On the Well-Posedness and Qualitative Behaviour of Compressible Fluids with Non-Local Interactions



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This thesis is dedicated to my mother
whose love and unwavering support have made all of this possible.

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Abstract

This thesis investigates the well-posedness and qualitative behaviour of the compressible Euler–Riesz equations (CEREs), a hyperbolic model for compressible fluids with non-local interactions. While the special case of the compressible Euler–Poisson equations (CEPEs) has been studied since the nineteenth century, the analysis of the CEREs is comparatively recent and has largely focused on well-posedness in the super-Newtonian regime. Our results provide a more comprehensive understanding of the mathematical structure and dynamics of these non-local fluid models.

In Chapter 2, we prove the global existence of finite-energy weak solutions for a wider class of Riesz interactions than previously known, including sub-Newtonian interactions and the limiting logarithmic potential, within the class of radially symmetric solutions. The proof is based on viscous approximations by the compressible Navier–Stokes–Riesz equations (CNSREs) and a compensated compactness argument for the vanishing viscosity limit. Since the Riesz potential does not admit the local representation available in the Newtonian/Coulomb case, the analysis requires new estimates to control the singularity and regularity of the non-local interaction near the origin.

Using a variational approach, we study the existence and stability of steady states of the attractive CEREs. In Chapter 3, we first consider stationary states in the mass-subcritical regime, proving their existence and nonlinear stability via a simplified concentration compactness argument. Moreover, we obtain a quantitative finite-time stability result by establishing a relative entropy bound for finite-energy weak solutions, without requiring uniform pointwise bounds on the density away from zero and infinity. In the mass-supercritical polytropic regime, we prove the existence of stationary states and establish their nonlinear instability by analysing the free-energy functional under mass-preserving scalings.

In Chapter 4, we study rotating axisymmetric steady states of the attractive CEREs, which we term rotating Riesz star solutions. In the mass-

subcritical regime, we prove existence and nonlinear stability under suitable sub-homogeneity assumptions on the angular momentum profile. The rotational setting introduces additional compactness difficulties, including the possible concentration of minimising sequences along rings with unbounded radii. In the mass-supercritical regime, we prove the existence of rotating Riesz star solutions in the polytropic setting under suitable super-homogeneity assumptions on the angular momentum, extending the theory beyond the small angular velocity regime. This requires a careful analysis of the relevant mass-preserving scalings, which are more delicate than in the non-rotating setting. Finally, using a mechanism analogous to that developed in Chapter 3, we prove instability of mass-supercritical rotating Riesz star solutions. These results show that rotation may have either a stabilising or destabilising effect, depending on the singularity of the Riesz interaction.

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Chapter 1

Introduction

This thesis is concerned with the analytical theory of compressible fluids with non-local interactions. In the absence of viscosity, such fluids can be modelled by the compressible Euler–Riesz equations (CEREs). We begin the introduction by presenting the structure and key properties of these equations. We then review the development of the theory of hyperbolic balance laws, of which the CEREs form a particular class. Subsequently, we examine the historical and theoretical context of the compressible Euler–Poisson equations (CEPEs), a fundamental special case, before turning to the more recent study of the CEREs themselves. We then focus on the nonlinear stability of steady state solutions, including stationary configurations and rotating star solutions. Finally, we outline the main contributions of this work, highlighting the principal challenges encountered and the novel aspects of the results.

1.1 The compressible Euler-Riesz equations

For a compressible fluid, the multi-dimensional (M-D) compressible Euler-Riesz equations are a system of $n+1$ PDEs given by

$$\begin{cases} \partial_t \rho + \nabla \cdot \boldsymbol{\mathcal{M}} = 0, \\ \partial_t \boldsymbol{\mathcal{M}} + \nabla \cdot \Big(\dfrac{\boldsymbol{\mathcal{M}} \otimes \boldsymbol{\mathcal{M}}}{\rho} \Big) + \nabla p + \kappa \rho \nabla \mathcal{W}_\alpha = \mathbf{0}, \end{cases} \tag{1.1}$$

where for each point in time and space $(t, \boldsymbol{x}) \in [0,\infty) \times \mathbb{R}^n$, $\rho : [0,\infty) \times \mathbb{R}^n \to [0,\infty)$ denotes the density of the fluid, $\boldsymbol{\mathcal{M}} : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ the momentum, $p : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ the pressure, and $\mathcal{W}_\alpha : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ the Riesz potential for $\alpha > 0$ and the logarithmic potential for $\alpha = 0$. Here, the Riesz potential is a non-local interaction between fluid particles and is given by the convolution $\mathcal{W}_\alpha = \Phi_\alpha * \rho$, where $\Phi_\alpha : \mathbb{R}^n \to \mathbb{R}$ is a Riesz kernel for $\alpha \in (0, n)$ and a logarithmic kernel for $\alpha = 0$; that

is, the kernel given by

$$\Phi_\alpha(\boldsymbol{x}) = \begin{cases} -\dfrac{|\boldsymbol{x}|^{-\alpha}}{\alpha} & \text{for } \alpha \neq 0, \\ \log|\boldsymbol{x}| & \text{for } \alpha = 0. \end{cases}$$

Here, the logarithmic kernel can be seen as a limit of the Riesz kernel, as $\alpha \to 0$, in the sense that

$$\lim_{\alpha\to 0}\left(\Phi_\alpha(\boldsymbol{x}) + \frac{1}{\alpha}\right) = \lim_{\alpha\to 0}\frac{1-|\boldsymbol{x}|^{-\alpha}}{\alpha} = \log|\boldsymbol{x}|. \tag{1.2}$$

Thus, whenever α appears, we use the convention that $\alpha = 0$ represents the logarithmic case. When $\alpha < n$, the kernel Φ_α is integrable near 0 and hence can be analysed via mean-field theory and potential theory. However, for $\alpha \geq n$, the kernel Φ_α is not integrable at the origin and thus, it is no longer possible to apply such theories to the study of the potential, in which case the kernel is called hyper-singular and behaves more like a short-range interaction problem; see [75]. Thus, we consider α in the range $\alpha \in (-1, n)$, which, for $n \geq 2$, includes the important Coulomb/Newtonian (repulsive/attractive) case $\alpha = n-2$, where the Riesz kernel represents a multiple of the fundamental solution of the Poisson equation. Since, in the more singular range, when $\alpha \geq n-1$, it is not possible to pass the gradient $\nabla\mathcal{W}_\alpha$ onto the Riesz kernel, to make sense of $\nabla\mathcal{W}_\alpha$, we define

$$\nabla\mathcal{W}_\alpha(\boldsymbol{x}) := \begin{cases} (\nabla\Phi_\alpha * \rho)(\boldsymbol{x}) & \text{for } \alpha \in (-1, n-1), \\ \displaystyle\int_{\mathbb{R}^n} \nabla\Phi_\alpha(\boldsymbol{x}-\boldsymbol{y})(\rho(\boldsymbol{y})-\rho(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{y} & \text{for } \alpha \in [n-1, n). \end{cases}$$

Thus, for $\alpha \in [n-1, n)$, if $\rho \in C^{0,\beta}(\mathbb{R}^n)$ for some $\beta \in (1+\alpha-n, 1)$, the difference $\rho(\boldsymbol{y}) - \rho(\boldsymbol{x})$, controls the singularity of $\nabla\Phi_\alpha$, meaning that $\nabla\mathcal{W}_\alpha$ is well-defined. In (1.1), $\kappa > 0$ represents the attractive potential case, and $\kappa < 0$ represents the repulsive potential case. By scaling, without loss of generality, we can choose $\kappa \in \{-1, 1\}$.

In the case of the super-Newtonian Riesz potential, that is, for $\alpha \in (n-2, n)$, $\mathcal{W}_\alpha$ is known to be proportional to the inverse of the fractional Laplacian operator:

$$(-\Delta)^{\frac{n-\alpha}{2}}\mathcal{W}_\alpha = -c_{n,\alpha}\rho,$$

in the sense of distributions, where

$$c_{n,\alpha} = \frac{2^{n-\alpha}\pi^{\frac{n}{2}}\Gamma\left(\frac{n-\alpha}{2}\right)}{\alpha\Gamma\left(\frac{\alpha}{2}\right)},$$

see [63] for the details of the calculation. The fractional Laplacian is a non-local operator that can be defined either via Fourier multipliers or through a singular integral representation in physical space. Serfaty [121] first showed that the pressure-less Euler

equations of the form (1.1), in the repulsive regime with Newtonian or super-Newtonian Riesz potentials, can be obtained as mean-field limits of interacting particle systems. Subsequently, Nguyen–Rosenzweig–Serfaty [116] extended the result to the full range $\alpha \in [0, n)$ by introducing a modulated energy method; see also Serfaty [122].

Riesz gases appear in a wide range of physical contexts. Examples include plasma physics, stellar and galactic dynamics, solid-state physics, ferrofluids, elasticity theory, and random matrix theory; see [17]. The Riesz interaction has also been investigated in the settings of jellium and the uniform electron gas; see [92, 47]. For a recent overview, including numerous open problems, we refer to [91]. Applications also arise in mathematical biology, particularly in swarming models; see [20, 21].

When $\alpha = 0$, one has $\Phi_0(\boldsymbol{x}) = \log|\boldsymbol{x}|$, which is known as the logarithmic potential. This case arises frequently and plays a central role in numerous areas, including statistical and quantum mechanics, random matrix theory, self-avoiding random walks, random tilings, and the study of functional inequalities; see [91, 4] for further discussion.

When $\alpha = n-2$ and $n \geq 3$, the interaction potential is known as the Coulomb/Newtonian potential. The corresponding Coulomb gas serves as a fundamental toy model for studying classical matter in the absence of quantum effects. For instance, Gamow's liquid drop model provides a simplified framework for describing atomic nuclei, electrons, and atoms; see [122]. In regimes where one phase dominates, the system can be reduced to particles interacting through a Coulomb potential; see [1]. Since the 1970s, Coulomb gases have played a central role in statistical mechanics; see [89] and the references therein. The Coulomb interaction is also fundamental in density functional theory; see [95, 92, 46, 47]. In particular, the notion of indirect Coulomb energy and its bounds arise from the Lieb–Oxford inequality and from n-marginal optimal transport problems with Coulomb cost. In dimension $n = 2$, the Coulomb/Newtonian potential coincides with the logarithmic potential. In the physics literature, this case is referred to as the log gas, two-dimensional one-component plasma, two-dimensional jellium, or Dyson gas. The two-dimensional one-component plasma serves as a natural model in two-dimensional statistical physics, with connections to the fractional quantum Hall effect and random matrix theory.

In this thesis, we will make the assumption that the gas is barotropic, that is, that the pressure is given as a function of the density $p = p(\rho)$. A pressure that is commonly used in the study of physical systems is that of a power-law for a polytropic gas, where the pressure is given by

$$p = p(\rho) = a_0 \rho^\gamma,$$

for the adiabatic exponent $\gamma \geq 1$. When $\gamma = 1$, we call this the isothermal case. By scaling, for $\gamma > 1$, we may choose $a_0 = \frac{(\gamma-1)^2}{4\gamma} > 0$. The value $\gamma = \frac{n+\alpha}{n}$ is referred to as the mass-critical exponent, arising from the balance between the internal and potential energies. Another important scaling is the energy-critical exponent $\gamma = \frac{2n}{2n-\alpha}$, which is associated with the scaling of the potential energy. In the classical case $n = 3$ and $\alpha = n - 2 = 1$, the range $\frac{6}{5} < \gamma < \frac{4}{3}$ is known as the mass-supercritical regime, while $\gamma > \frac{4}{3}$ corresponds to the mass-subcritical regime. In the context of the CEREs, these regimes generalise to $\frac{2n}{2n-\alpha} < \gamma < \frac{n+\alpha}{n}$ for the mass-supercritical regime, and $\gamma > \frac{n+\alpha}{n}$ for the mass-subcritical regime. A power-law is known to be a good model for gases when the density is small. However, when the density is large, we may observe different behaviour in the pressure. As an example, consider a white dwarf star in dimension $n = 3$ with $\alpha = n - 2 = 1$. In this case, the pressure takes the form

$$p(\rho) = A \int_0^{B\rho^{\frac{1}{3}}} \frac{\sigma^4}{\sqrt{D + \sigma^2}} \, \mathrm{d}\sigma, \tag{1.3}$$

for $\rho > 0$, where $A, B, D > 0$ are constants. The pressure law (1.3) exhibits the following asymptotic behaviour for small and large densities:

$$\begin{cases} p(\rho) \sim \frac{AB^5}{5\sqrt{D}} \rho^{\frac{5}{3}}, & \text{as } \rho \to 0, \\ p(\rho) \sim \frac{AB^4}{4} \rho^{\frac{4}{3}}, & \text{as } \rho \to \infty. \end{cases}$$

As such, it is of interest to study a general pressure $p : [0, \infty) \to [0, \infty)$, $p \in C^1([0, \infty))$ under the condition that

$$p'(\rho) > 0, \tag{1.4}$$

for $\rho > 0$.

Throughout the thesis, we consider the Cauchy problem for (1.1) with the initial data $\rho_0 : \mathbb{R}^n \to [0, \infty)$ and $\boldsymbol{\mathcal{M}}_0 : \mathbb{R}^n \to \mathbb{R}^n$ that satisfy

$$(\rho, \boldsymbol{\mathcal{M}})|_{t=0} = (\rho_0, \mathcal{M}_0)(\boldsymbol{x}) \longrightarrow (0, \mathbf{0}), \qquad \text{as } |\boldsymbol{x}| \to \infty. \tag{1.5}$$

Since global solutions of the CEREs system may include vacuum regions

$$\{(t, \boldsymbol{x}) \,|\, \rho(t, \boldsymbol{x}) = 0\},$$

where the fluid velocity is not well-defined, we work instead with the momentum variable $\boldsymbol{\mathcal{M}}(t, \boldsymbol{x})$. This quantity remains globally well-defined and therefore provides a more suitable formulation than the velocity field $\boldsymbol{u}(t, \boldsymbol{x})$. Under these assumptions, the energy of the CEREs is given by

$$E(\rho, \boldsymbol{\mathcal{M}})(t) := \int_{\mathbb{R}^n} \left(\rho(t, \boldsymbol{x}) e(\rho(t, \boldsymbol{x})) + \frac{1}{2} \left| \frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}} \right|^2 (t, \boldsymbol{x}) + \frac{\kappa}{2} \rho(\boldsymbol{x}) (\Phi_\alpha * \rho)(t, \boldsymbol{x}) \right) \mathrm{d}\boldsymbol{x}, \tag{1.6}$$

where

$$e(\rho) = \int_0^\rho \frac{p(\tau)}{\tau^2}\,\mathrm{d}\tau,$$

is the internal energy density. Here, the first term corresponds to the internal energy of the fluid, the second represents the kinetic energy of the system, and the final term accounts for the potential energy.

1.2 Hyperbolic systems of balance laws

The CEREs are hyperbolic balance laws. The theory of such equations, including systems of 1-D conservation laws given by

$$\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U})_x = \boldsymbol{0}, \tag{1.7}$$

is not well understood in comparison to the theory of elliptic and parabolic equations. For such equations, we can generally expect the breakdown of smooth solutions in finite time. Some of the more classical theory in the area began with the pioneering work of Glimm [65]. He proved global in time existence for entropy solutions with bounded variation to (1.7) by employing what is now known as the Glimm scheme. Further, Bianchini-Bressan [8] proved a similar result but with use of a vanishing viscosity limit. The next major area of work was for the polytropic equations of gas dynamics, where $\boldsymbol{U} = (\rho, m)$,

$$\boldsymbol{F}(\rho, m) = \left(m, \frac{m^2}{\rho} + p(\rho)\right),$$

with (1.7) becoming

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + \left(\dfrac{m^2}{\rho} + p(\rho)\right)_x = 0. \end{cases} \tag{1.8}$$

Owing to the breakdown of smooth solutions in finite time, we are forced to consider well-posedness of weak solutions. However, we find that in such a class of solutions, there may no longer be a unique solution to the corresponding Cauchy problem. To single out the unique physically relevant solution, we employ entropy pairs to define entropy solutions, which satisfy the desired well-posedness properties. We say that a pair $(\eta, q) : \mathbb{R}^2_+ \to \mathbb{R}^2$ is an entropy/entropy-flux pair, if for any smooth solution of (1.8),

$$\nabla q(\rho, m) = \nabla \eta(\rho, m) \nabla \boldsymbol{F}(\rho, m).$$

We say that an entropy pair (η, q) is a weak entropy pair if

$$\eta|_{\rho=0} = 0, \qquad \text{for all fixed } u = \frac{m}{\rho}.$$

Moreover, we say that (ρ, m) is an entropy solution of (1.8), if it satisfies

$$\eta(\rho, m)_t + q(\rho, m)_x \leq 0,$$

in the sense of distributions for all convex weak entropy pairs (η, q). DiPerna [54] developed a compensated compactness argument (see Murat [115] and Tartar [128]) to pass the viscosity limit and prove existence of solutions with $\gamma = 1 + \frac{2}{2k+1}$ for $k \geq 2$. Chen-Ding-Luo [51, 53, 52], and Chen [31] employed the Lax-Friedrichs scheme and Godunov scheme with a careful analysis of the entropy and compensated argument to prove existence of L^∞ entropy solutions for $\gamma \in (1, \frac{5}{3}]$. The existence result for general L^∞ entropy solutions with $\gamma \in [1, \infty)$ was proved in a further series of papers, with Lions-Perthame-Tadmor [100] proving $\gamma \in [3, \infty)$, Lions-Perthame-Sougnaidis [101] covering $\gamma \in (\frac{5}{3}, 3)$ and Huang-Wang [78] proving the remaining case when $\gamma = 1$.

1.3 Solutions of the compressible Euler-Poisson and Euler-Riesz equations

Much work has been done on the CEREs in the Coulomb/Newtonian potential case ($\alpha = n - 2$), known as the CEPEs. In the attractive case ($\kappa = 1$), the local classical well-posedness theory was studied by Makino [111] and Luo-Xin-Zeng [109]. On the other hand, in the repulsive case ($\kappa = -1$), in a series of work by Guo [67], Guo-Pausader [72], Ionescu-Pausader [81], and Guo-Ionescu-Pausader [71], it was shown that close to some constant background with small amplitude, there existed irrotational global solutions. This was recently extended to the attractive case ($\kappa = 1$) with an extension to vacuum-free boundaries in Hadžić-Jang [73].

However, because the CEPEs have a strongly hyperbolic and nonlinear structure, we can expect smooth solutions in general to break down in finite time, especially in the presence of large initial data. This behaviour can happen for a multitude of reasons, some of which include the formation of shock waves in the solution, the cavitation of the density and the concentration of the density. In particular for the attractive potential, in the case of radially symmetric solutions, it is expected that concentration of the density at the origin can occur for $\gamma \in [1, \frac{4}{3})$. Such behaviour is representative of the collapse of large mass stars to singularities under the force of self-gravitation. This phenomenon has been observed by constructing special solutions as in Guo-Hadžić-Jang [69, 68], Guo-Hadžić-Jang-Schrecker [70], and Huang-Yao [79]. When $\gamma = \frac{4}{3}$, it has been observed that there exists a critical mass, such that solutions with mass in the subcritical range express dissipation of the density. While, in the supercritical range, concentration of the density is observed. This was originally studied in Goldreich-Weber [66], where they constructed a family of smooth compact polytropic

collapsing star solutions, and further studied by Makino [112], and Fu-Lin [61]. Moreover, the stability of such Goldreich-Weber solutions under radial perturbations was studied by Hadžić-Jang [74]. A series of works by Lax [88], John [87], and Liu [104] established that, for a broad class of one-dimensional hyperbolic systems, including the one-dimensional Euler equations, suitably chosen smooth initial data can lead to finite-time blow-up of the first derivatives of the solution. In the multi-dimensional setting, Sideris [124] proved that smooth solutions of the three-dimensional compressible Euler equations arising from sufficiently compressed and localised initial data cannot exist globally in time. Furthermore, the monographs of Christodoulou [44] and Christodoulou-Miao [45] provide a detailed analysis of shock formation for the three-dimensional relativistic and classical compressible Euler equations, respectively. More recently, Buckmaster-Shkoller-Vicol [13] established the formation of self-similar shocks in finite time for the two-dimensional polytropic compressible Euler equations with suitable smooth initial data. Furthermore, Merle-Raphaël-Rodnianski-Szeftel [114] proved the occurrence of finite-time density singularities for the compressible polytropic Euler equations.

This leads us to the study of the existence of weak solutions of the CEPEs. While there is still no general theory for the well-posedness of the CEPEs, we highlight some of the key results towards this. In the radially symmetric setting, it was proved by Xiao [132] that using a fractional step Godunov scheme, one can prove the existence of L^∞ solutions for $\gamma \in (1, \frac{5}{3}]$. This was done in an exterior domain, as such, the concentration of the density at the origin was avoided. However, the majority of focus of the research community has been towards the use of vanishing viscosity limits to prove the existence of solutions. The formal idea of solutions of the Euler equations being the limit of solutions of the Navier-Stokes equations under a vanishing viscosity dates back to the work of Stokes [125]. It was not until the work of Gilbarg [64], where the first rigorous proof of the passing of a vanishing viscosity limit from shock layers of the steady compressible Navier-Stokes equations to shock waves of the steady compressible Euler equations was achieved. Hoff-Liu [77] proved that the inviscid limit of solutions of the 1-D polytropic compressible Navier-Stokes equations with initial shock data was indeed the corresponding shock solution for the compressible Euler equations. Chen-Perepelitsa [37] were the first to prove the rigorous inviscid limit of solutions of the 1-D polytropic compressible Navier-Stokes equations to solutions of the compressible polytropic Euler equations consisting of large initial data for $\gamma > 1$. This was achieved by extending the L^p compensated compactness framework of Lefloch-Westdickenberg [90], where they proved existence of solutions for $\gamma \in (1, \frac{5}{3})$. Schrecker-Schulz [119, 120] extended this result to the study of general asymptotically isothermal pressures. Chen-Perepelitsa [35] proved that, under a compensated compactness framework, one

could employ the 1-D theory to solutions of radial symmetry to pass the viscosity limit for the M-D compressible Euler equations, proving existence of solutions. Chen-He-Wang-Yuan [36] carefully constructed approximating solutions for the polytropic compressible Navier-Stokes equations and used a vanishing viscosity argument to prove existence of radially symmetric solutions to the CEPEs with $\gamma > 1$ in the repulsive case and $\gamma > \frac{6}{5}$ in the attractive case. Note that for $\gamma \in (\frac{6}{5}, \frac{4}{3}]$, concentration of the density does not occur in the attractive case, due to the imposition of the mass being smaller than some critical mass based on the Chandrasekhar limit; see [29]. By employing a careful analysis of the entropy and entropy-flux pair, as studied in Chen-Lefloch [32, 33] for the L^∞ setting, one can extend the result to general pressures with certain asymptotic power-law behaviours, as was done in Chen-Huang-Li-Wang-Wang [34].

Thus, to prove the global existence of finite-energy weak solutions with radial symmetry to the CEREs, we will take advantage of a vanishing viscosity limit. Hence, we first study solutions of the so-called compressible Navier-Stokes-Riesz equations (CNSREs), of the form:

$$\begin{cases} \partial_t \rho + \nabla \cdot \boldsymbol{\mathcal{M}} = 0, \\ \partial_t \boldsymbol{\mathcal{M}} + \nabla \cdot \Big(\dfrac{\boldsymbol{\mathcal{M}} \otimes \boldsymbol{\mathcal{M}}}{\rho} \Big) + \nabla p + \kappa \rho \nabla \mathcal{W}_\alpha = \varepsilon \nabla \cdot \Big(\mu(\rho) D\Big(\dfrac{\boldsymbol{\mathcal{M}}}{\rho} \Big) \Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad + \varepsilon \nabla \Big(\lambda(\rho) \nabla \cdot \Big(\dfrac{\boldsymbol{\mathcal{M}}}{\rho} \Big) \Big), \end{cases} \tag{1.9}$$

where $D\big(\frac{\boldsymbol{\mathcal{M}}}{\rho}\big) = \frac{1}{2}\big(\nabla(\frac{\boldsymbol{\mathcal{M}}}{\rho}) + \big(\nabla(\frac{\boldsymbol{\mathcal{M}}}{\rho})\big)^\intercal\big)$ is the stress tensor, $\varepsilon > 0$ is the inverse of the Reynolds number, and $\mu(\rho)$ and $\lambda(\rho)$ are the shear and bulk viscosity coefficients respectively that satisfy

$$\mu(\rho) \geq 0, \quad \mu(\rho) + n\lambda(\rho) \geq 0, \qquad \text{for } \rho \geq 0.$$

It was observed in Bresch-Desjardins [11] and Bresch-Desjardins-Lin [12], that if the shear and bulk viscosity satisfy the so-called Bresch-Desjardins (BD) relation

$$\lambda(\rho) = \rho\mu'(\rho) - \mu(\rho),$$

then we can obtain control of the spatial derivative of the density.

Since the Coulomb/Newtonian potential is given as a solution of the Poisson equations, it can be seen as a local operator. However, in the case of the general Riesz potential, we lose this locality property. The non-locality poses problems in the analysis of the system when trying to generalise from the CEPEs to the CEREs. In fact, one can observe new phenomena not present in the theory of the CEPEs. Thus, new analytical techniques are required to study the new behaviour presented by the non-locality. It is only recently that preliminary results have been produced for the Euler-Riesz

equations, this started with the work of Choi-Jeong [40], where they proved local well-posedness away from the vacuum for both the attractive and repulsive potential in the super-Newtonian range $\alpha \in (n-2, n)$. They also proved that under sufficient conditions, the expected result of the breakdown of classical solutions in finite-time is observed. Danchin-Ducomet [48] proved that, under some dispersive spectral estimates on the initial velocity field, there exists a global classical solution for $\alpha \in [n-2, n-1]$. Choi-Jung [42] proved that for the pressure-less and damped system, that given small enough initial data in some regular enough Sobolev space, there exists a unique global classical solution for the attractive potential in the super-Newtonian regime $\alpha \in (n-2, n)$. Choi-Jung-Lee [43] extended this work to non-damped polytropic flows in both the attractive and repulsive case in the super-Newtonian range. For the Manev potential $\alpha = n-1$, see [10, 27, 80] for work on the Vlasov-Manev system, which shares major similarities in behaviour and structure with the CEREs.

1.4 Stability of steady states

It is of great interest to understand the behaviour around special solutions of the CEREs, in particular steady solutions, solutions that do not evolve in time. For instance, for the attractive CEPEs in 3-D, such steady states may refer to steady stars. As such, it is of physical interest to study the nonlinear stability of such steady states. That is, if the configuration of a star starts close to a steady state in some sense, then does it remain close or even converge to the steady state? If this is true, we refer to the steady state as nonlinearly stable and nonlinearly unstable if there exist configurations starting arbitrarily close to the steady state that grow apart from the steady state for large time. This question was first answered by Rein [117], where it was proven that minimisers of the free-energy functional

$$\mathcal{G}(\varrho) = \int_{\mathbb{R}^n} \left(\varrho(\boldsymbol{x}) e(\varrho(\boldsymbol{x})) + \frac{1}{2} \varrho(\boldsymbol{x}) (\Phi_\alpha * \varrho)(\boldsymbol{x}) \right) \mathrm{d}\boldsymbol{x}, \tag{1.10}$$

for $\alpha = n-2$, are stationary states, steady states with no velocity field, of the attractive 3-D CEPEs and have compact support. Moreover, it was shown under sufficient conditions for a general pressure, corresponding to $\gamma > \frac{4}{3}$ in the polytropic case, that the minimisers were, in fact, nonlinearly stable. Such stationary states refer to non-rotating stars, that is, stationary solutions. It was recently shown by Lin-Zeng [99], that a turning point principle applied to the linear stability of non-rotating stars. Moreover, based on linear stability criterion, Lin-Wang-Zhu [98] were able to obtain an unconditional stability result for the attractive CEPEs with a general pressure. On the instability side, Deng-Liu-Yang-Yao [49] proved instability of non-rotating stars for the 3-D polytropic CEPEs with $\gamma = \frac{4}{3}$. This was achieved by proving growth of the

support of solutions with positive initial energy, which, owing to the compact support of steady states, implies instability. Jang [82, 83] proved nonlinear instability of Lane-Emden stars for $\gamma \in [\frac{6}{5}, \frac{4}{3})$. More recently, Cheng-Cheng-Lin [39] proved nonlinear instability for non-rotating polytropic stars with $\gamma \in (\frac{6}{5}, \frac{4}{3}]$, by taking advantage of the concavity of the energy with respect to the mass-preserving scaling parameter and using this to prove growth of the support of solutions. Note, by observing the corresponding free-energy functional, it can be seen that stationary states of the CEREs relate to stationary states of the aggregation-diffusion equation with a Riesz potential. As such, results such as Carrillo-Hittmeir-Volzone-Yao [18], where they prove the radial symmetry of stationary states in a large class of aggregation-diffusion equations known as the Keller-Segel equations, apply to the CEREs. For further properties of the aggregation-diffusion equations, see [24].

Alternatively, another steady state of great interest is that of rotating star solutions of the CEPEs in 3-D, where the velocity field is axisymmetric with respect to the z-axis. Originally, Auchmuty-Beals [6] proved existence of rotating star solutions as minimisers of a free-energy functional over a class of axisymmetric density distributions, and showed that these minimisers satisfy the corresponding steady state equations. Subsequent works further developed the variational framework and analysed structural properties of rotating stars under various assumptions on the equation of state and admissible classes; see Caffarelli-Friedman [14], Friedman-Turkington [59, 60], Auchmuty [5], Li [93], Chanillo-Yan [30], Luo-Smoller [106], McCann [113] and Wu [131, 130]. Lions [102] proved existence of rotating star solutions under a prescribed axisymmetric potential using a variational approach. More recently, Strauss-Wu [126, 127] proved existence of rotating star solutions for a prescribed angular velocity and under a different set of admissible pressures requiring less decay close to a vacuum and smaller growth for large density. This was further extended to non-constant entropy in Jang-Strauss-Wu [86]. An alternative approach based on the Implicit Function Theorem has also been employed to construct rotating solutions; see Lichtenstein [94], Heilig [76] and Jang-Makino [84, 85]. While the previous results pertain only to the existence and structure of rotating star solutions, Luo-Smoller [107, 108] proved nonlinear stability of rotating star solutions via a concentration compactness argument. Furthermore, Lin-Wang [97] proved linear spectral stability of rotating star solutions. For an overview of results and open problems in the theory of steady states for the CEPEs, see Luo [105].

1.5 Focus of Chapter 2: Existence of finite-energy weak solutions of the compressible Euler-Riesz equations

The bulk of this chapter is devoted to establishing the existence of solutions to the polytropic compressible Euler–Riesz equations. In §2.3 we begin by constructing a free boundary problem that approximates the Cauchy problem for the CEREs. In §2.4, we then construct approximate initial data to the free boundary problem, with associated convergence results. In §2.5 we derive local estimates for the radial representation of the Riesz potential and its derivatives. In §2.6 we then prove uniform energy estimates for solutions to the approximate free boundary problem, which yield uniform L^p bounds. This gives us the tools we need to prove the growth of the domain of the free boundary problem to the whole space $\mathbb{R}^n$ in §2.7. Next, in §2.8, we establish a BD-type entropy inequality in order to obtain uniform L^p estimates for the spatial derivatives of the density. In §2.9, using the local estimates for the Riesz potential, we derive uniform local higher-integrability estimates for both the density and the velocity. In §2.10, by combining the BD-type estimates with the Aubin–Lions Lemma, we pass from solutions of the free boundary problem to global weak solutions of the CNSREs system, which inherit the same uniform bounds. We then employ the higher-integrability estimates together with the compensated compactness framework of Chen–Perepelitsa [37] to pass from global weak solutions of the CNSREs to global weak solutions of the CEREs in §2.11. Finally, in §2.12, we extend the analysis to more general pressure laws.

A principal difficulty in the study of the CEREs stems from the non-local nature of the interaction potential. In the Coulomb/Newtonian case $\alpha = n - 2$, the potential admits the local radial representation

$$(\Phi_{n-2} * \rho)_r(t, r) = \frac{\omega_n}{r^{n-1}} \int_0^r \rho(t, \eta)\eta^{n-1} \, \mathrm{d}\eta,$$

where r denotes the radial variable and ω_n is the surface area of the unit sphere in $\mathbb{R}^n$. This formula, derived from the Poisson equation

$$\Delta(\Phi_{n-2} * \rho) = \omega_n \rho,$$

plays a crucial role in establishing uniform energy estimates, BD-type entropy bounds, and higher integrability properties in [36]. For general α, however, such a local representation is no longer available. Even when $\alpha \in (n - 2, n)$, the potential satisfies a fractional Poisson-type equation, which remains non-local and thus cannot be used to derive local estimates. To overcome this difficulty, we instead employ a radial representation of the Riesz potential to obtain suitable bounds; see Lemma 2.5.1. These

estimates are particularly important for establishing higher integrability properties, where local control is essential despite the non-local character of the interaction.

Another major issue arises in analysing the convergence of the potential energy as solutions pass from the free boundary problem to the Cauchy problem for global weak solutions. In the absence of a local representation formula, the argument used in [36] cannot be directly applied. To address this, we adopt a different strategy to study the behaviour near the origin. While previous works established only local convergence of the density away from the origin, we prove global convergence of the density; see Lemma 2.10.7. The key ingredient is the combination of local convergence with conservation of mass in the limit, which rules out both concentration of mass at the origin and loss of mass at infinity, thereby yielding global convergence.

Further technical novelties concern the BD-type entropy estimates. Due to the non-local nature of the potential, integration by parts produces a non-zero boundary term in the case $\alpha < n-2$; see (2.77) and Remark 2.8.2. In the repulsive regime ($\kappa = -1$), this boundary term has a favourable sign and can be controlled. In the attractive case ($\kappa = 1$), however, it cannot be handled directly. We therefore employ a Grönwall-type argument to avoid the problematic integration by parts step and obtain the desired BD-type estimate; see Lemma 2.8.3. When $\alpha > n-2$ (the super-Newtonian case), the potential is no longer twice differentiable, which prevents a direct application of the integration by parts method used in [36]. Here again, the Grönwall-type argument allows us to derive BD-type entropy bounds for both the attractive and repulsive cases.

