^2\sigma)\lambda^2 \\ &+ \frac{X_i^4(\delta + 3\sigma) - (1 + b)X_i^2\delta^2 + (b + X_i^2 + b\sigma)\delta^3}{\delta}\lambda \\ &+ (bX_i^2 - 3X_i^2 + 4X_i^2\delta - b\delta)\sigma\lambda + 2X_i^2(2X_i^2 - \delta - b\delta + \delta^2)\sigma = 0. \end{aligned} \quad (6)$$

In order to consider stability of equilibrium, we write

$$\lambda^5 + \delta_1\lambda^4 + \delta_2\lambda^3 + \delta_3\lambda^2 + \delta_4\lambda + \delta_5 = 0. \quad (7)$$

Constructing the Routh array, we find that the real parts of all the roots λ are negative if and only if

$$\begin{aligned} \Delta_1 = \delta_1 > 0, \Delta_2 = \begin{vmatrix} \delta_1 & \delta_3 \\ 1 & \delta_2 \end{vmatrix} > 0, \Delta_3 = \begin{vmatrix} \delta_1 & \delta_3 & \delta_5 \\ 1 & \delta_2 & \delta_4 \\ 0 & \delta_1 & \delta_3 \end{vmatrix} > 0, \\ \Delta_4 = \begin{vmatrix} \delta_1 & \delta_3 & \delta_5 & 0 \\ 1 & \delta_2 & \delta_4 & 0 \\ 0 & \delta_1 & \delta_3 & \delta_5 \\ 0 & \delta_1 & \delta_2 & \delta_4 \end{vmatrix} > 0, \delta_5 > 0. \end{aligned} \quad (8)$$

Therefore, the equilibrium E_i is asymptotically stable when the above conditions (8) are met.

To simplify, we consider only the effect of the parameters r and δ and set the other parameters $\sigma = 10, b = 8/3$. Equilibrium O is asymptotically stable if r and δ lie in the green region of Fig. 2(a). Equilibria $E_{1,2}$ are asymptotically stable if r

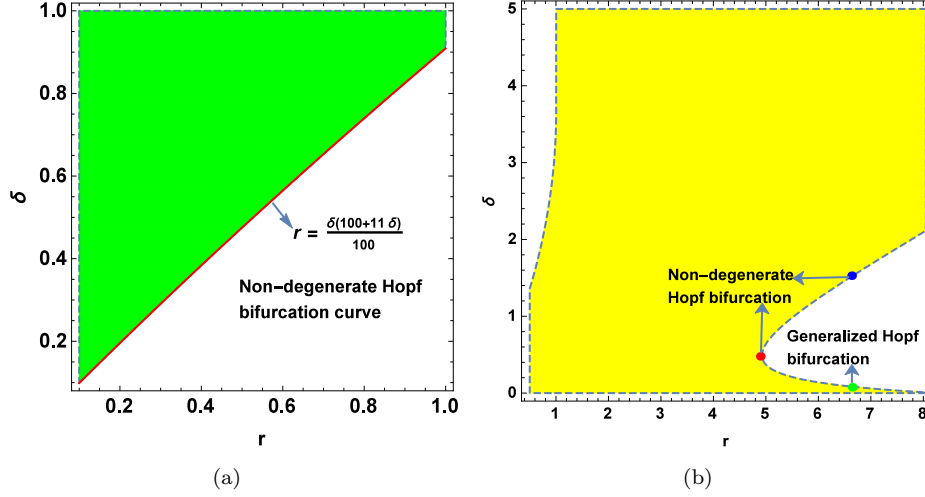


Fig. 2. (a) Let $(\sigma, b) = (10, 8/3)$. The equilibrium O of system (1) is asymptotically stable in the green region; the equilibria $E_{1,2}$ of system (1) is asymptotically stable in the yellow region.

and δ lie in the yellow region of Fig. 2(b). Equilibria $E_{3,4}$ will exist and be unstable for $1 + b - \delta > \sqrt{1 - 2b + b^2 + 4br - 2\delta - 2b\delta + \delta^2}$, which means δ_5 will be negative.

3. Hopf bifurcation analysis. We now summarise the key steps of the projection method described in [9, 10, 11] for the calculation of the first Lyapunov coefficient l_1 for the stability of Hopf bifurcations.

