On Some Non-Periodic Branch Groups

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"It is good to have an end to journey toward;
but it is the journey that matters, in the end."

Ernest Hemingway

Contents

List of Figures

Abstract

In this thesis we study groups of automorphisms acting on rooted trees. We will define in Chapter 1 rooted trees, automorphisms acting on them and state selected theorems about them. The main ideas of the constructions in this thesis are coming from [LS03].

In Chapter 2 we describe an explicit construction of a group G. The first Section describes the generators of this group and some of its properties. In Section 2.2 we determine the abelianisation, which is the largest abelian quotient, of G and some of its subgroups. In the last Section we prove further algebraic properties of G. In particular, we give an algorithm how we can for any g, h in G construct a non-trivial word $w_{g,h}(x, y)$ in the free group $F(x, y)$ such that $w_{g,h}(g, h) = 1$ in G.

In Chapter 3 we alter the construction of Chapter 2 to obtain more refined results. In the first Section we describe this new construction and show that we have some flexibility in its definition. In particular, we will see in Section 3.2 that we can control some of the algebraic properties that G will have by a group P that this construction uses and that embeds into G.

The fourth Chapter is about the presentation and word growth of the group defined in Chapter 2. We will show in Section 4.1 that G cannot be finitely presented. In Section 4.2 we describe a recursive set of relators that hold in G. In Section 4.3 we show that G has exponential word growth under certain assumptions. This provides the first known example of a branch group of exponential growth which does not contain free subgroups.

In Chapter 5 we describe the notion of Hausdorff dimensions for groups acting on rooted trees. In the first two Sections we discuss the terminology and give some background. In Section 5.3 we explicitly calculate the Hausdorff dimensions of the groups from Chapter 2 and Chapter 3 and generalise those results. In the last Section we construct branch groups Γ_α with finitely generated subgroups of irrational Hausdorff dimension α in Γ_α. This answers an open question about the finitely generated Hausdorff spectrum of a finitely generated branch group.

Chapter 1

Preliminaries

In this Chapter we will recall some of the notation and definitions from [BGv03] and [Seg01]. We will start with defining graphs and automorphisms on trees. We discuss different ways a group of such automorphisms can act on a tree and give important results about such groups.

1.1 Trees

A *graph* G is an ordered pair of disjoint sets (V, E) such that E is a subset of the set of pairs of V. The set V is the set of *vertices* and E is the set of *edges*. If $(x, y) \in E(G)$ then x and y are *adjacent*. Two edges are *adjacent* if they have exactly one common endvertex. We observe that writing the edge $e = (x, y)$ as a pair determines a direction on e, which leads to the notion of a *directed graph*. We define an equivalence relation on E via

$$e_1 = (x, y) \sim e_2 = (u, v) \text{ if and only if } (x, y) = (v, u) \text{ or } (x, y) = (u, v).$$

The graph $(V, E/\sim)$ is then called the *undirected graph* with vertex set V and edge set $E/\sim$.

A *path* is a graph P of the form

$$V(P) = \{x_0, x_1, \ldots, x_l\}, \quad E(P) = \{(x_0, x_1), (x_1, x_2), \ldots, (x_{l-1}, x_l)\}$$

and $l = |E(P)|$ is the *length* of P. A *cycle* is a path where $x_0 = x_l$. A graph is *connected* if for every pair (x, y) of distinct vertices there is a path from x to y.

A *tree* is an undirected connected graph T which has no non-trivial cycles. If T has a distinguished *root* vertex r it is called a *rooted tree*. The distance of a vertex v from the root is given by the length of the path from r to v and called the *norm* of v. The number

$$d_v = |\{e \in E(T) : e = (v_1, v_2), v = v_1 \text{ or } v = v_2\}|$$

is called the *degree* of $v \in V(T)$.

The tree is called *spherically homogeneous* if vertices of the same norm have the same degree. Let $\Omega(n)$ denote the set of vertices of distance n from the root. This set is called the *n-th level* of T. We will further consider exclusively spherically homogeneous trees. Figure 1.1 shows an example of a spherically homogeneous rooted tree. A spherically homogeneous tree T with

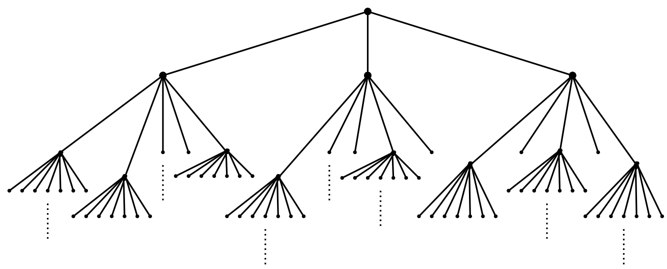

Figure 1.1: An example of a spherically homogeneous rooted tree.

root r is determined up to isomorphism by, depending on the tree, a finite or infinite sequence $\bar{l} = \{l_n\}_{n=0}^{\infty}$ where $l_n + 1$ is the degree of the vertices on level n for $n \geq 1$ and l_0 is degree of the root. We denote the dependence of T on $\bar{l}$ by $T_{\bar{l}}$ where needed and call $\bar{l}$ the *defining sequence of* $T_{\bar{l}}$. Each level $\Omega(n)$ has $\prod_{i=0}^{n-1} l_i$ vertices. We denote this number by $m_n = |\Omega(n)|$, which is explicitly given as

$$m_n = \prod_{i=0}^{n-1} l_i. \tag{1.1}$$

The tree determined by the sequence $\bar{l}$ is unique up to isomorphism. If we want to specify a particular tree, we also need to choose an order of the vertices of T on each level. If the tree is spherically homogeneous it is enough to specify an order among the l_n subvertices of each vertex v on level $n-1$.

A tree is called *regular* if the defining sequence $\bar{l}$ is given such that $l_i = l_{i+1}$ for all $i \in \mathbb{N}$. Let $T[n]$ denote the finite tree where all vertices have norm less than or equal to n and write T_v for the subtree of T with root v. For all vertices $u, v \in \Omega(n)$ we have that $T_u \simeq T_v$ because T is spherically homogeneous. Denote a tree isomorphic to T_v for $v \in \Omega(n)$ by T_n. This will be the tree with defining sequence $(l_n, l_{n+1}, \dots)$. To each sequence $\bar{l}$ we associate a sequence $\{X_n\}_{n\in\mathbb{N}}$ of alphabets where $X_n = \left\{v_1^{(n)}, \dots, v_{l_n}^{(n)}\right\}$. A path beginning at the root of length n in $T_{\bar{l}}$ is identified with the sequence $x_1, \dots, x_i, \dots, x_n$ where $x_i \in X_i$ and infinite paths are identified in a natural way with infinite sequences. Vertices will be identified with finite strings in the alphabets

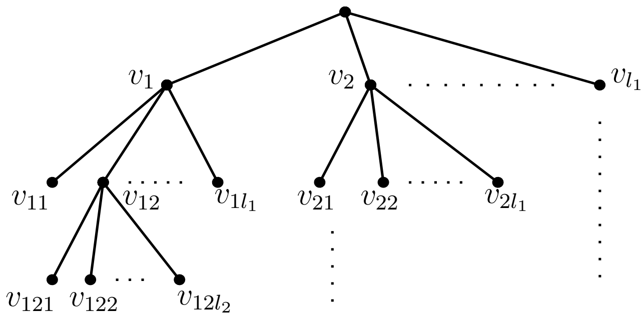

Figure 1.2: Example of a labelling of a tree.

X_i. Vertices on level n can be written as elements of $Y_n = X_0 \times \cdots \times X_{n-1}$. Alphabets induce the lexicographic order on the paths of a tree and therefore on the vertices. We can identify paths in an infinite tree with infinite sequences in $\mathcal{Y} = \lim_{n\to\infty} Y_n = \prod_{i=0}^{\infty} X_i$. We denote by $Y_{i,n}$ the infinite sequences $Y_{i,n} = X_i \times \cdots \times X_{i+n}$ and by $\mathcal{Y}_i$ the set $\mathcal{Y}_i = \lim_{n\to\infty} Y_{i,n}$. We can describe the sets just defined as follows:

- X_i : alphabet at level i
- Y_i : words up to level i
- $\mathcal{Y}$: infinite paths
- $\mathcal{Y}_i$: infinite paths starting at level i

1.2 Automorphism Groups

An *automorphism* of a rooted tree $T = T_{\bar{l}}$ is a bijection from the set of vertices $V(T)$ to itself that preserves edge incidence and the distinguished root vertex r. The set of all such bijections on T is denoted by $\mathrm{Aut}(T)$ and is called the *full automorphism group* of the tree T. This group induces an imprimitive permutation on $\Omega(n)$ for each $n \geq 2$. It is given by

$$\mathrm{Aut}(T) = \varprojlim_{n} \mathrm{Sym}\left(l_{n-1}\right) \wr \cdots \wr \mathrm{Sym}\left(l_0\right).$$

Let $g \in \mathrm{Aut}(T)$ be an element of the full automorphism group. The *section* of g at a vertex $u \in \Omega(n)$ is the automorphism g_u of T_u defined via the vertex permutations

$$\bar{g}(u) \cdot g_u(w) = g(uw)$$

for $w \in T_u$ where $\bar{g}$ denotes the projection of g onto $T[n]$.

Assume that $X_n = X$ for a fixed alphabet X for all n, then $\mathcal{Y}_i = \mathcal{Y}$ for all $i \geq 0$. We say a group $G \leq \mathrm{Aut}(T)$ acts *self-similarly* if for every $x \in X$, $y \in \mathcal{Y}$ and $g \in G$ there exists a $t \in X$ and $h \in G$ such that

$$g(xy) = th(y).$$

In this thesis we will encounter groups that are defined via a sequence $\{A_i\}_{i\geq 0}$ of finite groups A_i. We denote this by $G = G\left(\{A_i\}_{i\geq 0}\right)$. We say a group G_n is *morally similar* to $G\left(\{A_i\}_{i\geq 0}\right)$ if G_n is the group defined via $G_n = \left(\{A_i\}_{i\geq n}\right)$. This allows us to generalise the definition of self-similarity.

We say a group $G\left(\{A_i\}_{i\geq 0}\right) \leq \mathrm{Aut}(T)$ acts *morally self-similarly* if for every $x \in X_j$, $y \in \mathcal{Y}_{j+1}$ there exists a group $G_j\left(\{A_i\}_{i\geq j}\right)$ with $g \in G_j$ and elements $t \in X_i$ and $h \in G_{i+1}$ such that

$$g(xy) = th(y).$$

We now introduce important subgroups of groups acting on rooted trees. Those will allow us to deduce structural properties such as residual finiteness or being a branch group in some important cases.

With any group $G \leq \mathrm{Aut}(T)$ we associate the subgroups

$$\mathrm{St}_G(u) = \{g \in G : g(u) = u\},$$

the *stabilizer* of a vertex u. Then the subgroup

$$\mathrm{St}_G(n) = \bigcap_{u \in \Omega(n)} \mathrm{St}_G(u)$$

is called the *n-th level stabilizer* and it fixes all vertices on the n-th level. The following Lemma is an easy consequence of the above definitions, however it is an important fact that we will often refer to.

Lemma 1.2.1. $\mathrm{St}_G(n)$ *is a normal subgroup of finite index in* G*.*

Proof. Denote by $\mathrm{Aut}(T[n])$ the automorphism group acting on the finite tree $T[n]$, which is a finite group. The subgroup $\mathrm{St}_G(n)$ is exactly the kernel of the map $G \longrightarrow \mathrm{Aut}(T[n])$, hence has finite index in G. □

A group G of tree automorphisms satisfies the *congruence subgroup property* if for every finite index subgroup H of G there exists an integer n such that $\mathrm{St}_G(n) \leq H$.

Another important class of subgroups associated with $G \leq \mathrm{Aut}(T)$ are the *rigid vertex stabilizers*

$$\mathrm{rst}_G(u) = \{g \in G : \forall v \in V(T) \setminus V(T_u) : g(v) = v\}.$$

The subgroup

$$\mathrm{rst}_G(n) = \mathrm{rst}_G(u_1) \times \cdots \times \mathrm{rst}_G(u_{m_n})$$

for $u_i \in \Omega(n)$ is called the *n-th level rigid stabilizer.* Obviously $\mathrm{rst}_G(n) \leq \mathrm{St}_G(n)$. Depending on $\mathrm{rst}_G(n)$ we can classify groups acting on rooted trees. Let G be a subgroup of $\mathrm{Aut}(T)$ where T is as above. As defined in [Bv01] we say that G acts on T as

1. a *branch group* if it acts transitively on the vertices of each level of T and $\mathrm{rst}_G(n)$ has finite index for all $n \in \mathbb{N}$,
2. as a *weakly branch group* if all rigid level stabilizers are non-trivial (which implies that they are infinite)
3. as a *rough group* if all rigid level stabilizers are trivial.

This definition implies that branch groups are infinite and residually finite groups. The examples in Section 1.3 will demonstrate that it is rather easy to obtain a non-trivial group of tree automorphisms that is not a branch group.

Although not used in this thesis, we also give a purely algebraic definition of a branch group as can be found in [Bv01].

The group G is a *branch group* if it has trivial centre and contains descending chains of subgroups $\{H_n\}_{n=1}^{\infty}$, $\{L_n\}_{n=1}^{\infty}$, where each L_n is a subgroup of H_n such that the following hold:

(a) (i) H_n is normal in G, the index $|G/H_n|$ is finite and $\bigcap_{n=1}^{\infty} H_n = 1$.

(ii) There is a sequence $\{N_n\}_{n=1}^{\infty}$ of natural numbers such that N_n divides N_{n+1}, and there are subgroups $L_n^{(1)}, \ldots, L_n^{(N_n)}$ of G, each isomorphic to $L_n = L_n^{(1)}$, such that H_n can be represented as the direct product

$$H_n = L_n^{(1)} \times \cdots \times L_n^{(N_n)}, \tag{1.2}$$

in such a way that the product decomposition (1.2) of H_{n+1} refines that of H_n; that is, each factor $L_n^{(i)}$ from (1.2) contains a product of $m_{n+1} = N_{n+1}/N_n$ factors $L_{n+1}^{(j)}$ with $(i-1) \cdot m_{n+1} + 1 \leq j \leq i \cdot m_{n+1}$, from the corresponding decomposition of H_{n+1}.

(iii) When G acts on itself by conjugation, for each $n = 1, 2, \ldots$ the factors in (1.2) are permuted transitively among themselves.

(b) The profinite group G is a branch group if it satisfies the conditions (a), with the additional proviso that all the subgroups under discussion are closed.

We can specify an automorphism g of T that fixes all vertices of level n by writing

$$g = (g_1, g_2, \ldots, g_{m_n})_n$$

with $g_i \in \mathrm{Aut}\,(T_n)$ where the subscript n of the bracket indicates that we are on level n. Each automorphism can be written as

$$g = (g_1, g_2, \ldots, g_{m_n})_n \cdot \alpha$$

with $g_i \in \mathrm{Aut}\,(T_n)$ and α an element of $\mathrm{Sym}\,(l_{n-1}) \wr \cdots \wr \mathrm{Sym}\,(l_0)$. If an automorphism g is such that we can write g as $g = (1, \ldots, 1)_1 \cdot \sigma$ with $\sigma \in \mathrm{Sym}\,(l_0)$, then we call g a *rooted automorphisms.* We have just seen that we can identify those with the elements of $\mathrm{Sym}\,(l_0)$.
We will often use the *portrait* on the tree of an automorphism g to describe how g acts on the tree. This is a decoration of the tree T, where the decoration of the vertex v belongs to $\mathrm{Sym}\left(X_{|v|}\right)$, and is defined inductively as follows: first, there is $\pi_\emptyset \in \mathrm{Sym}\,(X_0)$ such that $g = h\pi_\emptyset$ and $h \in G$

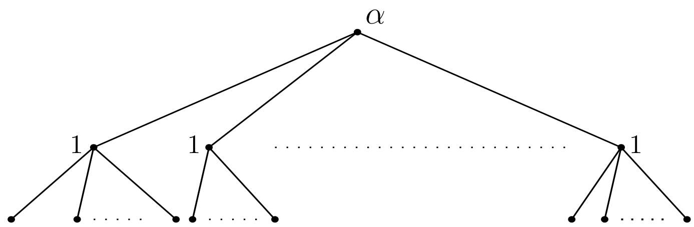

Figure 1.3: The portrait of a rooted automorphism α.

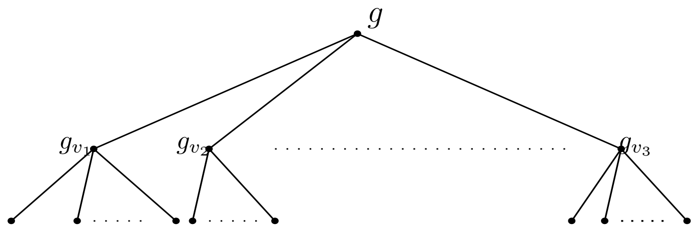

Figure 1.4: The portrait of g and its sections.

stabilises the first level. This $\pi_\emptyset$ is the label of the root vertex. Then, for all $x \in X_0$, label the tree below x with the portrait of the section of g at x.

A vertex v is in the *support* of an automorphism g if g is not in the stabilizer $\mathrm{St}_G(v)$. Let P be an infinite path $\{p_i\}$ with vertices p_i in T. Following the definition in [FA11], if we consider, for every $n \geq 1$, an immediate descendant s_n of p_{n-1} not lying in P, we say that the sequence $S = (s_n)_{n\geq 1}$ is a *spine* of the tree T. An element $g \in G$ is a *spinal automorphism* if the support of g is a finite union of spines. Figure 1.5 depicts an example of the portrait of a spinal automorphism b that acts as b_{ji} on the first l_i vertices of level i and on all unlabelled vertices as the identity.

An automorphism group G is called a *spinal group* if G is generated by a set A of rooted automorphisms and a set B of spinal automorphisms which are defined via the same path. As in [BGv03, Section 2], this can be more formally described in the following way:

Let ω be a triple consisting of a group of rooted automorphisms A_ω, a group B and a doubly indexed family $\bar{\omega}$ of homomorphisms

$$\omega_{ij} : B \longrightarrow \mathrm{Sym}\,(X_i)\,, \quad i \in \mathbb{N}, j \in \{2, \ldots, l_i\}\,. \tag{1.3}$$

Such a triple is called *defining triple*. Each $b \in B$ defines a spinal automorphism b_ω whose portrait is depicted in the diagram in Figure 1.6.

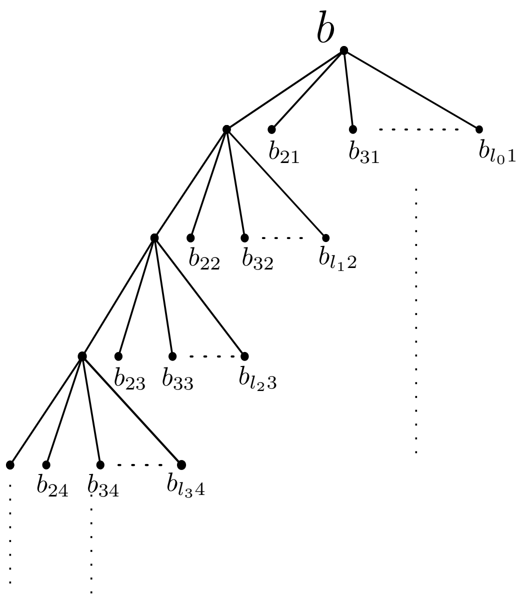

Figure 1.5: Example of a portrait of a spinal automorphism b.

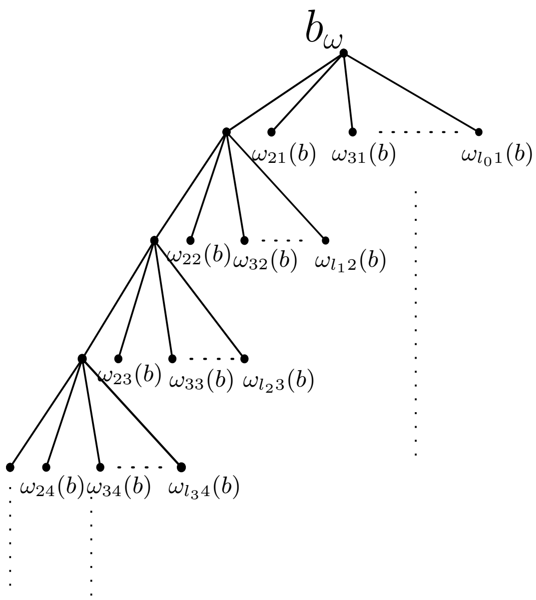

Figure 1.6: Portrait of a spinal automorphism b_ω.

Remark 1.2.2. *We note that the notation in this thesis is slightly different from the one used in [BGv03]. A spinal group as defined there, requires j in* (1.3) *to be between* $1 \le j \le l_i - 1$.

This means that the spines go along the right-most path of the tree rather than the left-most path as depicted in all figures of this thesis. It can easily be verified that the resulting groups are isomorphic.

Therefore, $B_\omega = \{b_\omega | b \in B\}$ is a set of spinal automorphisms. We can think of B as acting on the tree T by automorphisms. We define now the group G_ω, where ω is a defining triple, as the group of tree automorphisms generated by A_ω and B_ω, namely $G_\omega = \langle A_\omega \cup B_\omega \rangle$.

1.3 Selected Results and Examples

In this Section we will quote important Theorems about branch groups that are often referred to throughout this thesis.

Theorem 1.3.1 ([Gri84])**.** *Let* $\Gamma \leq \mathrm{Aut}(T)$ *be a spherically transitive subgroup of the full automorphism group on* T*. If* $1 \neq N \lhd \Gamma$*, then there exists an* n *with* $\mathrm{rst}_\Gamma(n)' \leq N$*.*

The next result gives a criterion for branch groups to be just infinite.

Theorem 1.3.2 ([Gri00], Theorem 4)**.** *If* Γ *is a branch group, then the group* Γ *is just infinite if and only if for each* $n \geq 1$ *the index of the commutator subgroup* $\mathrm{rst}_G(n)'$ *in* $\mathrm{rst}_G(n)$ *is finite.*

Šuniḱ originally proved in his thesis [Š00] that a wider class of automorphism groups on trees cannot be finitely presented. The Theorem is also stated and proved in [BGv03].

Let us define the *shifted triple* $\sigma^r\omega$, for $r \in \mathbb{N}_0$. The triple $\sigma^r\omega$ consists of the group $A_{\sigma^r\omega}$ of rooted automorphisms of T_r defined by

$$A_{\sigma^r\omega} = \left\langle \bigcup_{j=2}^{l_r} (B)\omega_{rj} \right\rangle,$$

the same group B as in ω, and the shifted family $\sigma^r\bar{\omega}$ of homomorphisms

$$\omega_{i+r,j} : B \to \mathrm{Sym}\left(Y_{i+r}\right), \quad i \in \mathbb{N}, \quad j \in \{2, \dots, l_{i+r}\}.$$

Theorem 1.3.3 ([BGv03, 4.4])**.** *Let* $\mathcal{C}$ *be a class of groups that is closed under homomorphic images and subgroups (of finite index) and* $\omega = (A_\omega, S, \bar{\omega})$ *be a sequence that defines a spinal group in* $\mathcal{C}$*. Further, assume that, for every* r*, there exists a triple* $\eta^{(r)}$ *of the form* $\eta^{(r)} = (A_{\sigma^r\omega}, S, \bar{\eta})$*, where* $\bar{\eta}^{(r)}$ *is a doubly indexed family of homomorphisms*

$$\eta_{ij} : S \longrightarrow \mathrm{Sym}\left(X_j\right), \quad i \in \{r+1, r+2, \dots\}, \quad j \in \{2, \dots, l_i\}$$

defining a group of tree automorphisms (not necessarily spinal) $G_{\eta^{(r)}}$ *that acts on the shifted tree* $T^{\sigma^r(Y)}$ *and is not in* $\mathcal{C}$*. Then the spinal group* G_ω *is not finitely presented.*

We will now see that not all groups of automorphisms on a rooted tree are branch groups. As the first example demonstrates, it is rather simple to even get a group acting roughly on a tree. Many other such examples can be defined in a similar way.

A Roughly Acting Group

We take the regular binary tree T defined by the defining sequence $\bar{l} = (2, 2, 2, \dots)$. Assume all vertices apart from the root are labelled by finite sequences $x_1 \dots x_r$ with $x_i \in \{0, 1\}$. Let a be an automorphism that swaps the two vertices on the first level and acts as the identity on all vertices below. Let b be another automorphism such that b acts in the following way: if v is such that $v = (0 \dots 01)$ then let b swap the two subvertices of v. In addition, let b act in a way such that if u is given by a sequence $u = (1 \dots 10)$ then it swaps the two subvertices of u. Figure 1.3 depicts this automorphism b, where the decoration ϵ on a vertex denotes a swap of the two subvertices. Now define $G = \langle a, b \rangle$. Because b acts on two subtrees of the root simultaneously we deduce that $\mathrm{rst}_G(n)$ is empty for all $n \geq 1$.

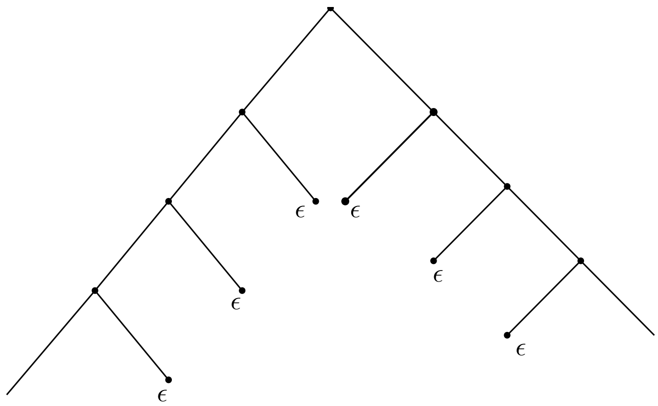

Figure 1.7: The spinal automorphism b.

A Weakly Branch Group

An example of a weakly branch group is the Basilica group as introduced and studied by Grigorchuk and Zuk in [GZ02]. It is an automaton group and can be defined on the binary tree as a group $G = \langle a, b \rangle$ where $a = (1, b)$ and $b = (1, a)\epsilon$ where ϵ denotes a swap of the two subvertices.

This group shares some fundamental properties with the group constructed in Chapter 2 as the following Theorem from [GZ02] compared with Theorems 2.1.29, 4.3.5 and 2.3.12 shows.

Theorem 1.3.4 ([GZ02]). *The basilica group*

(a) is torsion free,

(b) has exponential growth (with containing a free semigroup),

(c) every of its proper quotients is soluble and

(d) has no free non-abelian subgroups.

Branch Groups

Probably the most well-studied and well-known branch groups are the Grigorchuk group and the class of Gupta-Sidki-Groups. Both of them are acting on a binary respectively a p-regular tree and are infinite torsion groups. Further examples of branch groups can for example be found in [BGv03].

Chapter 2

A Class of Groups

In this Chapter we introduce a construction of a branch group G using a sequence $\{A_i\}$ of finite abelian groups. The determining factor will be a sequence of coprime integers $\{l_i\}_{i=0}^{\infty}$ such that each group A_i has order l_i.

Such a construction provides new examples of groups acting on rooted trees similar to the ones in [LS03]. We will see that it also provides the first example of a branch group without free subgroups which has exponential word growth.

In the first Section we describe how we construct this group G and prove various technical results. The derived subgroup G' of G is of particular interest and will be shown to be finitely generated. We then apply some well-known theorems about branch groups to deduce that in fact every normal subgroup of G is finitely generated. Similar to some other examples of branch groups we will see that every proper quotient of G is soluble.

In Section 2.2 we compute the abelianisation of G and of some of its subgroups. We will in particular see that G is not just infinite. We will also prove that it has a sequence of subgroups of finite index whose abelianisation rank grows unboundedly, which is expressed as its virtual first Betti number being infinite.

In the last Section we show that the group G is not large for any choice of defining sequence $\{l_i\}$. A more refined argument then allows us to construct non-trivial relations between any two elements in G if $\{l_i\}$ satisfies a certain condition. Hence G can then not contain a non-abelian free subgroup.

2.1 Main Construction

In this Section we describe the construction of G and deduce various technical results.

2.1.1 Basic Properties

A group in this class of groups depends on a sequence of finite d-generated abelian groups $\{A_i\}_{i\geq 0}$. To simplify statements we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We let the groups A_i have coprime orders $l_i = |A_i|$, where l_i is odd for every $i \in \mathbb{N}$. We will later require that $l_i \geq 3$ when $d = 1$ and even $l_i \geq 9$ in some cases when $d > 1$ for some more refined results. According to the definitions in Chapter 1 we consider the rooted tree $T = T_{\{l_i\}}$ defined by the sequence $\{l_i\}_{i\geq 0}$ and recall the definition of

$$m_n = \prod_{i=0}^{n-1} l_i.$$

We fix an order for the generators of each group A_i and write $A_i = \langle a_{1,i}, \dots, a_{d,i}\rangle$. Denote by $q_{j,i}$ the order of the j-th generator of A_i, where $j = 1, \dots, d$. We also fix an order for the elements in A_i and write $A_i = \{g_{1,i}, \dots, g_{l_i,i}\}$ and assume without loss of generality that $g_{1,i} = 1$. Further, we fix an order of the subvertices of a vertex $v \in \Omega(i-1)$ as follows:

1. fix a first vertex $v_{1,i} \in \Omega(i)$,
2. let the j-th vertex be such that $v_{j,i} = v_{1,i} g_{j,i}$.