Finally, we consider the range $\alpha \in (-1, 0]$, including the logarithmic case $\alpha = 0$, which arises frequently in physics and mathematical biology. This regime presents additional challenges. Unlike the Riesz potential with $\alpha > 0$, the logarithmic kernel does not decay at infinity and does not have a definite sign. For $\alpha \in (0, n)$, the Hardy–Littlewood–Sobolev (HLS) inequality provides L^p bounds for $\Phi_\alpha * \rho$. In the logarithmic case, however, the corresponding log-HLS inequality yields only a one-sided bound and does not provide an L^p estimate. For $\alpha \in (-1, 0)$, the reverse HLS inequality gives a lower bound in terms of an L^q norm with $q \in (0, 1)$, which is insufficient for our purposes. To address these difficulties, we analyse separately the behaviour of the logarithmic potential near the origin and in the far field. This allows us to obtain local L^∞ control of $\Phi_0 * \rho$, thereby overcoming the limitations of relying solely on the log-HLS inequality. Moreover, we show, at least formally, that the energy estimates for the logarithmic case arise as the limit of the corresponding Riesz energy estimates as $\alpha \to 0$, highlighting the connection between the Riesz and logarithmic interactions; see Remark 2.6.2.

1.6 Focus of Chapter 3: Existence, stability and instability of stationary states of the compressible Euler-Riesz equations

The first part of the chapter is devoted to establishing the existence and stability of stationary states for the attractive CEREs with general pressure laws in the mass-subcritical regime. The second part addresses the existence and instability of stationary states for the attractive polytropic CEREs in the mass-supercritical regime. In §3.3 and §3.4, we develop a simplified variational approach to deal with the nonlinear stability. We employ the concentration compactness argument from Lions [103] to prove the existence of minimisers of (1.10) and their nonlinear stability. The appendices contain details on the convolution inequalities used in the main text. In §3.5 we employ a relative entropy type estimate to quantify the stability. In §3.6, by proving the strong convexity of the second moment of the density, we prove the growth of the support of solutions with positive initial energy, with conditions on the pressure similar to those in the previous sections. This shows that indeed the stability is a local result. In §3.7 and §3.8 we prove instability of stationary states of the polytropic CEREs in the range $\frac{2n}{2n-\alpha} < \gamma < \frac{n+\alpha}{n}$. This is done by proving the concavity of the free-energy with respect to the scaling parameter for mass-preserving dilations. We then prove that minimisers of the free-energy are indeed stationary states of the CEREs using the compact embedding of Sobolev spaces and fractional Sobolev spaces. By carefully considering the behaviour of solutions with suitable initial conditions as $t \to \infty$, we can prove that the radius of the support of such solutions grows to infinity as $t \to \infty$.

Previous results have primarily established conditional stability for radial perturbations of stationary solutions, relying on the a priori assumption of the global existence of weak solutions; see [117, 107, 108, 98] for analyses of nonlinear stability in the CEPEs setting. A key novelty of our work is that, by invoking the results of Chapter 2, we show that this assumption, namely, the global existence of finite-energy weak solutions in a neighbourhood of such stationary states, is non-empty for a large class of the CEREs. Consequently, our stability results for the CEREs are unconditional within the radially symmetric finite-energy weak solution class constructed in Chapter 2. Moreover, we characterise stationary states through variational arguments and establish a connection with the aggregation–diffusion equations; see Remark 3.2.3.

We establish the nonlinear stability of minimisers in the mass-subcritical regime for general pressure laws. Under suitable assumptions on p, including (1.4) and appropriate asymptotic conditions, we prove the existence of minimisers via the concentration compactness principle [103], and then establish nonlinear orbital stability of the corresponding stationary states; see Theorem 3.2.2 and §3.4. These assumptions are

more general than those in previous works, as they allow for different behaviour of the pressure near vacuum and at infinity. Furthermore, we obtain quantitative stability estimates using a refined relative entropy method for finite-energy weak solutions. Going beyond purely variational stability, we impose natural higher-integrability conditions on the density and momentum in order to control the relative entropy and entropy flux terms. This leads to the need to estimate the non-local quantity

$$\int_{\mathbb{R}^n} |\boldsymbol{\mathcal{M}}|\big(|\nabla \mathcal{W}_\alpha - \nabla \bar{\mathcal{W}}_\alpha| + |\nabla \mathcal{W}_\alpha|\big)\, \mathrm{d}\boldsymbol{x},$$

see (3.43). In contrast to [3], where it is assumed that the stationary state is uniformly bounded above and below away from zero in order to control this term via the relative entropy, we instead perform a careful analysis of the difference between the density $\rho(t)$ and the stationary state $\bar{\rho}$. This allows us to control the non-local term without imposing such restrictive assumptions; see Lemmas 3.5.2 and 3.5.3. Additionally, we introduce a simplified spatial cut-off function to derive the relative entropy estimates. Our results improve upon [110] by avoiding the strong assumptions that $\rho, \boldsymbol{\mathcal{M}} \in L^\infty$ and $\gamma \leq 2$.

One of the main difficulties in this chapter is that of proving linear growth of positive energy solutions for $\alpha > 2$. In the work of [49], they proved growth of solutions for the CEPEs in the 3-D case, where $\alpha = 1$. However, the approach cannot be extended to the case that $\alpha > 2$, due to the competition of the potential energy with the kinetic energy, as we obtain a lower bound for the second derivative of the second moment of the density involving

$$(2-\alpha)\int_{\Omega(t)} \left|\frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}}\right|^2 \mathrm{d}\boldsymbol{x}.$$

For $\alpha > 2$ (which includes the Newtonian potential for $n > 4$), this term has a bad sign and cannot be controlled by the positivity of the other terms. In particular, we establish lower bounds on the asymptotic growth rate of the support radius for global classical solutions with positive energy. For $\alpha > 2$, we overcome the presence of the term with unfavourable sign by proving that, under a suitable mass threshold, the internal energy dominates the potential energy. This yields a uniform positive lower bound on the second derivative of the second moment; see Theorem 3.2.6. This allows us to show that the support of positive-energy solutions exhibits linear-in-time growth.

In this chapter, we establish nonlinear instability of stationary states in the mass-supercritical regime. That is, for the attractive polytropic CEREs, we prove nonlinear instability of stationary states in the range

$$\frac{2n}{2n-\alpha} < \gamma < \frac{n+\alpha}{n},$$

by exploiting concavity properties of the free-energy along mass-preserving dilations; see Theorem 3.2.8 and §3.8. In addition, a careful analysis of the asymptotic behaviour of key critical scalings enables us to prove instability in cases not covered by [39]. While insights have often transferred from the aggregation–diffusion equations to the study of the CEREs, the instability of stationary states in the corresponding regime remains poorly understood for aggregation–diffusion models. The current literature primarily establishes finite-time blow-up for solutions with large initial data in the Newtonian case; see, for example, Chen–Wang [38]. Our results suggest that analogous instability phenomena, possibly driven by different mechanisms, should also hold for the aggregation–diffusion equations.

Another technical novelty is that of the convergence of the potential for minimising sequences of the free-energy in §3.7. Here, the minimising sequences are bounded in $L^1_+(\mathbb{R}^n) \cap L^\gamma(\mathbb{R}^n)$ and hence have a weakly convergent subsequence. In fact, the assumption of weak convergence of a sequence ϱ_m is not strong enough to prove the convergence of

$$\int_{\mathbb{R}^n} \varrho_m \Phi_\alpha * \varrho_m \, \mathrm{d}\boldsymbol{x},$$

as $m \to \infty$. Using the compact embedding of Sobolev spaces and fractional Sobolev spaces, we are able to conclude the local convergence of the potential. Owing to the fact that we can consider a radially symmetric minimising sequence, we can take advantage of the decay estimates in Lemma 2.5.1 of the potential in order to show that local convergence is indeed enough to prove global convergence of the potential.

1.7 Focus of Chapter 4: Existence, stability and instability of rotating Riesz star solutions of the compressible Euler-Riesz equations

This chapter is devoted to the study of rotating Riesz star solutions of the attractive CEREs, a generalisation of rotating star solutions of the attractive CEPEs. In §4.3 and §4.4, we establish both the existence and stability of such solutions in the mass-subcritical regime. Our approach is different to that of [107, 108] and is based on an adaptation of Lions concentration compactness argument in [103]. By considering minimisers of the free-energy in axisymmetric coordinates, we prove tightness of the mass along minimising sequences, up to vertical translations. Furthermore, we show that this tightness occurs around the origin, which enables the passage to a strong limit and thus yields the existence of rotating Riesz star solutions. To address stability, we introduce two distance functionals between general solutions and rotating Riesz star configurations. Using a contradiction argument, we then prove that these rotating

Riesz star solutions are nonlinearly stable. In §4.5, we turn to the polytropic mass-supercritical regime and prove the existence of rotating Riesz star solutions, a result that, to the best of the author's knowledge, has not been established beyond the small angular velocity regime, even for the more extensively studied CEPEs. This is achieved by adapting the variational framework developed in [39] to the rotational setting, constructing an energy minimisation problem whose minimisers correspond to rotating Riesz star solutions. However, the inclusion of the rotational term introduces additional difficulties. In particular, this term is not amenable to radially decreasing rearrangement techniques. Consequently, unlike in Lemma 3.7.3, a different approach is required to establish the compactness of the potential term. Finally, in §4.6, as in Chapter 3, a careful analysis of the concavity properties of the associated free-energy functional under mass-preserving scalings allows us to show growth of the support for solutions arbitrarily close to any rotating Riesz star solution, which proves their instability.

In the mass-subcritical regime, one of the main difficulties arises when considering minimisers of the associated free-energy functional $\mathcal{F}^a$ over the admissible set of axisymmetric densities $\mathcal{A}_M^a$. In contrast to the stationary state setting, one cannot freely apply translations to minimising sequences, as such translations may destroy axisymmetry and hence take the sequence outside $\mathcal{A}_M^a$. To overcome this, we develop a careful adaptation of the concentration compactness framework of [103], tailored to the axisymmetric setting, in order to characterise the possible behaviour of minimising sequences. We show that either the sequence exhibits a splitting phenomenon into two axisymmetric profiles whose supports drift apart, leading to a contradiction, or the mass is tight along rings in $\mathbb{R}^3$, up to vertical translations. A further difficulty arises if the radii of these rings become unbounded, in which case concentration compactness arguments cannot be directly applied to ensure convergence of the potential energy. We prove that unless the mass is tight around the origin, equivalently, unless the radii of the rings remain uniformly bounded, the potential energy vanishes in the limit. This would imply that the infimum of the energy is non-negative, contradicting the variational structure of the problem.

In the mass-supercritical regime, we encounter a range of difficulties that do not arise in the stationary setting. In particular, unlike in §3.7, it is no longer true that for an arbitrary density there exists a unique mass-preserving critical scaling such that the rescaled density belongs to the natural admissible set; see Lemma 4.5.2. In fact, such a critical scaling may fail to exist altogether, which creates significant obstacles, especially in establishing growth properties of solutions. These complications are most pronounced when $\alpha \in (0, 2)$, whereas for $\alpha \in [2, 3)$, both the existence of rotating Riesz star solutions and their instability are comparatively more tractable. In particular, the

case $\alpha = 2$ appears to be the natural setting for the variational problem, due to the scaling behaviour of the rotational kinetic energy. Nevertheless, in both regimes, the mass-preserving critical scaling that places a given density in the natural admissible set is no longer given explicitly, in contrast to (3.70), which introduces an additional layer of difficulty.

To overcome these obstacles, we exploit a particular feature of the mass-preserving scaling which allows us to effectively disregard the contribution of the rotational kinetic term when classifying the concavity properties of the free-energy functional with respect to such scalings; see Lemma 4.6.3. In the range $\alpha \in (0, 2)$, this leads to the introduction of a distinguished quantity $\kappa(\varrho)$, which separates the two possible mass-preserving critical scalings of a given density ϱ into the natural admissible set. Importantly, while the mass-preserving critical scalings themselves are only known implicitly, the quantity $\kappa(\varrho)$ admits an explicit expression. This enables us to circumvent the difficulties associated with the lack of an explicit formula for the critical scalings. Through a careful analysis of the behaviour of the free-energy along each critical scaling, we are able to select the appropriate branch that yields both the existence of minimisers and their instability. This approach allows us to restrict attention to a suitable subset of the natural admissible set, which we then adopt as the effective admissible class for our variational problem.

In the range $\alpha \in (0, 2)$, additional difficulties arise when proving that minimisers of the free-energy give rise to rotating Riesz star solutions, as well as in establishing growth of solutions arbitrarily close to such states. The core issue is the lack of uniqueness of the mass-preserving critical scaling into the natural admissible set. This non-uniqueness leads to potentially delicate behaviour, for instance when the two admissible critical scalings approach one another, or when a solution of the CEREs transitions from one critical scaling branch to the other. To address these issues, we impose a suitable super-homogeneity condition on the square of the angular momentum. Under this assumption, we show that branch switching cannot occur, and that the regime in which the two critical scalings approach each other does not lead to any pathological behaviour. This ensures that the variational framework remains well-posed and allows us to complete the analysis.

Chapter 2

Existence of finite-energy weak solutions of the compressible Euler-Riesz equations

2.1 Introduction

The aim of this chapter is to establish the existence of global finite-energy weak solutions to the Cauchy problem (1.1) and (1.5). As noted previously, the CEREs form a strongly hyperbolic system and remain poorly understood in many respects. In particular, classical solutions are expected to break down in finite time. Consequently, the appropriate framework for global existence is that of weak solutions, and in particular finite-energy weak solutions, which we define below.

Definition 2.1.1. A measurable vector-valued function $(\rho, \boldsymbol{\mathcal{M}})(t, \boldsymbol{x})$ is a finite-energy weak solution of the Cauchy problem (1.1) and (1.5) with pressure p for $T > 0$ if $(\rho, \boldsymbol{\mathcal{M}})$ satisfies the following conditions:

(a) $\rho(t, \boldsymbol{x}) \geq 0$ *a.e.* and $(\boldsymbol{\mathcal{M}}, \frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}})(t, \boldsymbol{x}) = \mathbf{0}$ *a.e.* on the vacuum states $\{(t, \boldsymbol{x}) \,|\, \rho(t, \boldsymbol{x}) = 0\}$,

$$\rho \in L^\infty(0, T; L^1(\mathbb{R}^n)), \quad \rho e(\rho) \in L^\infty(0, T; L^1(\mathbb{R}^n)), \quad \frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}} \in L^\infty(0, T; L^2(\mathbb{R}^n)).$$

Moreover, for $\alpha \in (0, n)$,

$$\Phi_\alpha * \rho \in L^\infty(0, T; L^{\frac{2n}{\alpha}}(\mathbb{R}^n)), \quad \nabla \Phi_\alpha * \rho \in L^\infty(0, T; L^{\frac{2n}{\alpha+2}}(\mathbb{R}^n)),$$

with, for $\alpha \in [n-1, n)$,

$$\rho \in L^\infty(0, T; C^{0,\beta}(\mathbb{R}^n)),$$

for some $\beta \in (1 + \alpha - n, 1)$, while for $\alpha \in (-1, 0]$,

$$\Phi_\alpha * \rho \in L^\infty(0, T; L^\infty_{\mathrm{loc}}(\mathbb{R}^n)), \quad \nabla \Phi_\alpha * \rho \in L^\infty(0, T; L^{\frac{2n}{\alpha+2}}_{\mathrm{loc}}(\mathbb{R}^n)).$$

(b) For *a.e.* $t > 0$, the total energy is finite:

- For $\kappa = -1$,

$$\int_{\mathbb{R}^n} \Big(\frac{1}{2} \Big| \frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}} \Big|^2 + \rho e(\rho) - \frac{1}{2} \rho (\Phi_\alpha * \rho) \Big)(t, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \leq E_0.$$

- For $\kappa = 1$,

$$\begin{cases} \displaystyle\int_{\mathbb{R}^n} \Big(\frac{1}{2} \Big| \frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}} \Big|^2 + \rho e(\rho) - \frac{1}{2} \rho (\Phi_\alpha * \rho) \Big)(t, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \leq C(E_0, M), \\ \displaystyle\int_{\mathbb{R}^n} \Big(\frac{1}{2} \Big| \frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}} \Big|^2 + \rho e(\rho) + \frac{1}{2} \rho (\Phi_\alpha * \rho) \Big)(t, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \leq E_0. \end{cases}$$

(c) For any $\zeta \in C_0^1(\mathbb{R}_+ \times \mathbb{R}^n)$,

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^n} \big(\rho \zeta_t + \boldsymbol{\mathcal{M}} \cdot \nabla \zeta \big) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t + \int_{\mathbb{R}^n} \rho_0 \zeta(0, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = 0.$$

(d) For any $\boldsymbol{\psi} \in (C_0^1(\mathbb{R} \times \mathbb{R}^n))^n$,

$$\begin{aligned} &\int_{\mathbb{R}_+} \int_{\mathbb{R}^n} \Big(\boldsymbol{\mathcal{M}} \cdot \boldsymbol{\psi}_t + \frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}} \cdot \Big(\frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}} \cdot \nabla \Big) \boldsymbol{\psi} + p(\rho) \nabla \cdot \boldsymbol{\psi} \Big)(t, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\ &\qquad + \int_{\mathbb{R}^n} \boldsymbol{\mathcal{M}}_0(\boldsymbol{x}) \cdot \boldsymbol{\psi}(0, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \\ &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^n} \kappa (\rho \nabla \mathcal{W}_\alpha \cdot \boldsymbol{\psi})(t, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t. \end{aligned}$$

A lot of the known theory relating to these equations is in the case of 1-D. To take advantage of this, we consider radially symmetric solutions of the form:

$$\rho(t, \boldsymbol{x}) = \rho(t, r), \quad \boldsymbol{\mathcal{M}}(t, \boldsymbol{x}) = m(t, r) \frac{\boldsymbol{x}}{r}, \qquad \text{for } r = |\boldsymbol{x}|, \tag{2.1}$$

with initial conditions:

$$(\rho, \boldsymbol{\mathcal{M}})|_{t=0} = (\rho_0, \boldsymbol{\mathcal{M}}_0)(\boldsymbol{x}), \tag{2.2}$$

where $\rho_0 \in L^1_+(\mathbb{R}^n)$ and $\frac{\boldsymbol{\mathcal{M}}_0}{\sqrt{\rho_0}} \in L^2(\mathbb{R}^n)$. Under these assumptions, (1.1) can be written in the form:

$$\begin{cases} \rho_t + m_r + \dfrac{n-1}{r} m = 0, \\ m_t + \Big(\dfrac{m^2}{\rho} + p(\rho) \Big)_r + \dfrac{n-1}{r} \dfrac{m^2}{\rho} + \kappa \rho \, (\mathcal{W}_\alpha)_r = 0. \end{cases}$$

Using the form of radially symmetric solutions allows us to reduce our equations from a system of $n+1$ equations for $n+1$ variables to a system of 2 equations with 2 variables. This allows us to employ techniques used in the analysis of 1-D equations

of gas dynamics, such as the well understood form of the entropy and entropy-flux kernels. However, this system is still tricky to work with.

In order to prove existence of solutions to the CEREs, we employ a vanishing viscosity method. That is, we first prove existence of sufficiently well-behaved solutions to the CNSREs (1.9), which under radial symmetry assumptions becomes

$$\begin{cases} \rho_t + m_r + \dfrac{n-1}{r} m = 0, \\ m_t + \left(\dfrac{m^2}{\rho} + p(\rho)\right)_r + \dfrac{n-1}{r}\dfrac{m^2}{\rho} + \kappa\rho\left(\mathcal{W}_\alpha\right)_r = \varepsilon\Big((\mu+\lambda)\Big(\Big(\dfrac{m}{\rho}\Big)_r + \dfrac{n-1}{r}\dfrac{m}{\rho}\Big)\Big)_r \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\varepsilon\dfrac{n-1}{r}\mu_r\dfrac{m}{\rho}. \end{cases} \tag{2.3}$$

For the Coulomb/Newtonian case ($\alpha = n-2$), considerable work has been done toward the global existence of the compressible Navier-Stokes-Poisson equations (CNSPEs). It was proven by Duan-Li [56] that, for radially symmetric compactly supported arbitrarily large initial data, there exist global classical radially symmetric solutions of the CNSPEs with a stress-free boundary condition and $\gamma \in (\frac{6}{5}, \frac{4}{3}]$. This can further be extended to the CNSREs with a larger range of γ. Considering strong approximate solutions on some bounded free-boundary first allows us to derive powerful estimates, such as higher integrability for the density and velocity of the approximate solutions, these are conserved in the limit as we allow the domain to grow to $\mathbb{R}^n$. Such estimates are crucial in the application of the compensated compactness argument required to pass the viscosity limit, allowing us to prove the following result on the existence of finite-energy weak solutions.

Theorem 2.1.2 (Existence of radially symmetric solutions of the CEREs)**.** *Consider $\alpha \in (-1, n-1)$ and the initial data of radial symmetry as defined in (2.1)–(2.2). Assume that the pressure is polytropic with adiabatic exponent γ, $(\rho_0, \boldsymbol{\mathcal{M}}_0)$ satisfy (2.4)–(2.5) and additionally (2.6) for $\alpha \in (-1, 0]$, $\gamma > \frac{1}{n-1-\alpha}$, and*

- *When $\kappa = -1$, $\rho_0 \in L^{\frac{2n}{2n-\alpha}}(\mathbb{R}^n)$ for $\alpha \in (0, n-1)$ and*
  - (a) *$\gamma > 1$ for $\alpha \in (-1, n-2]$,*
  - (b) *$\gamma \geq \frac{3n}{3n-2(1+\alpha)}$ for $\alpha \in (n-2, n-1)$.*
- *When $\kappa = 1$, $\gamma \geq \frac{3n}{3n-2(1+\alpha)}$ when $\alpha \neq n-2$ and*
  - (c) *$\gamma > 1$ for $\alpha \in (-1, 0]$,*
  - (d) *$\gamma > \frac{n+\alpha}{n}$ for $\alpha \in (0, n-1)$,*
  - (e) *$M < M_{\mathrm{c}}^{\alpha}(\gamma)$ for $\gamma \in (\frac{2n}{2n-\alpha}, \frac{n+\alpha}{n}]$ and $\alpha \in (0, n-1)$.*

Then there exists a global radially symmetric finite-energy weak solution $(\rho, \boldsymbol{\mathcal{M}})$ of the Cauchy problem (1.1) and (1.5) in the sense of Definition 2.1.1.

2.2 Preliminaries and results

We first assume that the given initial data function $(\rho_0, \boldsymbol{\mathcal{M}}_0)(\boldsymbol{x})$ has finite total energy and total mass:

$$E_0 := \int_{\mathbb{R}^n} \rho_0 \Big(\frac{1}{2} \Big| \frac{\boldsymbol{\mathcal{M}}_0}{\rho_0} \Big|^2 + e(\rho_0) + \frac{\kappa}{2} (\Phi_\alpha * \rho_0) \Big)(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} < \infty, \tag{2.4}$$

$$M := \int_{\mathbb{R}^n} \rho_0(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = \omega_n \int_0^\infty \rho_0(r) r^{n-1} \, \mathrm{d}r < \infty. \tag{2.5}$$

When $\alpha \in (-1, 0]$, since we can no longer employ the HLS inequality to obtain an L^p estimate for the potential energy, we assume that $(\rho_0, \boldsymbol{\mathcal{M}}_0)(\boldsymbol{x})$ has finite $(-\alpha)$–moment:

$$\int_{\mathbb{R}^n} \rho_0(\boldsymbol{x}) k_\alpha (1 + |\boldsymbol{x}|^2) \, \mathrm{d}\boldsymbol{x} = \omega_n \int_0^\infty \rho_0(r) k_\alpha (1 + r^2) r^{n-1} \, \mathrm{d}r < \infty, \tag{2.6}$$

where

$$k_\alpha(r) = \begin{cases} r^{-\frac{\alpha}{2}} & \text{for } \alpha \in (-1, 0), \\ \log(r) & \text{for } \alpha = 0, \end{cases}$$

ω_n represents the surface area of the unit sphere in $\mathbb{R}^n$. Such a condition is propagated by solutions of the equations through time. Here we require the initial data such that the convolution $\Phi_\alpha * \rho_0$ is well-defined and differentiable. We consider a polytropic pressure with adiabatic exponent $\gamma > 1$, that is,

$$p(\rho) = a_0 \rho^\gamma.$$

We first construct global weak solutions of the CNSREs with appropriately adapted density-dependent viscosity terms and approximate initial conditions, constructed in §2.4 with the properties as in Lemmas 2.4.1–2.4.8, given by

$$(\rho^\varepsilon, \boldsymbol{\mathcal{M}}^\varepsilon)(0, \boldsymbol{x}) = (\rho_0^\varepsilon, \boldsymbol{\mathcal{M}}_0^\varepsilon)(\boldsymbol{x}). \tag{2.7}$$

Now we provide the definition of global weak solutions of the Cauchy problem (1.9) and (2.7).

Definition 2.2.1. A measurable vector function $(\rho^\varepsilon, \boldsymbol{\mathcal{M}}^\varepsilon)(t, \boldsymbol{x})$ is a weak solution of the Cauchy problem (1.9) and (2.7) with $(\mu, \lambda) = (\rho, 0)$ if $(\rho^\varepsilon, \boldsymbol{\mathcal{M}}^\varepsilon)(t, \boldsymbol{x})$ satisfies the following conditions:

(a) $\rho^\varepsilon(t, \boldsymbol{x}) \geq 0$ *a.e.* and $(\boldsymbol{\mathcal{M}}^\varepsilon, \frac{\boldsymbol{\mathcal{M}}^\varepsilon}{\sqrt{\rho^\varepsilon}})(t, \boldsymbol{x}) = \mathbf{0}$ *a.e.* on the vacuum states $\{(t, \boldsymbol{x}) \,|\, \rho^\varepsilon(t, \boldsymbol{x}) = 0\}$,

$$\rho^\varepsilon \in L^\infty(0, T; L^1(\mathbb{R}^n)), \qquad \rho^\varepsilon e(\rho^\varepsilon) \in L^\infty(0, T; L^1(\mathbb{R}^n)),$$

$$\frac{\boldsymbol{\mathcal{M}}^\varepsilon}{\sqrt{\rho^\varepsilon}} \in L^\infty(0,T;L^2(\mathbb{R}^n)), \qquad \nabla\sqrt{\rho^\varepsilon} \in L^\infty(0,T;L^2(\mathbb{R}^n)).$$

Moreover, for $\alpha \in (0,n)$,

$$\Phi_\alpha * \rho^\varepsilon \in L^\infty(0,T;L^{\frac{2n}{\alpha}}(\mathbb{R}^n)), \quad \nabla\Phi_\alpha * \rho^\varepsilon \in L^\infty(0,T;L^{\frac{2n}{\alpha+2}}(\mathbb{R}^n)),$$

with, for $\alpha \in [n-1,n)$,

$$\rho^\varepsilon \in L^\infty(0,T;C^{0,\beta}(\mathbb{R}^n)),$$

for some $\beta \in (1+\alpha-n,1)$, while for $\alpha \in (-1,0]$,

$$\Phi_\alpha * \rho^\varepsilon \in L^\infty(0,T;L^\infty_{\mathrm{loc}}(\mathbb{R}^n)), \quad \nabla\Phi_\alpha * \rho^\varepsilon \in L^\infty(0,T;L^{\frac{2n}{\alpha+2}}_{\mathrm{loc}}(\mathbb{R}^n)).$$

(b) For any $\zeta \in C_0^1(\mathbb{R}\times\mathbb{R}^n)$ and $t_2 \geq t_1 \geq 0$,

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^n} (\rho^\varepsilon\zeta_t + \boldsymbol{\mathcal{M}}^\varepsilon\cdot\nabla\zeta)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_{\mathbb{R}^n}(\rho^\varepsilon\zeta)(t_2,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \int_{\mathbb{R}^n}(\rho^\varepsilon\zeta)(t_1,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

(c) For any $\boldsymbol{\psi} \in (C_0^1(\mathbb{R}\times\mathbb{R}^n))^n$,

$$\begin{aligned}
&\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}\Big(\boldsymbol{\mathcal{M}}^\varepsilon\cdot\boldsymbol{\psi}_t + \frac{\boldsymbol{\mathcal{M}}^\varepsilon}{\sqrt{\rho^\varepsilon}}\cdot\Big(\frac{\boldsymbol{\mathcal{M}}^\varepsilon}{\sqrt{\rho^\varepsilon}}\cdot\nabla\Big)\boldsymbol{\psi} + p(\rho^\varepsilon)\nabla\cdot\boldsymbol{\psi}\Big)(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&\qquad + \int_{\mathbb{R}^n}\boldsymbol{\mathcal{M}}_0^\varepsilon(\boldsymbol{x})\cdot\boldsymbol{\psi}(0,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
&= -\varepsilon\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}\Big(\frac{1}{2}\boldsymbol{\mathcal{M}}^\varepsilon\cdot\big(\Delta\boldsymbol{\psi}+\nabla(\nabla\cdot\boldsymbol{\psi})\big) + \frac{\boldsymbol{\mathcal{M}}^\varepsilon}{\sqrt{\rho^\varepsilon}}\cdot(\nabla\sqrt{\rho^\varepsilon}\cdot\nabla)\boldsymbol{\psi} \\
&\qquad + \nabla\sqrt{\rho^\varepsilon}\cdot\Big(\frac{\boldsymbol{\mathcal{M}}^\varepsilon}{\sqrt{\rho^\varepsilon}}\cdot\nabla\Big)\boldsymbol{\psi}\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_{\mathbb{R}_+}\int_{\mathbb{R}^n}\kappa(\rho^\varepsilon\nabla\mathcal{W}_\alpha^\varepsilon\cdot\boldsymbol{\psi})(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,
\end{aligned}$$

where

$$\nabla\mathcal{W}_\alpha^\varepsilon(\boldsymbol{x}) = \begin{cases} (\nabla\Phi_\alpha * \rho^\varepsilon)(\boldsymbol{x}) & \text{for } \alpha \in (-1,n-1), \\ \displaystyle\int_{\mathbb{R}^n}\nabla\Phi_\alpha(\boldsymbol{x}-\boldsymbol{y})\big(\rho^\varepsilon(\boldsymbol{y})-\rho^\varepsilon(\boldsymbol{x})\big)\,\mathrm{d}\boldsymbol{y} & \text{for } \alpha \in [n-1,n). \end{cases}$$

The main result of this chapter is the existence of finite-energy weak solutions of the polytropic CEREs given by Theorem 2.1.2. In Theorem 2.1.2, the quantity $M_{\mathrm{c}}^\alpha\big(\frac{n+\alpha}{n}\big)$ for $\alpha \in (0,n)$ represents the sharp critical mass. This is analogous to the Chandrasekhar limit $M_{\mathrm{ch}} := M_{\mathrm{c}}^{n-2}\Big(\frac{2(n-1)}{n}\Big)$, which arises in the theory of steady gaseous stars; see [29]. It is well known that no steady white dwarf star exists with total mass $M > M_{\mathrm{ch}}$, in other words, there is no stationary solution of the form $(\bar\rho(|\boldsymbol{x}|),\boldsymbol{0})$ when $p(\rho) = a_0\rho^{\frac{2n-2}{n}}$ in this regime. An analogous dichotomy result and the corresponding

sharp critical mass for all $\alpha \in (0, n)$ were established in [25] in connection with variants of the Hardy–Littlewood–Sobolev inequalities; see also [9] for the Newtonian case $\alpha = n - 2$.

The critical mass for $\alpha > 0$ and $\gamma \in (\frac{2n}{2n-\alpha}, \frac{n+\alpha}{n}]$ is defined by

$$M_{\mathrm{c}}^{\alpha}(\gamma) = \begin{cases} B_{n,\gamma,\alpha}^{-\frac{n}{n-\alpha}}, & \text{for } \gamma = \frac{n+\alpha}{n}, \\ \Big(\dfrac{\alpha B_{n,\gamma,\alpha}}{n(\gamma-1)}\Big)^{\frac{n(\gamma-1)}{2n-\gamma(2n-\alpha)}} \Big(\dfrac{\alpha E_0}{\alpha - n(\gamma-1)}\Big)^{\frac{\alpha-n(\gamma-1)}{2n-\gamma(2n-\alpha)}}, & \text{for } \gamma \in (\frac{2n}{2n-\alpha}, \frac{n+\alpha}{n}), \end{cases}$$

where

$$B_{n,\gamma,\alpha} = \frac{C_*(\alpha,\gamma,n)}{2\alpha} \omega_n^{\frac{\alpha}{n(\gamma-1)}-1} \Big(\frac{\gamma-1}{a_0}\Big)^{\frac{\alpha}{n(\gamma-1)}},$$

and $C_*(\alpha, \gamma, n) > 0$ is the sharp constant given by the variation of the HLS inequality from Lemma A.0.3. For the Newtonian case $\alpha = n-2$, the bound on the critical mass

$$M_{\mathrm{c}}^{n-2}(\gamma) = B_{n,\gamma,n-2}^{-\frac{n}{2}}, \qquad \gamma < \frac{2(n-1)}{n},$$

was established in [36].

Remark 2.2.2. In Theorem 2.1.2, one observes that

$$\frac{2n}{2n-\alpha} < \frac{3n}{3n-2(1+\alpha)}, \qquad \text{for all } \alpha \in (-1, n-1).$$

Moreover, the inequality

$$\frac{n+\alpha}{n} > \frac{3n}{3n-2(1+\alpha)},$$

holds only for certain values of α when $n \geq 20$. In this case, the above inequality is satisfied precisely for $\alpha \in (\alpha_-, \alpha_+)$, where

$$\alpha_\pm = \frac{n-2 \pm \sqrt{n^2-20n+4}}{4}.$$

Furthermore, as $n \to \infty$,

$$\frac{\alpha_-}{n} \to 0, \qquad \frac{\alpha_+}{n} \to \frac{1}{2}.$$

Consequently, assumption (e) in Theorem 2.1.2 is relevant only in dimensions $n \geq 20$ and for $\alpha \in (\alpha_-, \alpha_+)$.