Consider the differential equation

$$\dot{x} = f(x, \mu), \quad (9)$$

where $x \in R^5$ and $\mu \in R$ are respectively vectors representing phase variables and control parameters. Assume that f is a class of C^∞ and in $R^5 \times R$. Suppose that (16) has an equilibrium point $x = x_0$ at $\mu = \mu_0$, and denoting the variable $x - x_0$ also by x , write

$$F(x) = f(x, \mu_0), \quad (10)$$

as

$$F(x) = Ax + \frac{1}{2}B(x, x) + \frac{1}{6}C(x, x, x) + O(\|x\|^4), \quad (11)$$

where $A = f_x(0, \mu_0)$ and

$$B_i(x, y) = \sum_{j, k=1}^5 \frac{\partial^2 F_i(\xi)}{\partial \xi_j \partial \xi_k} \Big|_{\xi=0} x_j y_k,$$

$$C_i(x, y, z) = \sum_{j, k, l=1}^5 \frac{\partial^3 F_i(\xi)}{\partial \xi_j \partial \xi_k \partial \xi_l} \Big|_{\xi=0} x_j y_k z_l,$$

$$D_i(x, y, z, u) = \sum_{j, k, l, m=1}^5 \frac{\partial^4 F_i(\xi)}{\partial \xi_j \partial \xi_k \partial \xi_l \partial \xi_m} \Big|_{\xi=0} x_j y_k z_l u_m,$$

$$E_i(x, y, z, u, v) = \sum_{j, k, l, m, h=1}^5 \frac{\partial^5 F_i(\xi)}{\partial \xi_j \partial \xi_k \partial \xi_l \partial \xi_m \partial \xi_h} \Big|_{\xi=0} x_j y_k z_l u_m v_h,$$

where $i = 1, 2, 3, 4, 5$.

We assume that A has a pair of complex conjugate eigenvalues on the imaginary axis: $\lambda_{4,5} = \pm i\omega_0$ ($\omega_0 > 0$), with no other eigenvalues having $\operatorname{Re} \lambda = 0$. Let T^c be the generalized eigenspace of A corresponding to $\lambda_{4,5}$. Let $p, q \in C^5$ be vectors such that

$$Aq = i\omega_0 q, A^T p = -i\omega_0 p, \langle p, q \rangle = 1, \quad (12)$$

where A^T is the transpose of matrix A . Any vector $y \in T^c$ can be written as $y = wq + \bar{w}\bar{q}$, where $w = \langle q, y \rangle \in C$. The two-dimensional center manifold associated with the eigenvalues $\lambda_{2,3}$ can be parameterized by w and \bar{w} , by an immersion of the form $x = H(w, \bar{w})$, where $H : C \rightarrow R^5$ has a Taylor expansion of the form

$$H(w, \bar{w}) = wq + \bar{w}\bar{q} + \sum_{2 \leq j+k \leq 5} \frac{1}{j!k!} h_{jk} w^j \bar{w}^k + O(|w|^6), \quad (13)$$

with $h_{jk} \in C^5$ and $h_{jk} = \bar{h}_{kj}$. Substituting $H(w, \bar{w})$ into (10) yields the following differential equation

$$H_w w' + H_{\bar{w}} \bar{w}' = F(H(w, \bar{w})),$$

where F is given by (10). The complex vectors h_{ij} are obtained to solve the system of linear equations defined by the coefficients of (13). For the second-order coefficients

$$h_{20} = (2i\omega_0 I_5 - A)^{-1} B(q, q),$$

$$h_{11} = -A^{-1} B(q, \bar{q}).$$

For the third-order coefficients, we have

$$h_{30} = (3i\omega_0 I_5 - A)^{-1} (C(q, q, q) + 3B(q, h_{20})) \quad (14)$$

and following singular equation

$$(i\omega_0 I_5 - A)h_{21} = (C(q, q, \bar{q}) + B(\bar{q}, h_{20}) + 2B(q, h_{11}) - G_{21}q). \quad (15)$$

h_{21} can be obtained from the (15) if and only if

$$\langle p, C(q, q, \bar{q}) + B(\bar{q}, h_{20}) + 2B(q, h_{11}) - G_{21}q \rangle = 0.$$

Taking into account the coefficients of F , so that system (10), on the chart w for a central manifold, writes as follows

$$\dot{w} = iw_0 w + \frac{1}{2} G_{21} w |w|^2 + O(|w|^4),$$

where $G_{ij} \in C$ and first Lyapunov coefficient is then

$$l_1 = \frac{1}{2} \operatorname{Re} G_{21}, \quad (16)$$

where $G_{21} = \langle p, C(q, q, \bar{q}) + B(\bar{q}, h_{20}) + 2B(q, h_{11}) \rangle$.