This means that A_i is acting regularly on the $l_i = |A_i|$ vertices of level $i+1$. Identifying vertices with group elements then gives that this action is given by right multiplication.

Where unambiguous, we will speak of position i to mean a subvertex of $v \in \Omega(n-1)$ with decoration $g_{i,n}$ on a level n. The elements $a_{1,0}, \dots, a_{d,0}$ act as rooted automorphisms on the vertices $\Omega(1)$ of the first level of the tree T. We recursively define spinal automorphisms b_i for $i = 1, \dots, d$ on each level as

$$b_{i,n} = (b_{i,n+1}, a_{i,n+1}, 1, \dots, 1)_{n+1}.$$

Figure 2.1 depicts the action of b_i on the tree T, where we let it act as the identity on all unlabelled vertices.

Some of the latter results will depend on d. We define a group

$$G_A = \langle a_{1,0}, \dots, a_{d,0}, b_{1,0}, \dots, b_{d,0}\rangle$$

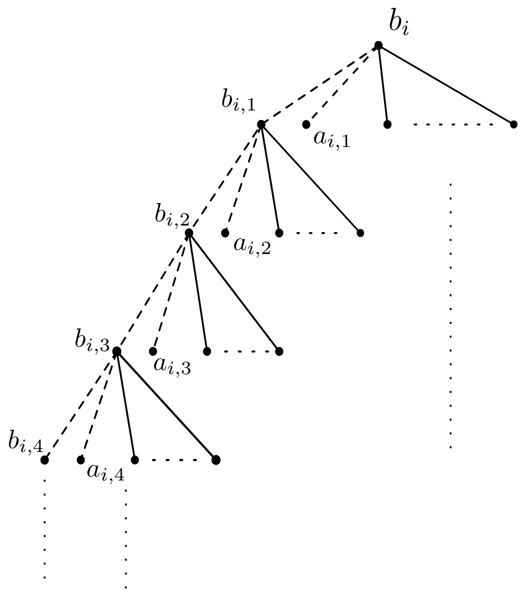

Figure 2.1: The spinal automorphism b_i.

if $d \geq 2$ and

$$G_C = \langle a_{1,0}, b_{1,0} \rangle$$

if $d = 1$ which is the case when all groups A_i are cyclic. We then use the abbreviated notation $a = a_{1,0}$ and $b = b_{1,0}$, so the definition of G_C then simplifies to $G_C = \langle a, b \rangle$. If we do not distinguish between G_C and G_A then we will speak of G to mean both groups.

Denote by S_n the symmetric group on n points. The action of G_A and G_C on the vertices of level n of the tree $T[n]$ induces a map $\pi_n : G \longrightarrow S_{m_n}$. The following Proposition 2.1.1 will help to interpret several later results.

Proposition 2.1.1. *The group G acts as the iterated wreath product $A_{n-1} \wr \cdots \wr A_0$ on the set $\Omega(n)$ of m_n vertices of each level n.*

Proof. This follows by induction on the level n. The statement is clear for $n = 1$, because the action there is given by A_0. Now assume G acts as $A_{n-2} \wr \cdots \wr A_0$ on level $n-1$. For every $a \in A_{n-1}$ there exists an integer $q \in \mathbb{Z}$ and hence an element $c \in G$ with $c = b^{m_{n-1}q}$ such that $c = (s^{m_{n-1}q}, a, 1, \ldots, 1)_{n-1}$ where $b \in \langle b_{1,0}, \ldots, b_{d,0} \rangle$ and $s \in \langle b_{1,n-1}, \ldots, b_{d,n-2} \rangle$. By Lemma A.2.3 and by induction hypothesis we now get the result. □

For our further considerations it is useful to define the groups

$$B = \left\langle b_{1,0}^{A_0}, \ldots, b_{d,0}^{A_0} \right\rangle \quad \text{and similarly} \quad B_n = \left\langle b_{1,n}^{A_n}, \ldots, b_{d,n}^{A_n} \right\rangle. \tag{2.1}$$

We denote by $G' = [G, G]$ the derived subgroup of G.

Lemma 2.1.2. *The group G can be written as $G = B \rtimes A_0$ and so $G' = B' \cdot \langle [B, A_0] \rangle$.*

Proof. We observe $B \cap A_0 = 1$ and $B \lhd G$. Every element $g \in G$ can be written as

$$g = \prod_{i=1}^{r_g} a_{h_i,0}^{s_i} \cdot b_{k_i}^{t_i},$$

$h_i, k_i \in \{1, \ldots, d\}$, $s_i \in \{0, \ldots, q_{h_i,0} - 1\}$ and $t_i \in \mathbb{Z}$ and $r_g \in \mathbb{N}$. □

We define $G_n = \langle A_n, B_n \rangle$. This group is then morally similar to G as defined in Section 1.2. The notation G_n will therefore be useful to study the morally self-similar properties of G.

We now fix a notation for the rest of this thesis. If we write $\prod_{i=1}^{m_n} G_n$, we mean the subgroup H of $\mathrm{Aut}(T)$ that acts as G_n on each of the subtrees $T|_{v_i}$ for all vertices $v_i \in \Omega(n)$.

Proposition 2.1.3. *(a)* $\mathrm{St}_G(1) \leq G_1 \times \cdots \times G_1$

(b) $\mathrm{St}_G(n) = G \cap (G_n \times \cdots \times G_n)$.

(c) $B = \mathrm{St}_G(1)$.

Proof. The stabilizer $\mathrm{St}_G(1)$ is generated by a-conjugates of $b_{i,0}$ for $a \in A_0$. On level 1 these act as $b_{i,1}$ and $a_{i,1}$. Both $b_{i,1}$ and $a_{i,1}$ are in G_1 for all $i \in \{1, \ldots, d\}$. Part (b) now follows immediately. For the last claim we observe $B \leq \mathrm{St}_G(1)$. Using $G = B \cdot A_0$ and the modular law (A.3.2) with $B \leq \mathrm{St}_G(1)$ gives us

$$\mathrm{St}_G(1) = B\,(A_0 \cap \mathrm{St}_G(1)) = B$$

because $A_0 \cap \mathrm{St}_G(1) = 1$. □

Denote by B^q the subgroup $\langle b_1^q, \ldots, b_d^q \rangle^{A_0}$ of B for some $q \in \mathbb{Z}$ and by k_n the product of orders $k_n = \prod_{j=1}^{n} l_j$.

Lemma 2.1.4. *$b_{i,n-1}^{k_n} \in G$ and hence $B_{i,n-1}^{k_n} \leq G$.*

Proof. This follows immediately from observing that $b_i^{k_n}$ can be identified with $b_{i,n-1}^{k_n} \in B_{i,n-1}$ because $a_{j,r}^{k_n} = 1$ for all $r \leq n$ and $j \in \{1, \ldots, d\}$. □

Lemma 2.1.5. *If $l_0 \geq 3$ and l_0 is odd then we have $B'B^{l_1} \leq \mathrm{rst}_G(1)$.*

Proof. We first show $B' \leq \mathrm{rst}_G(1)$. Let $u, v \in A_0$ and b_k, b_l be two generators of B for $k, l \in \{1, \dots, d\}$. We claim

$$[b_k^u, b_l^v] = \begin{cases} (1, \dots, 1, [a_{k,1}, b_{l,1}], 1, \dots, 1) & \text{if} \quad g_{2,0}u = g_{1,0}v \ \text{ and } \ g_{2,0}v \neq g_{1,0}u \\ (1, \dots, 1, [b_{k,1}, a_{l,1}], 1, \dots, 1) & \text{if} \quad g_{2,0}v = g_{1,0}u \ \text{ and } \ g_{2,0}u \neq g_{1,0}v \\ 1 & \quad \text{else.} \end{cases}$$

Evaluating the commutator gives

$$[b_k^u, b_l^v] = u^{-1}b_k^{-1}uv^{-1}b_l^{-1}vu^{-1}b_kuv^{-1}b_lv.$$

We consider three cases and compute the action on the first layer, where unfilled positions indicate trivial action.

1. the elements $g_{1,0}u, g_{2,0}u, g_{1,0}v, g_{2,0}v$ are pairwise distinct:

$$[b_k^u, b_l^v] = \left(\dots, b_{k,1}^{-1}b_{k,1}, \dots, a_{k,1}^{-1}a_{k,1}, \dots, b_{l,1}^{-1}b_{l,1}, \dots, a_{l,1}^{-1}a_{l,1}, \dots\right)_1 = 1.$$

2. $g_{2,0}u = g_{1,0}v$:

$$[b_k^u, b_l^v] = \left(\dots, b_{k,1}^{-1}b_{k,1}, \dots, a_{k,1}^{-1}b_{l,1}^{-1}a_{k,1}b_{l,1}, \dots, a_{l,1}^{-1}a_{l,1}, \dots\right)_1 = (\dots, [a_{k,1}, b_{l,1}], \dots)_1 .$$

3. $g_{2,0}v = g_{1,0}u$:

$$[b_k^u, b_l^v] = \left(\dots, b_{l,1}^{-1}b_{l,1}, \dots, b_{k,1}^{-1}a_{l,1}^{-1}b_{k,1}a_{l,1}, \dots, a_{k,1}^{-1}a_{k,1}, \dots\right)_1 = (\dots, [b_{k,1}, a_{l,1}], \dots,)_1 .$$

The inclusion $B^{l_1} \leq \mathrm{rst}_G(1)$ follows straight from Lemma 2.1.4. □

Remark 2.1.6. *The above cases do not include $g_{2,0}u = g_{1,0}v$ and $g_{1,0}u = g_{2,0}v$. Easy transformation shows that this can only happen if we allow elements of order 2, which is excluded by the requirement that l_0 is odd.*

Corollary 2.1.7. *If $l_n \geq 3$ and l_n is odd then we have $B_n'B_n^{k_{n+1}} \times \dots \times B_n'B_n^{k_{n+1}} \leq \mathrm{rst}_G(n+1) \leq G$.*

Remark 2.1.8. *The conclusion $B^{l_1} \leq \mathrm{rst}_G(1)$ in the above proof can be improved by considering the factors of l_1. We recall that $q_{1,1}, \dots, q_{d,1}$ are the orders of the generators $a_{1,1}, \dots, a_{d,1}$ of A_1. It would be sufficient to consider $\left\langle b_{1,0}^{q_{1,1}}, \dots, b_{d,0}^{q_{d,1}} \right\rangle$. However, the statement $B^{l_1} \leq \mathrm{rst}_G(1)$ as given in the Lemma will be enough to deduce useful results later on.*

2.1.2 A Description for G'

We now aim towards finding explicit generators for G'. It will be shown that G' is finitely generated. This allows us to deduce results about the moral self-similarity of G.

We denote by $F_{d\cdot l_0} = F\left(x_{1,1}, \ldots, x_{d,l_0}\right)$ the free group on $d \cdot l_0$ generators. The map

$$f : \begin{cases} F\left(x_{1,1}, \ldots, x_{1,l_0}, x_{2,1}, \ldots, x_{d,l_0}\right) & \longrightarrow F_d^{ab} = \mathbb{Z}^{(d)} \\ x_{i,j} & \mapsto \bar{x}_i, \quad \text{for all } i \in \{1, \ldots, d\} \end{cases}$$

is surjective. The kernel of f describes all words for which the sum of exponents of each block $x_{j,1}, x_{j,2}, \ldots, x_{j,l_0}$ of generators is zero. We will denote it by $\mathcal{N}\left(x_{1,1}, \ldots, x_{d,l_0}\right) = \ker(f)$.

Lemma 2.1.9. $\mathcal{N}\left(x_{1,1}, \ldots, x_{d,l_0}\right) = \left\langle x_{i,p}^{-1} x_{j,q}^{-1} x_{i,s} x_{j,t} \middle|\ p, q, s, t \in \{1, \ldots, l_0\}, i, j \in \{1, \ldots, d\} \right\rangle^F$.

Proof. We set $X = \left\langle x_{i,p}^{-1} x_{j,q}^{-1} x_{i,s} x_{j,t} \ \middle|\ p, q, s, t \in \{1, \ldots, l_0\}, i, j \in \{1, \ldots, d\} \right\rangle^F$ and observe that we have $F' \leq X$ and $X \leq \mathcal{N}$. We compare the quotients $\mathcal{N}/F'$ and X/F'. The first one consists of all products $\left\{ x_{1,1}^{z_{1,1}} \cdots x_{1,l_0}^{z_{1,l_0}} \cdot \cdots \cdot x_{d,1}^{z_{d,1}} \cdots x_{d,l_0}^{z_{d,l_0}} \right\}$ with $z_{j,i} \in \mathbb{Z}$ and where the sum $\sum_{k=1}^{l_0} z_{i,k} = 0$ for all $i = 1, \ldots, d$. The second quotient consists of products of the form $\prod_{k=1}^{r} \left(x_{p_k,i_k}^{-z_k} x_{q_k,j_k}^{-t_k} x_{s_k,i_k}^{z_k} x_{t_k,j_k}^{t_k} \right)^{f_k}$ with $z_k, t_k, r \in \mathbb{Z}$ and $f_k \in F, p_k, q_k, s_k, t_k \in \{1, \ldots, d\}, i_k, j_k \in \{1, \ldots, l_0\}$. Slight rearranging now shows equality, because the sum of exponents of each block is zero in the latter expression. □

Recall that we fixed an order for the elements of A_i as $\{g_{1,i}, \ldots, g_{l_i,i}\}$ with $g_{1,i} = 1$ and define

$$N_0 = \mathcal{N}\left(b_{1,0}^{g_{1,0}}, \ldots, b_{1,0}^{g_{l_0,0}}, \ldots, b_{d,0}^{g_{1,0}}, \ldots, b_{d,0}^{g_{l_0,0}} \right)$$

and similarly

$$N_n = \mathcal{N}\left(b_{1,n}^{g_{1,n}}, \ldots, b_{1,n}^{g_{l_n,n}}, \ldots, b_{d,n}^{g_{1,n}}, \ldots, b_{d,n}^{g_{l_n,n}} \right).$$

We have a natural map π from the free group $F_{d\cdot l_0}$ on $d \cdot l_0$ generators onto B. We identify the group N_0 with the image of $\mathcal{N}$ under π and similarly for N_i. We will further write N for N_0.

Lemma 2.1.10. *Assume that $\{l_i\}$ is given such that $l_i \geq 3$ and l_i is odd for all $i \in \mathbb{N}$. Then $G' = N_0$ and hence $G_i' = N_i$ for all $i \in \mathbb{N}$.*

Proof. Lemma 2.1.5 and the proof of Lemma 2.1.9 give $B' \leq N_0$. Lemma 2.1.2 gives $G' \leq N_0$ because $[B, A_0] \leq N_0$. Using that

$$N_0 = \left\langle b_{i,0}^{-a_u} \cdot b_{j,0}^{-a_v} \cdot b_{i,0}^{a_w} \cdot b_{j,0} \right\rangle^G$$

with $a_u, a_v, a_w \in A_0$ now gives triviality of the quotient N_0/G'. □

The following two Lemmas show that contrary to $\mathcal{N} \leq F_{d \cdot l_0}$ the subgroup N of G is finitely generated.

Lemma 2.1.11. *If $d = 1$, $l_0 \geq 5$ and l_0 is odd, then N is finitely generated.*

Proof. We denote

$$b(k) = b^{a^{k-1}} \tag{2.2}$$

and observe that

$$b\left(i + l_0 \cdot z\right) = b(i) \text{ for } i \in \{1, \ldots, l_0\}, z \in \mathbb{Z}.$$

We prove that

$$\left\{b(2)^{-1}b(1), b(3)^{-1}b(2), \ldots, b\left(l_0\right)^{-1} b\left(l_0 - 1\right), b(1)^{-1}b\left(l_0\right)\right\}$$

is a generating set for $N = \left\langle b(2)^{-1}b(1)\right\rangle^G$. Set $D = \left\langle b(2)^{-1}b(1), \ldots, b(1)^{-1}b\left(l_0\right)\right\rangle$. We show that $\left(b(2)^{-1}b(1)\right)^{b(k)} \in D$. Concatenation yields that $b(j)^{-1}b(i) \in D$ for all $i, j \in \{1, \ldots, l_0\}$. Using that $l_0 \geq 5$ we can see that

$$b(i)b(i-1)^{-1} = b(i+2)^{-1}b(i) \cdot b(i-1)^{-1}b(i+2)$$

because $[b(i), b(j)] = 1$ if i, j are such that $|i-j| \mod l_0 > 1$ by Lemma 2.1.5. Hence all elements $b(j)b(k)^{-1}$ for all j, k are in $\langle D\rangle$. We can write

$$\left(b(i)^{-1}b(i-1)\right)^{b(k)} = b(k)^{-1}b(i)^{-1}b(i-1)b(k)$$

$$= b(k)^{-1}b(i+2) \cdot b(i+2)b(i)^{-1} \cdot b(i+2)^{-1}b(i-1) \cdot b(i+2)^{-1}b(k)$$

because $b(i+2)$ commutes with $b(i)$ and $b(i-1)$. The latter is a product of four elements of D. This yields $D^b \leq D$ for all $b \in B$ and so $D^B = D$ which gives $N = D$ and so N is finitely generated. □

We now use the same idea for $d > 1$ and adapt it to multiple generators.

Lemma 2.1.12. *If $d > 1$, $l_0 \geq 9$ and l_0 is odd then N is finitely generated.*

To simplify notation we set $b_i\left(a_u\right) = b_i^{a_u}, a_u \in A_0, i \in \{1, \ldots, d\}$ for the rest of this Chapter.

Proof. We again set

$$D = \left\langle b_i\left(a_u\right)^{-1} \cdot b_j\left(a_v\right)^{-1} \cdot b_i\left(a_w\right) \cdot b_j(1)\right\rangle, a_u, a_v, a_w \in A_0, i, j \in \{1, \ldots, d\}.$$

Without loss of generality it is enough to show that $\left(b_i\left(a_u\right)^{-1} b_j\left(a_v\right)^{-1} b_i\left(a_w\right) b_j(1)\right)^{b_k\left(a_z\right)} \in D$ for $a_z \in A_0$. Again we insert three elements of the form $b_{h_t}\left(a_r\right), h_t \in \{1, \ldots, d\}, t \in \{1,2,3\}, a_r \in A_0$, which commute with all factors in the product

$$b_k\left(a_z\right)^{-1} \cdot b_i\left(a_u\right)^{-1} b_j\left(a_v\right)^{-1} b_i\left(a_w\right) b_j(1) \cdot b_k\left(a_z\right).$$

The elements in the product decorate five pairs of places $\left(x_c, y_c\right), c \in \{1,2,3,4,5\}$. The elements $b_{h_t}\left(a_r\right)$ must be such that $\left[b_h\left(a_r\right), b_s\left(a_x\right)\right]=1$ for all $s \in \{j, i, k\}$ and all $x \in \{u, v, w, 1\}$. Let f, g be the places that the three elements $b_{h_t}\left(a_r\right)$ decorate. The four elements

$$b_i\left(a_u\right), b_j\left(a_v\right), b_i\left(a_w\right), b_j(1)$$

decorate a most 8 places with the pairs $\left(x_c, y_c\right), c \in \{1,2,3,4\}$. This is the worst case and then we can choose $(f, g)=\left(x_{i_0}, y_{i_0}\right)$ for some $i_0 \in \{1,2,3,4,5\}$. Otherwise, if the a_u, a_v, a_w are such that none of the $b_i\left(a_u\right), b_j\left(a_v\right), b_i\left(a_w\right), b_j(1)$ commute, then they decorate at most 5 places and we can choose a_r such that $b_{h_t}\left(a_r\right)$ commutes with all of them. In either case 8 places suffice. This is given by the assumption that $l_i \geq 9$ for all $i \in \mathbb{N}$. We can then write the product as

$$\begin{gathered}
b_k\left(a_z\right)^{-1} b_{h_1}\left(a_r\right)^{-1} b_{h_2}\left(a_r\right) b_{h_3}\left(a_r\right) \\
\cdot b_i\left(a_u\right)^{-1} b_j\left(a_v\right)^{-1} b_i\left(a_w\right) b_j(1) \\
\cdot b_{h_3}\left(a_r\right)^{-1} b_{h_2}\left(a_r\right)^{-1} b_{h_1}\left(a_r\right) b_k\left(a_z\right)
\end{gathered}$$

which is a product of three elements of D if $h_1=h_3$ and $h_2=k$. □

Lemma 2.1.13. *We have the equalities $N^{\prime}=B^{\prime}$ and $N_i^{\prime}=B_i^{\prime}$ for all $i \in \mathbb{N}$ if $l_0 \geq 5$ and l_0 is odd.*

Proof. The inclusion $N \leq B$ gives $N^{\prime} \leq B^{\prime}$. We now aim to write the generators of $B^{\prime}$ as a commutator of elements in N. Using the same idea as in the proof of Lemma 2.1.12 we get

$$\begin{gathered}
\left[b_i^{a_u}, b_j^{a_v}\right]=b_i\left(a_u\right)^{-1} b_j\left(a_v\right)^{-1} b_i\left(a_u\right) b_j\left(a_v\right) \\
=b_i\left(a_u\right)^{-1} b_k\left(a_\alpha\right)^{-1} b_i\left(a_\alpha\right) b_k\left(a_\alpha\right) \\
\cdot b_j\left(a_v\right)^{-1} b_k\left(a_\alpha\right)^{-1} b_j\left(a_\alpha\right) b_k\left(a_\alpha\right) \\
\cdot b_k\left(a_\alpha\right)^{-1} b_i\left(a_\alpha\right)^{-1} b_k\left(a_\alpha\right) b_i\left(a_u\right) \\
\cdot b_k\left(a_\alpha\right)^{-1} b_j\left(a_\alpha\right)^{-1} b_k\left(a_\alpha\right) b_j\left(a_v\right)
\end{gathered}$$

for an appropriate position of a_α such that $[b_i(a_v), b_k(a_\alpha)] = 1$ and $[b_j(a_u), b_k(a_\alpha)] = 1$. This also implies that $[b_i(a_v), b_i(a_\alpha)] = 1$ and $[b_j(a_v), b_j(a_\alpha)] = 1$. It is possible to choose such an $a_\alpha \in A_0$ because of the assumption that $l_0 \geq 5$. ☐

Remark 2.1.14. *In the above Lemma we only require $l_0 \geq 5$. However, to also have the assertion of Lemma 2.1.12 we will sometimes have to assume $l_0 \geq 9$.*

We will now further explore the morally self-similar structure of G.

Lemma 2.1.15. *(a) $N'_n = N_{n+1} \times \cdots \times N_{n+1}$ with m_n factors in the direct product.*

(b) $B'_{n-1} = N_n \times \cdots \times N_n \leq B_n \times \cdots \times B_n$ where we have m_{n-1} factors in the direct product.

(c) $B''_n = B'_{n+1} \times \cdots \times B'_{n+1}$ with m_n factors in the direct product.

Proof. Using $B \leq G_1 \times \cdots \times G_1$ immediately gives $N' = B' \leq N_1 \times \cdots \times N_1$ by Lemma 2.1.10. For the other inclusion we prove $G'_1 \times \cdots \times G'_1 \leq B'$ which then gives $N_1 \times \cdots \times N_1 \leq N'$. The group G_1 can be written as $G_1 = B_1 \cdot A_1$ and hence $G'_1 = B'_1 \cdot [B_1, A_1]$. We first show $B'_1 \leq B'$. For this we write a generator $[b_{i,1}(a_{v,1}), b_{j,1}(a_{w,1})]$ of B_1 as a commutator of elements in B. Assume we want to reconstruct B_1 on position y of level 1. Choose $a_y \in A_0$ such that $g_{2,0}a_y = y$ and $s_y \in A_0$ such that $g_{1,0}s_y = y$. Assume $g_{1,0}a_y = x$ and $g_{2,0}s_y = z$ with $x, z \neq y$. Denote by $b_v \in B$ the element with $b_v = (b_{v,1}, a_{v,1}, 1, \ldots, 1)_1$ and by $b_w \in B$ the element with $b_w = (b_{w,1}, a_{w,1}, 1, \ldots, 1)_1$. Writing out the terms and cancellation then show

$$[b_{i,1}(a_{v,1}), b_{j,1}(a_{w,1})] = \left[\left(b_i(s_y)^{-1}\right)^{b_v^{-1}(a_y)}, \left(b_{j,1}(s_y)^{-1}\right)^{b_w^{-1}(a_y)}\right].$$

For the subgroup $[B_1, A_1]$ we write

$$[b_{i,1}(a_{q,1}), a_{t,1}] = b_{i,1}^{-1}(a_{q,1}) \cdot (b_{i,1}(a_{q,1}))^{a_{t,1}}.$$

Then similar considerations give

$$[b_{i,1}(a_{q,1}), a_{t,1}] = \left[b_i(s_y)^{b_q(a_y)}, b_t(a_y)\right] \in B'.$$

The statement for N_i and parts (a) and (c) are now immediate consequences of this and Lemma 2.1.13. ☐

This allows us to sharpen the result from Lemma 2.1.5 in the case when $d = 1$, when G is a 2-generator group.

Lemma 2.1.16. *If $d = 1$, $l_0 \geq 3$ and l_0 is odd then we have*

$$\mathrm{rst}_G(1) = B' \cdot B^{l_1} = \left\langle b(1)^{l_1}, \ldots, b\left(l_0\right)^{l_1}, b_1(2)^{-1}b_1(1), \ldots, b_1(1)^{-1}b_1\left(l_1\right)\right\rangle^{A_1 \times \cdots \times A_1}.$$

Proof. The inclusion $B'B^{l_1} \leq \mathrm{rst}_G(1)$ is the statement of Lemma 2.1.5. We observe that the last l_1 generators are precisely the generators of $B' = N' = N_1 \times \cdots \times N_1$. Assume $g \in G$ is an element that acts trivially outside a subtree $T|_{v_1}$ for a vertex $v_1 \in \Omega(1)$. The element g must be such that it can be written as

$$g = \prod_{j=1}^{l_0} \prod_{k=1}^{q} b\left(i_{j,k}\right)^{q_{i,k}}, \quad t \in \mathbb{N}, q_{i,k} \in \mathbb{Z}, i_{j,k} \in \{1, \ldots, l_0\},$$

where we require that each product $\prod_{k=1}^{t} b\left(i_{j,k}\right)^{q_{i,k}}$ only acts on exactly one vertex of $\Omega(1)$. Then g can only be either of the form b^{l_1} or a word in the generators $b_1(i)$ of B_1 such that the sum of the exponents of the $b_1(i)$ is zero, which is equivalent to $g \in N_1 \times \cdots \times N_1$. □

Remark 2.1.17. *The above Lemma 2.1.16 can be generalised for the case $d > 1$ with*

$$\mathrm{rst}_G(1) = B' \cdot \prod_{i=1}^{d} \left\langle b_i^{q_{i,0}} \right\rangle^{A_0} = \left(N_1 \times \cdots \times N_1\right) \cdot \prod_{i=1}^{d} \left\langle b_i^{q_{i,0}} \right\rangle^{A_0},$$

where we have l_0 factors in the direct product of the N_1. Then the generators of $\mathrm{rst}_G(1)$ *are given by*

$$x_{i,k} = \left(b_{i,0}^{q_{i,0}}\right)^{g_{k,0}}, \quad g_{k,0} \in A_0,$$

and by $y_{i,j}$ which are given by the finitely many generators of $N_1 \times \cdots \times N_1$, where we have l_0 factors in the direct product.

Denote in the case $d = 1$ the $2l_0$ generators of $\mathrm{rst}_G(1)$ by

$$x_i = b(i)^{l_1} \quad \text{and} \quad y_{i,j} = \left(b_1(i+1)^{-1}b_1(i)\right)^{a^j}.$$

Lemma 2.1.18. *If $d = 1$, then we have $\mathrm{rst}_G(i) \leq \mathrm{St}_G(i+1)$.*

Proof. Easy computation verifies that each x_i and $y_{i,j}$ stabilises level 2, hence $\mathrm{rst}_G(1) \leq \mathrm{St}_G(2)$. We now use the self-similar structure of G to deduce that

$$\mathrm{rst}_G(i) = \mathrm{rst}_{G_{i-1}}(1) \cap G \leq \mathrm{St}_{G_{i-1}}(2) \cap G \leq \mathrm{St}_G(i+1).$$

□

We will see in the following Lemma that this inclusion is in general strict.