2.3 Approximate PDEs

Some of the principal difficulties in studying problem (2.3) on the whole space $\mathbb{R}^n$ arise from the possible formation of vacuum states, mass concentration, and shock waves in the solutions in the viscosity limit. In particular, in the case of radial solutions, the

formation of singularities is expected to occur at the origin. To circumvent these issues, we introduce the following free boundary formulation for the approximate solutions

$$\begin{cases} \rho_t^{\varepsilon,b} + (\rho^{\varepsilon,b}u^{\varepsilon,b})_r + \dfrac{n-1}{r}\rho^{\varepsilon,b}u^{\varepsilon,b} = 0, \\ (\rho^{\varepsilon,b}u^{\varepsilon,b})_t + (\rho^{\varepsilon,b}(u^{\varepsilon,b})^2 + p(\rho^{\varepsilon,b}))_r + \dfrac{n-1}{r}\rho^{\varepsilon,b}(u^{\varepsilon,b})^2 + \kappa\rho^{\varepsilon,b}(\Phi_\alpha * \rho^{\varepsilon,b})_r \\ \qquad\qquad = \varepsilon\Big(\rho^{\varepsilon,b}\big(u_r^{\varepsilon,b} + \dfrac{n-1}{r}u^{\varepsilon,b}\big)\Big)_r - \varepsilon\dfrac{n-1}{r}\rho_r^{\varepsilon,b}u^{\varepsilon,b}. \end{cases} \tag{2.8}$$

This is defined on the moving domain

$$\Omega^T = \big\{(t,r)\,|\,0 \le t \le T,\ a \le r \le b(t)\big\},$$

where $b(t)$ is a free boundary that is subject to

$$\begin{cases} b'(t) = u^{\varepsilon,b}(t,b(t)), & \text{for } t > 0, \\ b(0) = b, \end{cases} \tag{2.9}$$

with boundary conditions

$$u^{\varepsilon,b}(t,a) = 0, \quad \Big(p(\rho^{\varepsilon,b}) - \varepsilon\rho^{\varepsilon,b}\Big(u_r^{\varepsilon,b} + \frac{n-1}{r}u^{\varepsilon,b}\Big)\Big)(t,b(t)) = 0, \qquad \text{for } t > 0, \tag{2.10}$$

for $a := b^{-1}$ with $b \gg 1$. The initial conditions are prescribed by

$$(\rho^{\varepsilon,b}, u^{\varepsilon,b})|_{t=0}(r) = (\rho_0^{\varepsilon,b}, u_0^{\varepsilon,b})(r), \qquad \text{for } r \in [a,b], \tag{2.11}$$

where $(\rho_0^{\varepsilon,b}, u_0^{\varepsilon,b})(r)$ are defined in the next section. For each fixed pair (ε, b), these functions satisfy the estimate that there exists a constant $C_{\varepsilon,b} > 0$ such that

$$0 < C_{\varepsilon,b}^{-1} \le \rho_0^{\varepsilon,b}(r) \le C_{\varepsilon,b} < \infty. \tag{2.12}$$

We define the spatial domain Ω_t of the free boundary problem at time $t \ge 0$ to be

$$\Omega_t = \big\{\boldsymbol{x} \in \mathbb{R}^n\,|\,a \le |\boldsymbol{x}| \le b(t)\big\}.$$

Note that the exclusion of the ball of radius a in the definition of Ω_t allows us to avoid the formation of singularities at the origin, we will see later that this is even true in the limit as $b \to \infty$. Here, system (2.8) corresponds to system (2.3) with $(\mu, \lambda) = (\rho, 0)$, written in terms of the velocity $u = \frac{m}{\rho}$. Under the present free boundary formulation, one can establish the existence of smooth solutions $(\rho^{\varepsilon,b}, u^{\varepsilon,b})$ by adapting the approach of [56]. Moreover, by virtue of (2.12), the density satisfies $\rho^{\varepsilon,b} > 0$, ensuring that the velocity $u^{\varepsilon,b}$ is well-defined. Hence, this formulation is equivalent to the original system. More precisely, there exists a constant $\tilde{C}_{\varepsilon,b,T} > 0$ such that, for all $t \in [0,T]$,

$$0 < \tilde{C}_{\varepsilon,b,T}^{-1} \le \rho^{\varepsilon,b}(t,r) \le \tilde{C}_{\varepsilon,b,T} < \infty. \tag{2.13}$$

We emphasize that the boundary condition in (2.10) imposed on the free boundary is a stress-free condition. This choice is physically natural, as it ensures that the boundary moves with the fluid velocity, consistent with the free boundary formulation of the problem. We denote the solution of the free boundary problem (2.8)–(2.11) by $(\rho^{\varepsilon,b}, u^{\varepsilon,b})$. For simplicity, we suppress the superscripts whenever no confusion arises, up to §2.10.

Remark 2.3.1. From $(2.8)_1$, (2.9) and (2.10), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t,b(t)) = -\rho\Big(u_r + \frac{n-1}{r}u\Big)(t,b(t)) = -\frac{p(\rho(t,b(t)))}{\varepsilon} \leq 0.$$

Thus,

$$\rho(t,b(t)) \leq \rho_0(b). \tag{2.14}$$

The critical mass, as before, is defined for $\alpha > 0$ as

$$M_{\mathrm{c}}^{\varepsilon,b,\alpha}(\gamma) = \begin{cases} B_{n,\gamma,\alpha}^{-\frac{n}{n-\alpha}}, & \text{for } \gamma = \frac{n+\alpha}{n}, \\ \Big(\frac{\alpha B_{n,\gamma,\alpha}}{n(\gamma-1)}\Big)^{\frac{n(\gamma-1)}{2n-\gamma(2n-\alpha)}} \Big(\frac{\alpha E_0^{\varepsilon,b}}{\alpha - n(\gamma-1)}\Big)^{\frac{\alpha-n(\gamma-1)}{2n-\gamma(2n-\alpha)}}, & \text{for } \gamma \in (\frac{2n}{2n-\alpha}, \frac{n+\alpha}{n}). \end{cases} \tag{2.15}$$

Here, the total energy $E_0^{\varepsilon,b}$ will be defined in Definition 2.4.7. The critical mass $M_{\mathrm{c}}^{\varepsilon,\alpha}(\gamma)$ is defined analogously to $M_{\mathrm{c}}^{\varepsilon,b,\alpha}(\gamma)$ in (2.15), with the only difference being that $E_0^{\varepsilon,b}$ is replaced by E_0^{ε}, where E_0^{ε} will also be given in Definition 2.4.7. Moreover, by the construction of $\rho_0^{\varepsilon,b}$, the following mass identity holds

$$\int_a^b \rho_0^{\varepsilon,b}(r)\, r^{n-1}\, \mathrm{d}r = \frac{M}{\omega_n}.$$

The conservation of mass follows directly from $(2.8)_1$ together with (2.9)–(2.10).

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^{b(t)} \rho(t,r)\, r^{n-1}\, \mathrm{d}r = (\rho u)(t,b(t))b(t)^{n-1} - \int_a^{b(t)} (\rho u r^{n-1})_r(t,r)\, \mathrm{d}r = 0.$$

Thus, for all $t \geq 0$,

$$\int_a^{b(t)} \rho(t,r)\, r^{n-1}\, \mathrm{d}r = \frac{M}{\omega_n}.$$

Using this, we define the Lagrangian coordinates $(\tau, x) = (t, x(t,r))$ by

$$x(t,r) = \int_a^r \rho(t,y)\, y^{n-1}\, \mathrm{d}y.$$

It is straightforward to verify that this defines a smooth bijective mapping from $[0,T]\times[a,b(t)]$ onto $[0,T]\times[0,\frac{M}{\omega_n}]$. This transformation is particularly useful since the latter domain is fixed in time. We then obtain

$$\nabla_{(t,r)}x = (-\rho u r^{n-1}, \rho r^{n-1}), \quad \nabla_{(t,r)}\tau = (1,0), \quad \nabla_{(\tau,x)}r = (u, \rho^{-1}r^{1-n}), \quad \nabla_{(\tau,x)}t = (1,0). \tag{2.16}$$

Under the change of variables, the boundary conditions become

$$u(\tau,0)=0,\quad \big(p-\varepsilon\rho^2(r^{n-1}u)_x\big)(\tau,\frac{M}{\omega_n})=0,\qquad \text{for } \tau\in[0,T]. \tag{2.17}$$

Under this transformation, system (2.8) can be rewritten in the form

$$\begin{cases}\rho_\tau+\rho^2(r^{n-1}u)_x=0,\\ u_\tau+r^{n-1}p_x+\kappa\rho r^{n-1}(\Phi_\alpha*\rho)_x=\varepsilon r^{n-1}\big(\rho^2(r^{n-1}u)_x\big)_x-(n-1)\varepsilon r^{n-2}\rho_x u.\end{cases} \tag{2.18}$$

2.4 Construction of the approximate initial data

As in [36], we approximate the initial density by a strictly positive density profile, bounded away from the vacuum. Consequently, the corresponding approximate solutions remain bounded away from the vacuum, thereby avoiding the difficulties associated with cavitation in the approximation scheme. This enables us to formulate the equations in terms of the velocity and the density, rather than working with the momentum and the density formulation.

Consider a standard radially symmetric mollifier $\varphi\in C_c^\infty(\mathbb{R}^n)$, $\operatorname{supp}(\varphi)\subset B_1$,

$$\int_{\mathbb{R}^n}\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=1,$$

and the associated mollifier:

$$\varphi_\delta(\boldsymbol{x}):=\frac{1}{\delta^n}\varphi\Big(\frac{\boldsymbol{x}}{\delta}\Big). \tag{2.19}$$

We now define the approximation

$$\tilde{\rho}_0^\varepsilon(\boldsymbol{x}):=\Big(\int_{\mathbb{R}^n}\sqrt{\rho_0(\boldsymbol{x}-\boldsymbol{y})}\varphi_{\sqrt{\varepsilon}}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}+\varepsilon e^{-|\boldsymbol{x}|^2}\Big)^2.$$

We construct the approximation in this manner in order to obtain a convenient expression for $\sqrt{\tilde{\rho}_0^\varepsilon}(\boldsymbol{x})$. The first component smooths the square root of the density, while the second prevents the occurrence of vacuum regions in the initial data. Moreover, we normalise the approximation so that the total mass of the initial density remains independent of ε. We define the approximate initial density as follows

$$\rho_0^\varepsilon(\boldsymbol{x}):=\frac{M}{\int_{\mathbb{R}^n}\tilde{\rho}_0^\varepsilon(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}}\tilde{\rho}_0^\varepsilon(\boldsymbol{x}). \tag{2.20}$$

We have the following lemma.

Lemma 2.4.1. *Let $q\in\{1,\gamma,\frac{2n}{2n-\alpha}\}$ for $\kappa=-1$, $\alpha>0$ and $q\in\{1,\gamma\}$ otherwise. Then*

$$\begin{aligned}&\int_{\mathbb{R}^n}\rho_0^\varepsilon(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=M,\quad \|\rho_0^\varepsilon\|_{L^q}\le C\big(\|\rho_0\|_{L^q}+1\big),\qquad \text{for all } \varepsilon\in(0,1],\\ &\lim_{\varepsilon\to0}\big(\|\rho_0^\varepsilon-\rho_0\|_{L^q}+\|\sqrt{\rho_0^\varepsilon}-\sqrt{\rho_0}\|_{L^{2q}}\big)=0,\\ &\varepsilon\int_{\mathbb{R}^n}\Big|\nabla_{\boldsymbol{x}}\sqrt{\rho_0^\varepsilon(\boldsymbol{x})}\Big|^2\,\mathrm{d}\boldsymbol{x}\le C\big(\|\rho_0\|_{L^1}+1\big)\longrightarrow0,\qquad \text{as } \varepsilon\to0.\end{aligned} \tag{2.21}$$

Note, in the case that $\alpha \in (-1, 0]$, $\Phi_\alpha * \rho_0^\varepsilon$ is well-defined:

$$\begin{aligned}
&\int_{\mathbb{R}^n} \rho_0^\varepsilon(\boldsymbol{y})|\Phi_\alpha(\boldsymbol{x}-\boldsymbol{y})|\,\mathrm{d}\boldsymbol{y} \\
&\leq \frac{2M}{\int_{\mathbb{R}^n} \tilde{\rho}_0^\varepsilon(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}} \int_{\mathbb{R}^n} \tilde{\rho}_0^\varepsilon(\boldsymbol{y})|\Phi_\alpha(\boldsymbol{x}-\boldsymbol{y})|\,\mathrm{d}\boldsymbol{y} \\
&\leq \frac{4M}{\int_{\mathbb{R}^n} \tilde{\rho}_0^\varepsilon(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}} \int_{\mathbb{R}^n} \left(|\sqrt{\rho_0} * \varphi_{\sqrt{\varepsilon}}|^2(\boldsymbol{y}) + \varepsilon^2 e^{-2|\boldsymbol{y}|^2}\right)|\Phi_\alpha(\boldsymbol{x}-\boldsymbol{y})|\,\mathrm{d}\boldsymbol{y} \\
&= \frac{4M}{\int_{\mathbb{R}^n} \tilde{\rho}_0^\varepsilon(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}} \int_{\mathbb{R}^n} \left(\left|\int_{\mathbb{R}^n} \sqrt{\rho_0}(\boldsymbol{y}-\boldsymbol{z})\varphi_{\sqrt{\varepsilon}}(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\right|^2 + \varepsilon^2 e^{-2|\boldsymbol{y}|^2}\right)|\Phi_\alpha(\boldsymbol{x}-\boldsymbol{y})|\,\mathrm{d}\boldsymbol{y} \\
&\leq \frac{4M}{\int_{\mathbb{R}^n} \tilde{\rho}_0^\varepsilon(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}} \Bigg(\left(\int_{\mathbb{R}^n} \varphi_{\sqrt{\varepsilon}}(\boldsymbol{z}) \sqrt{(|\Phi_\alpha| * \rho_0)(\boldsymbol{x}-\boldsymbol{z})}\,\mathrm{d}\boldsymbol{z} \right)^2 \\
&\qquad\qquad + \int_{\mathbb{R}^n} \varepsilon^2 e^{-2|\boldsymbol{y}|^2} |\Phi_\alpha(\boldsymbol{x}-\boldsymbol{y})|\,\mathrm{d}\boldsymbol{y} \Bigg) \\
&< \infty,
\end{aligned} \tag{2.22}$$

where we employ the Minkowski integral inequality in the penultimate inequality. We also obtain the following convergence result for the potential.

Lemma 2.4.2. *$\Phi_\alpha * \rho_0^\varepsilon$ with $\alpha \in (0, n-1)$ satisfies the following:*

$$\|\Phi_\alpha * \rho_0^\varepsilon - \Phi_\alpha * \rho_0\|_{L^{\frac{2n}{\alpha}}} \leq C\|\rho_0^\varepsilon - \rho_0\|_{L^{\frac{2n}{2n-\alpha}}} \longrightarrow 0, \qquad \text{as } \varepsilon \to 0,$$

$$\|\nabla(\Phi_\alpha * \rho_0^\varepsilon - \Phi_\alpha * \rho_0)\|_{L^{\frac{2n}{\alpha+2}}} \leq C\|\rho_0^\varepsilon - \rho_0\|_{L^{\frac{2n}{2n-\alpha}}} \longrightarrow 0, \qquad \text{as } \varepsilon \to 0.$$

For $\alpha \in (-1, 0]$ and $K \Subset \mathbb{R}^n$,

$$\|\Phi_\alpha * \rho_0^\varepsilon - \Phi_\alpha * \rho_0\|_{L^\infty(K)} + \|\nabla(\Phi_\alpha * \rho_0^\varepsilon - \Phi_\alpha * \rho_0)\|_{L^{\frac{2n}{\alpha+2}}(K)} \longrightarrow 0, \qquad \text{as } \varepsilon \to 0. \tag{2.23}$$

Moreover, there exists a constant $C > 0$ such that, for $\varepsilon > 0$ sufficiently small,

$$\int_{\mathbb{R}^n} \rho_0^\varepsilon(\boldsymbol{x}) k_\alpha (1 + |\boldsymbol{x}|^2)\,\mathrm{d}\boldsymbol{x} \leq C. \tag{2.24}$$

Proof. For $\alpha \in (0, n-1)$, the first part is a direct consequence of the HLS inequality from Lemma A.0.2.

For $\alpha \in (-1, 0]$, for simplicity, we may assume that $K = B_D$ for some $D > 0$. Then, for $\hat{\sigma} > 0$,

$$\begin{aligned}
&\left\|\Phi_\alpha * \rho_0^\varepsilon - \Phi_\alpha * \rho_0\right\|_{L^\infty(K)} \\
&\leq \left\|(\mathbb{1}_{B_{\hat{\sigma}}}|\Phi_\alpha|) * (\rho_0^\varepsilon - \rho_0)\right\|_{L^\infty(K)} + \left\|(\mathbb{1}_{B_{\hat{\sigma}}^c}|\Phi_\alpha|) * (\rho_0^\varepsilon - \rho_0)\right\|_{L^\infty(K)} \\
&=: I_1 + I_2.
\end{aligned}$$

Since $\mathbb{1}_{B_{\hat{\sigma}}}\Phi_\alpha \in L^{\frac{\gamma}{\gamma-1}}$, we estimate I_1 using Hölder's inequality as follows

$$I_1 \leq \|\rho_0^\varepsilon - \rho_0\|_{L^\gamma}\|\mathbb{1}_{B_{\hat{\sigma}}}\Phi_\alpha\|_{L^{\frac{\gamma}{\gamma-1}}} \longrightarrow 0, \qquad \text{as } \varepsilon \to 0. \tag{2.25}$$

We now want to show that $\|(\mathbb{1}_{B^c_{\hat\sigma}}|\Phi_\alpha|) * \rho_0\|_{L^\infty(B_{D+1})} \to 0$ as $\hat\sigma \to \infty$. We take $\hat\sigma$ large enough so that, for all $\boldsymbol{x} \in B_{D+1}$ and $\boldsymbol{y} \in B^c_{\hat\sigma - D-1}$, $1 \le |\boldsymbol{x}-\boldsymbol{y}| \le 2|\boldsymbol{y}|$. Then, for any $\boldsymbol{x} \in B_{D+1}$,

$$\begin{aligned}\big|(\mathbb{1}_{B^c_{\hat\sigma}}|\Phi_\alpha|) * \rho_0\big|(\boldsymbol{x}) &\le 2\int_{\mathbb{R}^n} \rho_0(\boldsymbol{y})\mathbb{1}_{B^c_{\hat\sigma}}(\boldsymbol{x}-\boldsymbol{y})|k_\alpha(|\boldsymbol{x}-\boldsymbol{y}|^2)|\,\mathrm{d}\boldsymbol{y}\\ &\le \int_{B^c_{\hat\sigma-D-1}} \rho_0(\boldsymbol{y})|k_\alpha(|\boldsymbol{x}-\boldsymbol{y}|^2)|\,\mathrm{d}\boldsymbol{y}\\ &\le C\int_{B^c_{\hat\sigma-D-1}} \rho_0(\boldsymbol{y})\big(1+k_\alpha(1+|\boldsymbol{y}|^2)\big)\,\mathrm{d}\boldsymbol{y}.\end{aligned}$$

This implies

$$\big\|(\mathbb{1}_{B^c_{\hat\sigma}}|\Phi_\alpha|) * \rho_0\big\|_{L^\infty(B_{D+1})} \le C\int_{B^c_{\hat\sigma-D-1}} \rho_0(\boldsymbol{y})\big(1+k_\alpha(1+|\boldsymbol{y}|^2)\big)\,\mathrm{d}\boldsymbol{y} \longrightarrow 0, \qquad \text{as } \hat\sigma \to \infty.$$

By the Lebesgue dominated convergence Theorem, we obtain

$$\int_{\mathbb{R}^n} \tilde\rho_0^\varepsilon(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \longrightarrow M, \qquad \text{as } \varepsilon \to 0.$$

Then, for ε small enough, $\int_{\mathbb{R}^n} \tilde\rho_0^\varepsilon(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \ge \frac{M}{2}$. Moreover, for $\varepsilon \in (0,1)$ small enough, if $\boldsymbol{x} \in K$, by a similar calculation to (2.22), we have

$$\begin{aligned}&\big|(\mathbb{1}_{B^c_{\hat\sigma}}|\Phi_\alpha|) * \rho_0^\varepsilon\big|(\boldsymbol{x})\\ &\le \frac{2M}{\int_{\mathbb{R}^n}\tilde\rho_0^\varepsilon(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}}\bigg(\Big(\int_{\mathbb{R}^n}\varphi_{\sqrt\varepsilon}(\boldsymbol{z})\sqrt{((\mathbb{1}_{B^c_{\hat\sigma}}|\Phi_\alpha|)*\rho_0)(\boldsymbol{x}-\boldsymbol{z})}\,\mathrm{d}\boldsymbol{z}\Big)^2\\ &\qquad\qquad + \int_{\mathbb{R}^n}\varepsilon^2 e^{-2|\boldsymbol{y}|^2}(\mathbb{1}_{B^c_{\hat\sigma}}|\Phi_\alpha|)(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\bigg)\\ &\le 4\Big(\big\|(\mathbb{1}_{B^c_{\hat\sigma}}|\Phi_\alpha|)*\rho_0\big\|_{L^\infty(B_{D+1})} + \big\|(\mathbb{1}_{B^c_{\hat\sigma}}|\Phi_\alpha|)*e^{-2|\cdot|^2}\big\|_{L^\infty(B_{D+1})}\Big).\end{aligned}$$

This yields

$$I_2 \le 5\big\|(\mathbb{1}_{B^c_{\hat\sigma}}|\Phi_\alpha|)*\rho_0\big\|_{L^\infty(B_{D+1})} + 4\big\|(\mathbb{1}_{B^c_{\hat\sigma}}|\Phi_\alpha|)*e^{-2|\cdot|^2}\big\|_{L^\infty(B_{D+1})} \longrightarrow 0, \qquad \text{as } \hat\sigma \to \infty. \tag{2.26}$$

Thus, combining (2.25) with (2.26), we obtain

$$\big\|\Phi_\alpha * \rho_0^\varepsilon - \Phi_\alpha * \rho_0\big\|_{L^\infty(K)} \longrightarrow 0, \qquad \text{as } \varepsilon \to 0.$$

For the final part, take $1 < p < \min\{\frac{2n}{\alpha+2}, \gamma\}$, by applying the HLS inequality from Lemma A.0.2 and an interpolation argument, we have

$$\begin{aligned}&\big\|\nabla(\Phi_\alpha * \rho_0^\varepsilon - \Phi_\alpha * \rho_0)\big\|_{L^{\frac{2n}{\alpha+2}}(K)}\\ &\le \big\|(|\cdot|^{-(\alpha+1)}\mathbb{1}_{B_1}) * (\rho_0^\varepsilon - \rho_0)\big\|_{L^{\frac{2n}{\alpha+2}}(K)} + \big\|(|\cdot|^{-(\alpha+1)}\mathbb{1}_{B_1^c}) * (\rho_0^\varepsilon - \rho_0)\big\|_{L^{\frac{2n}{\alpha+2}}(K)}\\ &\le \big\|(|\cdot|^{-(1+\alpha/2+n(1-1/p))}) * (\rho_0^\varepsilon - \rho_0)\big\|_{L^{\frac{2n}{\alpha+2}}(K)} + \big\|\mathbb{1}_{B_1^c} * (\rho_0^\varepsilon - \rho_0)\big\|_{L^{\frac{2n}{\alpha+2}}(K)}\\ &\le C\|\rho_0^\varepsilon - \rho_0\|_{L^p(\mathbb{R}^n)} + |K|^{\frac{\alpha+2}{2n}}\|\rho_0^\varepsilon - \rho_0\|_{L^1(\mathbb{R}^n)}\\ &\le C\|\rho_0^\varepsilon - \rho_0\|_{L^1(\mathbb{R}^n)}^{\frac{\gamma-p}{p(\gamma-1)}}\|\rho_0^\varepsilon - \rho_0\|_{L^\gamma(\mathbb{R}^n)}^{\frac{\gamma(p-1)}{p(\gamma-1)}} + |K|^{\frac{\alpha+2}{2n}}\|\rho_0^\varepsilon - \rho_0\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

Thus, we conclude

$$\|\nabla(\Phi_\alpha * \rho_0^\varepsilon - \Phi_\alpha * \rho_0)\|_{L^{\frac{2n}{\alpha+2}}(K)} \longrightarrow 0, \qquad \text{as } \varepsilon \to 0.$$

Now, by using the initial assumption (2.6), there exists $C > 0$ such that

$$\int_{\mathbb{R}^n} \rho_0(\boldsymbol{x}) k_\alpha (1 + |\boldsymbol{x}|^2)\, \mathrm{d}\boldsymbol{x} \le C.$$

Thus, applying the Hölder inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \rho_0^\varepsilon(\boldsymbol{x}) k_\alpha (1 + |\boldsymbol{x}|^2)\, \mathrm{d}\boldsymbol{x} \\
&\le \int_{\mathbb{R}^n} \rho_0(\boldsymbol{x}) k_\alpha (1 + |\boldsymbol{x}|^2)\, \mathrm{d}\boldsymbol{x} + \Big| \int_{\mathbb{R}^n} \big(\rho_0^\varepsilon(\boldsymbol{x}) - \rho_0(\boldsymbol{x})\big) k_\alpha (1 + |\boldsymbol{x}|^2)\, \mathrm{d}\boldsymbol{x} \Big| \\
&\le C + \Big| \int_{B_1} \big(\rho_0^\varepsilon(\boldsymbol{x}) - \rho_0(\boldsymbol{x})\big) k_\alpha (1 + |\boldsymbol{x}|^2)\, \mathrm{d}\boldsymbol{x} \Big| \\
&\qquad\qquad + \Big| \int_{B_1^c} \big(\rho_0^\varepsilon(\boldsymbol{x}) - \rho_0(\boldsymbol{x})\big) k_\alpha (1 + |\boldsymbol{x}|^2)\, \mathrm{d}\boldsymbol{x} \Big| \\
&\le C + k_\alpha(2) \|\rho_0^\varepsilon - \rho_0\|_{L^1(B_1)} + \Big| \int_{B_1^c} \big(\rho_0^\varepsilon(\boldsymbol{x}) - \rho_0(\boldsymbol{x})\big) k_\alpha (2|\boldsymbol{x}|^2)\, \mathrm{d}\boldsymbol{x} \Big| \\
&\le C + k_\alpha(2) \|\rho_0^\varepsilon - \rho_0\|_{L^1(B_1)} + k_\alpha(2) \|\rho_0^\varepsilon - \rho_0\|_{L^1(B_1^c)} \\
&\qquad\qquad + C \Big| \int_{B_1^c} \big(\rho_0^\varepsilon(\boldsymbol{x}) - \rho_0(\boldsymbol{x})\big) k_\alpha (|\boldsymbol{x}|^2)\, \mathrm{d}\boldsymbol{x} \Big| \\
&= C\big(1 + \|\rho_0^\varepsilon - \rho_0\|_{L^1(\mathbb{R}^n)} + |\Phi_\alpha * (\rho_0^\varepsilon - \rho_0)|(\boldsymbol{0})\big).
\end{aligned}$$

Now, by (2.21) and (2.23), we have

$$\|\rho_0^\varepsilon - \rho_0\|_{L^1(\mathbb{R}^n)} + |\Phi_\alpha * (\rho_0^\varepsilon - \rho_0)|(\boldsymbol{0}) \longrightarrow 0, \qquad \text{as } \varepsilon \to 0.$$

Then there exists some $C > 0$ such that, for small enough ε, (2.24) holds. □

As in the previous literature, in order to prove the BD entropy result and the convergence of Ω^T to $[0, T] \times (0, \infty)$ in some sense, we require that

$$\rho_0^{\varepsilon,b}(b) \propto b^{-(n-\beta)}, \tag{2.27}$$

where $\beta = \min\{\frac{1}{2}, (1 - \frac{1}{\gamma})n\}$. To do this, we choose a cut-off function $S \in C^\infty(\mathbb{R})$, such that

$$\begin{cases} S(z) = 0, & \text{if } z \in (-\infty, 0], \\ S(z) = 1, & \text{if } z \in [1, \infty), \\ S(z) \text{ is monotonic in } [0, 1]. \end{cases}$$

We define

$$\tilde{\rho}_0^{\varepsilon,b}(\boldsymbol{x}) \coloneqq \Big\{ \sqrt{\rho_0^\varepsilon(\boldsymbol{x})}\big(1 - S(2(|\boldsymbol{x}| - (b-1)))\big) + b^{-\frac{n-\beta}{2}} S(2(|\boldsymbol{x}| - (b-1))) \Big\}^2.$$

Then $\tilde{\rho}_0^{\varepsilon,b}$ is a radial function and satisfies $\tilde{\rho}_0^{\varepsilon,b}(b) = b^{-(n-\beta)}$. Now, as before, we normalise $\tilde{\rho}_0^{\varepsilon,b}$ (but on the corresponding domain from problem (2.8)–(2.11): the annulus $\{b^{-1} \le |\boldsymbol{x}| \le b\}$) to produce

$$\rho_0^{\varepsilon,b}(\boldsymbol{x}) \coloneqq \frac{M}{\int_{b^{-1}\le|\boldsymbol{x}|\le b} \tilde{\rho}_0^{\varepsilon,b}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}} \mathbb{1}_{\{b^{-1}\le|\boldsymbol{x}|\le b\}}(\boldsymbol{x})\tilde{\rho}_0^{\varepsilon,b}(\boldsymbol{x}). \tag{2.28}$$

As before, it is direct to see that (2.27) is satisfied. One can show the following lemma.

Lemma 2.4.3. *The defined function $\rho_0^{\varepsilon,b}(\boldsymbol{x})$ in (2.28) is smooth. Moreover, for any $q \in \{1, \gamma, \frac{2n}{2n-\alpha}\}$ when $\kappa = -1$, $\alpha > 0$ or $q \in \{1, \gamma\}$ otherwise, the following estimates hold*

$$\begin{aligned}
&\int_{b^{-1}\le|\boldsymbol{x}|\le b} \rho_0^{\varepsilon,b}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = M, \qquad \text{for all } \varepsilon \in (0,1] \text{ and } b > 1,\\
&\int_{|\boldsymbol{x}|\le b} \left(\left|\rho_0^{\varepsilon,b}(\boldsymbol{x}) - \rho_0^{\varepsilon}(\boldsymbol{x})\right|^q + \left|\sqrt{\rho_0^{\varepsilon,b}(\boldsymbol{x})} - \sqrt{\rho_0^{\varepsilon}(\boldsymbol{x})}\right|^{2q}\right)\mathrm{d}\boldsymbol{x} \longrightarrow 0, \qquad \text{as } b \to \infty,\\
&\varepsilon \int_{|\boldsymbol{x}|\le b} \left|\nabla_{\boldsymbol{x}}\sqrt{\rho_0^{\varepsilon,b}(\boldsymbol{x})}\right|^2 \mathrm{d}\boldsymbol{x} \le C\big(\|\rho_0\|_{L^1} + 1\big).
\end{aligned}$$

We also have the convergence of the initial potentials, represented by the following lemma.

Lemma 2.4.4. *$\Phi_\alpha * \rho_0^{\varepsilon,b}$ with $\alpha \in (0, n-1)$ satisfy*

$$\begin{aligned}
&\|\Phi_\alpha * \rho_0^{\varepsilon,b} - \Phi_\alpha * \rho_0^{\varepsilon}\|_{L^{\frac{2n}{\alpha}}} \le C\|\rho_0^{\varepsilon,b} - \rho_0^{\varepsilon}\|_{L^{\frac{2n}{2n-\alpha}}} \longrightarrow 0, \qquad \text{as } b \to \infty,\\
&\|\nabla(\Phi_\alpha * \rho_0^{\varepsilon,b} - \Phi_\alpha * \rho_0^{\varepsilon})\|_{L^{\frac{2n}{\alpha+2}}} \le C\|\rho_0^{\varepsilon,b} - \rho_0^{\varepsilon}\|_{L^{\frac{2n}{2n-\alpha}}} \longrightarrow 0, \qquad \text{as } b \to \infty.
\end{aligned}$$

For $\alpha \in (-1, 0]$ and $K \Subset \mathbb{R}^n$,

$$\|\Phi_\alpha * \rho_0^{\varepsilon,b} - \Phi_\alpha * \rho_0^{\varepsilon}\|_{L^\infty(K)} + \|\nabla(\Phi_\alpha * \rho_0^{\varepsilon,b} - \Phi_\alpha * \rho_0^{\varepsilon})\|_{L^{\frac{2n}{\alpha+2}}(K)} \longrightarrow 0, \qquad \text{as } b \to \infty.$$

Moreover, there exists a constant $C > 0$ such that, for $\varepsilon > 0$ sufficiently small and b sufficiently large,

$$\int_{\mathbb{R}^n} \rho_0^{\varepsilon,b}(\boldsymbol{x}) k_\alpha (1 + |\boldsymbol{x}|^2)\,\mathrm{d}\boldsymbol{x} \le C. \tag{2.29}$$

Proof. The first part is estimated by applying the HLS inequality from Lemma A.0.2. The second part can be handled by proceeding in the same manner as in the proof of Lemma 2.4.2. □

We now introduce an approximation of the initial velocity. To prevent difficulties arising from possible vacuum regions, we define the approximate initial velocity u_0^ε as follows:

$$u_0^\varepsilon(\boldsymbol{x}) \coloneqq \frac{1}{\sqrt{\rho_0^\varepsilon(\boldsymbol{x})}}\left(\frac{m_0}{\sqrt{\rho_0}}\right)(\boldsymbol{x}). \tag{2.30}$$

As in [36, Lemma A.8], we can prove the following lemma.

Lemma 2.4.5. *As defined above, u_0^ε satisfies*

$$\int_{\mathbb{R}^n} \rho_0^\varepsilon(\boldsymbol{x})|u_0^\varepsilon(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^n} \frac{|m_0(\boldsymbol{x})|^2}{\rho_0(\boldsymbol{x})} \,\mathrm{d}\boldsymbol{x},$$
$$\left\|\rho_0^\varepsilon u_0^\varepsilon - m_0\right\|_{L^1} \longrightarrow 0, \qquad \text{as } \varepsilon \to 0.$$

We can now define $\tilde{u}_0^{\varepsilon,b}$:

$$\tilde{u}_0^{\varepsilon,b}(\boldsymbol{x}) = \frac{1}{\sqrt{\rho_0^{\varepsilon,b}(\boldsymbol{x})}} \int_{\mathbb{R}^n} \left(\frac{m_0 \mathbb{1}_{[4b^{-1},b-2]}}{\sqrt{\rho_0}}\right)(\boldsymbol{x}-\boldsymbol{y})\varphi_{b^{-1}}(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y}.$$

Here, we have introduced the indicator function $\mathbb{1}_{[4b^{-1},\, b-2]}$ so that

$$\operatorname{supp}\left(\tilde{u}_0^{\varepsilon,b}\right) \subset [2b^{-1},\, b-1],$$

which ensures that the lower boundary condition is satisfied. We may now impose the stress-free condition at the upper boundary. To fulfil (2.10), we construct an initial velocity satisfying

$$\left(p(\rho_0^{\varepsilon,b}) - \varepsilon\rho_0^{\varepsilon,b}(u_{0,r}^{\varepsilon,b} + \frac{n-1}{r}u_0^{\varepsilon,b})\right)(r) = 0, \qquad \text{for } r \in (b-\tfrac{1}{4}, b].$$

Given a solution to the above, then

$$(u_0^{\varepsilon,b} r^{n-1})_r = \frac{1}{\varepsilon}\frac{p(\rho_0^{\varepsilon,b}(r))}{\rho_0^{\varepsilon,b}(r)} r^{n-1}, \qquad \text{for } r \in (b-\tfrac{1}{4}, b).$$

Integrating over (r, b) and choosing $u_0^{\varepsilon,b}(b) = 0$, we obtain

$$u_0^{\varepsilon,b}(r) = -\frac{1}{\varepsilon}\frac{1}{r^{n-1}} \int_r^b \frac{p(\rho_0^{\varepsilon,b}(z))}{\rho_0^{\varepsilon,b}(z)} z^{n-1} \,\mathrm{d}z.$$

Therefore, in order to incorporate this boundary condition into $\tilde{u}_0^{\varepsilon,b}$, we adopt the following construction

$$u_0^{\varepsilon,b}(\boldsymbol{x}) \coloneqq \tilde{u}_0^{\varepsilon,b}(\boldsymbol{x}) - \frac{1}{\varepsilon} S\left(4\left(|\boldsymbol{x}| - \left(b - \frac{1}{2}\right)\right)\right) \frac{1}{|\boldsymbol{x}|^{n-1}} \int_{|\boldsymbol{x}|}^b \frac{p(\rho_0^{\varepsilon,b}(z))}{\rho_0^{\varepsilon,b}(z)} z^{n-1} \,\mathrm{d}z. \tag{2.31}$$

Then $u_0^{\varepsilon,b}$ satisfies (2.10). As in [36, Lemma A.9], we obtain the following lemma.