The complex vector h_{21} can be found by solving the nonsingular six dimensional system

$$\begin{pmatrix} i\omega_0 I_5 & q \\ \bar{p} & 0 \end{pmatrix} \begin{pmatrix} h_{21} \\ s \end{pmatrix} = \begin{pmatrix} \mathcal{H}_{21} - G_{21}q \\ 0 \end{pmatrix}, \quad (17)$$

where $\mathcal{H}_{21} = C(q, q, \bar{q}) + B(\bar{q}, h_{20}) + 2B(q, h_{11})$ and $\langle p, h_{21} \rangle = 0$.

For the forth-order coefficients

$$\begin{aligned} h_{40} &= (4i\omega_0 I_5 - A)^{-1} (D(q, q, q, q) + 6C(q, q, h_{20}) + 4B(q, h_{30}) + 3B(h_{20}, h_{20})), \\ h_{31} &= (2i\omega_0 I_5 - A)^{-1} (D(q, q, q, \bar{q}) + 3C(q, q, h_{11}) + 3C(q, \bar{q}, h_{20}) + 3B(h_{20}, h_{11}) \\ &\quad + B(\bar{q}, h_{30}) + 3B(q, h_{21}) - 3G_{21}h_{20}), \\ h_{22} &= -A^{-1} (D(q, q, \bar{q}, \bar{q}) + 4C(q, \bar{q}, h_{11}) + C(\bar{q}, \bar{q}, h_{20}) + C(q, q, \bar{h}_{20}) + 2B(h_{11}, h_{11}) \\ &\quad + 2B(q, h_{21}) + 2B(\bar{q}, h_{21}) + B(h_{20}, h_{20}) - 2h_{11}(G_{21} + \bar{G}_{21})). \end{aligned}$$

Hopf bifurcation will be degenerate when l_1 vanishes. Defining \mathcal{H}_{32} as

$$\begin{aligned} \mathcal{H}_{32} &= 6B(h_{11}, h_{21}) + B(\bar{h}_{20}, h_{30}) + 3B(\bar{h}_{21}, h_{20}) + 3B(q, h_{22}) + 2B(\bar{q}, h_{31}) \\ &\quad + 6C(q, h_{11}, h_{11}) + 3C(q, \bar{h}_{20}, h_{20}) + 3C(q, q, \bar{h}_{21}) + 6C(q, \bar{q}, h_{21}) \\ &\quad + 6C(\bar{q}, h_{20}, h_{11}) + C(\bar{q}, \bar{q}, h_{30}) + D(q, q, q, \bar{h}_{20}) + 6D(q, q, \bar{q}, \bar{h}_{11}) \\ &\quad + 3D(q, \bar{q}, \bar{q}, h_{20}) + E(q, q, q, \bar{q}, \bar{q}) - 6G_{21}h_{21} - 3\bar{G}_{21}h_{21}, \end{aligned}$$

and $G_{32} = \langle p, \mathcal{H}_{32} \rangle$, the second Lyapunov coefficient l_2 is given by

$$l_2 = \frac{1}{2} \text{Re } G_{32}.$$

From system (1), we know

$$\begin{aligned} B(x, y) &= (0, -x_1y_3 - x_3y_1, x_1y_2 + x_2y_1, -x_1y_5 - x_5y_1, x_1y_4 + x_4y_1), \\ C(x, y, z) &= D(x, y, z, u) = E(x, y, z, u, v) = (0, 0, 0, 0, 0). \end{aligned} \quad (18)$$

We suppose that the characteristic equation (7) at O has a pair of pure imaginary roots $\pm i\omega$ ($\omega \in R^+$). For $4b\delta - \delta^2 > 0$ and $-\delta + \sigma - \delta\sigma > 0$, it is straightforward to show that when $r = r_0 = \frac{\delta(\delta + \delta\sigma + \sigma^2)}{\sigma^2}$, (7) yields

$$\lambda_1 = -\delta - \sigma < 0, \quad \lambda_{2,3} = \frac{1}{2}(-\delta \pm \sqrt{4b\delta - \delta^2})i, \quad \lambda_{4,5} = \pm \sqrt{\frac{\delta(-\delta + \sigma - \delta\sigma)}{\sigma}}i.$$

Here, we let $\sigma = 10, b = 8/3$ and take r as the Hopf bifurcation parameter. The transversality condition

$$\text{Re}(\lambda'(r_0))|_{\lambda = \sqrt{\frac{\delta(10-11\delta)}{10}}i} = \frac{500}{1000 + 210\delta - \delta^2} > 0$$

is also satisfied for $0 < \delta < 10/11$. Therefore, we have the following theorems.

