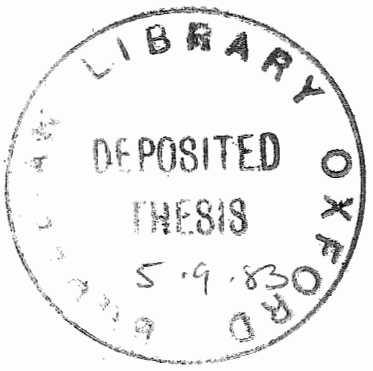
Finitely Presented Infinite Simple Groups

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A thesis submitted in partial fulfilment of the requirements for the degree of Doctor of Philosophy in the University of Oxford.

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To My Parents

Finitely Presented Infinite Simple Groups

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ABSTRACT

This thesis is mainly concerned with the development of a procedure to construct finitely presented infinite simple groups.

The finitely presented groups, $G_{n,1}$, constructed by G.Higman [5] are extended by certain subgroups, H, of the inverse limit of wreath products, $S_{n-1}(\text{wr } S_{n-1})^{\omega}$. It is shown that if H is defined by a finite set of expansion rules (see Chapter 2) then the group $\langle G_{n,1},H\rangle$ is finitely presented if H is finitely presented. The commutator subgroup, $\langle G_{n,1},H\rangle'$, is shown to be a simple group and it is proved that an integer m and a set of expansion rules defining H can be chosen such that $\langle G_{m,1},H\rangle'$ is a finitely presented infinite simple group.

Let $GL(n,\mathbb{Z})$ be the general linear group of $n\times n$ dimensional matrices over the integers. The finitely presented simple group, $\langle G_{m,1}, \mathbb{Z}^n.GL(n,\mathbb{Z})\rangle'$, is constructed, where $\mathbb{Z}^n$ is the free Abelian group of finite rank, n. So $GL(n,\mathbb{Z})$ can be embedded in a finitely presented simple group, for all n.

Let X_p be the ring of rationals whose denominators are divisible only by primes belonging to the finite set P. If A is a countable Abelian group whose torsion factor group is a subgroup of a finitely generated torsion free X_p-module, then it is shown that A can be embedded in a finitely presented simple group.

It is also noted, at the end of the discussion of linear groups, that if there exists a finitely presented linear group, over the integers, with unsolvable conjugacy or orbit problem, then a finitely presented infinite simple group with unsolvable conjugacy problem can be constructed.

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Chapter 1

Introduction

In 1974 W.W.Boone and G.Higman [2] proved that every finitely generated group with solvable word problem can be embedded in a simple subgroup of a finitely presented group. It is thus natural to ask whether every finitely generated group with solvable word problem can be embedded in a finitely presented simple group. Every finitely presented simple group has solvable word problem so if we could answer the above question positively then we would have a characterisation of the finitely generated subgroups of finitely presented simple groups. A finitely generated group would have solvable word problem precisely when it is a subgroup of a finitely presented simple group.

R.J.Thompson [9] has improved on the Boone-Higman result by showing that every finitely generated group with solvable word problem can be embedded in a finitely generated simple subgroup of a finitely presented group. The groups constructed in this thesis are based on the groups of permutations of infinite binary sequences constructed by R.J.Thompson. The approach adopted in this work does not require any knowledge of R.J.Thompson's methods but the following is a brief outline of his construction, using his notation.

Let K be a non-empty set of infinite binary sequences such that, for every finite binary sequence, σ, and for every infinite binary sequence, α, $\sigma\hat{}\alpha \in K$ if and only if $\alpha \in K$. [If $\sigma = (a_1,\ldots,a_n)$ and $\alpha = (b_1,\ldots,b_m,\ldots)$ where $a_i, b_j \in \{0,1\}$, then $\sigma\hat{}\alpha = (a_1,\ldots,a_n,b_1,\ldots,b_m,\ldots)$]. Define a partition set for K, Σ, to

be a set of finite binary sequences such that, for every $\alpha \in K$, there exists precisely one $\sigma \in \Sigma$ with the property that $\sigma\hat{}\beta = \alpha$, for some infinite binary sequence, β. If Σ_0 and Σ_1 are partition sets for K and if ψ is a one-to-one map from Σ_0 onto Σ_1 then we can associate a permutation, F, of K with the triple Σ_0, Σ_1, ψ by the rule $(\sigma\hat{}\beta)F = (\sigma\psi)\hat{}\beta$, for each $\sigma \in \Sigma_0$ and for each infinite binary sequence, β. The group of all permutations of K which can be defined in this way is denoted by Pa(K). The set of permutations of this type, whose associated partition sets, Σ_0 and Σ_1, are finite, forms a subgroup of Pa(K), denoted by Ft(K). R.J.Thompson proves that Ft(K) is simple and he also notes that if H is a group satisfying $Ft(K) \leq H \leq Pa(K)$ then the commutator subgroup, H', is simple and $Ft(K) \leq H'$. The analogue of this theorem, for the groups that we want to consider, is proved at the end of Chapter 2.

Returning to the question of embeddability in finitely presented simple groups, a major problem is the lack of examples of finitely presented infinite simple groups from which to gain insight into possible methods of attack. This thesis is mainly concerned with developing a construction procedure which will produce a fairly flexible class of such groups.

In 1974 G.Higman [5], starting with the work of R.J.Thompson described above, constructed an infinite family of finitely presented infinite simple groups. Some of these groups are constructed during the course of Chapter 2 but a general outline of the way in which they arise is given below.

Let $X = \{x_1,\ldots,x_r\}$ and $A = \{a_1,\ldots,a_n\}$ be distinct sets. Let W be the set of all finite words (or, equivalently, finite sequences) of the form $a_{i_1}a_{i_2}\ldots a_{i_k}$, on the letters A. Then W forms a semi-group under the operation of juxtaposition, the empty word is the identity of the semi-group. We define XW to be the set of all words of the form $x_i a_{i_1}\ldots a_{i_k}$ and $U \subseteq XW$ is said to be a subspace of XW if, for all words $x_i u \in U$ and for all $w \in W$, $x_i uw \in U$. The subspace, U, is said to be cofinite if the set $XW \backslash U$ is finite. The finitely presented simple groups constructed by G.Higman are commutator subgroups of groups of isomorphisms, $G_{n,r}$, between cofinite subspaces. Roughly speaking, R.J.Thompson's groups, Ft(K), correspond to the groups $G_{2,r}$.

In the present work we take $|X|=1$ and extend the groups $G_{n,1}$ by certain finitely presented subgroups, H, of the infinite wreath product, $S_{n-1}(\text{wr } S_{n-1})^{\omega}$. The main result of this thesis is that if H satisfies certain conditions (i.e. can be defined by a finite set of expansion rules, see Chapter 2) then the group $\langle G_{n,1},H\rangle$ is a finitely presented group. The analogue of the R.J.Thompson result gives that the commutator subgroup, $\langle G_{n,1},H\rangle'$, is simple. If H is defined by a finite set of expansion rules we show that we can find a set of expansion rules, also defining H, so that $\langle G_{m,1},H\rangle'$ has finite index in $\langle G_{m,1}, H\rangle$. Then the groups $\langle G_{m,1},H\rangle'$ are finitely presented infinite simple groups.

In Chapter 2 the groups $\langle G_{n,1},H\rangle$ are constructed using the concept of expansion rules and extending the definition of symbols introduced by G.Higman [5]. In Chapter 3 we show that if H is finitely presented then so is the group $\langle G_{n,1},H\rangle$. This is basically because every element of $\langle G_{n,1},H\rangle$ can be represented by a finite symbol and the rules for manipulating the symbols are essentially finite.

G.Higman has shown, in particular, that the linear group $GL(3,\mathbb{Z})$ cannot be embedded in any of the groups $G_{n,r}$. In Chapter 4 we show that we can embed $GL(n,\mathbb{Z})$ in a finitely presented simple group for any integer, n. Thus some of the groups constructed here are different from any of those constructed by G.Higman. By the orbit problem for a linear group, L, over the integers we mean the problem of deciding, for two given elements $t,s \in \mathbb{Z}^n$ whether or not there exists an $\ell \in L$ such that $s\ell = t$. We show at the end of Chapter 4 that if we can construct a finitely presented linear group, over the integers, which has unsolvable conjugacy problem then we can construct a finitely presented linear group with unsolvable orbit problem. We also show that if L is a finitely presented linear group with unsolvable orbit problem then the commutator subgroup, $\langle G_{m,1}, \mathbb{Z}^n L' \rangle$ is a finitely presented simple group with unsolvable conjugacy problem. Thus we have the theorem that if we can construct a finitely presented linear group with unsolvable conjugacy or orbit problem then we can construct a finitely presented simple group with unsolvable conjugacy problem.

Let X_P be the ring of rationals whose denominators contain only primes belonging to the finite set, P. In Chapter 5 we show that if A is a countable Abelian group whose torsion factor group is a subgroup of a finitely generated torsion free X_P-module, then A can be embedded in a finitely presented simple group. In particular, the additive group of rationals $\{a/_m b \mid a,b \in \mathbb{Z}\}$ can be embedded in a finitely presented simple group for each fixed integer, m. We would like to embed the whole of $\mathbb{Q}^+$, the additive group of the rationals, in a finitely presented simple group, but it is not at all clear how this could be done using the current methods.

Finally, we remark that recently D.Djoković [3] has constructed a finitely presented infinite simple group, not connected with the work described above. The finite subgroups of this group are all of bounded order, so it is quite different from any of the groups constructed here.

Chapter 2

The Construction of the Groups $\langle G_{n,1}, H \rangle$

The aim of this work is to construct finitely presented simple groups with 'interesting' subgroups. These are obtained by extending the groups $G_{n,1}$ described in [5]. In this chapter we introduce the idea of expansion rules and use finite sets of expansion rules to define finitely generated subgroups, H, of the infinite wreath product $S_{n-1}(wrS_{n-1})^{\omega}$. It is these subgroups which are used to extend the groups $G_{n,1}$. We then expand the idea of a symbol, introduced in [5], by adding a middle row containing elements from $F\langle T \rangle$, the free group on n generators. These new symbols represent the elements of $\langle G_{n,1}, H \rangle$, the extension of $G_{n,1}$.

2.1 Elementary definitions and constructions.

Take an alphabet $A=\{a_1,\ldots,a_n\}$ and assume that $n\geq 2$ to avoid trivialities. Let $W=W(A)$ be the set of all finite words in the a_i, including the empty word, 1, then W is a semi-group under the operation of juxtaposition. The empty word is the identity of the semi-group and two words $a_{i_1}\ldots a_{i_m}$ and $a_{j_1}\ldots a_{j_p}$ in W are equal in the semi-group if and only if $m=p$ and $i_k=j_k$ for all $1\leq k\leq m$.

A subset, Y, of W is said to be a subspace (of W) if it is closed under right multiplication by elements of W (i.e. if, for all $y\in Y$ and for all $w\in W$, $yw\in Y$). A subspace, Y, of W is inescapable (in W) if, given any $u\in W$, there exists some $w\in W$ such that $uw\in Y$. For example,

$Y = \{w \in W \mid w$ contains at least one $a_1\}$ is an inescapable subspace of W. The intersection of two inescapable subspaces of W is an inescapable subspace of W and if $Y \subseteq Z$ are subspaces of W then Z is inescapable in W if Y is.

A subspace, Z, of W is said to be cofinite (in W) if $W \backslash Z$ is finite. For any $z \in W$ if $y \in W$ satisfies $yw=z$ for some $w \in W$ then we call y an initial segment of z. If $w \neq 1$, then y is said to be a proper initial segment of z. For $z \in W$ the set $Z = \{y \in W \mid y$ is not a proper initial segment of $z\}$ is a cofinite subspace of W containing z. Any cofinite subspace of W is inescapable in W since there is a finite bound on the length (as a word in the a_i) of any element not in a given cofinite subspace. The intersection of two cofinite subspaces of W is also a cofinite subspace of W.

We will also need the concept of a cofinite basis. A finite set $U = \{u_1,\ldots,u_s\} \subseteq W$ is said to be a cofinite basis if $UW = \{u_i w \mid w \in W, 1 \leq i \leq s\}$ is a cofinite subspace and no non-trivial subset of U generates a cofinite space. Note that if Z is a cofinite subspace then the set $U = \{z \in Z \mid$ no proper initial segment of z belongs to $Z\}$ is a cofinite basis, so every cofinite subspace contains a cofinite basis, and in fact $UW = Z$. If $\{u_1,\ldots,u_s\}$ is a cofinite basis then so is $\{u_1,\ldots,u_{i-1},u_ia_1,\ldots,u_ia_n,u_{i+1},\ldots,u_s\}$ and given $z \in W$ there exists a cofinite basis containing z.†

A W-space homomorphism, ϕ, is a map between subspaces of W such that if $z\phi=y$ then $(zw)\phi=yw$ for all $w \in W$. A W-space isomorphism, θ, is a bijective W-space homomorphism. If the domain and range of θ are

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†Cofinite bases are discussed in more detail in [5], where these and other results are proved.

inescapable (cofinite) then θ is said to be an inescapable (cofinite) W-space isomorphism.

The groups $\mathcal{G}_{n,1}$ and $G_{n,1}$ which we will now construct are sets of inescapable and cofinite W-space isomorphisms.

Lemma 2.1 Every inescapable W-space isomorphism, θ^*, can be extended to a unique maximal such isomorphism, θ.

Proof: Let θ^* be an inescapable isomorphism with domain Y. Define the subset $Y' \subseteq W$ by

$$Y' = \{z \in W \mid \text{for some } y \in W, \text{ for all } w \in W, (zw)\theta^* = yw \text{ if } zw \in Y\},$$

then $Y \subseteq Y'$.

Y' is a subspace of W since, for $z \in Y'$ and $u \in W$, $zuw \in Y$ implies that $(zuw)\theta^* = yuw$ and so $zu \in Y'$. Thus, since Y is inescapable and $Y \subseteq Y'$, Y' is an inescapable subspace.

Define a map, θ, on Y' by $z\theta = y$ where $(zw)\theta^* = yw$ whenever $(zw)\theta^*$ is defined. If $z\theta = y = z'\theta$ choose w so that $zw, z'w \in Y$ and then, by definition of θ, $(zw)\theta^* = yw = (z'w)\theta^*$. Since θ^* is a bijection we must have $zw = z'w$ and hence $z = z'$. Thus θ is injective. If $z\theta = y$ then, for all $u \in W$, $(zuw)\theta^* = yuw$ whenever the left hand side is defined, and thus $(zu)\theta = yu$. The range of θ is a subspace and the range of θ^* lies within the range of θ. So the range of θ is inescapable and θ is an inescapable isomorphism.

Clearly, θ is an extension of θ^* so suppose that ϕ is an inescapable isomorphism which also extends θ^*. If $x\phi = z$ then for all w

such that $(xw)\theta^*$ is defined, $(xw)\theta^*=(xw)\phi=zw$. Hence, by definition, $x\epsilon Y'$ and $x\theta=z$. Thus θ extends ϕ and so θ is the required unique maximal extension of θ^*. ◀

If θ and ϕ are inescapable isomorphisms we define the composition $\theta\circ\phi$ by $y(\theta\circ\phi)=(y\theta)\phi$ whenever the right hand side is defined. Thus if θ and ϕ have domains Y and Z respectively then $\theta\circ\phi$ has domain $Y\wedge(Y\theta\wedge Z)\theta^{-1}$ and range $(Y\theta\wedge Z)\phi$. These are both inescapable subspaces of W and $\theta\circ\phi$ is an inescapable isomorphism. Thus we have that the set of maximal inescapable isomorphisms is closed under the operation $\theta\phi=(\theta\circ\phi)'$ where $(\theta\circ\phi)'$ is the unique maximal extension of $\theta\circ\phi$. It is easy to see that the set forms a group under this operation. Call this group $\mathcal{G}_{n,1}$.

The unique maximal extension of a cofinite isomorphism is still a cofinite isomorphism and the set of maximal cofinite isomorphisms forms a subgroup of $\mathcal{G}_{n,1}$. We call this group $G_{n,1}$ and it is the same as the group $G_{n,1}$ described in [5] (see Lemma 2.7).

Note, if n=1 then $G_{1,1}=\mathcal{G}_{1,1}$ and since we are interested in extending $G_{n,1}$ in $\mathcal{G}_{n,1}$ we will only consider the non-trivial case n>2.

2.2 Groups defined by expansion rules.

We now describe the groups which we are going to use to extend $G_{n,1}$. This is most easily done using wreath products but the fact that these groups are subgroups of a wreath product will not be used anywhere

else in this work.

Let A and B be two sets, B finite. Let G and H be groups of permutations of A and B respectively. The wreath product GwrH is the set of all permutations of the set $A\times B$ which can be defined in the following way. With each $f \in GwrH$ there is associated some $h\in H$ and some family $\{g_b \in G \mid b\in B\}$ such that for all $(a,b) \in A\times B$

$$(a,b)f = (ag_b, bh).$$

The wreath product is associative (see, for example, [6]) and so if G_k is a permutation group on the set A_k, $1\leq k\leq m$ (A_k finite for $k<m$) we can define the wreath product $G_m wr \ldots wr G_2 wr G_1$ as a permutation group on the set $A_m\times\ldots\times A_2\times A_1$. With each $f \in G_m wr \ldots wr G_1$ there is associated some $g \in G_1$ and $m-1$ families of the form $\{g_{b_k} \in G_k \mid b_k \in A_{k-1}\times\ldots\times A_1\}$, $1\leq k\leq m$. For all $(a_m,\ldots a_2,a_1) \in A_m\times\ldots\times A_2\times A_1$

$$(a_m,\ldots,a_2,a_1)f = (a_m g_{b_m},\ldots,a_2 g_{b_2}, a_1 g)$$

where $b_k = (a_{k-1},\ldots,a_1)$.

Finally, if we have countably many finite sets $A_1,A_2,\ldots,A_m,\ldots$ with groups of permutations $G_1,G_2,\ldots,G_m,\ldots$ respectively, then we can define the infinite wreath product $\ldots wr G_m wr \ldots wr G_2 wr G_1$ as a set of permutations of the set $\ldots\times A_m\times\ldots\times A_2\times A_1$. With each $f\in \ldots wr G_m wr \ldots wr G_1$ there is associated some $g\in G_1$ and infinite family of sets $\{g_{b_k}\in G_k \mid b_k\in A_{k-1}\times\ldots\times A_1\}$. For all $(\ldots,a_m,\ldots,a_2,a_1) \in \ldots\times A_m\times\ldots A_2\times A_1$

$$(\ldots,a_m,\ldots a_2,a_1)f = (\ldots,a_m g_{b_m},\ldots,a_2 g_{b_2}, a_1 g)$$

where $b_k = (a_{k-1},\ldots,a_1)$.

Let F be the free group of rank k acting on itself by right multiplication and let S_{n-1} be the symmetric group on the set $\{a_2,\ldots,a_n\}$. We consider the wreath products $S_{n-1}wr\ldots wrS_{n-1}$, $FwrS_{n-1}wr\ldots wrS_{n-1}$ and $\ldots wrS_{n-1}wr\ldots wrS_{n-1}$ which will be written $S_{n-1}(wrS_{n-1})^{i-1}$, $F(wrS_{n-1})^{i}$ and $S_{n-1}(wrS_{n-1})^{\omega}$ respectively.

We can write each element f of $FwrS_{n-1}$ in the form

$$\begin{pmatrix} a_2 & a_3 & \ldots & a_n \\ \ell_2 & \ell_3 & \ldots & \ell_n \\ a_{2\pi} & a_{3\pi} & \ldots & a_{n\pi} \end{pmatrix}$$

where, for each $(h,a_i) \in F^{\times}\{a_2,\ldots,a_n\}$, $(h,a_i)f = (h\ell_i, a_{i\pi})$ ($\pi \in S_{n-1}$, $\ell_i \in F$). We can write each element $g \in FwrS_{n-1}wrS_{n-1}$ in the form

$$\begin{pmatrix} a_2a_2 & a_2a_3 & \ldots & a_2a_n & a_3a_2 & \ldots & a_na_2 & \ldots & a_na_n \\ \ell_{2,2} & \ell_{2,3} & \ldots & \ell_{2,n} & \ell_{3,2} & \ldots & \ell_{n,2} & \ldots & \ell_{n,n} \\ a_{2\pi}a_{2\rho_2} & a_{2\pi}a_{3\rho_2} & \ldots & a_{2\pi}a_{n\rho_2} & a_{3\pi}a_{2\rho_3} & \ldots & a_{n\pi}a_{2\rho_n} & \ldots & a_{n\pi}a_{n\rho_n} \end{pmatrix}$$

where for each $(h,a_j,a_i) \in F^{\times}\{a_2,\ldots,a_n\}\times\{a_2,\ldots,a_n\}$

$$(h,a_j,a_i)g = (h^{\ell_{i,j}}, a_{j\rho_i}, a_{i\pi})$$

($\pi,\rho_i \in S_{n-1}$ and $\ell_{i,j} \in F$). In general, we can write each $k \in F(wrS_{n-1})^{m}$ in the form

$$\begin{pmatrix} y_1 & y_2 & \ldots & y_t \\ \ell(1) & \ell(2) & \ldots & \ell(t) \\ z_1 & z_2 & \ldots & z_t \end{pmatrix}$$

where $\{y_1,y_2,\ldots,y_t\} = \{a_{j_1}a_{j_2}\ldots a_{j_m} \mid 2\leq j_1,\ldots,j_m\leq n\}$ and if we think of y_i and z_i as reverse order m-tuples, for each $h\in F$ $(h,y_i)k = (h^{\ell(i)},z_i)$.

If $f,g \in FwrS_{n-1}$ are the elements

$$\begin{pmatrix} a_2 & a_3 & \dots & a_n \\ \ell_2 & \ell_3 & \dots & \ell_n \\ a_{2\pi} & a_{3\pi} & \dots & a_{n\pi} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_2 & a_3 & \dots & a_n \\ h_2 & h_3 & \dots & h_n \\ a_{2\rho} & a_{3\rho} & \dots & a_{n\rho} \end{pmatrix}$$

respectively, then their product, fg, is the element

$$\begin{pmatrix} a_2 & a_3 & \dots & a_n \\ \ell_2 h_{2\pi} & \ell_3 h_{3\pi} & \dots & \ell_n h_{n\pi} \\ a_{2\pi\rho} & a_{3\pi\rho} & \dots & a_{n\pi\rho} \end{pmatrix}.$$

Products between elements of $F(wrS_{n-1})^m$ are formed in the same way.

We now suppose that F is finitely generated by $T=\{t_1,\dots,t_k\}$. Any map, ψ', from T into $FwrS_{n-1}$ can be extended to a homomorphism $\psi: F \to FwrS_{n-1}$. We say that the set $\{\ell \to \ell\psi \mid \ell\in F\}$ is a set of expansion rules and that $\ell \to \ell\psi$ is an expansion rule for ℓ.