Lemma 2.4.6. *For given $\varepsilon > 0$,*

$$\lim_{b\to\infty} \int_{|\boldsymbol{x}|\le b} \left|\sqrt{\rho_0^{\varepsilon,b}(\boldsymbol{x})} u_0^{\varepsilon,b}(\boldsymbol{x})\right|^2 \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^n} \left|\sqrt{\rho_0^{\varepsilon}(\boldsymbol{x})} u_0^{\varepsilon}(\boldsymbol{x})\right|^2 \mathrm{d}\boldsymbol{x},$$
$$\sqrt{\rho_0^{\varepsilon,b}(\boldsymbol{x})} u_0^{\varepsilon,b}(\boldsymbol{x}) \longrightarrow \sqrt{\rho_0^{\varepsilon}(\boldsymbol{x})} u_0^{\varepsilon}(\boldsymbol{x}), \qquad \text{in } L^2(\{|\boldsymbol{x}| \le b\}) \text{ as } b \to \infty.$$

We define the approximate initial energy as follows.

Definition 2.4.7. Define the energy for the approximate initial data by

$$E_0^\varepsilon := \int_{\mathbb{R}^n} \left(\frac{1}{2} \left| \frac{\boldsymbol{\mathcal{M}}_0^\varepsilon}{\sqrt{\rho_0^\varepsilon}} \right|^2 + \rho_0^\varepsilon e(\rho_0^\varepsilon) + \frac{\kappa}{2} \rho_0^\varepsilon (\Phi_\alpha * \rho_0^\varepsilon) \right)(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} < \infty,$$

and

$$E_0^{\varepsilon,b} := \int_{\mathbb{R}^n} \left(\frac{1}{2} \left| \frac{\boldsymbol{\mathcal{M}}_0^{\varepsilon,b}}{\sqrt{\rho_0^{\varepsilon,b}}} \right|^2 + \rho_0^{\varepsilon,b} e(\rho_0^{\varepsilon,b}) + \frac{\kappa}{2} \rho_0^{\varepsilon,b} (\Phi_\alpha * \rho_0^{\varepsilon,b}) \right)(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} < \infty.$$

Combining all together, we obtain the following lemma

Lemma 2.4.8. *Take $\rho_0^\varepsilon, \rho_0^{\varepsilon,b}, u_0^\varepsilon$, and $u_0^{\varepsilon,b}$ to be defined as in (2.20), (2.28), (2.30), and (2.31) respectively. Suppose $M < M_{\mathrm{c}}^\alpha(\gamma)$, then $(\rho_0^{\varepsilon,b}, u_0^{\varepsilon,b}) \in C^\infty([b^{-1}, b])$ and satisfies (2.10). For $q \in \{1, \gamma, \frac{2n}{2n-\alpha}\}$ for $\kappa = -1$, $\alpha > 0$ and $q \in \{1, \gamma\}$ otherwise, the following statements hold:*

(a) *For all $\varepsilon \in (0, 1]$,*

$$\int_0^\infty \rho_0^\varepsilon(r) \, \omega_n r^{n-1} \, \mathrm{d}r = M, \qquad \varepsilon \int_0^\infty \left| \partial_r \sqrt{\rho_0^\varepsilon(r)} \right|^2 r^{n-1} \, \mathrm{d}r \le C(M+1).$$

As $\varepsilon \to 0$,

$$E_0^\varepsilon \longrightarrow E_0,$$
$$(\rho_0^\varepsilon, m_0^\varepsilon)(r) \longrightarrow (\rho_0, m_0)(r), \qquad \text{in } L^q([0,\infty); r^{n-1}\,\mathrm{d}r) \times L^1([0,\infty); r^{n-1}\,\mathrm{d}r).$$

Then there exists $\varepsilon_0 > 0$ such that

$$M < M_{\mathrm{c}}^{\varepsilon,\alpha}(\gamma), \qquad \text{for all } \varepsilon \in (0, \varepsilon_0] \text{ and } \gamma \in (\tfrac{2n}{2n-\alpha}, \tfrac{n+\alpha}{n}].$$

(b) *For all $\varepsilon \in (0, 1]$,*

$$\int_{b^{-1}}^b \rho_0^{\varepsilon,b}(r) \, \omega_n r^{n-1} \, \mathrm{d}r = M, \qquad \varepsilon \int_{b^{-1}}^b \left| \partial_r \sqrt{\rho_0^{\varepsilon,b}(r)} \right|^2 r^{n-1} \, \mathrm{d}r \le C(M+1).$$

Moreover, for any fixed $\varepsilon > 0$, as $b \to \infty$,

$$E_0^{\varepsilon,b} \longrightarrow E_0^\varepsilon,$$
$$(\rho_0^{\varepsilon,b}, m_0^{\varepsilon,b})(r) \longrightarrow (\rho_0^\varepsilon, m_0^\varepsilon)(r), \qquad \text{in } L^q([0,b]; r^{n-1}\,\mathrm{d}r) \times L^1([0,b]; r^{n-1}\,\mathrm{d}r).$$

Then there exists $\mathfrak{B}(\varepsilon) > 0$ such that

$$M < M_{\mathrm{c}}^{\varepsilon,b,\alpha}(\gamma), \qquad \text{for all } \varepsilon \in (0, \varepsilon_0],\ b \ge \mathfrak{B}(\varepsilon), \text{ and } \gamma \in (\tfrac{2n}{2n-\alpha}, \tfrac{n+\alpha}{n}].$$

2.5 Estimates on the radial non-local potential

To handle the potential term, we first establish several of its key properties. In particular, we show that $\Phi_\alpha * f$ is differentiable whenever f is sufficiently regular. Exploiting the radial symmetry of the solutions, we also derive a radial representation of the kernel together with sharp bounds. This representation enables us to obtain L^∞ estimates for the potential, as required in the higher-integrability analysis; see Lemma 2.9.1. Moreover, it allows us to prove sufficient decay of the potential energy of radial functions, which is essential for the compactness argument used in Lemma 3.7.3 in Chapter 3.

Given a radial function f, we have that $\Phi_\alpha * f$ is a radial function. Like in [7], we can represent the convolution by

$$(\Phi_\alpha * f)(\boldsymbol{x}) = \int_a^\infty K(|\boldsymbol{x}|, \eta) f(\eta) \eta^{n-1} \, \mathrm{d}\eta,$$

where

$$K(r,\eta) \coloneqq \begin{cases} \displaystyle\int_{\mathbb{S}^{n-1}} \frac{|r\boldsymbol{e_1} - \eta \boldsymbol{y}|^{-\alpha}}{-\alpha} \, \mathrm{d}S(\boldsymbol{y}), & \text{for } \alpha \neq 0, \\ \displaystyle\int_{\mathbb{S}^{n-1}} \log(|r\boldsymbol{e_1} - \eta \boldsymbol{y}|) \, \mathrm{d}S(\boldsymbol{y}), & \text{for } \alpha = 0, \end{cases}$$

with $\boldsymbol{e}_1 = (1, 0, \cdots, 0)$. We now consider the differentiability of such a convolution and the radial representation for the potential and its derivative.

Lemma 2.5.1 (Continuity of the Potential)**.** *Let $\alpha \in (-1, n-1)$. Given a radial function $f \in L^1(\mathbb{R}^n) \cap L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{\mathbf{0}\})$ for some $q \in (\frac{n}{n-1-\alpha}, \infty)$ such that $\Phi_\alpha * f$ and $\nabla\Phi_\alpha * f$ are well-defined, then $(\Phi_\alpha * f)(\boldsymbol{x})$ is continuously differentiable on $\mathbb{R}^n \setminus \{\mathbf{0}\}$, and*

$$(\nabla\Phi_\alpha * f)(\boldsymbol{x}) \cdot \frac{\boldsymbol{x}}{|\boldsymbol{x}|} = (\Phi_\alpha * f)_r(|\boldsymbol{x}|) = \int_0^\infty \omega(|\boldsymbol{x}|, \eta) f(\eta) \, \eta^{n-1} \, \mathrm{d}\eta,$$

is a radial function, where

$$\omega(r,\eta) \coloneqq \int_{\mathbb{S}^{n-1}} \nabla\Phi_\alpha(r\boldsymbol{e_1} - \eta\boldsymbol{y}) \cdot \boldsymbol{e_1} \, \mathrm{d}S(\boldsymbol{y}).$$

Moreover, there exists $C = C(\alpha, n) > 0$ such that

$$\begin{cases} |K(r,\eta)| \le C(r\eta)^{-\frac{\alpha}{2}}, & \text{for } \alpha \in (-1, n-1), \\ |K(r,\eta)| \le C(r\eta)^{-\frac{n-1}{2}} |r-\eta|^{n-1-\alpha}, & \text{for } \alpha \in (n-1, n), \end{cases} \tag{2.32}$$

and

$$\begin{cases} |\omega(r,\eta)| \le C(r\eta)^{-\frac{\alpha+1}{2}} & \text{for } \alpha \in (-1, n-2), \\ \omega(r,\eta) = \dfrac{\omega_n \mathbb{1}_{(0,r)}(\eta)}{r^{n-1}} & \text{for } \alpha = n-2, \\ |\omega(r,\eta)| \le C(r\eta)^{-\frac{n-1}{2}} |r-\eta|^{n-2-\alpha} & \text{for } \alpha \in (n-2, n-1), \end{cases} \tag{2.33}$$

where $\mathbb{1}_{(0,r)}$ is the indicator function on the interval $(0, r)$.

Proof. We divide the proof into five steps.

Step 1. For $\alpha \in (0, n-1)$, since $q > \frac{n}{n-1-\alpha}$, Lemma A.0.1 guarantees the existence of a constant $C_1 > 0$ such that, for every $\varepsilon > 0$, the function

$$\boldsymbol{z} \mapsto \int_{B_1(\boldsymbol{x})} \Phi_\alpha(\boldsymbol{z}-\boldsymbol{y}) \mathbb{1}_{B_\varepsilon(\boldsymbol{x})}(\boldsymbol{y}) f(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y},$$

belongs to $C^{0,1}\big(\overline{B_1(\boldsymbol{x})}\big)$ for $|\boldsymbol{x}| > 0$, with Lipschitz constant bounded by

$$C_1 \|f\|_{L^q(B_\varepsilon(\boldsymbol{x}))},$$

for sufficiently small $\varepsilon > 0$. Here, $\overline{B_1(\boldsymbol{x})}$ denotes the closure of the open ball $B_1(\boldsymbol{x})$ of radius 1 centred at $\boldsymbol{x}$. For $\alpha \in (0, n)$ and $0 < |\boldsymbol{z}| \le 1$,

$$\begin{aligned}
&\frac{1}{|\boldsymbol{z}|}\left| \int_{B_\varepsilon(\boldsymbol{x})} (\Phi_\alpha(\boldsymbol{x}+\boldsymbol{z}-\boldsymbol{y}) - \Phi_\alpha(\boldsymbol{x}-\boldsymbol{y})) f(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y} \right| \\
&= \frac{1}{\alpha|\boldsymbol{z}|}\left| \int_{B_1(\boldsymbol{x})} \left(|\boldsymbol{x}+\boldsymbol{z}-\boldsymbol{y}|^{-\alpha} - |\boldsymbol{x}-\boldsymbol{y}|^{-\alpha} \right) \mathbb{1}_{B_\varepsilon(\boldsymbol{x})}(\boldsymbol{y}) f(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y} \right| \\
&\le \frac{C_1}{\alpha} \|f\|_{L^q(B_\varepsilon(\boldsymbol{x}))} \longrightarrow 0, \qquad \text{uniformly in } \boldsymbol{z} \text{ as } \varepsilon \to 0.
\end{aligned}$$

Step 2. For $\alpha \in (-1, 0]$, since $q > \frac{n}{n-1-\alpha}$, Lemma A.0.1 ensures that there exists a constant $C_2 > 0$ such that, for sufficiently small $\varepsilon > 0$, the function

$$\boldsymbol{z} \mapsto \int_{B_1(\boldsymbol{x})} |\boldsymbol{z}-\boldsymbol{y}|^{-(\alpha+1)} \mathbb{1}_{B_\varepsilon(\boldsymbol{x})}(\boldsymbol{y}) f(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y},$$

belongs to $C\big(\overline{B_1(\boldsymbol{x})}\big)$ for $|\boldsymbol{x}| > 0$ and

$$\left\| \int_{B_1(\boldsymbol{x})} |\cdot - \boldsymbol{y}|^{-(\alpha+1)} \mathbb{1}_{B_\varepsilon(\boldsymbol{x})}(\boldsymbol{y}) f(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y} \right\|_{C(B_1(\boldsymbol{x}))} \le C_2 \|f\|_{L^q(B_\varepsilon(\boldsymbol{x}))}.$$

For $0 < |\boldsymbol{z}| \le 1$, we have

$$\boldsymbol{0} \notin \{\boldsymbol{x} + s\boldsymbol{z} - \boldsymbol{y} : s \in [0,1]\},$$

for almost every $\boldsymbol{y} \in B_1(\boldsymbol{x})$. Hence, for almost every $\boldsymbol{y} \in B_1(\boldsymbol{x})$, that is, for those $\boldsymbol{y}$ such that $\boldsymbol{0} \notin \{\boldsymbol{x} + s\boldsymbol{z} - \boldsymbol{y} : s \in [0,1]\}$, we obtain

$$|\Phi_\alpha(\boldsymbol{x}+\boldsymbol{z}-\boldsymbol{y}) - \Phi_\alpha(\boldsymbol{x}-\boldsymbol{y})| = \left| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} \Phi_\alpha(\boldsymbol{x}+s\boldsymbol{z}-\boldsymbol{y}) \,\mathrm{d}s \right| = \left| \boldsymbol{z} \cdot \int_0^1 \frac{\boldsymbol{x}+s\boldsymbol{z}-\boldsymbol{y}}{|\boldsymbol{x}+s\boldsymbol{z}-\boldsymbol{y}|^{\alpha+2}} \,\mathrm{d}s \right|.$$

Therefore, for $\alpha \in (-1, 0]$ and $0 < |\boldsymbol{z}| \le 1$,

$$\begin{aligned}
&\frac{1}{|\boldsymbol{z}|}\left| \int_{B_\varepsilon(\boldsymbol{x})} (\Phi_\alpha(\boldsymbol{x}+\boldsymbol{z}-\boldsymbol{y}) - \Phi_\alpha(\boldsymbol{x}-\boldsymbol{y})) f(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y} \right| \\
&\le \int_0^1 \int_{B_1(\boldsymbol{x})} |\boldsymbol{x}+s\boldsymbol{z}-\boldsymbol{y}|^{-(\alpha+1)} \mathbb{1}_{B_\varepsilon(\boldsymbol{x})}(\boldsymbol{y}) |f(\boldsymbol{y})| \,\mathrm{d}\boldsymbol{y} \,\mathrm{d}s \\
&\le C_2 \|f\|_{L^q(B_\varepsilon(\boldsymbol{x}))} \longrightarrow 0, \qquad \text{uniformly in } \boldsymbol{z} \text{ as } \varepsilon \to 0.
\end{aligned}$$

Step 3. For $\alpha \in (-1, n-1)$, we observe that, for any fixed $\varepsilon > 0$,

$$\frac{|\Phi_\alpha(\boldsymbol{x}+\boldsymbol{z}-\boldsymbol{y}) - \Phi_\alpha(\boldsymbol{x}-\boldsymbol{y}) - \boldsymbol{z}\cdot\nabla\Phi_\alpha(\boldsymbol{x}-\boldsymbol{y})|}{|\boldsymbol{z}|} \longrightarrow 0,$$

uniformly for $\boldsymbol{y} \in \mathbb{R}^n \setminus B_\varepsilon(\boldsymbol{x})$ as $\boldsymbol{z} \to \mathbf{0}$. Since $f \in L^1(\mathbb{R}^n)$, it follows that

$$\frac{1}{|\boldsymbol{z}|}\left| \int_{B_\varepsilon^c(\boldsymbol{x})} \big(\Phi_\alpha(\boldsymbol{x}+\boldsymbol{z}-\boldsymbol{y}) - \Phi_\alpha(\boldsymbol{x}-\boldsymbol{y})\big) f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} - \boldsymbol{z}\cdot\int_{B_\varepsilon^c(\boldsymbol{x})} \nabla\Phi_\alpha(\boldsymbol{x}-\boldsymbol{y}) f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right|$$
$$\longrightarrow 0, \qquad \text{as } \boldsymbol{z} \to 0.$$

We therefore conclude that $(\Phi_\alpha * f)(\boldsymbol{x})$ is differentiable on $\mathbb{R}^n \setminus \{\mathbf{0}\}$, and that $\nabla(\Phi_\alpha * f)(\boldsymbol{x}) = (\nabla\Phi_\alpha * f)(\boldsymbol{x})$.

Step 4. As in [7], we can write

$$(\nabla\Phi_\alpha * f)(\boldsymbol{x}) = \int_0^\infty \omega(|\boldsymbol{x}|,\eta) f(\eta)\eta^{n-1}\,\mathrm{d}\eta\,\frac{\boldsymbol{x}}{|\boldsymbol{x}|}.$$

It follows that $(\nabla\Phi_\alpha * f)(\boldsymbol{x})\cdot\frac{\boldsymbol{x}}{|\boldsymbol{x}|}$ depends only on $r = |\boldsymbol{x}|$, and hence is a radial function given by

$$(\nabla\Phi_\alpha * f)(\boldsymbol{x})\cdot\frac{\boldsymbol{x}}{|\boldsymbol{x}|} = (\Phi_\alpha * f)_r(|\boldsymbol{x}|).$$

Proceeding as in [22], we obtain, for $r > 0$,

$$(\Phi_\alpha * f)_r(r) = \int_0^\infty \Big(\int_{\mathbb{S}^{n-1}} \frac{r\boldsymbol{e_1} - \eta\boldsymbol{y}}{|r\boldsymbol{e_1} - \eta\boldsymbol{y}|^{\alpha+2}}\cdot\boldsymbol{e_1}\,\mathrm{d}S(\boldsymbol{y})\Big) f(\eta)\eta^{n-1}\,\mathrm{d}\eta.$$

We now focus on the inner integral and perform a change of variables to poly-spherical coordinates, defined by

$$\boldsymbol{y} = (\cos\theta, \sin\theta\,\hat{\boldsymbol{z}}),$$

with $\theta \in [0,\pi)$ and $\hat{\boldsymbol{z}} \in \mathbb{S}^{n-2}$, we have

$$\begin{aligned}
&\int_{\mathbb{S}^{n-1}} \frac{r\boldsymbol{e_1} - \eta\boldsymbol{y}}{|r\boldsymbol{e_1} - \eta\boldsymbol{y}|^{\alpha+2}}\cdot\boldsymbol{e_1}\,\mathrm{d}S(\boldsymbol{y}) \\
&= \omega_{n-1}\int_0^\pi \big((r-\eta\cos\theta)^2 + (\eta\sin\theta)^2\big)^{-\frac{\alpha+2}{2}}(r-\eta\cos\theta)|\sin\theta|^{n-2}\,\mathrm{d}\theta.
\end{aligned}$$

There exists $c > 0$ such that $2(1-\cos\theta) \geq c\theta^2$. Thus, looking at the integrand, we have

$$\begin{aligned}
&\Big|\big((r-\eta\cos\theta)^2 + (\eta\sin\theta)^2\big)^{-\frac{\alpha+2}{2}}(r-\eta\cos\theta)|\sin\theta|^{n-2}\Big| \\
&\leq \big((r-\eta)^2 + 2r\eta(1-\cos\theta)\big)^{-\frac{\alpha+1}{2}}|\sin\theta|^{n-2} \\
&\leq \big((r-\eta)^2 + cr\eta\theta^2\big)^{-\frac{\alpha+1}{2}}\theta^{n-2}. \qquad (2.34)
\end{aligned}$$

For $\zeta > 0$, by applying (2.34), we deduce that, in the case $\alpha > n-2$,

$$\begin{aligned}
&\Big| \int_0^\pi \big((r-\eta\cos\theta)^2 + (\eta\sin\theta)^2\big)^{-\frac{\alpha+2}{2}} (r-\eta\cos\theta)|\sin\theta|^{n-2}\,\mathrm{d}\theta\Big| \\
&\le \int_0^\infty \big((r-\eta)^2 + cr\eta\theta^2\big)^{-\frac{\alpha+1}{2}} \theta^{n-2}\,\mathrm{d}\theta \\
&\le \int_\zeta^\infty \big(cr\eta\theta^2\big)^{-\frac{\alpha+1}{2}} \theta^{n-2}\,\mathrm{d}\theta + \int_0^\zeta |r-\eta|^{-(\alpha+1)}\theta^{n-2}\,\mathrm{d}\theta \\
&= \big(cr\eta\big)^{-\frac{\alpha+1}{2}} \Big[\frac{\theta^{n-2-\alpha}}{n-2-\alpha}\Big]_\zeta^\infty + |r-\eta|^{-(\alpha+1)}\Big[\frac{\theta^{n-1}}{n-1}\Big]_0^\zeta. \qquad (2.35)
\end{aligned}$$

This is minimised by $\zeta \propto \frac{|r-\eta|}{\sqrt{r\eta}}$. Thus, there exists $C > 0$ such that

$$\begin{aligned}
&\Big| \int_0^\pi \big((r-\eta\cos\theta)^2 + (\eta\sin\theta)^2\big)^{-\frac{\alpha+2}{2}} (r-\eta\cos\theta)|\sin\theta|^{n-2}\,\mathrm{d}\theta\Big| \\
&\le C(r\eta)^{-\frac{n-1}{2}} |r-\eta|^{n-2-\alpha}. \qquad (2.36)
\end{aligned}$$

Step 5. For $r \gg \varepsilon > 0$, we apply (2.34) to deduce that, for $s \in \big(r-\frac{\varepsilon}{2}, r+\frac{\varepsilon}{2}\big)$ and $\eta \in (0,\infty)\setminus(r-\varepsilon, r+\varepsilon)$,

$$\Big|\big((s-\eta\cos\theta)^2 + (\eta\sin\theta)^2\big)^{-\frac{\alpha+2}{2}} (s-\eta\cos\theta)|\sin\theta|^{n-2}\Big| \le C(\varepsilon).$$

Therefore, by the Lebesgue dominated convergence Theorem, we conclude that

$$\lim_{s\to r} \int_{(0,\infty)\setminus(r-\varepsilon,r+\varepsilon)} \omega(s,\eta) f(\eta)\,\eta^{n-1}\,\mathrm{d}\eta = \int_{(0,\infty)\setminus(r-\varepsilon,r+\varepsilon)} \omega(r,\eta) f(\eta)\,\eta^{n-1}\,\mathrm{d}\eta.$$

By (2.36) and Lemma A.0.1, it follows that

$$\Big|\int_{r-\varepsilon}^{r+\varepsilon} \omega(s,\eta) f(\eta)\eta^{n-1}\,\mathrm{d}\eta\Big| \le C\int_{r-\varepsilon}^{r+\varepsilon} |s-\eta|^{n-2-\alpha} f(\eta)\eta^{n-1}\,\mathrm{d}\eta \le C\|f\|_{L^q(B_\varepsilon(r))}.$$

Since $\|f\|_{L^q(B_\varepsilon(r))} \to 0$ as $\varepsilon \to 0$, and

$$\lim_{\varepsilon\to 0} \int_{(0,\infty)\setminus(r-\varepsilon,r+\varepsilon)} \omega(r,\eta) f(\eta)\,\eta^{n-1}\,\mathrm{d}\eta = \int_0^\infty \omega(r,\eta) f(\eta)\,\eta^{n-1}\,\mathrm{d}\eta,$$

we conclude that $\lim_{s\to r}(\Phi_\alpha * f)_r(s) = (\Phi_\alpha * f)_r(r)$.

For $\alpha \in (-1, n-2)$, proceeding in the same manner as in (2.35), we obtain

$$|\omega(r,\eta)| \le C\int_0^\pi \big(r\eta\theta^2\big)^{-\frac{\alpha+1}{2}} \theta^{n-2}\,\mathrm{d}\theta \le C(r\eta)^{-\frac{\alpha+1}{2}}.$$

Therefore, by the same argument used in the case $\alpha > n-2$, we conclude that $(\Phi_\alpha * f)_r$ is continuous for $\alpha \in (-1, n-2)$.

For $\alpha = n-2$, since Φ_{n-2} is (up to a multiplicative constant) the fundamental solution of the Laplace equation, it follows that, for any radial function $f \in C_c^\infty(\mathbb{R}^n)$,

$$\big(r^{n-1}(\Phi_{n-2} * f)_r\big)_r(r) = r^{n-1}\Delta(\Phi_{n-2} * f)(r) = \omega_n r^{n-1} f(r).$$

Since it is classically differentiable, then $r^{n-1}(\Phi_{n-2} * f)_r$ and hence $(\Phi_{n-2} * f)_r$ are continuous. Furthermore, integrating over r we obtain

$$\int_0^\infty \omega(r,\eta) f(\eta)\eta^{n-1}\,\mathrm{d}\eta = (\Phi_{n-2} * f)_r(r) = \frac{\omega_n}{r^{n-1}} \int_0^r f(\eta)\eta^{n-1}\,\mathrm{d}\eta.$$

This implies

$$\int_0^\infty \Big(\omega(r,\eta) - \frac{\omega_n \mathbb{1}_{(0,r)}(\eta)}{r^{n-1}}\Big) f(\eta)\eta^{n-1}\,\mathrm{d}\eta = 0,$$

for all radial functions $f \in C_c^\infty(\mathbb{R}^n)$. Then, by the fundamental Lemma of the calculus of variations, we conclude that

$$\omega(r,\eta) = \frac{\omega_n \mathbb{1}_{(0,r)}(\eta)}{r^{n-1}}.$$

We note that formula (2.32) can be obtained by the same argument used to derive (2.33). □

Remark 2.5.2. For each fixed $t > 0$, the density $\rho(t, \boldsymbol{x})$ satisfies the assumptions imposed on f in Lemma 2.5.1. Indeed, $\rho(t, \boldsymbol{x})$ is smooth in Ω_t and, by convention, vanishes outside this region. Moreover,

$$\rho(t,\cdot) \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{\mathbf{0}\}), \quad \text{for all } q \in [1,\infty],$$

and

$$\rho(t,\cdot) \in L^1(\mathbb{R}^n),$$

since the total mass is finite and equal to M. Therefore, Lemma 2.5.1 applies to $\rho(t, \boldsymbol{x})$ in the context of the free boundary problem.

2.6 Energy estimates

We now establish uniform energy estimates for solutions of (2.8)–(2.11). These estimates play a crucial role in the subsequent analysis, particularly in proving convergence in the boundary limit $b \to \infty$ and in the viscosity limit $\varepsilon \to 0$.

Lemma 2.6.1 (Energy Estimates)**.** *Let $\alpha \in (-1, n-1)$, then any smooth solution $(\rho, u)(t, r)$ of the free boundary problem (2.8)–(2.11) satisfies the following energy estimate:*

$$\begin{aligned}
&\int_a^{b(t)} \rho\Big(\frac{1}{2}u^2 + e(\rho) + \frac{\kappa}{2}(\Phi_\alpha * \rho)\Big) r^{n-1}\,\mathrm{d}r + \varepsilon \int_0^t \int_a^{b(s)} \rho\Big(u_r^2 + (n-1)\frac{u^2}{r^2}\Big) r^{n-1}\,\mathrm{d}r\,\mathrm{d}s \\
&\quad + \varepsilon(n-1)\int_0^t (\rho u^2)(s, b(s)) b(s)^{n-2}\,\mathrm{d}s \\
&= \int_a^b \rho_0\Big(\frac{1}{2}u_0^2 + e(\rho_0) + \frac{\kappa}{2}(\Phi_\alpha * \rho_0)\Big) r^{n-1}\,\mathrm{d}r. \qquad (2.37)
\end{aligned}$$

In particular, we have the following estimates:

Case 1 ($\kappa=-1$, $\alpha\in(0,n-1)$ and $\gamma>1$)**.**

$$\begin{aligned}&\int_a^{b(t)}\rho\Big(\frac{1}{2}u^2+e(\rho)-\frac{1}{2}(\Phi_\alpha*\rho)\Big)r^{n-1}\,\mathrm{d}r+\varepsilon\int_0^t\int_a^{b(s)}\rho\Big(u_r^2+(n-1)\frac{u^2}{r^2}\Big)r^{n-1}\,\mathrm{d}r\,\mathrm{d}s\\&\quad+\varepsilon(n-1)\int_0^t(\rho u^2)(s,b(s))b(s)^{n-2}\,\mathrm{d}s\\&=\int_a^b\rho_0\Big(\frac{1}{2}u_0^2+e(\rho_0)-\frac{1}{2}(\Phi_\alpha*\rho_0)\Big)r^{n-1}\,\mathrm{d}r=E_0^{\varepsilon,b}.\end{aligned}$$

Case 2 ($\kappa=1$, $\alpha\in(0,n-1)$, $\gamma\in(\frac{2n}{2n-\alpha},\frac{n+\alpha}{n}]$ and $M<M_{\mathrm{c}}^{\varepsilon,b,\alpha}(\gamma)$)**.**

$$\begin{aligned}&\int_a^{b(t)}\rho\Big(\frac{1}{2}u^2+C_\gamma e(\rho)\Big)r^{n-1}\,\mathrm{d}r+\varepsilon\int_0^t\int_a^{b(s)}\rho\Big(u_r^2+(n-1)\frac{u^2}{r^2}\Big)r^{n-1}\,\mathrm{d}r\,\mathrm{d}s\\&\quad+\varepsilon(n-1)\int_0^t(\rho u^2)(s,b(s))b(s)^{n-2}\,\mathrm{d}s\\&\le\int_a^b\rho_0\Big(\frac{1}{2}u_0^2+e(\rho_0)+\frac{\kappa}{2}(\Phi_\alpha*\rho_0)\Big)r^{n-1}\,\mathrm{d}r=E_0^{\varepsilon,b},\end{aligned}\tag{2.38}$$

where the positive constant $C_\gamma>0$ is defined as

$$C_\gamma:=\begin{cases}1-B_{n,\gamma,\alpha}M^{\frac{n-\alpha}{n}}, & \text{for } \gamma=\frac{n+\alpha}{n},\\ \frac{\alpha-n(\gamma-1)}{\alpha}, & \text{for } \gamma\in(\frac{2n}{2n-\alpha},\frac{n+\alpha}{n}).\end{cases}$$

Case 3 ($\kappa=1$, $\alpha\in(0,n-1)$ and $\gamma>\frac{n+\alpha}{n}$)**.**

$$\begin{aligned}&\frac{1}{2}\int_a^{b(t)}\rho\left(u^2+e(\rho)\right)r^{n-1}\,\mathrm{d}r+\varepsilon\int_0^t\int_a^{b(s)}\rho\Big(u_r^2+(n-1)\frac{u^2}{r^2}\Big)r^{n-1}\,\mathrm{d}r\,\mathrm{d}s\\&\quad+\varepsilon(n-1)\int_0^t(\rho u^2)(s,b(s))b(s)^{n-2}\,\mathrm{d}s\le C(M,E_0^{\varepsilon,b}).\end{aligned}$$

Case 4 ($\kappa\in\{-1,1\}$, $\alpha\in(-1,0]$ and $\gamma>1$)**.**

$$\begin{aligned}&\frac{1}{2}\int_a^{b(t)}\rho\left(u^2+e(\rho)+k_\alpha(1+r^2)\right)r^{n-1}\,\mathrm{d}r+\varepsilon\int_0^t\int_a^{b(s)}\rho\Big(u_r^2+\frac{u^2}{r^2}\Big)r^{n-1}\,\mathrm{d}r\,\mathrm{d}s\\&\quad+\varepsilon(n-1)\int_0^t(\rho u^2)(s,b(s))b(s)^{n-2}\,\mathrm{d}s\le C(M,E_0^{\varepsilon,b},T).\end{aligned}\tag{2.39}$$

Proof. We divide the proof into six steps.