Theorem 3.1. *If $\sigma = 10, b = 8/3$, as r passes through the critical value $r_0 = \frac{\delta(100+11\delta)}{100}$, system (1) undergoes a Hopf bifurcation at the equilibrium $O(0, 0, 0, 0, 0)$ when $0 < \delta < 10/11$. Moreover, the first Lyapunov coefficient associated with the equilibrium O is given by*

$$l_1 = -\frac{10000(-8000 + 2845\delta + 312\delta^2)}{\delta(100 + 11\delta)(-1000 - 210\delta + \delta^2)(200 - 870\delta + 1683\delta^2)}. \quad (19)$$

It is easy to know first Lyapunov coefficient l_1 will be negative from Fig.3. The non-degenerate Hopf point at O is stable (weak stable focus) and for $r > r_0$, but close to r_0 , there exists a stable limit cycle near the unstable equilibrium O .

Here, the non-degenerate Hopf bifurcation curve corresponds to the red line in Fig. 2(a). Now we also consider bifurcation analysis about the non trivial equilibrium point E_1 for $\sigma = 10, b = 8/3$.

Theorem 3.2. *If $\sigma = 10, b = 8/3, \delta = 0.5$ and r varies and passes through the critical value $r_1 = 4.8969$, system (1) undergoes a Hopf bifurcation at the equilibrium E_1 . Moreover, the first Lyapunov coefficient associated with the equilibrium E_1 is given by*

$$l_1 = -0.010311 < 0. \quad (20)$$

Therefore, the Hopf point at E_1 is stable (weak stable focus) and for $r > r_1$, but close to r_1 , a stable limit cycle bifurcates off the unstable equilibrium E_1 .

Transversality condition

$$\operatorname{Re}(\lambda'(r_1))|_{\lambda=2.9648i} = 0.0121 > 0$$

is also satisfied. This bifurcation point corresponds to the red dot in Fig. 2(b).

Now we choose $r = 6.638712$, it can be found that system (1) undergoes a nondegenerate Hopf bifurcation or generalized Hopf bifurcation for different bifurcation value δ_1 at the equilibrium E_1 . Transversality condition is also satisfied.

Theorem 3.3. *If $\sigma = 10, b = 8/3, r = 6.638712$ and δ varies and passes through the critical value $\delta_1 = 1.53698$, system (1) undergoes a Hopf bifurcation at the equilibrium E_1 . Moreover, the first Lyapunov coefficient associated with the equilibrium E_1 is given by*

$$l_1 = -0.00123 < 0. \quad (21)$$

Therefore, the Hopf point at E_1 is stable (weak stable focus) and for $\delta < \delta_1$, but close to δ_1 , a stable limit cycle bifurcates off the unstable equilibrium E_1 .

Transversality condition

$$\operatorname{Re}(\lambda'(\delta_1))|_{\lambda=4.9023i} < 0.$$

This bifurcation point corresponds to the blue dot in Fig. 2(b).

Here we can obtain some vectors for the computation of Lyapunov coefficient in Theorem 3.3:

$$\begin{aligned} q &= \{0.16665 + 0.17223i, 0.08222 + 0.25392i, 0.24304 - 0.30533i, \\ &\quad -0.11566 - 0.25313i, -0.80099\}, \\ p &= \{0.26271 - 0.06083i, 0.59957 + 0.88800i, 0.55945 - 0.63749i, \\ &\quad 0.10539 - 0.35695i, -0.35349 - 0.063397i\}, \\ h_{11} &= \{-0.00698, -0.00698, -0.00413, 0.06421, 0.07585\}, \\ h_{20} &= \{0.01614 + 0.02062i, -0.00408 + 0.03645i, 0.02899 - 0.00157i, \\ &\quad 0.02417 - 0.02184i, -0.04174 - 0.01171i\}. \end{aligned}$$

Theorem 3.4. *If $\sigma = 10, b = 8/3, r = 6.638712$ and δ varies and passes through the critical value $\delta_2 = 0.081613$, system (1) undergoes a Hopf bifurcation at the equilibrium E_1 . Moreover, the first Lyapunov coefficient will vanish and the second Lyapunov coefficient associated with the equilibrium E_1 is given by*

$$l_2 = -0.00011 < 0. \quad (22)$$

Therefore, the Hopf point at E_1 is stable (weak stable focus). But when we choose proper perturbations around $(r, \delta) = (6.638712, 0.081613)$, system (1) can exhibit two small-amplitude limit cycles.