We can use the homomorphism ψ to generate a homomorphism $\psi^{(2)}: F \to F(wrS_{n-1})^2$. Suppose that

$$\ell\psi = \begin{pmatrix} a_2 & a_3 & \dots & a_n \\ \ell_2 & \ell_3 & \dots & \ell_n \\ a_{2\pi} & a_{3\pi} & \dots & a_{n\pi} \end{pmatrix}$$

and that

$$\ell_i\psi = \begin{pmatrix} a_2 & a_3 & \dots & a_n \\ \ell_{i,2} & \ell_{i,3} & \dots & \ell_{i,n} \\ a_{2\pi_i} & a_{3\pi_i} & \dots & a_{n\pi_i} \end{pmatrix}$$

for $2\le i\le n$. Then define $\psi^{(2)}: F \to F(wrS_{n-1})^2$ by

$$\ell\psi^{(2)} = \begin{pmatrix} a_2a_2 & \dots & a_2a_n & a_3a_2 & \dots & a_na_n \\ \ell_{2,2} & \dots & \ell_{2,n} & \ell_{3,2} & \dots & \ell_{n,n} \\ a_{2\pi}a_{2\pi_2} & \dots & a_{2\pi}a_{n\pi_2} & a_{3\pi}a_{2\pi_3} & \dots & a_{n\pi}a_{n\pi_n} \end{pmatrix}$$

It can be seen from the discussion about products of elements above that

$\psi^{(2)}$ is a homomorphism.

Inductively, we can define a homomorphism $\psi^{(i)}: F \to F(wrS_{n-1})^i$ using ψ. If

$$\ell\psi^{(i-1)} = \begin{pmatrix} y_1 & y_2 & \cdots & y_m \\ h_1 & h_2 & \cdots & h_m \\ z_1 & z_2 & \cdots & z_m \end{pmatrix}$$

and if

$$h_j\psi = \begin{pmatrix} a_2 & a_3 & \cdots & a_n \\ h_{j,2} & h_{j,3} & \cdots & h_{j,n} \\ a_{2\rho_j} & a_{3\rho_j} & \cdots & a_{n\rho_j} \end{pmatrix}$$

then define

$$\ell\psi^{(i)} = \begin{pmatrix} y_1a_2 & \cdots & y_1a_n & y_2a_2 & \cdots & y_ma_2 & \cdots & y_ma_n \\ h_{1,2} & \cdots & h_{1,n} & h_{2,2} & \cdots & h_{m,2} & \cdots & h_{m,n} \\ z_1a_{2\rho_1} & \cdots & z_1a_{n\rho_1} & z_2a_{2\rho_2} & \cdots & z_ma_{2\rho_m} & \cdots & z_ma_{n\rho_m} \end{pmatrix}.$$

If we let $p^{(i)}: F(wrS_{n-1})^i \to S_{n-1}(wrS_{n-1})^{i-1}$ be the natural projection then $\phi^{(i)}=\psi^{(i)}p^{(i)}$ is a homomorphism from F to $S_{n-1}(wrS_{n-1})^{i-1}$. We can define a map $\phi: F \to S_{n-1}(wrS_{n-1})^{\omega}$ by the rule

$$(\ldots a_{i_m},\ldots,a_{i_2},a_{i_1})(\ell\phi) = (\ldots b_{i_m},\ldots,b_{i_2},b_{i_1})$$

where $(a_{i_r},\ldots,a_{i_1})(\ell\phi^{(r)}) = (b_{i_r},\ldots,b_{i_1})$. This map is well defined by the inductive construction of the $\phi^{(i)}$ and ϕ is a homomorphism since each $\phi^{(i)}$ is.

A map, ϕ, constructed in this way depends only on ψ and hence on the initial choice of ψ'. Let $\Psi=\ker\phi$, then F/Ψ is a subgroup of $S_{n-1}(wrS_{n-1})^{\omega}$. We call F/Ψ the group defined by the finite set of expansion rules $\{t_i \to t_i\psi \mid F=F\langle t_1,\ldots,t_k\rangle\}$.

Lemma 2.2 Every group defined by a finite set of expansion rules can be embedded in $\mathcal{G}_{n,1}$.

Proof: Let $\psi: F \to FwrS_{n-1}$ be a homomorphism, let F/ψ be the group defined by the expansion rules $t_i \to t_i\psi$ and let ϕ be the homomorphism from F into $S_{n-1}wr(S_{n-1})^{\omega}$, as constructed above. Associate F/ψ with its image in the wreath product. For each $\ell\phi \in F/\psi$ define $\sigma_\ell \in \mathcal{G}_{n,1}$ to be the unique maximal extension of the map σ_ℓ' given by

$$a_i w \to a_i w \qquad \text{for } 1 \le i \le n-1,\ w \in W.$$

$$a_n u a_1 w \to a_n(u(\ell\phi^{(r)}))a_1 w \qquad \text{where u has length r and does not contain } a_1,\ w \in W.$$

This map is clearly a W-space homomorphism. We show that it is an inescapable isomorphism.

For $1 \le i \le n$, if $(a_i y)\sigma_\ell'$ is defined then $(a_i y)\sigma_\ell' = a_i z$, for some z. If $1 \le i \le n-1$ then y=z and so $(a_i x)\sigma_\ell' = (a_i y)\sigma_\ell'$ if and only if $a_i x = a_i y$. For i=n, $(a_n u a_1 w)\sigma_\ell' = (a_n v a_1 y)\sigma_\ell'$ if and only if $a_n(u(\ell\phi^{(r)})a_1 w) = a_n(v(\ell\phi^{(s)})a_1 y)$. Thus, since each $\ell\phi^{(i)}$ preserves lengths of words, we must have s=r, $v(\ell\phi^{(r)}) = u(\ell\phi^{(r)})$ and w=y. Since $\ell\phi^{(r)}$ is a permutation we must have u=v. Thus σ_ℓ' is injective. From the definition of σ_ℓ' we see that for all $w \in W$ $(wa_1)\sigma_\ell'$ is defined so the domain of σ_ℓ' is inescapable and, since $\ell\phi^{(r)}$ is a permutation, wa_1 lies in the range of σ_ℓ. Hence σ_ℓ' is an inescapable isomorphism.

Thus we define a map $\Sigma: F/\psi \to \mathcal{G}_{n,1}$ by $\Sigma: \ell\phi \to \sigma_\ell$. If $h\phi=\ell\phi$ then $\ell\phi^{(r)}=h\phi^{(r)}$ for all r and so $\sigma_\ell=\sigma_h$, thus Σ is well defined. If u doesn't contain a_1 then nor does $u(\ell\phi^{(r)})$ and each $\ell\phi^{(r)}$ is a homomorphism so Σ is a homomorphism. If $\sigma_\ell=\sigma_h$ then $\ell\phi^{(r)} = h\phi^{(r)}$ for all r and hence $\ell\phi=h\phi$, thus Σ is an embedding of F/ψ in $\mathcal{G}_{n,1}$. We call Σ the natural embedding or standard embedding. ◀

From now on we identify all groups given by a finite set of expansion rules with their images in $\mathcal{G}_{n,1}$ under the natural embedding.

The following theorem is proved in the next chapter.

If H is a finitely presented group defined by a finite set of expansion rules then the group $\langle G_{n,1}, H\rangle$ is finitely presented.

2.3 Representation of group elements by symbols.

To prove the theorem stated at the end of Section 2.2, we need to be able to write out explicitly any given element of $\langle G_{n,1}, H\rangle$. Thus, we extend the idea of symbols introduced in [5]. Whenever we write $\langle G_{n,1}, H\rangle$ we will assume that we have an explicit set of expansion rules which define H and that $\langle G_{n,1}, H\rangle$ is a subgroup of $\mathcal{G}_{n,1}$ containing the image of H in $\mathcal{G}_{n,1}$, with respect to these expansion rules, under the natural embedding.

Given a group $\langle G_{n,1}, H\rangle$, where H is defined by a finite set of expansion rules $\{t_i \to t_i\psi \mid 1 \leq i \leq k\}$, we define a symbol to be an object of the form

$$\begin{pmatrix} u_1 & u_2 & \dots & u_s \\ \ell_1 & \ell_2 & \dots & \ell_s \\ v_1 & v_2 & \dots & v_s \end{pmatrix}$$

where $\{u_1,\dots,u_s\}$ and $\{v_1,\dots,v_s\}$ are cofinite bases and $\ell_i \in F\langle t_1,\dots,t_k\rangle = F\langle T\rangle$. If $\{y_1,\dots,y_s\}$ is a permutation of $\{u_1,\dots,u_s\}$ and $\{z_1,\dots,z_s\}$ is the corresponding permutation of $\{v_1,\dots,v_s\}$, and if $\ell_i h_j^{-1} \in \Psi$ (i.e. $\ell_i\phi = h_j\phi$) for $u_i = y_j$ then the symbol

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_s \\ h_1 & h_2 & \cdots & h_s \\ z_1 & z_2 & \cdots & z_s \end{pmatrix}$$

is considered to be the same symbol as the symbol above. Thus we can reorder columns and replace any element of F by another in the same coset of Ψ without changing the symbol. From the discussion that follows it is clear that these symbols represent the same element of $\mathcal{G}_{n,1}$ and so we are justified in considering them as the same symbol.

If $\ell_i\psi = \begin{pmatrix} a_2 & a_3 & \cdots & a_n \\ \ell_{i,2} & \ell_{i,3} & \cdots & \ell_{i,n} \\ a_{2\pi} & a_{3\pi} & \cdots & a_{n\pi} \end{pmatrix}$ then

$$\begin{pmatrix} u_1 & \cdots & u_{i-1} & u_i a_1 & u_i a_2 & \cdots & u_i a_n & u_{i+1} & \cdots & u_s \\ \ell_1 & \cdots & \ell_{i-1} & 1 & \ell_{i,2} & \cdots & \ell_{i,n} & \ell_{i+1} & \cdots & \ell_s \\ v_1 & \cdots & v_{i-1} & v_i a_1 & v_i a_{2\pi} & \cdots & v_i a_{n\pi} & v_{i+1} & \cdots & v_s \end{pmatrix}$$

is also a symbol and we say that it is an elementary expansion of the symbol

$$\begin{pmatrix} u_1 & u_2 & \cdots & u_s \\ \ell_1 & \ell_2 & \cdots & \ell_s \\ v_1 & v_2 & \cdots & v_s \end{pmatrix}$$

If one symbol can be obtained from another by a finite series of elementary expansions then we say that the former is an expansion of the latter symbol. Conversely, we say the latter symbol is a contraction of the former. We also say that the column $\begin{pmatrix} u_i \\ \ell_i \\ v_i \end{pmatrix}$ belongs to the symbol above and that the symbol contains the column.

Lemma 2.3 For a given set of expansion rules, with each symbol we can associate a unique element of $\mathcal{G}_{n,1}$.

Proof: Given a symbol

$$\begin{pmatrix} u_1 & u_2 & \cdots & u_s \\ \ell_1 & \ell_2 & \cdots & \ell_s \\ v_1 & v_2 & \cdots & v_s \end{pmatrix}$$

define τ^* by $(u_i w)\tau^* = v_i w'$ (where if σ_{ℓ_i} is as defined in Lemma 2.2 then $(a_n w)\sigma_{\ell_i} = a_n w'$) for all w such that $(a_n w)\sigma_{\ell_i}$ is defined.

Since $\{u_1,\ldots,u_s\}_\wedge$ and $\{v_1,\ldots,v_s\}$ is a cofinite basis and σ_{ℓ_i} is an inescapable isomorphism, τ^* is an inescapable isomorphism and hence has a unique maximal extension, τ, in $\mathcal{G}_{n,1}$. We associate this τ with the above symbol and say that the symbol is a symbol for τ and that τ has this symbol. ◀

Lemma 2.4 Every expansion of a given symbol is a symbol for the same element of $\mathcal{G}_{n,1}$.

Proof: We only need to show that if

$$\begin{pmatrix} u_1 & u_2 & \cdots & u_s \\ \ell_1 & \ell_2 & \cdots & \ell_s \\ v_1 & v_2 & \cdots & v_s \end{pmatrix}$$

is a symbol for $\tau \in \mathcal{G}_{n,1}$ then so is

$$\begin{pmatrix} u_1 \cdots u_{i-1} & u_i a_1 & u_i a_2 & \cdots & u_i a_n & u_{i+1} \cdots u_s \\ \ell_1 \cdots \ell_{i-1} & 1 & \ell_{i,2} & \cdots & \ell_{i,n} & \ell_{i+1} \cdots \ell_s \\ v_1 \cdots v_{i-1} & v_i a_1 & v_i a_{2\pi} & \cdots & v_i a_{n\pi} & v_{i+1} \cdots v_s \end{pmatrix}$$

If $\varepsilon \in \mathcal{G}_{n,1}$ is the element associated with the second symbol above then

$$(u_j w)\varepsilon^* = v_j w' = (u_j w)\tau^* \qquad \text{for } j \neq i$$

and

$$(u_i a_1 w)\varepsilon^* = v_i a_1 w = (u_i a_1 w)\tau^*.$$

By the construction of $\phi^{(r)}$,

$$(a_j u)(\ell_i \phi^{(r)}) = a_{j\pi}(u(\ell_{i,j}\phi^{(r-1)})) \qquad (2 \leq j \leq n)$$

for any u and so $(u_i a_j w)\varepsilon^* = v_i a_{j\pi} w' = v_i (a_j w)\sigma_{\ell_i} = (u_i a_j w)\tau^*$ whenever $(u_i a_j w)\varepsilon^*$ is defined. ($(a_n w)\sigma_{\ell_{ij}} = a_n w'$ and $(a_n a_j w)\sigma_{\ell_i} = a_n (a_j w)'$ whenever defined). Thus τ^* is an extension of ε^* and, by the uniqueness of maximal elements, $\tau = \varepsilon$. ◀

Lemma 2.5 If $\tau, \varepsilon \in \mathcal{G}_{n,1}$ have symbols then so does their product, $\tau\varepsilon$.

Proof: Suppose that τ and ε have symbols

$$\begin{pmatrix} u_1 & u_2 & \dots & u_s \\ \ell_1 & \ell_2 & \dots & \ell_s \\ v_1 & v_2 & \dots & v_s \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y_1 & y_2 & \dots & y_m \\ h_1 & h_2 & \dots & h_m \\ z_1 & z_2 & \dots & z_m \end{pmatrix},$$

first and last

respectively. Because the rows of the symbols are cofinite bases, by reordering the columns if necessary, we have that either $v_1 w = y_1$ or $y_1 w = v_1$ for some $w \in W$.

If $v_1 w = y_1$ then, since $\{y_1, \dots, y_m\}$ is a cofinite basis, we can find some cofinite basis $\{w_1, \dots, w_p\}$ such that $v_1 w_i \in \{y_1, \dots, y_m\}$ for $1 \leq i \leq p$. By reordering the columns we can suppose that $v_1 w_i = y_i$. Inspecting the expansion rules we see that we can expand the first symbol above to

$$\begin{pmatrix} u_1 w_1' & \dots & u_1 w_p' & u_2 & \dots & u_s \\ \ell_1' & \dots & \ell_p' & \ell_2 & \dots & \ell_s \\ v_1 w_1 & \dots & v_1 w_p & v_2 & \dots & v_s \end{pmatrix} = \begin{pmatrix} u_1 w_1' & \dots & u_1 w_p' & u_2 & \dots & u_s \\ \ell_1' & \dots & \ell_p' & \ell_2 & \dots & \ell_s \\ y_1 & \dots & y_p & v_2 & \dots & v_s \end{pmatrix}$$

where $\{w_1', \dots, w_p'\}$ is also a cofinite basis and w_i' has the same length as w_i.

If $v_1=y_1w$ then the same argument shows that we can find a cofinite basis $\{w_1,\ldots,w_p\}$ such that $v_i=y_1w_i$ and that the second symbol above can be expanded to

$$\begin{pmatrix} v_1 \ldots v_p & y_2 \ldots y_m \\ h_1' \ldots h_p' & h_2 \ldots h_m \\ z_1w_1' \ldots z_1w_p' & z_2 \ldots z_m \end{pmatrix}$$

for some cofinite basis $\{w_1',\ldots,w_p'\}$.

Considering each of the columns of the above symbols in this way we see that we can expand both symbols to have the form

$$\begin{pmatrix} u_1' & u_2' & \ldots & u_p' \\ \ell_1' & \ell_2' & \ldots & \ell_p' \\ v_1' & v_2' & \ldots & v_p' \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_1' & v_2' & \ldots & v_p' \\ h_1' & h_2' & \ldots & h_p' \\ z_1' & z_2' & \ldots & z_p' \end{pmatrix}$$

respectively. We show that $\tau\varepsilon$ has symbol

$$\begin{pmatrix} u_1' & u_2' & \ldots & u_p' \\ \ell_1'h_1' & \ell_2'h_2' & \ldots & \ell_p'h_p' \\ z_1' & z_2' & \ldots & z_p' \end{pmatrix}. \qquad (*)$$

For any $w\in W$ such that $(a_nw)\sigma_{\ell_i'}$ and $(a_nw)\sigma_{\ell_i'h_i'}$ are both defined $(u_i'w)\tau^* = v_i'x$ and $(v_i'x)\varepsilon^* = z_i'y$, where $(a_nw)\sigma_{\ell_i'} = a_nx$ and $(a_nx)\sigma_{h_i'} = a_ny$. Thus $(u_i'w)(\tau^*\circ\varepsilon^*) = z_i'y$. So the map defined by the symbol (*) is an extension of $\tau^*\circ\varepsilon^*$ and hence, by the uniqueness of maximal elements, (*) is a symbol for $\tau\varepsilon$. ◀

Note: For each $\ell \in F\langle T\rangle$ σ_ℓ has symbol

$$\begin{pmatrix} a_1 \ldots a_{n-1} & a_n \\ 1 \ldots 1 & \ell \\ a_1 \ldots a_{n-1} & a_n \end{pmatrix}.$$

We can now describe the group $\langle G_{n,1}, H\rangle$, for a given set of expansion

rules.

Theorem 2.6 The group $\langle G_{n,1}, H\rangle$ is precisely the subset of $\mathcal{G}_{n,1}$ of elements which have symbols.

Proof: If $\alpha \in G_{n,1}$ then there is some cofinite basis $\{u_1,\ldots,u_s\}$ in the domain of α. Thus

$$\begin{pmatrix} u_1 & u_2 & \ldots & u_s \\ 1 & 1 & \ldots & 1 \\ u_1\alpha & u_2\alpha & \ldots & u_s\alpha \end{pmatrix}$$

is a symbol and clearly it is a symbol for α.

Conversely, if $\alpha \in \mathcal{G}_{n,1}$ has symbol

$$\begin{pmatrix} u_1 & u_2 & \ldots & u_s \\ 1 & 1 & \ldots & 1 \\ v_1 & v_2 & \ldots & v_s \end{pmatrix}$$

then $\alpha \in G_{n,1}$, so $G_{n,1}$ is the set of elements with symbols of this form.

Any $\tau \in \langle G_{n,1}, H\rangle$ is of the form $\alpha_1\sigma_{\ell_1}\alpha_2\sigma_{\ell_2}\ldots\alpha_m\sigma_{\ell_m}$ for some $\alpha_i \in G_{n,1}$, $\sigma_{\ell_i} \in H$. Each α_i and σ_{ℓ_i} has a symbol so, by Lemma 2.5, τ also has a symbol.

Again by Lemma 2.5, the symbol

$$\begin{pmatrix} u_1 & u_2 & \ldots & u_s \\ \ell_1 & \ell_2 & \ldots & \ell_s \\ v_1 & v_2 & \ldots & v_s \end{pmatrix}$$

is a symbol for the element $\alpha_1\sigma_{\ell_1}\beta_1\sigma_{\ell_2}\ldots\beta_{s-1}\sigma_{\ell_s}\alpha_s = \tau$ where α_1, β_i, α_2 and σ_{ℓ_i} have symbols respectively

$$\begin{pmatrix} u_1 & u_2 & \dots & u_s \\ 1 & 1 & \dots & 1 \\ a_n & y_2 & \dots & y_s \end{pmatrix}, \qquad \begin{pmatrix} y_2 & \dots & y_i & a_n & y_{i+1} & y_{i+2} & \dots & y_s \\ 1 & \dots & 1 & 1 & 1 & 1 & \dots & 1 \\ y_2 & \dots & y_i & y_{i+1} & a_n & y_{i+2} & \dots & y_s \end{pmatrix},$$

$$\begin{pmatrix} y_2 & \dots & y_s & a_n \\ 1 & \dots & 1 & 1 \\ v_1 & \dots & v_{s-1} & v_s \end{pmatrix} \text{ and } \begin{pmatrix} y_2 & \dots & y_i & a_n & y_{i+1} & \dots & y_s \\ 1 & \dots & 1 & \ell_i & 1 & \dots & 1 \\ y_2 & \dots & y_i & a_n & y_{i+1} & \dots & y_s \end{pmatrix}.$$

So $\tau \in \langle G_{n,1}, H \rangle$ and the theorem is proved. ◀

We now define some useful notation. For each $\ell \in F\langle T \rangle$ and $w \in W$ define $w_\ell \in W$ to be the word satisfying $(a_n w a_1)\sigma_\ell = a_n w_\ell a_1$. Then w_ℓ is a word of the same length as w.

If we take a column $\begin{pmatrix} u \\ \ell \\ v \end{pmatrix}$ we can expand it, using the expansion rule for ℓ, to the set of columns

$$\begin{pmatrix} ua_1 & ua_2 & \dots & ua_n \\ \ell_1 & \ell_2 & \dots & \ell_n \\ va_1 & va_{2\pi} & \dots & va_{n\pi} \end{pmatrix}.$$

Note that $a_{i\pi} = (a_i)\ell$. In general we can expand the column above to any set of columns

$$\begin{pmatrix} uw_1 & uw_2 & \dots & uw_p \\ h_1 & h_2 & \dots & h_p \\ vz_1 & vz_2 & \dots & vz_p \end{pmatrix}$$

where $\{w_1, \dots w_p\}$ is a cofinite basis. We say that the column above expands to the column $\begin{pmatrix} uw_i \\ h_i \\ vz_i \end{pmatrix}$ for $1 \leq i \leq p$ and that this column can be

obtained by expanding the one above.

If $\begin{bmatrix} u \\ \ell \\ v \end{bmatrix}$ belongs to a symbol for $\tau \in \mathcal{G}_{n,1}$ then $(uwa_1)\tau$ is defined,

and equal to $vw_\ell a_1$ by definition. If the column $\begin{bmatrix} uw \\ h \\ vz \end{bmatrix}$ can be obtained by

expanding this column then, by Lemma 2.4, it also belongs to some symbol for τ and $(uwa_1)\tau = vza_1$ gives that $z=w_\ell$. Thus for w_i and z_i as above, $(w_i)_\ell = z_i$.