Step 1. Multiplying $(2.18)_2$ by u and integrating with respect to x yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^{\frac{M}{\omega_n}}u^2\,\mathrm{d}x+\int_0^{\frac{M}{\omega_n}}\left(p-\varepsilon\rho^2(r^{n-1}u)_x\right)_x ur^{n-1}\,\mathrm{d}x\\&=-\varepsilon(n-1)\int_0^{\frac{M}{\omega_n}}\rho_x u^2r^{n-2}\,\mathrm{d}x-\kappa\int_0^{\frac{M}{\omega_n}}\rho ur^{n-1}(\Phi_\alpha*\rho)_x\,\mathrm{d}x.\end{aligned}\tag{2.40}$$

For the last term on the left-hand side of (2.40), we apply the change of variables (2.16), the boundary conditions (2.17), the conservation of mass $(2.18)_1$, and integration by parts to obtain

$$\begin{aligned}
&\int_0^{\frac{M}{\omega_n}} \big(p(\rho) - \varepsilon\rho^2(r^{n-1}u)_x\big)_x u\, r^{n-1}\,\mathrm{d}x \\
&= -\int_0^{\frac{M}{\omega_n}} \big(p(\rho) - \varepsilon\rho^2(r^{n-1}u)_x\big)(r^{n-1}u)_x\,\mathrm{d}x \\
&= -\int_0^{\frac{M}{\omega_n}} p(\rho)(r^{n-1}u)_x\,\mathrm{d}x + \varepsilon\int_0^{\frac{M}{\omega_n}} \rho^2\big((r^{n-1}u)_x\big)^2\,\mathrm{d}x \\
&= a_0\int_0^{\frac{M}{\omega_n}} \rho^{\gamma-2}\rho_\tau\,\mathrm{d}x + \varepsilon\int_0^{\frac{M}{\omega_n}} \rho^2\big(r^{n-1}u_x + (n-1)r^{n-2}r_x u\big)^2\,\mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^{\frac{M}{\omega_n}} e(\rho)\,\mathrm{d}x + \varepsilon\int_0^{\frac{M}{\omega_n}} \Big(\rho^2(r^{n-1}u_x)^2 + (n-1)^2\frac{u^2}{r^2} + 2(n-1)r^{n-2}\rho u u_x\Big)\,\mathrm{d}x.
\end{aligned} \tag{2.41}$$

For the first term on the right-hand side of (2.40), an integration by parts yields

$$\begin{aligned}
&(n-1)\varepsilon\int_0^{\frac{M}{\omega_n}} \rho_x u^2 r^{n-2}\,\mathrm{d}x \\
&= -(n-1)\varepsilon\int_0^{\frac{M}{\omega_n}} \big(2r^{n-2}\rho u u_x + (n-2)\frac{u^2}{r^2}\big)\,\mathrm{d}x + \varepsilon(n-1)(\rho u^2 r^{n-2})(\tau, \frac{M}{\omega_n}).
\end{aligned} \tag{2.42}$$

For the last term on the right-hand side of (2.40), we convert back to Eulerian coordinates and apply the boundary condition (2.17) together with the conservation of mass $(2.8)_1$ to obtain

$$\begin{aligned}
&\kappa\int_0^{\frac{M}{\omega_n}} \rho u\,(\Phi_\alpha * \rho)_x\, r^{n-1}\,\mathrm{d}x \\
&= \kappa\int_a^{b(t)} \rho u\,(\Phi_\alpha * \rho)_r\, r^{n-1}\,\mathrm{d}r \\
&= \kappa\big(\rho u r^{n-1}(\Phi_\alpha * \rho)\big)(t, b(t)) - \kappa\int_a^{b(t)} (\rho u r^{n-1})_r(\Phi_\alpha * \rho)\,\mathrm{d}r \\
&= \kappa\big(\rho u r^{n-1}(\Phi_\alpha * \rho)\big)(t, b(t)) + \kappa\int_a^{b(t)} \rho_t(\Phi_\alpha * \rho)\, r^{n-1}\,\mathrm{d}r \\
&= \frac{\kappa}{2}\big(\rho u r^{n-1}(\Phi_\alpha * \rho)\big)(t, b(t)) + \frac{\kappa}{2}\int_a^{b(t)} \frac{\partial}{\partial t}\big(\rho(\Phi_\alpha * \rho)\big)\, r^{n-1}\,\mathrm{d}r \\
&= \frac{\kappa}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^{b(t)} \rho(\Phi_\alpha * \rho)\, r^{n-1}\,\mathrm{d}r \\
&= \frac{\kappa}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^{\frac{M}{\omega_n}} \Phi_\alpha * \rho\,\mathrm{d}x,
\end{aligned} \tag{2.43}$$

where the fifth line follows from the symmetry property $K(r,\eta)=K(\eta,r)$, the Fubini Theorem, and

$$\begin{aligned}
&\int_a^{b(t)} \rho(\Phi_\alpha * \rho)_t\, r^{n-1}\,\mathrm{d}r\\
&= \int_a^{b(t)} \rho\Big(\int_a^{b(t)} K(r,\eta)\rho(t,\eta)\eta^{n-1}\,\mathrm{d}\eta\Big)_t r^{n-1}\,\mathrm{d}r\\
&= (\rho u)(t,b(t))b(t)^{n-1}\int_a^{b(t)} K(b(t),r)\rho(t,r)\, r^{n-1}\,\mathrm{d}r\\
&\quad + \int_a^{b(t)} \rho\Big(\int_a^{b(t)} K(r,\eta)\rho_t(t,\eta)\eta^{n-1}\,\mathrm{d}\eta\Big) r^{n-1}\,\mathrm{d}r\\
&= \big(\rho u r^{n-1}(\Phi_\alpha * \rho)\big)(t,b(t)) + \int_a^{b(t)} \rho_t\Big(\int_a^{b(t)} K(\eta,r)\rho(t,r) r^{n-1}\,\mathrm{d}r\Big)\eta^{n-1}\,\mathrm{d}\eta\\
&= \big(\rho u r^{n-1}(\Phi_\alpha * \rho)\big)(t,b(t)) + \int_a^{b(t)} \rho_t(\Phi_\alpha * \rho)\, r^{n-1}\,\mathrm{d}r.
\end{aligned}$$

Substituting (2.41)–(2.43) into (2.40), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\Big\{\int_0^{\frac{M}{\omega_n}} \Big(\frac{1}{2}u^2 + e(\rho) + \frac{\kappa}{2}(\Phi_\alpha * \rho)\Big)\,\mathrm{d}x\Big\} + \varepsilon\int_0^{\frac{M}{\omega_n}} \big(\rho^2(r^{n-1}u_x)^2 + (n-1)\frac{u^2}{r^2}\big)\,\mathrm{d}x\\
&+ \varepsilon(n-1)(\rho u^2 r^{n-2})(\tau,\frac{M}{\omega_n}) = 0.
\end{aligned}\tag{2.44}$$

Rewriting (2.44) in Eulerian coordinates, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_a^{b(t)} \rho\Big(\frac{1}{2}u^2 + e(\rho) + \frac{\kappa}{2}(\Phi_\alpha * \rho)\Big) r^{n-1}\,\mathrm{d}r + \varepsilon\int_a^{b(t)} \rho\Big(u_r^2 + (n-1)\frac{u^2}{r^2}\Big) r^{n-1}\,\mathrm{d}r\\
&+ \varepsilon(n-1)(\rho u^2)(t,b(t))b(t)^{n-2} = 0.
\end{aligned}$$

Integrating over $[0,t]$, we derive the first part of this lemma:

$$\begin{aligned}
&\int_a^{b(t)} \rho\Big(\frac{1}{2}u^2 + e(\rho) + \frac{\kappa}{2}(\Phi_\alpha * \rho)\Big) r^{n-1}\,\mathrm{d}r + \varepsilon\int_0^t\int_a^{b(s)} \rho\Big(u_r^2 + (n-1)\frac{u^2}{r^2}\Big) r^{n-1}\,\mathrm{d}r\,\mathrm{d}s\\
&\quad + (n-1)\varepsilon\int_0^t (\rho u^2)(s,b(s))b(s)^{n-2}\,\mathrm{d}s\\
&= \int_a^b \rho_0\Big(\frac{1}{2}u_0^2 + e(\rho_0) + \frac{\kappa}{2}(\Phi_\alpha * \rho_0)\Big) r^{n-1}\,\mathrm{d}r.
\end{aligned}\tag{2.45}$$

Then Case 1 follows directly from (2.45).

Step 2. In Cases 2 and 3 (i.e. $\kappa = 1$), we must control the potential energy term by the internal energy term. To this end, we use the variant of the HLS inequality given in Lemma A.0.3 as follows

$$\frac{1}{2}\Big|\int_a^{b(t)} \rho(\Phi_\alpha * \rho) r^{n-1}\,\mathrm{d}r\Big|$$

$$
\begin{aligned}
&= \frac{1}{2\omega_n}\left|\int_{\Omega_t}\big(\rho(\Phi_\alpha * \rho)\big)(t,\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right| \\
&\le \frac{1}{2\alpha\omega_n}\left|\int_{\Omega_t}\big(\rho(|\cdot|^{-\alpha} * \rho)\big)(t,\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right| \\
&\le \frac{C_*(\alpha,\gamma,n)}{2\alpha\omega_n}\|\rho\|_{L^1(\Omega_t)}^{\frac{\gamma(2n-\alpha)-2n}{n(\gamma-1)}}\|\rho\|_{L^\gamma(\Omega_t)}^{\frac{\alpha\gamma}{n(\gamma-1)}} \\
&= \frac{C_*(\alpha,\gamma,n)}{2\alpha}\omega_n^{\frac{\alpha}{n(\gamma-1)}-1}M^{\frac{\gamma(2n-\alpha)-2n}{n(\gamma-1)}}\Big(\int_a^{b(t)}\rho^\gamma r^{n-1}\,\mathrm{d}r\Big)^{\frac{\alpha}{n(\gamma-1)}} \\
&= \frac{C_*(\alpha,\gamma,n)}{2\alpha}\omega_n^{\frac{\alpha}{n(\gamma-1)}-1}\Big(\frac{\gamma-1}{a_0}\Big)^{\frac{\alpha}{n(\gamma-1)}}M^{\frac{\gamma(2n-\alpha)-2n}{n(\gamma-1)}}\Big(\int_a^{b(t)}\rho e(\rho)r^{n-1}\,\mathrm{d}r\Big)^{\frac{\alpha}{n(\gamma-1)}} \\
&= B_{n,\gamma,\alpha}M^{\frac{\gamma(2n-\alpha)-2n}{n(\gamma-1)}}\Big(\int_a^{b(t)}\rho e(\rho)r^{n-1}\,\mathrm{d}r\Big)^{\frac{\alpha}{n(\gamma-1)}}. \qquad (2.46)
\end{aligned}
$$

Thus, by applying (2.46), we obtain

$$
\begin{aligned}
&\int_a^{b(t)}\rho\Big(e(\rho)+\frac{1}{2}(\Phi_\alpha*\rho)\Big)r^{n-1}\,\mathrm{d}r \\
&\ge \int_a^{b(t)}\rho e(\rho)r^{n-1}\,\mathrm{d}r - B_{n,\gamma,\alpha}M^{\frac{\gamma(2n-\alpha)-2n}{n(\gamma-1)}}\Big(\int_a^{b(t)}\rho e(\rho)r^{n-1}\,\mathrm{d}r\Big)^{\frac{\alpha}{n(\gamma-1)}}. \qquad (2.47)
\end{aligned}
$$

Step 3. In Case 3, we assume that $\gamma > \frac{n+\alpha}{n}$, which implies that $\frac{\alpha}{n(\gamma-1)} < 1$. Hence, combining (2.47) with Young's inequality, we obtain

$$
\int_a^{b(t)}\rho\Big(e(\rho)+\frac{1}{2}(\Phi_\alpha*\rho)\Big)r^{n-1}\,\mathrm{d}r \ge \frac{1}{2}\int_a^{b(t)}\rho e(\rho)r^{n-1}\,\mathrm{d}r - C(M).
$$

Step 4. We now turn to Case 2. First, consider the critical case $\gamma = \frac{n+\alpha}{n}$. Then, it follows from (2.47) that

$$
\begin{aligned}
&\int_a^{b(t)}\rho\Big(e(\rho)+\frac{1}{2}(\Phi_\alpha*\rho)\Big)r^{n-1}\,\mathrm{d}r \\
&\ge \int_a^{b(t)}\rho e(\rho)r^{n-1}\,\mathrm{d}r - B_{n,\gamma,\alpha}M^{\frac{n-\alpha}{n}}\int_a^{b(t)}\rho e(\rho)r^{n-1}\,\mathrm{d}r \\
&= C_\gamma\int_a^{b(t)}\rho e(\rho)r^{n-1}\,\mathrm{d}r,
\end{aligned}
$$

where $C_\gamma > 0$, since we have assumed that $M < M_{\mathrm{c}}^{\varepsilon,b,\alpha}(\gamma)$. We now consider the remaining case $\gamma \in \big(\frac{2n}{2n-\alpha}, \frac{n+\alpha}{n}\big)$. Motivated by (2.47), we introduce the quantity

$$
F(s) = s - B_{n,\gamma,\alpha}M^{\frac{\gamma(2n-\alpha)-2n}{n(\gamma-1)}}s^{\frac{\alpha}{n(\gamma-1)}}.
$$

Then

$$
\begin{cases}
F'(s) = 1 - \dfrac{\alpha}{n(\gamma-1)}B_{n,\gamma,\alpha}M^{\frac{\gamma(2n-\alpha)-2n}{n(\gamma-1)}}s^{\frac{\alpha-n(\gamma-1)}{n(\gamma-1)}}, \\
F''(s) = -\dfrac{\alpha(\alpha-n(\gamma-1))}{n^2(\gamma-1)^2}B_{n,\gamma,\alpha}M^{\frac{\gamma(2n-\alpha)-2n}{n(\gamma-1)}}s^{\frac{\alpha-2n(\gamma-1)}{n(\gamma-1)}},
\end{cases}
$$

which, for $\gamma \in \left(\frac{2n}{2n-\alpha}, \frac{n+\alpha}{n}\right)$, satisfies $F''(s) < 0$ for all $s > 0$. Hence, $F(s)$ is concave on $(0, \infty)$. We denote

$$s_* = \left(\frac{\alpha B_{n,\gamma,\alpha}}{n(\gamma-1)}\right)^{\frac{n(\gamma-1)}{n(\gamma-1)-\alpha}} M^{\frac{\gamma(2n-\alpha)-2n}{n(\gamma-1)-\alpha}},$$

the critical point of $F(s)$. Since $F''(s) < 0$ for all $s > 0$, it follows that the maximum of $F(s)$ on $[0, \infty)$ is attained by

$$F(s_*) = \frac{\alpha - n(\gamma-1)}{\alpha} s_* > 0.$$

Therefore, since $M < M_{\mathrm{c}}^{\varepsilon,b,\alpha}(\gamma)$, by (2.15), and because

$$\frac{\gamma(2n-\alpha)-2n}{n(\gamma-1)-\alpha} < 0, \quad \text{for } \gamma \in \left(\frac{2n}{2n-\alpha}, \frac{n+\alpha}{n}\right),$$

we conclude that

$$F(s_*) > \frac{\alpha - n(\gamma-1)}{\alpha}\left(\frac{\alpha B_{n,\gamma,\alpha}}{n(\gamma-1)}\right)^{\frac{n(\gamma-1)}{n(\gamma-1)-\alpha}} M_{\mathrm{c}}^{\varepsilon,b,\alpha}(\gamma)^{\frac{\gamma(2n-\alpha)-2n}{n(\gamma-1)-\alpha}} = E_0^{\varepsilon,b}, \tag{2.48}$$

and

$$s_* = \frac{\alpha}{\alpha - n(\gamma-1)} F(s_*) > \frac{\alpha}{\alpha - n(\gamma-1)} E_0^{\varepsilon,b} > 2E_0^{\varepsilon,b}. \tag{2.49}$$

Now, using (2.45) and (2.47)–(2.49), we obtain

$$\begin{aligned} &F\left(\int_a^{b(t)} \rho e(\rho) r^{n-1}\,\mathrm{d}r\right) \le E_0^{\varepsilon,b} < F(s_*), \\ &\int_a^b \rho_0 e(\rho_0) r^{n-1}\,\mathrm{d}r \le E_0^{\varepsilon,b} < s_*. \end{aligned} \tag{2.50}$$

Therefore, since the mapping

$$t \longmapsto \int_a^{b(t)} \rho e(\rho) r^{n-1}\,\mathrm{d}r,$$

is continuous in t, it follows that

$$\int_a^{b(t)} \rho e(\rho) r^{n-1}\,\mathrm{d}r < s_*,$$

as otherwise we can apply the intermediate value Theorem to derive a contradiction to (2.50). This implies that

$$\begin{aligned} &F\left(\int_a^{b(t)} \rho e(\rho) r^{n-1}\,\mathrm{d}r\right) \\ &\ge \left(1 - B_{n,\gamma,\alpha} M^{\frac{\gamma(2n-\alpha)-2n}{n(\gamma-1)}} s_*^{\frac{\alpha-n(\gamma-1)}{n(\gamma-1)}}\right) \int_a^{b(t)} \rho e(\rho) r^{n-1}\,\mathrm{d}r \\ &= \frac{\alpha - n(\gamma-1)}{\alpha} \int_a^{b(t)} \rho e(\rho) r^{n-1}\,\mathrm{d}r. \end{aligned} \tag{2.51}$$

Therefore, using (2.45), (2.47), and (2.51), we derive (2.38).

Step 5. For Case 4, we first consider $\alpha = 0$. For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ with $|\boldsymbol{x} - \boldsymbol{y}| \geq 1$, we know that $\log(|\boldsymbol{x}| + |\boldsymbol{y}|) \geq \log(|\boldsymbol{x} - \boldsymbol{y}|) \geq 0$. Then, if $|\boldsymbol{y}| \geq |\boldsymbol{x}|$, we obtain

$$\log(|\boldsymbol{x}| + |\boldsymbol{y}|) = \frac{1}{2}\log((|\boldsymbol{x}| + |\boldsymbol{y}|)^2) \leq \frac{1}{2}\log(4 + 4|\boldsymbol{y}|^2) = \log 2 + \frac{1}{2}\log(1 + |\boldsymbol{y}|^2).$$

If $|\boldsymbol{x}| \geq |\boldsymbol{y}|$, then

$$\log(|\boldsymbol{x}| + |\boldsymbol{y}|) \leq \log 2 + \frac{1}{2}\log(1 + |\boldsymbol{x}|^2).$$

Hence, for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ satisfying $|\boldsymbol{x} - \boldsymbol{y}| \geq 1$, we obtain

$$\log(|\boldsymbol{x}| + |\boldsymbol{y}|) \leq \log 4 + \frac{1}{2}\log(1 + |\boldsymbol{x}|^2) + \frac{1}{2}\log(1 + |\boldsymbol{y}|^2). \tag{2.52}$$

Then, by (2.52) together with Hölder's inequality, we obtain

$$\begin{aligned}
&\left| \int_a^{b(t)} \rho(\Phi_0 * \rho) r^{n-1} \,\mathrm{d}r \right| \\
&= \frac{1}{\omega_n} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho(\boldsymbol{x}) \log(|\boldsymbol{x} - \boldsymbol{y}|) \rho(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}\boldsymbol{y} \right| \\
&\leq \frac{1}{\omega_n} \int_{\mathbb{R}^n} \rho(\boldsymbol{y}) \Big(\int_{B_1(\boldsymbol{y})} \rho(\boldsymbol{x}) \log(|\boldsymbol{x} - \boldsymbol{y}|^{-1}) \,\mathrm{d}\boldsymbol{x} + \int_{B_1(\boldsymbol{y})^c} \rho(\boldsymbol{x}) \log(|\boldsymbol{x} - \boldsymbol{y}|) \,\mathrm{d}\boldsymbol{x} \Big) \,\mathrm{d}\boldsymbol{y} \\
&\leq \frac{1}{\omega_n} \int_{\mathbb{R}^n} \rho(\boldsymbol{y}) \Big(\|\rho\|_{L^\gamma(B_1(\boldsymbol{y}))} \|\log(|\cdot|)\|_{L^{\frac{\gamma}{\gamma-1}}(B_1)} \\
&\qquad\qquad + \int_{|\boldsymbol{x}-\boldsymbol{y}|\geq 1} \rho(\boldsymbol{x}) \Big(\log 4 + \frac{1}{2}\log(1 + |\boldsymbol{x}|^2) + \frac{1}{2}\log(1 + |\boldsymbol{y}|^2) \Big) \,\mathrm{d}\boldsymbol{x} \Big) \,\mathrm{d}\boldsymbol{y} \\
&\leq C(M) \Big(1 + \|\rho\|_{L^\gamma(\Omega_t)} + \int_{\Omega_t} \rho(\boldsymbol{x}) \log(1 + |\boldsymbol{x}|^2) \,\mathrm{d}\boldsymbol{x} \Big) \\
&\leq \frac{1}{2} \int_a^{b(t)} \rho e(\rho) r^{n-1} \,\mathrm{d}r + C(M) \Big(1 + \int_a^{b(t)} \rho \log(1 + r^2) r^{n-1} \,\mathrm{d}r \Big),
\end{aligned} \tag{2.53}$$

where we have applied Young's inequality in the final line.

Step 6. For $\alpha \in (-1, 0)$, we observe that, for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$,

$$|\Phi_\alpha(\boldsymbol{x} - \boldsymbol{y})| \leq \frac{2^{-\alpha}}{-\alpha} \big(k_\alpha(1 + |\boldsymbol{x}|^2) + k_\alpha(1 + |\boldsymbol{y}|^2) \big). \tag{2.54}$$

Hence, by proceeding in the same manner as in (2.53) and applying (2.54), we obtain

$$\left| \int_a^{b(t)} \rho(\Phi_\alpha * \rho) r^{n-1} \,\mathrm{d}r \right| \leq \frac{2^{1-\alpha} M}{-\alpha} \int_a^{b(t)} \rho k_\alpha (1 + r^2) r^{n-1} \,\mathrm{d}r. \tag{2.55}$$

We now examine the latter integral appearing in (2.53) and (2.55), in the general setting $\alpha \in (-1, 0]$. Using system (2.8) together with integration by parts, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_a^{b(t)} \rho k_\alpha (1 + r^2) r^{n-1} \,\mathrm{d}r$$

$$
\begin{aligned}
&= \big(\rho u k_\alpha(1+r^2)r\big)\,(t,b(t)) + \int_a^{b(t)} \rho_t k_\alpha(1+r^2)r^{n-1}\,\mathrm{d}r \\
&= \big(\rho u k_\alpha(1+r^2)r\big)\,(t,b(t)) - \int_a^{b(t)} k_\alpha(1+r^2)(\rho u r^{n-1})_r\,\mathrm{d}r \\
&= \int_a^{b(t)} (k_\alpha(1+r^2))_r \rho u r^{n-1}\,\mathrm{d}r \\
&\le \int_a^{b(t)} \rho u r^{n-1}\,\mathrm{d}r \\
&\le \frac{1}{2}\int_a^{b(t)} \rho(u^2+1)r^{n-1}\,\mathrm{d}r \\
&= \frac{1}{2}\int_a^{b(t)} \rho u^2 r^{n-1}\,\mathrm{d}r + \frac{M}{2\omega_n}.
\end{aligned} \tag{2.56}
$$

Integrating (2.56) with respect to t, we deduce that, for $\alpha \in (-1,0]$,

$$
\begin{aligned}
\int_a^{b(t)} \rho k_\alpha(1+r^2)r^{n-1}\,\mathrm{d}r \le \int_a^{b} \rho_0 k_\alpha(1+r^2)r^{n-1}\,\mathrm{d}r + \frac{Mt}{2\omega_n} \\
+ \frac{1}{2}\int_0^t \int_a^{b(s)} \rho u^2 r^{n-1}\,\mathrm{d}r\,\mathrm{d}s.
\end{aligned} \tag{2.57}
$$

Hence, combining (2.53), (2.55), and (2.57), we obtain

$$
\begin{aligned}
&\Big|\int_a^{b(t)} \rho(\Phi_\alpha * \rho)r^{n-1}\,\mathrm{d}r\Big| + \int_a^{b(t)} \rho k_\alpha(1+r^2)r^{n-1}\,\mathrm{d}r \\
&\le \frac{1}{2}\int_a^{b(t)} \rho e(\rho) r^{n-1}\,\mathrm{d}r + C(M,T)\Big(1 + \int_a^b \rho_0 k_\alpha(1+r^2)r^{n-1}\,\mathrm{d}r \\
&\qquad + \int_0^t \int_a^{b(s)} \rho u^2 r^{n-1}\,\mathrm{d}r\,\mathrm{d}s\Big).
\end{aligned} \tag{2.58}
$$

Combining (2.45) and (2.58), we conclude that, for $\alpha \in (-1,0]$,

$$
\begin{aligned}
&\frac{1}{2}\int_a^{b(t)} \rho\big(u^2 + e(\rho) + k_\alpha(1+r^2)\big)r^{n-1}\,\mathrm{d}r + \varepsilon\int_0^t\int_a^{b(s)} \rho\Big(u_r^2 + \frac{u^2}{r^2}\Big)r^{n-1}\,\mathrm{d}r\,\mathrm{d}s \\
&\quad + \varepsilon\int_0^t (\rho u^2)(s,b(s))b(s)^{n-2}\,\mathrm{d}s \\
&\le \int_a^b \rho_0\Big(\frac{1}{2}u_0^2 + e(\rho_0) + \frac{\kappa}{2}(\Phi_\alpha * \rho_0)\Big)r^{n-1}\,\mathrm{d}r \\
&\quad + C(M,T)\Big(1 + \int_a^b \rho_0 k_\alpha(1+r^2)r^{n-1}\,\mathrm{d}r + \int_0^t\int_a^{b(s)} \rho u^2 r^{n-1}\,\mathrm{d}r\,\mathrm{d}s\Big).
\end{aligned} \tag{2.59}
$$

Applying Grönwall's inequality to (2.59), and subsequently using (2.29), we conclude that

$$
\frac{1}{2}\int_a^{b(t)} \rho\big(u^2 + e(\rho) + k_\alpha(1+r^2)\big)r^{n-1}\,\mathrm{d}r + \varepsilon\int_0^t\int_a^{b(s)} \rho\big(u_r^2 + \frac{u^2}{r^2}\big)r^{n-1}\,\mathrm{d}r\,\mathrm{d}s
$$

$$
\begin{aligned}
&+\varepsilon\int_0^t(\rho u^2)(s,b(s))b(s)^{n-2}\,\mathrm{d}s\\
&\le C(M,T)\Big(1+\int_a^b\rho_0\Big(\frac{1}{2}u_0^2+e(\rho_0)+\frac{\kappa}{2}(\Phi_\alpha*\rho_0)+k_\alpha(1+r^2)\Big)r^{n-1}\,\mathrm{d}r\Big)\\
&\le C(M,E_0^{\varepsilon,b},T).
\end{aligned}
$$

□

Remark 2.6.2. Adding $\frac{\kappa M^2}{2\alpha}=\frac{\kappa}{2}\int_a^{b(t)}\rho(\frac{1}{\alpha}*\rho)r^{n-1}\,\mathrm{d}r=\frac{\kappa}{2}\int_a^b\rho_0(\frac{1}{\alpha}*\rho_0)r^{n-1}\,\mathrm{d}r$ to both sides of (2.37) yields

$$
\begin{aligned}
&\int_a^{b(t)}\rho\Big(\frac{1}{2}u^2+e(\rho)+\frac{\kappa}{2}\Big(\Big(\frac{1}{\alpha}+\Phi_\alpha\Big)*\rho\Big)\Big)r^{n-1}\,\mathrm{d}r\\
&+\varepsilon\int_0^t\int_a^{b(s)}\rho\Big(u_r^2+(n-1)\frac{u^2}{r^2}\Big)r^{n-1}\,\mathrm{d}r\,\mathrm{d}s+(n-1)\varepsilon\int_0^t(\rho u^2)(s,b(s))b(s)^{n-2}\,\mathrm{d}s\\
&=\int_a^b\rho_0\Big(\frac{1}{2}u_0^2+e(\rho_0)+\frac{\kappa}{2}\big((\frac{1}{\alpha}+\Phi_\alpha)*\rho_0\big)\Big)r^{n-1}\,\mathrm{d}r. \qquad (2.60)
\end{aligned}
$$

We can formally see that, using (1.2), as $\alpha\to 0$, (2.60) converges to

$$
\begin{aligned}
&\int_a^{b(t)}\rho\Big(\frac{1}{2}u^2+e(\rho)+\frac{\kappa}{2}(\Phi_0*\rho)\Big)r^{n-1}\,\mathrm{d}r+\varepsilon\int_0^t\int_a^{b(s)}\rho\Big(u_r^2+(n-1)\frac{u^2}{r^2}\Big)r^{n-1}\,\mathrm{d}r\,\mathrm{d}s\\
&\quad+(n-1)\varepsilon\int_0^t(\rho u^2)(s,b(s))b(s)^{n-2}\,\mathrm{d}s\\
&=\int_a^b\rho_0\Big(\frac{1}{2}u_0^2+e(\rho_0)+\frac{\kappa}{2}(\Phi_0*\rho_0)\Big)r^{n-1}\,\mathrm{d}r.
\end{aligned}
$$

The following corollary makes precise the uniform estimates that follow as a direct consequence of Lemma 2.6.1.

Corollary 2.6.3 (Uniform Estimates)**.** *Under the conditions of* Lemma 2.6.1*, the following uniform estimates in* (ε,b) *(for* ε *small enough and* b *large enough) hold*:

Case 1 ($\kappa=-1$ with $\gamma>1$ for $\alpha\in(0,n-2]$ or $\gamma>\frac{2n}{2n-\alpha}$ for $\alpha\in(n-2,n-1)$)**.**

$$
\int_{\mathbb{R}^n}\rho\big(1+u^2+\rho^{\gamma-1}-\Phi_\alpha*\rho\big)\,\mathrm{d}\boldsymbol{x}\le C(M,E_0), \qquad (2.61)
$$

Case 2 ($\kappa=1$, $\alpha\in(0,n-1)$ and $\gamma>\frac{2n}{2n-\alpha}$)**.**

$$
\int_{\mathbb{R}^n}\rho\big(1+u^2+\rho^{\gamma-1}\big)\,\mathrm{d}\boldsymbol{x}\le C(M,E_0), \qquad (2.62)
$$

Case 3 ($\kappa\in\{-1,1\}$, $\alpha\in(-1,0]$ and $\gamma>1$)**.**

$$
\int_{\mathbb{R}^n}\rho\big(1+u^2+\rho^{\gamma-1}+k_\alpha(1+|\boldsymbol{x}|^2)\big)\,\mathrm{d}\boldsymbol{x}\le C(M,E_0,T), \qquad (2.63)
$$

where we have used the convention that ρ and u are zero outside of Ω_t. In particular, the following uniform estimates in (ε,b) (for ε small enough and b large enough) hold:

For Cases 1–2,

$$\|(\Phi_\alpha * \rho)(t)\|_{L^{\frac{2n}{\alpha}}(\mathbb{R}^n)} + \|(\nabla\Phi_\alpha * \rho)(t)\|_{L^{\frac{2n}{\alpha+2}}(\mathbb{R}^n)} \leq C(M, E_0);$$

and for Case 3, *for any* $K \Subset \mathbb{R}^n$, *there exists* $C(M, E_0, T, K) > 0$ *such that*

$$\|(\Phi_\alpha * \rho)(t)\|_{L^\infty(K)} + \|(\nabla\Phi_\alpha * \rho)(t)\|_{L^{\frac{2n}{\alpha+2}}(K)} \leq C(M, E_0, T, K).$$

Proof. Estimates (2.61)–(2.63) follow directly from Lemma 2.6.1, together with the identities

$$\int_a^{b(t)} \rho\, r^{n-1}\,\mathrm{d}r = \frac{M}{\omega_n}, \qquad e(\rho) = \frac{a_0}{\gamma-1}\rho^{\gamma-1},$$

and the convergence $E_0^{\varepsilon,b} \to E_0$ as $b \to \infty$ and $\varepsilon \to 0$, which is guaranteed by Lemma 2.4.8. For the second part of the corollary, consider Case 2 or $\kappa = -1$ with $\gamma > \frac{2n}{2n-\alpha}$. By applying the HLS inequality from Lemma A.0.2 together with interpolation, we obtain

$$\begin{aligned}
\|(\Phi_\alpha * \rho)(t)\|_{L^{\frac{2n}{\alpha}}(\mathbb{R}^n)} &\leq \frac{C_{n,\alpha}}{\alpha}\|\rho(t)\|_{L^1(\mathbb{R}^n)}^{\frac{\gamma(2n-\alpha)-2n}{n(\gamma-1)}}\|\rho(t)\|_{L^\gamma(\mathbb{R}^n)}^{\frac{\alpha\gamma}{n(\gamma-1)}} \leq C(M, E_0),\\
\|(\nabla\Phi_\alpha * \rho)(t)\|_{L^{\frac{2n}{\alpha+2}}(\mathbb{R}^n)} &\leq C_{n,\alpha+1}\|\rho(t)\|_{L^1(\mathbb{R}^n)}^{\frac{\gamma(2n-\alpha)-2n}{n(\gamma-1)}}\|\rho(t)\|_{L^\gamma(\mathbb{R}^n)}^{\frac{\alpha\gamma}{n(\gamma-1)}} \leq C(M, E_0),
\end{aligned}$$

where the final inequalities follow from the first part of this corollary. For Case 1, we apply the Riesz composition formula given in Lemma A.0.6, together with the HLS inequality from Lemma A.0.2 and Tonelli's Theorem, to obtain

$$\begin{aligned}
\|\Phi_\alpha * \rho\|_{L^{\frac{2n}{\alpha}}(\mathbb{R}^n)} &= \frac{1}{\alpha}\big\||\cdot|^{-\alpha} * \rho\big\|_{L^{\frac{2n}{\alpha}}(\mathbb{R}^n)}\\
&= \frac{1}{\alpha\kappa_{\frac{n-\alpha}{2},\frac{n-\alpha}{2}}}\big\||\cdot|^{-\frac{n+\alpha}{2}} * \big(|\cdot|^{-\frac{n+\alpha}{2}} * \rho\big)\big\|_{L^{\frac{2n}{\alpha}}(\mathbb{R}^n)}\\
&\leq \frac{C_{n,\frac{n+\alpha}{2}}}{\alpha\kappa_{\frac{n-\alpha}{2},\frac{n-\alpha}{2}}}\big\||\cdot|^{-\frac{n+\alpha}{2}} * \rho\big\|_{L^2(\mathbb{R}^n)}\\
&= \frac{C_{n,\frac{n+\alpha}{2}}}{\alpha\kappa_{\frac{n-\alpha}{2},\frac{n-\alpha}{2}}}\int_{\mathbb{R}^n}\rho\left(|\cdot|^{-\frac{n+\alpha}{2}} * \big(|\cdot|^{-\frac{n+\alpha}{2}} * \rho\big)\right)\mathrm{d}\boldsymbol{x}\\
&= -C_{n,\frac{n+\alpha}{2}}\int_{\mathbb{R}^n}\rho(\Phi_\alpha * \rho)\,\mathrm{d}\boldsymbol{x}\\
&\leq C(M, E_0),
\end{aligned}$$

and, for $\alpha \in (0, n-2)$,

$$\begin{aligned}
\|\nabla\Phi_\alpha * \rho\|_{L^{\frac{2n}{\alpha+2}}(\mathbb{R}^n)} &\leq \frac{1}{\alpha}\big\||\cdot|^{-(\alpha+1)} * \rho\big\|_{L^{\frac{2n}{\alpha+2}}(\mathbb{R}^n)}\\
&= \frac{1}{\alpha\kappa_{\frac{n-\alpha}{2}-1,\frac{n-\alpha}{2}}}\big\||\cdot|^{-\left(1+\frac{n+\alpha}{2}\right)} * \big(|\cdot|^{-\frac{n+\alpha}{2}} * \rho\big)\big\|_{L^{\frac{2n}{\alpha+2}}(\mathbb{R}^n)}
\end{aligned}$$

$$\leq \frac{C_{n,\frac{n+\alpha}{2}+1}}{\alpha\kappa_{\frac{n-\alpha}{2}-1,\frac{n-\alpha}{2}}} \left\| |\cdot|^{-\frac{n+\alpha}{2}} * \rho \right\|_{L^2(\mathbb{R}^n)}, \tag{2.64}$$

where $\kappa_{\alpha,\beta}$ is defined in (A.3).