Transversality condition

$$\operatorname{Re}(\lambda'(\delta_2))|_{\lambda=1.3034i} > 0.$$

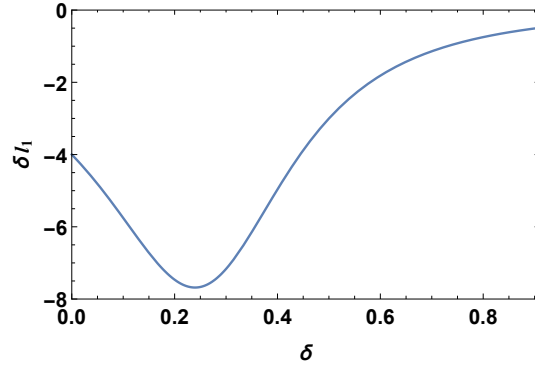


Fig. 3. First Lyapunov coefficient l_1 will be negative for $\sigma = 10, b = 8/3, 0 < \delta < 10/11$.

This bifurcation point corresponds to the red dot in Fig. 2(b).

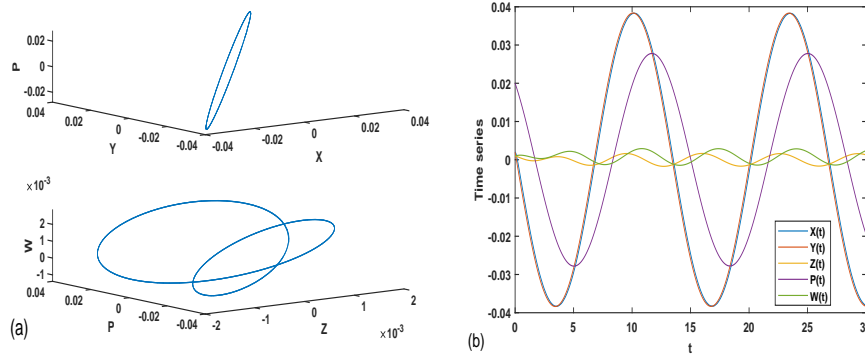


Fig.4. Stable periodic orbit near O of system (1) from Hopf bifurcation with parameter values $(\sigma, b, r, \delta) = (10, 8/3, 0.53, 0.5)$, and initial values $(0.002, 0.002, 0.001, 0.02, 0.001)$: (a) stable periodic orbit; (b) time series of state variables.

For $\sigma = 10, b = 8/3$, we have the bifurcation value $r_0 = \frac{\delta(100+11\delta)}{100}$. According to Theorem 3.1, when $\delta = 0.5$, system (1) undergoes a Hopf bifurcation when the parameter r crosses the critical value $r = r_0 = 0.5275$ and first Lyapunov coefficient $l_1 = -0.0061 < 0$: a stable periodic orbit therefore emerges from O with $r > r_0$ in the neighborhood $r = r_0$. Choosing initial values $(0.002, 0.002, 0.001, 0.02, 0.001)$ near the equilibrium O , we take $r = 0.53$ in Fig. 4, a stable periodic orbit exists near the unstable equilibrium O . According to Theorem 3.2, choosing initial values $(1.65, 1.6, 9, -6.5, 2.6)$, we take $r = 4.92$ in Fig. 5, a stable periodic orbit exists near the unstable equilibrium $E_1(1.6120, 1.6120, 9.1539, -6.7655, 2.5985)$. According to Theorem 3.3, if we take $r = 6.638712$ in Fig. 6, and δ crosses the critical value $\delta = \delta_2 = 1.536980$, an stable periodic orbit exists near the unstable equilibrium $E_1(2.79592, 2.79591, 0.7248 - 11.4244, 7.8172)$.