Lemma 2.7 There exists a unique, uncontractable symbol for each $\tau \in \langle G_{n,1}, H \rangle$.

Proof: As above, we consider two symbols to be the same if permuting the columns, and substituting elements of $F\langle T \rangle$ for elements in the same Ψ-coset, of one symbol gives the other.

Let the two symbols

$$\begin{bmatrix} u_1 & \dots & u_s \\ \ell_1 & \dots & \ell_s \\ v_1 & \dots & v_s \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} y_1 & \dots & y_m \\ h_1 & \dots & h_m \\ z_1 & \dots & z_m \end{bmatrix}$$

both be uncontractable symbols for τ. Since $\{u_1,\dots,u_s\}$ and $\{y_1,\dots,y_m\}$ are cofinite bases, by reordering the columns, we can suppose that $u_1w' = y_1$ for some $w' \in W$. If $w' \neq 1$ then we can suppose that $w' = wa_1$ and $u_1wa_i = y_i$ for $1<i<n$.

We can expand the column $\begin{pmatrix} u_1 \\ \ell_1 \\ v_1 \end{pmatrix}$ to $\begin{pmatrix} u_1 w \\ k \\ u_1 w_{\ell 1} \end{pmatrix}$ for some $k \in F\langle T\rangle$ and if

$$k\psi = \begin{pmatrix} a_2 & \dots & a_n \\ k_2 & \dots & k_n \\ a_{2\rho} & \dots & a_{n\rho} \end{pmatrix}$$

then $(u_1 w a_j a_1)\tau = v_1 w_{\ell 1} a_{j\rho} a_1$. But $(y_j a_1)\tau = z_j a_1$ so $z_j = v_1 w_{\ell 1} a_{j\rho}$ and the second symbol above can be written

$$\begin{pmatrix} u_1 w a_1 & u_1 w a_2 & \dots & u_1 w a_n & y_{n+1} & \dots & y_m \\ 1 & h_2 & \dots & h_n & h_{n+1} & \dots & h_m \\ v_1 w_{\ell_1} a_1 & v_1 w_{\ell_1} a_{2\rho} & \dots & v_1 w_{\ell_1} a_{n\rho} & z_{n+1} & \dots & z_m \end{pmatrix} \cdot \qquad (*)$$

If $h_i^{-1} k_i \notin \Psi$ then for some r, $h_i \phi^{(r)} \neq k_i \phi^{(r)}$ and so for some $x \in W$ $x_{h_i} \neq x_{k_i}$. But $(u_1 w a_i x_{h_i} a_1)\tau = (u_1 w a_i x_{k_i} a_1)\tau$, since both symbols above are supposed to be symbols for τ. So, since τ is a bijection, we must have $h_i^{-1} k_i \in \Psi$. Thus (*) can be written

$$\begin{pmatrix} u_1 w a_1 & u_1 w a_2 & \dots & u_1 w a_n & y_{n+1} & \dots & y_m \\ 1 & k_2 & \dots & k_n & h_{n+1} & \dots & h_m \\ v_1 w_{\ell 1} a_1 & v_1 w_{\ell 1} a_{2\rho} & \dots & v_1 w_{\ell 1} a_{n\rho} & z_{n+1} & \dots & z_m \end{pmatrix}$$

which contracts to

$$\begin{pmatrix} u_1 w & y_{n+1} & \dots & y_m \\ k & h_{n+1} & \dots & h_m \\ v_1 w_{\ell 1} & z_{n+1} & \dots & z_m \end{pmatrix}$$

contrary to assumption. Thus we must have $w'=1$, $v_1 = z_1$ and, as for $h_i^{-1} k_i$ above, $\ell_1^{-1} h_1 \in \Psi$.

Repeating the argument for each of the columns of the two symbols gives the result. ◀

In particular, each of the elements $\alpha \in G_{n,1}$ has a unique shortest symbol of the form

$$\begin{pmatrix} u_1 & \dots & u_s \\ 1 & \dots & 1 \\ v_1 & \dots & v_s \end{pmatrix}.$$

Each element of the group $G_{n,1}$ as constructed in [5] has a unique symbol of exactly the same form whence the two groups are isomorphic in a natural way.

2.4 Results about $G_{n,1}$.

In proving results about $\langle G_{n,1}, H \rangle$ we assume the results about $G_{n,1}$ proved in [5] so these results are stated here, without proof.

Theorem 2.8 The groups $G_{n,1}$ are finitely presented.

Theorem 2.9 The commutator subgroup, $G'_{n,1}$, of $G_{n,1}$ is simple and

(i) if n is even $G_{n,1} = G'_{n,1}$,

(ii) if n is odd $G'_{n,1}$ has index 2 in $G_{n,1}$.

The simplicity of $G'_{n,1}$ is used in the proof of a theorem of R.J.Thompson [9] given at the end of this section. Since $G'_{n,1}$ is of finite index in the finitely presented $G_{n,1}$, we have that $G'_{n,1}$ is a finitely presented simple group.

Theorem 2.10 Any countable locally finite group can be embedded in $G_{n,1}$.

Theorem 2.11 If A is a torsion free Abelian group of finite rank in $G_{n,1}=G$ then

(i) A is free Abelian of finite rank,

(ii) the centralizer, $C_G(A)$, is of finite index in the normalizer, $N_G(A)$

and (iii) $N_G(A)$ has a direct factor B which has a free Abelian subgroup C such that C is of finite index in AB.

The general aim is to construct finitely presented simple groups which contain wide classes of finitely generated groups with solvable word problem and the above theorem shows that there are strong restrictions on even the Abelian subgroups of $G_{n,1}$ of finite rank. We note the corollary to this theorem proved in [5].

Corollary 2.12 The group $GL(3,\mathbb{Z})$, which is finitely presented with solvable word problem, cannot be embedded in any of the groups $G_{n,r}$.

Theorem 2.13 The groups $G_{n,1}$ and $G_{m,1}$ are not isomorphic unless n=m.

Theorem 2.14 There are only finitely many conjugacy classes of elements of any given finite order in $G'_{n,1}$ and if p is a prime which doesnot divide n-1 then there are precisely n conjugacy classes of elements of order p in $G'_{n,1}$ (and $G_{n,1}$).

Finally we prove a theorem of R.J.Thompson [9] which enables us to construct finitely presented infinite simple groups.

Theorem 2.15 If K is a group satisfying $G'_{n,1} \lhd K \lhd \mathcal{G}_{n,1}$ then the commutator subgroup, K', of K is simple.

Proof: The method of proof is to show that every non-trivial normal subgroup, N, of K contains K'. Then, since $G'_{n,1}$ is simple and non-Abelian $G''_{n,1} = G'_{n,1} \lhd K'$ and thus K' satisfies the restrictions of K, so K" is contained in every non-trivial normal subgroup of K'. But $G'_{n,1} \lhd K"$ so K" is non-trivial and hence $K' \lhd K"$ and so K' is simple.

Suppose that N is a nontrivial normal subgroup of K and that $\theta \in N$ is nontrivial. If $u \neq u\theta$ and $y\theta$ is defined, where $yw=u$, then $y\theta \neq y$ so we choose $u \in W$ such that $u\theta \neq u$ and we suppose that θ is not defined on any proper initial segment of u. Let $v=u\theta$. We use u to construct a nontrivial element in $G'_{n,1} \cap N$. Then, by Theorem 2.9, $G'_{n,1} \lhd N$. We then use this to show that K/N is Abelian.

Let U be the cofinite subspace of all elements of W which are not initial segments of ua_1. Let V be the corresponding cofinite subspace for va_1. Define $\alpha \in G'_{n,1}$† by

$$y\alpha = y \qquad \text{if } y \in U \text{ and } uw \neq y \text{ for any } w \in W$$

$$(ua_1a_1)\alpha = ua_1a_2$$

$$(ua_1a_2)\alpha = ua_2$$

$$(ua_2)\alpha = ua_1a_1$$

$$(ua_1a_i)\alpha = ua_1a_i \quad \Big\} \quad \text{for } i>3.$$

$$(ua_i)\alpha = ua_i$$

Similarly define $\beta \in G'_{n,1}$† by

$$y\beta = y \qquad \text{if } y \in Y \text{ and } vw \neq y \text{ for any } w \in W$$

$$(va_1a_1)\beta = va_1a_2$$

$$(va_1a_2)\beta = va_2$$

† $\alpha, \beta \in G_{n,1}$ are even elements and hence belong to $G'_{n,1}$ (see [5], pp 34-37).

$$\left.\begin{aligned}(va_2)\beta &= va_1a_1\\ (va_1a_i)\beta &= va_1a_i\\ (va_i)\beta &= va_i\end{aligned}\right\} \quad \text{for } i\geq 3.$$

Let Y be the domain of θ^{-1} and take $y \in Y \cap V$. If y does not have initial segment v then $y\theta^{-1}$ does not have initial segment u and so $(y\theta^{-1})\alpha = y\theta^{-1}$. Thus $y(\theta^{-1}\alpha\theta) = y = y\beta$. If $y = va_1a_1w$ then $y(\theta^{-1}\alpha\theta) = (ua_1a_1w)\alpha\theta = va_1a_2w = y\beta$. If $y = va_1a_2w$ then $y(\theta^{-1}\alpha\theta) = va_2w = y\beta$ and if $y = va_1w$ then $y(\theta^{-1}\alpha\theta) = va_1a_1w = y\beta$. If $y = va_{1i}w$ or va_iw for $i\geq 3$ then $y(\theta^{-1}\alpha\theta) = y\beta$. So $\theta^{-1}\alpha\theta$ and β are both maximal extensions of the same inescapable isomorphism on $Y \cap V$ and thus $\theta^{-1}\alpha\theta = \beta \in G'_{n,1}$. Hence $\alpha^{-1}\theta^{-1}\alpha\theta \in G'_{n,1}$ and, since $N \lhd K$, $\alpha^{-1}\theta^{-1}\alpha\theta \in N$. This element is non-trivial since $(ua_2a_1^i)\alpha^{-1}\theta^{-1}\alpha\theta = (ua_1a_2a_1^i)\beta = ua_2a_1^i$ if and only if u=v, but u≠v. Thus $G'_{n,1} < N$.

Take $\theta,\phi \in K$, we want to show that $N\theta\phi = N\phi\theta$. If $u'\theta = u$ and $v'\phi = v$ choose $\alpha,\beta \in G'_{n,1}$ so that $u\alpha = u'$ and $v\beta = v'$.† Since $G'_{n,1} < N$ it is sufficient to show that $N\theta'\phi' = N\phi'\theta'$ where $\theta' = \alpha\theta$ and $\phi' = \beta\phi$. If α and β have symbols respectively

$$\begin{pmatrix} u & u_2 & \dots & u_s \\ 1 & 1 & \dots & 1 \\ u' & u'_2 & \dots & u'_s \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v & v_2 & \dots & v_m \\ 1 & 1 & \dots & 1 \\ v' & v'_2 & \dots & v'_m \end{pmatrix}$$

let $\gamma,\delta \in G'_{n,1}$ be elements with symbols respectively

$$\begin{pmatrix} a_1 & a_2 & \dots & a_{n-1} & a_na_1 & a_ny_2 & \dots & a_ny_s \\ 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\ ua_1 & ua_2 & \dots & ua_{n-1} & ua_n & u_2 & \dots & u_s \end{pmatrix}$$

† Again, it can be seen from [5] that this can be done.

and

$$\begin{pmatrix} a_2 & \cdots & a_n & a_1a_1 & a_1z_2 & \cdots & a_1z_m \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ va_2 & \cdots & va_n & va_1 & v_2 & \cdots & v_m \end{pmatrix}$$

where $\{a_1,y_1,\ldots,y_s\}$ and $\{a_1,z_2,\ldots,z_m\}$ are arbitrary cofinite bases. We show that $\sigma = \gamma\theta'\gamma^{-1}$ and $\tau = \delta\phi'\delta^{-1}$ commute, and then $\gamma\theta'\gamma^{-1}N = N\theta'$ and $\delta\phi'\delta^{-1}N = N\phi'$ give $N\theta'\phi' = N\phi'\theta'$ as required.

If $y=a_iw$ for $2\leq i\leq n-1$ then $y\sigma=y=y\tau$ (θ' fixes u and ϕ' fixes v). If $y=a_1w$ then, whenever it is defined, $y\tau = a_1w'$ for some w' (again, because ϕ' fixes v) and so $y(\sigma\tau\sigma^{-1}\tau^{-1}) = y$. If $y=a_nw$ then $y\sigma = a_nw'$, for some w', whenever it is defined (θ' fixes u) and so $y(\sigma\tau\sigma^{-1}\tau^{-1}) = y$. Thus $\sigma\tau\sigma^{-1}\tau^{-1}$ acts identically on an inescapable subspace and hence $\sigma\tau\sigma^{-1}\tau^{-1} = 1$. So $H' < N$ as required. ◀

Chapter 3

Properties of the Groups $\langle G_{n,1}, H \rangle$

If H is a finitely presented subgroup of $\mathcal{G}_{n,1}$ given by a finite set of expansion rules $\{t_i \to t_i\psi \mid 1 \leq i \leq k\}$, as described in Chapter 2, we show that $\langle G_{n,1}, H \rangle$ is finitely presented. The proof of this is in the form of a series of lemmas and relies on the manipulation of symbols. We can choose a set of expansion rules defining H such that, if we associate H with its image in $\mathcal{G}_{m,1}$, under the natural embedding with respect to these expansion rules, the derived subgroup, $\langle G_{m,1}, H \rangle'$, has finite index in $\langle G_{m,1}, H \rangle$. Hence it is finitely presented[†] and $\langle G_{m,1}, H \rangle'$ is a finitely presented simple group. We also use symbols to show that if the word and order problems are solvable in H then they are solvable in $\langle G_{n,1}, H \rangle$.

We assume, with the notation of Chapter 2, that $H = \langle \sigma_{t_1}, \ldots, \sigma_{t_k} \rangle$ and whenever we write $\langle G_{n,1}, H \rangle$ we will assume that we have an explicit set of expansion rules, defining H, in mind.

3.1 A finite presentation for $\langle G_{n,1}, H \rangle$.

If H is a finitely presented group given by a finite set of expansion rules we show that $\langle G_{n,1}, H \rangle$ is defined by the following, essentially finite set of relations.

A. The multiplication tables of $G_{n,1}$ and H.

[†]See, for example, Magnus, Karrass and Solitar [7]

B. Relations of the form $\sigma_{\ell}\alpha = \alpha\sigma_{\ell}$, where $\alpha \in G_{n,1}$ fixes a_n and $\ell \in F\langle T\rangle$.

C. Relations of the form $\gamma\sigma_{\ell}\gamma\sigma_h = \sigma_h\gamma\sigma_{\ell}\gamma$, where $\ell, h \in F\langle T\rangle$ and $\gamma \in G_{n,1}$ has symbol

$$\begin{pmatrix} a_1 & \cdots & a_{n-2} & a_{n-1} & a_n \\ 1 & \cdots & 1 & 1 & 1 \\ a_1 & \cdots & a_{n-2} & a_n & a_{n-1} \end{pmatrix}.$$

*D. Relations of the form $\delta\,\sigma_{\xi_{i,2}}\delta_3 \cdots \sigma_{\xi_{i,n}}\delta_{n,i} = \sigma_{t_i}$ and $\delta\,\sigma_{\eta_{i,2}}\delta_3 \cdots \sigma_{\eta_{i,n}}\delta'_{i,n} = \sigma_{t_i^{-1}}$, where t_i has expansion rule

$$t_i \to \begin{pmatrix} a_2 & a_3 & \cdots & a_n \\ \xi_{i,2} & \xi_{i,3} & \cdots & \xi_{i,n} \\ a_{2\pi_i} & a_{3\pi_i} & \cdots & a_{n\pi_i} \end{pmatrix},$$

$\eta_{i,j} = \xi_{i,j\pi_i^{-1}}$ and δ, $\delta_{n,i}$, δ_j and $\delta'_{n,i}$ have symbols

$$\begin{pmatrix} a_2..a_{n-1} & a_n a_2 & a_n a_1 \cdots a_n a_n & a_1 \\ 1\,..\,1 & 1 & 1 \cdots 1 & 1 \\ a_2..a_{n-1} & a_n & a_1 a_2 \cdots a_1 a_n & a_1 a_1 \end{pmatrix}, \quad \begin{pmatrix} a_2..a_{n-1} & a_1 a_2 & a_1 a_3 \cdots & a_n & a_1 a_1 \\ 1\,..\,1 & 1 & 1 \cdots & 1 & 1 \\ a_2..a_{n-1} & a_n a_1 & a_n a_{2\pi_i} \cdots & a_n a_{n\pi_i} & a_1 \end{pmatrix},$$

$$\begin{pmatrix} a_2 & \cdots & a_{n-1} & a_n & a_1 a_2 & \cdots & a_1 a_{j-1} & a_1 a_j & a_1 a_{j+1} & \cdots & a_1 a_n & a_1 a_1 \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 \\ a_2 & \cdots & a_{n-1} & a_1 a_j & a_1 a_2 & \cdots & a_1 a_{j-1} & a_n & a_1 a_{j+1} & \cdots & a_1 a_n & a_1 a_1 \end{pmatrix}$$

and $$\begin{pmatrix} a_2 & \cdots & a_{n-1} & a_1 a_2 & a_1 a_3 & \cdots & a_1 a_n & a_n & a_1 a_1 \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ a_2 & \cdots & a_{n-1} & a_n a_1 & a_n a_{2\pi_i^{-1}} & \cdots & a_n a_{(n-1)\pi_i^{-1}} & a_n a_{n\pi_i^{-1}} & a_1 \end{pmatrix}$$

respectively.

Relations A are essentially finite since $G_{n,1}$ and H are finitely presented. Relations B hold in $\langle G_{n,1}, H\rangle$ since σ_ℓ fixes everything not of the form $a_n w$. If $G_{n,n-1}$ is the group of maximal isomorphisms between cofinite subspaces of $\{x_i a_{i_1} \ldots a_{i_m} \mid 1 \leq i \leq n-1,\ 1 \leq i_j \leq n\}$ (see page 3 and [5]) then $G_{n,n-1}$ is finitely presented [5]. Let $K \leq G_{n,1}$ be the group of all elements which fix $a_n w$ for all $w \in W$. If $\tau \in G_{n,n-1}$ and $(x_i z)\tau = x_{i\pi} y$, $1 \leq i \leq n-1$, $\pi \in S_{n-1}$, $z, y \in W$ then define a map $f: G_{n,n-1} \to K$ by $Jf = \alpha$ where

$$(a_i w)\alpha = a_{i\pi} y \quad 1 \leq i \leq n-1, \text{ when } (x_i w)\tau \text{ is defined}$$
$$\text{and} \quad (a_n w)\alpha = a_n w \quad \text{for all } w \in W.$$

Then f is in fact an isomorphism, so K is finitely generated. The relations B are generated by the relations $\beta_j \sigma_{t_i} = \sigma_{t_i} \beta_j$ where $K = \langle \beta_1, \ldots, \beta_m \rangle$, so relations B are essentially finite. The elements $\gamma\sigma_\ell\gamma\sigma_h$ and $\sigma_h\gamma\sigma_\ell\gamma$ have the same symbol and so relations C hold in $\langle G_{n,1}, H\rangle$. These relations are generated by the relations $\sigma_{t_i}\gamma\sigma_{t_j}\gamma = \gamma\sigma_{t_j}\gamma\sigma_{t_i}$ $(1 \leq i, j < k)$ and so are essentially finite. There are only finitely many relations D and consideration of symbols shows that these also hold in $\langle G_{n,1}, H\rangle$.

We show that every relation which holds in $\langle G_{n,1}, H\rangle$ is a consequence of relations A, B, C and D. In the following lemma μ_1, η_i, μ_2, μ_3, μ_4, $\beta \in G_{n,1}$ are the elements with symbols respectively

$$\begin{pmatrix} u_1 & u_2 \ldots u_s \\ 1 & 1 \ldots 1 \\ a_n & w_2 \ldots w_s \end{pmatrix}, \quad \begin{pmatrix} w_2 \ldots w_{i-1} & a_n & w_i & w_{i+1} \ldots w_s \\ 1 \ldots 1 & 1 & 1 & 1 \ldots 1 \\ w_2 \ldots w_{i-1} & w_i & a_n & w_{i+1} \ldots w_s \end{pmatrix}, \quad \begin{pmatrix} w_2 \ldots & w_s & a_n \\ 1 \ldots & 1 & 1 \\ v_1 \ldots v_{s-1} & v_s & \end{pmatrix},$$

$$\begin{pmatrix} y_1 & y_2 \ldots y_s \\ 1 & 1 \ldots 1 \\ a_n & w_2 \ldots w_s \end{pmatrix}, \quad \begin{pmatrix} w_2 \ldots & w_s & a_n \\ 1 \ldots & 1 & 1 \\ z_1 \ldots z_{s-1} & z_s & \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_1 & v_2 \ldots v_s \\ 1 & 1 \ldots 1 \\ y_1 & y_2 \ldots y_s \end{pmatrix}.$$

Lemma 3.1 The relation

$$\mu_1\sigma_{\ell_1}\eta_2\sigma_{\ell_2}\cdots\eta_s\sigma_{\ell_s}\mu_2\beta\mu_3\sigma_{h_1}\eta_2\cdots\eta_s\sigma_{h_s}\mu_4 = \mu_1\,\sigma_{\ell_1 h_1}\,\eta_2\,\sigma_{\ell_2 h_2}\cdots\eta_s\,\sigma_{\ell_s h_s}\,\mu_4$$

is a consequence of relations A,B,C, and D ($\ell_i, h_j \in F$).