Lemma 2.1.19. *If $d = 1$, then the inclusion $\mathrm{rst}_G(2) \leq \mathrm{St}_G(3)$ is strict.*

Proof. The element $z = b(2)b(2)^b b(2)^{l_2-1}\left(b(2)^b\right)^{l_2-1} b(2)^{-2l_2}$ is an element of $\mathrm{St}_G(3)$:

$$\begin{aligned} z &= \left(1, b_1 b_1^{a_1} b_1^{l_2-1} (b_1^{a_1})^{l_2-1} b_1^{-2l_2}, a_1 a_1 a_1^{l_2-1} a_1^{l_2-1} a_1^{-2l_2}, 1, \dots, 1\right)_1 \\ &= \left(1, \dots, 1, b_2 b_2^{l_2-1} b_2^{-2l_2}, a_2 b_2 a_2^{l_2-1} b_2^{l_2-1} a_2^{-2l_2}, a_2 a_2^{l_2-1}, 1 \dots, 1\right)_2 \\ &= \left(1, \dots, 1, b_2^{-l_2}, a_2 b_2 a_2^{l_2-1} b_2^{l_2-1} a_2^{-2l_2}, 1, \dots, 1\right)_2 . \end{aligned}$$

The sum of the exponents of a_2 in the second term is $-l_2$, hence it stabilises level 3 and we conclude that $z \in \mathrm{St}_G(3)$. Assume that $z \in \mathrm{rst}_G(2)$. Then $k = \left(b_2^{-l_2}, 1, \dots, 1\right)_2$ is in $\mathrm{rst}_G(2)$ and in particular in G. Hence also $b^{l_2}k = \left(1, a_1^{l_2}, 1, \dots, 1\right)_1 \in G$ and so $(1, a_1, 1, \dots, 1)_1 \in G$ which would give $G_1 \times \cdots \times G_1 \leq G$, a contradiction. ☐

We have chosen to show $\mathrm{rst}_G(2)$ is strictly included in $\mathrm{St}_G(3)$ because we have $\mathrm{rst}_G(1) = \mathrm{St}_G(2)$. Although this property of course holds for any of the groups G_i, we cannot deduce that we have $\mathrm{rst}_G(i) = \mathrm{St}_G(i+1)$. This is because

$$\mathrm{rst}_G(i) = \prod \left(\mathrm{rst}_{G_{i-1}}(1) \cap G\right),$$

but on the other hand

$$\mathrm{St}_G(i+1) = \left(\prod \mathrm{St}_{G_{i-1}}(2)\right) \cap G.$$

The subtle difference comes from the fact that if $g = (h, k) \in H \times K$ is an element of a subgroup $T \leq H \times K$ then we cannot deduce that $(h, 1)$ or $(k, 1)$ are elements of G as the following example demonstrates:

Example 2.1.20. Let H and K both be $\mathbb{Z} = \langle 1 \rangle$ in additive notation and choose $T = \langle (2, 2) \rangle$, a subgroup of $H \times K$. Then $(2, 2) \in T$, but neither $(2, 0)$ nor $(0, 2)$ are in T.

Remark 2.1.21. *Lemma 2.1.18 and Lemma 2.1.19 can be generalised to $d > 1$.*

Proposition 2.1.22. *If $l_i \geq 3$ and odd for all $i \in \mathbb{N}_0$ then the subgroup $G^{(n+1)}$ can be written as*

$$G^{(n+1)} = G_n' \times \cdots \times G_n'$$

for $n \geq 0$.

Proof. We note

$$G^{(n+1)} = (G')^{(n)} = N^{(n)} = (N')^{(n-1)} = N_1^{(n-1)} \times \cdots \times N_1^{(n-1)} = N_n \times \cdots \times N_n$$

by using Lemma 2.1.15 iteratively. With Lemma 2.1.10 we get $G^{(n+1)} = G'_n \times \cdots \times G'_n$. □

Lemma 2.1.23. *If $l_i \geq 3$ and odd for all $i \in \mathbb{N}_0$ then $G^{(n+1)} \leq \operatorname{rst}_G(n)$.*

Proof. Proposition 2.1.22 gives $G^{(n+1)} = N_n \times \cdots \times N_n \leq (G \cap G_n) \times \cdots \times (G \cap G_n) = \operatorname{rst}_G(n)$. □

Corollary 2.1.24. *If $l_i \geq 9$ and odd for all $i \in \mathbb{N}$ then we have $\operatorname{rst}_G(n)' = N_{n+1} \times \cdots \times N_{n+1}$ with m_n factors in the direct product and hence $\operatorname{rst}_G(n)'$ is finitely generated for all $n \in \mathbb{N}$.*

Proof. The proof of Lemma 2.1.23 shows $N_n \times \cdots \times N_n \leq \operatorname{rst}_G(n)$ and so by Lemma 2.1.15 and Lemma 2.1.13 we get that $B'_n \times \cdots \times B'_n = N'_n \times \cdots \times N'_n = N_{n+1} \times \cdots \times N_{n+1} \leq \operatorname{rst}_G(n)'$. With $\operatorname{rst}_G(n) \leq B_n \times \cdots \times B_n$ this gives $N_{n+1} \times \cdots \times N_{n+1} = \operatorname{rst}_G(n)'$ and hence $\operatorname{rst}_G(n)'$ is finitely generated. □

Lemma 2.1.25. $\operatorname{St}_G(n)' = ((G_n \times \cdots \times G_n) \cap G)' = G'_n \times \cdots \times G'_n$.

Proof. The n-th level stabilizer $\operatorname{St}_G(n)$ is given by $G \cap (G_n \times \cdots \times G_n)$ which gives the first equality. By using $G'_{n-1} \times \cdots \times G'_{n-1} \leq (G_n \times \cdots \times G_n) \cap G$ we get

$$G'_n \times \cdots \times G'_n = G''_{n-1} \times \cdots \times G''_{n-1} \leq ((G_n \times \cdots \times G_n) \cap G)'$$

$$\leq (G'_n \times \cdots \times G'_n) \cap G' = G^{(n+1)} \cap G' = G^{(n+1)} = G'_n \times \cdots \times G'_n.$$

□

Theorem 2.1.26. *Assume $l_i \geq 3$ and odd for all $i \in \mathbb{N}_0$. Then the quotient $\frac{\operatorname{St}_G(n)}{\operatorname{rst}_G(n)}$ is abelian and has exponent less than or equal to $l_1 \cdot \cdots \cdot l_n$. Consequently $\operatorname{rst}_G(n)$ is of finite index in G, so G is a branch group.*

Proof. We get $\operatorname{St}_G(n)' = G' \cap (G'_n \times \cdots \times G'_n) = G^{(n+1)}$ by Lemma 2.1.25 and Proposition 2.1.22 and further $G^{(n+1)} \leq \operatorname{rst}_G(n)$ by Lemma 2.1.23. Hence the quotient $\operatorname{St}_G(n)/\operatorname{rst}_G(n)$ is abelian. Further we see that

$$\operatorname{St}_G(n)^{l_1 \cdots l_n} = \left(\operatorname{St}_G(n)^{l_n}\right)^{l_1 \cdot \cdots \cdot l_{n-1}} \leq \left(B_n^{l_n} \times \cdots \times B_n^{l_n}\right)^{l_1 \cdot \cdots \cdot l_{n-1}} \leq \operatorname{rst}_G(n)$$

by Lemma 2.1.4. □

Proposition 2.1.27. *The quotient* $\frac{\mathrm{St}_G(n+1)}{\mathrm{rst}_G(n)}$ *has rank less than or equal to* $l_0 \cdot \cdots \cdot l_{n-1} \cdot d$ *if* $l_i \geq 3$ *and odd for all* $i \in \mathbb{N}_0$.

Proof. Lemma 2.1.18 and Remark 2.1.21 show that $\mathrm{rst}_G(n) \leq \mathrm{St}_G(n+1)$. Together with

$$\mathrm{St}_G(n+1) \leq B_n \times \cdots \times B_n$$

and

$$N_n \times \cdots \times N_n = G^{(n+1)} \leq \mathrm{rst}_G(n)$$

this gives that

$$\frac{\mathrm{St}_G(n+1)}{\mathrm{rst}_G(n)} \quad \text{is a section of} \quad \frac{B_n \times \cdots \times B_n}{N_n \times \cdots \times N_n}.$$

The latter is the image of $F_{l_n \cdot d}/N_{F_{l_n \cdot d}}$ under the identification $x_{i,j} \mapsto b_{i,0}^{a_j}$. Hence $\frac{B_n}{N_n}$ is the image of $\prod_{i=1}^{d} \mathbb{Z}$ and so $\frac{B_n \times \cdots \times B_n}{N_n \times \cdots \times N_n}$ has rank less or equal than $l_0 \cdots l_{n-1} \cdot d$. □

2.1.3 Finite Generation of Normal Subgroups and Proper Quotients

We show that the group G has the property that if $H \lhd G$ then H is finitely generated. This is a non-trivial observation because we will see later in Theorem 2.2.4 that G is not just infinite.

Theorem 2.1.28. *If* $H \lhd G$ *is a non-trivial normal subgroup of* G *and* $\{l_i\}$ *is such that* $l_i \geq 9$ *and odd for all* $i \in \mathbb{N}$*, then* H *is finitely generated.*

Proof. Corollary 2.1.24 states that $\mathrm{rst}_G(n)'$ is finitely generated for all n. By Theorem 1.3.1 from Chapter 1 every non-trivial normal subgroup $K \lhd G$ contains $\mathrm{rst}_G(n)'$ for some n. This yields that $(K \cap \mathrm{rst}_G(n)) / \mathrm{rst}_G(n)'$ is a subgroup of the finitely generated abelian group $\mathrm{rst}_G(n)/\mathrm{rst}_G(n)'$, therefore finitely generated and abelian. On the other hand, $(K \cap \mathrm{rst}_G(n)) / \mathrm{rst}_G(n)'$ is a normal subgroup of finite index in $K/\mathrm{rst}_G(n)'$ and hence $K/\mathrm{rst}_G(n)'$ is a finite extension of a finitely generated abelian group, hence finitely generated. Now $\mathrm{rst}_G(n)'$ is finitely generated, hence so is K. □

Theorem 2.1.29. *If* $l_i \geq 3$ *and odd for all* $i \in \mathbb{N}_0$ *and* K *is a non-trivial normal subgroup of* G*, then* G/K *is soluble.*

Proof. Lemma 2.1.23 implies $G^{(n+2)} \leq \mathrm{rst}_G(n)'$. By Theorem 1.3.1 we therefore deduce that if K is a non-trivial normal subgroup of G, then there exists an n such that $G^{(n+2)} \leq K$. Hence G/K is soluble. □

The property shown above in Theorem 2.1.29 is shared by many other examples of branch groups.

Lemma 2.1.30. *Let Γ be a just infinite p-group. Then every proper quotient of Γ is nilpotent, in particular soluble.*

Proof. Let N be a normal subgroup of Γ. Then Γ/N is a finite p-group, hence nilpotent. □

In particular, it was shown by Grigorchuk and Zuk in [GZ02] that every proper quotient of the weakly branch basilica group is soluble.

2.2 Abelianisations

This Section is about the abelianisation of G and of some of its subgroups. Using the rich structure that branch groups provide, we will be able to deduce that G is not just infinite and that it has infinite virtual first Betti number.

We start with determining the abelianisation of G and write for it $G^{ab} = G/G'$ respectively $B^{ab} = B/B'$ and $\mathrm{rst}_G(n)^{ab} = \mathrm{rst}_G(n)/\mathrm{rst}_G(n)'$. We recall that we denoted the order of the j-th generator $a_{i,j}$ of A_i by $q_{i,j}$.

Theorem 2.2.1. *Let G be generated by $\langle a_1, \dots, a_d, b_1, \dots, b_d \rangle$ with orders $q_{i,0}$ for a_i. Then*

$$\prod_{i=1}^{d} C_{q_{i,0}} \times C_\infty \leq G^{ab} \leq \prod_{i=1}^{d} C_{q_{i,0}} \times \prod_{i=1}^{d} C_\infty$$

and

$$\prod_{i=1}^{m_n \cdot d} C_{q_{i,n}} \times \prod_{i=1}^{m_n} C_\infty \leq G_n^{ab} \leq \prod_{i=1}^{m_n \cdot d} C_{q_{i,n}} \times \prod_{i=1}^{m_n \cdot d} C_\infty.$$

Proof. We define the group $W(n)$ to be $W(n) = A_{n-1} \wr \cdots \wr A_0$. We have a surjective map

$$\varphi : G \twoheadrightarrow \frac{G}{G'} \twoheadrightarrow \frac{W(n)}{W(n)'} = A_{n-1} \times \cdots \times A_0.$$

The order of $\varphi(b_i)$ in $W(n)^{ab}$ is $\prod_{j=1}^{n-1} q_{i,j}$. This grows unboundedly with n hence the order of $\varphi(b_i)$ must be infinite in G^{ab}. The generators $a_1, \dots, a_d$ have orders $q_{1,0}, \dots, q_{d,0}$ in the abelianisation by construction. The other inclusions are given by the bounds from the generators of G and G_n. □

As a straight forward Corollary we get the abelianisation of B and of all groups B_n for each level $n \geq 0$.

Corollary 2.2.2. *The abelianisation of B satisfies*

$$\prod_{i=1}^{l_0} C_\infty \leq B^{ab} \leq \prod_{i=1}^{l_0 \cdot d} C_\infty.$$

Similarly, for all $n \geq 0$, the abelianisation of B_n satisfies

$$\prod_{i=1}^{m_{n+1}} C_\infty \leq B_n^{ab} \leq \prod_{i=1}^{m_{n+1} \cdot d} C_\infty.$$

Proof. The images of b_i have infinite order in G^{ab}. By Lemma 2.1.13 we have a sequence of surjections

$$B \twoheadrightarrow \frac{B}{B'} \twoheadrightarrow \frac{B}{G'} \leq \frac{G}{G'}.$$

Hence the images of b_i in B^{ab} also have infinite order. Now look at the elements $(b_i^u)^{l_1}$ and $(b_i^v)^{l_1}$ for $i, j = 1, \ldots, d$ and $u, v \in A_0$. If $u \neq v$, then $(b_i^u)^{l_1}$ and $(b_i^v)^{l_1}$ decorate disjoint subtrees. Hence their images in B^{ab} must be distinct as well. □

The rigid stabilizers play an important role in determining the structure of branch groups. It is therefore of particular interest to calculate their abelianisations. Knowing about them allows us to deduce whether a branch group is just infinite or not.

Theorem 2.2.3. *The abelianisation of the rigid stabilizer* $\mathrm{rst}_G(n)$ *of level* n *satisfies*

$$\mathrm{rst}_G(n)^{ab} \geq \prod_{i=1}^{m_{n+1}} C_\infty.$$

Proof. By Lemma 2.1.15 and Lemma 2.1.10 we get

$$\prod_{i=0}^{m_n} B_n' \leq \mathrm{rst}_G(n)' \leq \mathrm{rst}_G(n) \leq \prod_{i=1}^{m_n} B_n$$

and hence find that

$$\frac{\mathrm{rst}_G(n)}{\mathrm{rst}_G(n)'} \quad \text{is a section of} \quad \prod_{i=1}^{m_n} \frac{B_n}{B_n'}, \quad B^{ab} \geq \prod_{i=1}^{m_{n+1}} C_\infty$$

by Corollary 2.2.2. The elements $\left(b_{i,n}^{m_{n+1}}\right)^t$ for $t \in A_n$ are all in $\mathrm{rst}_G(n)$ by Lemma 2.1.4 because $\prod_{i=1}^{n-1} l_i$ divides m_n. The proof of Theorem 2.2.1 again gives that each $\left(b_{i,n}^{m_{n+1}}\right)^t$ has infinite order in $\mathrm{rst}_G(n)^{ab}$ for all $i \in \{1, \ldots, d\}$ and $t \in A_n$. By the same arguments as above the images of $\left(b_{i,n}^{m_{n+1}}\right)^t$ and $\left(b_{i,n}^{m_{n+1}}\right)^s$ are different for $s, t \in A_n$ if $s \neq t$. We deduce that the abelianisation of $\mathrm{rst}_G(n)$ contains $\prod_{i=1}^{m_{n+1}} C_\infty$. □

The following Theorem is now an immediate consequence of Theorem 2.2.1.

Theorem 2.2.4. *G is not just infinite.*

2.2.1 Virtual First Betti Number

The *first Betti number* $b_1(\Gamma)$ of a group Γ is the torsion-free rank of $H_1(\Gamma; \mathbb{Z}) \otimes \mathbb{Q}$. This is the rank of Γ^{ab}.

The *virtual first Betti number* $vb_1(\Gamma)$ of Γ is defined [LLR08] to be

$$vb_1(\Gamma) = \sup\{b_1(H) : |\Gamma/H| < \infty\}.$$

Theorem 2.2.5. *The virtual first Betti number $vb_1(G)$ of G is infinite.*

Proof. Theorem 2.2.3 states that the rank of $\mathrm{rst}_G(n)^{ab}$ is greater than or equal to m_{n+1} for all n. By Theorem 2.1.26 the subgroups $\mathrm{rst}_G(n)$ all have finite index. We have therefore found a sequence of subgroups for which the above supremum is infinite. □

2.3 Largeness and Free Subgroups

We will first see that G cannot be large because all its proper quotients are soluble. We will then prove the absence of free subgroups for the case of the 2-generator group G_C, given by a sequence of finite cyclic groups. It is to be expected that G has similar properties in the $2d$-generator case.

We will construct for any two $g, h \in G$ a non-trivial word $w_{g,h}(x, y) \in F_2$ in the free group F_2 on two generators such that $w_{g,h}(g, h) = 1$ in G. This construction is rather technical. Example 2.3.13 demonstrates the idea behind it with two explicit elements of G.

2.3.1 Largeness

Denote by F_2 the free group of rank 2. A group Γ is called *large* if there exists a finite index subgroup $H \leq_f \Gamma$ such that H surjects onto F_2.

We observe that if a group Γ is large, then Γ also has infinite virtual first Betti number as defined in Subsection 2.2.1. It is an open question, whether there exist finitely presented groups with infinite virtual first Betti number but which are not large. We will see in the following that there at least exist finitely generated groups which can have infinite virtual first Betti number but are not large.

Example 2.3.1. The group $\mathbb{Z} \wr \mathbb{Z}$ as described in Section A.2.3 of Section A.2 has infinite virtual first Betti number but is not large.

The following Theorem 2.3.2 shows that there also exist more complicated examples.

Theorem 2.3.2. *The group G is not large.*

Proof. Assume that there exists a subgroup H of finite index in G that maps onto F_2. Hence H also maps onto A_5, the alternating group of order 60. Write N for the kernel of $H \twoheadrightarrow A_5$. If we define $N_0 = \bigcap_{g \in G} N^g$, then H/N_0 is an insoluble quotient of H. On the other hand, H has finite index in G and hence G/N_0 is finite. By Theorem 2.1.29 it must also be soluble, hence cannot have a section isomorphic to A_5. □

The above Theorem 2.3.2 together with Theorem 2.2.5 gives an example of a finitely generated group that is not large, yet has infinite virtual first Betti number. This and Example 2.3.1 are the only known examples of such groups.

2.3.2 Free Subgroups

Under certain assumptions we can sharpen this result and deduce that G_C does not contain any free subgroups. Our aim is to show that if the defining sequence is such that all l_i are distinct primes and the sequence is ascending and it satisfies $l_i \geq 36^i$ for all $i \in \mathbb{N}$ then the group constructed above has no free subgroups of rank 2. Indeed, given any two elements g_1, g_2 of the group we construct a non-trivial word $w_{g_1,g_2}(x,y) \in F(x,y)$ in the free group of rank 2 such that $w_{g_1,g_2}(g_1,g_2) = 1$ in G. We will for all further considerations write G for G_C and denote the two generators simply by a and b.

It follows from Lemma 2.1.2 that we can write every $g \in G$ as $g = a^r \prod_{i=1}^t b(k_i)^{q_i}$ with $r, q_i \in \mathbb{Z}$, $k_i \in \{1, \ldots, l_0\}$ and $t \in \mathbb{N}$.

Definition 2.3.3. A *spine* $s = g^{-1}b^q g$ is a power of a g-conjugate of b with $g \in G$ and some $q \in \mathbb{Z}\setminus\{0\}$. Denote by

$$\xi(g) = \min\left\{t \mid g = a^r \prod_{i=1}^{t} b(k_i)^{q_i}\right\}$$

the *number of spines* of g and by

$$\lambda(g) = r + \sum_{i=1}^{t} (q_i + 2k_i - 2)$$

the *length* of g. This is the usual notation for the word length of an element which can be seen by considering the definition of $b(i) = a^{-i+1}ba^{i-1}$

Remark 2.3.4. *The number of spines $\lambda(g)$ should not be confused with the word length of $g \in B$ as a word in the generators of B.*

Lemma 2.3.5. $\lambda(gh) \leq \lambda(g) + \lambda(h)$ *and hence* $\lambda\left(g^h\right) \leq \lambda(g) + 2\lambda(h)$ *for any* $g, h \in G$.

Proof. This follows immediately from the definition. □

Let $g_1, g_2 \in G$ be fixed for the rest of this Section. Recursively define commutators

$$c_1 = [g_1, g_2] \quad \text{and} \quad c_i = \left[c_{i-1}, c_{i-1}^{c_{i-2}}\right], \text{ for } i \geq 2 \text{ with } c_0 = g_1. \tag{2.3}$$

With those definitions we get the following Lemma:

Lemma 2.3.6. *If $g_1, g_2 \in G$, then the length $\lambda(c_i)$ of the commutator c_i defined as above is bounded by $\lambda(c_i) \leq 5^i (\lambda(g_1) + \lambda(g_2))$ for all $i \geq 0$.*

Proof. We use induction. It follows from Lemma 2.3.5 that $\lambda(c_1) \leq 2\lambda(g_1) + 2\lambda(g_2) \leq 5 \cdot (\lambda(g_1) + \lambda(g_2))$. Then by induction hypothesis

$$\lambda(c_i) \leq 4\lambda(c_{i-1}) + 4\lambda(c_{i-2})$$

$$\leq 4 \cdot 5^{i-1} (\lambda(g_1) + \lambda(g_2)) + 4 \cdot 5^{i-2} (\lambda(g_1) + \lambda(g_2)) \leq 5^i (\lambda(g_1) + \lambda(g_2)).$$

□

The strategy for constructing a word is to observe that the number of spines of the commutator c_i grows more slowly with i than the number of vertices on each level. We note the position of the spines of c_i and aim to move them by conjugation such that none of the conjugated spines is at an old position. This new element will then commute with c_i.

The following Lemma 2.3.7 gives one of the key properties of the commutators defined in (2.3).

Lemma 2.3.7. *For every $i \geq 1$ we have $c_i \in \mathrm{rst}_G(i-1) \leq \mathrm{St}_G(i)$.*

Proof. We have $c_2 \in G'' \leq \mathrm{rst}_G(1) \lhd G$. Hence $c_2^{c_1} \in \mathrm{rst}_G(1)$ and so

$$c_3 = [c_2, c_2^{c_1}] \in \mathrm{rst}_G(1)' \leq \mathrm{St}_G(1)' \leq \mathrm{rst}_G(2)$$

by Lemma 2.1.25 and Lemma 2.1.23. Now assume $c_{n-1} \in \mathrm{rst}_G(n-2) \lhd G$. Then $c_{n-1}^{c_{n-2}} \in \mathrm{rst}_G(n-2)$ and hence again

$$c_n = \left[c_{n-1}, c_{n-1}^{c_{n-2}}\right] \in \mathrm{rst}_G(n-2)' \leq \mathrm{St}_G(n-2)' \leq \mathrm{rst}_G(n-1).$$

The last statement $\mathrm{rst}_G(n-1) \leq \mathrm{St}_G(n)$ is given by Lemma 2.1.18. □

We will now describe the commutators c_i. We will see below that after some finite level, their non-trivial decorations can only be one of two rather simple types of elements.

Proposition 2.3.8. *The commutators c_i have the recursive form $c_i = (d_{i,k,1}, \ldots, d_{i,k,m_k})_k$ on level k where each $d_{i,k,j}$ falls into one of the four cases:*

1. *$d_{i,k,j} = 1$,*
2. *$d_{i,k,j} = b_k^t$ for $t \in \mathbb{Z}, b_k$ the generator of G_k, $b_k \in B_k$,*
3. *$d_{i,k,j} = a_k^q \cdot z$ with $q \not\equiv 0 \mod l_i, z \in B_k, a_k \in A_k$ or*

4. $d_{i,k,j} = \left(d_{i,k+1,1+(j-1)l_i}, \ldots, d_{i,k+1,j\cdot l_i}\right)_{k+1}$.

Further, there exists some level n such that all $d_{i,n,j}$ will have fallen into one of the first three cases.

The indices in $d_{i,k,j}$ are as follows:

- i: means that $d_{i,k,j}$ is coming from the commutator c_i,
- k: describes the level k on which we look at the commutator c_i,
- j: indicates the position j on level k on which $d_{i,k,j}$ acts.

Write $\lfloor i/j \rfloor$ for the biggest integer q such that $q \leq \frac{i}{j}$.

Proof. From Lemma 2.3.7 we have $c_i = (d_{i,i,1}, \ldots, d_{i,i,m_i})_i \in G_i \times \cdots \times G_i$ with

$$d_{i,i,j} = a_i^{q_j} \prod_{k=0}^{u_j} b_i\left(r_{i,j,k}\right)^{f_{i,j,k}}$$

and $q_j, f_{i,j,k} \in \mathbb{Z}, u_j \in \mathbb{N}, r_{i,j,k} \in \{1, \ldots, l_i\}$. As an element of G_i, $d_{i,i,j}$ can only either be of the form $a_i^q z$ with $a_i \in A_i, z \in B_i$ which is the third case, or $d_{i,i,j} \in B_i$. If $d_{i,i,j}$ is an element of $B_i \times \cdots \times B_i \leq G_{i+1} \times \cdots \times G_{i+1}$ then it can be written as

$$d_{i,i,j} = \prod_{k=1}^{r} b_i\left(r_{i,j,k}\right)^{f_{i,j,k}}.$$

Now assume that $r > 1$, hence $d_{i,i,j}$ is not of the form b_i^t for some $t \in \mathbb{Z}$, hence does not fall into the second case. Then $d_{i,i,j}$ can be expressed as

$$d_{i,i,j} = \left(d_{i,i+1,1+(j-1)l_i}, \ldots, d_{i,i+1,j\cdot l_i}\right)_{i+1} \in \mathrm{St}_G(i+1),$$

where each of the $d_{i,i+1,h}$ is an element of G_{i+1} for $h \in \{1, \ldots, m_{i+1}\}$. Assume that at least one $d_{i,i+1,h}$ is again of this form, which is the forth case. Then

$$d_{i,i+1,h} = a_{i+1}^{q_h} \prod_{s=1}^{y_h} b_{i+1}\left(f_{h,s}\right)^{z_{h,s}}$$

with $q_h, z_{h,s} \in \mathbb{Z}$, $y_h \in \mathbb{N}$ and $f_{h,s} \in \{1, \ldots, l_{i+1}\}$. We assume that not all $f_{h,s}$ are equal to 1 and that $q_h \equiv 0 \mod l_i$ to eliminate cases 2 and 3. However, if there exists an $f_{h,s_0} \neq 1$ then $d_{i,i,j}$ was such that $b_{i+1}\left(f_{h,s_0}\right) = b_i(c)^{b_i(c-1)^q}$ for some $c \in \{1, \ldots, l_i\}$ and some $q \in \mathbb{Z}\backslash\{0\}$. This yields that the word lengths satisfy $\lambda\left(d_{i,i+1,h}\right) < \lambda\left(d_{i,i,j}\right) - 1$ if $j = \lfloor h/l_i \rfloor$ and hence there exists a level n such that all $d_{i,n,m}$ fall into one of the first three cases. $\square$

The following Proposition deduces from the decoration of a particular vertex v what the decoration of the vertex directly above v must look like.

Proposition 2.3.9. *Let x, y be elements of $\mathrm{rst}_G(n-1)$. If $[x, y] \notin \mathrm{rst}_G(n+1)$ then either $x \notin \mathrm{rst}_G(n)$ or $y \notin \mathrm{rst}_G(n)$.*

Proof. The elements x and y can also be seen as elements of G_{n-1}. The word

$$[x, y] = (h_1, \dots, h_{m_n})_n$$

is a commutator and hence by Proposition 2.3.8 we get that each h_j falls into one of the four cases above. Now assume that both x and y are of the form 1,2 or 4. Then $x, y \in B_n \times \cdots \times B_n$ and so

$$[h, k] \in B_n' \times \cdots \times B_n' = N_n' \times \cdots \times N_n' = \mathrm{rst}_G(n)' \leq \mathrm{rst}_G(n+1) \leq \mathrm{St}_G(n+2)$$

and hence h_j cannot be of the third type because this case does not stabilise level n. $\square$

Corollary 2.3.10. *For every $d_{i,i,j}$ in c_i that is of the second or third type we have that either*

$$d_{i,i-1,\lfloor j/l_i \rfloor} = a_{i-1}^q z \quad \text{or} \quad d_{i,i-1,\lfloor j/l_i \rfloor}^{d_{i-2,\lfloor j/(l_{i-1}l_i) \rfloor}} = a_{i-1}^q z$$

with $q \not\equiv 0 \mod l_i$, $z \in B_{i-1}$, hence at least one of the two was of type 3.

Figure 2.2 depicts where the elements $d_{i,i,j}$, $d_{i,i-1,\lfloor j/l_i \rfloor}$ and $d_{i,i-1,\lfloor j/l_i \rfloor}^{d_{i,i-2,\lfloor j/(l_{i-1}l_i) \rfloor}}$ act relative to each other.

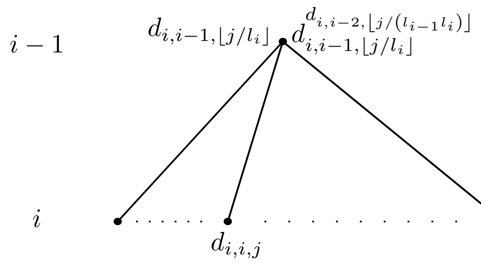

Figure 2.2: The commutator c_i in detail.