For $\alpha = n-2$, it follows directly that

$$\|\nabla\Phi_\alpha * \rho\|_{L^{\frac{2n}{\alpha+2}}(\mathbb{R}^n)} \leq \left\| |\cdot|^{-\frac{n+\alpha}{2}} * \rho \right\|_{L^2(\mathbb{R}^n)}. \tag{2.65}$$

We now apply Lemma A.0.6 together with Tonelli's Theorem to obtain

$$\begin{aligned} \left\| |\cdot|^{-\frac{n+\alpha}{2}} * \rho \right\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \rho \left(|\cdot|^{-\frac{n+\alpha}{2}} * \left(|\cdot|^{-\frac{n+\alpha}{2}} * \rho \right) \right) \mathrm{d}\boldsymbol{x} \\ &= -\kappa_{\frac{n-\alpha}{2},\frac{n-\alpha}{2}} \int_{\mathbb{R}^n} \rho(\Phi_\alpha * \rho)\, \mathrm{d}\boldsymbol{x} \\ &\leq C(M, E_0). \end{aligned} \tag{2.66}$$

Combining (2.64)–(2.66), we conclude that, for $\kappa = -1$ and $\alpha \in (0, n-2]$,

$$\|\nabla\Phi_\alpha * \rho\|_{L^{\frac{2n}{\alpha+2}}(\mathbb{R}^n)} \leq C(M, E_0).$$

We now consider the case $\alpha = 0$. For simplicity, assume that $K = \overline{B_D(\mathbf{0})}$ for some $D > 0$, where $B_D(\mathbf{0})$ denotes the open ball centred at the origin with radius D. Observe that, for any $\boldsymbol{x} \in K$ and $\boldsymbol{y} \in \mathbb{R}^n$ satisfying $|\boldsymbol{x} - \boldsymbol{y}| \geq 1$, we have

$$\log\left(|\boldsymbol{x}| + |\boldsymbol{y}|\right) \geq \log|\boldsymbol{x} - \boldsymbol{y}| \geq 0.$$

In particular, if $\boldsymbol{y} \in K$,

$$\log(|\boldsymbol{x}| + |\boldsymbol{y}|) \leq \log(1 + 2D).$$

If $\boldsymbol{y} \in \mathbb{R}^n \setminus K$, then

$$\log(|\boldsymbol{x}| + |\boldsymbol{y}|) = \frac{1}{2}\log((|\boldsymbol{x}| + |\boldsymbol{y}|)^2) \leq \frac{1}{2}\log(4 + 4|\boldsymbol{y}|^2) = \log 2 + \frac{1}{2}\log(1 + |\boldsymbol{y}|^2).$$

Hence, for any $\boldsymbol{y} \in \mathbb{R}^n$, we obtain

$$\log(|\boldsymbol{x}|+|\boldsymbol{y}|) \leq \log(1+2D)+\log 2+\frac{1}{2}\log(1+|\boldsymbol{y}|^2) = \log(2+4D)+\frac{1}{2}\log(1+|\boldsymbol{y}|^2). \tag{2.67}$$

Then, for $\boldsymbol{x} \in K$, we apply Hölder's inequality together with (2.67) and (2.39) to obtain

$$\begin{aligned} &|(\Phi_0 * \rho)(\boldsymbol{x})| \\ &\leq \int_{B_1(\boldsymbol{x})} \log(|\boldsymbol{x} - \boldsymbol{y}|^{-1})\rho(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} + \int_{B_1(\boldsymbol{x})^c} \log(|\boldsymbol{x} - \boldsymbol{y}|)\rho(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} \\ &\leq \|\log(|\cdot|)\|_{L^{\frac{\gamma}{\gamma-1}}(B_1)} \|\rho\|_{L^\gamma(B_1(\boldsymbol{x}))} + \int_{B_1(\boldsymbol{x})^c} \left(\log(2+4D) + \frac{1}{2}\log(1+|\boldsymbol{y}|^2)\right)\rho(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} \end{aligned}$$

$$\leq C\|\rho\|_{L^\gamma(B_1(\boldsymbol{x}))} + \log(2+4D)\|\rho\|_{L^1(B_1(\boldsymbol{x})^c)} + \frac{1}{2}\int_{B_1(\boldsymbol{x})^c} \log(1+|\boldsymbol{y}|^2)\rho(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$$
$$\leq C(K, M, E_0, T). \tag{2.68}$$

Therefore, we conclude that

$$\|\Phi_0 * \rho\|_{L^\infty(K)} \leq C(K, M, E_0, T).$$

We now consider the case $\alpha \in (-1, 0)$, and again assume that $K = \overline{B_D}$ for some $D > 0$. By applying (2.54), we obtain that, for any $\boldsymbol{x} \in K$ and $\boldsymbol{y} \in \mathbb{R}^n$,

$$|\Phi_\alpha(\boldsymbol{x}-\boldsymbol{y})| \leq \frac{2^{-\alpha}}{-\alpha}\left(k_\alpha(1+D^2) + k_\alpha(1+|\boldsymbol{y}|^2)\right). \tag{2.69}$$

Therefore, proceeding as in (2.68) and applying (2.69) together with (2.39), we obtain

$$\begin{aligned}|(\Phi_\alpha * \rho)(\boldsymbol{x})| &\leq \frac{2^{-\alpha}}{-\alpha}\int_{\mathbb{R}^n}\left(k_\alpha(1+D^2) + k_\alpha(1+|\boldsymbol{y}|^2)\right)\rho(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\\ &\leq C(K, M, E_0, T).\end{aligned}$$

Then, we obtain

$$\|\Phi_\alpha * \rho\|_{L^\infty(K)} \leq C(K, M, E_0, T).$$

For $\alpha \in (-1, 0]$, let $1 < p < \min\left\{\frac{2n}{\alpha+2}, \gamma\right\}$. Applying the HLS inequality from Lemma A.0.2, together with interpolation and (2.39), we obtain

$$\begin{aligned}&\|\nabla\Phi_\alpha * \rho\|_{L^{\frac{2n}{\alpha+2}}(K)}\\ &\leq \|(|\cdot|^{-(\alpha+1)}\mathbb{1}_{B_1}) * \rho\|_{L^{\frac{2n}{\alpha+2}}(K)} + \|(|\cdot|^{-(\alpha+1)}\mathbb{1}_{B_1^c}) * \rho\|_{L^{\frac{2n}{\alpha+2}}(K)}\\ &\leq \|(|\cdot|^{-(1+\frac{\alpha}{2}+n(1-\frac{1}{p}))}) * \rho\|_{L^{\frac{2n}{\alpha+2}}(K)} + \|\mathbb{1}_{B_1^c} * \rho\|_{L^{\frac{2n}{\alpha+2}}(K)}\\ &\leq C_{n,1+\frac{\alpha}{2}+n(1-\frac{1}{p})}\|\rho\|_{L^p(\mathbb{R}^n)} + |K|^{\frac{\alpha+2}{2n}}\|\rho\|_{L^1(\mathbb{R}^n)}\\ &\leq C_{n,1+\frac{\alpha}{2}+n(1-\frac{1}{p})}\|\rho\|_{L^1(\mathbb{R}^n)}^{\frac{\gamma-p}{p(\gamma-1)}}\|\rho\|_{L^\gamma(\mathbb{R}^n)}^{\frac{\gamma(p-1)}{p(\gamma-1)}} + |K|^{\frac{\alpha+2}{2n}}\|\rho\|_{L^1(\mathbb{R}^n)}\\ &\leq C(K, M, E_0, T).\end{aligned}$$

□

Remark 2.6.4. In the repulsive case $\kappa = -1$ with $\alpha \in (n-2, n-1)$, it remains an open question whether the previous result holds without imposing any restriction on $\gamma > 1$. The difficulty arises from the fact that the $L^{\frac{2n}{\alpha+2}}$-norm of the gradient of the potential cannot be controlled by the potential energy of the system.

2.7 Expanding the domain

To establish the existence of global weak solutions to (2.8), it is necessary to show that, as $b \to \infty$, the domain Ω^T expands to the entire space $[0,T] \times \mathbb{R}^n$. This expansion enables us to first prove local convergence in the limit, which can subsequently be extended to global convergence.

Lemma 2.7.1 (Expansion of the domain Ω^T)**.** *Given $T > 0$ and $\varepsilon \in (0, \varepsilon_0]$, there exists a positive constant $C_1(M, E_0, T, \varepsilon) > 0$ such that, if $b \geq \max\{C_1(M, E_0, T, \varepsilon), \mathfrak{B}(\varepsilon)\}$ (as defined in* Lemma 2.4.8*),*

$$b(t) \geq \frac{b}{2}, \qquad \text{for } t \in [0,T]. \tag{2.70}$$

This lemma can be proved by following the same argument as in [36, Lemma 3.4]. Therefore, as $b \to \infty$, we have

$$\lim_{b\to\infty} \inf_{t\in[0,T]} b(t) = \infty,$$

as desired. Equivalently, this means that Ω^T expands to the entire space $[0,T] \times \mathbb{R}^n$ as $b \to \infty$.

2.8 BD entropy estimates

We next derive uniform estimates on the spatial derivatives of the density in order to apply the Aubin–Lions Lemma and prove the convergence of $(\rho^{\varepsilon,b}, \rho^{\varepsilon,b}u^{\varepsilon,b})$ as $b \to \infty$ toward global weak solutions $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)$ of (2.8). To this end, we establish a BD-type entropy estimate, making use of the structural relation satisfied by the viscosity coefficients,

$$\lambda(\rho) = \rho\mu'(\rho) - \mu(\rho).$$

We obtain two different BD-type estimates. The first follows the method of [36] for $\alpha \in (-1, n-2]$. This restriction arises because, for $\alpha > n-2$, the term $\Phi_\alpha * \rho$ is no longer twice differentiable, so the integration by parts argument cannot be justified rigorously. However, even in the range $\alpha \in (-1, n-2)$, this approach does not yield a satisfactory estimate in the attractive case $\kappa = 1$. The difficulty stems from the boundary contribution

$$a^{n-1}\big(\rho(\Phi_\alpha * \rho)_r\big)(t,a),$$

which cannot be controlled due to the absence of *a priori* information on the density at the lower boundary $r = a$, apart from (2.13). Moreover, the resulting bound is not uniform in b, which is essential for our purposes. In the Coulomb/Newtonian case

$\alpha = n-2$, this issue does not arise, as the boundary term can be shown to vanish. Furthermore, in the repulsive case $\kappa = -1$, the boundary contribution has a favourable sign and can therefore be controlled.

Lemma 2.8.1 (BD-Type Entropy Estimate I)**.** *Under the conditions of* Lemma 2.6.1, *for* $\kappa = -1$ *with* $\alpha \in (-1, n-2]$ *or* $\kappa = 1$ *with* $\alpha = n-2$, *any given* $T > 0$, $\varepsilon \in (0, \varepsilon_0]$, *and* $b \geq \max\{C_1(M, E_0, T, \varepsilon), \mathfrak{B}(\varepsilon)\}$ (*as defined in* Lemmas 2.4.8 and 2.7.1), *the following estimate holds*:

$$\begin{aligned}
&\frac{\varepsilon^2}{4}\int_a^{b(t)} \left|(\sqrt{\rho(t,r)})_r\right|^2 r^{n-1}\,\mathrm{d}r + \frac{4a_0\varepsilon}{\gamma}\int_0^t\int_a^{b(s)} |(\rho^{\frac{\gamma}{2}})_r|^2\, r^{n-1}\,\mathrm{d}r\,\mathrm{d}s \\
&\quad + \frac{1}{n}p(\rho(t,b(t)))b(t)^n + \frac{1}{n\varepsilon}\int_0^t p(\rho(s,b(s)))p'(\rho(s,b(s)))b(s)^n\,\mathrm{d}s \\
&\leq C(E_0, M, T), \qquad \text{for all } t \in [0,T].
\end{aligned}$$

Proof. We divide the proof into four steps.

Step 1. Differentiating the mass equation $(2.18)_1$ with respect to x, we obtain

$$\rho_{x\tau} = -\big(\rho^2(r^{n-1}u)_x\big)_x.$$

Substituting this expression into $(2.18)_2$, we obtain

$$u_\tau + r^{n-1}p_x = -\varepsilon r^{n-1}\rho_{x\tau} - \varepsilon(n-1)r^{n-2}\rho_x u - \kappa\rho r^{n-1}(\Phi_\alpha * \rho)_x. \tag{2.71}$$

Moreover, (2.71) can be rewritten in the simplified form

$$(u + \varepsilon r^{n-1}\rho_x)_\tau + r^{n-1}p_x = -\kappa\rho r^{n-1}(\Phi_\alpha * \rho)_x. \tag{2.72}$$

Next, multiplying (2.72) by $u + \varepsilon r^{n-1}\rho_x$ and integrating with respect to x over the interval $[0, \frac{M}{\omega_n}]$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^{\frac{M}{\omega_n}} (u + \varepsilon r^{n-1}\rho_x)^2\,\mathrm{d}x + \varepsilon\int_0^{\frac{M}{\omega_n}} r^{2n-2}p(\rho)_x\rho_x\,\mathrm{d}x + \int_0^{\frac{M}{\omega_n}} r^{n-1}p(\rho)_x u\,\mathrm{d}x \\
&= -\kappa\int_0^{\frac{M}{\omega_n}} u\rho r^{n-1}(\Phi_\alpha * \rho)_x\,\mathrm{d}x - \kappa\varepsilon\int_0^{\frac{M}{\omega_n}} \rho\rho_x r^{2(n-1)}(\Phi_\alpha * \rho)_x\,\mathrm{d}x.
\end{aligned} \tag{2.73}$$

Using (2.43), we can rewrite (2.73) in the form

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^{\frac{M}{\omega_n}} (u + \varepsilon r^{n-1}\rho_x)^2\,\mathrm{d}x + \varepsilon\int_0^{\frac{M}{\omega_n}} r^{2n-2}p(\rho)_x\rho_x\,\mathrm{d}x + \int_0^{\frac{M}{\omega_n}} r^{n-1}p(\rho)_x u\,\mathrm{d}x \\
&= -\frac{\kappa}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^{\frac{M}{\omega_n}} (\Phi_\alpha * \rho)\,\mathrm{d}x - \kappa\varepsilon\int_0^{\frac{M}{\omega_n}} \rho\rho_x r^{2(n-1)}(\Phi_\alpha * \rho)_x\,\mathrm{d}x.
\end{aligned} \tag{2.74}$$

Using $(2.18)_1$ together with the boundary condition (2.17), we obtain

$$\rho_\tau(\tau, \frac{M}{\omega_n}) = -(\rho^2(r^{n-1}u)_x)(\tau, \frac{M}{\omega_n}) = -\frac{a_0\rho^\gamma}{\varepsilon}(\tau, \frac{M}{\omega_n}). \tag{2.75}$$

Now, applying (2.16), $(2.18)_1$, and (2.75), we obtain

$$\begin{aligned}
&\int_0^{\frac{M}{\omega_n}} p(\rho)_x u\, r^{n-1}\,\mathrm{d}x \\
&= -\int_0^{\frac{M}{\omega_n}} p(\rho)(r^{n-1}u)_x\,\mathrm{d}x + (p(\rho)ur^{n-1})(\tau, \frac{M}{\omega_n}) \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^{\frac{M}{\omega_n}} e(\rho)\,\mathrm{d}x + (p(\rho)r^{n-1})(\tau, \frac{M}{\omega_n})\, r_\tau(\tau, \frac{M}{\omega_n}) \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^{\frac{M}{\omega_n}} e(\rho)\,\mathrm{d}x + \Big(\frac{1}{n}p(\rho(\tau, \frac{M}{\omega_n}))b(\tau)^n\Big)_\tau - \frac{1}{n}p'(\rho(\tau, \frac{M}{\omega_n}))\rho_\tau(\tau, \frac{M}{\omega_n})b(\tau)^n \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^{\frac{M}{\omega_n}} e(\rho)\,\mathrm{d}x + \Big(\frac{1}{n}p(\rho(\tau, \frac{M}{\omega_n}))b(\tau)^n\Big)_\tau + \frac{1}{n\varepsilon}p(\rho(\tau, \frac{M}{\omega_n}))p'(\rho(\tau, \frac{M}{\omega_n}))b(\tau)^n.
\end{aligned} \tag{2.76}$$

Substituting (2.76) into (2.74), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\Big\{\int_0^{\frac{M}{\omega_n}} \frac{1}{2}(u + \varepsilon r^{n-1}\rho_x)^2\,\mathrm{d}x + \int_0^{\frac{M}{\omega_n}} e(\rho)\,\mathrm{d}x + \frac{\kappa}{2}\int_0^{\frac{M}{\omega_n}} \Phi_\alpha * \rho\,\mathrm{d}x\Big\} \\
&\quad + \varepsilon\int_0^{\frac{M}{\omega_n}} p'(\rho)\rho_x^2\, r^{2n-2}\,\mathrm{d}x + \Big(\frac{1}{n}p(\rho(\tau, \frac{M}{\omega_n}))b(\tau)^n\Big)_\tau \\
&\quad + \frac{1}{n\varepsilon}p(\rho(\tau, \frac{M}{\omega_n}))p'(\rho(\tau, \frac{M}{\omega_n}))b(\tau)^n \\
&= -\kappa\varepsilon\int_0^{\frac{M}{\omega_n}} \rho\rho_x r^{2(n-1)}(\Phi_\alpha * \rho)_x\,\mathrm{d}x \\
&= -\kappa\varepsilon\Big[\rho^2 r^{2(n-1)}(\Phi_\alpha * \rho)_x\Big]_0^{\frac{M}{\omega_n}} + \kappa\varepsilon\int_0^{\frac{M}{\omega_n}} \rho(\rho r^{2(n-1)}(\Phi_\alpha * \rho)_x)_x\,\mathrm{d}x.
\end{aligned} \tag{2.77}$$

Step 2. We now examine the last integral in (2.77). Applying the HLS inequality from Lemma A.0.2, we obtain the following estimate for $\alpha \in (-1, n-2]$,

$$\begin{aligned}
&\int_0^{\frac{M}{\omega_n}} \rho\big(\rho r^{2(n-1)}(\Phi_\alpha * \rho)_x\big)_x\,\mathrm{d}x \\
&= \frac{1}{\omega_n}\int_{\Omega_t} (\rho\Delta(\Phi_\alpha * \rho))(t, \boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \\
&\le \frac{1}{\omega_n}\|\rho\|_{L^{\frac{2n}{2n-(\alpha+2)}}(\Omega_t)}\|\Delta(\Phi_\alpha * \rho)\|_{L^{\frac{2n}{\alpha+2}}(\mathbb{R}^n)} \\
&= \frac{1}{\omega_n}\|\rho\|_{L^{\frac{2n}{2n-(\alpha+2)}}(\Omega_t)}\big\|\Delta\Phi_\alpha * \rho\big\|_{L^{\frac{2n}{\alpha+2}}(\mathbb{R}^n)} \\
&\le C\|\rho\|^2_{L^{\frac{2n}{2n-(\alpha+2)}}(\Omega_t)},
\end{aligned}$$

where the last line follows from the fact that

$$\Delta\Phi_\alpha = \begin{cases} \frac{(n-2)-\alpha}{|\cdot|^{\alpha+2}} & \text{for } \alpha \in (-1, n-2), \\ \omega_n \delta_0 & \text{for } \alpha = n-2, \end{cases}$$

in the sense of distributions, where δ_0 denotes the Dirac mass centred at 0.

Case 1 ($\kappa = -1$)**.** We now show that the last term on the right-hand side of (2.77) is non-positive, we obtain

$$\begin{aligned} \rho(\rho r^{2(n-1)}(\Phi_\alpha * \rho)_x)_x &= \Delta(\Phi_\alpha * \rho) \\ &= \begin{cases} (n-2-\alpha)(|\cdot|^{-(\alpha+2)} * \rho) & \text{for } \alpha \in (-1, n-2), \\ \omega_n \rho & \text{for } \alpha = n-2 \end{cases} \\ &\geq 0. \end{aligned} \tag{2.78}$$

Thus, we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\tau}\Big\{ \int_0^{\frac{M}{\omega_n}} \frac{1}{2}(u + \varepsilon r^{n-1}\rho_x)^2 \,\mathrm{d}x + \int_0^{\frac{M}{\omega_n}} e(\rho)\,\mathrm{d}x + \frac{\kappa}{2}\int_0^{\frac{M}{\omega_n}} \Phi_\alpha * \rho \,\mathrm{d}x \Big\} \\ &\quad + \varepsilon \int_0^{\frac{M}{\omega_n}} p'(\rho)\rho_x^2 \, r^{2n-2}\,\mathrm{d}x + \Big(\frac{1}{n} p(\rho(\tau, \frac{M}{\omega_n}))b(\tau)^n\Big)_\tau \\ &\quad + \frac{1}{n\varepsilon} p(\rho(\tau, \frac{M}{\omega_n}))p'(\rho(\tau, \frac{M}{\omega_n}))b(\tau)^n \\ &\leq -\kappa\varepsilon \Big[\rho^2 r^{2(n-1)}(\Phi_\alpha * \rho)_x\Big]_0^{\frac{M}{\omega_n}}. \end{aligned}$$

Case 2 ($\kappa = 1$ and $\gamma \geq \frac{2n}{2n-\alpha-2}$)**.** It follows that the last integral in (2.77) can be controlled by the internal energy term, namely,

$$\|\rho\|^2_{L^{\frac{2n}{2n-\alpha-2}}(\Omega_t)} \leq C\Big(\int_a^{b(t)} \rho\big(1 + e(\rho)\big) r^{n-1}\,\mathrm{d}r \Big)^{\frac{2n-\alpha-2}{n}}.$$

Case 3 ($\kappa = 1$ and $\gamma \in (\frac{2n}{2n-\alpha}, \frac{2n}{2n-\alpha-2})$)**.** Applying interpolation yields

$$\|\rho\|_{L^{\frac{2n}{2n-\alpha-2}}(\Omega_t)} = \|\rho\|^{\phi}_{L^{\frac{n\gamma}{n-2}}(\Omega_t)} \|\rho\|^{1-\phi}_{L^\gamma(\Omega_t)},$$

with

$$\phi = \frac{2n + (\alpha + 2 - 2n)\gamma}{4}.$$

This can be controlled as in [36].

Step 3. We now turn to the second-to-last term in (2.77), for which we obtain

$$\Big[\rho^2 r^{2(n-1)}(\Phi_\alpha * \rho)_x\Big]_0^{\frac{M}{\omega_n}} = \Big[\rho r^{n-1}(\Phi_\alpha * \rho)_r\Big]_a^{b(t)}.$$

From (2.78), we observe that the function $r \mapsto r^{n-1}(\Phi_\alpha * \rho)_r$ is increasing in r. Moreover,

$$\lim_{r \to 0^+} r^{n-1}(\Phi_\alpha * \rho)_r = 0.$$

Hence,

$$a^{n-1}\big(\rho(\Phi_\alpha * \rho)_r\big)(t,a) \geq 0,$$

since $\rho(t,a) > 0$. In the repulsive case $\kappa = -1$, this boundary term can be discarded, as it appears with a negative sign. In the Coulomb/Newtonian case $\alpha = n-2$, we obtain

$$(r^{n-1}(\Phi_{n-2} * \rho)_r)_r = r^{n-1}\Delta(\Phi_{n-2} * \rho) = \omega_n r^{n-1}\rho(t,r) = 0,$$

for all $r \in (0,a)$. Hence, the function $r \mapsto r^{n-1}(\Phi_{n-2} * \rho)_r$ is constant for $r \in (0,a)$. Since

$$\lim_{r\to 0^+} r^{n-1}(\Phi_{n-2} * \rho)_r = 0,$$

it follows that

$$a^{n-1}\big(\rho(\Phi_{n-2} * \rho)_r\big)(t,a) = 0.$$

Step 4. We now turn to the remaining boundary term in the case $\alpha \in (-1, n-2)$, in order to derive

$$\left| \int_{\mathbb{S}^{n-1}} \frac{b(s)\boldsymbol{e_1} - \eta \boldsymbol{y}}{|b(s)\boldsymbol{e_1} - \eta \boldsymbol{y}|^{\alpha+2}} \cdot \boldsymbol{e_1} \, \mathrm{d}S(\boldsymbol{y}) \right| \leq \omega_n |b(s) - \eta|^{-(\alpha+1)}. \tag{2.79}$$

Applying (2.79), (2.70), (2.33), and using $a = \frac{1}{b}$, we obtain that, for $\eta \in (a, b(s))$,

$$\begin{aligned} \left| \int_{\mathbb{S}^{n-1}} \frac{b(s)\boldsymbol{e_1} - \eta \boldsymbol{y}}{|b(s)\boldsymbol{e_1} - \eta \boldsymbol{y}|^{\alpha+2}} \cdot \boldsymbol{e_1} \, \mathrm{d}S(\boldsymbol{y}) \right| &\leq C \min\{|b(s) - \eta|^{-(\alpha+1)}, \, (\eta b(s))^{-\frac{\alpha+1}{2}}\} \\ &\leq C \min\{|b(s) - \eta|^{-(\alpha+1)}, \, 1\} \leq C. \end{aligned} \tag{2.80}$$

From [36, Lemma 3.4], it follows that, for $b \geq \max\{C_1(M, E_0, T, \varepsilon), \mathfrak{B}(\varepsilon)\}$, we have

$$\left| \int_0^t u(s, b(s)) \, \mathrm{d}s \right| \leq \frac{1}{4} b. \tag{2.81}$$

Using (2.14), (2.80), and (2.81), we obtain

$$\begin{aligned} &\left| \int_0^t (\rho r^{n-1}(\Phi_\alpha * \rho)_r)(s, b(s)) \, \mathrm{d}s \right| \\ &\leq \int_0^t \rho_0(b) \Big(\int_a^{b(s)} \left| \int_{\mathbb{S}^{n-1}} \frac{b(s)\boldsymbol{e_1} - \eta \boldsymbol{y}}{|b(s)\boldsymbol{e_1} - \eta \boldsymbol{y}|^{\alpha+2}} \cdot \boldsymbol{e_1} \, \mathrm{d}S(\boldsymbol{y}) \right| \rho(s,\eta)\eta^{n-1} \, \mathrm{d}\eta \Big) b(s)^{n-1} \, \mathrm{d}s \\ &\leq C \int_0^t \rho_0(b) \Big(\int_a^{b(s)} \rho(s,\eta)\eta^{n-1} \, \mathrm{d}\eta \Big) b(s)^{n-1} \, \mathrm{d}s \\ &\leq C(M) \int_0^t \rho_0(b) b(s)^{n-1} \, \mathrm{d}s \\ &= C(M) \int_0^t \rho_0(b) \Big(b + \int_0^s u(\tau, b(\tau)) \, \mathrm{d}\tau \Big)^{n-1} \, \mathrm{d}s \\ &\leq C(M) \int_0^t \rho_0(b) b^{n-1} \, \mathrm{d}s \\ &\leq C(M,T) b^{-(n-\beta)} b^{n-1} \\ &\leq C(M,T). \end{aligned}$$

Here we have used the asymptotic behaviour $\rho_0(b) \propto b^{-(n-\beta)}$, where

$$\beta = \min\left\{\frac{1}{2}, \left(1 - \frac{1}{\gamma}\right) n\right\}.$$

In the Coulomb/Newtonian case $\alpha = n-2$, the same bound can also be derived from [36]. Consequently, arguing as in [36], we conclude that, for the Coulomb case $\kappa = -1$ with $\alpha \in (-1, n-2]$, or for the Newtonian case $\kappa = 1$ with $\alpha = n-2$,

$$\begin{aligned}
&\frac{\varepsilon^2}{4}\int_a^{b(t)} \left|(\sqrt{\rho(t,r)})_r\right|^2 r^{n-1}\,\mathrm{d}r + \frac{4a_0\varepsilon}{\gamma}\int_0^t\int_a^{b(s)} \left|(\rho^{\frac{\gamma}{2}})_r\right|^2 r^{n-1}\,\mathrm{d}r\,\mathrm{d}s \\
&\quad + \frac{1}{n}p(\rho(t,b(t)))b(t)^n + \frac{1}{n\varepsilon}\int_0^t p(\rho(s,b(s)))p'(\rho(s,b(s)))b(s)^n\,\mathrm{d}s \\
&\leq C(E_0, M, T).
\end{aligned}$$

□

Remark 2.8.2. In Step 2, when $\kappa = 1$, it follows from (2.78) that the boundary contribution

$$\kappa\varepsilon a^{n-1}\big(\rho(\Phi_\alpha * \rho)_r\big)(t,a),$$

has an unfavourable sign. When $\alpha \in (-1, n-2)$, we have

$$\begin{aligned}
a^{n-1}(\Phi_\alpha * \rho)_r(t,a) &= \int_0^a (r^{n-1}(\Phi_\alpha * \rho)_r)_r\,\mathrm{d}r \\
&= \frac{(n-2)-\alpha}{\omega_n}\int_{B_a} (|\cdot|^{-(\alpha+2)} * \rho)(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} > 0.
\end{aligned} \tag{2.82}$$

For $\gamma < \frac{n}{(n-2)-\alpha}$, by the HLS inequality in Lemma A.0.2 together with (2.82), we obtain

$$a^{n-1}(\Phi_\alpha * \rho)_r(t,a) \leq C\|\rho\|_{L^\gamma(\Omega_t)} \leq C(E_0).$$

For $\gamma \geq \frac{n}{(n-2)-\alpha}$, combining the previous estimate with an interpolation argument, we obtain

$$a^{n-1}(\Phi_\alpha * \rho)_r(t,a) \leq C(M, E_0).$$

Therefore, in order to control the boundary contribution

$$\kappa\varepsilon a^{n-1}\big(\rho(\Phi_\alpha * \rho)_r\big)(t,a),$$

it is sufficient to obtain control of $\rho(t,a)$. However, no suitable *a priori* estimate for the density at the lower boundary $r = a$ is currently available.

We now establish the second BD-type estimate. This approach applies to both the repulsive and attractive cases $\kappa \in \{-1, 1\}$, and allows us to treat the more singular Riesz potentials $\alpha \in (n-2, n-1)$. However, the bound obtained via Grönwall's inequality is less precise than the estimate derived through integration by parts in the Coulomb/Newtonian case. Consequently, the admissible range of γ differs from that obtained previously. This limitation reflects the non-local nature of the Riesz potential, in contrast with the Coulomb/Newtonian case $\alpha = n-2$, where the potential operator is local.