Proof: First we show that for all $1\leq i<j\leq s$ the relation $\eta_j\cdots\eta_{i+1}\sigma_{\ell_i}\eta_{i+1}\cdots\eta_j\sigma_{\ell_j} = \sigma_{\ell_j}\eta_j\cdots\eta_{i+1}\sigma_{\ell_i}\eta_{i+1}\cdots\eta_j$ is a consequence of relations A,B,C and D. If $\alpha_1, \alpha_2 \in G_{n,1}$ have symbols

$$\begin{pmatrix} w_2 & \cdots & w_i & w_{i+1} & w_{i+2} & \cdots & w_j & a_n & w_{j+1} & \cdots & w_s \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ x_2 & \cdots & x_i & a_{n-1} & x_{i+1} & \cdots & x_{j-1} & a_n & x_j & \cdots & x_{s-1} \end{pmatrix}$$

and

$$\begin{pmatrix} x_2 & \cdots & x_i & a_n & x_{i+1} & \cdots & x_{j-1} & a_{n-1} & x_j & \cdots & x_{s-1} \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ w_2 & \cdots & w_i & a_n & w_{i+1} & \cdots & w_{j-1} & w_j & w_{j+1} & \cdots & w_s \end{pmatrix}$$

respectively, where $\{x_1,\ldots,x_{s-1}\}$ is an arbitrary cofinite basis, then $\alpha_1\gamma\alpha_2 = \eta_j\cdots\eta_{i+1}$ and $\alpha_2^{-1}\gamma\alpha_1^{-1} = \eta_{i+1}\cdots\eta_j$ are consequences of relations A, (γ as in relations C). Hence the relations

$$\begin{aligned} \eta_j\cdots\eta_{i+1}\sigma_\ell\ \eta_{i+1}\cdots\eta_j\sigma_{\ell'} &= \alpha_1\gamma\alpha_2\sigma_\ell\ \alpha_2^{-1}\gamma\alpha_1^{-1}\sigma_{\ell'} \\ &= \alpha_1\gamma\sigma_\ell\ \gamma\sigma_{\ell'}\alpha_1^{-1} \\ &= \alpha_1\sigma_{\ell'}\gamma\sigma_\ell\ \gamma\alpha_1^{-1} \\ &= \sigma_{\ell'}\alpha_1\gamma\alpha_2\sigma_\ell\ \alpha_2^{-1}\gamma\alpha_1^{-1} \\ &= \sigma_{\ell'}\eta_j\cdots\eta_{i+1}\sigma_\ell\ \eta_{i+1}\cdots\eta_j \end{aligned}$$

are consequences of relations A,B and C.

Thus the relations

$$\mu_1\sigma_{\ell_1}\eta_2\sigma_{\ell_2}\cdots\eta_s\sigma_{\ell_s}\mu_2 = \mu_1\eta_2\sigma_{\ell_2}\eta_2\sigma_{\ell_1}\eta_2\eta_3\sigma_{\ell_3}\eta_4\cdots\sigma_{\ell_s}\mu_2$$
$$= \mu_1\eta_2\sigma_{\ell_2}\eta_3\sigma_{\ell_3}\eta_3\eta_2\sigma_{\ell_1}\eta_2\eta_3\eta_4\ \sigma_{\ell_4}\eta_5\cdots\ \sigma_{\ell_s}\mu_2$$
$$\vdots$$
$$= \mu_1\eta_2\sigma_{\ell_2}\eta_3\sigma_{\ell_3}\eta_4\sigma_{\ell_4}\cdots\eta_s\sigma_{\ell_s}\eta_s\cdots\eta_2\sigma_{\ell_1}\eta_2\cdots\eta_s\mu_2$$

are consequences of relations A,B,C and D. The relation $\eta_s\cdots\eta_2 = \mu_2\beta\mu_3$ is a consequence of relations A, and thus the relation $\sigma_{\ell_1}\eta_2\cdots\eta_s\mu_2\beta\mu_3\sigma_{h_1} = \sigma_{\ell_1 h_1}$ is a consequence of relations A.

The relation

$$\mu_1\eta_2\sigma_{\ell_2}\eta_3\sigma_{\ell_3}\eta_4\sigma_{\ell_4}\cdots\eta_s\sigma_{\ell_s} = \mu_1\eta_2\eta_3\sigma_{\ell_3}\eta_4\sigma_{\ell_4}\cdots\eta_s\sigma_{\ell_s}\eta_s\cdots\eta_3\sigma_{\ell_2}\eta_3\cdots\eta_s$$

is a consequence of relations A,B,C and D as above, the relation $\sigma_{\ell_2}\eta_2\sigma_{\ell_1}h_1\eta_2 = \eta_2\sigma_{\ell_1}h_1\eta_2\sigma_{\ell_2}$ is a consequence of the relations at the beginning of this proof and hence the relation

$$\mu_1\sigma_{\ell_1}\eta_2\sigma_{\ell_2}\cdots\eta_s\sigma_{\ell_s}\mu_2\beta\mu_3\sigma_{h_1}\eta_2\sigma_{h_2} = \mu_1\eta_2\eta_3\sigma_{\ell_3}\eta_4\cdots\sigma_{\ell_s}\eta_s\cdots\eta_2\sigma_{\ell_1 h_1}\eta_2\sigma_{\ell_2 h_2}$$

is a consequence of relations A,B,C and D. Finally, ↑ carrying on in this way, the relation

$$\mu_1\sigma_{\ell_1}\eta_2\cdots\eta_s\sigma_{\ell_s}\mu_2\beta\mu_3\sigma_{h_1}\cdots\sigma_{h_s}\mu_4 = \mu_1\sigma_{\ell_1 h_1}\eta_2\sigma_{\ell_2 h_2}\cdots\eta_s\sigma_{\ell_s h_s}\mu_4$$

is a consequence of relations A,B,C and D as required. ◀

In the next lemma the elements δ_i are as defined in relations D, $\ell \in F$ has expansion rule

$$\ell \to \begin{pmatrix} a_2 & \cdots & a_n \\ \ell_2 & \cdots & \ell_n \\ a_{2\rho} & \cdots & a_{n\rho} \end{pmatrix}$$

and $\delta_{n\ell}$ is the element with symbol

$$\begin{pmatrix} a_2 & \cdots & a_{n-1} & a_1a_2 & \cdots & a_1a_n & a_n & a_1a_1 \\ 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & 1 \\ a_2 & \cdots & a_{n-1} & a_na_1 & \cdots & a_na_{(n-1)\rho} & a_na_{n\rho} & a_1 \end{pmatrix}.$$

Lemma 3.2 For each $\ell \in F$ the relation $\delta\ \sigma_{\ell_2}\delta_3\sigma_{\ell_3}\cdots\delta_n\ \sigma_{\ell_n}\delta_n\ell = \sigma_\ell$ is a consequence of relations A,B,C and D.

Proof: By induction on the length of ℓ as a word in $t_1,\ldots,t_k$.

If $\ell=1$, $t_i^{\pm 1}$ then the relation is a consequence of relations A and D.

If $\ell=ht_i$, $h\neq 1$, the induction hypothesis gives that

$$\sigma_\ell = \delta\ \sigma_{h_2}\delta_3\sigma_{h_3}\cdots\delta_n\ \sigma_{h_n}\delta_{nh}\delta\bar{\delta}_\mu\bar{\sigma}\xi_{i,2\rho}\cdots\delta_n\ \sigma_{\xi_{i,n\rho}}\delta_\mu\delta_{n,i\rho}$$

is a consequence of relations A,B,C and D.

The δ_i satisfy the conditions of the η_i in Lemma 3.1 so the relation

$$\sigma_\ell = \delta\ \sigma_{h_2\xi_{i,2\rho}}\delta_3\sigma_{h_3\xi_{i,3\rho}}\cdots\delta_n\ \sigma_{h_n\xi_{i,n\rho}}\delta_{n\ell}$$

is a consequence of relations A,B,C and D as required. The case $\ell=ht_i^{-1}$ follows in exactly the same way. ◀

Suppose that σ_ℓ has symbol

$$\begin{pmatrix} u_1 & u_2 & \cdots & u_s \\ \ell_1 & \ell_2 & \cdots & \ell_s \\ v_1 & v_2 & \cdots & v_s \end{pmatrix}$$

and let β_1, β_2, γ_i have symbols

$$\begin{pmatrix} u_1 & u_2\ldots u_s \\ 1 & 1\ldots 1 \\ a_n & w_2\ldots w_s \end{pmatrix},\quad \begin{pmatrix} w_2\ldots & w_s & a_n \\ 1\ldots & 1 & 1 \\ v_1\ldots v_{s-1} & & v_s \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} w_2\ldots w_{i-1} & a_n & w_i & w_{i+1}\ldots w_s \\ 1\ldots & 1 & 1\ 1\ 1 & \ldots 1 \\ w_2\ldots w_{i-1} & w_i & a_n & w_{i+1}\ldots w_s \end{pmatrix}$$

respectively, where $\{a_n, w_2,\ldots,w_s\}$ is a cofinite basis. Then the relation $\sigma_\ell = \beta_1\sigma_{\ell_1}\gamma_2\sigma_{\ell_2}\cdots\gamma_s\ \sigma_{\ell_s}\beta_2$ holds in $\langle G_{n,1},H\rangle$.

Lemma 3.3 If σ_ℓ, β_1, β_2, γ_i are as above, then the relation $\sigma_\ell = \beta_1\sigma_{\ell_1}\gamma_2\sigma_{\ell_2}\ldots\gamma_s\sigma_{\ell_s}\beta_2$ is a consequence of relations A,B,C and D.

Proof: By Lemma 2.7 every symbol for a given element of $\langle G_{n,1}, H\rangle$ is an expansion of the unique shortest symbol for this element. Hence the symbol

$$\begin{pmatrix} u_1 & u_2 & \ldots & u_s \\ \ell_1 & \ell_2 & \ldots & \ell_s \\ v_1 & v_2 & \ldots & v_s \end{pmatrix}$$

is an expansion of the symbol

$$\begin{pmatrix} a_1 & \ldots & a_{n-1} & a_n \\ 1 & \ldots & 1 & \ell \\ a_1 & \ldots & a_{n-1} & a_n \end{pmatrix}.$$

The proof is by induction on the number of columns of the form $\begin{pmatrix} a_n w \\ h \\ a_n w_\ell \end{pmatrix}$ in the symbol for σ_ℓ.

If there is only one such column then the symbol for σ_ℓ must be of the form

$$\begin{pmatrix} u_1 & \ldots & u_{s-1} & a_n \\ 1 & \ldots & 1 & \ell \\ u_1 & \ldots & u_{s-1} & a_n \end{pmatrix}$$

and $\sigma_\ell = \beta_1\sigma_1\gamma_2\sigma_1\gamma_3\ldots\sigma_1\gamma_s\sigma_\ell\beta_2 = \beta_1\gamma_2\ldots\gamma_s\sigma_\ell\beta_2$ is the relation we are considering. In this case $v_s = a_n$ and so β_2 fixes a_n, thus the relation $\beta_1\gamma_2\ldots\overset{\sigma_\ell}{\downarrow}\gamma_s\beta_2 = \beta_1\gamma_2\ldots\gamma_s\beta_2\sigma_\ell$ is a consequence of relations B. But $\beta_1\gamma_2\ldots\gamma_s\beta_2 = 1$ and so $\sigma_\ell = \beta_1\gamma_2\ldots\gamma_s\sigma_\ell\beta_2$ is a consequence of relations A and B.

For the inductive step suppose that the symbol for σ_ℓ is of the form

$$\begin{pmatrix} u_1 \cdots u_{q-1}\ a_n x_1 \cdots a_n x_{r-1}\ a_n x_r a_1\ a_n x_r a_2 \cdots a_n x_r a_n \\ 1 \cdots 1 \quad \ell_1 \cdots \ell_{r-1} \quad 1 \quad \ell_{r,2} \cdots \ell_{r,n} \\ u_1 \cdots u_{q-1}\ a_n v_1 \cdots a_n v_{r-1}\ a_n v_r a_1\ a_n v_r a_{2\rho} \cdots a_n v_r a_{n\rho} \end{pmatrix}$$

and that the lemma is true for symbols with less than $r+n-1$ columns of the type above.

The relation that we are considering is

$$\sigma_\ell = \beta_1 \gamma_2 \cdots \gamma_q \quad \sigma_{\ell_1} \gamma_{q+1} \cdots \sigma_{\ell_{r-1}} \gamma_{r+q-1} \gamma_{r+q} \sigma_{\ell_{r,2}} \gamma_{r+q+1} \cdots \sigma_{\ell_{r,n}} \beta_2.$$

If ε_1, ε_2 have symbols respectively

$$\begin{pmatrix} z_2 \cdots z_{r+q-2}\ a_n a_1\ a_n a_2 \cdots a_n a_n \quad a_1 \\ 1 \cdots 1 \quad 1 \quad 1 \cdots 1 \quad 1 \\ w_2 \cdots w_{r+q-2}\ a_n\ w_{r+q} \cdots w_{r+q+n-2}\ w_{q+r-1} \end{pmatrix}$$

and

$$\begin{pmatrix} w_2 \cdots w_{r+q-2}\ w_{r+q}\ w_{r+q+1} \cdots w_{r+q+n-2} \quad a_n \quad w_{r+q-1} \\ 1 \cdots 1 \quad 1 \quad 1 \cdots 1 \quad 1 \quad 1 \\ z_2 \cdots z_{r+q-2}\ a_n a_1\ a_n a_{2\rho} \cdots a_n a_{(n-1)\rho}\ a_n a_{n\rho} \quad a_1 \end{pmatrix}$$

we have the relation $\varepsilon_1 \gamma_{r+q} \sigma_{\ell_{r,2}} \cdots \gamma_{q+r+n-2} \sigma_{\ell_{r,n}} \varepsilon_2 = \sigma_{\ell_r}$. If α_r has symbol

$$\begin{pmatrix} w_2 \cdots w_{q+r-2}\ a_n\ w_{r+q} \cdots w_{r+q+n-2}\ w_{r+q-1} \\ 1 \cdots 1 \quad 1 \quad 1 \cdots 1 \quad 1 \\ z_2 \cdots z_{q+r-2}\ a_n\ a_1 a_2 \cdots a_1 a_n \quad a_1 a_1 \end{pmatrix}$$

then $\gamma_{r+q} = \alpha_r \delta_2 \alpha_r^{-1}$ (δ_i, $2 \leq i \leq n$, is as defined in relations D). Since α_r fixes a_n the relation $\varepsilon_1 \alpha_r \delta_2 \sigma_{\ell_{r,2}} \alpha_r^{-1} = \varepsilon_1 \gamma_{r+q} \sigma_{\ell_{r,2}}$ is a consequence of relations A and B. We also have that $\alpha_r \delta_3 \alpha_r^{-1} = \gamma_{r+q+1}$ so the relation $\sigma_{\ell_{r,2}} \alpha_r^{-1} \gamma_{r+q+1} \sigma_{\ell_{r,3}} = \sigma_{\ell_{r,2}} \delta_3 \sigma_{\ell_{r,3}} \alpha_r^{-1}$ is a consequence of relations A and B. In general, $\alpha_r \delta_i \alpha_r^{-1} = \gamma_{r+q+i-2}$ and so the relation

$\varepsilon_1\alpha_r\delta_2\sigma_{\ell_{r,2}}\delta_3\cdots\sigma_{\ell_{r,n}}\alpha_r^{-1}\varepsilon_2 = \varepsilon_1\gamma_{r+q}\sigma_{\ell_{r,2}}\cdots\gamma_{r+q+n-2}\sigma_{\ell_{r,n}}\varepsilon_2$ is a consequence of relations A and B.

Examination of the symbols for $\varepsilon_1\alpha_r$ and $\alpha_r^{-1}\varepsilon_2$ shows that $\varepsilon_1\alpha_r\delta_\lambda = \delta$ and $\alpha_r^{-1}\varepsilon_2 = \delta_{n\ell}$ where $\delta_{n\ell}$ is as in Lemma 3.2. Thus, by Lemma 3.2, the relation $\sigma_{\ell_r} = \varepsilon_1\gamma_{r+q}\sigma_{\ell_{r,2}}\cdots\gamma_{r+q+n-2}\sigma_{\ell_{n,r}}\varepsilon_2$ is a consequence of relations A,B,C and D. It is thus sufficient to show that $\sigma_\ell = \beta_1\gamma_2\cdots\gamma_q\ \sigma_{\ell_1}\gamma_{q+1}\cdots\gamma_{q+r-1}\varepsilon_1^{-1}\sigma_{\ell_r}\varepsilon_2^{-1}\beta_2$ is a consequence of A,B,C and D.

Let α_q, η_i, μ_1 and μ_2 be the elements with symbols respectively

$$\begin{pmatrix} z_2 & \cdots & z_{q-1} & a_n & z_q & \cdots & z_{r+q-2} & a_1a_1 & \cdots & a_1a_n \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ w_2 & \cdots & w_{q-1} & a_n & w_q & \cdots & w_{r+q-2} & w_{r+q-1} & \cdots & w_{r+q+n-2} \end{pmatrix},$$

$$\begin{pmatrix} z_2 & \cdots & z_{i-1} & a_n & z_i & \cdots & z_{r+q-2} & a_1 \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 \\ z_2 & \cdots & z_{i-1} & z_i & a_n & \cdots & z_{r+q-2} & a_1 \end{pmatrix},$$

$$\begin{pmatrix} u_1 & u_2 & \cdots & u_{q-1} & a_nY_1 & \cdots & a_nY_{r-1} & a_nY_r \\ 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 \\ a_n & z_2 & \cdots & z_{q-1} & z_q & \cdots & z_{q+r-2} & a_1 \end{pmatrix}$$

and $$\begin{pmatrix} z_2 & \cdots & z_q & z_{q+1} & \cdots & z_{q+r-2} & a_1 & a_n \\ 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & 1 \\ u_1 & \cdots & u_{q-1} & a_nv_1 & \cdots & a_nv_{r-2} & a_nv_{r-1} & a_nv_r \end{pmatrix}.$$

Then the relations $\alpha_q^{-1}\sigma_{\ell_i}\alpha_q = \sigma_{\ell_i}$, $\alpha_q\gamma_i\alpha_q^{-1} = \eta_i$, $\mu_1 = \beta_1\alpha_q^{-1}$ and $\mu_2 = \varepsilon_2^{-1}\beta_2$ are consequences of relations A and B. If η_{r+q-1} is the element with symbol

$$\begin{pmatrix} z_2 & \cdots & z_{r+q-2} & a_1 & a_n \\ 1 & \cdots & 1 & 1 & 1 \\ z_2 & \cdots & z_{r+q-2} & a_n & a_1 \end{pmatrix}$$

then $\eta_{r+q-1} = \alpha_q \gamma_{r+q-1} \varepsilon_1^{-1}$ and hence it is sufficient to show that the relation $\sigma_\ell = \mu_1 \sigma_{\ell_1} \eta_2 \sigma_{\ell_2} \cdots \eta_{r+q-2} \sigma_{\ell_{r-1}} \eta_{r+q-1} \sigma_{\ell_r} \mu_2$ is a consequence of relations A, B, C and D. But this is the induction hypothesis so the lemma is proved. ◀

If $\alpha_1 \sigma_{\ell_1} \alpha_2 \ldots \sigma_{\ell_m}$ is an element of $\langle G_{n,1}, H \rangle$ then, from Chapter 2, we see that there exist symbols

$$\begin{pmatrix} u_{i,1} & u_{i,2} & \cdots & u_{i,s} \\ \ell_{i,1} & \ell_{i,2} & \cdots & \ell_{i,s} \\ u_{i+1,1} & u_{i+1,2} & \cdots & u_{i+1,s} \end{pmatrix}$$

for the elements $\alpha_i \sigma_{\ell_i}$ such that

$$\begin{pmatrix} u_{1,1} & \cdots\cdots & u_{1,s} \\ \ell_{1,1} \cdots \ell_{m,1} & \cdots\cdots & \ell_{1,s} \cdots \ell_{m,s} \\ u_{m+1,1} & \cdots\cdots & u_{m+1,s} \end{pmatrix}$$

is a symbol for $\alpha_1 \sigma_{\ell_1} \alpha_2 \ldots \sigma_{\ell_m}$. Also, from Lemma 3.3, we have that the relation $\sigma_{\ell_i} = \mu_1 \sigma_{\ell_{i,1}} \eta_2 \ldots \sigma_{\ell_{i,s}} \mu_2$ is a consequence of relations A, B, C and D.

Lemma 3.4 Every relation in $\langle G_{n,1}, H \rangle$ is a consequence of relations A, B, C and D.

Proof: Suppose that $\alpha_1 \sigma_{\ell_1} \ldots \alpha_m \sigma_{\ell_m} = 1$ and let μ_{2i-1}, μ_{2i}, η_j $(2 \leqslant j \leqslant s,\ 1 \leqslant i \leqslant m)$ have symbols respectively

$$\begin{pmatrix} u_{i,1} & u_{i,2} & \cdots & u_{i,s} \\ 1 & 1 & \cdots & 1 \\ a_n & w_2 & \cdots & w_s \end{pmatrix}, \quad \begin{pmatrix} w_2 & \cdots & w_s & a_n \\ 1 & \cdots & 1 & 1 \\ u_{i+1,1} & \cdots & u_{i+1,s-1} & u_{i+1,s} \end{pmatrix}$$

and $$\begin{pmatrix} w_2 & \cdots & w_{j-1} & a_n & w_j & w_{j+1} & \cdots & w_{s-1} \\ 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 \\ w_2 & \cdots & w_{j-1} & w_j & a_n & w_{j+1} & \cdots & w_{s-1} \end{pmatrix}.$$

Then, by Lemma 3.3, the relations $\sigma_{\ell_i} = \alpha_i^{-1}\mu_{2i-1}\sigma_{\ell_{i,1}}\eta_2\sigma_{\ell_{i,2}}\cdots\eta_s\sigma_{\ell_{i,s}}\mu_{2i}$ are consequences of relations A, B, C and D and so, therefore, is the relation $\mu_1\sigma_{\ell_{1,1}}\eta_2\cdots\sigma_{\ell_{1,s}}\mu_2\mu_3\sigma_{\ell_{2,1}}\cdots\sigma_{\ell_{2,s}}\mu_4\cdots\mu_{2m-1}\sigma_{\ell_{m,1}}\cdots\eta_s\sigma_{\ell_{m,s}}\mu_{2m} = \alpha_1\sigma_{\ell_1}\cdots\alpha_m\sigma_{\ell_m}$. But μ_j and η_i have the form required for Lemma 3.1 so the relation $\mu_1\sigma_{\ell_{1,1}\cdots\ell_{m,1}}\eta_2\sigma_{\ell_{1,2}\cdots\ell_{m,2}}\cdots\eta_s\sigma_{\ell_{1,s}\cdots\ell_{m,s}}\mu_{2m} = \alpha_1\sigma_{\ell_1}\cdots\alpha_m\sigma_{\ell_m}$ is a consequence of relations A, B, C and D.

Because $\alpha_1\sigma_{\ell_1}\cdots\alpha_m\sigma_{\ell_m} = 1$ from the symbol above for this element we see that we must have $\sigma_{\ell_{1,i}\cdots\ell_{m,i}} = 1$ for all $1 \leq i \leq s$. Thus the relation $\mu_1\eta_2\cdots\eta_s\mu_{2m} = \alpha_1\sigma_{\ell_1}\cdots\alpha_m\sigma_{\ell_m}$ is a consequence of relations A, B, C and D. Then $\mu_1\eta_2\cdots\eta_s\mu_{2m} = 1$ is a relation in A so the relation $\alpha_1\sigma_{\ell_1}\cdots\alpha_m\sigma_{\ell_m} = 1$ is a consequence of relations A, B, C and D. ◀

Lemma 3.4 gives that the relations A, B, C and D form an essentially finite set of relations which defines the group $\langle G_{n,1}, H\rangle$ on the generators of $G_{n,1}$ and the elements $\sigma_{t_i} \in H$. Thus we have the following theorem.