Proof. This is an application of Proposition 2.3.9 with $x = d_{i,i-1,\lfloor j/l_i \rfloor}$ and $y = d_{i,i-1,\lfloor j/l_i \rfloor}^{d_{i,i-2,\lfloor j/(l_{i-1}l_i) \rfloor}}$. Figure 2.2 depicts that in that case both x and y decorate the vertex immediately above $d_{i,i,j}$. $\square$

Example 2.3.11. We illustrate Proposition 2.3.9 with an example for which we assume $l_1 > 3$. We compute the commutator of

$$x = [b, b^a] = (1, [a_1, b_1], 1, \ldots, 1)_1 \tag{2.4}$$

and

$$y = b^3 = \left(b_1^3, a_1^3, 1, \ldots, 1\right)_1 \tag{2.5}$$

given by $\left[[b, b^a], b^3\right]$. Writing out the terms shows that on the first level this commutator is given as

$$\left[[b, b^a], b^3\right] = \left(1, b_1^{-1} a_1^{-1} b_1 a_1 a_1^{-3} a_1^{-1} b_1^{-1} a_1 b_1 a_1^3, 1, \ldots, 1\right)_1.$$

After evaluating the term on the second position we get on level 2

$$\left(1, \ldots, 1, b_2^{-1}, a_2^{-1} b_2, b_2, b_2^{-1} a_2, a_2^{-1}, 1, \ldots, 1\right)_2$$

where the first non-trivial term b_2^{-1} is at position $l_1 + 1$, which is the first subvertex of the second vertex of level 1. Figure 2.3 shows this.

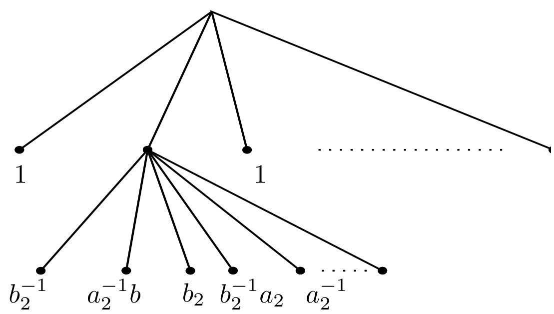

Figure 2.3: The commutator $\left[[b, b^a], b^3\right]$.

Analysing the example shows that we have some vertex decorations of type 2 or 3 on the second level. The second position of y in (2.5) is given by a_1^3 and decorates the vertex immediately above any non-trivial terms of the second level of $[x, y]$.

Corollary 2.3.10 allows us to deduce that if we have a vertex with non-trivial decoration, then there must be a decoration above it with a non-trivial rooted part, a power of some $a_i \in A_i$.

Theorem 2.3.12. *Assume that the defining sequence $\{l_i\}$ satisfies $l_i \geq 36^i$ where the l_i are pairwise coprime odd integers and we have $l_0 \geq 3$. Then G has no free subgroup of rank 2.*

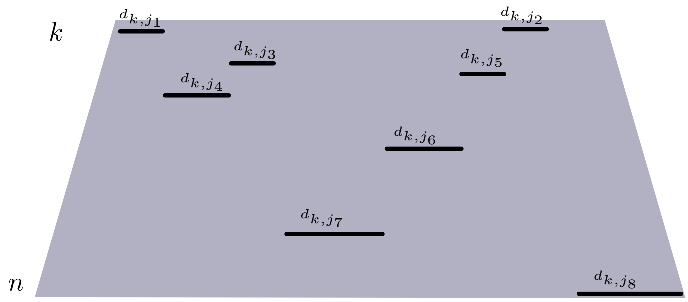

Figure 2.4: The elements $d_{k,i_t} = \left(d_{k,i_t,1}, \ldots, d_{k,i_t,y_j}\right)_{i_t}, y_j \in \mathbb{N}$, lie on different layers i_t.

Proof. Let $g_1, g_2 \in G$. From these we construct a non-trivial word $w_{g_1,g_2}(x,y)$ such that we have $w_{g_1,g_2}(g_1, g_2) = 1$. Find a level k such that

$$\lambda(c_k) \leq 6^k. \tag{2.6}$$

Such a k exists because we had that $\lambda(c_k) \leq 5^k \cdot (\lambda(g_1) + \lambda(g_2))$ by Lemma 2.3.6.

Write $c_k = (d_{k,k,1}, \ldots, d_{k,k,m_k})_k$. Some of the $d_{k,k,t}$ might be of the fourth, recursive, case. Proposition 2.3.8 states that this case only occurs down to some finite level n. Every $d_{k,k,t}$ that is of the recursive case will then satisfy that there exists a level i_t such that d_{k,i_t,j_t} is of case 1,2 or 3, decorating a vertex v_{j_t} of level i_t with $k \leq i_t \leq n$ and j_t with $\left\lfloor j_t / \left(\prod_{r=k}^{i_t-1} l_r\right)\right\rfloor = t$. In this situation we have that d_{k,i_t,j_t} decorates a subtree $T_{v_{j_t}}$, with root $v_{j_t} \in \Omega(i_t)$. We need to look at those vertices which have a non-recursive decoration coming from c_k. Those will as just argued lie on different levels i_t between n and k which Figure 2.4 illustrates.

We aim to form commutator words. Let $v_{j_t} \in \Omega(i_t)$ be the vertex of level i_t on which d_{k,i_t,j_t} acts. By Corollary 2.3.10 we now have either

$$c_{k-1}|_{v_{j_t}} = a_{i_t}^q z \quad \text{or} \quad c_{k-1}^{c_{k-2}}|_{v_{j_t}} = a_{i_t}^q z \tag{2.7}$$

with $q \not\equiv 0 \mod l_{i_t}$ and $z \in B_{i_t}$. We now shift the spines of $c_k|_{v_{j_t}}$ by conjugation such that their new position does not overlap with their previous one assuming l_{i_t} is large enough which we will justify in the second half of this proof. We are using the non-trivial power q of a_{i_t} from (2.7) to conjugate the spines of d_{k,i_t,j_t} to empty positions. This conjugated element of d_{k,i_t,j_t} will then commute with d_{k,i_t,j_t}. The non-trivially decorated subvertices of v_{j_t} can only have positions p with $-\lambda(c_k) < p < \lambda(c_k)$ because in order to decorate a subvertex of v_{j_t} at position p on level $i_t + 1$ we need to have a power $a_{i_t}^p$, which will add p to the word length of c_k. We have found 6^k

to be an upper bound for the word length of c_k in (2.6). With these considerations we conclude that either

$$d = \left[d_{k,i_t,j_t}, d_{k,i_t,j_t}^{d_{k,i_t-1,\lfloor j/l_{i_t} \rfloor}^{6^k}} \right] \quad \text{or} \quad e = \left[d_{k,i_t,j_t}, d_{k,i_t,j_t}^{\left(d_{k,i_t-1,\lfloor j/l_{i_t} \rfloor}^{d_{k,i_t-2,\lfloor j/(l_k l_{i_t-1}) \rfloor}}\right)^{6^k}} \right] \tag{2.8}$$

is such that $d|_{v_{j_t}} = 1$ or $e|_{v_{j_t}} = 1$ for all vertices $v_{j_t} \in \Omega(i_t)$. We have the case that $d|_{v_{j_t}} = 1$ if we had $c_{k-1}|_{v_{j_t}} = a_{i_t}^q z$ in equation (2.7) and we have $e|_{v_{j_t}} = 1$ if we had $c_{k-1}^{c_{k-2}}|_{v_{j_t}} = a_{i_t}^q z$.

We now describe how this will yield a word $w_{g_1,g_2}(x,y) \in F(x,y)$. Similarly to the definition of c_k recursively define commutators in $F(x,y)$ as

$$\gamma_0 = x, \quad \gamma_1 = [x,y], \quad \gamma_i = \left[\gamma_{i-1}, \gamma_{i-1}^{\gamma_{i-2}}\right]$$

with $\gamma_i \in F(x,y)^{(i)}$, the i-th derived group of $F(x,y)$. We begin our word w by $w_1 = \left[\gamma_k, \left(\gamma_k^{\gamma_{k-1}}\right)^{6^k}\right]$. This will give identity on the vertices v_{j_t} with $d|_{v_{j_t}} = 1$ with d as in (2.8). We will then proceed with $w_2 = \left[w_1, w_1^{\left(c_{k-1}^{c_{k-2}}\right)^{6^k}}\right]$ to get identity for the vertices in which only the case $e|_{v_{j_t}} = 1$ applied.

The two cases in (2.8) are the only ones possible for any vertex v_{j_t}. Hence we either have $w_1(g_1,g_2)|_{v_{j_t}} = 1$ after the first step or $w_2(g_1,g_2)|_{v_{j_t}} = 1$ after the second step. Assume $w_1(g_1,g_2)|_{v_{j_t}} \neq 1$ for some vertex v_{j_t}. Then this means that $d_{k,i_t-1,\lfloor j/l_{i_t} \rfloor} \in B_{i_t-1}$, hence there is no rooted action and conjugating by $d_{k,i_t-1,\lfloor j/l_{i_t} \rfloor}$ will not move any spines. We then get that $w_2(g_1,g_2)|_{v_{j_t}} = 1$ in the second step.

We now have to justify that l_i is big enough in each step. We are conjugating by the power 6^k in (2.8). This applies to an element $a_{i_t}^q$, where $|q| < \lambda(c_k) \leq 6^k$. We get that this moves spines by at most $6^{2k} - 6^k$. Hence we need at most 6^{2k} places to fit these at most 6^k spines in and so each l_i for $i \geq k$ must be such that $6^{2k} \leq l_i$. It is hence enough to require that the sequence $\{l_i\}_{i \in \mathbb{N}_0}$ is such that $l_i \geq 36^i$ and $l_0 \geq 3$.

The procedure described above will result in a non-trivial word $w_{g_1,g_2}(x,y)$ in the free group $F(x,y)$ on the two generators x and y. This word has the form of a nested commutator and is an element of $F(x,y)^{(k)}$, the k-th derived group of $F(x,y)$, where k depends on the word length of the two chosen elements g_1 and g_2. This now implies that G cannot contain a non-abelian free subgroup under the given assumptions on the defining sequence $\{l_i\}$. □

The proof of Theorem 2.3.12 at least allows slight improvements on the bound for l_i. However, it seems likely that the construction of a non-trivial word will require the sequence l_i to grow at least as fast as $l_i \geq C^i$ where $C > 0$.

Private communication with Nekrashevych suggests that the result from his paper [Nek10] could possibly be used to prove the absence of free subgroups for all defining sequences $\{l_i\}$.

We now illustrate Theorem 2.3.12 by computing an example.

Example 2.3.13. We choose two elements

$$g_1 = a^{-4}(ab)^4 \quad \text{and} \quad g_2 = [b, b^a]$$

and will construct a non-trivial word $w_{g_1,g_2}(x, y) \in F_2, w \neq 1$ such that $w_{g_1,g_2}(g_1, g_2) = 1$ in G. We chose g_1 and g_2 from the subgroup $B \leq_f G$ to simplify the calculation.

$$\begin{aligned} c_1 = [g_1, g_2] &= \left[a^{-4}(ab)^4, [(b_1, a_1, 1, \ldots, 1)_1, (1, b_1, a_1, 1, \ldots, 1)_1]\right] \\ &= \left[(b_1, b_1a_1, b_1a_1, b_1a_1, a_1, 1, \ldots, 1)_1, \left(b_1^{-1}, b_1a_1^{-1}, a_1, 1, \ldots, 1\right)_1\right] \\ &= \left(1, \left[b_1a_1, b_1a_1^{-1}\right], [b_1a_1, a_1], 1, \ldots, 1\right)_1 \\ &= \left((1, \ldots, 1), \left(a_2, b_2^{-1}, a_2^{-1}, 1, \ldots, 1, b_2\right), \left(1, b_2^{-1}, a_2^{-1}b_2, a_2, 1, \ldots, 1\right), 1, \ldots, 1\right)_2 \end{aligned}$$

where there are l_2 elements in each of the brackets, grouped by being subvertices of the same vertex. We observe that every vertex that c_1 decorates with a spinal part has a term b_1a_1 directly above it as the recursive form of Proposition 2.3.8 states. We assume at this step that l_2 already fulfills the hypothesis of the proof. We then conjugate c_1 by g_1^4 and get

$$\begin{aligned} c_1^{g_1^4} &= \left((1, \ldots, 1), \left(a_2, b_2^{-1}, a_2^{-1}, 1, \ldots, b_2\right)^{(b_1a_1)^4}, \left(1, b_2^{-1}, a_2^{-1}b_2, a_2, 1, \ldots, 1\right)^{(b_1a_1)^4}, 1, \ldots, 1\right)_2 \\ &= \left((1, \ldots, 1), \left(1, 1, 1, b_2, a_2^{-1}b_2^{-1}a_2b_2a_2, a_2^{-1}b_2^{-1}a_2, a_2, 1, \ldots, 1\right), \right. \\ &\qquad \left. \left(1, 1, 1, 1, 1, a_2^{-1}b_2^{-1}a_2, a_2^{-1}b_2, a_2, 1, \ldots, 1\right), 1, \ldots, 1\right)_2. \end{aligned}$$

We now see that

$$\left[c_1, c_1^{g_1^4}\right] = 1$$

because none of the non-trivial positions of c_1 and $c_1^{g_1^4}$ overlap anymore.

We look at the word that this process generated. The first step was the commutator $[g_1, g_2]$, hence $w_1(x, y) = [x, y] \in F(x, y)$ as a word in the free group. We then conjugated by g_1^4, hence $w_2(x, y) = w_1^{x^4}$. Finally we had $w_3(x, y) = [w_1, w_2]$. This gives the word

$$w_{g_1,g_2}(x, y) = \left[[x, y], [x, y]^{x^4}\right] = y^{-1}x^{-1}yx^{-3}y^{-1}x^{-1}yx^4y^{-1}xyx^{-5}y^{-1}xyx^4.$$

2.4 Open Questions

Question 2.4.1. Is every finitely presented group that surjects onto G large?

We use the relators as described in Chapter 4 and take only a finite subset of them. For small such subsets this can be verified directly. However, the situation becomes unclear for bigger subsets. It might be possible to use the Reidemeister-Schreier algorithm to rewrite the relations that hold in finite index subgroups and then apply known results about largeness and deficiency.

Question 2.4.2. Can we embed G in a finitely presented HNN-extension of G?

This has been done by Grigorchuk in [Gri98] to produce a finitely presented amenable but not elementary amenable group. This could eventually lead to a finitely presented group with infinite virtual first Betti number which is not large.

Question 2.4.3. Can G contain a free subgroup if the defining sequence grows very slowly?

A result by Brieussel [Bri09] states that groups similar to the ones described here are amenable for rather slowly growing sequences.

Chapter 3

Morally Self-Similar Groups

In this Chapter we study a different group on the same trees as in Chapter 2. Such a construction has already been studied in [LS03] to provide examples of subgroup growth. It was shown in [SW03] by Sidki and Wilson that there exist branch groups containing the free group. We will see here, that with the construction in this Chapter, it is also easily possible to obtain branch groups containing the free group.

The first Section describes the construction of G. It will be shown that this construction allows us to control the structure of G by a chosen group P which can be chosen with certain conditions.

In the second Section we compute the abelianisation of G, which will turn out to be completely dependent on the group P that we chose to construct G.

We will use this construction in Chapter 4 to deduce that the group presented in Chapter 2 is not finitely presented. We further build upon it in Chapter 5 to construct a branch group Γ with finitely generated subgroups of arbitrary Hausdorff dimension $\alpha \in [0, 1]$ within Γ.

3.1 The Construction

In this Section we will describe the construction of G and deduce some basic properties. We first specify a generating set of elements but we will see later that we can vary this quite a bit and will still obtain the same group.

3.1.1 Definition

Assume we have a countable infinite family $\{A_n\}$ of finite, perfect d-generator groups

$$A_n = \langle \alpha_{1,n}, \ldots, \alpha_{d,n} \rangle \, .$$

Let A_n act non-regularly and transitively on sets L_n and denote by l_n the number of elements in L_n. For convenience we choose a fixed order of the elements in L_n as

$$L_n = \{s_{1,n}, \dots, s_{l_n,n}\}. \tag{3.1}$$

Remark 3.1.1. *The requirement that the action of A_n is non-regular and transitive is in particular fulfilled when each A_n acts 2-transitively on L_n. An example, that we will refer to later and which is also used in Chapter 5, is given for $d = 2$ by the alternating groups* $\mathrm{Alt}(l_n)$ *of chosen orders l_n with $l_n = |L_n|$.*

Further, let P be a d-generator group with $P = \langle t_1, \dots, t_d \rangle$ and suppose that for each $n \geq 0$ there exists an epimorphism

$$\phi_n : P \longrightarrow A_n, \quad n \geq 1$$

where

$$\bigcap_{n=i}^{\infty} \ker(\phi_n) = 1 \quad \text{for all } i \in \mathbb{N} \tag{3.2}$$

and we additionally require that

$$\phi_n(t_j) = \alpha_{j,n}. \tag{3.3}$$

Remark 3.1.2. *If P is torsion free it is enough to assume $\bigcap_{n=1}^{\infty} \ker(\phi_n) = 1$.*

Similarly to Chapter 2, we define for such a sequence of groups $\{A_n\}$ an automorphism group acting on the rooted tree with defining sequence $\{l_n\}$. We label the vertices on the first level by the elements in L_0, hence by $s_{1,0}, \dots, s_{l_0,0}$. We choose the natural labelling via the elements $s_{1,n}, \dots, s_{l_n,n}$ for a fixed order of the subvertices of a vertex v with $v \in \Omega(n-1)$ on each level n. The automorphism group G will be defined by a set of d rooted and d spinal automorphisms which are defined as follows:

- To define the group we take α_i to be the rooted automorphism acting on the first level as $\alpha_{i,0}$ acts on the set L_0.
- We recursively define spinal automorphisms β_i for $i = 1, \dots, d$ as

  $$\beta_{i,n} = (\beta_{i,n+1}, \alpha_{i,n+1}, 1, \dots, 1)_{n+1} \tag{3.4}$$

  where the position of $\beta_{i,n+1}$ is labelled by $s_{1,n+1}$ and the position of $\alpha_{i,n+1}$ by $s_{2,n+1}$ from the set L_{n+1}. Figures 3.1 depicts this.

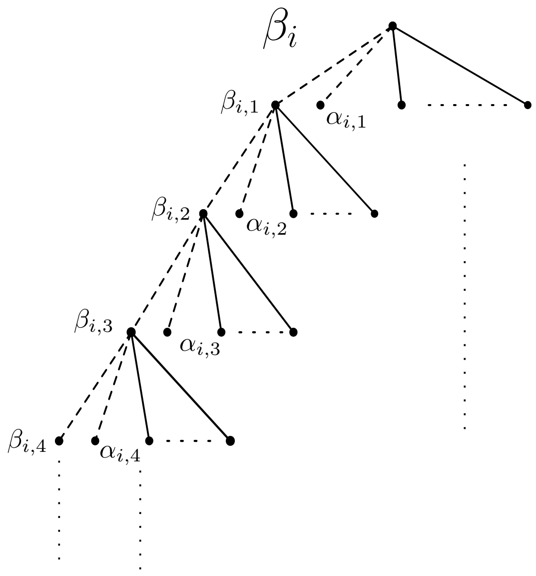

Figure 3.1: Action of β_i.

With the above notation, we define

$$G = \langle \alpha_1, \ldots, \alpha_d, \beta_1, \ldots, \beta_d \rangle , \quad P_0 = \langle \beta_1, \ldots, \beta_d \rangle , \quad A = \langle \alpha_1, \ldots, \alpha_d \rangle . \tag{3.5}$$

The assumption in (3.3) now gives that P_0 is the image of P under the map

$$\pi : P \longrightarrow \prod_{n=1}^{\infty} A_n$$

which is defined via

$$\pi(t_i) = (\alpha_{i,n})_{n\in\mathbb{N}} .$$

In fact, $P = \langle t_1, \ldots, t_d \rangle$ embeds into G because we assumed that $\bigcap_{n=i}^{\infty} \ker(P \to A_n) = 1$ in (3.2). The embedding is given by the map $t_i \mapsto \alpha_i$.

Similarly to G and P_0 we define for all $n \in \mathbb{N}_0$

$$G_n = \langle \alpha_{1,n}, \ldots, \alpha_{d,n}, \beta_{1,n}, \ldots, \beta_{d,n} \rangle \quad \text{and} \quad P_n = \langle \beta_{1,n}, \ldots, \beta_{d,n} \rangle .$$

Those definitions coincide with G and P_0 for $n = 0$.

The following Lemma is purely combinatorial.

Lemma 3.1.3. *Let A be a group acting non-regularly and transitively on a set L of order n. Then there exists an element $x \in A$ and a labelling $\{s_1, \ldots, s_n\}$ of the elements in L such that $s_1x = s_1$ and $s_2x = s_i$ with $i \in \{3, \ldots, n\}$.*

Proof. We show first that we can order the elements in L such that the Lemma holds for a given non-trivial $x \in A$ and then relabel. The group A acts non-regularly, hence there exists an element $x \in A$ with $s_j \in \mathrm{Fix}(x)$ for some $j \in \{1, \dots, n\}$. Assume without loss of generality that $j = 1$. Because $x \neq 1$, there also exists $s_k \in \mathrm{supp}(x)$ for some $k \in \{2, \dots, n\}$, hence $s_k x = s_d$ for some $d \in \{3, 4, \dots, n\}$ where $d \neq 1$, because x leaves s_1 fixed. Hence we deduce that $s_k x \neq s_1$ and of course also $s_k x \neq s_k$ by the choice of k. We can now relabel the elements in L such that we can choose $k = 2$ which concludes the proof. □

The above Lemma 3.1.3 allows us to relabel the vertices of the tree in such a way, that we can find elements $x_n \in A_n$ with $s_{2,n} x_n = s_{2,n}$ and $s_{1,n} x_n \neq s_{1,n}$. We will for the rest of this Chapter assume this labelling. For the following Proposition 3.1.4 we remind the reader of the notation $m_n = \prod_{i=0}^{n-1} l_i$ which has been defined in (1.1).

Proposition 3.1.4. *Assume $l_n \geq 3$ for each $n \in \mathbb{N}$. Then the direct product $G_n \times \cdots \times G_n$ is contained in G where we have m_n factors in the direct product.*

Proof. We assumed that all groups A_n act non-regularly and transitively, in particular A_0 acts non-regularly and transitively on the set of vertices of the first level which are labelled by

$$\{s_{1,0}, \dots, s_{l_0,0}\}$$

as in (3.1). By Lemma 3.1.3 there exists an element $x \in A_0$ such that $s_{2,0} x = s_{2,0}$ and $s_{1,0} x = s_{j,0}$ for some $j \in \{3, \dots, l_0\}$. This gives us that the following elements are in G:

$$\beta_i = (\beta_{i,1}, \alpha_{i,1}, 1, \dots, 1)_1 \quad \text{and} \quad \tau_i = (1, \alpha_{i,1}, 1, \dots, 1, \beta_{i,1}, 1, \dots, 1)_1$$

where $\beta_{i,1}$ in τ_i is at position $s_{j,0}$ for some $j \in \{3, \dots, l_0\}$ and $i = 1, \dots, d$. The group A_1 is perfect, hence for every $y \in A_1$ there exist $r_y \in \mathbb{N}$ and words $w_{y,1,k}(x_1, \dots, x_d)$ and $w_{y,2,k}(x_1, \dots, x_d)$ for $k = 1, \dots, r_y$ such that

$$y = \prod_{k=1}^{r_y} \left[w_{y,1,k}(\alpha_{1,1}, \dots, \alpha_{d,1}), w_{y,2,k}(\alpha_{1,1}, \dots, \alpha_{d,1}) \right].$$

In particular there exist an $r_{\alpha_{i,1}} \in \mathbb{N}$ and words $w_{\alpha_{i,1},1,k}, w_{\alpha_{i,1},2,k}$ for each $i = 1, \dots, d$ and $k = 1, \dots, r_{\alpha_{i,1}}$ with

$$\alpha_{i,1} = \prod_{k=1}^{r_{\alpha_{i,1}}} \left[w_{\alpha_{i,1},1,k}(\alpha_{1,1}, \dots, \alpha_{d,1}), w_{\alpha_{i,1},2,k}(\alpha_{1,1}, \dots, \alpha_{d,1}) \right].$$

We can now use those words to get

$$\gamma_i = \prod_{k=1}^{r_{\alpha_{i,1}}} \left[w_{\alpha_{i,1},1,k}\left(\beta_1,\ldots,\beta_d\right), w_{\alpha_{i,1},2,k}\left(\tau_1,\ldots,\tau_d\right)\right]$$

$$= \left(1, \prod_{k=1}^{r_{\alpha_{i,1}}} \left[w_{\alpha_{i,1},1,k}\left(\alpha_{1,1},\ldots,\alpha_{d,1}\right), w_{\alpha_{i,1},2,k}\left(\alpha_{1,1},\ldots,\alpha_{d,1}\right)\right], 1, \ldots, 1\right)_1 = \left(1, \alpha_{i,1}, 1, \ldots, 1\right)_1.$$

This can be done for each generator $\alpha_{i,1}$ of A_1 and hence $1 \times A_1 \times 1 \times \cdots \times 1 \leq G$ and by conjugation we assert that

$$A_1 \times \cdots \times A_1 \leq G.$$

It follows from this that

$$\gamma_i^{-1}\beta_i = \left(\beta_{i,1}, 1, \ldots, 1\right)_1 \in G$$

and so $G_1 \times \cdots \times G_1 \leq G$. Repeatedly applying the same argument to each G_1 in the direct product $G_1 \times \cdots \times G_1$ with l_0 factors then gives

$$G_n \times \cdots \times G_n \leq G$$

with m_n factors in the direct product. □

3.1.2 The Action of G

In this Subsection we concentrate on how the group constructed above acts on the rooted tree T with defining sequence $\{l_i\}$. It will be shown that G has a morally self-similar structure which makes it easy to deduce properties such as: G is a branch group.

Lemma 3.1.5. *The group G acts as the iterated wreath product $A_{n-1} \wr \cdots \wr A_0$ on the set $\Omega(n)$ of m_n vertices of every level n.*

Proof. We argue by induction. The action on level 1 is given by A_0 and there is nothing to prove. Proposition 3.1.4 shows that $A_1 \times \cdots \times A_1 \leq G$ and hence $A_n \times \cdots \times A_n \leq G$ for all levels $n \geq 0$ with m_n factors in the direct product. The action of G on level $n-1$ is by induction hypothesis given by $A_{n-2} \wr \cdots \wr A_0$. Denote by H the restriction $G|_{T_v}$ for some vertex $v \in \Omega(n-1)$. Then $A_{n-1} \leq H$ by the above and together with the transitivity we deduce that G acts as $A_{n-1} \wr \cdots \wr A_0$ on the m_n vertices of level n and with Lemma A.2.3 this gives the result. □

The above Lemma 3.1.5 shows that G is morally self-similar. This implies that the action of the group on the rooted tree T is rather easy to describe and understand.

Lemma 3.1.6. *The n-th level stabilizer of G is* $\mathrm{St}_G(n) = G_n \times \cdots \times G_n$ *and hence the rigid stabilizer is also given as* $\mathrm{rst}_G(n) = G_n \times \cdots \times G_n$*, in particular* $\mathrm{rst}_G(n) = \mathrm{St}_G(n)$.

Proof. The subgroup $G_n \times \cdots \times G_n$ is contained in G and hence $G_n \times \cdots \times G_n \leq \mathrm{St}_G(n)$ respectively $G_n \times \cdots \times G_n \leq \mathrm{rst}_G(n)$. The other inclusion is trivial. □

With this Lemma 3.1.6 we can easily deduce that G is a branch group.

Theorem 3.1.7. *The group G as defined above is a branch group.*

Proof. Lemma 3.1.6 asserts that $\mathrm{rst}_G(n) = \mathrm{St}_G(n)$. The subgroup $\mathrm{St}_G(n)$ is of finite index in G for all $n \geq 0$ by Lemma 1.2.1 from Chapter 1. Hence $\mathrm{rst}_G(n)$ has finite index in G for all $n \geq 0$ and we deduce that G is a branch group. □

The group G was constructed with additional requirements upon the existence of a d-generator group P with certain properties. The following Lemma 3.1.8 shows that some properties from P transfer to G and its subgroups.

Lemma 3.1.8 ([LS03, Theorem 13.4])**.** *If P is perfect, then G is perfect and so are all P_n, G_n and* $\mathrm{rst}_G(n)$.

Proof. Each one of the generators β_i is such that it acts on different subtrees as $\alpha_{i,n}$ on each level n. This can be seen directly from the definition of β_i in (3.4) and Figure 3.1 also depicts this. It follows that any relation satisfied by a generator t_i of P is satisfied by each of the tuples $\beta_{i,n}$ and hence by β_i. So the map $t_i \mapsto \beta_i$ defines an epimorphism from P onto $P_0 = \langle \beta_1, \ldots, \beta_d \rangle$. Thus G is generated by two images of the perfect group P and hence is perfect as claimed. By moral self-similarity this also yields that each G_n is perfect for $n \geq 0$ and hence so are all $\mathrm{rst}_G(n)$. □

Lemma 3.1.9. *If P is perfect, then G has the congruence subgroup property.*

Proof. By Theorem 1.3.1 there exists for every normal subgroup N of G an integer n such that $\mathrm{rst}_G(n)' \leq N$. By Lemma 3.1.8 the subgroup $\mathrm{rst}_G(n)$ is perfect for each n, hence $\mathrm{rst}_G(n) \leq N$. This yields $\mathrm{St}_G(n) \leq N$ which concludes the proof. □

We also see that with this construction it is rather easy to obtain examples of branch groups containing free subgroups.

Theorem 3.1.10. *If P satisfies the criteria for this construction and is such that it contains a free subgroup, then G is a branch group containing free subgroups.*

Example 3.1.11. The above theorem gives us an explicit example. A result by Katz and Magnus [KM68] states that free groups are residually alternating. Hence we can choose $P = F_2$, the free group on 2-generators. This group is torsion free and so qualifies for the construction described in this Chapter. We obtain a branch group which contains F_2 and hence also F_d for $d \geq 1$.