Lemma 2.8.3 (BD-Type Entropy Estimate II)**.** *Under the conditions of* Lemma 2.6.1, *for* $\alpha \in (-1, n-1)$, $\gamma \geq \frac{3n}{3n-2(1+\alpha)}$, *any given* $T > 0$, $\varepsilon \in (0, \varepsilon_0]$, *and* $b \geq \max\{C_1(M, E_0, T, \varepsilon), \mathfrak{B}(\varepsilon)\}$ (*as defined in* Lemmas 2.4.8 and 2.7.1), *the following uniform estimate holds for* $t \in [0, T]$:

$$\begin{aligned}
&\frac{\varepsilon^2}{4}\int_a^{b(t)} \left|(\sqrt{\rho(t,r)})_r\right|^2 r^{n-1}\,\mathrm{d}r + \frac{4a_0\varepsilon}{\gamma}\int_0^t\int_a^{b(s)} \left|(\rho^{\frac{\gamma}{2}})_r\right|^2 r^{n-1}\,\mathrm{d}r\,\mathrm{d}s \\
&\quad + \frac{1}{n}p(\rho(t,b(t)))b(t)^n + \frac{1}{n\varepsilon}\int_0^t p(\rho(s,b(s)))p'(\rho(s,b(s)))b(s)^n\,\mathrm{d}s \\
&\leq C(E_0, M, T).
\end{aligned}$$

Proof. We make use of the computations (2.71)–(2.76) carried out in Lemma 2.8.1. Substituting (2.76) into (2.74), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\int_0^{\frac{M}{\omega_n}} \frac{1}{2}(u + \varepsilon r^{n-1}\rho_x)^2\,\mathrm{d}x + \int_0^{\frac{M}{\omega_n}} e(\rho)\,\mathrm{d}x\right\} + \varepsilon\int_0^{\frac{M}{\omega_n}} p'(\rho)\rho_x^2\, r^{2n-2}\,\mathrm{d}x \\
&\quad + \left(\frac{1}{n}p(\rho(\tau, \frac{M}{\omega_n}))b(\tau)^n\right)_\tau + \frac{1}{n\varepsilon}p(\rho(\tau, \frac{M}{\omega_n}))p'(\rho(\tau, \frac{M}{\omega_n}))b(\tau)^n \\
&= -\kappa\int_0^{\frac{M}{\omega_n}} (u + \varepsilon r^{n-1}\rho_x)\rho r^{n-1}(\Phi_\alpha * \rho)_x\,\mathrm{d}x. \qquad (2.83)
\end{aligned}$$

In the case $\gamma \geq \frac{3n}{3n-2(1+\alpha)}$, we first apply Hölder's inequality and then use the HLS inequality from Lemma A.0.2 to obtain

$$\begin{aligned}
&\int_a^{b(t)} |(\Phi_\alpha * \rho)_r(t,r)|^2\rho(t,r)r^{n-1}\,\mathrm{d}r \\
&\leq \frac{1}{\omega_n}\int_{\Omega_t} |\nabla(\Phi_\alpha * \rho)(t, \boldsymbol{y})|^2\rho(t, \boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \\
&\leq C\|\nabla(\Phi_\alpha * \rho)\|^2_{L^{\frac{3n}{1+\alpha}}(\Omega_t)}\|\rho\|_{L^{\frac{3n}{3n-2(1+\alpha)}}(\Omega_t)} \\
&\leq C\||\cdot|^{-(\alpha+1)} * \rho\|^2_{L^{\frac{3n}{1+\alpha}}(\Omega_t)}\|\rho\|_{L^{\frac{3n}{3n-2(1+\alpha)}}(\Omega_t)} \\
&\leq C\|\rho\|^3_{L^{\frac{3n}{3n-2(1+\alpha)}}(\Omega_t)} \\
&\leq C\|\rho\|^{3\phi}_{L^\gamma(\Omega_t)}\|\rho\|^{3(1-\phi)}_{L^1(\Omega_t)}
\end{aligned}$$

$$\leq C(M, E_0), \tag{2.84}$$

where

$$\phi = \frac{2\gamma(1+\alpha)}{3n(\gamma-1)}.$$

Applying the Cauchy–Schwarz inequality to (2.83) and using (2.84), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\Big\{\int_0^{\frac{M}{\omega_n}} \frac{1}{2}(u+\varepsilon r^{n-1}\rho_x)^2\,\mathrm{d}x + \int_0^{\frac{M}{\omega_n}} e(\rho)\,\mathrm{d}x\Big\} + \varepsilon\int_0^{\frac{M}{\omega_n}} p'(\rho)\rho_x^2\, r^{2n-2}\,\mathrm{d}x \\
&\quad + \Big(\frac{1}{n}p(\rho(\tau,\frac{M}{\omega_n}))b(\tau)^n\Big)_\tau + \frac{1}{n\varepsilon}p(\rho(\tau,\frac{M}{\omega_n}))p'(\rho(\tau,\frac{M}{\omega_n}))b(\tau)^n \\
&\leq C(M,E_0) + \int_0^{\frac{M}{\omega_n}} \frac{1}{2}\big(u+\varepsilon r^{n-1}\rho_x\big)^2\,\mathrm{d}x.
\end{aligned} \tag{2.85}$$

Applying a Grönwall-type inequality to (2.85), we obtain

$$\begin{aligned}
&\int_0^{\frac{M}{\omega_n}} \frac{1}{2}(u+\varepsilon r^{n-1}\rho_x)^2\,\mathrm{d}x \\
&\quad + \int_0^\tau e^{\tau-s}\Big\{\Big(\int_0^{\frac{M}{\omega_n}} e(\rho)\,\mathrm{d}x\Big)_s + \varepsilon\int_0^{\frac{M}{\omega_n}} p'(\rho)\rho_x^2\, r^{2n-2}\,\mathrm{d}x + \Big(\frac{1}{n}p(\rho(s,\frac{M}{\omega_n}))b(s)^n\Big)_s \\
&\qquad\qquad + \frac{1}{n\varepsilon}p(\rho(s,\frac{M}{\omega_n}))p'(\rho(s,\frac{M}{\omega_n}))b(s)^n\Big\}\,\mathrm{d}s \\
&\leq C(M,E_0)\int_0^\tau e^{\tau-s}\,\mathrm{d}s + e^\tau\int_0^{\frac{M}{\omega_n}} \frac{1}{2}(u_0+\varepsilon r^{n-1}\rho_{0,x})^2\,\mathrm{d}x.
\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}
&\int_0^{\frac{M}{\omega_n}} \frac{1}{2}(u+\varepsilon r^{n-1}\rho_x)^2\,\mathrm{d}x + \int_0^{\frac{M}{\omega_n}} e(\rho)\,\mathrm{d}x + \frac{1}{n}p(\rho(\tau,\frac{M}{\omega_n}))b(\tau)^n \\
&+ \int_0^\tau e^{\tau-s}\Big\{\int_0^{\frac{M}{\omega_n}} e(\rho)\,\mathrm{d}x + \varepsilon\int_0^{\frac{M}{\omega_n}} p'(\rho)\rho_x^2\, r^{2n-2}\,\mathrm{d}x + \frac{1}{n}p(\rho(s,\frac{M}{\omega_n}))b(s)^n \\
&\qquad\qquad + \frac{1}{n\varepsilon}p(\rho(s,\frac{M}{\omega_n}))p'(\rho(s,\frac{M}{\omega_n}))b(s)^n\Big\}\,\mathrm{d}s \\
&\leq e^\tau\Big\{\int_0^{\frac{M}{\omega_n}} \frac{1}{2}(u_0+\varepsilon r^{n-1}\rho_{0,x})^2\,\mathrm{d}x + \int_0^{\frac{M}{\omega_n}} e(\rho_0)\,\mathrm{d}x\Big\} \\
&\qquad\qquad + C(M,E_0)(e^\tau-1) + \frac{e^\tau}{n}p(\rho_0(\frac{M}{\omega_n}))b^n.
\end{aligned} \tag{2.86}$$

Using the non-negativity of the integrands appearing in the second line of (2.86), we

deduce the following inequality

$$\begin{aligned}
&\int_0^{\frac{M}{\omega_n}} \frac{1}{2}(u+\varepsilon r^{n-1}\rho_x)^2\,\mathrm{d}x + \int_0^{\frac{M}{\omega_n}} e(\rho)\,\mathrm{d}x + \frac{1}{n}p(\rho(\tau,\frac{M}{\omega_n}))b(\tau)^n \\
&\quad + \int_0^{\tau}\left\{\varepsilon\int_0^{\frac{M}{\omega_n}} p'(\rho)\rho_x^2 r^{2n-2}\,\mathrm{d}x + \frac{1}{n\varepsilon}p(\rho(s,\frac{M}{\omega_n}))p'(\rho(s,\frac{M}{\omega_n}))b(s)^n\right\}\mathrm{d}s \\
&\leq e^{\tau}\left\{\int_0^{\frac{M}{\omega_n}} \frac{1}{2}(u_0+\varepsilon r^{n-1}\rho_{0,x})^2\,\mathrm{d}x + \int_0^{\frac{M}{\omega_n}} e(\rho_0)\,\mathrm{d}x\right\} \\
&\qquad\qquad + C(M,E_0)(e^{\tau}-1) + \frac{e^{\tau}}{n}p(\rho_0(\frac{M}{\omega_n}))b^n.
\end{aligned}$$

□

2.9 Higher integrability

In order to apply the compensated compactness framework of Chen–Perepelitsa [37] to prove the convergence of weak solutions $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)$ of (2.8) to finite-energy weak solutions of the Cauchy problem (1.1) and (1.5) as $\varepsilon \to 0$, we require uniform higher-integrability estimates for both the density and the velocity. In the following lemma, we assume that $\gamma > \frac{1}{n-1-\alpha}$. This condition is needed in order to obtain a pointwise estimate for the derivative of the convolution between the kernel and the density.

Lemma 2.9.1 (Higher Integrability of the Density)**.** *Let (ρ, u) be the smooth solution of (2.8). Then, under the assumptions of* Theorem 2.1.2*, $T > 0$, $\varepsilon \in (0, \varepsilon_0]$, and $b \geq \max\{C_1(M, E_0, T, \varepsilon), \mathfrak{B}(\varepsilon)\}$ (as defined in* Lemmas 2.4.8 and 2.7.1*),*

$$\int_0^T \int_K \rho^{\gamma+1}(t,r)\,\mathrm{d}r\,\mathrm{d}t \leq C(K, M, E_0, T),$$

for any $K \Subset (a, b(t))$ for all $t \in [0, T]$.

Proof. Given $K \Subset (a, b(t))$ for any $t \in [0, T]$, there exist constants $a < d < D$ such that $K \Subset (d, D) \Subset (a, b(t))$. Let $w \in C_c^\infty(\mathbb{R})$ be such that $\mathrm{supp}(w) \subset (d, D)$ and $w(r) = 1$ for all $r \in K$. Multiplying $(2.8)_2$ by w, we obtain the equation

$$\begin{aligned}
&(\rho u w)_t + \big((\rho u^2 + p(\rho))w\big)_r + \frac{n-1}{r}\rho u^2 w + \kappa\rho(\Phi_\alpha * \rho)_r w \\
&= \varepsilon\Big(\rho\big(u_r + \frac{n-1}{r}u\big)w\Big)_r - \varepsilon\frac{n-1}{r}\rho_r u w + \Big(\rho u^2 + p(\rho) - \varepsilon\rho\big(u_r + \frac{n-1}{r}u\big)\Big)\Big)w_r.
\end{aligned} \tag{2.87}$$

We integrate (2.87) over the interval $[d, r)$ and then multiply the resulting expression by ρw to obtain

$$\rho p(\rho)w^2 = \varepsilon\rho^2\big(u_r + \frac{n-1}{r}u\big)w^2 - \varepsilon\rho w\int_d^r \frac{n-1}{z}u\rho_z w\,\mathrm{d}z - \Big(\rho w\int_d^r \rho u w\,\mathrm{d}z\Big)_t$$

$$
\begin{aligned}
&- \Big(\rho u w \int_d^r \rho u w \,\mathrm{d}z\Big)_r + \rho u w_r \int_d^r \rho u w \,\mathrm{d}z - \frac{n-1}{r}\rho u w \int_d^r \rho u w \,\mathrm{d}z \\
&- \rho w \int_d^r \frac{n-1}{z}\rho u^2 w \,\mathrm{d}z + \rho w \int_d^r \Big(\rho u^2 + p(\rho) - \varepsilon\rho(u_z + \frac{n-1}{z}u)\Big) w_z \,\mathrm{d}z \\
&- \kappa\rho w \int_d^r \rho(\Phi_\alpha * \rho)_z w \,\mathrm{d}z =: \sum_{i=1}^{9} I_i. \qquad (2.88)
\end{aligned}
$$

As in [36], we have

$$
\Big| \int_0^T \int_d^D \Big(\sum_{i=1}^{8} I_i\Big) \,\mathrm{d}r \,\mathrm{d}t \Big| \le C(d, M, E_0, T). \qquad (2.89)
$$

When $\alpha \in (n-2, n-1)$, applying (2.33) together with Fubini's Theorem, yields

$$
\begin{aligned}
&\Big| \int_0^T \int_d^D I_9 \,\mathrm{d}r \,\mathrm{d}t \Big| \\
&\le C \int_0^T \int_d^D (\rho w)(t,r) \int_d^r (\rho w)(t,z) \Big(\int_{(a,b(t))\cap(z-\frac{d}{2},z+\frac{d}{2})} \omega(z,\eta)\rho(t,\eta)\eta^{n-1} \,\mathrm{d}\eta \\
&\qquad + \int_{(a,b(t))\cap(z-\frac{d}{2},z+\frac{d}{2})^c} \omega(z,\eta)\rho(t,\eta)\eta^{n-1} \,\mathrm{d}\eta \Big) \,\mathrm{d}z \,\mathrm{d}r \,\mathrm{d}t \\
&\le C \int_0^T \int_d^D (\rho w)(t,r) \int_d^r (\rho w)(t,z) \Big(\int_{(a,b(t))\cap(z-\frac{d}{2},z+\frac{d}{2})} \frac{|z-\eta|^{n-2-\alpha}}{(z\eta)^{\frac{n-1}{2}}} \rho(t,\eta)\eta^{n-1} \,\mathrm{d}\eta \\
&\qquad + \int_{(a,b(t))\cap(z-\frac{d}{2},z+\frac{d}{2})^c} |z-\eta|^{-(\alpha+1)} \rho(t,\eta)\eta^{n-1} \,\mathrm{d}\eta \Big) \,\mathrm{d}z \,\mathrm{d}r \,\mathrm{d}t \\
&\le C(d) \int_0^T \int_d^D (\rho w)(t,r) \int_d^r (\rho w)(t,z) \\
&\qquad \times \int_a^{b(t)} \big(1 + |z-\eta|^{n-2-\alpha}\big) \rho(t,\eta)\eta^{n-1} \,\mathrm{d}\eta \,\mathrm{d}z \,\mathrm{d}r \,\mathrm{d}t. \\
&\le C(d,M,T) + C(d,D) \int_0^T \int_d^D (\rho w)(t,r) \Big(\int_a^{b(t)} \rho(t,\eta)\eta^{n-1} \\
&\qquad \times \int_d^D |z-\eta|^{n-2-\alpha} (\rho w)(t,z) \,\mathrm{d}z \,\mathrm{d}\eta \Big) \,\mathrm{d}r \,\mathrm{d}t. \qquad (2.90)
\end{aligned}
$$

For $\gamma > \frac{1}{n-1-\alpha}$, Lemma A.0.1 yields

$$
\begin{aligned}
&\Big| \int_0^T \int_d^D I_9 \,\mathrm{d}r \,\mathrm{d}t \Big| \\
&\le C(d,M,T) + C(d,D) \int_0^T \int_d^D (\rho w)(t,r) \Big(\int_a^{b(t)} \rho(t,\eta)\eta^{n-1} \\
&\qquad \times \sup_{s\in\mathbb{R}_+} \Big\{ \int_d^D |z-s|^{n-2-\alpha} (\rho w)(t,z) \,\mathrm{d}z \Big\} \,\mathrm{d}\eta \Big) \,\mathrm{d}r \,\mathrm{d}t \\
&= C(d,M,T) + C(d,D) \int_0^T \int_d^D (\rho w)(t,r) \Big(\int_a^{b(t)} \rho(t,\eta)\eta^{n-1}
\end{aligned}
$$

$$\times \sup_{s\in[d,D]}\Big\{\int_d^D |z-s|^{n-2-\alpha}(\rho w)(t,z)\,\mathrm{d}z\Big\}\,\mathrm{d}\eta\Big)\,\mathrm{d}r\,\mathrm{d}t$$
$$\le C(d,M,T)+C(d,D)\|\rho\|_{L^\gamma(d,D)}\int_0^T\int_d^D(\rho w)(t,r)\Big(\int_a^{b(t)}\rho(t,\eta)\eta^{n-1}\,\mathrm{d}\eta\Big)\,\mathrm{d}r\,\mathrm{d}t.$$
$$\le C(d,D,M,E_0,T). \tag{2.91}$$

When $\alpha\in(-1,n-2)$, a similar argument to that used in (2.90), together with (2.33), yields

$$\begin{aligned}
&\Big|\int_0^T\int_d^D I_9\,\mathrm{d}r\,\mathrm{d}t\Big|\\
&\le C\int_0^T\int_d^D(\rho w)(t,r)\int_d^r(\rho w)(t,z)\Big(\int_{(a,b(t))\cap(z-\frac{d}{2},z+\frac{d}{2})}\omega(z,\eta)\rho(t,\eta)\eta^{n-1}\,\mathrm{d}\eta\\
&\qquad+\int_{(a,b(t))\cap(z-\frac{d}{2},z+\frac{d}{2})^c}\omega(z,\eta)\rho(t,\eta)\eta^{n-1}\,\mathrm{d}\eta\Big)\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}t\\
&\le C(d,M,T)+C\int_0^T\int_d^D(\rho w)(t,r)\Big(\int_d^r(\rho w)(t,z)\\
&\qquad\times\int_{(a,b(t))\cap(z-\frac{d}{2},z+\frac{d}{2})}(z\eta)^{-\frac{\alpha+1}{2}}\rho(t,\eta)\eta^{n-1}\,\mathrm{d}\eta\,\mathrm{d}z\Big)\,\mathrm{d}r\,\mathrm{d}t\\
&\le C(d,M,T)+C(d)\int_0^T\int_d^D(\rho w)(t,r)\Big(\int_d^r(\rho w)(t,z)\int_a^{b(t)}\rho(t,\eta)\eta^{n-1}\,\mathrm{d}\eta\,\mathrm{d}z\Big)\,\mathrm{d}r\,\mathrm{d}t\\
&\le C(d,M,T).
\end{aligned}\tag{2.92}$$

When $\alpha=n-2$, applying (2.33) yields

$$\begin{aligned}
\Big|\int_0^T\int_d^D I_9\,\mathrm{d}r\,\mathrm{d}t\Big|&=\int_0^T\int_d^D(\rho w)(t,r)\Big(\int_d^r(\rho w)(t,z)\int_a^r\frac{\omega_n}{z^{n-1}}\rho(t,\eta)\eta^{n-1}\,\mathrm{d}\eta\,\mathrm{d}z\Big)\,\mathrm{d}r\,\mathrm{d}t\\
&\le C(d,M,T).
\end{aligned}\tag{2.93}$$

Therefore, integrating (2.88) over $[0,T]\times[d,D]$ and using (2.89)–(2.93), we conclude the desired result. □

To establish higher integrability for the velocity, we make use of the theory of entropy pairs for the one-dimensional polytropic Euler system as introduced in §1.2. As shown in [100], any weak entropy pair (η,q) can be represented by

$$\begin{cases}
\eta^\psi(\rho,m)=\eta^\psi(\rho,\rho u)=\displaystyle\int_{\mathbb{R}}\chi(\rho;s-u)\psi(s)\,\mathrm{d}s,\\
q^\psi(\rho,m)=q^\psi(\rho,\rho u)=\displaystyle\int_{\mathbb{R}}(\theta s+(1-\theta)u)\chi(\rho;s-u)\psi(s)\,\mathrm{d}s,
\end{cases}\tag{2.94}$$

where

$$\chi(\rho;s-u)=\big(\rho^{2\theta}-(s-u)^2\big)_+^\lambda,$$

with

$$\lambda = \frac{3-\gamma}{2(\gamma-1)}, \qquad \theta = \frac{\gamma-1}{2},$$

and $(\cdot)_+ = \max\{\cdot, 0\}$. Choosing $\psi(s) = \frac{1}{2}s|s|$ as in [36], we obtain the corresponding entropy pair $(\eta^\#(\rho,m), q^\#(\rho,m))$, which can be represented as

$$\begin{cases} \eta^\#(\rho,m) = \eta^\#(\rho,\rho u) = \dfrac{1}{2}\rho\displaystyle\int_{-1}^{1}(u+\rho^\theta s)|u+\rho^\theta s|(1-s^2)_+^\lambda \,\mathrm{d}s, \\ q^\#(\rho,m) = q^\#(\rho,\rho u) = \dfrac{1}{2}\rho\displaystyle\int_{-1}^{1}(u+\theta\rho^\theta s)(u+\rho^\theta s)|u+\rho^\theta s|(1-s^2)_+^\lambda \,\mathrm{d}s. \end{cases} \tag{2.95}$$

Using (2.95), one can show that there exists a constant $C_\gamma > 0$ such that the following inequalities hold

$$\begin{aligned} &|\eta^\#(\rho,\rho u)| \le C_\gamma(\rho|u|^2+\rho^\gamma), \quad && q^\#(\rho,\rho u) \ge C_\gamma^{-1}(\rho|u|^3+\rho^{\gamma+\theta}), \\ &|\eta_m^\#(\rho,\rho u)| \le C_\gamma(|u|+\rho^\theta), && |\eta_\rho^\#(\rho,\rho u)| \le C_\gamma(|u|^2+\rho^{2\theta}). \end{aligned} \tag{2.96}$$

As in [35, 37], we obtain

$$\rho u\partial_\rho\eta^\# + \rho u^2\partial_m\eta^\# - q^\# = \frac{\theta}{2}\rho^{1+\theta}\int_{-1}^{1}(u-\rho^\theta s)s|u+\rho^\theta s|\,(1-s^2)_+^\lambda \,\mathrm{d}s \le 0. \tag{2.97}$$

Lemma 2.9.2 (Higher Integrability of the Velocity)**.** *Let (ρ,u) be the smooth solution of (2.8). Then, under the assumption of* Lemma 2.9.1*,*

$$\int_0^T\int_d^D \big(\rho|u|^3+\rho^{\gamma+\theta}\big)(t,r)\,\mathrm{d}r\,\mathrm{d}t \le C(d,D,M,E_0,T),$$

for any $(d,D) \Subset [a,b(t)]$ for all $t \in [0,T]$.

Proof. First, multiplying $(2.8)_1$ by $r^{n-1}\eta_\rho^\#$ and $(2.8)_2$ by $r^{n-1}\eta_m^\#$, we obtain

$$\begin{aligned} &(\eta^\# r^{n-1})_t + (q^\# r^{n-1})_r + (n-1)\big(-q^\# + \rho u\eta_\rho^\# + \rho u^2\eta_m^\#\big)r^{n-2} \\ &= \varepsilon\eta_m^\#\big((\rho u_r)_r + (n-1)\rho\big(\frac{u}{r}\big)_r\big)r^{n-1} - \kappa\eta_m^\#\rho(\Phi_\alpha * \rho)_r r^{n-1}. \end{aligned} \tag{2.98}$$

Integrating (2.98) over $[r,b(t))$ and using (2.9) and (2.97), we obtain

$$\begin{aligned} q^\#(t,r)r^{n-1} \le{}& -\varepsilon\int_r^{b(t)}\eta_m^\#(t,z)(\rho u_z)_z\,z^{n-1}\,\mathrm{d}z - (n-1)\varepsilon\int_r^{b(t)}\eta_m^\#(t,z)\left(\frac{u}{z}\right)_z\rho\,z^{n-1}\,\mathrm{d}z \\ &+ \Big(\int_r^{b(t)}\eta^\#(t,z)\,z^{n-1}\,\mathrm{d}z\Big)_t + \big(q^\# - u\eta^\#\big)(t,b(t))b(t)^{n-1} \\ &+ \kappa\int_r^{b(t)}\eta_m^\#(t,z)\rho(\Phi_\alpha*\rho)_z z^{n-1}\,\mathrm{d}z =: \sum_{i=1}^5 I_i. \end{aligned} \tag{2.99}$$

Now, as in [36, Lemma 3.7], we obtain

$$\Big|\int_0^T\int_d^D\Big(\sum_{i=1}^4 I_i\Big)\,\mathrm{d}r\,\mathrm{d}t\Big| \le C(d,D,M,E_0,T).$$

To estimate the potential term I_5, we use (2.96), Young's inequality, and the estimates in (2.33). Applying the same approach as in (2.90) and (2.91)–(2.92), we obtain

$$\begin{aligned}
&\left| \int_0^T \int_d^D \int_r^{b(t)} \kappa \eta_m^{\#} \rho (\Phi_\alpha * \rho)_z z^{n-1} \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}t \right| \\
&\leq C(d, D, M, E_0, T) \int_0^T \int_d^D \int_r^{b(t)} \big((|u| + \rho^\theta) \rho \big)(t, z) z^{n-1} \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}t \\
&\leq C(d, D, M, E_0, T) \int_0^T \int_d^D \int_r^{b(t)} \big(\rho |u| + \rho^{\frac{\gamma}{2}} \rho^{\frac{1}{2}} \big)(t, z) z^{n-1} \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}t \\
&\leq C(d, D, M, E_0, T) \int_0^T \int_d^D \int_r^{b(t)} \big(\rho u^2 + \rho + \rho^\gamma \big)(t, z) z^{n-1} \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}t \\
&\leq C(d, D, M, E_0, T).
\end{aligned} \tag{2.100}$$

Combining (2.96) with (2.99)–(2.100), we obtain the desired result. □

2.10 Global solutions to the compressible Navier-Stokes-Riesz equations

We now establish the convergence of the solutions $(\rho^{\varepsilon,b}, \rho^{\varepsilon,b} u^{\varepsilon,b})$ of the free boundary problem (2.8)–(2.11) to global weak solutions $(\rho^\varepsilon, m^\varepsilon)$ of the CNSREs system (2.8) as $b \to \infty$. In the limit, cavitation (vacuum states) may occur and must therefore be handled carefully. For this purpose, we extend $(\rho^{\varepsilon,b}, u^{\varepsilon,b})$ by zero on $([0,T] \times (0,\infty)) \setminus \Omega^T$.

Lemma 2.10.1. *Under the assumptions of* Theorem 2.1.2, *for fixed* $\varepsilon \in (0, \varepsilon_0]$ *(as defined in* Lemma 2.4.8*), there exists a non-negative function* $\rho^\varepsilon(t,r)$ *a.e. such that, as* $b \to \infty$ *(up to a subsequence),*

$$(\sqrt{\rho^{\varepsilon,b}}, \rho^{\varepsilon,b}) \longrightarrow (\sqrt{\rho^\varepsilon}, \rho^\varepsilon), \qquad \text{a.e. and strongly in } C(0,T; L^q_{\mathrm{loc}}), \tag{2.101}$$

for any $q \in [1, \infty)$, *where* L^q_{loc} *denotes* $L^q(K)$ *for* $K \Subset (0,\infty)$.

The proof is the same as for [36, Lemma 4.1].

Corollary 2.10.2. *Under the assumptions of* Lemma 2.10.1, *the pressure function sequence* $p(\rho^{\varepsilon,b})$ *is uniformly bounded in* $L^\infty(0,T; L^q_{\mathrm{loc}})$ *for all* $q \in [1,\infty]$ *and, as* $b \to \infty$ *(up to a subsequence),*

$$p(\rho^{\varepsilon,b}) \longrightarrow p(\rho^\varepsilon), \qquad \text{strongly in } L^q(0,T; L^q_{\mathrm{loc}}) \text{ for all } q \in [1,\infty).$$

Lemma 2.10.3. *Under the assumptions of* Lemma 2.10.1, *as* $b \to \infty$ (*up to a subsequence*), *the momentum function sequence* $m^{\varepsilon,b} := \rho^{\varepsilon,b}u^{\varepsilon,b}$ *converges strongly in* $L^2(0,T;L^q_{\mathrm{loc}})$ *to some function* $m^\varepsilon(t,r)$ *for all* $q \in [1,\infty)$, *which particularly implies that*

$$m^{\varepsilon,b} = \rho^{\varepsilon,b}u^{\varepsilon,b} \longrightarrow m^\varepsilon(t,r), \qquad a.e.\ in\ [0,T]\times(0,\infty).$$

Proof. From Lemmas 2.6.1, 2.8.1, and 2.8.3, together with the Sobolev embedding Theorem, we observe that $\sqrt{\rho^{\varepsilon,b}}$ is uniformly bounded in

$$L^\infty(0,T;H^1_{\mathrm{loc}}) \hookrightarrow L^\infty(0,T;L^\infty_{\mathrm{loc}}),$$

for $b>0$. Consequently,

$$\rho^{\varepsilon,b}u^{\varepsilon,b} = \sqrt{\rho^{\varepsilon,b}}\big(\sqrt{\rho^{\varepsilon,b}}u^{\varepsilon,b}\big),$$

is uniformly bounded in $L^\infty(0,T;L^2_{\mathrm{loc}})$ for $b>0$. Since

$$\big(\rho^{\varepsilon,b}u^{\varepsilon,b}\big)_r = \rho^{\varepsilon,b}_r u^{\varepsilon,b} + \rho^{\varepsilon,b}u^{\varepsilon,b}_r = 2\big(\sqrt{\rho^{\varepsilon,b}}\big)_r\big(\sqrt{\rho^{\varepsilon,b}}u^{\varepsilon,b}\big) + \sqrt{\rho^{\varepsilon,b}}\big(\sqrt{\rho^{\varepsilon,b}}u^{\varepsilon,b}_r\big),$$

it follows from Lemmas 2.6.1, 2.8.1, and 2.8.3 that $(\rho^{\varepsilon,b}u^{\varepsilon,b})_r$ is uniformly bounded in $L^2(0,T;L^1_{\mathrm{loc}})$ for $b>0$. Consequently,

$$\rho^{\varepsilon,b}u^{\varepsilon,b} \in L^2(0,T;W^{1,1}_{\mathrm{loc}}), \qquad \text{uniformly for } b>0. \tag{2.102}$$

From Lemma 2.6.1 and the HLS inequality in Lemma A.0.2, we obtain

$$\begin{cases} \partial_r\big((\sqrt{\rho^{\varepsilon,b}}u^{\varepsilon,b})^2\big) \in L^\infty(0,T;W^{-1,1}_{\mathrm{loc}}),\\ \dfrac{n-1}{r}\big(\sqrt{\rho^{\varepsilon,b}}u^{\varepsilon,b}\big)^2 \in L^\infty(0,T;L^1_{\mathrm{loc}}),\\ \partial_r p(\rho^{\varepsilon,b}) \in L^2(0,T;H^{-1}_{\mathrm{loc}}),\\ \rho^{\varepsilon,b}\big(\Phi_\alpha * \rho^{\varepsilon,b}\big)_r \in L^\infty(0,T;L^\infty_{\mathrm{loc}}), \end{cases} \tag{2.103}$$

and

$$\sqrt{\rho^{\varepsilon,b}}\left(\sqrt{\rho^{\varepsilon,b}}u^{\varepsilon,b}_r\right) + \frac{n-1}{r}\sqrt{\rho^{\varepsilon,b}}\left(\sqrt{\rho^{\varepsilon,b}}u^{\varepsilon,b}\right) \in L^2(0,T;L^2_{\mathrm{loc}}), \tag{2.104}$$

uniformly for $b>0$. Hence, it follows from (2.104) that

$$\partial_r\big(\rho^{\varepsilon,b}(u^{\varepsilon,b}_r + \frac{n-1}{r}u^{\varepsilon,b})\big) \in L^2(0,T;H^{-1}_{\mathrm{loc}}), \qquad \text{uniformly for } b>0. \tag{2.105}$$

Using Lemmas 2.6.1, 2.8.1, and 2.8.3, we obtain

$$\frac{n-1}{r}u^{\varepsilon,b}\partial_r\rho^{\varepsilon,b} = \frac{2(n-1)}{r}(\sqrt{\rho^{\varepsilon,b}})_r\left(\sqrt{\rho^{\varepsilon,b}}u^{\varepsilon,b}\right) \in L^2(0,T;L^1_{\mathrm{loc}}), \quad \text{uniformly for } b>0. \tag{2.106}$$

Then, substituting (2.103), (2.105) and (2.106) into $(2.8)_2$, and using the Sobolev embedding Theorem, we see that

$$\partial_t(\rho^{\varepsilon,b}u^{\varepsilon,b}) \in L^2(0,T;W^{-1,q}_{\mathrm{loc}}), \qquad \text{uniformly for } b>0 \text{ and } q\in[1,\tfrac{n}{n-1}).$$

Using the Aubin–Lions Lemma together with (2.102), we deduce that $\rho^{\varepsilon,b}u^{\varepsilon,b}$ is compact in $L^2(0,T;L^q_{\mathrm{loc}})$ for all $q\in[1,\infty)$. □

Lemma 2.10.4. *Under the assumptions of* Lemma 2.10.1, *the limit function* $m^\varepsilon(t,r)$ *in* Lemma 2.10.3 *satisfies that* $m^\varepsilon(t,r)=0$ *a.e. on* $\{(t,r)\,|\,\rho^\varepsilon(t,r)=0\}$. *Furthermore, there exists a function* $u^\varepsilon(t,r)$ *such that* $m^\varepsilon(t,r)=\rho^\varepsilon(t,r)u^\varepsilon(t,r)$ *a.e., and* $u^\varepsilon(t,r)=0$ *a.e. on* $\{(t,r)\,|\,\rho^\varepsilon(t,r)=0\}$. *Moreover, as* $b\to\infty$ *(up to a subsequence),*

$$\begin{aligned} &m^{\varepsilon,b}\longrightarrow m^\varepsilon=\rho^\varepsilon u^\varepsilon, \qquad \text{strongly in } L^2(0,T;L^q_{\mathrm{loc}}) \text{ for } q\in[1,\infty),\\ &\frac{m^{\varepsilon,b}}{\sqrt{\rho^{\varepsilon,b}}}\longrightarrow\sqrt{\rho^\varepsilon}u^\varepsilon=:\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}, \qquad \text{strongly in } L^2(0,T;L^2_{\mathrm{loc}}).\end{aligned}$$

The proof of Lemma 2.10.4 is the same as [36, Lemma 4.4].

Lemma 2.10.5. *Under the assumptions of* Lemma 2.10.1, *let* $0\le t_1<t_2\le T$, *and let* $\zeta(t,\boldsymbol{x})\in C^1_{\mathrm{c}}([0,T]\times\mathbb{R}^n)$ *be any smooth function with compact support. Then*

$$\int_{\mathbb{R}^n}\rho^\varepsilon(t_2,\boldsymbol{x})\zeta(t_2,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=\int_{\mathbb{R}^n}\rho^\varepsilon(t_1,\boldsymbol{x})\zeta(t_1,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}+\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\left(\rho^\varepsilon\zeta_t+\boldsymbol{\mathcal{M}}^\varepsilon\cdot\nabla\zeta\right)\mathrm{d}\boldsymbol{x}\,\mathrm{d}t. \tag{2.107}$$

Moreover, the total mass is conserved:

$$\int_{\mathbb{R}^n}\rho^\varepsilon(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=\int_{\mathbb{R}^n}\rho_0^\varepsilon(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=M, \qquad \text{for } t\ge0. \tag{2.108}$$

The proof of Lemma 2.10.5 is the same as [36, Lemma 4.9].

We now establish an *a priori* estimate for the energy terms.

Lemma 2.10.6. *Under the assumptions of* Lemma 2.10.1, *the following estimates hold:*

(i) *When* $\alpha\in(0,n-1)$,

$$\begin{aligned}&\int_0^\infty\Big(\frac{1}{2}\Big|\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}\Big|^2+\rho^\varepsilon e(\rho^\varepsilon)\Big)(t,r)\,r^{n-1}\,\mathrm{d}r+\varepsilon(n-1)\int_0^t\int_0^\infty\Big|\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}\Big|^2r^{n-3}\,\mathrm{d}r\,\mathrm{d}s\\&\le C(M,E_0).\end{aligned}$$

(ii) *When* $\alpha\in(-1,0]$,

$$\begin{aligned}&\int_0^\infty\Big(\frac{1}{2}\Big|\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}\Big|^2+\rho^\varepsilon\big(e(\rho^\varepsilon)+k_\alpha(1+r^2)\big)\Big)(t,r)\,r^{n-1}\,\mathrm{d}r\\&\qquad+\varepsilon(n-1)\int_0^t\int_0^\infty\Big|\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}\Big|^2r^{n-3}\,\mathrm{d}r\,\mathrm{d}s\\&\le C(M,E_0,T).\end{aligned}$$

Proof. It follows directly from Lemma 2.6.1 that, for $\alpha \in (0, n-1)$,

$$\int_a^{b(t)} \Big(\frac{1}{2}\Big|\frac{m^{\varepsilon,b}}{\sqrt{\rho^{\varepsilon,b}}}\Big|^2 + \rho^{\varepsilon,b} e(\rho^{\varepsilon,b})\Big)(t,r)\, r^{n-1}\,\mathrm{d}r + \varepsilon(n-1)\int_0^t \int_a^{b(s)} \Big|\frac{m^{\varepsilon,b}}{\sqrt{\rho^{\varepsilon,b}}}\Big|^2 r^{n-3}\,\mathrm{d}r\,\mathrm{d}s$$
$$\leq C(n, M, E_0, T).$$

Now, applying Fatou's Lemma together with Lemmas 2.7.1, 2.10.1, and 2.10.4, we obtain the desired result for $\alpha \in (0, n-1)$. The argument for $\alpha \in (-1, 0]$ follows in the same way. □

Note that the estimates obtained so far do not involve the potential term, since the convergence of the potential has not yet been established. To address this, several additional steps are required. Since the total mass is conserved in the limit and the local convergence of the density has already been verified, neither concentration of mass at the origin nor loss of mass at infinity can occur. This observation enables us to establish certain global convergence results.

Lemma 2.10.7. *Under the assumptions of* Lemma 2.10.1, *as* $b \to \infty$ (*up to a subsequence*),

$$\rho^{\varepsilon,b} \longrightarrow \rho^{\varepsilon}, \qquad \textit{strongly in } C(0,T; L^q(\mathbb{R}^n)) \textit{ for any } q \in [1, \gamma).$$

That is, the convergence of $\rho^{\varepsilon,b}$ *is global sub-sequentially in* L^q.