Theorem 3.5 If H is a finitely presented group defined by a finite set of expansion rules, then $\langle G_{n,1}, H\rangle$ is a finitely presented subgroup of $\mathcal{G}_{n,1}$ with simple derived subgroup.

3.2 Commutator subgroups.

If H is defined by a finite set of expansion rules of the form

$$t_i \to \begin{pmatrix} a_2 & \cdots & a_n \\ \xi_{i,2} & \cdots & \xi_{i,n} \\ a_{2\pi_i} & \cdots & a_{n\pi_i} \end{pmatrix} \qquad (1 \leq i \leq k)$$

then H is also defined by any set of expansion rules of the form

$$t_i \to \begin{pmatrix} a_2 & \cdots & a_n & a_{n+1} & \cdots & a_{n+p} & a_{n+p+1} & \cdots & a_{n+p+r} \\ \xi_{i,2} & \cdots & \xi_{i,n} & t_i & \cdots & t_i & 1 & \cdots & 1 \\ a_{2\pi_i} & \cdots & a_{n\pi_i} & a_{n+1} & \cdots & a_{n+p} & a_{n+p+1} & \cdots & a_{n+p+r} \end{pmatrix}$$

for $1 \leq i \leq k$, for any positive integers p and r. We show that we can choose p and r so that $\langle G_{n+p+r,1}, H\rangle'$ has finite index in $\langle G_{n+p+r,1}, H\rangle = G$.

We will choose r so that n+p+r is even and thus $G_{n+p+r}' = G_{n+p+r} \leq G'$.

From relations D we have that

$$\sigma_{t_i} = \delta\, \sigma_{\xi_{i,2}} \cdots \delta_n \sigma_{\xi_{i,n}} \delta_{n+1} \sigma_{t_i} \cdots \delta_{n+p} \sigma_{t_i} \delta_{n,i}$$

and so $\sigma_{t_i} = \tau_i \sigma_{\xi_{i,2}} \cdots \xi_{i,n} \sigma_{t_i}^p$, for some $\tau_i \in G'$. Furthermore, if m_{ij} is the sum of powers of t_j in $\xi_{i,2} \cdots \xi_{i,n} t_i^{-1}$ we have that $\varepsilon_i = \sigma_{t_1}^{m_{i1}} \cdots \sigma_{t_k}^{m_{ik}} \sigma_{t_i}^p$, for some $\varepsilon_i \in G'$.

Consider the integer matrix

$$M = \begin{pmatrix} m_{11}+p & m_{12} & \dots & m_{1k} \\ m_{21} & m_{22}+p & \dots & m_{2k} \\ \vdots & \vdots & & \vdots \\ m_{k1} & m_{k2} & \dots & m_{kk}+p \end{pmatrix}$$

and choose p large enough so that detM >0. Then, using elementary row and column operations on M, we can construct the matrix

$$M' = \begin{pmatrix} b_1 & 0 & \dots & 0 \\ \cdot & b_2 & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \dots & b_k \end{pmatrix}.$$

If we carry out the corresponding manipulations on the set of relations $\{\varepsilon_i = \sigma_{t_1}^{m_{i1}} \dots \sigma_{t_k}^{m_{ik}} \sigma_{t_i}^{p} \mid 1 \leq i \leq k \}$ we obtain k relations of the form $\sigma_{t_i}^{b_i} = \varepsilon_i'$, for some $\varepsilon_i' \in G'$. Thus G' has index at most $b_1 b_2 \dots b_k$ in G.

Theorem 3.6 If H is a finitely presented group defined by a finite set of expansion rules then we can choose an integer m and then a set of expansion rules defining H such that the commutator subgroup, $\langle G_{m,1}, H \rangle'$, is a finitely presented simple group.

3.3 The word and order problems.

We now show that if the word and order problem are solvable in H they are solvable in $\langle G_{n,1}, H \rangle$. The former is easier to prove than the latter.

Theorem 3.7 If the word problem is solvable in H then it is solvable in $\langle G_{n,1}, H\rangle$.

Proof: If $\tau \in \langle G_{n,1}, H\rangle$ then we can find, constructively, a symbol for τ by writing it as a product of elements of $G_{n,1}$ and H and using Lemma 2.5.

Let

$$\begin{pmatrix} u_1 & u_2 & \dots & u_s \\ \ell_1 & \ell_2 & \dots & \ell_s \\ v_1 & v_2 & \dots & v_s \end{pmatrix}$$

be a symbol for τ. If $u_i \neq v_i$ then $(u_i a_1)\tau = v_i a_1 \neq u_i a_1$ so $\tau \neq 1$. Suppose $u_i = v_i$ for all $1 \leq i \leq s$.

If $\sigma_{\ell_i} = 1$ for all $1 \leq i \leq s$ then $\tau = 1$ since the symbol above is the same as the symbol

$$\begin{pmatrix} u_1 & \dots & u_s \\ 1 & \dots & 1 \\ u_1 & \dots & u_s \end{pmatrix}.$$

If $\sigma_{\ell_i} \neq 1$ for some i then, for some $x \in W$, $x\sigma_{\ell_i} \neq x$. By the definition of σ_{ℓ_i}, $x = a_n w a_1 u$ for some u and for some non-trivial w which does not contain a_1. Because $x\sigma_{\ell_i} = a_n w_{\ell_i} a_1 u \neq x$ we must have $w \neq w_{\ell_i}$. But $(u_i w a_1)\tau = u_i w_{\ell_i} a_1$ (see proof of Lemma 2.2) and so $\tau \neq 1$.

Thus, if we can decide whether $\sigma_{\ell_i} = 1$ for each $1 \leq i \leq s$, the word problem is solvable in $\langle G_{n,1}, H\rangle$. ◀

To solve the order problem we need some definitions and preliminary discussions.

If $\tau \in \langle G_{n,1}, H \rangle$ has unique shortest symbol

$$\begin{pmatrix} u_1 & u_2 & \dots & u_s \\ \ell_1 & \ell_2 & \dots & \ell_s \\ v_1 & v_2 & \dots & v_s \end{pmatrix}$$

then we define a map $\bar{\tau}: UW \to VW$ $(U=\{u_1,\dots,u_s\}, V=\{v_1,\dots,v_s\})$ by $(u_i w)\bar{\tau} = v_i w_{\ell_i}$. This map is not a W-space homomorphism, but it is an extension of τ as a map. Define an operation on the set $\{\bar{\tau} \mid \tau \in \langle G_{n,1}, H \rangle\}$ by the rule $\bar{\tau}.\bar{\varepsilon} = \overline{\tau\varepsilon}$. Then $\bar{\tau}^m = \overline{(\tau^m)}$.

We define a pseudo-orbit (p-orbit) of τ, $O(w)$, to be a set of the form $\{\dots w\bar{\tau}^{-m},\dots,w\bar{\tau}^{-1}, w, w\bar{\tau},\dots,w\bar{\tau}^{k},\dots\}$. For each $w \in W$, $O(w)$ is called the p-orbit of τ generated by w. If $w\bar{\tau}$, $w\bar{\tau}^{-1}$ are defined then $O(w) = O(w\bar{\tau}) = O(w\bar{\tau}^{-1})$. However, if $w\bar{\tau}$ is not defined but $w\bar{\tau}^i$ is defined for some $i>1$ then $O(w) \neq O(w\bar{\tau}^i)$.

There are five types of p-orbit:

A. Finite Complete. If $w\bar{\tau}^i$ is defined for all $i \in \mathbb{Z}$ and, for some finite m, $w\bar{\tau}^m = w$ then $O(w)$ is finite complete.

B. Infinite Complete. If $w\bar{\tau}^i$ is defined for all $i \in \mathbb{Z}$ and $w\bar{\tau}^m \neq w$ for any m then $O(w)$ is infinite complete.

C. Right Semi-infinite. If $w\bar{\tau}^i$ is defined for all $i \in \mathbb{N}$ but, for some $j<0$, $w\bar{\tau}^j$ is not defined then $O(w)$ is right semi-infinite.

D. Left Semi-infinite. If $w\bar{\tau}^{i}$ is defined for all $-i \in \mathbb{N}$ but $w\bar{\tau}^{j}$ is not defined, for some $j>0$, then $O(w)$ is left semi-infinite.

E. Finite Incomplete. If, for some $m,k \in \mathbb{N}$, $w\bar{\tau}^{-k}$, $w\bar{\tau}^{m}$ are not defined then $O(w)$ is finite incomplete.

Lemma 3.8 Each $\tau \in \langle G_{n,1}, H \rangle$ has only finitely many p-orbits of types C, D and E.

Proof: If $O(w) = \{w\bar{\tau}^{-i}, \ldots, w, w\bar{\tau}, \ldots\}$ is a p-orbit of τ of type C then $w\bar{\tau}^{-(i+1)}$ is not defined and hence $w\bar{\tau}^{i}$ must be a proper initial segment of some v_i in the shortest symbol for τ. Any element completely defines its p-orbit and there are only finitely many elements in $W \backslash VW$ so there are only finitely many p-orbits of type C.

Similarly, τ has only finitely many p-orbits of type D since $W \backslash UW$ is finite.

A p-orbit of τ of type E must contain an element from both $W \backslash UW$ and $W \backslash VW$ and hence τ has only finitely many p-orbits of type E. ◀

Note: For fixed τ there are only finitely many p-orbits of type E each containing only finitely many elements. Thus there are only finitely many elements of w which belong to p-orbits of τ of type E.

We say that a symbol for τ is in quasinormal form if none of the elements of the top row of the symbol lie in p-orbits of type E and if no proper contraction of the symbol is in quasinormal form.

Lemma 3.9 Given $\tau \in \langle G_{n,1}, H\rangle$ we can construct a symbol for τ which is in quasinormal form.

Proof: We can construct the unique shortest symbol for τ,

$$\begin{pmatrix} u_1 & u_2 & \dots & u_s \\ \ell_1 & \ell_2 & \dots & \ell_s \\ v_1 & v_2 & \dots & v_s \end{pmatrix}.$$

For each u_i construct $u_i\bar{\tau}^j$ for $j=1,2,\dots$. Eventually, either we reach k such that $u_i\bar{\tau}^k$ is not defined or, because each u_q has only finitely many initial segments, for some $z,x \in W$, $u_i\tau^k = u_pz$ and $u_i\bar{\tau}^m = u_px$, for $k<m$ and $1\leq p\leq s$. In the latter case we have $(u_pz)\bar{\tau}^{(m-k)} = u_px$. If neither $u_p\bar{\tau}^{(m-k)}$ nor $u_p\bar{\tau}^{(k-m)}$ is defined then $O(u_p)$ is of type E, so we expand the above symbol so that it has top row $\{u_1,\dots, u_{p-1}, u_pa_1,\dots,u_pa_n,u_{p+1},\dots,u_s\}$ and begin again. If $u_p\bar{\tau}^{(m-k)}$ is defined then $u_px = u_p\bar{\tau}^{(m-k)}z_h$, for some h, and so $u_py = u_p\bar{\tau}^{(m-k)}$ or $u_p = u_p\bar{\tau}^{(m-k)}y$, for some y. In the first case $u_i\bar{\tau}^j$ is defined for all $j\geq 0$ and in the second case $u_p\bar{\tau}^{(k-m)} = u_pyh$ and $u_i\bar{\tau}^j$ is defined for all $j\leq k$. In either case $O(u_i)$ is not of the type E. If $u_p\bar{\tau}^{(k-m)}$ is defined then $u_pz = u_p\bar{\tau}^{(k-m)}x_h$ for some h and again $O(u_i)$ is infinite.

In the latter case, $u_i\bar{\tau}^k$ not defined, construct $u_i\bar{\tau}^j$ for $j= -1,-2,\dots$. Eventually we either reach $m<0$ such that $u_i\bar{\tau}^m$ is not defined or for some $z,x \in W$, $u_i\bar{\tau}^k = u_pz$ and $u_i\bar{\tau}^m = u_px$ for $k<m\leq 0$ and

$1 \leq p \leq s$. In the first case $O(u_i)$ is of type E so we expand the above symbol so that the top row contains $u_i a_q$ for $1 \leq q \leq n$ and begin again. The second case works as for $j>0$ and either $O(u_p)$ is of type E, in which case we expand and start again, or $O(u_i)$ is infinite and therefore not of type E.

This process stops eventually since we know there are only finitely many elements in p-orbits of type E. Thus we are left with a symbol in which none of the elements of the top row lie in p-orbits of type E. We then apply the process to all the finitely many contractions of this symbol to test if they have the required form. ◀

Lemma 3.10 If $\tau \in \langle G_{n,1}, H \rangle$ has a p-orbit of type B, C or D then τ has infinite order.

Proof: If τ has a p-orbit of type B or C then for some $w \in W$, $w\bar{\tau}^i$ is defined for all $i>0$, but $w\bar{\tau}^i \neq w$ for any such i. If $w\bar{\tau}$ is defined then $(wa_1)\tau$ is defined and $(wa_1)\tau = w\bar{\tau}a_1$. Thus $(wa_1)\tau^i = w\bar{\tau}^i a_1$ for all $i \geq 0$ and hence τ does not have finite order.

If τ has a p-orbit of type D then $w\bar{\tau}^i$ defined for some w and for all $i \leq 0$, with $w\bar{\tau}^i \neq w$ for any i. Then $(wa_1)\tau^i = w\bar{\tau}^i a_1$ and so τ has infinite order. ◀

Lemma 3.11 For $\tau \in \langle G_{n,1}, H \rangle$ either τ has a symbol whose top row is a permutation of its bottom row or we can tell that τ has infinite order.

Proof: Construct a symbol,

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_r \\ h_1 & h_2 & \cdots & h_r \\ z_1 & z_2 & \cdots & z_r \end{pmatrix},$$

for τ in quasinormal form. Let M be the number of words $w \in W$ which satisfy $wz = y_i$, some $z \in W \setminus \{1\}$, $1 \leq i \leq r$. For each y_i construct $y_i \bar{\tau}^j$ for $j = 1,2,..,M$. If $y_i \bar{\tau}^j = y_i$ for some j then $O(y_i)$ is finite complete. If, for some $j \leq M$, $y_i \bar{\tau}^j$ is not defined construct $y_i \bar{\tau}^j$ for $j = -1,-2,\ldots$. Since $O(y_i)$ is not of type E, $y_i \bar{\tau}^j$ must be defined for all $j<0$ and hence there exist $z,x \in W$ such that $y_i \bar{\tau}^k = y_q z$ and $y_i \bar{\tau}^m = y_q x$ for some $k<m<0$ and $1 \leq q \leq s$ (because each y_p has only finitely many initial segments and there are infinitely many $y_i \bar{\tau}^j$). Thus we have that $(y_q x) \bar{\tau}^{(k-m)} = y_q z$. But $O(y_q)$ is not of type E and thus $y_q \bar{\tau}^p$ is defined for all $(k-m) \leq p < 0$ or $0 \leq p < (m-k)$. In the first case $O(y_q)$ is of type B or D and in the second case, by Lemma 3.9, τ has infinite order.

If $y_i \bar{\tau}^j$ is defined for sufficiently large $j > 0$ then we can find $x,z \in W$ such that $y_i \bar{\tau}^k = y_q z$ and $y_i \bar{\tau}^m = y_q x$ for some $0 \leq k < m$ and $1 \leq q \leq s$. Then, as for $j<0$ above, $O(y_q)$ is of type B, C or D and thus τ has infinite order.

Thus, either we can tell that τ has infinite order or $O(y_i)$ is of type A for all $1 \leq i \leq s$.

Suppose z_1 is a proper initial segment of y_i. Since $\{y_1,\ldots,y_r\}$ is a cofinite basis we have that $z_1 w_j \in \{y_1,\ldots,y_r\}$ for some cofinite basis $\{w_1,\ldots,w_k\}$. Since $O(y_1) = O(z_1)$ is assumed to be of type A, $z_1 \bar{\tau}$ is defined and hence the columns

$$\begin{pmatrix} z_1 w_1 & \cdots & z_1 w_k \\ h_{i_1} & \cdots & h_{i_k} \\ z_1\bar{\tau}(w_1)_{\ell} & \cdots & z_1\bar{\tau}(w_k)_{\ell} \end{pmatrix} \qquad (*)$$

belong to the above symbol for τ.

The column $\begin{pmatrix} z_1 \\ \ell \\ z_1\bar{\tau} \end{pmatrix}$ belongs to some symbol for τ, since $z_1\bar{\tau}$ is defined, thus the set of columns (*) must contract to this column. So the symbol

$$\begin{pmatrix} z_1 & y_{i_{k+1}} & \cdots & y_{i_r} \\ \ell & h_{i_{k+1}} & \cdots & h_{i_r} \\ z_1\bar{\tau} & z_{i_{k+1}} & \cdots & z_{i_r} \end{pmatrix}$$

is also a symbol for τ. But $O(z_1) = O(y_1)$ is not of type E and so this contradicts the assumption that the first symbol for τ above is in quasinormal form. Thus z_1 (and similarly any z_j) cannot be a proper initial segment of any y_i.

If some y_i is a proper initial segment of z_1 then we proceed as above to show that the column $\begin{pmatrix} y_i\tau^{-1} \\ \ell \\ y_i \end{pmatrix}$ belongs to some symbol for τ and that the columns

$$\begin{pmatrix} y_i\bar{\tau}^{-1}(w_1)_{\ell^{-1}} & \cdots & y_i\bar{\tau}^{-1}(w_k)_{\ell^{-1}} \\ h_{i_1} & \cdots & h_{i_k} \\ y_i w_1 & \cdots & y_i w_k \end{pmatrix}$$

belong to the first symbol above. This also shows that the symbol contracts in a manner contrary to assumption. Thus no y_i can be a proper initial segment of z_1 (or, similarly, any z_j).

Thus, since $\{y_1,\ldots,y_z\}$ is a cofinite basis, we must have $z_1=y_i$ for some $1<i<r$. Similarly we must have $z_j = y_{i_r}$ for each $1<j<r$ and since $z_i \neq z_j$ unless $i=j$ we have $\{z_1,\ldots,z_r\} = \{y_1,\ldots,y_r\}$. ◀

The above lemma shows that either τ can be seen to have infinite order or else we can construct a symbol for τ of the form.

$$\begin{pmatrix} y_1 \cdots y_b \cdots & y_p & \cdots y_s \\ h_1 \cdots h_b \cdots & h_p & \cdots h_s \\ y_2 \cdots y_1 \cdots & y_{p+1} & \cdots y_p \end{pmatrix}.$$

Thus we have the following theorem.

Theorem 3.12 The order problem is solvable in $\langle G_{n,1},H\rangle$ if it is solvable in H.

Proof: Suppose τ has the above symbol. If m is the lowest common multiple of the lengths, $b,\ldots,(s-p)$, of the cycles in this symbol then τ^m has symbol

$$\begin{pmatrix} y_1 & \cdots\cdots & y_s \\ (h_1\ldots h_b)^{\frac{m}{b}} & \cdots\cdots & (h_p\ldots h_s)^{\frac{m}{s-p}} \\ y_1 & \cdots\cdots & y_s \end{pmatrix}.$$

Since τ has finite order if and if only if τ^m does, τ has finite order if and only if each of the elements $\sigma_{h_1\ldots h_b}$, ..., $\sigma_{h_p\ldots h_s} \in H$ do.

Thus the order problem in $\langle G_{n,1},H\rangle$ is solvable if and only it is solvable in H. ◀

3.4 Residual finiteness

If $H < \mathcal{G}_{n,1}$ is defined by a finite set of expansion rules let $\sigma_\ell \in H$ be non-trivial. From the standard embedding we see that $\sigma_\ell \neq 1$ gives that, for some r, $\ell\phi^{(r)} \neq 1$. The map $\theta: \sigma_h \to h\phi^{(r)}$ is a homomorphism from H to $S_{n-1}(wrS_{n-1})^{r-1}$. The kernel of this homomorphism has finite index in H and does not contain σ_ℓ, since $\sigma_\ell\theta = \ell\phi^{(r)} \neq 1$. Thus we have the following theorem.

Theorem 3.13 If $H < \mathcal{G}_{n,1}$ is given by a set of expansion rules then H is residually finite.

This theorem puts a restriction on the finitely generated groups with solvable word problem that we can hope to construct from a set of expansion rules of this type. For example, the group $\langle s,t \mid s^{-1}t^2s=t^3\rangle$ is a one relator group, and hence has solvable word problem, but is not residually finite [1].

Of course, this theorem does not put an immediate restriction on the subgroups of $\langle G_{n,1},H\rangle$ itself but the strength of this method lies in the fact that we can make H the group that we want to embed in a finitely presented simple group. It seems to be very hard to find out what general subgroups of $\langle G_{n,1},H\rangle$ can look like.

Chapter 4

Linear Groups

We now use the construction described in the previous chapters to show that we can embed $GL(n,\mathbb{Z})$, the linear group of invertible integer matrices, in a finitely presented simple group. We do this by showing that the semidirect product, $\mathbb{Z}^n.GL(n,\mathbb{Z})$, is a group defined by a finite set of expansion rules, as defined in Chapter 2. The nature of the expansion rules obtained makes it clear that for any finitely generated subgroup, L, of $GL(n,\mathbb{Z})$, the semidirect product, $\mathbb{Z}^nL$, is a group defined by a finite set of expansion rules.

We also show that two elements of L are conjugate in $\langle G_{m,1}, \mathbb{Z}^nL\rangle$ only if they are conjugate in L and that two elements of $\mathbb{Z}^n$ are conjugate in $\langle G_{m,1}, \mathbb{Z}^nL\rangle$ if and only if the conjugating element can be chosen from L. From this we show that, if there exists a finitely presented linear group with unsolvable conjugacy problem or unsolvable orbit problem, we can construct a finitely presented simple group with unsolvable conjugacy problem.

4.1 The construction of $\langle G_{2n+2,1}, \mathbb{Z}^n.GL(n,\mathbb{Z})\rangle$ and $\langle G_{m,1}, \mathbb{Z}^nL\rangle$.

Lemma 4.1 The free Abelian group of rank n, $\mathbb{Z}^n$, is a group defined by a finite set of expansion rules.