3.2 Abelianisation

In this Section we tackle the more complex question of what the abelianisation of G looks like. It will be shown that this depends purely on the choice of P.

Lemma 3.2.1. *For each $g \in G$ there exists an integer $n_g \in \mathbb{N}$ such that if we write $g = \left(h_1, \ldots, h_{m_{n_g}}\right)_{n_g} \cdot w$ where $w \in A_{n_g-1} \wr \cdots \wr A_0$, then for each $i = 1, \ldots, m_{n_g}$ we have*

$$h_i \in A_{n_g} \text{ or } h_i \in P_{n_g}.$$

This says that each element $g \in G$ splits into spinal parts below level n_g. Figure 3.2 depicts this.

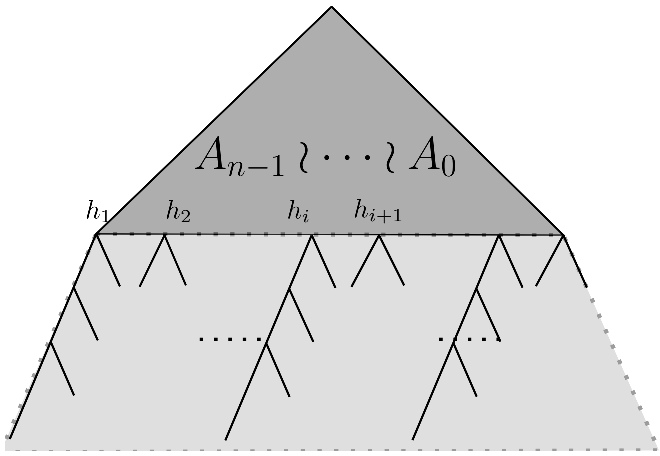

Figure 3.2: Spinal form of elements in G.

Proof. We argue by induction on the length of g. If $|g| = 1$ then either $g = \alpha_i$ or $g = \beta_i$ for some i with $1 \leq i \leq d$. Let us now assume the statement holds for words g up to length n. If $v_i \in \Omega(1)$ is a vertex of level 1 we have

$$|g|_{v_i}| \leq n - 1 \quad \text{or} \quad g = \beta_i^t \quad \text{or} \quad g = \alpha_i^t$$

for some integer t because $a|_u = 1$ for any $a \in A_0$ and any $u \in \Omega(1)$. By induction there exists k_i with $g|_{v_i} \in G_{k_i} \cup P_{k_i}$. Now take $n_g = \max_{i=1}^{l_0} k_i$. □

Proposition 3.2.2. *Let $\bar{\beta}_{i,n}$ be the image of $\beta_{i,n} \in P_n$ in P_n^{ab} and respectively $\bar{t}_i$ the image of $t_i \in P$ in P^{ab}. Then the map*

$$\psi_n = \begin{cases} P_n^{ab} & \longrightarrow P^{ab} \\ \bar{\beta}_{i,n} & \mapsto \bar{t}_i \end{cases}$$

is an isomorphism.

Proof. It follows from equation (3.2) that $P \simeq P_n$ and hence $P^{ab} \simeq P_n^{ab}$ for all $n \geq 0$. □

We construct maps

$$\varphi_n(g) = \begin{cases} G & \longrightarrow P_n^{ab} \\ g & \mapsto \left(\prod_{h_i \in P_n} h_i\right) P_n' \in P_n^{ab} \end{cases}$$

where $g = (h_1, \ldots, h_{m_n})_n \cdot w, w \in A_{n-1} \wr \cdots \wr A_0$, and the product is over those h_i which are in P_n.

Lemma 3.2.3. *Let $g \in G$ and assume $k \geq n_g$ for some $k \in \mathbb{N}$. Then*

$$\psi_k\left(\varphi_k(g)\right) = \psi_{n_g}\left(\varphi_{n_g}(g)\right).$$

In other words, $\varphi_k = \varphi_{n_g}$ for all $k \geq n_g$.

Proof. Let $g \in G$ and choose a vertex $v \in \Omega(n_g)$ such that if $g = \left(h_1, \ldots, h_{m_{n_g}}\right)_{n_g} \cdot w$ as in Lemma 3.2.1, then h_s is the decoration at position v with $h_s \in P_n$. If no such v exists, then $\psi_k\left(\varphi_k(g)\right) = \psi_{n_g}\left(\varphi_{n_g}(g)\right) = 1$ and we are done. Let $g = (f_1, \ldots, f_{m_k})_k \cdot w_1$ be the decomposition of g on level k. We observe that the number of factors in each of the products $\varphi_k(g) = \left(\prod h_i\right) P_k'$ and $\varphi_{n_g}(g) = \left(\prod h_j\right) P_{n_g}'$ is the same because of the assumption that $k \geq n_g$. Proposition 3.2.2 yields that $P_k^{ab} \simeq P_{n_g}^{ab}$ via the natural identification of generators $\bar{\beta}_{i,k} \mapsto \bar{\beta}_{i,n_g}$. This yields that we can identify $\varphi_k(g)$ with $\varphi_{n_g}(g) \in P_n^{ab}$ and both map to the same element in P^{ab}. □

From Lemma 3.2.3 we see that the following map

$$\varphi_\infty(g) = \begin{cases} G & \longrightarrow P^{ab} \\ g & \mapsto \psi_{n_g}\left(\varphi_{n_g}(g)\right) \end{cases}$$

is well defined for n_g from Lemma 3.2.1.

Lemma 3.2.4. *The map φ_∞ is a homomorphism.*

Proof. The image of gh under φ_∞ is given as $\psi_{n_{gh}}\left(\varphi_{n_{gh}}(gh)\right)$. As in the proof of Lemma 3.2.3 we can directly identify elements of P_{n_g} and P_{n_h} with elements of $P_{n_{gh}}$. Hence assume $n \geq \max\{n_{gh}, n_g, n_h\}$, then $\varphi_n(gh) = \varphi_n(g)\varphi_n(h)$. The map ψ_n is a homomorphism, so

$$\varphi_\infty(gh) = \psi_n\left(\varphi_n(gh)\right) = \psi_n\left(\varphi_n(g)\right) \cdot \psi_n\left(\varphi_n(h)\right) = \varphi_\infty(g) \cdot \varphi_\infty(h).$$

□

Proposition 3.2.5. *The restricted kernel of φ_∞ is given by* $\ker\left(\varphi_\infty|_{P_0}\right) = P_0'$.

Proof. The image of $\varphi_\infty|_{P_0}$ is abelian and hence $K = \ker(\varphi_\infty|_{P_0})$ contains P_0'. On the other hand, let $g \in G$ be such that $\varphi_\infty(g) = 1$. Then $\varphi_{n_g}(g) = 1$, which means the product $\prod_{h_i \in P_{n_g}} h_i$ is an element of P_{n_g}'. We have restricted the map to P_0. This means that the product only contains one factor h_0 and hence $h_0 \in P_{n_g}'$. We can directly identify elements of P_{n_g} with elements of P_0, hence h_0 with g, which now implies that $g \in P_0'$. □

Lemma 3.2.6. *The kernel of φ_∞ is given by* $\ker(\varphi_\infty) = G'$.

Proof. Denote $K = \ker(\varphi_\infty)$. The image of φ_∞ is abelian and hence $G' \leq K$. For the other inclusion we write $G = P_0 \cdot A_0$ and because A_0 is perfect we get $G = P_0 \cdot G'$. We apply the modular law (A.3.2) to deduce

$$(P_0 \cdot G') \cap K = (P_0 \cap K) \cdot G'.$$

Now we have $(P_0 \cdot G') \cap K = G \cap K = K$ and $P_0' \cdot G' = G'$ and hence $K = G'$. □

Theorem 3.2.7. *The abelianisation of G is given by $G^{ab} = P^{ab}$.*

Proof. This is a straight forward consequence of the previous Lemma 3.2.6 which yields that the map $G^{ab} \longrightarrow P^{ab}$ is a bijection. □

Chapter 4

Presentation and Growth

This Chapter is written in three parts. In the first Section 4.1 we will show that the group G from Chapter 2 cannot be finitely presented. We discuss in Section 4.2 explicit sets of relators that hold in G. In the last part, in Section 4.3, it will be shown that G has exponential word growth under certain assumptions.

4.1 Presentation

A group Γ is very often described as a quotient group of a free group: $\Gamma = F/N$. If F is free with basis $X = \{x_1, \ldots, x_n\}$ and N is the normal closure in F of a set $R \subset F$, we say that the pair (X, R) is a *presentation* for Γ and denote this by $\Gamma = \langle X|R\rangle$. If $\bar{X}$ is the set of images $\bar{x}$ in Γ of the elements $x \in X$, then $\bar{X}$ generates Γ. If $r = r(x_1, \ldots, x_n)$ is in R, then the equation $r(\bar{x_1}, \ldots, \bar{x_n}) = 1$ holds in G. The elements $r \in R$ are called *relators.* A presentation (X, R) is *finitely generated* if X is finite, and is *finitely related* if R is finite. If both, X and R are finite, then the group $\Gamma = \langle X|R\rangle$ is *finitely presented.* It is easy to see that any finitely presented group can also be written as an infinitely presented group. However, if a finitely generated group Γ cannot be presented by (X, R) for any finite set R, then Γ is *infinitely presented.* A set S is *recursively enumerable* if there exists an algorithm such that the set of input numbers for which the algorithm halts is exactly S. If R is infinite, but is recursively enumerable, then we say that the presentation (X, R) is *recursive.*

For all examples of branch groups where it is known, it can be proved that they are infinitely presented. In fact, there are general statements in [BGv03] about the infinite presentability of groups acting on rooted trees. In a recent paper, Benli [Ben12] has shown that also the profinite completion of the Grigorchuk group is not finitely presented. However, the case of a non-regular tree does not seem to have been considered. We will show that a Theorem in [BGv03] also applies in the case of the groups of Chapter 2 to show that those groups are not finitely presented.

4.1.1 G is Infinitely Presented

Theorem 4.1.1. *Let G be the group constructed in Chapter 2 from a sequence $\{l_i\}$ of distinct, odd, coprime integers such that $l_i \geq 5$. Then G is not finitely presented.*

We show that it is possible to apply Šuniḱ's Theorem (4.1.2, [BGv03, Theorem 4.4]) as already stated in Chapter 1. We refer to the paragraph before 1.3.3 for the terminology used.

Theorem 4.1.2 ([BGv03, 4.4])**.** *Let $\mathcal{C}$ be a class of groups that is closed under homomorphic images and subgroups of finite index and $\omega = (A_\omega, S, \bar{\omega})$ be a sequence that defines a spinal group in $\mathcal{C}$. Further, assume that, for every r, there exists a triple $\eta^{(r)}$ of the form $\eta^{(r)} = (A_{\sigma^r\omega}, S, \bar{\eta})$, where $\bar{\eta}^{(r)}$ is a doubly indexed family of homomorphisms*

$$\eta_{ij} : S \longrightarrow \mathrm{Sym}\,(X_j)\,, \quad i \in \{r+1, r+2, \dots\}\,, \quad j \in \{2, \dots, l_i\}$$

defining a group of tree automorphisms (not necessarily spinal) $G_{\eta^{(r)}}$ that acts on the shifted tree $T^{\sigma^r(Y)}$ and is not in $\mathcal{C}$. Then the spinal group G_ω is not finitely presented.

Now we will apply this to the groups in Chapter 2.

As class $\mathcal{C}$ we take all groups which have the property that every proper quotient is soluble.

Lemma 4.1.3. *The class $\mathcal{C}$ is closed under homomorphic images and subgroups of finite index.*

Proof. Let H be a group in $\mathcal{C}$. A homomorphic image of H is isomorphic to H/N for some $N \lhd H$. A finite quotient $(H/N)/K$ of H/N is isomorphic to a finite quotient of H, hence $(H/N)/K$ is soluble and so $H/N \in \mathcal{C}$. Finally, let K be a subgroup of finite index in H and $N \lhd_f K$. Then $N_0 = \bigcup_{g \in H} N^g$ is an intersection of finitely many subgroups of finite index N^g and $N_0 \lhd H$, hence H/N_0 is soluble. The quotient K/N is a section of H/N_0 and hence soluble again. □

Proof of Theorem 4.1.1. We take as S the infinite cyclic group $S = \langle h \rangle$. Denote for each $i \geq 0$ by σ_{l_i} and τ_{l_i} the cyclic permutations

$$\sigma_{l_i} = (123 \dots l_i)\,, \quad \tau_{l_i} = (123).$$

The element τ_{l_i} does not depend on l_i and we will write for convenience $\tau = \tau_{l_i}$ for all i and mean the respective $\tau_{l_i} \in \mathrm{Sym}\,(l_i)$.

We now show the existence of a triple $\eta^{(r)} = (A_{\sigma^r\omega}, S, \bar{\eta})$ for every r. We choose

$$\eta_{ij}(h) = \begin{cases} \sigma_{l_i} & \text{if } j = 2, \\ \tau & \text{if } j = 3, \\ 1 & \text{else.} \end{cases}$$

for all $i \geq 1$. Figure 4.1 shows the portrait of the spinal automorphism defined by η_{ij}.

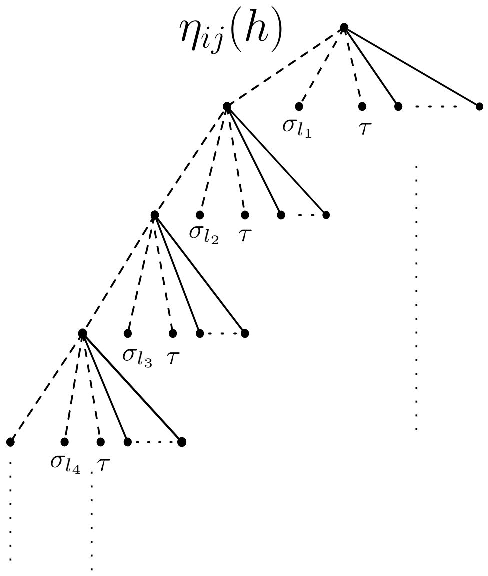

Figure 4.1: The portrait of $\eta_{ij}(h)$.

We further assumed that each l_i is odd, which now yields that $\sigma_{l_i} \in \mathrm{Alt}\,(l_i)$ for all $i \geq 0$. We chose each $l_i \geq 5$. This now implies that the two cyclic permutations $(12 \ldots l_i), (123) \in \mathrm{Sym}\,(l_i)$ have coprime orders 3 and l_i. If z is a spinal automorphism defined by the family $\{\eta_{ij}\}$, then z is a power of h and has infinite order. The shifted group $A_{\sigma^i\omega}$ is the abelian group A_i of order l_i, which acts transitively on the l_i vertices below the new root v_i, a vertex of level i. We get that the spinal group $(A_{\sigma^r\omega}, S, \eta_{ij})$ acts as $\mathrm{Alt}\,(l_i) \wr A_{l_{i-1}}$ on the vertices of the second level of T_{v_i}. It therefore has an insoluble finite quotient and hence is not in the class $\mathcal{C}$, which satisfies the hypothesis of Theorem 4.1.2 to deduce that G is not finitely presented. $\square$

4.2 Relators in G

In this Subsection we will present a set of recursive relators for the group G as defined in Chapter 2. We remind the reader of the notation of Chapter 2, that the finite abelian groups A_i are generated by $\{a_{1,i}, \ldots, a_{d,i}\}$, which have orders $q_{j,i} = 1$. The group B was generated by the spinal automorphisms $\{b_1, \ldots, b_d\}$. We first discuss a common relator in G_A and G_C and then an additional case for G_C.

It has been shown in [BGDLH12] that every finitely presented group mapping onto the Grigorchuk group is large. Here we give relations that hold in G which emerged from a similar investigation.

We will now explore complicated commutator words in G. We remember that we chose an order for the l_n elements of A_n and denoted them by

$$A_n = \{g_{1,n}, \ldots, g_{l_n,n}\}$$

where $g_{1,n} = 1$. Fix for this Section an element $u_n \in A_n$ with

$$g_{1,n} u_n = g_{2,n}.$$

In words, u_n moves the first vertex of level n to the second position.

To simplify some of the statements we will set the following notation: $a^{-t} = t^{-1} a^{-1} t$.

Proposition 4.2.1. *Let u_0 be as above. Then the relation $\left[b_i, \left(b_j^{-u_0}\right)^{q_{j,0}}\right]$ holds in G for all $i, j = 1, \ldots, d$.*

Proof. By the assumption on u_0 the element $\left(b_j^{-u_0}\right)^{q_{j,0}}$ is given as

$$\left(b_j^{-u_0}\right)^{q_{j,0}} = \left(1, \ldots, 1, b_{j,1}^{q_{j,0}}, 1, \ldots, 1\right)_1$$

where $b_{j,1}^{q_{j,0}}$ is on a position different from $g_{1,1}$ and $g_{2,1}$. On the other hand, b_i is given as

$$b_i = (b_{i,1}, a_{i,1}, 1, \ldots, 1)_1 .$$

Those now commute because the only non-trivial positions are distinct. □

We can now generalise this to all levels of the tree. Let p_n be the word such that $u_n^{-1} = p_n(a_{1,n}, \ldots, a_{d,n})$. We see that $b_i^{-u_0}$ acts rooted as $a_{i,1}$ on the first vertex $v_{1,1}$ of level 1. Repeating this process yields that $p_1\left(b_1^{-u_0}, \ldots, b_d^{-u_0}\right)$ acts as u_1^{-1} on $v_{1,1}$. We recursively build conjugators

$$\mu_0 = u_0^{-1}, \;\; \mu_1 = p_1(b_1, \ldots, b_d)^{\mu_0}, \;\; \mu_n = p_n(b_1, \ldots, b_d)^{\mu_{n-1}} .$$

The element μ_n now acts as u_n^{-1} on the first vertex $v_{1,n}$ of each level n as shown in Figure 4.2.

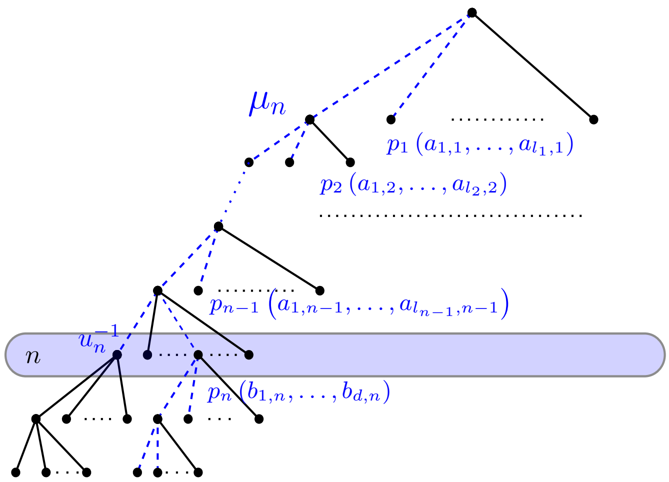

Figure 4.2: Portrait of μ_n.

We can formulate some more commutator relations that hold in G.

Theorem 4.2.2. *The set of relations*

$$\left\{\left[b_{j,0},\left(b_{k,0}^{\mu_{n-1}}\right)^{q_{k,n}}\right]=1 \quad \text{where} \quad j,k=1,\ldots,d \quad \text{and} \quad n\in\mathbb{N}\right\}$$

with μ_i defined as above, holds in G.

Figure 4.3 depicts how $\left(b_{k,0}^{\mu_{n-1}}\right)^{q_{k,n}}$ acts on the tree.

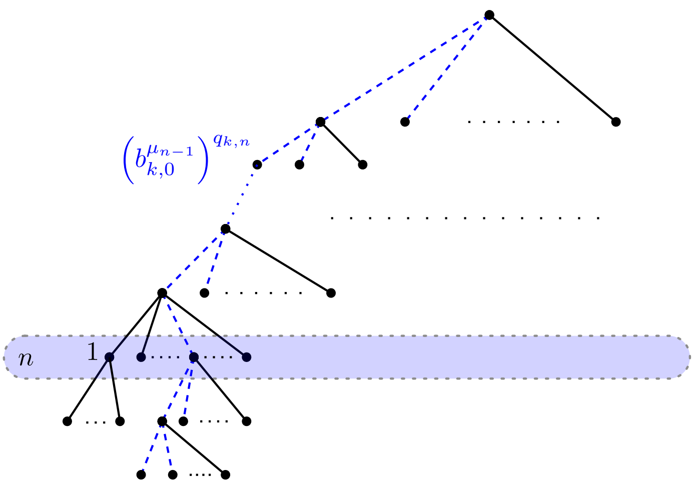

Figure 4.3: Portrait of $\left(b_{k,0}^{\mu_{n-1}}\right)^{q_{k,n}}$.

Proof. The elements $b_{j,0}$ and $\left(b_{k,0}^{\mu_{n-1}}\right)^{q_{k,n}}$ can be written as

$$b_{j,0} = (b_{j,n}, a_{j,n}, 1, \dots, 1)_n \cdot c_1, \tag{4.1}$$

$$\left(b_{k,0}^{\mu_{n-1}}\right)^{q_{k,n}} = \left(a_{k,n}^{q_{k,n}}, 1, \dots, 1, b_{k,n}^{q_{k,n}}, 1, \dots, 1\right)_n \cdot c_2 = \left(1, \dots, 1, b_{k,n}^{q_{k,n}}, 1, \dots, 1\right)_n \cdot c_2, \tag{4.2}$$

where $c_i \in A_{n-1} \wr \cdots \wr A_0$ and $b_{k,n}^{q_{k,n}}$ in (4.2) is at a position $g \in A_n$ with $g \neq g_{1,n}, g \neq g_{2,n}$. The spinal automorphism μ_n is given on each level i by a rooted automorphism $p_i\left(a_{1,i}, \dots, a_{l_i,i}\right)$ acting on the second vertex $v_{2,i}$ for all levels $i < n$, as depicted in Figure 4.2. The element $b_{j,0}$ in (4.1) decorates exactly the same vertices $v_{2,i}$ with the rooted automorphisms $a_{k,i}$ for $i \leq n$. Both, $p_i\left(a_{1,i}, \dots, a_{l_i,i}\right)$ and $a_{j,i}$, are elements of the abelian group A_i, hence they commute. We further observe that $p_i\left(a_{1,i}, \dots, a_{l_i,i}\right)$ and $p_k\left(a_{1,k}, \dots, a_{l_k,k}\right)$ act on disjoint subtrees for $i \neq k$. Let π_n be the natural homomorphism $G \longrightarrow G/\operatorname{St}_G(n)$. The above implies that the images $\pi_n\left(\mu_n\right)$ and $\pi_n\left(b_{k,0}\right)$ commute. The position of $b_{k,n}^{q_{k,n}}$ in (4.2) is chosen to be different from the non-trivial positions in (4.1). We deduce that

$$\left[(b_{k,n}, a_{k,n}, 1, \dots, 1)_n \cdot c_1, \left(1, \dots, 1, b_{k,n}^{q_{k,n}}, 1, \dots, 1\right)_n \cdot c_2\right] = 1.$$

□

We now give more commutator relators. First we will discuss these on level 1 before we generalise them for the case $d = 1$ for all levels.

Proposition 4.2.3. *The set of identities*

$$\left\{\left[b_{k,0}, b_{j,0}^{a}\right] = 1, a \in A_0, \quad a \neq u_0, u_0^{-1}\right\}$$

holds in G.

Proof. The assumption that $a \neq u_0$ and $a \neq u_0^{-1}$ gives that

$$b_{k,0} = (b_1, a_1, 1, \dots, 1)_1,$$

$$(b_{j,0})^{a} = (1, \dots, 1, b_1, 1, \dots, 1, a_1, 1, \dots, 1)_1$$

where b_1 is at position $g_{1,0}a \neq g_{2,0}$ and a_1 is at position $g_{2,0}a \neq g_{1,0}$. The regularity of the groups A_i acting on themselves by multiplication further gives that $g_{i,0}a \neq g_{i,0}$. We can see that $b_{k,0}$ and $b_{j,0}^{a}$ only act non-trivially on disjoint subtrees, hence commute as Figure 4.4 illustrates.

□

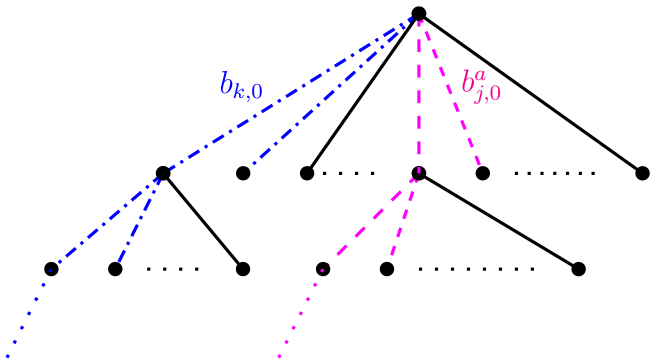

Figure 4.4: Action of $b_{k,0}$ and $b^a_{j,0}$ on the tree.

As before, graphical visualisation on the tree now suggests that some similar relations holds on every level n of the tree. We will see in Theorem 4.2.4 that we can indeed give such at least for the case $d = 1$. The same can probably be done for $d > 1$ with a much more complicated notation.

As earlier in Chapter 2, we denote the generators of G_C by $a = a_{1,0}$ and $b = b_{1,0}$.

Define $\kappa : G_C \longrightarrow G_C$ by

$$\kappa(x) = b^{x^{-1}} \qquad \text{for } x \in G_C$$

for the generator b of B. In Chapter 2 we have chosen an order of the vertices on each level corresponding to the elements of each A_i. We assume the same order here. Further, let $u \in A_0$ be such that $g_{1,0}u = g_{2,0}$, in words, u moves the first vertex of level 1 to the second position. Then the n-th iteration $\kappa^n(u)$ of $\kappa(u)$ is a conjugate of b as depicted in Figure 4.5.
The restriction of $\kappa^n(u)$ on $v_{1,n} \in \Omega(n)$ is given by

$$(a_{1,n+1}, 1, \ldots, 1, b_{1,n+1})_{n+1} \tag{4.3}$$

where $b_{1,n+1}$ is at position $g_{l_n,n}$.

Theorem 4.2.4. *The identity* $\left[b^{(\kappa^n(u))^t}, b\right] = 1$ *for* $t = 2, \ldots, l_{n-1} - 2$ *holds in* G_C *for all* $n \geq 0$.

Proof. The element b acts as $(b_{1,n}, a_{1,n}, 1, \ldots, 1)_n$ if restricted to the vertex $v_{1,n-1}$ of level $n-1$. The element $b^{(\kappa^n(u))^t}$ acts as

$$b^{(\kappa^n(u))^t} = (1, \ldots, 1, b_{1,n}, a_{1,n}, 1, \ldots, 1)_n$$

on level n where $b_{1,n}$ is at position $g_{t+1,n}$. Each of the elements b and $b^{(\kappa^n(u))^t}$ can be written as

$$b = (b_{1,n}, a_{1,n}, \ldots, 1)_n \cdot \pi_n(b) \tag{4.4}$$

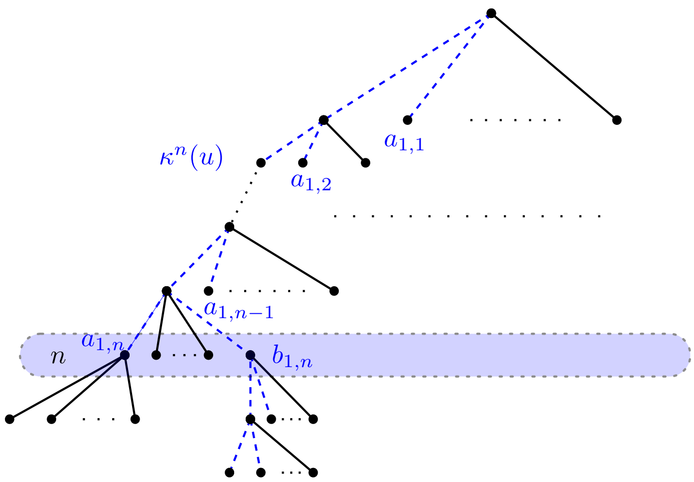

Figure 4.5: The n-th iterated conjugate $\kappa^n(u)$.

respectively

$$b^{(\kappa^n(u))^t} = (1,\dots,1,b_{1,n},a_{1,n},1,\dots,1)_n \cdot \pi_n\left(\kappa^n(u)\right), \tag{4.5}$$

with again $b_{1,n}$ in (4.5) at position $g_{t+1,n}$ with $t = 2,\dots,l_{n-1}-2$. It is left to show that $\pi_n(b)$ and $\pi_n\left(b^{(\kappa^n(u))^t}\right)$ commute. The conjugator $\kappa^n(u)^t$ acts as b^t on all levels above n. Hence

$$\pi_n\left(b^{(\kappa^n(u))^t}\right) = \pi_n\left(\kappa^n(u)^{-t}\right)\cdot\pi_n\left(b^t\right)\cdot\pi_n\left(\kappa^n(u)^t\right) = \pi_n\left(b^t\right)$$

which clearly commutes with $\pi_n(b)$. □

4.3 Growth

In this Section we will prove that the group G from Chapter 2 has exponential word growth under certain assumptions.

In 1968 there were no known examples of groups of intermediate growth type. As stated in [CWM$^+$07] Milnor raised the question in the 60s, whether any finitely generated group must be of either exponential or polynomial type. In his important paper [Gri84] Grigorchuk answered this question negatively. He provided a counterexample, the first Grigorchuk group. This group is a branch group acting on a rooted binary tree and initiated the study of these. As shown later, there also exist branch groups of exponential growth. An example constructed by Sidki and Wilson [SW03] even contains a free subgroup and therefore has exponential growth. Hence it seems a natural question to ask of which growth type a given branch group is. We will prove in this Section that the growth type of G from Chapter 2 is exponential under certain assumptions.