4. Bogdanov-Takens bifurcation analysis at non trivial equilibrium point.

We now apply the techniques of [12] to analyse the Bogdanov-Takens bifurcation of

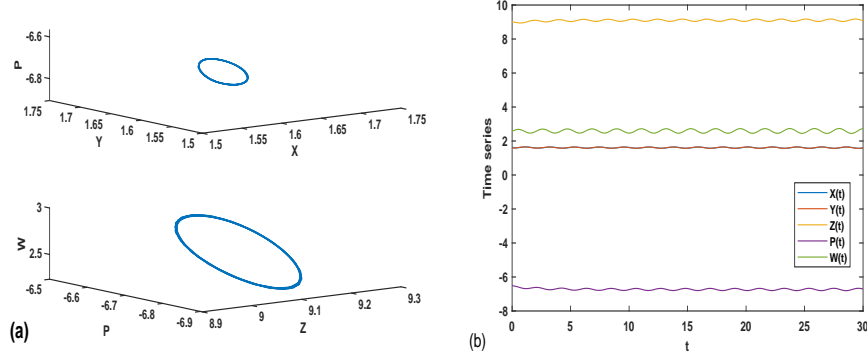


Fig.5. Stable periodic orbit near E_1 of system (1) from Hopf bifurcation with parameter values $(\sigma, b, r, \delta) = (10, 8/3, 4.92, 0.5)$, and initial values $(1.65, 1.6, 9, -6.5, 2.6)$: (a) stable periodic orbit; (b) time series of state variables.

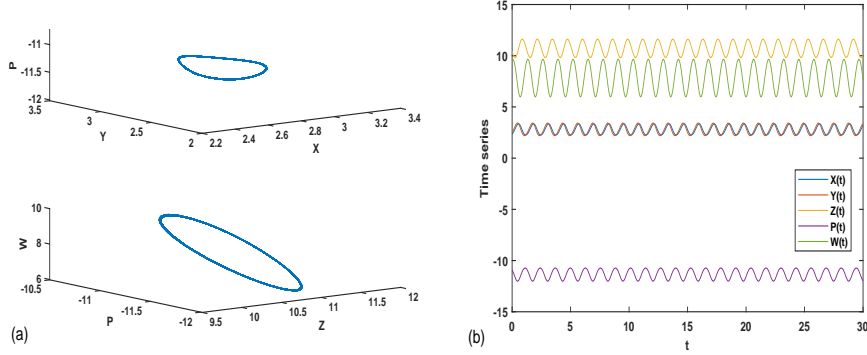


Fig.6. Stable periodic orbit near E_1 of system (1) from Hopf bifurcation with parameter values $(\sigma, b, r, \delta) = (10, 8/3, 6.638712, 1.5268)$, and initial values $(2.33, 2.46, 10.41, -10.90, 9.59)$: (a) stable periodic orbit; (b) time series of state variables.

(1) near E_1 . There are two zero eigenvalues for

$$r = \frac{-1 + 2b - b^2 + 2\delta + 2b\delta - \delta^2}{4b}, \sigma = \frac{\delta(1 - 2b + b^2 - 2\delta - 2b\delta + \delta^2)}{-3 - 2b + 5b^2 + 8\delta + 4b\delta - 5\delta^2}.$$

Here, we let $b = 0.5, \delta = 0.3$ [7].

We reconsider system (1) and have

$$F(x, \mu) = \begin{cases} \sigma(Y - X), \\ rX - P - XZ, \\ XY - W, \\ -XW - 0.3(P - Y), \\ XP - 0.3(W - 0.5Z). \end{cases} \quad (23)$$

where $x \in R^5$ and $\mu = (r, \sigma)^T$ are respectively vectors.

When $\mu = \mu_0$, x_0 should be $E_1(0.4243, 0.4243, 0.1800, 0.1697, -0.1200)$. Then matrix $A = DF(E_1, \mu_0)$ has only two zero eigenvalues and other three eigenvalues

don't have zero real parts. Denoting generalized eigenvectors of A corresponding to the eigenvalues $\lambda_{1,2} = 0$ are $q, q_1 \in R^5$, which satisfy

$$Aq = 0, Aq_1 = q.$$

Furthermore, there exists matrix $P_0 \in R^{5 \times 3}$ and corresponding matrix $G = (q, q_1, P_0)$. Then we have $G^{-1}DF(x_0, \mu_0)G = \begin{pmatrix} J_0 & 0 \\ 0 & J_1 \end{pmatrix}$. Here we write $G^{-1} = (p_1, p, Q_0^T)^T$, where matrix $Q_0 \in R^{3 \times 5}$ and

$$p^T A = 0, p_1^T A = p^T.$$

p_1, p are the left generalized eigenvectors of A corresponding to the eigenvalues $\lambda_{1,2} = 0$.

Before we establish the main result, we summarise the essential theory in the following theorem.