Proof: Consider the n expansion rules

$$t_1 \to \begin{pmatrix} a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & \cdots & a_{2n} & a_{2n+1} & a_{2n+2} \\ t_1 & 1 & t_1 & 1 & t_1 & 1 & \cdots & t_1 & 1 & 1 \\ a_3 & a_2 & a_5 & a_4 & a_7 & a_6 & \cdots & a_{2n+1} & a_{2n} & a_{2n+2} \end{pmatrix}$$

$$t_2 \to \begin{pmatrix} a_2 & a_3 & a_4 & a_5 & \cdots & a_{2n-2} & a_{2n-1} & a_{2n} & a_{2n+1} & a_{2n+2} \\ t_2 & t_2 & 1 & 1 & \cdots & t_2 & t_2 & 1 & 1 & 1 \\ a_4 & a_5 & a_2 & a_3 & \cdots & a_{2n} & a_{2n+1} & a_{2n-2} & a_{2n-1} & a_{2n+2} \end{pmatrix}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$t_n \to \begin{pmatrix} a_2 & a_3 & \cdots & a_{2n-1+1} & a_{2n-1+2} & a_{2n-1+3} & \cdots & a_{2n+1} & a_{2n+2} \\ t_n & t_n & \cdots & t_n & 1 & 1 & \cdots & 1 & 1 \\ a_{2n-1+2} & a_{2n-1+3} & \cdots & a_{2n+1} & a_2 & a_3 & \cdots & a_{2n-1+1} & a_{2n+2} \end{pmatrix}.$$

If σ_{t_i} is the unique element of $\mathcal{G}_{2n+2,1}$ corresponding to t_i, under the standard embedding of $F\langle t_1,\ldots,t_n\rangle$ in $\mathcal{G}_{2n+2,1}$ (see Chapter 2), then we show that $\langle \sigma_{t_1},\ldots,\sigma_{t_n} \rangle$ is the free Abelian group of rank n.

The symbol

$$\begin{pmatrix} a_1 & \cdots & a_{2n+1} & a_{2n+2}a_1 & a_{2n+2}a_2 & \cdots & a_{2n+2}a_{2n+1} & a_{2n+2}a_{2n+2} \\ 1 & \cdots & 1 & 1 & t_i & \cdots & 1 & 1 \\ a_1 & \cdots & a_{2n+1} & a_{2n+2}a_1 & a_{2n+2}a_{2i-1+2} & \cdots & a_{2n+2}a_{2n-2i-1+1} & a_{2n+2}a_{2n+2} \end{pmatrix}$$

is a symbol for σ_{t_i} and so $\sigma_{t_i} \neq 1$. In fact, σ_{t_i} cannot have odd order since $(a_{2n+2}a_2a_1)\sigma_{t_i}^{m} = a_{2n+2}a_{2i-1+2}a_1$ for all odd integers, m. If $\sigma_{t_i}^{2m} = 1$ then the symbol

$$\begin{pmatrix} a_1 & \cdots & a_{2n+1} & a_{2n+2}a_1 & a_{2n+2}a_2 & a_{2n+2}a_3 & \cdots & a_{2n+2}a_{2n+1} & a_{2n+2}a_{2n+2} \\ 1 & \cdots & 1 & 1 & t_i^m & t_i^m & \cdots & t_i^m & 1 \\ a_1 & \cdots & a_{2n+1} & a_{2n+2}a_1 & a_{2n+2}a_2 & a_{2n+2}a_3 & \cdots & a_{2n+2}a_{2n+1} & a_{2n+2}a_{2n+2} \end{pmatrix}$$

is a symbol for the identity. For all $w \in W$, $(a_{2n+2}a_2wa_1)\sigma_{t_i}^{2m} = a_{2n+2}a_2wt_i^m a_1$ and so $wt_i^m = w$. Thus $(a_{2n+2}w)\sigma_{t_i}^{m} = a_{2n+2}wt_i^m = a_{2n+2}w$ whenever it is defined and so, if $\sigma_{t_i}^{2m} = 1$, then $\sigma_{t_i}^{m} = 1$. But $\sigma_{t_i}^{m} \neq 1$ for any odd m (including m=1) and so we deduce that each σ_{t_i} must have infinite order.

The element $\sigma_{t_i^{-1}t_j^{-1}t_it_j}$ has symbol

$$\begin{pmatrix} a_1 & \cdots & a_{2n+1} & a_{2n+2}a_1 & a_{2n+2}a_2 & \cdots & a_{2n+2}a_{2n+1} & a_{2n+2}a_{2n+2} \\ 1 & \cdots & 1 & 1 & \delta_1 & \cdots & \delta_{2n+1} & 1 \\ a_1 & \cdots & a_{2n+1} & a_{2n+2}a_1 & a_{2n+2}a_2 & \cdots & a_{2n+2}a_{2n+1} & a_{2n+2}a_{2n+2} \end{pmatrix}$$

where $\delta_k = 1$ or $t_i^{-1}t_j^{-1}t_it_j$. Let $\ell = [t_i, t_j]$ and let σ_ℓ^* be the element defined by $u\sigma_\ell^* = v$ if and only if the column $\begin{pmatrix} u \\ 1 \\ v \end{pmatrix}$ belongs to some symbol for σ_ℓ. Then, as in Lemma 2.3, σ_ℓ is the unique maximal extension of σ_ℓ^*.

By inspection of the symbols for σ_ℓ we see that, for words u of length 1 or 2, one of the columns $\begin{pmatrix} u \\ 1 \\ u \end{pmatrix}$ and $\begin{pmatrix} u \\ \ell \\ u \end{pmatrix}$ belongs to a symbol for σ_ℓ. Suppose that this is true for words u of length less than k. From the expansion rule for ℓ we have that either the column $\begin{pmatrix} ua_i \\ 1 \\ ua_i \end{pmatrix}$ or the column $\begin{pmatrix} ua_i \\ \ell \\ ua_i \end{pmatrix}$ belongs to a symbol for σ_ℓ, $1 \leq i \leq 2^{n+2}$. Thus, by induction, all the columns of all symbols for σ_ℓ are of one of the two types above.

So, if $v\sigma_\ell^*$ is defined, we must have $v\sigma_\ell^* = v$. Thus, σ_ℓ is the unique maximal extension of the trivial element and so $\sigma_\ell=1$.

We have that $\langle\sigma_{t_1},\ldots,\sigma_{t_n}\rangle$ is an Abelian group, all of whose generators have infinite order. If $\sigma_{t_1}^{q_1}\ldots\sigma_{t_n}^{q_n} = 1$, for some integers, $q_1,\ldots,q_n$, then all of these integers must be even. (If not, then the permutation of the bottom row of a symbol for $\sigma_{t_1}^{q_1}\ldots\sigma_{t_n}^{q_n}$, with respect to the top row, is non-trivial, thus the symbol cannot be a symbol for the identity). Suppose $\sigma_{t_1}^{2q_1}\ldots\sigma_{t_n}^{2q_n} = 1$. From the symbol,

$$\begin{pmatrix} a_1 & \ldots & a_{2n+1} & a_{2n+2}a_1 & a_{2n+2}a_2 & \ldots & a_{2n+2}a_{2n+1} & a_{2n+2}a_{2n+2} \\ 1 & \ldots & 1 & 1 & h & \ldots & h & 1 \\ a_1 & \ldots & a_{2n+1} & a_{2n+2}a_1 & a_{2n+2}a_2 & \ldots & a_{2n+2}a_{2n+1} & a_{2n+2}a_{2n+2} \end{pmatrix},$$

where $h = t_1^{q_1}\ldots t_n^{q_n}$, for $\sigma_{t_1}^{2q_1}\ldots\sigma_{t_n}^{2q_n} = \sigma_h^2$ we see that, if $\sigma_h^2 = 1$, then $\sigma_h = 1$. Thus, $\sigma_{t_1}^{q_1}\ldots\sigma_{t_n}^{q_n} \neq 1$ for any $q_1,\ldots,q_n \neq 0$, so $\langle\sigma_{t_1},\ldots,\sigma_{t_n}\rangle$ is the free Abelian group of rank n. ◀

We now show that we can obtain an expansion rule for each automorphism of $\mathbb{Z}^n$, given the expansion rules for the t_i described in Lemma 4.1.

The automorphisms of $\mathbb{Z}^n = \langle T_1,\ldots,T_n\rangle$ are generated by the elementary automorphisms X, Y and Z_i given by

$T_1X = T_1^{-1}$	$T_1Y = T_1T_2$	$T_1Z_i = T_i$	for $2\leq i\leq n$, $2\leq j\neq i\leq n$.
$T_iX = T_i$	$T_iY = T_i$	$T_iZ_i = T_1$	
		$T_jZ_i = T_j$	

Consider the expansion rule

$$x \to \begin{pmatrix} a_2 & a_3 & a_4 & a_5 & \dots & a_{2n} & a_{2n+1} & a_{2n+2} \\ x & xt_1 & x & xt_1 & \dots & x & xt_1 & 1 \\ a_2 & a_3 & a_4 & a_5 & \dots & a_{2n} & a_{2n+1} & a_{2n+2} \end{pmatrix}.$$

Let σ_x be the image of x in $\mathcal{G}_{2n+2,1}$ under the standard embedding. The element $x^{-1}t_1xt_1 = \ell$ has expansion rule

$$\ell \to \begin{pmatrix} a_2 & a_3 & a_4 & a_5 & \dots & a_{2n} & a_{2n+1} & a_{2n+2} \\ \ell & 1 & \ell & 1 & \dots & \ell & 1 & 1 \\ a_2 & a_3 & a_4 & a_5 & \dots & a_{2n} & a_{2n+1} & a_{2n+2} \end{pmatrix}.$$

If σ^*_ℓ is defined as in Lemma 4.1 then the above expansion rule shows that $v\sigma^*_\ell = v$ whenever it is defined. So, as in the proof of Lemma 4.1, $\sigma_\ell = 1$. Thus, $\sigma_x^{-1}\sigma_{t_1}\sigma_x = \sigma_{t_1}^{-1}$.

If $i \neq 1$, let $h = [x, t_i^{-1}]$. Then, since $\langle \sigma_{t_1}, \dots, \sigma_{t_n} \rangle$ is Abelian and

$$\sigma_{t_1}^{-1}\sigma_x^{-1}\sigma_{t_i}\sigma_x\sigma_{t_i}^{-1}\sigma_{t_1} = \sigma_x^{-1}\sigma_{t_1}\sigma_{t_i}\sigma_x\sigma_{t_i}^{-1}\sigma_{t_1} = \sigma_x^{-1}\sigma_{t_i}\sigma_x\sigma_{t_i}^{-1} = \sigma_h,$$

the expansion rule for h is equivalent to the expansion rule

$$h \to \begin{pmatrix} a_2 & a_3 & a_4 & a_5 & \dots & a_{2n} & a_{2n+1} & a_{2n+2} \\ h & h & h_4 & h_5 & \dots & h_{2n} & 1 & 1 \\ a_2 & a_3 & a_4 & a_5 & \dots & a_{2n} & a_{2n+1} & a_{2n+2} \end{pmatrix}$$

where $h_j = 1, h$ for $4 \leq j \leq 2^n$. Again, consideration of σ^*_h shows that $\sigma_h = \sigma_x^{-1}\sigma_{t_i}\sigma_x\sigma_{t_i}^{-1} = 1$.

Now consider the expansion rule

$$y \to \begin{pmatrix} a_2 & a_3 & a_4 & a_5 & \dots\dots & a_{2n-1} & a_{2n} & a_{2n+1} & a_{2n+2} \\ y & y & y & yt_2^{-1} & \dots\dots & y & y & yt_2^{-1} & 1 \\ a_2 & a_5 & a_4 & a_3 & \dots\dots & a_{2n+1} & a_{2n} & a_{2n-1} & a_{2n+2} \end{pmatrix}.$$

Then, if $\ell = y^{-1}t_1yt_2^{-1}t_1^{-1}$, ℓ has expansion rule

$$\ell \to \begin{pmatrix} a_2 & a_3 & a_4 & a_5 & \cdots & a_{2n} & a_{2n+1} & a_{2n+2} \\ \ell & 1 & \ell & 1 & \cdots & \ell & 1 & 1 \\ a_2 & a_3 & a_4 & a_5 & \cdots & a_{2n} & a_{2n+1} & a_{2n+2} \end{pmatrix}$$

and so $\sigma_\ell = 1$. If $k = y^{-1}t_2 y t_2^{-1}$ then k has expansion rule

$$k \to \begin{pmatrix} a_2 & a_3 & a_4 & a_5 & \cdots & a_{2n} & a_{2n+1} & a_{2n+2} \\ k & 1 & 1 & k & \cdots & 1 & k & 1 \\ a_2 & a_3 & a_4 & a_5 & \cdots & a_{2n} & a_{2n+1} & a_{2n+2} \end{pmatrix}$$

and so $\sigma_y^{-1}\sigma_{t_2}\sigma_y = \sigma_{t_2}$. Then, if $h = y^{-1}t_i y t_i^{-1}$ for $3 \leq i \leq n$, we have that $\sigma_{t_2}\sigma_y^{-1}\sigma_{t_i}\sigma_y\sigma_{t_i}^{-1}\sigma_{t_2}^{-1} = \sigma_h$ and so we can assume that h has expansion rule

$$h \to \begin{pmatrix} a_2 & a_3 & a_4 & a_5 & \cdots & a_{2n} & a_{2n+1} & a_{2n+2} \\ h & h & h_4 & h_5 & \cdots & h_{2n} & 1 & 1 \\ a_2 & a_3 & a_4 & a_5 & \cdots & a_{2n} & a_{2n+1} & a_{2n+2} \end{pmatrix}$$

where $h_j = 1$ or h, for $4 \leq j \leq 2^n$. Thus $\sigma_y^{-1}\sigma_{t_i}\sigma_y = \sigma_{t_i}$.

Finally, consider the expansion rules

$$z_2 \to \begin{pmatrix} a_2 & a_3 & a_4 & a_5 & \cdots & a_{2n-2} & a_{2n-1} & a_{2n} & a_{2n+1} & a_{2n+2} \\ z_2 & z_2 & z_2 & z_2 & \cdots & z_2 & z_2 & z_2 & z_2 & 1 \\ a_2 & a_4 & a_3 & a_5 & \cdots & a_{2n-2} & a_{2n} & a_{2n-1} & a_{2n+1} & a_{2n+2} \end{pmatrix}$$

$$z_3 \to \begin{pmatrix} a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & \cdots & a_{2n} & a_{2n+1} & a_{2n+2} \\ z_3 & z_3 & z_3 & z_3 & z_3 & z_3 & z_3 & z_3 & \cdots & z_3 & z_3 & 1 \\ a_2 & a_6 & a_4 & a_8 & a_3 & a_7 & a_5 & a_9 & \cdots & a_{2n-3} & a_{2n+1} & a_{2n+2} \end{pmatrix}$$

$$\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots$$

$$z_n \to \begin{pmatrix} a_2 & a_3 & a_4 & a_5 & \cdots & a_{2n-1+1} & a_{2n-1+2} & \cdots & a_{2n+1} & a_{2n+2} \\ z_n & z_n & z_n & z_n & \cdots & z_n & z_n & \cdots & z_n & 1 \\ a_2 & a_{2n-1+2} & a_4 & a_{2n-1+4} & \cdots & a_{2n} & a_3 & \cdots & a_{2n+1} & a_{2n+2} \end{pmatrix}.$$

By considering, in turn, the expansion rules for $z_i^{-1}t_1z_it_i^{-1}$, $z_i^{-1}t_iz_it_1^{-1}$, $z_i^{-1}t_2z_it_2^{-1}$,..., $z_i^{-1}t_nz_it_n^{-1}$ as for σ_x and σ_y, we see that

$\sigma_{z_i}^{-1}\sigma_{t_1}\sigma_{z_i}\sigma_{t_i}^{-1} = \sigma_{z_i}^{-1}\sigma_{t_i}\sigma_{z_i}\sigma_{t_1}^{-1} = \sigma_{z_i}^{-1}\sigma_{t_j}\sigma_{z_i}\sigma_{t_j}^{-1}$ for $2\leq i\leq n$, $2\leq j\neq i\leq n$.

So, if $S = \{x,y,z_2,\ldots,z_n,t_1,\ldots,t_n\}$, we have defined an expansion rule for each element of $F\langle S\rangle$, the group freely generated by S. This allows us to prove the following lemma.

Lemma 4.2 The semidirect product of the free Abelian group by its automorphism group, $\mathbb{Z}^n.GL(n,\mathbb{Z})$, is a group defined by a finite set of expansion rules.

Proof: Let $S = \{x,y,z_2,\ldots,z_n,t_1,\ldots,t_n\}$, where each of the elements of S has the associated expansion rule above. Let $H = \{\sigma_\ell \mid \ell \in F\langle S\rangle\}$ be the image of $F\langle S\rangle/\psi$ under the standard embedding. We show that H is isomorphic to $\mathbb{Z}^n.GL(n,\mathbb{Z})$.

Let $T_1,\ldots,T_n$ freely generate $\mathbb{Z}^n$ and let X,Y,Z_i be the elements of $GL(n,\mathbb{Z})$ defined by

$$T_1X = T_1^{-1} \qquad T_1Y = T_1T_2 \qquad T_1Z_i = T_i \qquad \text{for } 2\leq i\neq j\leq n.$$
$$T_iX = T_i \qquad T_iY = T_i \qquad T_iZ_i = T_1$$
$$T_jZ_i = T_j$$

Then $\mathbb{Z}^n.GL(n,\mathbb{Z})$ is generated by $X,Y,Z_2,\ldots,Z_n,T_1,\ldots,T_n$. Define a homomorphism $\theta: \mathbb{Z}^n.GL(n,\mathbb{Z}) \to H$ by $T_i\theta = \sigma_{t_i}$, $X\theta = \sigma_x$, $Y\theta = \sigma_y$ and $z_j\theta = \sigma_{z_j}$ for $2\leq j\leq n$, $1\leq i\leq n$. We will always write σ_ℓ for the image of $L \in \mathbb{Z}^n.GL(n,\mathbb{Z})$ under θ and 1 for σ_1.

To show that θ is well defined we want to show that if $L=K$ then $\sigma_\ell = \sigma_k$. Each element of $\mathbb{Z}^n.GL(n,\mathbb{Z})$ can be uniquely written as a product of an element of $\mathbb{Z}^n$ and an element of $GL(n,\mathbb{Z})$. So, if $L=K$, then $L=TL'$ and $K=SK'$ for some $L'=K' \in GL(n,\mathbb{Z})$ and $T=S \in \mathbb{Z}^n$. By definition of θ, $\sigma_\ell = \sigma_{t\ell'} = \sigma_t\sigma_{\ell'}$ and $\sigma_k = \sigma_s\sigma_{k'}$. By Lemma 4.1, if $S=T$ then $\sigma_t=\sigma_s$ and so we can suppose that $L,K \in GL(n,\mathbb{Z})$.

From the generating elements σ_x,σ_y and σ_{z_i} we see that any $\sigma_\ell \in \langle\sigma_x, \sigma_y, \sigma_{z_2},\ldots,\sigma_{z_n}\rangle$ has a symbol of the form

$$\begin{pmatrix} a_1 & \ldots & a_{2n+1} & a_{2n+2}a_1 & a_{2n+2}a_2 & a_{2n+2}a_3 & \ldots & a_{2n+2}a_{2n+1} & a_{2n+2}a_{2n+2} \\ 1 & \ldots & 1 & 1 & \ell & \ell u_3 & \ldots & \ell u_{2n+1} & 1 \\ a_1 & \ldots & a_{2n+1} & a_{2n+2}a_1 & a_{2n+2}a_2 & a_{2n+2}a_{3\tau} & \ldots & a_{2n+2}a_{(2n+1)\tau} & a_{2n+2}a_{2n+2} \end{pmatrix}$$

where $\tau \in S_{2n-1}$ and each u_i is a word in $t_1,\ldots,t_n$. Let σ_k have symbol

$$\begin{pmatrix} a_1 & \ldots & a_{2n+1} & a_{2n+2}a_1 & a_{2n+2}a_2 & a_{2n+2}a_3 & \ldots & a_{2n+2}a_{2n+1} & a_{2n+2}a_{2n+2} \\ 1 & \ldots & 1 & 1 & k & kv_3 & \ldots & kv_{2n+1} & 1 \\ a_1 & \ldots & a_{2n+1} & a_{2n+2}a_1 & a_{2n+2}a_2 & a_{2n+2}a_{3\varepsilon} & \ldots & a_{2n+2}a_{(2n+1)\varepsilon} & a_{2n+2}a_{2n+2} \end{pmatrix}.$$

We show that $\tau=\varepsilon$ and $\sigma_{u_i} = \sigma_{v_i}$. Then, by arguments with σ^*_ℓ and σ^*_k as for Lemma 4.1, we must have $\sigma_\ell=\sigma_k$.

Since $L=K$, $L^{-1}T_iL = K^{-1}T_iK \in \mathbb{Z}^n$ and, by Lemma 4.1, $\sigma_\ell^{-1}\sigma_{t_i}\sigma_\ell = \sigma_k^{-1}\sigma_{t_i}\sigma_k$. By considering the permutations of the top and bottom rows of the symbols above, we see that we must have

$$\begin{array}{llll} \tau^{-1}(23)(45) & \ldots\ldots & (2^n\ 2^{n+1})\tau = \varepsilon^{-1}(23)(45) & \ldots\ldots\ (2^n\ 2^{n+1})\varepsilon \\ \tau^{-1}(24)(35) & \ldots\ldots & (2^{n-1}\ 2^{n+1})\tau = \varepsilon^{-1}(24)(35) & \ldots\ldots\ (2^{n-1}\ 2^{n+1})\varepsilon \\ \vdots & & \vdots & \vdots \\ \tau^{-1}(2\ 2^{n-1+2}) & \ldots & (2^{n-1+1}\ 2^{n+1})\tau = \varepsilon^{-1}(2\ 2^{n-1+2}) & \ldots\ (2^{n-1+1}\ 2^{n+1})\varepsilon. \end{array}$$

But the n products of transpositions in these equations form the regular representation of $\mathbb{Z}_2^n = \langle g_1,\ldots,g_n \mid g_i^2 = [g_i,g_j] = 1 \rangle$ where

correspond to
$$\begin{matrix} 1, & g_1, & g_2, & g_1g_2, & g_3, & g_1g_3, & g_2g_3, & g_1g_2g_3, & g_4, & \cdots, & g_1g_2\ldots g_n \\ 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & \ldots, & 2^{n+1}. \end{matrix}$$

Then τ and ε generate the same automorphism of $\mathbb{Z}_2^n$, and hence the same permutation of $\{1,g_1,g_2,\ldots,g_1\ldots g_n\}$, and hence the same permutation of $\{2,3,\ldots,2^{n+1}\}$. In other words, $\tau=\varepsilon$.