4.3.1 Definitions and Basics

Let Γ be a finitely generated group, generated by the set $X = \{x_1, \ldots, x_d\}$. Each element $x \in \Gamma$ can be written as a product $x = \prod_{i=1}^{m_\alpha} x_{j_i}^{\pm 1}$, $x_{j_i} \in X$. We define a *length* function $\lambda : \Gamma \to \mathbb{N}$ by

$$\lambda(\alpha) = \min \left\{ m_\alpha : \alpha = \prod_{i=1}^{m_\alpha} x_{j_i}^{\pm 1}, \ \ x_{j_i} \in X \right\}$$

and call $\lambda(\alpha)$ the word length of α in the generators X of Γ. We are now concerned with the question of how many distinct elements in Γ have a given length n. More generally, we will ask how many distinct elements of length less than or equal to n there are in Γ. For this we define

$$\gamma_\Gamma(n) = |\{\alpha \in \Gamma : \lambda(\alpha) \leq n\}|$$

as the *growth function* of Γ. Basic considerations about the free group give an upper bound for $\gamma_\Gamma(n)$.

Proposition 4.3.1. *If Γ is generated by d elements, then for $n > 0$ we have*

$$\gamma_\Gamma(n) \leq 2d(2d-1)^{n-1}.$$

A lower bound for $\gamma_\Gamma(n)$ can be obtained from $\gamma_\Gamma(n) \geq \gamma_{\mathbb{Z}^d}(n)$.

We now look at bit closer into the notion of growth. First, note that by writing a word of length $m+n$ as a product of one of length m and one of length n, we get that $\gamma_\Gamma(m+n) \leq \gamma_\Gamma(m)\gamma_\Gamma(n)$. Therefore

$$\omega(\Gamma) = \lim_{n\to\infty} \gamma_\Gamma(n)^{1/n}$$

exists and is finite. If we assume that Γ is infinite, then $\gamma_\Gamma(n) \geq 1$ for all $n \in \mathbb{N}$ and hence $\omega(\Gamma) \geq 1$. We already know from Proposition 4.3.1 that if Γ is finitely generated by d elements, then $\omega(\Gamma) \leq 2d-1$. The exact value of $\omega(\Gamma)$ depends on the group Γ, but also on the set of generators X. We denote $\omega(\Gamma)$ by $\omega_X(\Gamma)$ if we want to emphasize the dependence on the choice of X. Depending on $\omega(\Gamma)$ we can talk about different growth behaviours of Γ:

(i) We say Γ has *exponential growth* if $\omega(\Gamma) \geq 1$ and *subexponential growth* if $\omega(\Gamma) = 1$. The number $\omega(\Gamma)$ is called the *exponential growth rate* of Γ.

(ii) We say Γ has *uniform exponential growth* if $\inf_X \omega_X > 1$.

(iii) The group Γ has *polynomial growth*, if there exist numbers $c \in \mathbb{R}$ and $s \in \mathbb{N}$ which generally depend on X such that $\gamma_\Gamma(n) \leq cn^s$, for all $n \in \mathbb{N}$. We call s the *degree* of the polynomial growth of Γ.

(iv) If the growth of Γ is neither exponential nor polynomial, then Γ has *intermediate growth.*

As we have seen in Proposition 4.3.1, the free group F_d on d generators is an example of a group of exponential growth and the infinite cyclic group $\mathbb{Z}$ has polynomial growth. As already mentioned in the introduction, the Grigorchuk group [Gri84] was the first example of a group of intermediate growth.

4.3.2 Growth of G

We will now estimate the word growth of G from Chapter 2 by using its partial morally self-similar structure. The first Proposition is an easy observation. However, it is important as similar statements are true for a much wider class of groups acting on rooted trees.

Proposition 4.3.2. *The group G does not have polynomial growth.*

Proof. The free abelian group F_n^{ab} of rank n embeds into G for all $n \in \mathbb{N}$ as the proof of Theorem 2.2.3 shows. □

The following statements and proofs will only deal with the case when G is defined via a sequence of cyclic groups. We will then use this as shown in Theorem 4.3.5 to deduce exponential growth for the case where G is defined via abelian groups under certain assumptions on the defining sequence. The next Lemma underlines that even though G is not self-similar, we can still treat it somehow as if it were.

As in Chapter 2, we denote by $G_n \times \cdots \times G_n$ the subgroup of $\mathrm{Aut}(T)$ that acts as G_n on each vertex of level n. If we say a group G acts as G_n on a vertex v of level n, then we mean that the projection of G onto a vertex v of level n acts as G_n on v.

Lemma 4.3.3. *Let $v_i \in \Omega(1)$ be the i-th vertex on level 1. Then the projection of G acts as G_1 on v_i for $i \in \{1, \ldots, l_0\}$.*

Proof. The action of the projection of G onto v_2 is given by $b(1) = (b_1, a_1, 1, \ldots, 1)_1$ and $b(2) = a^{-1}ba = (1, b_1, a_1, \ldots, 1)_1$. Hence $b(1)$ and $b(2)$ generate G_1 on v_2. The same follows for every vertex v_i on level 1 with $b(i-1)$ and $b(i)$. This is depicted in Figure 4.6. □

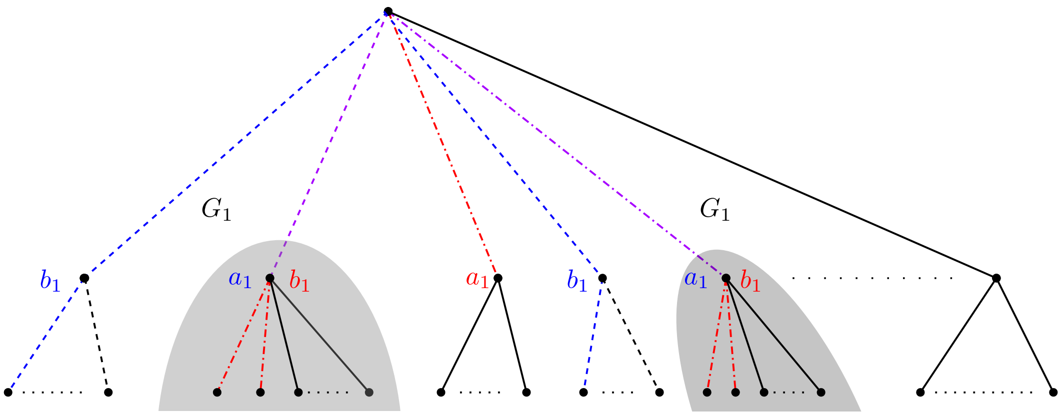

Figure 4.6: Partial self-similar structure of G

The last Lemma 4.3.3 allows us to make assumptions about G akin to assumptions that can be made about self-similar groups. In the next Lemma 4.3.4 we only decorate a third of the vertices of level i with G_i and then distribute them among the m_i vertices to obtain an estimate for the word growth.

Lemma 4.3.4. $\gamma_G\left(m_i(n+i)\right) \geq \gamma_{G_i}(n)^{\left\lfloor \frac{m_i}{3^i} \right\rfloor} \cdot \left(\left\lfloor \frac{2l_{i-1}}{3} \right\rfloor\right)^{\left\lfloor \frac{m_i}{3^i} \right\rfloor}$.

Proof. Assume w is a word in the generators a, b of G such that $w(a,b)$ is a generator of G_1 on the second vertex v_2 of level 1. Then w can be chosen to have length at most 3, because a_1 can be obtained from $b(1)$ and b_1 from $b(2) = a^{-1}ba$. Every word in the generators a_1, b_1 of G_1 on $v_i \in \Omega(1)$ can be obtained by one in the generators a_1, b_1 of G_1 on v_2 and then conjugated by at most $a^{\pm\lfloor \frac{l_0}{2} \rfloor}$, which adds l_0 to its word length in $\{a, b\}$. In order to get $\frac{l_0}{3}$ words $w_j \in G_1$ on positions $1 \leq j \leq l_0$ it is enough to construct $\prod_{j=1}^{\lfloor l_0/3 \rfloor} w_j \cdot a^{q_j}$ with $1 \leq q_j \leq l_0$, depending on where we place the $\lfloor \frac{l_0}{3} \rfloor$ words w_j and $\sum_{j=1}^{\lfloor l_0/3 \rfloor} q_j = l_0$. Hence we get a recursion

$$\gamma_G\left(\left\lfloor \frac{l_0}{3} \right\rfloor 3n + l_0\right) \geq \gamma_{G_1}(n)^{\lfloor \frac{l_0}{3} \rfloor} \cdot \left(\left\lfloor \frac{2l_0}{3} \right\rfloor\right)^{\lfloor \frac{l_0}{3} \rfloor}.$$

We can estimate this expression by $\frac{l_0}{3} \cdot 3n + l_0 \leq l_0 n + l_0$. Iterating this we get

$$(l_0 n + l_0)\, l_1 + l_1 = l_0 l_1 n + l_0 l_1 + l_1$$

and so for the i-th step we can see that $m_i\,(n+i)$ gives an upper bound for this expression. Hence we get

$$\gamma_G\left(m_i(n+i)\right) \geq \gamma_{G_i}(n)^{\lfloor \frac{m_i}{3^i} \rfloor} \cdot \left(\left\lfloor \frac{2l_{i-1}}{3} \right\rfloor\right)^{\lfloor \frac{l_{i-1}}{3} \rfloor \cdot \lfloor \frac{m_{i-1}}{3^{i-1}} \rfloor}.$$

□

We are now able to show that G has exponential growth under certain assumptions.

Theorem 4.3.5. *(a) The 2-generator group G has exponential growth rate if $\{l_i\}_{i\in\mathbb{N}_0}$ is a sequence of distinct primes and is such that $l_i \geq C^{3^{i+1}\cdot(2+i)}$ for some $C > 1$.*

(b) Assume we have $d > 1$ and a sequence $\{l_i\}_{i\in\mathbb{N}_0}$ of coprime integers. Let $q_{i,j}$ be the factors of l_i for $j = 1, \ldots, d$. Then the $2d$-generator group G has exponential growth rate if there exists a $j \in \{1, \ldots, d\}$ such that $q_{i,j} \geq C^{3^{i+1}\cdot(2+i)}$ for some $C > 1$, for all $i \in \mathbb{N}_0$.

Proof. We first prove (a) and then use it to deduce statement (b). We want an estimate for $\gamma_G(r)$. We can assume that $r = m_i(1+i)$, as the latter term grows unboundedly with i. Lemma 4.3.4 then gives

$$\gamma_G(r) = \gamma_G\left(m_i(1+i)\right) \geq \gamma_{G_i}(1)^{\frac{m_i}{3^i}} \cdot \left(\frac{2l_{i-1}}{3}\right)^{\frac{l_{i-1}}{3} \cdot \frac{m_{i-1}}{3^{i-1}}}.$$

We want to find an $\alpha > 0$ such that $\gamma_G(r) \geq e^{\alpha r}$. With $\gamma_{G_i}(1) = 2$, we get that we will need

$$e^{\alpha m_i(1+i)} \leq 2^{\frac{m_i}{3^i}} \cdot \left(\frac{2l_{i-1}}{3}\right)^{\frac{m_i}{3^i}}$$

which can be transformed into

$$\alpha \cdot m_i(1+i) \leq \frac{m_i}{3^i}\log(2) + \frac{m_i}{3^i}\log\left(\frac{2l_{i-1}}{3}\right).$$

It is enough to find a bound for $\alpha > 0$, so we focus on the second term of the sum on the right hand side which gives

$$\alpha \leq \frac{2\log(2) + \log(l_{i-1}) - \log(3)}{3^i \cdot (1+i)}$$

and it is further enough to have

$$\alpha \leq \frac{\log(l_{i-1})}{3^i \cdot (1+i)}.$$

We see that in order to be able to find such an $\alpha > 0$ independent of i we have to require that $l_{i-1} \geq C^{3^i \cdot (1+i)}$ for some $C > 1$ which gives that any $\alpha \leq \log(C)$ can be chosen.

We now apply this to show (b). In this case, G is given as a $2d$-generator group $G_{2d} = \langle a_1, \ldots, a_d, b_1, \ldots, b_d \rangle$ where each a_h has order $q_{h,0}$. We see that G_{2d} has a 2-generator subgroup $G_2 = \langle a_1, b_j \rangle$ for the j as given, where now $a_{i,j}$ has order $q_{i,j}$. The sequence $q_{i,j}$ is again coprime, hence ascending and so the same considerations as above apply with $q_{i,j}$ in place of l_i. □

Remark 4.3.6. *In private communication with Grigorchuk and Šunić it has occurred that the Hanoi group on* 3*-pegs also has exponential growth but does not contain any free subgroups. This result has not been published yet.*

4.4 Open Questions

Question 4.4.1. Is every morally, or partially morally, self-similar group infinitely presented?

Extending an approach by Grigorchuk in [Gri99] allows us to also show in the case of the groups of Chapter 2 and 3 that G is not finitely presented. This essentially only uses that G acts at least partially self-similarly. It hence seems possible to further alter this construction to show that no partially morally self-similar groups can be finitely presented.

Question 4.4.2. Does the group constructed in Chapter 2 have exponential growth for any defining sequence?

A result by Brieussel [Bri09] shows that a similar type of group is amenable if the sequence grows so slow that it does not allow distinct coprime values.

Question 4.4.3. Is the set of relators given in Section 4.2 a defining set of relators for G_C?

Question 4.4.4. Does G contain a free semigroup?

There are very few known examples of groups of exponential growth that do not contain a free semigroups. Such are free Burnside groups and its relatives, Tarski monsters.

Chapter 5

Hausdorff Dimensions

In this Chapter we study fractal dimensions of branch groups. In Section 5.3 we address the question of how 'large' a certain group is inside the full automorphism group of the tree that it acts on. Later in the Chapter, in Section 5.4, we are interested in the dimension that subgroups of such branch groups can have within them.

5.1 Introduction

Addressing the question of how large such a group G is within $\mathrm{Aut}(T)$, where T is a rooted tree, Barnea and Shalev [BS97] have computed an explicit formula for the Hausdorff dimension with respect to a metric coming from a natural filtration. We will look at the dimensions of the closures of branch groups with respect to the congruence completion.

There are two different main aspects of the computation of Hausdorff dimensions. One is asking for the dimension of certain groups G acting on a tree T within $\mathrm{Aut}(T)$, the other one is focused on which dimensions subgroups of G can have within G.

Addressing the first question, M. Abért and B. Virág [AV05] have shown that there exist finitely generated subgroups of $\mathrm{Aut}(T)$ with arbitrary Hausdorff dimension. In [Sie08], O. Siegenthaler explicitly computed the Hausdorff dimension of level-transitive spinal groups. G. Fernandez-Alcober and A. Zugadi-Reizabal [FA11] give an explicit set of values for the dimension of certain spinal groups.

Working on the second question, in his thesis [Klo99] B. Klopsch has shown that profinite branch groups have full subgroup Hausdorff spectrum $[0, 1]$. He leaves the question open, whether the finitely generated Hausdorff spectrum can contain irrational values. Here we give for each

$\alpha \in [0,1]$ an explicit example of a branch group G_α that has a finitely generated subgroup H, such that H has dimension α in G_α.

Further considerations yield that the finitely generated Hausdorff spectrum of the group G_α contains $\mathcal{T}\left(\{l_i - 2\}_{i\in\mathbb{N}_0}\right)_\alpha \cup ([0,1] \cap \mathcal{T}\left(\{l_i - 2\}_{i\in\mathbb{N}_0}\right))$, where $\mathcal{T}\left(\{l_i - 2\}_{i\in\mathbb{N}_0}\right)$ is a countable subset of $\mathbb{Q}$ and $\mathcal{T}\left(\{l_i - 2\}_{i\in\mathbb{N}_0}\right)_\alpha$ is a certain set of countable many irrational numbers in the interval $[0,\alpha]$. We do not know whether the parameters of this construction can be chosen such that for all $\nu \in \mathcal{T}\left(\{l_i - 2\}_{i\in\mathbb{N}_0}\right)_\alpha \cup ([0,1] \cap \mathbb{Q})$ there exists a finitely generated subgroup H of Hausdorff dimension ν in G.

5.2 Fractal Dimensions

We give a quick introduction on Hausdorff dimensions and explain how they can be defined in the context of profinite groups. We refer the reader to Falconer [Fal03] for more information on fractal dimensions.

Let (X, d) be a metric space, let $Y \subset X$ and $\alpha, \rho \in \mathbb{R}^+$. Define

$$\mathcal{H}^\alpha_\rho(Y) = \inf \sum_i (\operatorname{diam} S_i)^\alpha ,$$

where $\{S_i\}_{i=0}^\infty$ is a cover of Y by sets of diameter at most ρ, and the infimum is taken over all such covers. Note that $\mathcal{H}^\alpha_\rho(Y)$ is non-increasing with ρ, and so the limit

$$\mathcal{H}^\alpha = \liminf_{\rho \to 0} \mathcal{H}^\alpha_\rho(Y)$$

exists. It can be verified that $\mathcal{H}^\alpha(Y)$ is an outer measure on X, the *α-dimensional Hausdorff measure.*

Lemma 5.2.1. *If $\mathcal{H}^\alpha(Y) < \infty$ and $\alpha < \alpha'$, then $\mathcal{H}^{\alpha'}(Y) = 0$.*

We can now define the *Hausdorff dimension* of a set $Y \subset X$:

$$\dim_H(Y) = \sup \{\alpha | \mathcal{H}^\alpha(Y) = \infty\} = \inf \{\alpha | \mathcal{H}^\alpha(Y) = 0\} .$$

Example 5.2.2. We can consider the Cantor set as a metric space because it is a closed subset of the reals. With the metric coming from the reals, this set has Hausdorff dimension $\frac{\ln 2}{\ln 3}$.

A *filtration* of G is a descending chain of open normal subgroups $G = G_0 \geq G_1 \geq \cdots \geq G_n \geq \ldots$ which forms a base of the neighborhoods of the identity. For such a series we have $\bigcap_{n=0}^\infty G_n = \{1\}$. Now, let G be a profinite group, equipped with a filtration G_n. Define an invariant metric d on G by

$$d(x, y) = \inf \left\{ |G/G_n|^{-1} : \ \ xy^{-1} \in G_n \right\} .$$

With respect to these definitions, Barnea and Shalev [BS97] proved the following theorem:

Theorem 5.2.3. *Let G be a profinite group with a filtration $\{G_n\}_{n=0}^\infty$ and let $H \leq G$ be a closed subgroup. Then*

$$\dim_G(H) = \liminf_{n \to \infty} \frac{\log |H/ (H \cap G_n)|}{\log |G/G_n|}, \tag{5.1}$$

where the Hausdorff dimension is computed with respect to the metric associated with the filtration $\{G_n\}$.

The Hausdorff dimension of $H \leq G$ depends in general on the chosen filtration $\{G_n\}$ as the following example shows.

Example 5.2.4. [BS97, 2.5] Let $G = \mathbb{Z}_p \oplus \mathbb{Z}_p$ and $H = \{0\} \oplus \mathbb{Z}_p$. Then if the Hausdorff dimension is computed relative to the filtration $G_n = p^n\mathbb{Z}_p \oplus p^n\mathbb{Z}_p$, we obtain $\dim_G(H) = \frac{1}{2}$. However, if the Hausdorff dimension is computed relative to the filtration $G_n = p^{2n}\mathbb{Z}_p \oplus p^n\mathbb{Z}_p$, we obtain $\dim_G(H) = \frac{1}{3}$.

A natural filtration for branch groups is given by the level stabilizers $\mathrm{St}_G(n)$. When we talk about the Hausdorff dimension of a subgroup H in G, we will from now on mean the dimension of its closure $\bar{H}$ with respect to the congruence topology in the closure $\bar{G}$ of G. This denotes the congruence completion of G (see [RZ10] for a definition), which is the completion with respect to the filtration $\mathrm{St}_G(n)$.

Example 5.2.5.

1. The first Grigorchuk group as described in Chapter 1 has Hausdorff dimension $\frac{5}{8}$ ([Gri00, Section 15]) in the full automorphism group $\mathrm{Aut}\,(T_2)$ of the binary rooted tree T_2.

2. The Hanoi group H on 3-pegs as described in [Gv06] is a branch group. As computed in [Š07] it has Hausdorff dimension $\dim(G) = 1 - \frac{1}{3}\log_6(2) \sim 0.871$ in $\mathrm{Aut}\,(T_3)$, the full automorphism group of a 3-regular rooted tree T_3.

The *Hausdorff spectrum* $\mathrm{spec}_H(G)$ of a group G is the set of all values $\alpha \in [0,1]$ for which there exists a subgroup H such that $\dim_G(H) = \alpha$:

$$\mathrm{spec}_H(G) = \{\dim_G(H) | \ H \leq G\}.$$

The *finitely generated Hausdorff spectrum* of a group G is defined as

$$\mathrm{spec}_H^{fg}(G) = \{\dim_G(H) | \ H \leq G, H \text{ is finitely generated}\}.$$

5.3 Dimension of G in $\mathrm{Aut}(T)$

In this Section we explicitly compute what dimensions the groups constructed in Chapter 2 and Chapter 3 can have in $\mathrm{Aut}(T)$. We then generalise this to wider classes of groups.

We begin by only considering groups G that act transitively on each level of the tree with ascending defining sequence $\{l_i\}$ similar to our construction in Chapter 2. It will be proved that the only possible dimensions of such groups within $\mathrm{Aut}(T)$ are 0 and 1. We recall that the full automorphism group on a tree T with defining sequence $\{l_i\}$ was given as $\mathrm{Aut}(T) = \varprojlim_n \mathrm{Sym}(l_n) \wr \cdots \wr \mathrm{Sym}(l_0)$.

Theorem 5.3.1. *Assume the sequence $\{l_i\}$ is given by distinct primes. Let $\{A_i\} = \langle a_{1,i}, \ldots, a_{d,i}\rangle$ be a sequence of d-generator transitive permutation groups acting on l_i points. Then the closure of the spinal group $G = \langle a_{1,0}, \ldots, a_{d,0}, b_{1,0}, \ldots, b_{d,0}\rangle$ in $\mathrm{Aut}(T)$ has either Hausdorff dimension 0 or 1 in $\mathrm{Aut}(T)$.*

Proof. Because all l_i are prime, any transitive subgroup must also act primitively. According to [Mar02] the only primitive subgroups of $\mathrm{Sym}(l_i)$ are

1. $\mathrm{Alt}(l_i)$, $\mathrm{Sym}(l_i)$,
2. subgroups H with $|H| \leq l_i^{\log_2 l_i + 1}$ and
3. Mathieu groups M_{11}, M_{12}, M_{23} or M_{24} with their 4-transitive action as described in Section A.4 of the Appendix.

We first show that if we choose all $A_i = \mathrm{Sym}(l_i)$ or $A_i = \mathrm{Alt}(l_i)$ then we get $\dim(G) = 1$. We do the following computation for the case of the alternating groups. It can immediately be seen that this will also imply the result if any of the $A_i = \mathrm{Sym}(l_i)$. Proposition 3.1.5 states that G acts as $A_n \wr \cdots \wr A_0$ on level $n+1$ and $\mathrm{Aut}(T)$ as $\mathrm{Sym}(l_n) \wr \cdots \wr \mathrm{Sym}(l_0)$, hence

$$\dim_{\mathrm{Aut}(T)}(G) = \liminf_{n\to\infty} \frac{\log\left(|A_n \wr \cdots \wr A_{l_0}|\right)}{\log\left(|\,\mathrm{Sym}(l_n) \wr \cdots \wr \mathrm{Sym}(l_0)\,|\right)}$$

$$= \liminf_{n\to\infty} \frac{\log\left(\left(\frac{l_n!}{2}\right)^{l_0\cdot\dots\cdot l_{n-1}} \cdot \left(\frac{l_{n-1}!}{2}\right)^{l_0\cdot\dots\cdot l_{n-2}} \cdot\dots\cdot \frac{l_0!}{2}\right)}{\log\left((l_n!)^{l_0\cdot\dots\cdot l_{n-1}} \cdot\dots\cdot l_0!\right)}$$

$$= \liminf_{n\to\infty} \frac{\log\left((l_n!)^{l_0\cdots l_{n-1}} \cdot\dots\cdot l_0!\right) - (1 + l_0 + l_0 l_1 + \cdots + l_0\cdot\dots\cdot l_{n-1})\cdot\log 2}{\log\left((l_n!)^{l_0\cdot\dots\cdot l_{n-1}} \cdot\dots\cdot l_0!\right)}$$

$$= \liminf_{n\to\infty} \left(1 - \frac{(1 + l_0 + l_0 l_1 + \cdots + l_0 \cdot \cdots \cdot l_{n-1}) \cdot \log 2}{\log\left((l_n!)^{l_0 \cdot \cdots \cdot l_{n-1}} \cdot \cdots \cdot l_0!\right)}\right).$$

Now we investigate the fraction separately:

$$\liminf_{n\to\infty} \frac{(1 + l_0 + l_0 l_1 + \cdots + l_0 \cdot \cdots \cdot l_{n-1}) \cdot \log 2}{\log\left((l_n!)^{l_0 \cdot \cdots \cdot l_{n-1}} \cdot \cdots \cdot l_0!\right)} \leq \liminf_{n\to\infty} \frac{(n+1) l_0 \cdot \cdots \cdot l_{n-1}}{l_0 \cdot \cdots \cdot l_{n-1} \cdot \log(l_n!)}$$

$$\leq \liminf_{n\to\infty} \frac{n+1}{\log\left(\frac{l_n}{e}\right)^{l_n}} \leq \liminf_{n\to\infty} \frac{2n}{l_n \cdot (\log l_n - 1)} \leq \liminf_{n\to\infty} \frac{2 \cdot 1.045 \frac{l_n}{\log l_n}}{l_n \frac{\log l_n}{2}} \leq \liminf_{n\to\infty} \frac{5}{\log^2 l_n} = 0.$$

We used the prime number theorem A.5.3 which states that the number of primes $\pi(x)$ which are less than or equal to x is bounded by $\pi(x) \leq 1.04423 \frac{x}{\log x}$. This gives us

$$\dim_{\mathrm{Aut}(T)}(G) = \liminf_{n\to\infty} \frac{\log\left(\left(\frac{l_n!}{2}\right)^{l_0 \cdot \cdots \cdot l_{n-1}} \cdot \cdots \cdot \frac{l_0!}{2}\right)}{\log\left((l_n!)^{l_0 \cdot \cdots \cdot l_{n-1}} \cdot \cdots \cdot l_0!\right)} = 1.$$

We get the following estimation for the case where $|A_i| \leq l_i^{\log_2 l_i + 1}$:

$$\dim_H(G) = \liminf_{n\to\infty} \frac{\log\left(l_n^{(1+\log_2) \cdot l_n \cdot l_0 \cdot \cdots \cdot l_{n-1}} \cdot \cdots \cdot l_0^{1+\log_2 l_0}\right)}{\log\left((l_n!)^{l_0 \cdot \cdots \cdot l_{n-1}} \cdot \cdots \cdot l_0!\right)}$$

$$= \liminf_{n\to\infty} \frac{\log\left(e^{\log l_n (1+\log_2 l_n) \cdot l_0 \cdot \cdots \cdot l_{n-1}} \cdot \cdots \cdot e^{\log l_0 (1+\log_2 l_0)}\right)}{\log\left((l_n!)^{l_0 \cdot \cdots \cdot l_{n-1}} \cdot \cdots \cdot l_0!\right)}$$

$$\leq \liminf_{n\to\infty} \frac{2\log^2 l_n \cdot l_{n-1} \cdot \cdots \cdot l_0 + 2\log^2 l_{n-1} \cdot l_{n-2} \cdot \cdots \cdot l_0 + \cdots + 2\log^2 l_0}{\log\left((l_n!)^{l_{n-1} \cdot \cdots \cdot l_0} \cdot \cdots \cdot l_0!\right)}.$$

We now again use $n \leq \pi(l_n) \leq 1.04423 \frac{l_n}{\log l_n}$ from Theorem A.5.3 and $x! \geq e\left(\frac{x}{e}\right)^x$ to get the following:

$$\leq \liminf_{n\to\infty} \frac{2\log^2 l_n \cdot l_{n-1} \cdot \cdots \cdot l_0}{\log\left((l_n!)^{l_{n-1} \cdot \cdots \cdot l_0}\right)} + \liminf_{n\to\infty} \frac{2n \log^2 l_{n-1} \cdot l_{n-2} \cdot \cdots \cdot l_0}{\log\left((l_n!)^{l_{n-1} \cdot \cdots \cdot l_0}\right)}$$

$$\leq \liminf_{n\to\infty} \frac{2\log^2 l_n}{l_n(\log l_n - 1)} + \liminf_{n\to\infty} \frac{3\frac{l_n}{\log l_n} \cdot \log^2 l_{n-1}}{l_{n-1} l_n (\log l_n - 1)}$$

$$\leq \liminf_{n\to\infty} \frac{4\log l_n}{l_n} + \liminf_{n\to\infty} \frac{6\log^2 l_{n-1}}{l_{n-1}\log^2 l_n} = 0.$$

Together this gives $\dim_H(G) = 0$.