Theorem 4.1. [12] *Given the nonlinear system $\dot{x} = F(x, \mu)$, where $x \in R^n, \mu \in R^m$ with $m \geq 2$, such that, there exists (x_0, μ_0) , that satisfies the conditions:*

(H1) $F(x_0, \mu_0) = 0$,

(H2) $DF(x_0, \mu_0) = 0$ has only two zero eigenvalues, and other eigenvalues has no zero real part,

(H3) $\bar{a}\bar{b} \neq 0$, (nondegeneracy)

(H4) S_1 and S_2 are linearly independent, (transversality)

where

$$\begin{aligned} \bar{a} &= \frac{1}{2}q^T(p \bullet D^2F(x_0, \mu_0))q, \\ \bar{b} &= q^T(p_1 \bullet D^2F(x_0, \mu_0)q + q^T(p \bullet D^2F(x_0, \mu_0))q_1), \\ S_1 &= F_\mu^T(x_0, \mu_0)p, \\ S_2 &= [\frac{2\bar{a}}{\bar{b}}(q^T(p_1 \bullet D^2F(x_0, \mu_0))q_1 + q_1^T(p \bullet D^2F(x_0, \mu_0))q_1) \\ &\quad - q^T(p \bullet D^2F(x_0, \mu_0))q_1]F_\mu^T(x_0, \mu_0)p_1 \\ &\quad - \frac{2\bar{a}}{\bar{b}} \sum_{i=1}^2 (q_i \bullet (F_{\mu x}(x_0, \mu_0) - (P_0 J_1^{-1} Q_0 F_\mu(x_0, \mu_0))^T \bullet D^2F(x_0, \mu_0)))p_i \\ &\quad + (p \bullet F_{\mu x}(x_0, \mu_0) - (P_0 J_1^{-1} Q_0 F_\mu(x_0, \mu_0))^T \bullet D^2F(x_0, \mu_0))q. \end{aligned}$$

Then, the dynamics on the center manifold at $x = x_0$ and $\mu \approx \mu_0$, is locally topologically equivalent to the versal deformation of the Bogdanov-Takens bifurcation

$$\begin{cases} \dot{z}_1 = z_2, \\ \dot{z}_2 = \eta_1 + \eta_2 z_1 + \bar{a} z_1^2 + \bar{b} z_1 z_2, \end{cases} \quad (24)$$

where $\eta_1 = S_1^T(\mu - \mu_0)$, and $\eta_2 = S_2^T(\mu - \mu_0)$.

Here, we know that Hopf bifurcation occurs on the curve $\eta_1 = 0, \eta_2 < 0$ and $\bar{a}\bar{b}$ shows the criticality of the Hopf bifurcation. If $\mu = \mu_0 = (0.28, 0.84)^T$, we have five eigenvalues at equilibrium E_1 :

$$0, 0, -1.12165, -0.159177 + 0.827999i, -0.159177 - 0.827999i.$$

According to the Theorem 4.1, we can get the following results:

$$\begin{aligned}
q &= (0.134079, 0.134079, 0.379233, -0.107263, 0.11377)^T, \\
q_1 &= (1, 1.15962, 0.255064, 0.157707, 0.537016)^T, \\
P_0 &= \begin{pmatrix} 1.1557372029157649 & -0.09620541885921932 & -4.241797494560007 \\ 0.8418981906762063 & -1.2171999921137975 & 1.4222443263549716 \\ 1.6239234529921647 & 1.3357701272332747 & 1.9580465924011123 \\ 0.9151735983005292 & -0.10186091334362253 & -0.9321931732932462 \\ 0 & 1 & 1 \end{pmatrix} \\
J_0 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
J_1 &= \begin{pmatrix} -0.159177 & -0.827999 & 0 \\ 0.827999 & -0.159177 & 0 \\ 0 & 0 & -1.12165 \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
p_1 &= (1.23603, 0.189394, 1.51906, -4.43065, -2.13098)^T, \\
p &= (0.117851, 0.353553, -0.4, 0.235702, 1)^T, \\
Q_0 &= \begin{pmatrix} -0.0677811 & 0.0336365 & 0.2781480 & 0.6537858 & -0.2705252 \\ -0.0169308 & -0.3171043 & 0.0127715 & 0.2904336 & 0.6249159 \\ -0.0169308 & -0.3171043 & 0.0127715 & 0.2904336 & 0.6249159 \end{pmatrix}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\bar{a} &= -0.0431452, \bar{b} = -0.160077, \\
S_1 &= (0.15, 0)^T, S_2 = (-0.0730955, -0.0581386).
\end{aligned}$$