Finally, since $\sigma_\ell^{-1}\sigma_t\sigma_\ell = \sigma_k^{-1}\sigma_{t_1}\sigma_k$, the symbols

$$\begin{pmatrix} a_1\ldots a_{2^{n+2}}a_1 & a_{2^{n+2}}a_2 & a_{2^{n+2}}a_{3\tau} & a_{2^{n+2}}a_{4\tau} & \ldots a_{2^{n+2}}a_{(2^{n+1})\tau} & a_{2^{n+2}}a_{2^{n+2}} \\ 1\ \ldots\ 1 & \ell^{-1}t_1\ell u_3 & u_3^{-1} & u_4^{-1}\ell^{-1}t_1\ell u_5\ldots(u_{2^{n+1}})^{-1}u_{2^n} & 1 & \\ a_1\ldots a_{2^{n+2}}a_1 & a_{2^{n+2}}a_{3\tau} & a_{2^{n+2}}a_2 & a_{2^{n+2}}a_{5\tau} & \ldots\ a_{2^{n+2}}a_{(2^n)\tau} & a_{2^{n+2}}a_{2^{n+2}} \end{pmatrix}$$

and

$$\begin{pmatrix} a_1\ldots a_{2^{n+2}}a_1 & a_{2^{n+2}}a_2 & a_{2^{n+2}}a_{3\varepsilon} & a_{2^{n+2}}a_{4\varepsilon} & \ldots a_{2^{n+2}}a_{(2^{n+1})\varepsilon} & a_{2^{n+2}}a_{2^{n+2}} \\ 1\ \ldots\ 1 & k^{-1}t_1kv_3 & v_3^{-1} & v_4^{-1}k^{-1}t_1kv_5\ldots(v_{2^{n+1}})^{-1}v_{2^n} & 1 & \\ a_1\ldots a_{2^{n+2}}a_1 & a_{2^{n+2}}a_{3\varepsilon} & a_{2^{n+2}}a_2 & a_{2^{n+2}}a_{5\varepsilon} & \ldots\ a_{2^{n+2}}a_{(2^n)\varepsilon} & a_{2^{n+2}}a_{2^{n+2}} \end{pmatrix}$$

must define the same element. Thus we must have $\sigma_{u_3} = \sigma_{v_3}$ and $\sigma_{u_{i+1}}^{-1}\sigma_{u_i} = \sigma_{v_{i+1}}^{-1}\sigma_{v_i}$ $(i=4,6,\ldots,2^n)$. Similarly, from $\sigma_\ell^{-1}\sigma_{t_2}\sigma_\ell = \sigma_k^{-1}\sigma_{t_2}\sigma_k$, we must have $\sigma_{u_4} = \sigma_{v_4}$ and $\sigma_{u_i}^{-1}\sigma_{u_{i+2}} = \sigma_{v_i}^{-1}\sigma_{v_{i+2}}$ $(i=3,6,7,10,11,\ldots,2^n)$. From $\sigma_{u_5}^{-1}\sigma_{u_4} = \sigma_{v_5}^{-1}\sigma_{v_4}$ and $\sigma_{u_4} = \sigma_{v_4}$ we get $\sigma_{u_5} = \sigma_{v_5}$. In general, from $\sigma_\ell^{-1}\sigma_{t_{j+1}}\sigma_\ell = \sigma_k^{-1}\sigma_{t_{j+1}}\sigma_k$ we have $\sigma_{u_{2^j+2}} = \sigma_{v_{2^j+2}}$ and $\sigma_{u_k}^{-1}\sigma_{u_{2^j+k}} = \sigma_{v_k}^{-1}\sigma_{v_{2^j+k}}$ for $3<k<(2^j+1)$. So, if we assume $\sigma_{u_m} = \sigma_{v_m}$ for all $3\leq m<(2^j+1)$, we get $\sigma_{u_{2^j+k}} = \sigma_{v_{2^j+k}}$ for $(2^j+2) < k < (2^{j+1}+1)$. So by induction, we have that $\sigma_{u_i} = \sigma_{v_i}$ for all $3 < i < 2^{n+1}$. Thus $\sigma_\ell = \sigma_k$ as required.

The homomorphism, θ, is clearly surjective and if $\sigma_\ell = \sigma_k$ then $\sigma_t\sigma_{\ell'} = \sigma_s\sigma_{k'}$. So $\sigma_{\ell'}^{-1}\sigma_t^{-1}\sigma_{t_i}\sigma_t\sigma_{\ell'} = \sigma_{k'}^{-1}\sigma_s^{-1}\sigma_{t_i}\sigma_s\sigma_{k'}$ gives that $\sigma_{\ell'}^{-1}\sigma_{t_i}\sigma_{\ell'} = \sigma_{k'}^{-1}\sigma_{t_i}\sigma_{k'}$ for $1\leq i\leq n$. The uniqueness argument above then gives that $\sigma_{\ell'} = \sigma_{k'}$ and so $\sigma_s = \sigma_t$. Thus, by Lemma 4.1, S=T. Since $\sigma_{\ell'}$ induces the same automorphism on $\langle\sigma_{t_1},\ldots,\sigma_{t_n}\rangle = \mathbb{Z}^n$ as L' induces on $\langle T_1,\ldots,T_n\rangle = \mathbb{Z}^n$, if $\sigma_{\ell'} = \sigma_{k'}$ then K'=L'. So L=K and θ is injective and therefore an isomorphism. ◀

Lemma 4.3 The group $G = \langle G_{2^n+2,1}, \mathbb{Z}^n.GL(n,\mathbb{Z})\rangle$ is a finitely presented simple group.

Proof: From Lemma 4.2 and Theorem 3.5 we see that it is sufficient to show that G=G'.

Since 2^n+2 is even, by Theorem 2.9, $G_{2^n+2,1} = G'_{2^n+2,1}$ and so $G_{2^n+2,1} \leq G'$.

For any $1\leq i\leq n$, there exists some $\sigma_\ell \in GL(n,\mathbb{Z})$ such that $\sigma_\ell^{-1}\sigma_{t_j}\sigma_\ell = \sigma_{t_j}\sigma_{t_i}$ for some $j\neq i$. Thus $\sigma_{t_i} \in (\mathbb{Z}^n.GL(n,\mathbb{Z}))'$ for all i and hence $\mathbb{Z}^n\leq G'$.

From the expansion rule for σ_x we see that the relation $\sigma_x = \gamma_1\sigma_x\gamma_2\sigma_{xt_1}\gamma_3\sigma_x\cdots\gamma_{2^n}\sigma_x\gamma_{2^n+1}\sigma_{xt_1}\gamma_{2^n+2}$ holds in G for some $\gamma_1,\ldots,\gamma_{2^n+1} \in G_{2^n+2,1}$. Thus $\sigma_x = \tau\gamma_1\cdots\gamma_{2^n+1}\sigma_{t_1}^{2^{n-1}}\sigma_x^{2^n}$ for some $\tau\in G'$. But $\sigma_x^2 = 1$ and $G_{2^n+2,1}, \mathbb{Z}^n \in G'$ and so $\sigma_x\in G'$.

Similarly, from the expansion rule for σ_{z_i}, we see that the relation $\sigma_{z_i} = \rho_i\delta_{i,1}\cdots\delta_{i,2^n+2}\sigma_{z_i}^{2^n}$ holds in G, for some $\rho_i \in G'$, $\delta_{i,1},\ldots,\delta_{i,2^n+2} \in G_{2^n+2,1}$. Thus, since $\sigma_{z_i}^2 = 1$, $\sigma_{z_i} \in G'$.

Finally, from the expansion rule for σ_y, we have that the relation $\sigma_y = \varepsilon\beta_1 \ldots \beta_{2n_{+2}} \sigma_{t_2}^{-2^{n-2}} \sigma_y^{2^n}$ holds in G. Now, $\sigma_y^2 = [\sigma_\ell, \sigma_y]$ where $\sigma_\ell \in GL(n,\mathbb{Z})$ satisfies $[\sigma_\ell, \sigma_{t_1}^{-1}] = \sigma_{t_2}^{-1}$, $[\sigma_\ell, \sigma_{t_2}^{-1}] = \sigma_{t_2}^{-2}$ and $[\sigma_\ell, \sigma_{t_i}] = 1$ for $i>3$. So $\sigma_y^2 \in G'$ and hence $\sigma_y \in G'$. Thus $GL(n,\mathbb{Z}) < G'$.

Hence, $\langle G_{2^{n_{+2},1}}, \mathbb{Z}^n.GL(n,\mathbb{Z}) \rangle < G'$ as required. ◀

Thus, we have the following theorem.

Theorem 4.4 $GL(n,\mathbb{Z})$ can be embedded in a finitely presented simple group.

In particular, we can embed $GL(3,\mathbb{Z})$ in a finitely presented simple group constructed in this way whereas $GL(3,\mathbb{Z})$ cannot be embedded in any of the groups constructed in [5].

Let $L=\langle \ell_1, \ldots, \ell_r \rangle$ be a finitely generated subgroup of $GL(n,\mathbb{Z})$. If we consider the expansion rules

$$\ell_i \to \begin{bmatrix} a_2 & a_3 & \cdots & a_{2n_{+1}} & a_{2n_{+2}} \\ \ell_i & \ell_i u_{3,i} & \cdots & \ell_i u_{2n_{+1},i} & 1 \\ a_2 & a_3{}^{\tau i} & \cdots & a_{(2n_{+1})\tau_i} & a_{2n_{+2}} \end{bmatrix}$$

constructed as in Lemma 4.2, we see that the semidirect product, $\mathbb{Z}^n L$, is a group defined by a finite set of expansion rules. Thus, from Theorem 3.6 we have

Theorem 4.5 If L is a finitely presented subgroup of the linear group, $GL(n,\mathbb{Z})$, then we can choose a set of expansion rules, defining $\mathbb{Z}^n L$, such that $\langle G_{m,1}, \mathbb{Z}^n L \rangle$ has a finitely presented simple derived group.

4.2 Conjugacy results for $\langle G_{m,1}, \mathbb{Z}^n L \rangle$.

The following two lemmas are directed towards the possible construction of a finitely presented simple group with unsolvable conjugacy problem.

Lemma 4.6 If L is a finitely presented linear group then two elements of L are conjugate in any group of the form $\langle G_{m,1}, \mathbb{Z}^n L \rangle$ only if they are conjugate in L.

Proof: Suppose that $\sigma_\ell, \sigma_k \in L$ and that $\tau^{-1}\sigma_\ell\tau = \sigma_k$, for some $\tau \in \langle G_{m,1}, \mathbb{Z}^n L \rangle$. Every symbol for σ_ℓ contains a column of the form $\begin{pmatrix} a_m a_2^b \\ \ell \\ a_m a_2^b \end{pmatrix}$, for some power, b, of a_2. The column $\begin{pmatrix} a_m a_2^b \\ ht \\ u \end{pmatrix}$ belongs to a symbol for τ, for some $u \in W$, $\sigma_h \in L$, $\sigma_t \in \mathbb{Z}^n$. Thus the column $\begin{pmatrix} u \\ t^{-1}h^{-1}\ell ht \\ u \end{pmatrix}$ belongs to a symbol for $\tau^{-1}\sigma_\ell\tau$ and, hence, to a symbol for σ_k. But columns of symbols for σ_k are of the form $\begin{pmatrix} u \\ kr \\ u \end{pmatrix}$ or $\begin{pmatrix} u \\ 1 \\ u \end{pmatrix}$, $\sigma_r \in \mathbb{Z}^n$, and so we must have $\sigma_t^{-1}\sigma_h^{-1}\sigma_\ell\sigma_h\sigma_t = \sigma_h^{-1}\sigma_\ell\sigma_h\sigma_s = \sigma_k\sigma_r$, for some $\sigma_s \in \mathbb{Z}^n$. So $\sigma_h^{-1}\sigma_\ell\sigma_h\sigma_k^{-1} \in \mathbb{Z}^n$ and hence $\sigma_h^{-1}\sigma_\ell\sigma_h = \sigma_k$ and σ_ℓ and σ_k are conjugate in L as required. ◀

Lemma 4.7 There exists a group of the form $\langle G_{m,1}, \mathbb{Z}^n L \rangle$ such that two elements of $\mathbb{Z}^n$ are conjugate in $\langle G_{m,1}, \mathbb{Z}^n L \rangle$ only if they are conjugate by an element of L.

Proof: In this proof we shall assume that, in the notation of Section 3.2, $p>1$. In other words, we will consider $\mathbb{Z}^n L$ as being defined by the expansion rules

$$t_i \to \begin{pmatrix} a_2 & \cdots & a_{2i-1+1} & \cdots & a_{2n+1} & a_{2n+2} & a_{2n+3} & \cdots & a_c & a_{c+1} & \cdots & a_m \\ t_i & \cdots & t_i & \cdots & 1 & 1 & t_i & \cdots & t_i & 1 & \cdots & 1 \\ a_{2i-1+1} & \cdots & a_2 & \cdots & a_{2n-2i-1+1} & a_{2n+2} & a_{2n+3} & \cdots & a_c & a_{c+1} & \cdots & a_m \end{pmatrix}$$

$$\ell_j \to \begin{pmatrix} a_2 & a_3 & \cdots & a_{2n+1} & a_{2n+2} & a_{2n+3} & \cdots & a_c & a_{c+1} & \cdots & a_m \\ \ell_j & \ell_j u_{3,j} & \cdots & \ell_j u_{2n+1,j} & 1 & \ell_j & \cdots & \ell_j & 1 & \cdots & 1 \\ a_2 & a_{3\tau j} & \cdots & a_{(2n+1)\tau j} & a_{2n+2} & a_{2n+3} & \cdots & a_c & a_{c+1} & \cdots & a_m \end{pmatrix}$$

for $1\leq i\leq n$, $1\leq j\leq r$, where $m > 2^n+3$. Note, in the proof of Theorem 3.6 we only required that p was chosen to be greater than a certain finite bound. In this proof we only require that $p>1$, so we can construct a group $\langle G_{m,1}, \mathbb{Z}^n L\rangle$ satisfying the requirements for this lemma and also having a finitely presented, simple derived subgroup.

Suppose $\tau^{-1}\sigma_s\tau = \sigma_r$, for some σ_s, $\sigma_r \in \mathbb{Z}^n$ and $\tau \in \langle G_{m,1}, \mathbb{Z}^n L\rangle$.

First, we define the symbol

$$\begin{pmatrix} u_1a_1 & u_1a_2 & \cdots & u_1a_m & \cdots & u_sa_1 & u_sa_2 & \cdots & u_sa_m \\ 1 & \ell_{1,2} & \cdots & \ell_{1,m} & \cdots & 1 & \ell_{s,2} & \cdots & \ell_{s,m} \\ v_1a_1 & v_1a_{i_2} & \cdots & v_1a_{i_m} & \cdots & v_sa_1 & v_sa_{j_2} & \cdots & v_sa_{j_m} \end{pmatrix}$$

to be a complete expansion of the symbol

$$\begin{pmatrix} u_1 & \cdots & u_s \\ \ell_1 & \cdots & \ell_s \\ v_1 & \cdots & v_s \end{pmatrix}$$

and we note that, for any $\sigma_s \in \mathbb{Z}^n$, s^{2k} has expansion rule

$$s^{2k} \to \begin{pmatrix} a_2 & \cdots & a_{2n+1} & a_{2n+2} & a_{2n+3} & \cdots & a_c & a_{c+1} & \cdots & a_m \\ s^k & \cdots & s^k & 1 & s^{2k} & \cdots & s^{2k} & 1 & \cdots & 1 \\ a_2 & \cdots & a_{2n+1} & a_{2n+2} & a_{2n+3} & \cdots & a_c & a_{c+1} & \cdots & a_m \end{pmatrix}.$$

Then, if we completely expand the symbol,

$$\begin{pmatrix} a_1 & \cdots & a_{m-1} & a_m \\ 1 & \cdots & 1 & s^{2^k} \\ a_1 & \cdots & a_{m-1} & a_m \end{pmatrix}$$

for $\sigma_s^{2^k}$, k times then we get a symbol of the form

$$\begin{pmatrix} u_1 & \cdots & u_p & u_{p+1} & \cdots & u_q & u_{q+1} & \cdots & u_d \\ 1 & \cdots & 1 & s & \cdots & s & s^{2^i} & \cdots & s^{2^j} \\ u_1 & \cdots & u_p & u_{p+1} & \cdots & u_q & u_{q+1} & \cdots & u_d \end{pmatrix}.$$

There are only finitely many words, x, such that $(xa_1)\tau$ is not defined so choose k so that $(xa_1)\tau$ is defined for all words, x, of length greater than, or equal to, k. Then $(u_j a_1)\tau$ is defined for all $1 \leq j \leq d$.

The column $\begin{pmatrix} u_q \\ h \\ v_q \end{pmatrix}$ belongs to some symbol for τ, for some h, v_q, and hence the column $\begin{pmatrix} v_q \\ h^{-1}sh \\ v_q \end{pmatrix}$ belongs to some symbol for $\tau^{-1}\sigma_s^{2^k}\tau = \sigma_r^{2^k}$.

Since the top and bottom elements of the column are the same we must have $\sigma_{h^{-1}sh} = \sigma_r^{2^i}$, for some i. But $\sigma_h \in \mathbb{Z}^n L$ so let $\sigma_h = \sigma_t \sigma_\ell$, for $\sigma_\ell \in L$, and then $\sigma_\ell^{-1}\sigma_s\sigma_\ell = \sigma_r^{2^i}$.

Since $m > 2n+3$, the column $\begin{pmatrix} u_g \\ s^{2^k} \\ u_g \end{pmatrix}$ belongs to the above symbol for $\sigma_s^{2^k}$, for some $g < d$. So, as above, $\sigma_{\ell'}^{-1}\sigma_s^{2^k}\sigma_{\ell'} = \sigma_r^{2^j}$, for some $\sigma_{\ell'} \in L$

and for some j. But no symbol for $\sigma_r^{2^k}$ contains a column $\begin{pmatrix} u \\ r^{2^j} \\ u \end{pmatrix}$ for $j>k$, so we must have $j<k$. Considering the two conjugacy relations together we get $\sigma_{\ell^{-1}\ell'}\sigma_r^{2^j}\sigma_{\ell^{-1}\ell'}^{-1} = \sigma_r^{2^{i+k}}$. But the automorphism $\sigma_{\ell^{-1}\ell'}^{-1} \in L$ cannot send any element of $\mathbb{Z}^n$ to a proper power of itself so we must have $i+k=j$. Since $j<k$, we have $k=j$ and $i=0$. Thus $\sigma_\ell^{-1}\sigma_s\sigma_\ell = \sigma_r$, as required. ◀

Let L be a linear group over the integers. By the orbit problem for L we mean the problem of deciding, for two given vectors $s,r \in \mathbb{Z}^n$, whether or not there exists an element, ℓ, of L such that $s\ell=r$.

In 1973 C.Miller [8] asked whether there exists a finitely presented linear group, over the integers, with unsolvable conjugacy problem. If such a group, L, exists, and if $\mathbb{Z}^n L$ is perfect, then Theorem 4.5 and Lemma 4.6 show that we can construct a finitely presented simple group with unsolvable conjugacy problem. In fact, it seems to be still unknown whether $GL(n,\mathbb{Z})$ itself has solvable conjugacy problem. In the case, perhaps unlikely, that the conjugacy problem is unsolvable for this group, Theorem 4.4 and Lemma 4.6 would automatically give us a finitely presented simple group with unsolvable conjugacy problem. However, there seems to be more hope of obtaining a finitely presented infinite simple group from the results of this work, by the following construction.

Let L be a finitely presented linear group, over the integers, with unsolvable orbit problem. From Lemma 4.7 we see that, for some $\sigma_s, \sigma_r \in \mathbb{Z}^n$, we cannot tell whether σ_s and σ_r are conjugate in $\langle G_{m,1}, \mathbb{Z}^n L \rangle = G$. If $\tau^{-1}\sigma_s^{2^{n-1}}\tau = \sigma_t^{2^{n-1}}$, for some $\tau \in G$, then $\sigma_\ell^{-1}\sigma_s^{2^{n-1}}\sigma_\ell = \sigma_r^{2^{n-1}}$ for some $\sigma_\ell \in L$. Since $\sigma_\ell^{-1}\sigma_s\sigma_\ell = \sigma_t$, for some $\sigma_t \in \mathbb{Z}^n$, $\sigma_t^{2^{n-1}} = \sigma_r^{2^{n-1}}$ gives that $\sigma_t = \sigma_r$. So if we cannot decide whether σ_s and σ_r are conjugate in G, we cannot decide whether $\sigma_s^{2^{n-1}}$ and $\sigma_r^{2^{n-1}}$ are conjugate in G. Thus we will suppose, by taking $\sigma_s^{2^{n-1}}$ and $\sigma_r^{2^{n-1}}$ if necessary, that $\sigma_s, \sigma_r \in G'$. Suppose that we could decide whether σ_s and σ_r were conjugate in G'. By Lemma 3.6 we can suppose that G' has finite index in G, so let $\{x_1, \ldots, x_q\}$ be a transversal for G' in G.

Claim: If the conjugacy problem is solvable in G' then we can tell whether σ_s and σ_r are conjugate in G.

Since $G_{m,1} < G'$ we can suppose that $x_i = \sigma_{r_i}\sigma_{\ell_i}$, for some unique $\sigma_{r_i} \in \mathbb{Z}^n$, $\sigma_{\ell_i} \in L$. If σ_s and σ_r are conjugate in G', then they are conjugate in G. If σ_s and σ_r are conjugate in G then $\tau^{-1}x_i^{-1}\sigma_s x_i \tau = \sigma_r$, for some $\tau \in G'$, $1 \leq i \leq q$. Thus $\tau^{-1}\sigma_{\ell_i}^{-1}\sigma_s\sigma_{\ell_i}\tau = \sigma_r$. But $\sigma_{\ell_i^{-1}s\ell_i} \in G'$ so we can test whether $\sigma_{\ell_i}^{-1}\sigma_s\sigma_{\ell_i}$ and σ_r are conjugate in G', for $1 \leq i \leq q$, and hence decide whether or not σ_s and σ_t are conjugate in G, so the claim above is valid.

We have, therefore, that if L is a finitely presented linear group with unsolvable orbit problem then $\langle G_{m,1}, \mathbb{Z}^n L' \rangle$ is a finitely presented simple group with unsolvable conjugacy problem.

Finally we show that if there exists a finitely presented linear group, L, with unsolvable conjugacy problem then we can construct a linear group with unsolvable orbit problem. We consider the free

Abelian group of rank n^2 whose elements we shall write as n×n integer matrices. Define a subgroup, K_L, of $GL(n^2,\mathbb{Z})$ as follows. For each $k_\ell \in K_L$ and for each $a \in \mathbb{Z}^{n^2}$, let $ak_\ell = \ell^{-1}a\ell$. If the orbit problem is solvable for K_L then, given any $a,b \in \mathbb{Z}^{n^2}$, we can tell whether $ak_\ell=b$ for any $k_\ell \in K_L$. Thus we can tell if $b=\ell^{-1}a\ell$ for any $\ell \in L$. Take $h,k \in L$, then h and k both represent elements of $\mathbb{Z}^{n^2}$ and hence we can tell whether $\ell^{-1}h\ell=k$ for any $\ell \in L$. So the conjugacy problem is solvable in L.