The above considerations are the two extreme cases, when all groups are either $\mathrm{Sym}(l_i)$, $\mathrm{Alt}(l_i)$ or small primitive groups. Let us now assume that G acts as a small primitive group A_i as given in the beginning of this proof on infinitely many places. Assume that the A_i which are small primitive groups are given by a sequence p_n. We compute the limit based on where p_0 is and will see that even as p_0 goes to infinity, the dimension will automatically go to 0. We use that $|A_{p_0}| \leq l_{p_0}^{1+\log l_{p_0}}$ if A_{p_0} is a small primitive group and assume all other A_i for $i \leq p_0$ are given by $A_i = \mathrm{Sym}\,(l_i)$.

In this case the limit looks as follows:

$$\liminf_{p_0\to\infty} \frac{\log\left(|A_{p_0}|^{l_{p_0-1}\cdot\dots\cdot l_0}\cdot|A_{p_0-1}|^{l_{p_0-2}\cdot\dots\cdot l_0}\cdot\dots\cdot|A_0|\right)}{\log\left((l_{p_0}!)^{l_0\cdot\dots\cdot l_{p_0-1}}\cdot\dots\cdot l_0!\right)}$$

$$= \liminf_{p_0\to\infty} \frac{\log\left(l_{p_0}^{(1+\log l_{p_0})\cdot l_{p_0-1}\cdot\dots\cdot l_0}\cdot(l_{p_0-1}!)^{l_{p_0-2}\cdot\dots\cdot l_0}\cdot\dots\cdot(l_0!)\right)}{\log\left((l_{p_0}!)^{l_0\cdot\dots\cdot l_{p_0-1}}\cdot\dots\cdot l_0!\right)}$$

$$\leq \liminf_{p_0\to\infty} \frac{(1+\log l_{p_0})\cdot l_{p_0-1}\cdot\dots\cdot l_0\cdot\log(l_{p_0})+\log\left((l_{p_0-1}!)^{l_{p_0-2}\cdot\dots\cdot l_0}\cdot\dots\cdot(l_0!)\right)}{\log\left((l_{p_0}!)^{l_0\cdot\dots\cdot l_{p_0-1}}\cdot\dots\cdot l_0!\right)}$$

$$\leq \liminf_{p_0\to\infty} \frac{(1+\log l_{p_0})\cdot l_{p_0-1}\cdot\dots\cdot l_0\cdot\log(l_{p_0})}{\log\left(\left(\frac{l_{p_0}}{e}\right)^{l_{p_0}\cdot l_0\cdot\dots\cdot l_{p_0-1}}\right)} + \liminf_{p_0\to\infty} \frac{\log\left((l_{p_0-1}!)^{l_{p_0-2}\cdot\dots\cdot l_0}\cdot\dots\cdot(l_0!)\right)}{\log\left((l_{p_0}!)^{l_0\cdot\dots\cdot l_{p_0-1}}\cdot\dots\cdot l_0!\right)}$$

$$\leq \liminf_{p_0\to\infty} \frac{2\log^2 l_{p_0}}{l_{p_0}} + \liminf_{p_0\to\infty} \frac{\log\left((l_{p_0-1}!)^{n\cdot m_{p_0-1}}\right)}{m_{p_0}\cdot\log(l_{p_0}!)}$$

$$\leq \liminf_{p_0\to\infty} \frac{2\log^2 l_{p_0}}{l_{p_0}} + \liminf_{p_0\to\infty} \frac{n}{l_{p_0-1}} \leq \liminf_{p_0\to\infty} \frac{2\log^2 l_{p_0}}{l_{p_0}} + \liminf_{p_0\to\infty} \frac{1}{\log l_{p_0}} = 0.$$

Finally, we look at the case where some A_i is one of the five Mathieu groups. Those are all subgroups of Sym(24) and hence will not influence the limit because we assumed the sequence l_i to be ascending. □

The next Theorem 5.3.2 shows that groups G which are constructed using transitive and soluble permutation groups are very small within Aut(T).

Theorem 5.3.2. *Let G be any group as in Chapter 2 where the A_i are soluble and transitive permutation groups on sets $\{1,\dots,l_i\}$ for a sequence $\{l_i\}$ of coprime numbers with $l_i \geq 3$ for all $i \in \mathbb{N}$. Then the Hausdorff dimension of G in* Aut(T) *is* 0.

Proof. According to [DM96, Theorem 5.8B] the order of a soluble subgroup H of $\mathrm{Sym}\,(l_i)$ is bounded by

$$|H| \leq 24^{\frac{l_i-1}{3}}.$$

As before, we get that G acts as the full wreath product $A_n \wr \cdots \wr A_0$ on the vertices of level $n+1$. Therefore $G/\operatorname{St}_G(n+1) = A_n \wr \cdots \wr A_0$. By the formula for the Hausdorff dimension (5.1) we get with $c = 24^{1/3}$

$$\dim_{\operatorname{Aut} T}(G) \leq \liminf_{n\to\infty} \frac{\log\left(c^{(l_n-1)\cdot l_{n-1}\cdots l_0} \cdot c^{(l_{n-1}-1)\cdot l_{n-2}\cdot\dots\cdot l_0} \cdot \dots \cdot c^{l_0-1}\right)}{\log\left((l_n!)^{l_{n-1}\cdot\dots\cdot l_0} \cdot \dots \cdot l_0!\right)}$$

$$\leq \liminf_{n\to\infty} \frac{\log\left(\left(c^{l_n-1}\right)^{l_{n-1}\cdot\dots\cdot l_0}\right)}{\log\left((l_n!)^{l_{n-1}\cdot\dots\cdot l_0}\right)} + \liminf_{n\to\infty} \frac{\log\left(\left(\left(c^{l_{n-1}-1}\right)^{l_{n-2}\cdot\dots\cdot l_0}\right) \cdot \dots \cdot c^{l_0-1}\right)}{\log\left((l_n!)^{l_{n-1}\cdot\dots\cdot l_0} \cdot \dots \cdot l_0!\right)}.$$

We show that the limit for both fractions is 0. For the first one we get:

$$\liminf_{n\to\infty} \frac{\log\left(\left(c^{l_n-1}\right)^{l_{n-1}\cdot\dots\cdot l_0}\right)}{\log\left((l_n!)^{l_{n-1}\cdot\dots\cdot l_0}\right)} \leq \liminf_{n\to\infty} \frac{\log\left(\left(c^{l_n}\right)^{l_{n-1}\cdot\dots\cdot l_0}\right)}{l_{n-1}\cdot\dots\cdot l_0 \cdot \log\left(\frac{l_n}{e}\right)^{l_n}}$$

$$\leq \liminf_{n\to\infty} \frac{l_n \cdot \dots \cdot l_0 \log c}{l_n \cdot \dots \cdot l_0 \left(\log l_n - 1\right)} \leq \liminf_{n\to\infty} \frac{2\log c}{\log l_n} = 0.$$

And for the second one:

$$\frac{\log\left(\left(\left(c^{l_{n-1}-1}\right)^{l_{n-2}\cdot\dots\cdot l_0}\right) \cdot \dots \cdot c^{l_0-1}\right)}{\log\left((l_n!)^{l_{n-1}\cdot\dots\cdot l_0} \cdot \dots \cdot l_0!\right)} \leq \liminf_{n\to\infty} \frac{n l_{n-1} \log(c) l_{n-2} \cdot \dots \cdot l_0}{\log\left((l_n!)^{l_{n-1}\cdot\dots\cdot l_0}\right)}$$

$$\leq \liminf_{n\to\infty} \frac{2\log(c) \cdot \frac{l_n}{\log l_n}}{\log\left(\frac{l_n}{e}\right)^{l_n}} \leq \liminf_{n\to\infty} \frac{2\log(c) \cdot \frac{l_n}{\log l_n}}{l_n\left(\log l_n - 1\right)} \leq \liminf_{n\to\infty} \frac{4\log c}{\log^2 l_n} = 0.$$

□

5.4 Hausdorff Spectrum

In this Section we look at what possible Hausdorff dimensions subgroups H of G can have in G. In particular, we consider the case when H is finitely generated. We will show that for each $\alpha \in [0,1]$ there exists a group G_α which has a finitely generated subgroup H such that H has dimension α in G_α.

5.4.1 The Construction of G

We explain the construction of the group G with the desired properties. We will then show in the next Subsection that G indeed has those properties. We essentially construct a group as described in Chapter 3, choosing the defining sequence of the tree carefully within certain bounds.

Denote by A_k the alternating group acting on the set $\{1,\dots,k\}$. If k is odd, then the group A_k is generated by a 3-cycle τ_k and a k-cycle σ_k ([DM96]):

$$\tau_k = ((k-2)(k-1)k), \quad \sigma_k = (1\dots k).$$

Let $\{l_i\}_{n\geq 0}$ be a sequence of natural numbers and let $\{A_{l_i}\}_{i\in\mathbb{N}_0}$ be a sequence of alternating groups acting on the sets $\{1,\dots,l_i\}$. We study the group

$$G = \langle \tau_{l_0}, \sigma_{l_0}, \zeta, \psi\rangle$$

where ζ and ψ are recursively defined on each level n by

$$\zeta_n = \left(\zeta_{n+1}, \tau_{l_{n+1}}, 1, \dots, 1\right)_{n+1},$$

$$\psi_n = \left(\psi_{n+1}, \sigma_{l_{n+1}}, 1, \dots, 1\right)_{n+1}.$$

This means that the action on the first vertex of level $n+1$ is given by ζ_{n+1} or ψ_{n+1} and the action on the second vertex by the rooted automorphism τ_{n+1} or σ_{n+1}. Figure 5.1 depicts the action of the automorphisms ζ and ψ on the tree. The action of ζ and ψ on all unlabelled vertices v in the Figure will be given by the identity on T_v.

5.4.2 The Finitely Generated Spectrum

We show that for every $\alpha \in [0,1]$ we can construct a group G_α similar to the ones in Chapter 3 such that there exists a finitely generated subgroup $H \leq G_\alpha$ with $\dim_G(H) = \alpha$. We fix for the rest of this Chapter the filtration $G_i = \mathrm{St}_G(i)$ of G.

First a simple Lemma on the approximation of a number in the interval $[0,1]$.

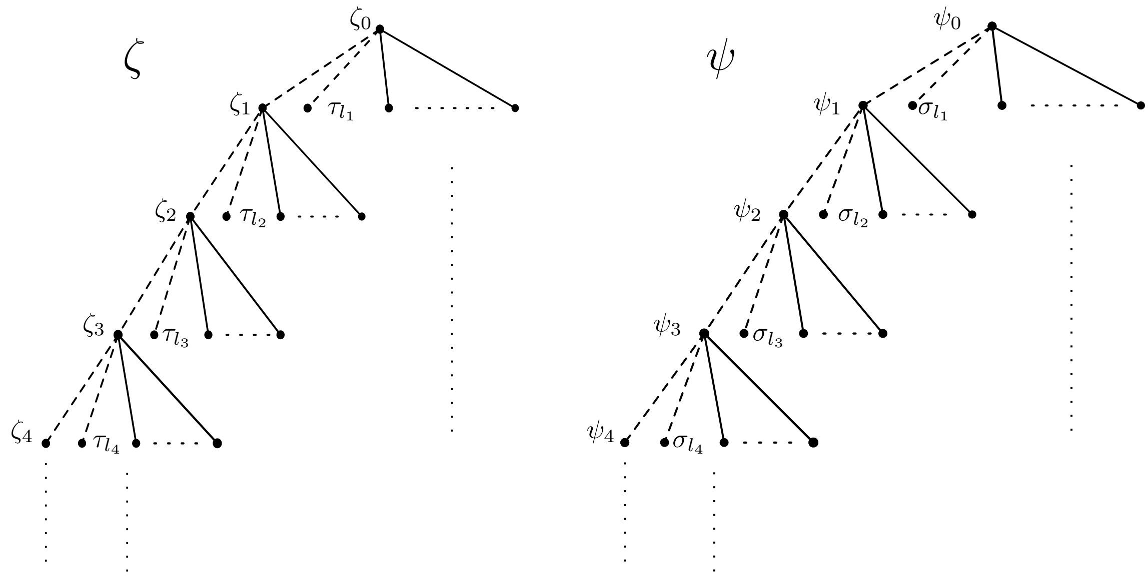

Figure 5.1: Portrait of the automorphisms ζ and ψ.

Lemma 5.4.1. *For every $\alpha \in (0,1)$ there exists an ascending sequence $\{l_i\}, l_i \in \mathbb{N}_0$ of distinct integers $l_i \geq 5$ such that*

$$\liminf_{i\to\infty} \prod_{j=0}^{i} \frac{l_j - 2}{l_j} = \alpha.$$

Proof. Choose l_0 such that

$$\frac{1}{7}(6+\alpha) > \frac{l_0 - 2}{l_0} > \alpha.$$

Further choose l_i for $i \geq 1$ such that

$$\frac{1}{7}\left(6 + \frac{\alpha}{\prod_{j=0}^{i-1} \frac{l_j-2}{l_j}}\right) > \frac{l_i - 2}{l_i} > \frac{\alpha}{\prod_{j=0}^{i-1} \frac{l_j-2}{l_j}}.$$

We can without loss of generality assume that the sequence fulfills $l_i \geq 5$ and is ascending. $\square$

Remark 5.4.2. *The approximating sequence $\{l_i\}_{i\geq 0}$ is not unique. It will be shown that if each l_i is such that it has many different prime factors, then we will be able to obtain a fuller Hausdorff spectrum.*

We can now show that we can construct a finitely generated subgroup H of dimension α in G if we choose the defining sequence $\{l_i\}$ for G depending on α as described in Lemma 5.4.1.

Theorem 5.4.3. *For every $\alpha \in [0,1]$, there exists a branch group G_α and a finitely generated subgroup $H \leq G_\alpha$ such that $\dim_{G_\alpha}(H) = \alpha$. Further, if H is non-trivial, it is again a finitely generated branch group.*

Proof. If $\alpha = 1$ set $H = G$ and if $\alpha = 0$ set $H = 1$. Otherwise let us choose the sequence $\{l_i\}$ as in Lemma 5.4.1 such that

$$\liminf_{i\to\infty} \prod_{j=0}^{i} \frac{l_i - 2}{l_i} = \alpha.$$

Let $G = \langle \tau_0, \psi_0, \zeta, \psi\rangle$ be as described in Section 5.4.1, constructed with the sequence $\{l_i\}$. By Theorem 3.1.7 it is a branch group. The elements $\kappa_n, \rho_n \in \mathrm{Alt}(n)$ with

$$\kappa_n = \sigma_n^{-2}\tau_n\sigma_n^2 = ((n-4)(n-3)(n-2)) \quad \text{and} \quad \rho_n = \tau_n^2\sigma_n = (1\dots(n-2))$$

generate the subgroup $\mathrm{Alt}(n-2) \leq \mathrm{Alt}(n)$. We use this to construct subgroups acting on $l_i - 2$ points of order $(l_i-2)!$ on each level. This gives us $\mathrm{Alt}\,(l_i-2) \leq \mathrm{Alt}\,(l_i)$ and we prove that the closure of the spinal subgroup

$$H = \langle \kappa_0, \rho_0, \xi, \theta\rangle$$

with $\xi_n = \left(\xi_{n+1}, \kappa_{l_{n+1}}, 1, \dots, 1\right)_{n+1}$ and $\theta_n = \left(\theta_{n+1}, \rho_{l_{n+1}}, 1, \dots, 1\right)_{n+1}$ has dimension

$$\alpha = \dim_G(H) = \liminf_{i\to\infty} \prod_{k=0}^{i} \frac{l_k - 2}{l_k}$$

in $\bar{G} = \varprojlim_i \mathrm{Alt}\,(l_i) \wr \cdots \wr \mathrm{Alt}\,(l_0)$. The subgroup H is obviously finitely generated. We saw above that $\kappa_{l_0} = \sigma_{l_0}^{-2}\tau_{l_0}\sigma_{l_0}^2$ and $\rho_{l_0} = \tau_{l_0}^2\sigma_{l_0}$. It follows that $\xi = \psi^{-2}\zeta\psi^2$ and $\theta = \zeta^2\psi$. We obtain from Proposition 3.1.5 that

$$G/\operatorname{St}_G(i+1) = A_{l_i} \wr \cdots \wr A_{l_0}.$$

Further, it is easy to see that then

$$H/\left(\operatorname{St}_G(i+1) \cap H\right) = H/\operatorname{St}_H(i+1) = A_{(l_i-2)} \wr \cdots \wr A_{(l_0-2)}.$$

The formula for the dimension $\dim_G(H)$ of the closure of H in $\bar{G}$ is hence given by

$$\dim_G(H) = \liminf_{i\to\infty} \frac{\log\left(\left(\frac{(l_i-2)!}{2}\right)^{\prod_{j=0}^{i-1} l_j - 2} \cdot \dots \cdot \left(\frac{(l_0-2)!}{2}\right)\right)}{\log\left(\left(\frac{l_i!}{2}\right)^{\prod_{j=0}^{i-1} l_j} \cdot \dots \cdot \frac{l_0!}{2}\right)}. \tag{5.2}$$

We separate this into

$$\dim_G(H) = \liminf_{i\to\infty} \frac{-\log\left(2^{\sum_{j=0}^{i-1}\prod_{k=0}^{j}(l_k-2)}\right) + \log\left((l_i-2)!^{\prod_{k=0}^{i-1}(l_k-2)}\cdot\dots\cdot(l_0-2)!\right)}{-\log\left(2^{\sum_{j=0}^{i-1}\prod_{k=0}^{j}l_k}\right) + \log\left(l_i!^{\prod_{j=0}^{i-1}l_j}\cdot\dots\cdot l_0!\right)}.$$

Let us denote this fraction to be of the form

$$\frac{-\log A(i) + \log B(i)}{-\log C(i) + \log D(i)}. \tag{5.3}$$

This can be estimated separately as

$$\le \frac{\log B(i)}{-\log C(i) + \log D(i)} = \frac{1}{-\frac{\log C(i)}{\log B(i)} + \frac{\log D(i)}{\log B(i)}}.$$

Simple estimations yield that $\liminf \frac{\log C(i)}{\log B(i)} = 0$. We denote $\alpha_i = \prod_{j=0}^{i-1}\frac{l_j-2}{l_j}$ and concentrate on $\frac{\log D(i)}{\log B(i)}$ by computing $\liminf_{i\to\infty}\frac{\log B(i)}{\log D(i)}$ which can be written as

$$\liminf_{i\to\infty}\frac{\log B(i)}{\log D(i)} = \liminf_{i\to\infty}\frac{\prod_{j=0}^{i-1}(l_j-2)\cdot\log\left((l_i-2)!\cdot\dots\cdot(l_0-2)!^{\prod_{j=0}^{i-1}\frac{1}{l_j-2}}\right)}{\prod_{j=0}^{i-1}l_j\cdot\log\left(l_i!\cdot\dots\cdot l_0!^{\prod_{j=0}^{i-1}\frac{1}{l_j}}\right)}$$

$$= \liminf_{i\to\infty}\alpha_i\cdot\frac{\log\left((l_i-2)!\cdot\dots\cdot(l_0-2)!^{\prod_{j=0}^{i-1}\frac{1}{l_j-2}}\right)}{\log\left(l_i!\cdot\dots\cdot l_0!^{\prod_{j=0}^{i-1}\frac{1}{l_j}}\right)}. \tag{5.4}$$

We now use that $l_i! = l_i\cdot(l_i-1)\,(l_i-2)!$. Hence the denominator has two extra factors l_i and l_i-1 compared to the numerator. We then use that $l_k \le l_i$ for all $k \le i$ and $l_{i-1}-2 \ge i$ and hence

$$(l_k-2)!^{\prod_{j=k}^{i-1}(l_j-2)^{-1}} \le e\left(\frac{l_k-1}{e}\right)^{(l_k-1)\cdot\prod_{j=k}^{i-1}(l_j-2)^{-1}}$$

$$\le (l_k-1)^{\prod_{j=k}^{i-1}(l_j-2)^{-1}}\cdot(l_k-1)^{\prod_{j=k+1}^{i-1}(l_j-2)^{-1}} \le (l_k-1)^{2\cdot\prod_{j=k+1}^{i-1}(l_j-2)^{-1}}$$

$$\le (l_k-1)^{\frac{1}{i}} \le l_i^{\frac{1}{i}}$$

if $k < i-1$. This gives us

$$\frac{\log\left((l_i-2)!\cdot(l_{i-1}-2)!^{\frac{1}{l_{i-1}-2}}\cdots(l_0-2)!^{\prod_{j=0}^{i-1}\frac{1}{l_j-2}}\right)}{\log\left(l_i!\cdot l_{i-1}!^{\frac{1}{l_{i-1}}}\cdot\dots\cdot l_0!^{\prod_{j=0}^{i-1}\frac{1}{l_j}}\right)}$$

$$\leq \frac{\log\left((l_i-2)!\cdot e\cdot\left(\frac{l_{i-1}-1}{e}\right)^{\frac{l_{i-1}-1}{l_{i-1}-2}}\cdot l_i^{1/i}\cdot\dots\cdot l_i^{1/i}\right)}{\log\left(\left(l_i^{\frac{1}{i}}\right)^i(l_i-1)\,(l_i-2)!\cdot e\cdot\left(\frac{l_{i-1}}{e}\right)^{\frac{l_{i-1}}{l_{i-1}}}\cdots l_0!^{\prod_{j=0}^{i-1}\frac{1}{l_j}}\right)}.$$

We now use that

$$e\cdot\left(\frac{l_{i-1}-1}{e}\right)^{\frac{l_{i-1}-1}{l_{i-1}-2}}\leq l_{i-1}$$

because using that $l_{i-1}\leq l_{i-1}^{\frac{l_{i-1}-1}{l_{i-1}-2}}$ gives

$$\left(\frac{l_{i-1}-1}{l_i}\right)^{\frac{l_{i-1}-1}{l_{i-1}-2}}=\left(1-\frac{1}{l_i}\right)^{\frac{l_{i-1}-1}{l_{i-1}-2}}\leq 1$$

which gives that the fraction is less than or equal to 1 and hence

$$\dim_G(H)\leq\liminf_{i\to\infty}\alpha_i=\alpha.$$

For the other inequality we see from (5.3) that

$$\frac{-\log A(i)+\log B(i)}{-\log C(i)+\log D(i)}\geq\frac{-\log A(i)}{\frac{1}{2}\cdot\log D(i)}+\frac{1}{-\frac{\log C(i)}{\log B(i)}+\frac{\log D(i)}{\log B(i)}}.$$

The first limit can be seen to be $\liminf_{i\to\infty}\frac{-2\log A(i)}{\log D(i)}=0$ and we focus our computations on the second term as before. We continue our estimation in (5.4) and see that

$$\liminf_{i\to\infty}\alpha_i\cdot\frac{\log\left((l_i-2)!\cdot\dots\cdot(l_0-2)!^{\prod_{j=0}^{i-1}\frac{1}{l_j-2}}\right)}{\log\left(l_i!\cdot\dots\cdot l_0!^{\prod_{j=0}^{i-1}\frac{1}{l_j}}\right)}$$

$$\geq\alpha_i\cdot\frac{\log\left((l_i-2)!\cdot(l_{i-1}-2)^{\frac{1}{l_{i-1}-2}}\cdot\dots\cdot(l_0-2)!^{\prod_{j=0}^{i-1}\frac{1}{l_j-2}}\right)}{\log\left(l_i!\cdot l_{i-1}!^{\frac{1}{l_{i-1}}}\cdot\dots\cdot l_0!^{\prod_{j=0}^{i-1}\frac{1}{l_j}}\right)}.$$

We split up $l_k!=l_k\cdot(l_k-1)\cdot(l_k-2)!$ for all terms in the denominator and split the logarithm into a sum

$$\log\left(l_i!\cdot l_{i-1}!^{\frac{1}{l_{i-1}}}\cdot\dots\cdot l_0!^{\prod_{j=0}^{i-1}\frac{1}{l_j}}\right)=\log\left(l_i\cdot l_{i-1}^{\frac{1}{l_{i-1}}}\cdot\dots\cdot l_0^{\prod_{j=0}^{i-1}\frac{1}{l_j}}\right)$$

$$+\log\left((l_i-1)\cdot(l_{i-1}-1)^{\frac{1}{l_{i-1}}}\cdot\dots\cdot(l_0-1)^{\prod_{j=0}^{i-1}\frac{1}{l_j}}\right)$$

$$+\log\left((l_i-2)!\cdot(l_{i-1}-2)!^{\frac{1}{l_{i-1}}}\cdot\dots\cdot(l_0-2)!^{\prod_{j=0}^{i-1}\frac{1}{l_j}}\right).$$

We divide all summands in the denominator by the numerator and get

$$\liminf_{i\to\infty} \alpha_i \cdot \frac{\log\left((l_i-2)!\cdot\dots\cdot(l_0-2)!^{\prod_{j=0}^{i-1}\frac{1}{l_j-2}}\right)}{\log\left(l_i!\cdot\dots\cdot l_0!^{\prod_{j=0}^{i-1}\frac{1}{l_j}}\right)} \geq \alpha_i \cdot \frac{1}{T_1+T_2+1}$$

where

$$T_1 = \frac{\log\left(l_i \cdot l_{i-1}^{\frac{1}{l_{i-1}}}\cdot\dots\cdot l_0^{\prod_{j=0}^{i-1}\frac{1}{l_j}}\right)}{\log\left((l_i-2)!\cdot(l_{i-1}-2)!^{\frac{1}{l_{i-1}-2}}\cdot\dots\cdot(l_0-2)!^{\prod_{j=0}^{i-1}\frac{1}{l_j-2}}\right)}$$

and

$$T_2 = \frac{\log\left((l_i-1)\cdot(l_{i-1}-1)^{\frac{1}{l_{i-1}}}\cdot\dots\cdot(l_0-1)^{\prod_{j=0}^{i-1}\frac{1}{l_j}}\right)}{\log\left((l_i-2)!\cdot(l_{i-1}-2)!^{\frac{1}{l_{i-1}-2}}\cdot\dots\cdot(l_0-2)!^{\prod_{j=0}^{i-1}\frac{1}{l_j-2}}\right)}.$$

Assuming $l_i \geq 5$ as stated in Lemma 5.4.1 for all $i \geq 0$ we can estimate

$$\sum_{j=0}^{n}\prod_{k=0}^{j}\frac{1}{l_k} \leq \sum_{k=1}^{n}\frac{1}{5^k} \leq \sum_{k=1}^{n}\frac{1}{2^k} = 1$$

with which we obtain the inequality

$$l_i \cdot l_{i-1}^{\frac{1}{l_{i-1}}}\cdot\dots\cdot l_0^{\prod_{j=0}^{i-1}\frac{1}{l_j}} \leq l_i\left(\cdot l_i^{\frac{1}{l_{i-1}}}\cdot\dots\cdot l_i^{\prod_{j=0}^{i-1}\frac{1}{l_j}}\right) \leq l_i \cdot l_i \leq l_i^2. \tag{5.5}$$

It is easy to see that $T_2 \leq T_1$ and so $T_1 + T_2 \leq 2T_1$. We use the estimate (5.5) in the numerator of T_1 and further

$$\log\left((l_i-2)!\cdot(l_{i-1}-2)!^{\frac{1}{l_{i-1}-2}}\cdot\dots\cdot(l_0-2)!^{\prod_{j=0}^{i-1}\frac{1}{l_j-2}}\right) \geq \log\left((l_i-2)!\right).$$

The assumption $l_i \geq 5$ further gives that $l_i - 2 \geq \frac{l_i}{2}$. Using

$$e\left(\frac{n}{e}\right)^n \leq n! \leq e\cdot\left(\frac{n+1}{e}\right)^{n+1}, \tag{5.6}$$

this gives us

$$T_1 \leq \frac{2\log l_i}{\log(l_i-2)!} \leq \frac{2\log l_i}{\log\left(e\left(\frac{l_i-2}{e}\right)^{l_i-2}\right)} \leq 2\cdot\frac{\log l_i}{(l_i-2)\log(l_i-2)} \leq \frac{8}{l_i} \longrightarrow 0.$$

Hence $\dim_G(H) \geq \alpha$, and so $\dim_G(H) = \alpha$. For the last part, we observe that H is a finitely generated branch group acting on the tree with defining sequence $\{l_n - 2\}_{n\geq 0}$. □

The proof of the Theorem 5.4.3 determines a sequence $\{l_i\}_{i\geq 0}$. We fix this sequence for the rest of this Chapter.

One of the results in the paper on Hausdorff dimensions by Barnea and Shalev [BS97] is the following:

Lemma 5.4.4. *Let G be an infinite profinite group. If H is an open subgroup of G, then $\dim_G(H) = 1$ and if H is a finite subgroup in G, then $\dim_G(H) = 0$.*

The compactness of the closure of G now yields that subgroups of finite index have full dimension in G.

Lemma 5.4.5. *Let H be a subgroup of finite index in a finitely generated branch group G. Then $\dim_G(H) = 1$.*

Proof. The closure of H is always open in G. Hence we can apply Lemma 5.4.4 to conclude that $\dim_G(H) = 1$. □

Lemma 5.4.6. *Let G be a finitely generated branch group and H and K subgroups such that*

$$\dim_G(H) = \alpha, \ \ \dim_H(K) = \beta.$$

If we assume that H is again a branch group and further assume that the limit inferiors in the dimension formulas are limits, then $\dim_G(K) = \alpha \cdot \beta$.