Then making a coordinate transformation

$$z_1 \rightarrow \frac{a}{b^2} z_1, z_2 \rightarrow -\frac{a^2}{b^3} z_2, t \rightarrow -\frac{b}{a} t,$$

we denote $\nu_1 = r - 0.28, \nu_2 = \sigma - 0.84$ and have that system (1) is locally topologically equivalent to

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = \beta_1 + \beta_2 z_1 + z_1^2 - z_1 z_2, \end{cases} \quad (25)$$

where

$$\beta_1 = -1.22632\nu_1, \beta_2 = -1.00619\nu_1 - 0.800304\nu_2.$$

The bifurcation diagram of system (25) is shown in Fig. 7. Curve $4\beta_1 - \beta_2^2 = 0$ is for a saddle-node bifurcation. Curve $\beta_1 = 0, \beta_2 < 0$ is for a non-degenerate Andronov-Hopf bifurcation. Furthermore, when $\beta_2 = 0$, (25) can have Bogdanov-Takens bifurcation, which reveals a curve in the parameter plane emanating from the codim 2 point and corresponding to a saddle homoclinic bifurcation (Curve $\beta_1 + \frac{6}{25}\beta_2^2 = o(\beta_2^2), \beta_2 < 0$). The unique and stable limit cycle from the Hopf bifurcation approaches the homoclinic orbit and finally disappears while its period gets bigger and bigger until infinity.

We therefore have the following result.

Corollary 4.2. *Let $r = \nu_1 + 0.28, \sigma = \nu_2 + 0.84$ and $b = 0.5, \delta = 0.3$. Then system (23) is locally topologically equivalent to the normal form (25), which has*

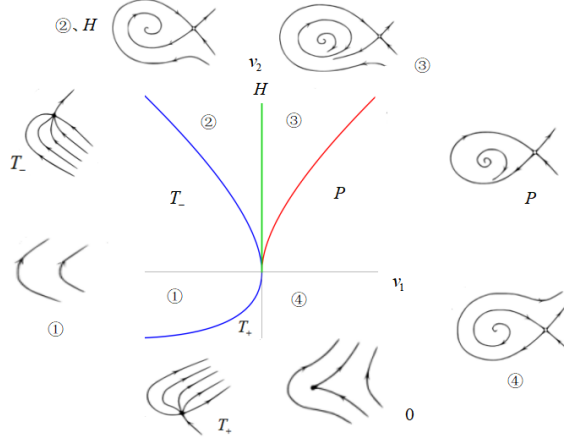


Fig.7. Bogdanov-Takens bifurcation of system (25).

the following local representations of the bifurcation curves in a small neighborhood of the origin:

- (1) there is a saddle-node bifurcation curve

$$T : -4.90529\nu_1 - 1.01243\nu_1^2 - 1.61052\nu_1\nu_2 - 0.640487\nu_2^2 = 0;$$

- (2) there is a nondegenerate Andronov-Hopf bifurcation curve

$$H : \nu_1 = 0, \nu_2 > 0;$$

- (3) there is a homoclinic bifurcation curve

$$\begin{aligned} P : & 0.242982\nu_1^2 + 0.386525\nu_1\nu_2 + 0.153717\nu_2^2 - 1.22632\nu_1 \\ & = o((-1.00619\nu_1 - 0.800304\nu_2)^2), \nu_2 > -1.25726\nu_1. \end{aligned}$$

5. Conclusion. The Cattaneo-Christov heat-flux model was proposed and studied with a focus on period-doubling bifurcations to chaos depending on the control parameters numerically [6]. However, there are some basic and important theoretical analyses that need to be further clarified analytically. In this paper, we returned to the five-dimensional Cattaneo-Christov heat-flux model, determined the sign of the first Lyapunov coefficient at Hopf bifurcation point, and showed that stable limit cycles bifurcate from the equilibrium. We then discussed the Bogdanov-Takens bifurcations, and obtained the curves of saddle-node, Hopf bifurcation and homoclinic bifurcations near the Bogdanov-Takens point. This study possibly will be useful to reveal the onset of oscillations for the amazing original Lorenz attractor. From this viewpoint, one is confident that there are still abundant complex properties and phenomena to be further investigated for the effects of thermal relaxation time on Rayleigh-Bnard convection of Boussinesq fluid layer.

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