Define a map $\theta: L \to K_L$ by $\ell\theta=k_\ell$. Since $k_\ell k_h = k_{\ell h}$, θ is a surjective homomorphism. If $k_\ell=1$ then $\ell^{-1}a\ell = a$ for all $a \in \mathbb{Z}^{n^2}$ and so $\ell^{-1}h\ell = h$ for all $h \in GL(n,\mathbb{Z})$. Thus $\ell = \pm I$, where I is the identity matrix. So $\ker\theta = \{\pm I\}$ or I depending on whether or not -I belongs to L. So K_L is isomorphic to a subgroup of L of index at most 2 in L. Thus, if L is finitely presented, so is K_L and if L has unsolvable conjugacy problem then K_L has unsolvable orbit problem.

We can summarise all of the above discussion into the following theorem.

Theorem 4.8 If we can construct a finitely presented linear group, over the integers, with unsolvable conjugacy problem or unsolvable orbit problem, then we can construct a finitely presented simple group with unsolvable conjugacy problem.

Chapter 5

Abelian Groups

Let A be a countable Abelian group whose torsion factor group, A/T, has finite rank. Let $X_P = \{a/m b \mid p \text{ prime}, p|m \text{ implies that } p \in P\}$, a where P is a finite set of primes. In this chapter we show that if, for some finite set of primes, P, A/T is a subgroup of a finitely generated torsion free X_P-module then A can be embedded in a finitely presented simple group.

The proof of this theorem is in two main steps. First we show that we can construct a group, H, defined by a finite set of expansion rules such that the commutator subgroup, H', of H has finite index in H and such that the group $\{a/m b \mid a,b \in \mathbb{Z}\}$ is contained in H' for a given integer, m. Then we show that, if $\langle G_{n,1}, H\rangle$ is a finitely presented group, so is $\langle G_{n,1}, \wr H^r\rangle$ for any finite r. It is easy to show that $G_{n,1} \wr H$ can be embedded in $\langle G_{n,1}, H\rangle$ and the theorem follows almost immediately from this.

Lemma 5.1 The group $\langle s,t \mid s^{-1}ts = t^m\rangle (m \in \mathbb{N})$ is a group defined by a finite set of expansion rules.

Proof: If m is odd, consider the expansion rules

$$t \to \begin{pmatrix} a_2 & a_3 & a_4 & a_5 & a_6 \\ t & 1 & t & 1 & 1 \\ a_3 & a_2 & a_5 & a_4 & a_6 \end{pmatrix}, \quad s \to \begin{pmatrix} a_2 & a_3 & a_4 & a_5 & a_6 \\ s & st^{-k} & s & st^{-k} & 1 \\ a_2 & a_3 & a_4 & a_5 & a_6 \end{pmatrix}$$

where $m=2k+1$. Let σ_t and σ_s be the images of t and s under the standard

embedding. As in the proof of Lemma 4.1, we see that σ_t has infinite order. Considering the expansion rule

$$s^{-1}tst^{-m} \to \begin{pmatrix} a_2 & a_3 & a_4 & a_5 & a_6 \\ s^{-1}tst^{-m} & 1 & s^{-1}tst^{-m} & 1 & 1 \\ a_2 & a_3 & a_4 & a_5 & a_6 \end{pmatrix}$$

and from the map $\sigma^*_{s^{-1}tst^{-m}}$, again as in Lemma 4.1, we see that $\sigma_{s^{-1}tst^{-m}} = 1$. Thus, given any $\sigma_\ell \in \langle \sigma_t, \sigma_s \rangle$, we can write $\sigma_\ell = \sigma_s^a \sigma_t^b \sigma_s^{-c}$, for some integers a,b,c with a,c>0.

Suppose that the relation $\sigma_s^a \sigma_t^b \sigma_s^{-c} = 1$ holds in $\langle \sigma_s, \sigma_t \rangle$. Since σ_t, and hence σ_s, has infinite order, we must have $\sigma_t^b = \sigma_s^{c-a}$, with b=0 if and only if a=c. From the expansion rules we see that if b is odd then $(a_2a_1)\sigma_t^b = a_3a_1 \neq (a_2a_1)\sigma_s^{c-a}$. Thus we can suppose that $\sigma_t^{2e} = \sigma_s^d$ for some integers e and d. But σ_t^{2e} and σ_s^d have symbols respectively

$$\begin{pmatrix} a_1 \dots a_5 & a_6a_1 & a_6a_2 & a_6a_3 & a_6a_4 & a_6a_5 & a_6a_6 \\ 1 \dots 1 & 1 & t^e & t^e & t^e & t^e & 1 \\ a_1 \dots a_5 & a_6a_1 & a_6a_2 & a_6a_3 & a_6a_4 & a_6a_5 & a_6a_6 \end{pmatrix}$$

and

$$\begin{pmatrix} a_1 \dots a_5 & a_6a_1 & a_6a_2 & a_6a_3 & a_6a_4 & a_6a_5 & a_6a_6 \\ 1 \dots 1 & 1 & s^d & (st^{-k})^d & s^d & (st^{-k})^d & 1 \\ a_1 \dots a_5 & a_6a_1 & a_6a_2 & a_5a_3 & a_6a_4 & a_6a_5 & a_6a_6 \end{pmatrix}$$

so if $\sigma_t^{2e} = \sigma_s^d$ then $\sigma_t^e = \sigma_s^d$ and, since σ_t and σ_s have infinite order, we must have e=d=0. So b=0 and a=c. Thus $\sigma_\ell = 1$ is a consequence of the relation $\sigma_s^{-1}\sigma_t\sigma_s = \sigma_t^m$, and the group generated by σ_s and σ_t is the group $\langle \sigma_s, \sigma_t \mid \sigma_s^{-1}\sigma_t\sigma_s = \sigma_t^m \rangle$.

For the case when m is even, consider the expansion rules

$$t \to \begin{pmatrix} a_2 & a_3 & a_4 & \dots & a_{m+1} & a_{m+2} \\ t & t & 1 & \dots & 1 & 1 \\ a_3 & a_4 & a_5 & \dots & a_{m+2} & a_2 \end{pmatrix}$$

$$s \to \begin{pmatrix} a_2 & a_3 & a_4 & a_5 & \dots & a_m & a_{m+1} & a_{m+2} \\ s & st^{-m+2} & st^{-2m+4} & st^{-2m+6} & \dots & st^{-4} & st^{-2} & st^{-1} \\ a_2 & a_{m+2} & a_{m+1} & a_m & \dots & a_5 & a_4 & a_3 \end{pmatrix}.$$

Again, σ_t has infinite order and from the expansion rule

$$s^{-1}tst^{-m} \to \begin{pmatrix} a_2 & a_3 & a_4 & a_5 & \dots & a_{m+1} & a_{m+2} \\ s^{-1}tst^{-m} & 1 & 1 & 1 & \dots & 1 & t^{m-2}s^{-1}tst^{-2m+2} \\ a_2 & a_3 & a_4 & a_5 & \dots & a_{m+1} & a_{m+2} \end{pmatrix}$$

we see that $u\sigma_{s^{-1}tst^{-m}}^{*} = u$, whenever it is defined. Thus $\sigma_s^{-1}\sigma_t\sigma_s = \sigma_t^m$ and σ_s also has infinite order. So, as for the case m odd, to see that σ_s and σ_t generate the group $\langle \sigma_s, \sigma_t \mid \sigma_s^{-1}\sigma_t\sigma_s = \sigma_t^m \rangle$, we only need show that if $\sigma_s^a = \sigma_t^b$, then a=b=0.

If $\sigma_s^b = \sigma_t^a$ then examination of the symbols for these elements shows that m+1 must divide b and that $\sigma_s^b = \sigma_t^{a/m+1}$. Thus $\sigma_t^{ma} = 1$ and, since $m \neq 0$, we must have a=0. Hence a=b=0, as required. ◀

Theorem 5.2 For a given $m \in \mathbb{N}$, the additive group of rationals, $\{a/_m b \mid a,b \in \mathbb{Z}\}$, can be embedded in a finitely presented simple group.

Proof: Let $H = \langle \sigma_s, \sigma_t \mid \sigma_s^{-1}\sigma_t\sigma_s = \sigma_t^m \rangle$ and let n=6, if m is odd, and n=m+2, if m is even. By Lemma 5.1 and Theorem 3.5, $G = \langle G_{n,1}, H \rangle$ is a finitely presented group with a simple derived subgroup, G'.

Since n is even, $G_{n,1} = G'_{n,1} < G'$.

If m is odd, from relations D in Chapter 3 we see that $\alpha\sigma_t\gamma\sigma_t\beta = \sigma_t$, where α, β and γ are elements of $G_{n,1}$. So $\tau\sigma_t^2 = \sigma_t$, where $\tau \in G'$, and hence $\sigma_t \in G'$.

From the corresponding relation for σ_s we have that $\sigma_s = \varepsilon\, \sigma_t^d\, \sigma_s^4$, for some d and for some $\varepsilon \in G'$. Thus $\sigma_s^3 \in G'$ and so $\langle G_{n,1}, \sigma_t, \sigma_s^3\rangle < G'$.

Since G' has index at most 3 in G, G' is a finitely presented simple group. We show that $X = \{{}^a/_{m^b} \mid a,b \in \mathbb{Z}\} < G'$. Write each $x \in X$ uniquely in the form ${}^a/_{m^b}$, where $b \geq 0$ and, if $b \neq 0$, m doesnot divide a. Define a map $\theta: X \to G'$ by $x\theta = \sigma_s^b\, \sigma_t^a\, \sigma_s^{-b}$. Suppose that $x = {}^a/_{m^b}$ and $y = {}^c/_{m^d}$, where $b \geq d$. Then

$$x\theta y\theta = \sigma_s^b\, \sigma_t^a\, \sigma_s^{-b}\, \sigma_s^d\, \sigma_t^c\, \sigma_s^{-d} = \sigma_s^b\, \sigma_s^d\, \sigma_t^{am^d}\, \sigma_t^{cm^b}\, \sigma_s^{-b}\, \sigma_s^{-d} = \sigma_s^b\, \sigma_t^{a+cm^{b-d}}\, \sigma_s^{-b}.$$

Since $xy = \dfrac{a+m^{b-d}c}{m^b}$, if $b \neq d$ then $(xy)\theta = x\theta y\theta$. If $b = d$ then $xy = \dfrac{a+c}{m^b}$

and $x\theta y\theta = \sigma_s^b\, \sigma_t^{a+c}\, \sigma_s^{-b}$. Let $a+c = fm^e$, where m does not divide f, then $(xy)\theta = \sigma_s^{b-e}\, \sigma_t^f\, \sigma_s^{e-b}$.

But $\sigma_s^b\, \sigma_t^{a+c}\, \sigma_s^{-b} = \sigma_s^{b-e}\, \sigma_s^e\, \sigma_t^{fm^e}\, \sigma_s^{-e}\, \sigma_s^{e-b} = \sigma_s^{b-e}\, \sigma_t^f\, \sigma_s^{e-b}$ and so θ is a homomorphism.

If $x\theta = y\theta$ then $\sigma_s^b\, \sigma_t^a\, \sigma_s^{-b} = \sigma_s^d\, \sigma_t^c\, \sigma_s^{-d}$. If $d = b$ then $a = c$, since σ_t has infinite order, and so $x = y$. Assume that $d < b$, then $\sigma_t^a = \sigma_s^{d-b}\, \sigma_t^c\, \sigma_s^{b-d} = \sigma_t^{cm^{b-d}}$, so $a = cm^{b-d}$ and $x = y$. Thus θ is an embedding and X can be embedded in a finitely presented simple group, G'.

If m is even, as for the case m odd, relations D give that $\tau\sigma_t^2 = \sigma_t$ and $\varepsilon\,\sigma_t^d\,\sigma_s^m = \sigma_s$ for some d and some $\tau, \varepsilon \in G'$. Thus $\langle G_{n,1}, \sigma_t, \sigma_s^{m-1}\rangle \lhd G'$. Defining $\theta: X \to G'$ by $(a/_m b)\theta = \sigma_s^b\,\sigma_t^a\,\sigma_s^{-b}$ we see, as for m odd, that θ is an embedding of X in the finitely presented group, G'.

So, X can be embedded in a finitely presented simple group for all m. ◀

We now suppose that $H = \langle \sigma_{t_1}, \ldots, \sigma_{t_k}\rangle$ is a finitely presented group defined by a finite set of expansion rules

$$t_i \to \begin{pmatrix} a_2 & a_3 & \cdots & a_n \\ \xi_{i,2} & \xi_{i,3} & \cdots & \xi_{i,n} \\ a_{2\pi_i} & a_{3\pi_i} & \cdots & a_{n\pi_i} \end{pmatrix} \qquad (1 \leq i \leq k).$$

We consider the direct product $H\times\ldots\times H = \langle \{(\sigma_{t_{i_1}}, \ldots, \sigma_{t_{i_r}}) \mid 1 \leq i_j \leq k\}\rangle$, and we will write $H\times\ldots\times H$ as $\times H^r$. Let $S = T\times\ldots\times T$ and define $s_{i,j} \in F\langle S\rangle$ to be the element $(1, \ldots, 1, t_i, 1, \ldots, 1)$, where the non-trivial element lies in the jth place. We also define $\eta_{i,j,q} \in F\langle S\rangle$ to be the word obtained by substituting $s_{h,i}$ for t_h in $\xi_{i,q}$, for $1 \leq h \leq k$.

Lemma 5.3 If H is a group defined by a finite set of expansion rules then so is $\times H^r$.

Proof: Consider the expansion rules

$$s_{i,j} \to \begin{pmatrix} a_2 \cdots a_n \cdots a_{jn} & a_{jn+1} & a_{jn+2} & \cdots & a_{jn+n} & a_{jn+n+1} \cdots a_{rn} \\ 1 \cdots 1 \cdots 1 & 1 & \eta_{i,j,2} & \cdots & \eta_{i,j,n} & 1 \cdots 1 \\ a_2 \cdots a_n \cdots a_{jn} & a_{jn+1} & a_{jn+2\pi_i} & \cdots & a_{jn+n\pi_i} & a_{jn+n+1} \cdots a_{rn} \end{pmatrix}$$

$(1 \leq i \leq k,\ 1 \leq j \leq r)$ and let $\sigma_{s_{i,j}}$ be the image of $s_{i,j}$ under the natural

embedding. We show that $\times H^r = \langle \{\sigma_{s_{i,j}} \mid 1 \leq i \leq k,\ 1 \leq j \leq r\} \rangle$.

Let $K = \times H^r = \langle \{\sigma_{s_{i,j}} \mid 1 \leq i \leq k,\ 1 \leq j \leq r\} \rangle$ and let $K_j = \langle \sigma_{s_{1,j}}, \ldots, \sigma_{s_{k,j}} \rangle$. Then every $\sigma_{s_j} \in K_j$ has a symbol of the form

$$\begin{pmatrix} a_1..a_{nr-1}\ a_{nr}a_1..a_{nr}a_{nj}\ a_{nr}a_{nj+1}\ a_{nr}a_{nj+2} \ldots a_{nr}a_{nj+n}\ a_{nj+n+1}..a_{nr}a_{nr} \\ 1\ ..\ 1\ \ 1\ ..\ 1\ \ 1\ \ \ell_2\ \ \ldots\ \ \ell_n\ \ 1\ ..\ 1 \\ a_1..a_{nr-1}\ a_{nr}a_1..a_{nr}a_{nj}\ a_{nr}a_{nj+1}\ a_{nr}a_{nj+i_2} \ldots a_{nr}a_{nj+i_n}\ a_{nj+n+1}..a_{nr}a_{nr} \end{pmatrix}$$

where ℓ_i is a word in the $s_{h,j}$, for $1 \leq h \leq k$. Thus, if $\sigma_{s_j} \in K_j$ and $\sigma_{s_q} \in K_q$, $\sigma_{s_j} = \sigma_{s_q}$ implies that $j=q$ or that $\sigma_{s_j} = \sigma_{s_q} = 1$. Clearly $K = \langle K_1, \ldots, K_r \rangle$, so $K = K_1 \times \ldots \times K_r$. We show that each K_j is isomorphic to H.

The map θ_j: $F\langle T \rangle \to F\langle S \rangle$, defined by $t_i\theta_j = s_{i,j}$ for some fixed j, is an embedding and $\xi_{i,q}\theta_j = \eta_{i,j,q}$ for $2 \leq q \leq n$. Thus the map ϕ_j: $H \to K$, defined by $\sigma_{t_i}\phi_j = \sigma_{s_{i,j}}$, is an embedding and, in fact, ϕ_j maps H onto K_j since $\{s_{1,j}, \ldots, s_{k,j}\} = \{t_1\phi_j, \ldots, t_k\phi_j\}$. So H is isomorphic to K_j and hence K is isomorphic to $\times H^r$.

So $\times H^r$ is defined by a finite set of expansion rules. ◀

If H is finitely presented then so is $\times H^r$ for any finite r and so we have the following theorem.

Theorem 5.4 If H is a finitely presented group, defined by a finite set of expansion rules, and if $K = \times H^r$ for any finite integer, r, then $\langle G_{nr,1}, K \rangle$ is a finitely presented group with a simple derived subgroup.

Before we can prove the final theorem we need the following lemma.

Lemma 5.5 If H is defined by a finite set of expansion rules then $G_{n,1} \times H$ can be embedded in $\langle G_{n,1}, H \rangle$.

Proof: If $(\alpha, \sigma_\ell) \in G_{n,1} \times H$, let α have symbol

$$\begin{pmatrix} u_1 & \cdots & u_s \\ 1 & \cdots & 1 \\ v_1 & \cdots & v_s \end{pmatrix}.$$

Define a map $\theta: G_{n,1} \times H \to \langle G_{n,1}, H \rangle$ by the rule $(\alpha, \sigma_\ell)\theta = \tau$, where τ has symbol

$$\begin{pmatrix} a_1 & \cdots & a_{n-2} & a_{n-1} & a_n u_1 & \cdots & a_n u_s \\ 1 & \cdots & 1 & \ell & 1 & \cdots & 1 \\ a_1 & \cdots & a_{n-2} & a_{n-1} & a_n v_1 & \cdots & a_n v_s \end{pmatrix}.$$

This map is well defined and is clearly a homomorphism. If $(\alpha, \sigma_\ell)\theta = (\beta, \sigma_h)\theta = \tau$ then let

$$\begin{pmatrix} a_1 & \cdots & a_{n-2} & a_{n-1} & a_n y_1 & \cdots & a_n y_r \\ 1 & \cdots & 1 & k & 1 & \cdots & 1 \\ a_1 & \cdots & a_{n-2} & a_{n-1} & a_n z_1 & \cdots & a_n z_r \end{pmatrix}$$

be the unique shortest symbol for τ. We must have $\sigma_\ell = \sigma_h = \sigma_k$ and the symbol

$$\begin{pmatrix} y_1 & \cdots & y_r \\ 1 & \cdots & 1 \\ z_1 & \cdots & z_r \end{pmatrix}$$

must be the unique shortest symbol for both α and β. Thus $\alpha = \beta$ and θ is the required embedding. ◀

Let A be an Abelian group with torsion subgroup $T<A>$. Suppose that $A/T<A>$ is a subgroup of a finitely generated torsion free X_P-module, M, where X_P is the ring of rationals whose denominators are products of primes belonging to the finite set P. Let C be the minimal divisible group containing A. If T is the torsion subgroup of C consider $B=<A,T>$. group, T is the torsion subgroup of B, T is divisible and $A/T<A>$ is isomorphic to B/T. Because T is divisible, B is isomorphic to the direct product, $T \times B/T$.†

Let $\{m_1,\ldots,m_r\}$ be a generating set for M and associate B/T with its image in M. For each $b \in B/T$ write b in the form $x_1m_1+\ldots+x_rm_r$, where each $x_i \in X_P$. Define a map $\theta: B/T \to X_P \times \ldots \times X_P$ by, $b\theta = (x_1,\ldots,x_r)$. If $b = y_1m_1+\ldots+y_rm_r$ then $(x_1-y_1)m_1+\ldots+(x_r-y_z)m_r = 0$. Since the m_i generate M, we must have $(x_i-y_i)m_i = 0$, $1 \leq i \leq r$. Each m_i has infinite order and so $x_i=y_i$, $1 \leq i \leq r$. Thus θ is well defined. Since θ is clearly an embedding, we can embed B/T in $\times X_P^r$.

If $H = <\sigma_s, \sigma_t \mid \sigma_s^{-1}\sigma_t\sigma_s = \sigma_t^p>$ we can embed $\times X_P^r$, and hence B/T, in the group $\times H^r = K$. By Lemma 5.1 and Theorem 5.4, $<G_{nr,1},K>$ is a finitely presented group.

If A is countable then T is countable, and locally finite. Thus, by Theorem 2.10, T can be embedded in $G_{nr,1}$. So $T \times B/T$ can be embedded in $G_{nr,1} \times K$ which, by Lemma 5.5, can be embedded in $<G_{nr,1},K>$.

Theorem 5.6 If A is a countable Abelian group whose torsion factor group is a subgroup of a finitely generated torsion free X_P-module, then

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†See, for example, Infinite Abelian Groups, Fuchs, pg.100 [4].

‡ This group exists, see [4] pg. 107

we can embed A in a finitely presented simple group.

Proof: As discussed above, we can embed A in a finitely presented group, $\langle G_{nr,1}, K\rangle$, where n is even. So $G'_{nr,1} = G_{nr,1}$ and, from the proof of Lemma 5.3, we see that we can embed the torsion factor group, $A/T\langle A\rangle$, of A in the derived subgroup, $H' \times \ldots \times H' = K'$, of K.

Since H' has index at most $p-1$ in H, K' has index at most $r(p-1)$ in K. Thus $\langle G_{nr,1}, K\rangle'$ has finite index in $\langle G_{nr,1}, K\rangle$ and hence is a finitely presented simple group. Of course, $G_{nr,1} \times K' \leq \langle G_{nr,1}, K\rangle'$.

Thus A can be embedded in a finitely presented simple group. ◀

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*Also include in D the relations

$$\sigma_{t_i} = \delta\delta_\mu^{-1}\sigma_{\xi_{i,2\mu}}\delta_3\sigma_{\xi_{i,3\mu}}\ldots\ldots\delta_n\sigma_{\xi_{i,n\mu}}\delta_\mu\delta_{n,i}$$

where $\mu \in S_{n-1}$ and δ_μ has symbol

$$\begin{pmatrix} a_2\ldots a_n & a_1a_1 & a_1a_2 & \ldots a_1a_n \\ 1\ldots 1 & 1 & 1 & \ldots\ 1 \\ a_2\ldots a_n & a_1a_1 & a_1a_{2\mu} & \ldots a_1a_{n\mu} \end{pmatrix}$$