Proof. This follows straight from

$$dim_G(K) = \lim_{n\to\infty} \frac{\log(|K/\operatorname{St}_K(n)|)}{\log(|G/\operatorname{St}_G(n)|)} = \lim_{n\to\infty} \frac{\log(|H/\operatorname{St}_H(n)|)}{\log(|G/\operatorname{St}_G(n)|)} \cdot \frac{\log(|K/\operatorname{St}_K(n)|)}{\log(|H/\operatorname{St}_H(n)|)}$$

$$= \lim_{n\to\infty} \frac{\log(|H/\operatorname{St}_H(n)|)}{\log(|G/\operatorname{St}_G(n)|)} \cdot \lim_{n\to\infty} \frac{\log(|K/\operatorname{St}_K(n)|)}{\log(|H/\operatorname{St}_H(n)|)} = \dim_G(H) \cdot \dim_H(K) = \alpha \cdot \beta.$$

because the limit of both products exists by assumption. □

Let $\mathcal{T}\left(\{l_i\}_{i\in\mathbb{N}_0}\right)$ be the set of rational numbers

$$\mathcal{T}\left(\{l_i\}_{i\in\mathbb{N}_0}\right) = \left\{ q \ \mid \ q \in [0,1]\cap\mathbb{Q}, \ \exists \left\{j_1, \ldots, j_{r_q}\right\} \subset \mathbb{N} \ \text{with} \ q \cdot \prod_{k=1}^{r_q} l_{j_k} \in \mathbb{Z} \right\}, \tag{5.7}$$

the set of all rational numbers $q \in [0,1]\cap\mathbb{Q}$ such that q can be written as a fraction with only prime factors that also occur in any of the l_{j_k}.

Proposition 5.4.7. *Let G be a finitely generated branch group defined as described by the sequence $\{l_i\}_{i\in\mathbb{N}_0}$. For every $\delta \in \mathcal{T}\left(\{l_i\}_{i\in\mathbb{N}_0}\right)$ there exists a finitely generated subgroup H with $\dim_G(H) = \delta$.*

Proof. We follow a similar idea as Klopsch in his thesis ([Klo99]), using the rigid level stabilizers of G. Those have, by the hypothesis that G is a branch group, finite index in G, hence are again finitely generated. The subgroup $\mathrm{rst}_G(n)$ is the direct product $\mathrm{rst}_G(n) = \prod_{i=1}^{m_n} \mathrm{rst}_G(v)$ where v is a vertex of level n. It follows straight from the notion of a Hausdorff dimension in branch groups that $H = \prod_{i=1}^{k} \mathrm{rst}_G(v)$ has dimension $\dim_G(H) = \frac{k}{m_n}$ in G. The desired dimension δ can be written as $\delta = \frac{p}{q} = \frac{1}{m_n} \cdot \frac{m_n p}{q} = \frac{\zeta \cdot p}{m_n}$ with $\zeta = \frac{m_n}{q}$ for every $n \geq 0$. By assumption there exists n_0 such that $\zeta \in \mathbb{Z}$ for all $n \geq n_0$. Choosing $H = \prod_{i=1}^{\zeta\cdot p} \mathrm{rst}_G(v)$ for a vertex v of level n_0 hence gives $\dim_G(H) = \delta$. □

We now see that a good choice of the sequence $\{l_i\}_{i\geq 0}$ allows the construction of a richer spectrum as remarked in 5.4.2. Using Proposition 5.4.7 we can then obtain a more detailed description of the finitely generated Hausdorff spectrum of G, where we again use the definition of $\mathcal{T}\left(\{l_i\}_{i\in\mathbb{N}_0}\right)$ from (5.7). Together with the construction of a subgroup of irrational dimension in G we can describe a set of values contained in the finitely generated Hausdorff spectrum of G.

Theorem 5.4.8. *For every $\alpha \in [0, 1]$ there exists a branch group G_α such that*

$$\mathcal{T}\left(\{l_i - 2\}_{i\in\mathbb{N}_0}\right)_\alpha \cup \mathcal{T}\left(\{l_i\}_{i\in\mathbb{N}_0}\right) \subseteq \mathrm{spec}_H^{fg}(G),$$

where $\mathcal{T}\left(\{l_i - 2\}_{i\in\mathbb{N}_0}\right)_\alpha = \left\{l \cdot \alpha \mid\ l \in \mathcal{T}\left(\{l_i - 2\}_{i\in\mathbb{N}_0}\right)\right\}$.

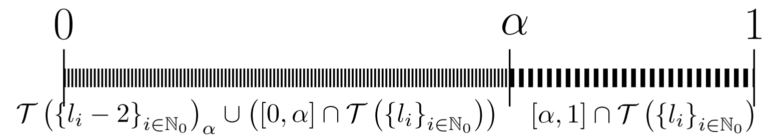

Figure 5.2: Hausdorff spectrum of G_α.

Proof. If $\alpha \in \mathcal{T}\left(\{l_i\}_{i\in\mathbb{N}_0}\right)$, then we apply Proposition 5.4.7. Otherwise, Theorem 5.4.3 yields that there exists a finitely generated subgroup H with $\dim_{G_\alpha}(H) = \alpha$, that is itself a branch group defined by the sequence $\{l_i - 2\}_{i\in\mathbb{N}_0}$. Therefore by Proposition 5.4.7 there exists $K \leq H$ with $\dim_H(K) = \delta$ for every $\delta \in \mathcal{T}\left(\{l_i - 2\}_{i\in\mathbb{N}_0}\right)$. Lemma 5.4.6 now asserts that $\dim_{G_\alpha}(K) = \alpha \cdot \delta \in \mathcal{T}\left(\{l_i - 2\}_{i\in\mathbb{N}_0}\right)_\alpha$. □

5.5 Open Questions

Question 5.5.1. Do there exist spinal groups whose finitely generated Hausdorff spectrum is contained in $[0,1] \cap \mathbb{Q}$?

We suspect that a construction as in Chapter 2 may give rise to groups whose Hausdorff spectrum is purely rational. In either case, it is however clear that the spectrum can only contain countably many values, as there are only countably many finitely generated subgroups of a finitely generated group.

Appendix A

Selected Topics

This Chapter is meant to give more information on the concepts used in this thesis. We will define and discuss profinite groups in Section A.1 before we move on to wreath products in Section A.2. We will briefly discuss the modular law and Mathieu groups (Sections A.3 and A.4) before we finish this Chapter with the prime number theorem in Section A.5.

A.1 Profinite Groups

In this Section we give the definition and a quick overview of profinite groups. A more detailed survey can be found in the books by Wilson [Wil98] and Ribes and Zalesskii [RZ10].

A.1.1 Inverse Limits

A *directed set* is a partially ordered set I such that for all $i_1, i_2 \in I$ there is an element $j \in I$ for which $i_1 \leq j$ and $i_2 \leq j$. An *inverse system* (X_i, φ_{ij}) of topological spaces indexed by a directed set I consists of a family $(X_i | i \in I)$ of topological spaces and a family

$$(\varphi_{ij} : X_j \to X_i | \ i, j \in I, i \leq j)$$

of continuous maps such that φ_{ii} is the identity map id_{X_i} for each i and $\varphi_{ij}\varphi_{jk} = \varphi_{ik}$ whenever $i \leq j \leq k$.

Now let (X_i, φ_{ij}) be an inverse system of topological spaces and let Y be a topological space. We shall call a family $(\psi_i : Y \to X_i | i \in I)$ of continuous maps *compatible* if $\varphi_{ij}\psi_j = \psi_i$ whenever $i \leq j$. This condition can be expressed diagrammatically as the requirement that each of the diagrams in Figure A.1 is commutative.

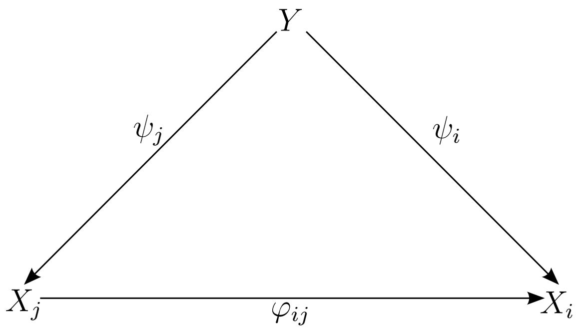

Figure A.1: Commutative diagram.

An *inverse limit* (X, φ_i) of an inverse system (X_i, φ_{ij}) of topological spaces (respective groups, rings) is a topological space (respective group, ring) X together with a compatible family $(\varphi_i : X \to X_i)$ of continuous maps (respective continuous homomorphisms) with the following universal property: whenever $(\psi_i : Y \to X_i)$ is a compatible family of continuous maps from a space Y (respective of continuous homomorphisms from a group or ring Y), there is a unique continuous map (respective continuous homomorphism) $\psi : Y \to X$ such that $\varphi_i \psi = \psi_i$ for each i. Thus we require that there is a unique ψ such that each of the diagrams in Figure A.2 is commutative.

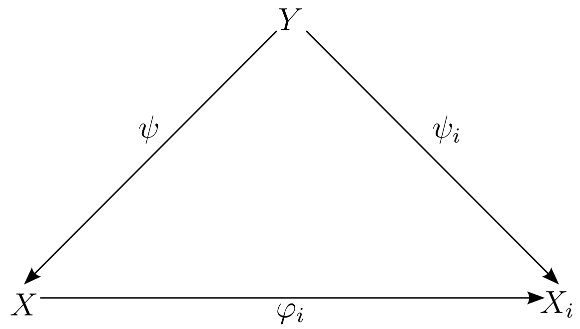

Figure A.2: Universal property of inverse limits.

The following result [Wil98] shows that inverse limits exist and are (in an appropriate sense) unique.

Proposition A.1.1. *Let (X_i, φ_{ij}) be an inverse system, indexed by I.*

(a) If $\left(X^{(1)}, \varphi_i^{(1)}\right)$ and $\left(X^{(2)}, \varphi_i^{(2)}\right)$ are inverse limits of the inverse system, then there is an isomorphism $\bar{\varphi} : X^{(1)} \to X^{(2)}$ such that $\varphi_i^{(2)} \bar{\varphi} = \varphi_i^{(1)}$ for each i. (If (X_i, φ_{ij}) is just an inverse system of spaces then $\bar{\varphi}$ is just a homeomorphism.)

(b) Write $C = \prod_{i \in I} X_i$ *and for each* i *write* π_i *for the projection map from* C *to* X_i*. Define*

$$X = \{c \in C | \ \varphi_{ij}\pi_j(c) = \pi_i(c) \text{ for all } i, j \text{ with } j \geq i\}$$

and $\varphi_i = \pi_i|_X$ *for each* i*. Then* (X, φ_i) *is an inverse limit of* (X_i, φ_{ij})*.*

(c) If (X_i, φ_{ij}) *is an inverse system of topological groups and continuous homomorphisms, then* X *is a topological group and the maps* φ_i *are continuous homomorphisms.*

We will now give further properties of inverse limits as can be found for example in [Wil98].

Proposition A.1.2. *[Wil98, 1.1.5] Let* (X_i, φ_{ij}) *be an inverse system, indexed by* I*, and write* $X = \varprojlim_i X_i$*.*

(a) If each X_i *is Hausdorff, so is* X*.*

(b) If each X_i *is totally disconnected, so is* X*.*

(c) If each X_i *is Hausdorff, then* X *is closed in the Cartesian product* $C = \prod_{i \in I} X_i$*.*

(d) If each X_i *is compact and Hausdorff, so is* X*.*

(e) If each X_i *is a non-empty compact Hausdorff space, then* X *is non-empty.*

The following Proposition will lead to the construction of profinite groups.

Proposition A.1.3. *[Wil98, 1.1.7] Let* X *be a compact Hausdorff totally disconnected space. Then* X *is the inverse limit of its discrete quotient spaces.*

A.1.2 Profinite Groups

We call a family I of normal subgroups of an arbitrary group G a *filter base* if for all $K_1, K_2 \in I$ there is a subgroup $K_3 \in I$ which is contained in $K_1 \cap K_2$. We quote two technical results from [Wil98] before we define profinite groups.

Proposition A.1.4. *[Wil98, 1.2.1] Let* (G, φ_i) *be an inverse limit of an inverse system* (G_i) *of compact Hausdorff topological groups and let* $L \lhd_o G$ *be an open normal subgroup of* G*. Then* $\ker \varphi_i \leq L$ *for some* i*. Consequently* G/L *is isomorphic (as a topological group) to a quotient group of a subgroup of some* G_i*, and if in addition each map* φ_i *is surjective then* G/L *is isomorphic to a quotient group of some* G_i*.*

Proposition A.1.5. *[Wil98, 1.2.2] Let G be a topological group and I a filter base of closed normal subgroups, and for $K, L \in I$ define $K \leq' L$ if and only if $L \leq K$. Thus I is directed with respect to the order $\leq'$ and the surjective homomorphisms $q_{KL} : G/L \to G/K$, defined for $K \leq' L$, make the groups G/K into an inverse system. Write $\left(\hat{G}, \varphi_K\right) = \varprojlim G/K$. There is a continuous homomorphism $\theta : G \to \hat{G}$ with kernel $\bigcap_{K \in I} K$, with image a dense subgroup of $\hat{G}$, and such that $\varphi_K \theta$ is the quotient map from G to G/K for each $K \in I$. If G is compact then θ is surjective; if G is compact and $\bigcap_{K \in I} K = 1$ then θ is an isomorphism of topological groups (it is an isomorphism of groups and a homeomorphism).*

We have now established the necessary setting for the definition of a profinite group.

Let $\mathcal{C}$ be a class groups. We assume that $\mathcal{C}$ is closed under taking normal subgroups, quotient groups and finite subdirect products, and that $\mathcal{C}$ contains a non-trivial group. Let $\mathcal{C}$ also be closed with respect to taking isomorphic images. Thus if $\mathcal{C}$ is a class and if $F_1 \in \mathcal{C}$ and $F_2 \simeq F_1$, then $F_2 \in \mathcal{C}$.

We call F a $\mathcal{C}$-group if $F \in \mathcal{C}$, and we call G a *pro-$\mathcal{C}$ group* if it is an inverse limit of $\mathcal{C}$-groups. Some important classes are the class of all finite groups, the class of all finite p-groups where p is a fixed prime and the class of all finite cyclic groups. An inverse limit of finite p-groups is called a *pro-p group*, and an inverse limit of finite cyclic groups is called a *procyclic group*. We have the following equivalences as proved in [Wil98]. We will write $H \leq_0 G$ to indicate that H is an open subgroup of G.

Theorem A.1.6. *[Wil98, 1.2.3] Let $\mathcal{C}$ be a class of finite groups which is closed for subgroups and direct products, and let G be a topological group. The following are equivalent:*

(i) *G is a pro-$\mathcal{C}$ group;*

(ii) *G is isomorphic (as a topological group) to a closed subgroup of a Cartesian product of $\mathcal{C}$-groups;*

(iii) *G is compact and $\bigcap (N|\ \ N \leq_o G,\ \ G/N \in \mathcal{C}) = 1$;*

(iv) *G is compact and totally disconnected, and for every $L \lhd_o G$ there is a subgroup $N \lhd_o G$ with $N \leq L$ and $G/N \in \mathcal{C}$.*

If in addition $\mathcal{C}$ is closed for quotients then (iv) can be replaced by
(iv') G is compact and totally disconnected, and $G/L \in \mathcal{C}$ for every $L \lhd_o G$.

The following Theorem describes how a given profinite group, and its subgroups and quotient groups, can be represented explicitly as inverse limits.

Theorem A.1.7. *[Wil98, 1.2.5]*

(a) Let G be a profinite group. If I is a filter base of closed normal subgroups of G such that $\bigcap(N|N \in I) = 1$ then

$$G \simeq \varprojlim_{N \in I} G/N.$$

Moreover

$$H \simeq \varprojlim_{N \in I} H/(H \cap N)$$

for each closed subgroup H and

$$G/K \simeq \varprojlim_{N \in I} G/KN$$

for each closed normal subgroup K.

(b) If $\mathcal{C}$ is a class of finite groups which is closed for subgroups and direct products, then closed subgroups, Cartesian products and inverses limits of pro-$\mathcal{C}$ groups are pro-$\mathcal{C}$ groups. If in addition $\mathcal{C}$ is closed for quotients, then quotient groups of pro-$\mathcal{C}$ groups by closed normal subgroups are pro-$\mathcal{C}$ groups.

A.2 Wreath Products

In this Section we give the definition of permutational wreath products as can be found for example in [Rob96].

A.2.1 Definition

Let H and K be permutation groups acting on sets X and Y respectively. We shall describe a very important way of constructing a new permutation group called the wreath product of H and K.

If $\gamma \in H, y \in Y$ and $\kappa \in K$ define permutations $\gamma(y)$ and κ^* of $Z = X \times Y$ by the rules

$$\gamma(y) : \begin{cases} (x,y) & \mapsto (x\gamma, y), \\ (x,u) & \mapsto (x,u) \ \text{ if } u \neq y, \end{cases}$$

and

$$\kappa^* : (x,y) \mapsto (x, y\kappa)$$

it can be easily verified that $\gamma^{-1}(y) = \gamma(y)^{-1}$ and $\left(\kappa^{-1}\right)^* = (\kappa^*)^{-1}$, so that $\gamma(y)$ and κ^* are in fact permutations. The functions

$$f_y : \begin{cases} H & \longrightarrow \mathrm{Sym}(Z) \\ \gamma & \mapsto \gamma(y), \end{cases}$$

with y a fixed element of Y, and

$$g : \begin{cases} K & \longrightarrow \mathrm{Sym}(Z) \\ \kappa & \mapsto \kappa^* \end{cases}$$

are monomorphisms. Denote their images by $f_y(H) = H(y)$ and $g(K) = K^*$. Then the *wreath product* of H and K is the permutation group on Z generated by K^* and all the $H(y), y \in Y$. This is written as

$$H \wr K = \langle H(y), K^* | y \in Y \rangle .$$

We observe that $(\kappa^*)^{-1} \gamma(y)\kappa^*$ maps $(x, y\kappa)$ to $(x\gamma, y\kappa)$ and fixes (u, v) if $v \neq y\kappa$. Hence by definition

$$(\kappa^*)^{-1} \gamma(y)\kappa^* = \gamma(y\kappa) \quad \text{and} \quad (\kappa^*)^{-1} H(y)\kappa^* = H(y\kappa). \tag{A.1}$$

In addition, when $y \neq v$, the permutations $\gamma(y)$ and $\gamma(v)$ cannot move the same element of Z. It follows that the elements in $H(y)$ generate their direct product. We denote this by

$$B = \prod_{y \in Y} H(y)$$

and call B the *base group* of the wreath product.

According to (A.1) conjugation by an element $\kappa^* \in K^*$ permutes the direct factors $H(y)$ in the same way as κ permutes the elements of Y. Since elements of K^* and B cannot move the same element of Z, we must have $K^* \cap B = 1$. Also of course $B \lhd H \wr K$ and $H \wr K = K \cdot B$. Thus $H \wr K$ is the semidirect product of B by K^* in which the automorphism of B produced by an element of K^* is given by (A.1).

Let G and H be permutation groups on sets X and Y respectively. A *similarity* from G to H is a pair (α, β) consisting of an isomorphism $\alpha : G \to H$ and a bijection $\beta : X \to Y$ which are related by the rules

$$g\beta = \beta g^{\alpha}, \quad g \in G.$$

When $X = Y$, this says that $g^{\alpha} = \beta^{-1} g \beta$ where now $\beta \in \mathrm{Sym}(X)$. Thus two permutation groups G and H on X are similar if and only if some $\beta \in \mathrm{Sym}(X)$ conjugates G into H.

Lemma A.2.1. *[Rob96, 1.6.4] Let H and K be groups acting on sets X and Y respectively.*

(a) If H and K are transitive, so is $H \wr K$.

(b) Let L be a permutation group on a set U. Let

$$\beta : (X \times Y) \times U \longrightarrow X \times (Y \times U)$$

be the bijection

$$((x, y), u) \mapsto (x, (y, u))$$

and let α be the function $\tau \mapsto \beta^{-1} \tau \beta$. Then (α, β) is a similarity from $(H \wr K) \wr L$ to $H \wr (K \wr L)$.

A.2.2 Standard Wreath Products

We will now concentrate on a special case of the wreath product. If H and K are arbitrary groups, we can think of them as permutation groups on their underlying sets via the right regular representation and form their wreath product $W = H \wr K$. This is called the *standard wreath product*. Its base group is $\prod_{k \in K} H_k$ where each $H_k \simeq H$ and $(H_k)_{k'} = H_{kk'}$.

The standard wreath product can for example be used to describe the Sylow subgroups of the symmetric group S_n.

Theorem A.2.2 (Kalužnin)**.** *[Rob96, 1.6.19]* *(a) A Sylow p-subgroup of S_{p^r} is isomorphic with the standard wreath product*

$$W(p,r) = (\dots(\mathbb{Z}_p \wr \mathbb{Z}_p)\dots) \wr \mathbb{Z}_p,$$

the number of factors being r.

(b) If the positive integer n is written in the form $a_0 + a_1 p + \cdots + a_{i-1}p^{i-1}$ where a_j is integral and $0 \leq a_j < p$, a Sylow p-subgroup of S_n is isomorphic with the direct product of a_1 copies of $W(p,1)$, a_2 copies of $W(p,2), \dots$ and a_{i-1} copies of $W(p, i-1)$.

We now describe iterated wreath products. The following lemma is a recursive criterion of when a group acts as the iterated wreath product on a set S.

Lemma A.2.3. *A group G acts on a set S of $m_i = \prod_{j=1}^{i} l_j$ points as the iterated wreath product $G_i \wr G_{i-1} \wr \cdots \wr G_1$ of finite groups G_i if*

1. *the action of G on S is imprimitive,*
2. *every group G_j acts transitively on a set of l_j points,*
3. *$G_{i-1} \wr \cdots \wr G_1$ acts on the m_i/l_i blocks of size l_i as the iterated wreath product.*

A.2.3 The Lamplighter Group

We briefly describe the group $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ and give a short sketch of why it cannot be finitely presented.

Lemma A.2.4. *The lamplighter group L does not contain a non-abelian free group.*

Proof. This can be seen immediately because L is metabelian. □

Lemma A.2.5. *The lamplighter group $L = (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ has a presentation*

$$L = \left\langle a, t \middle| \; a^2 = 1, \left[t^{-m}at^m, t^{-n}at^n\right] = 1, m, n \in \mathbb{N} \right\rangle.$$

We can use this presentation to give a short argument from [Osi02] why L cannot have a finite presentation.

Proposition A.2.6. *The lamplighter group L is not finitely presented.*

Proof. Set

$$H_n = \left\langle a, t | \ a^2 = 1, \left[t^{-i}at^i, t^{-j}at^j\right] = 1, |i| \leq n, |j| \leq n \right\rangle.$$

The groups H_n can be represented with $a_i = t^{-i}at$ as follows:

$$H_n = \left\langle a_{-n}, \dots, a_n, t \ | \ a_i^2 = [a_i, a_j] = 1, i, j = -n, \dots, n, t^{-1}a_it = a_{i+1}, i = -n, \dots, n-1 \right\rangle.$$

Thus, H_n is an HNN-extension of the finite group

$$\left\langle a_{-n}, \dots, a_n \ | \ a_i^2 = [a_i, a_j] = 1, i, j = -n, \dots, n \right\rangle \simeq (\mathbb{Z}/2\mathbb{Z})^{2n+1}.$$

By Brittons Lemma [LS70] on HNN-extensions the elements a_{-n} and $t^{-1}a_nt$ generate the free group of rank 2, contradicting Lemma A.2.4. Hence L cannot be finitely presented. □

We note that we can generalise the results from above to $\mathbb{Z} \wr \mathbb{Z}$, in particular this group is not large and not finitely presentable. We use that we can present $\mathbb{Z} \wr \mathbb{Z}$ as

$$\mathbb{Z} \wr \mathbb{Z} = \left\langle \dots, a_{-n}, \dots, a_n, \dots, t \ | \ [a_i, a_j] = 1, a_k^t = a_{k+1} \right\rangle.$$

Lemma A.2.7. *The group $\mathbb{Z} \wr \mathbb{Z}$ has infinite virtual first Betti number but is not large.*

Proof. It follows immediately from the definition of $W = \mathbb{Z} \wr \mathbb{Z}$ as a metabelian group that it cannot be large. It is left to find a sequence $H_n \leq_f W$ of subgroups of W such that the limit $\lim_{n\to\infty} rk\left(H_n^{ab}\right) = \infty$. We define subgroups

$$H_i = \left\langle \dots, a_{-n}, \dots, a_n, \dots, t^i \right\rangle.$$

Then each H_i has index i in W, but $H_i^{ab} = \mathbb{Z}^{i+1}$ and so $\text{rk}\left(H_i^{ab}\right) = i + 1$. Hence

$$\sup\left\{\text{rk}\left(H^{ab}\right) | \ H \leq_f W\right\} = \infty.$$

□

This Lemma shows that in addition to the group described in Chapter 2 the group $\mathbb{Z} \wr \mathbb{Z}$ is an example for a group that is not large but has infinite virtual first betti number. Such examples are in general hard to obtain and those two groups are the only two for which this property has been shown.

A.3 Modular Law

We say two subgroups H and K of a group G permute, if $[H, K] = 1$ or equivalently if $HK = KH$. This is in fact precisely the condition for their product HK to be a subgroup.

Lemma A.3.1. *[Rob96, 1.3.13] If H and K are subgroups of a group G, then HK is a subgroup if and only if H and K permute. In this event $HK = \langle H, K\rangle = KH$.*

Proposition A.3.2 (Dedekind's Modular Law)**.** *Let H, K, L be subgroups of a group and assume that $K \subseteq L$. Then*

$$(HK) \cap L = (H \cap L)K.$$

In particular, if H and K permute, then $\langle H, K\rangle \cap L = \langle H \cap L, K\rangle$.

Figure A.3 depicts this Proposition with $D = \langle H, K\rangle \cap L = \langle H \cap L, K\rangle$.

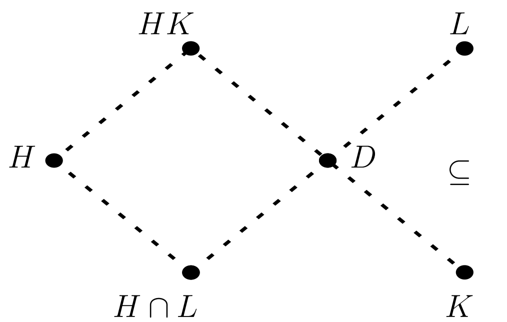

Figure A.3: Dedekind's Modular Law.

A.4 Mathieu Groups

In this Section we describe the five Mathieu groups $M_{11}, M_{12}, M_{22}, M_{23}$ and M_{24}. Those occur in Theorem 5.3.1 as one of the cases of primitive subgroups of the symmetric group. We refer the reader to [DM96, Chapter 6] for more details on them.

These groups were first described in papers of Mathieu in 1861 and 1873. They are the only finite 4- and 5-transitive groups which are not alternating or symmetric.

(i) M_{12}: this group is *sharply* 5-transitive of degree 12, which means that each 5-point stabilizer is the identity.

(ii) M_{11} occurs as a point stabilizer of M_{12}. So it is in turn sharply 4-transitive on 11 points.

(iii) M_{24} is 5-transitive of degree 24,

(iv) M_{23} is a 4-transitive one-point stabilizer of M_{24} and

(v) M_{22} is a 3-transitive two-point stabilizer of M_{24}.

Proposition A.4.1. *All five Mathieu groups are simple.*

In fact they constitute the earliest known examples of sporadic simple groups.

Lemma A.4.2. *The groups $M_{11}, M_{12}, M_{23}, M_{22}$ are all subgroups of M_{24}.*

Those groups can be constructed using Steiner systems as described in [DM96].

A.5 Prime Number Theorem

In this Section we state the Prime Number Theorem A.5.1 and give Tschebycheff's inequality (A.2). For more details and proofs we refer the reader to [Rib04, Chapter 4].

A.5.1 The Function $\pi(x)$

For every real number $x > 0$ the function $\pi(x)$ denotes the number of primes p such that $p \leq x$. We call $\pi(x)$ the *prime counting function.* The basic idea in the study of the function $\pi(x)$ is to compare it with functions that are both classical and computable, and such that their values are as close as possible to the values of $\pi(x)$.

The notation $f(x) \sim h(x)$ means that $\lim_{x\to\infty}(f(x)/h(x)) = 1$. The functions $f(x)$ and $h(x)$ are then said to be *asymptotically equal* as x tends to infinity. We have the following strong Theorem:

Theorem A.5.1 (Prime Number Theorem)**.** *The prime number function $\pi(x)$ is asymptotically equal to*

$$\pi(x) \sim \frac{x}{\log x}.$$

It was first conjectured by Gauss in 1792 and then by Legendre in 1796 and proved in 1896 by Jacques Hadamard and independently in 1898 by de la Vallée-Poussin.

Tschebycheff obtained the inequality (A.2) around 1850:

Theorem A.5.2 (Tschebycheff)**.** *The prime number function $\pi(x)$ can be bounded by*

$$(C' - \epsilon)\frac{x}{\log x} < \pi(x) < (C + \epsilon)\frac{x}{\log x}, \tag{A.2}$$

where

$$C' = \log \frac{2^{1/2}3^{1/3}5^{1/5}}{30^{1/30}} = 0.92129\ldots \quad \text{and} \quad C = \frac{6}{5}C' = 1.10555\ldots.$$

Moreover, Tschebycheff showed that if the limit of

$$\frac{\pi(x)}{x/\log x}$$

exists (as $x \to \infty$), it must be equal to 1.

In 1892, Sylvester was able to refine Tschebycheff's method and obtained explicit estimates which are used in Chapter 5.

Theorem A.5.3 (Sylvester)**.** *The prime number function $\pi(x)$ satisfies*

$$0.95695\frac{x}{\log x} < \pi(x) < 1.04423\frac{x}{\log x}$$

for every x sufficiently large.

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