Proof. For all $\alpha \in (-1, n-1)$, it follows from Lemma 2.10.1 and (2.108) that

$$\lim_{k\to\infty} \inf_{t\in[0,T]} \int_{\frac{1}{k}}^{k} \rho^{\varepsilon}(t,r) r^{n-1}\,\mathrm{d}r = \frac{M}{\omega_n}, \tag{2.109}$$

otherwise, there exist $\delta > 0$ and a sequence $t_k \in [0,T]$ such that

$$\int_{\frac{1}{k}}^{k} \rho^{\varepsilon}(t_k, r) r^{n-1}\,\mathrm{d}r < \frac{M}{\omega_n} - \delta, \tag{2.110}$$

for all $k \in \mathbb{N}$. Then there exists a subsequence $(t_{k_l})_l$ of $(t_k)_k$ and a point $t_* \in [0,T]$ such that $t_{k_l} \to t_*$ as $l \to \infty$. From (2.108), by the monotone convergence Theorem, there exists $L \in \mathbb{N}$ such that

$$\int_{\frac{1}{L}}^{L} \rho^{\varepsilon}(t_*, r) r^{n-1}\,\mathrm{d}r > \frac{M}{\omega_n} - \delta.$$

Note that for $k_l \geq L$, using (2.110), we have

$$\int_{\frac{1}{L}}^{L} \rho^{\varepsilon}(t_{k_l}, r) r^{n-1}\,\mathrm{d}r \leq \int_{\frac{1}{k_l}}^{k_l} \rho^{\varepsilon}(t_{k_l}, r) r^{n-1}\,\mathrm{d}r < \frac{M}{\omega_n} - \delta. \tag{2.111}$$

By Lemma 2.10.1, we have $\rho^\varepsilon \in C(0,T;L^1_{\rm loc})$, and hence

$$\int_{\frac{1}{L}}^{L} \rho^\varepsilon(t_{k_l},r) r^{n-1}\,\mathrm{d}r \longrightarrow \int_{\frac{1}{L}}^{L} \rho^\varepsilon(t_*,r) r^{n-1}\,\mathrm{d}r, \qquad \text{as } l\to\infty. \tag{2.112}$$

Thus, combining (2.111) with (2.112), we obtain

$$\int_{\frac{1}{L}}^{L} \rho^\varepsilon(t_*,r) r^{n-1}\,\mathrm{d}r \le \frac{M}{\omega_n} - \delta,$$

which is a contradiction. Hence, (2.109) holds. Consequently, there exists $k = k(\delta) \in \mathbb{N}$ such that

$$\inf_{t\in[0,T]} \int_{\frac{1}{k}}^{k} \rho^\varepsilon(t,r) r^{n-1}\,\mathrm{d}r > \frac{M}{\omega_n} - \frac{\delta}{4}, \tag{2.113}$$

which, by (2.108) and (2.113), implies

$$\sup_{t\in[0,T]} \Big(\int_{k}^{\infty} + \int_{0}^{\frac{1}{k}} \Big) \rho^\varepsilon(t,r) r^{n-1}\,\mathrm{d}r < \frac{\delta}{4}. \tag{2.114}$$

By (2.101), there exists $B(\delta) > 0$ such that, for $b \ge B(\delta)$ (up to a subsequence),

$$\sup_{t\in[0,T]} \int_{\frac{1}{k}}^{k} |\rho^\varepsilon - \rho^{\varepsilon,b}|(t,r) r^{n-1}\,\mathrm{d}r < \frac{\delta}{4}. \tag{2.115}$$

Then it follows from (2.113)–(2.115) that, for $b \ge B(\delta)$,

$$\begin{aligned} &\inf_{t\in[0,T]} \int_{\frac{1}{k}}^{k} \rho^{\varepsilon,b}(t,r) r^{n-1}\,\mathrm{d}r > \frac{M}{\omega_n} - \frac{\delta}{2}, \\ &\sup_{t\in[0,T]} \Big(\int_{k}^{\infty} + \int_{0}^{\frac{1}{k}} \Big) \rho^{\varepsilon,b}(t,r) r^{n-1}\,\mathrm{d}r < \frac{\delta}{2}. \end{aligned} \tag{2.116}$$

Thus, we obtain that, for $b > B(\delta)$,

$$\sup_{t\in[0,T]} \int_{0}^{\infty} |\rho^\varepsilon - \rho^{\varepsilon,b}|(t,r) r^{n-1}\,\mathrm{d}r < \delta.$$

Now, for $q \in [1,\gamma)$, using Lemmas 2.6.1 and 2.10.6, and applying interpolation, we obtain

$$\begin{aligned} &\|\rho^\varepsilon - \rho^{\varepsilon,b}\|_{C(0,T;L^q(\mathbb{R}^n))} \\ &\le \|\rho^\varepsilon - \rho^{\varepsilon,b}\|^{\frac{\gamma-q}{q(\gamma-1)}}_{C(0,T;L^1(\mathbb{R}^n))} \|\rho^\varepsilon - \rho^{\varepsilon,b}\|^{\frac{\gamma(q-1)}{q(\gamma-1)}}_{L^\infty(0,T;L^\gamma(\mathbb{R}^n))} \\ &\le C(n,M,E_0,T,q) \|\rho^\varepsilon - \rho^{\varepsilon,b}\|^{\frac{\gamma-q}{q(\gamma-1)}}_{C(0,T;L^1(\mathbb{R}^n))} \longrightarrow 0, \quad \text{as } b\to\infty \text{ (up to a subsequence).} \end{aligned}$$

□

Thus, with the strong global convergence of the density, we can now prove convergence of the potential term.

Lemma 2.10.8. *Under the assumptions of* Lemma 2.10.1*, the following statements hold as* $b \to \infty$ *(up to a subsequence)*:

$$\Phi_\alpha * \rho^{\varepsilon,b} \rightharpoonup \Phi_\alpha * \rho^\varepsilon, \quad \text{weak-}* \text{ in } L^\infty(0,T; W^{1,\frac{2n}{\alpha+2}}_{\mathrm{loc}}(\mathbb{R}^n)), \\ \text{and weakly in } L^2(0,T; W^{1,\frac{2n}{\alpha+2}}_{\mathrm{loc}}(\mathbb{R}^n)), \tag{2.117}$$

$$(\Phi_\alpha * \rho^{\varepsilon,b})_r(t,r) r^{n-1} \longrightarrow (\Phi_\alpha * \rho^\varepsilon)_r(t,r) r^{n-1}, \quad \text{in } C_{\mathrm{loc}}([0,T]\times[0,\infty)), \tag{2.118}$$

$$(\Phi_\alpha * \rho^{\varepsilon,b})(t,r) r^{n-1} \longrightarrow (\Phi_\alpha * \rho^\varepsilon)(t,r) r^{n-1}, \quad \text{in } C_{\mathrm{loc}}([0,T]\times[0,\infty)), \tag{2.119}$$

$$\int_0^\infty \left|\rho^{\varepsilon,b}(\Phi_\alpha * \rho^{\varepsilon,b}) - \rho^\varepsilon(\Phi_\alpha * \rho^\varepsilon)\right|(t,r)\, r^{n-1}\,\mathrm{d}r \longrightarrow 0. \tag{2.120}$$

Moreover, when $\alpha \in (0, n-1)$,

$$\|(\Phi_\alpha * \rho^\varepsilon)(t)\|_{L^{\frac{2n}{\alpha}}(\mathbb{R}^n)} + \|(\nabla\Phi_\alpha * \rho^\varepsilon)(t)\|_{L^{\frac{2n}{\alpha+2}}(\mathbb{R}^n)} \le C(M, E_0), \qquad \text{for } t \ge 0, \tag{2.121}$$

when $\alpha \in (-1, 0]$, *for any* $K \Subset \mathbb{R}^n$,

$$\|(\Phi_\alpha * \rho^\varepsilon)(t)\|_{L^\infty(K)} + \|(\nabla\Phi_\alpha * \rho^\varepsilon)(t)\|_{L^{\frac{2n}{\alpha+2}}(K)} \le C(M, E_0, T, K), \qquad \text{for } t \ge 0. \tag{2.122}$$

Proof. We divide the proof into five steps.

Step 1. We first observe that (2.117) and (2.121)–(2.122) follow directly from Corollary 2.6.3 together with weak-$*$ and weak compactness.

Step 2. To show that the weak limit of $\Phi_\alpha * \rho^{\varepsilon,b}$, denoted by Φ^ε_α, is indeed $\Phi_\alpha * \rho^\varepsilon$, we test against a smooth compactly supported function $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$. By shifting the convolution from $\rho^{\varepsilon,b}$ to φ, it can be shown that, as $b \to \infty$,

$$\int_{\mathbb{R}^n} \varphi(\Phi_\alpha * \rho^{\varepsilon,b})\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^n} \rho^{\varepsilon,b}(\Phi_\alpha * \varphi)\,\mathrm{d}\boldsymbol{x} \longrightarrow \int_{\mathbb{R}^n} \rho^{\varepsilon}(\Phi_\alpha * \varphi)\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^n} \varphi(\Phi_\alpha * \rho^{\varepsilon})\,\mathrm{d}\boldsymbol{x}.$$

Using (2.117), we obtain

$$\int_{\mathbb{R}^n} \varphi(\Phi_\alpha * \rho^{\varepsilon,b})\,\mathrm{d}\boldsymbol{x} \longrightarrow \int_{\mathbb{R}^n} \varphi\,\Phi^\varepsilon_\alpha\,\mathrm{d}\boldsymbol{x}.$$

Then, by the fundamental Theorem of the calculus of variations, it follows that $\Phi^\varepsilon_\alpha = \Phi_\alpha * \rho^\varepsilon$ almost everywhere. The result follows.

Step 3. We now prove (2.118). When $\alpha \in (-1, n-1)$, for $D > 0$, $(t, r) \in [0, T] \times [0, D]$, and b sufficiently large, and for $0 < \sigma < \hat{\sigma} < \infty$, we have

$$\begin{aligned}
&\big|(\Phi_\alpha * \rho^{\varepsilon,b})_r r^{n-1} - (\Phi_\alpha * \rho^{\varepsilon})_r r^{n-1}\big|(t, r) \\
&= r^{n-1}\Big| \int_0^\infty \omega(r, \eta)(\rho^{\varepsilon,b} - \rho^{\varepsilon})(t, \eta)\eta^{n-1}\, \mathrm{d}\eta\Big| \\
&\le r^{n-1}\Big| \int_0^\sigma \omega(r, \eta)(\rho^{\varepsilon,b} - \rho^{\varepsilon})(t, \eta)\eta^{n-1}\, \mathrm{d}\eta\Big| + r^{n-1}\Big| \int_\sigma^{\hat{\sigma}} \omega(r, \eta)(\rho^{\varepsilon,b} - \rho^{\varepsilon})(t, \eta)\eta^{n-1}\, \mathrm{d}\eta\Big| \\
&\quad + r^{n-1}\Big| \int_{\hat{\sigma}}^\infty \omega(r, \eta)(\rho^{\varepsilon,b} - \rho^{\varepsilon})(t, \eta)\eta^{n-1}\, \mathrm{d}\eta\Big| \\
&=: I_1 + I_2 + I_3. \qquad (2.123)
\end{aligned}$$

For I_1, applying Hölder's inequality, we obtain

$$\begin{aligned}
I_1 &\le r^{n-1}\Big(\int_0^\sigma \big((\rho^{\varepsilon,b})^\gamma + (\rho^b)^\gamma\big)\eta^{n-1}\, \mathrm{d}\eta\Big)\Big| \int_0^\sigma \omega^{\frac{\gamma}{\gamma-1}}(r, \eta)\eta^{n-1}\, \mathrm{d}\eta\Big|^{1-\frac{1}{\gamma}} \\
&\le C(M, E_0)r^{n-1}\Big| \int_0^\sigma \omega^{\frac{\gamma}{\gamma-1}}(r, \eta)\eta^{n-1}\, \mathrm{d}\eta\Big|^{1-\frac{1}{\gamma}}. \qquad (2.124)
\end{aligned}$$

When $\alpha \in (n-2, n-1)$, using (2.33), we obtain

$$\begin{aligned}
&r^{n-1}\Big| \int_0^\sigma \omega^{\frac{\gamma}{\gamma-1}}(r, \eta)\eta^{n-1}\, \mathrm{d}\eta\Big|^{1-\frac{1}{\gamma}} \\
&\le Cr^{n-1}\sigma^{(n-1)(1-\frac{1}{\gamma})}\Big(\int_{(0,\sigma)\cap(\frac{r}{2},\infty)} \omega^{\frac{\gamma}{\gamma-1}}(r, \eta)\, \mathrm{d}\eta + \int_{(0,\sigma)\cap(0,\frac{r}{2})} \omega^{\frac{\gamma}{\gamma-1}}(r, \eta)\, \mathrm{d}\eta\Big)^{1-\frac{1}{\gamma}} \\
&\le Cr^{n-1}\sigma^{(n-1)(1-\frac{1}{\gamma})}\Big(\int_{(0,\sigma)\cap(\frac{r}{2},\infty)} \Big(\frac{|r-\eta|^{n-2-\alpha}}{(r\eta)^{\frac{n-1}{2}}}\Big)^{\frac{\gamma}{\gamma-1}}\, \mathrm{d}\eta \\
&\qquad\qquad + \int_{(0,\sigma)\cap(0,\frac{r}{2})} |r-\eta|^{-(\alpha+1)\frac{\gamma}{\gamma-1}}\, \mathrm{d}\eta\Big)^{1-\frac{1}{\gamma}} \\
&\le Cr^{n-1}\sigma^{(n-1)(1-\frac{1}{\gamma})}\Big(r^{-(n-1)\frac{\gamma}{\gamma-1}} \int_{(0,\sigma)\cap(\frac{r}{2},\infty)} |r-\eta|^{(n-2-\alpha)\frac{\gamma}{\gamma-1}}\, \mathrm{d}\eta + r^{1-(\alpha+1)\frac{\gamma}{\gamma-1}}\Big)^{1-\frac{1}{\gamma}} \\
&\le C\sigma^{(n-1)(1-\frac{1}{\gamma})} + Cr^{n-1-\alpha-\frac{1}{\gamma}}\sigma^{(n-1)(1-\frac{1}{\gamma})} \\
&\le C(D)\sigma^{(n-1)(1-\frac{1}{\gamma})}, \qquad (2.125)
\end{aligned}$$

where the last inequality follows from the fact that $\gamma > \frac{1}{n-1-\alpha}$, and the penultimate inequality follows from the existence of a constant $C > 0$ such that, for sufficiently small $\sigma > 0$,

$$\int_{(0,\sigma)\cap(\frac{r}{2},\infty)} |r-\eta|^{(n-2-\alpha)\frac{\gamma}{\gamma-1}}\, \mathrm{d}\eta \le C,$$

uniformly in r, since $(n-2-\alpha)\frac{\gamma}{\gamma-1} > -1$. When $\alpha \in (-1, n-2]$, applying (2.33) and following a similar approach to that of (2.125), for sufficiently small σ, we obtain that

$$r^{n-1}\Big| \int_0^\sigma \omega^{\frac{\gamma}{\gamma-1}}(r, \eta)\eta^{n-1}\, \mathrm{d}\eta\Big|^{1-\frac{1}{\gamma}} \le C(D)\sigma^{n(1-\frac{1}{\gamma})} \le C(D)\sigma^{(n-1)(1-\frac{1}{\gamma})}. \qquad (2.126)$$

Thus, combining (2.124)–(2.126), we obtain

$$\sup_{[0,T]\times[0,D]} I_1 \le C(D,M,E_0)\sigma^{(n-1)(1-\frac{1}{\gamma})}\big(1+\sigma^{1-\frac{1}{\gamma}}\big) \longrightarrow 0, \qquad \text{as } \sigma \to 0. \tag{2.127}$$

Considering I_2, for $\alpha \in (-1, n-2]$, we use (2.33) to obtain

$$\begin{aligned} I_2 &\le Cr^{n-1}\int_\sigma^{\hat\sigma} (r\eta)^{-\frac{\alpha+1}{2}}|\rho^{\varepsilon,b}-\rho^\varepsilon|(t,\eta)\eta^{n-1}\,\mathrm{d}\eta \\ &\le C(\hat\sigma)r^{n-2-\alpha}\sigma^{-\frac{\alpha+1}{2}}\int_\sigma^{\hat\sigma}|\rho^{\varepsilon,b}-\rho^\varepsilon|(t,\eta)\,\mathrm{d}\eta \\ &\le C(D,\hat\sigma,\sigma)\|\rho^{\varepsilon,b}-\rho^\varepsilon\|_{L^1(\sigma,\hat\sigma)}. \end{aligned} \tag{2.128}$$

For $\alpha \in (n-2, n-1)$, using the HLS inequality from Lemma A.0.1 and (2.33) yields

$$\begin{aligned} I_2 &\le Cr^{n-1}\int_\sigma^{\hat\sigma}\frac{|r-\eta|^{n-2-\alpha}}{(r\eta)^{\frac{n-1}{2}}}|\rho^{\varepsilon,b}-\rho^\varepsilon|(t,\eta)\eta^{n-1}\,\mathrm{d}\eta \\ &\le C(\hat\sigma)\Big(\frac{r}{\sigma}\Big)^{\frac{n-1}{2}}\big\||\cdot|^{n-2-\alpha}*((\rho^{\varepsilon,b}-\rho^\varepsilon)\mathbb{1}_{(\sigma,\hat\sigma)})\big\|_{C[0,D]} \\ &\le C(D,\hat\sigma,\sigma)\|\rho^{\varepsilon,b}-\rho^\varepsilon\|_{L^\gamma(\sigma,\hat\sigma)}. \end{aligned} \tag{2.129}$$

Combining (2.128) and (2.129), and using Lemma 2.10.1, we obtain

$$\sup_{[0,T]\times[0,D]} I_2 \le C(D,\hat\sigma,\sigma)\sup_{[0,T]}\big(\|\rho^{\varepsilon,b}-\rho^\varepsilon\|_{L^1(\sigma,\hat\sigma)}+\|\rho^{\varepsilon,b}-\rho^\varepsilon\|_{L^\gamma(\sigma,\hat\sigma)}\big) \longrightarrow 0, \tag{2.130}$$

as $b\to\infty$. Considering I_3, since $\omega(r,\eta)\le C|r-\eta|^{-(\alpha+1)}$, we obtain

$$\begin{aligned} \sup_{[0,T]\times[0,D]} I_3 &\le C|D-\hat\sigma|^{-(\alpha+1)}\sup_{[0,T]}\int_{\hat\sigma}^\infty |\rho^{\varepsilon,b}-\rho^\varepsilon|(t,\eta)\eta^{n-1}\,\mathrm{d}\eta \\ &\le C(M)|D-\hat\sigma|^{-(\alpha+1)} \longrightarrow 0, \qquad \text{as } \hat\sigma\to\infty. \end{aligned} \tag{2.131}$$

Thus, combining (2.123) and (2.127) with (2.130)–(2.131), we conclude

$$\sup_{[0,T]\times[0,D]} |(\Phi_\alpha*\rho^{\varepsilon,b})_r r^{n-1}-(\Phi_\alpha*\rho^\varepsilon)_r r^{n-1}|(t,r) \longrightarrow 0, \qquad \text{as } b\to\infty.$$

Thus, (2.118) is proved.

Step 4. For $\alpha\in(0,n-1)$, employing the estimate for $K(r,\eta)$ in (2.32), the proof of (2.119) follows verbatim from the proof of (2.118) corresponding to the case $\alpha\in(0,n-2)$.

Let $\alpha\in(-1,0]$, $\hat\sigma>1$ and $\varphi\in C_c^\infty(\mathbb{R}^n)$ be a standard radially symmetric mollifier, with $\varphi_{\hat\sigma}$ defined in (2.19). Using (2.8) and integrating by parts, we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_a^{b(t)}\rho^{\varepsilon,b}(\mathbb{1}_{B^c_{2\hat\sigma}}*\varphi_{\hat\sigma})k_\alpha(1+r^2)r^{n-1}\,\mathrm{d}r \\ &=\int_a^{b(t)}\rho^{\varepsilon,b}u^{\varepsilon,b}\left((\mathbb{1}_{B^c_{2\hat\sigma}}*\varphi_{\hat\sigma})(k_\alpha(1+r^2))_r+(\mathbb{1}_{B^c_{2\hat\sigma}}*\varphi_{\hat\sigma})_r k_\alpha(1+r^2)\right)r^{n-1}\,\mathrm{d}r. \end{aligned}$$

Integrating over time, we obtain

$$\begin{aligned}
&\int_a^{b(t)} \rho^{\varepsilon,b}(\mathbb{1}_{B^c_{2\hat\sigma}} * \varphi_{\hat\sigma}) k_\alpha(1+r^2) r^{n-1}\,\mathrm{d}r \\
&= \int_a^{b} \rho_0^{\varepsilon,b}(\mathbb{1}_{B^c_{2\hat\sigma}} * \varphi_{\hat\sigma}) k_\alpha(1+r^2) r^{n-1}\,\mathrm{d}r \\
&\quad + \int_0^t \int_a^{b(s)} \left((\mathbb{1}_{B^c_{2\hat\sigma}} * \varphi_{\hat\sigma})(k_\alpha(1+r^2))_r + (\mathbb{1}_{B^c_{2\hat\sigma}} * \varphi_{\hat\sigma})_r k_\alpha(1+r^2)\right) \rho^{\varepsilon,b} u^{\varepsilon,b} r^{n-1}\,\mathrm{d}r\,\mathrm{d}s.
\end{aligned} \tag{2.132}$$

As in (2.68), for $\boldsymbol{x} \in K = \overline{B_D}$ and $q \in (1,\gamma)$, using (2.67) and (2.69), we obtain

$$\begin{aligned}
&\left|(\Phi_\alpha * \rho^{\varepsilon,b})(\boldsymbol{x}) - (\Phi_\alpha * \rho^{\varepsilon})(\boldsymbol{x})\right| \\
&\le \int_{B_{3\hat\sigma}(\boldsymbol{x})} k_\alpha(|\boldsymbol{x}-\boldsymbol{y}|)|\rho^{\varepsilon,b} - \rho^{\varepsilon}|(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} + \int_{B_{3\hat\sigma}(\boldsymbol{x})^c} k_\alpha(|\boldsymbol{x}-\boldsymbol{y}|)|\rho^{\varepsilon,b} - \rho^{\varepsilon}|(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \\
&\le \|k_\alpha(|\cdot|)\|_{L^{\frac{q}{q-1}}(B_{3\hat\sigma})} \|\rho^{\varepsilon,b} - \rho^{\varepsilon}\|_{L^q(B_{3\hat\sigma}(\boldsymbol{x}))} \\
&\quad + C\int_{B_{3\hat\sigma}(\boldsymbol{x})^c} \left(k_\alpha(1+D^2) + k_\alpha(1+|\boldsymbol{y}|^2)\right)|\rho^{\varepsilon,b} - \rho^{\varepsilon}|(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \\
&\le C(\hat\sigma)\|\rho^{\varepsilon,b} - \rho^{\varepsilon}\|_{L^q(\mathbb{R}^n)} + C(D)\|\rho^{\varepsilon,b} - \rho^{\varepsilon}\|_{L^1(\mathbb{R}^n)} \\
&\quad + C\int_{B_{3\hat\sigma}(\boldsymbol{x})^c} k_\alpha(1+|\boldsymbol{y}|^2)\rho^{\varepsilon}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} + C\int_{B_{3\hat\sigma}(\boldsymbol{x})^c} k_\alpha(1+|\boldsymbol{y}|^2)\rho^{\varepsilon,b}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

By Lemma 2.10.7, for fixed $\hat\sigma$, we obtain

$$\sup_{[0,T]\times[0,D]} |I_1 + I_2| \longrightarrow 0, \qquad \text{as } b \to \infty. \tag{2.133}$$

For I_3, by Lemma 2.10.6 and the Lebesgue dominated convergence Theorem, we obtain

$$\sup_{[0,T]\times[0,D]} |I_3| \longrightarrow 0, \qquad \text{as } \hat\sigma \to \infty, \text{ uniformly in } b.$$

For I_4, by (2.132), we obtain

$$\begin{aligned}
|I_4| &= C\int_{3\hat\sigma}^{b(t)} k_\alpha(1+r^2)\rho^{\varepsilon,b}(r) r^{n-1}\,\mathrm{d}r \\
&\le C\int_0^{b(t)} \rho^{\varepsilon,b}(\mathbb{1}_{B^c_{2\hat\sigma}} * \varphi_{\hat\sigma}) k_\alpha(1+r^2) r^{n-1}\,\mathrm{d}r \\
&= C\int_0^{b(t)} \rho_0^{\varepsilon,b}(\mathbb{1}_{B^c_{2\hat\sigma}} * \varphi_{\hat\sigma}) k_\alpha(1+r^2) r^{n-1}\,\mathrm{d}r \\
&\quad + C\int_0^t \int_0^{b(s)} \left((\mathbb{1}_{B^c_{2\hat\sigma}} * \varphi_{\hat\sigma})(k_\alpha(1+r^2))_r + (\mathbb{1}_{B^c_{2\hat\sigma}} * \varphi_{\hat\sigma})_r k_\alpha(1+r^2)\right) \rho^{\varepsilon,b} u^{\varepsilon,b} r^{n-1}\,\mathrm{d}r\,\mathrm{d}s \\
&\le C\int_{\hat\sigma}^{\beta(t)} \rho_0^{\varepsilon,b} k_\alpha(1+r^2) r^{n-1}\,\mathrm{d}r + \frac{C}{\hat\sigma^{\alpha+1}} \int_0^t \int_{\hat\sigma}^{b(s)} \rho^{\varepsilon,b} u^{\varepsilon,b} r^{n-1}\,\mathrm{d}r\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+\frac{Ck_\alpha(1+\hat\sigma^2)}{\hat\sigma}\int_0^t\int_{\hat\sigma}^{3\hat\sigma}\rho^{\varepsilon,b}u^{\varepsilon,b}r^{n-1}\,\mathrm{d}r\,\mathrm{d}s\\
&\le C\int_{\hat\sigma}^{\infty}\rho_0^{\varepsilon,b}k_\alpha(1+r^2)r^{n-1}\,\mathrm{d}r+\frac{Ck_\alpha(1+\hat\sigma^2)}{\hat\sigma}\int_0^t\int_a^{b(s)}\rho^{\varepsilon,b}u^{\varepsilon,b}r^{n-1}\,\mathrm{d}r\,\mathrm{d}s\\
&=: I_{4,1}+I_{4,2}.
\end{aligned}
$$

By Lemma 2.4.4 and the Lebesgue dominated convergence Theorem, we obtain

$$
\sup_{[0,T]\times[0,D]}|I_{4,1}|\longrightarrow 0,\qquad \text{as } \hat\sigma\to\infty, \text{ uniformly in } b. \tag{2.134}
$$

By Lemma 2.6.1 and the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
\sup_{[0,T]\times[0,D]}|I_{4,2}|&\le\frac{Ck_\alpha(1+\hat\sigma^2)}{\hat\sigma}\int_0^t\int_a^{b(s)}\big(\rho^{\varepsilon,b}+\rho^{\varepsilon,b}(u^{\varepsilon,b})^2\big)r^{n-1}\,\mathrm{d}r\,\mathrm{d}s\\
&\le\frac{C(M,E_0,T)k_\alpha(1+\hat\sigma^2)}{\hat\sigma}\longrightarrow 0,\qquad \text{as } \hat\sigma\to\infty, \text{ uniformly in } b.
\end{aligned} \tag{2.135}
$$

Thus, combining (2.133)–(2.135), we obtain that, for $\alpha\in(-1,0]$,

$$
(\Phi_\alpha*\rho^{\varepsilon,b})(t,r)r^{n-1}\longrightarrow(\Phi_\alpha*\rho^{\varepsilon})(t,r)r^{n-1},\qquad \text{in } C_{\mathrm{loc}}([0,T]\times[0,\infty)) \text{ as } b\to\infty.
$$

Step 5. For (2.120), it follows from Lemma 2.10.7 and (2.119) that, when $\alpha\in(-1,n-1)$ and $\hat\sigma>0$,

$$
\int_0^{\hat\sigma}\big|\rho^{\varepsilon,b}(\Phi_\alpha*\rho^{\varepsilon,b})-\rho^{\varepsilon}(\Phi_\alpha*\rho^{\varepsilon})\big|(t,r)\,r^{n-1}\,\mathrm{d}r\longrightarrow 0,\qquad \text{as } b\to\infty. \tag{2.136}
$$

When $\alpha\in(0,n-1)$, using (2.32), we have

$$
\begin{aligned}
&\int_{\hat\sigma}^{\infty}\big|\rho^{\varepsilon,b}(\Phi_\alpha*\rho^{\varepsilon,b})\big|(t,r)\,r^{n-1}\,\mathrm{d}r\\
&\le C\int_{\hat\sigma}^{\infty}\rho^{\varepsilon,b}(t,r)\Big(\Big(\int_0^{r-1}+\int_{r+1}^{\infty}\Big)\rho^{\varepsilon,b}(t,\eta)\eta^{n-1}\,\mathrm{d}\eta\\
&\qquad\qquad+\int_{r-1}^{r+1}\frac{1}{(r\eta)^{\frac{\alpha}{2}}}\rho^{\varepsilon,b}(t,\eta)\eta^{n-1}\,\mathrm{d}\eta\Big)r^{n-1}\,\mathrm{d}r\\
&\le C\int_{\hat\sigma}^{\infty}\rho^{\varepsilon,b}(t,r)\Big(\Big(\int_0^{r-1}+\int_{r+1}^{\infty}\Big)\rho^{\varepsilon,b}(t,\eta)\eta^{n-1}\,\mathrm{d}\eta\\
&\qquad\qquad+\frac{1}{(\hat\sigma-1)^{\alpha}}\int_{r-1}^{r+1}\rho^{\varepsilon,b}(t,\eta)\eta^{n-1}\,\mathrm{d}\eta\Big)r^{n-1}\,\mathrm{d}r\\
&\le C(M)\int_{\hat\sigma}^{\infty}\rho^{\varepsilon,b}(t,r)r^{n-1}\,\mathrm{d}r.
\end{aligned} \tag{2.137}
$$

We may also replace $\rho^{\varepsilon,b}$ by ρ^{ε} in (2.137). Hence, for $\delta>0$, using (2.114), (2.116), and (2.137), we obtain that, for b and $\hat\sigma$ sufficiently large,

$$
\int_{\hat\sigma}^{\infty}\big|\rho^{\varepsilon,b}(\Phi_\alpha*\rho^{\varepsilon,b})-\rho^{\varepsilon}(\Phi_\alpha*\rho^{\varepsilon})\big|(t,r)\,r^{n-1}\,\mathrm{d}r\le C(M)\delta. \tag{2.138}
$$

When $\alpha \in (-1, 0]$, using (2.52), (2.54), and Lemma 2.10.6, we obtain

$$
\begin{aligned}
&\int_{\hat{\sigma}}^{\infty} |\rho^{\varepsilon,b}(\Phi_\alpha * \rho^{\varepsilon,b})|(t,r)\, r^{n-1}\,\mathrm{d}r \\
&= \frac{1}{\omega_n}\Big| \int_{B^c_{\hat{\sigma}}} \int_{\mathbb{R}^n} \rho^{\varepsilon,b}(\boldsymbol{x}) k_\alpha(|\boldsymbol{x}-\boldsymbol{y}|)\rho^{\varepsilon,b}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\Big| \\
&\le \frac{1}{\omega_n}\int_{B^c_{\hat{\sigma}}} \rho^{\varepsilon,b}(\boldsymbol{y})\Big(\int_{B_1(\boldsymbol{y})} \rho^{\varepsilon,b}(\boldsymbol{x})|k_\alpha(|\boldsymbol{x}-\boldsymbol{y}|)|\,\mathrm{d}\boldsymbol{x} + \int_{B_1(\boldsymbol{y})^c} \rho^{\varepsilon,b}(\boldsymbol{x}) k_\alpha(|\boldsymbol{x}-\boldsymbol{y}|)\,\mathrm{d}\boldsymbol{x}\Big)\,\mathrm{d}\boldsymbol{y} \\
&\le \frac{1}{\omega_n}\int_{B^c_{\hat{\sigma}}} \rho^{\varepsilon,b}(\boldsymbol{y})\Big(\|\rho^{\varepsilon,b}\|_{L^\gamma(B_1(\boldsymbol{y}))} \|k_\alpha(|\cdot|)\|_{L^{\frac{\gamma}{\gamma-1}}(B_1)} \\
&\qquad\qquad + C\int_{|\boldsymbol{x}-\boldsymbol{y}|\ge 1} \rho^{\varepsilon,b}(\boldsymbol{x})\left(1 + k_\alpha(1+|\boldsymbol{x}|^2) + k_\alpha(1+|\boldsymbol{y}|^2)\right)\,\mathrm{d}\boldsymbol{x}\Big)\,\mathrm{d}\boldsymbol{y} \\
&\le C(M, E_0, T)\int_{B^c_{\hat{\sigma}}} \rho^{\varepsilon,b}(\boldsymbol{x})\left(1 + k_\alpha(1+|\boldsymbol{x}|^2)\right)\,\mathrm{d}\boldsymbol{x} \\
&\le C(M, E_0, T)\Big(\|\rho^{\varepsilon,b} - \rho^{\varepsilon,b}\|_{L^1(B^c_{\hat{\sigma}})} + \int_{B^c_{\hat{\sigma}}} \rho^{\varepsilon}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_{B^c_{\hat{\sigma}}} \rho^{\varepsilon,b}(\boldsymbol{x}) k_\alpha(1+|\boldsymbol{x}|^2)\,\mathrm{d}\boldsymbol{x}\Big).
\end{aligned}
$$

Notice that, by (2.134)–(2.135), we obtain

$$
\sup_{[0,T]}\Big| \int_{B^c_{\hat{\sigma}}} \rho^{\varepsilon,b}(\boldsymbol{x}) k_\alpha(1+|\boldsymbol{x}|^2)\,\mathrm{d}\boldsymbol{x}\Big| \longrightarrow 0, \qquad \text{as } b\to\infty, \text{ uniformly in } b. \tag{2.139}
$$

Thus, for $\delta > 0$, by Lemmas 2.10.6 and 2.10.7, the Lebesgue dominated convergence Theorem, and (2.139), we obtain that, for b and $\hat{\sigma}$ sufficiently large,

$$
\int_{\hat{\sigma}}^{\infty} \left|\rho^{\varepsilon,b}(\Phi_\alpha * \rho^{\varepsilon,b}) - \rho^{\varepsilon}(\Phi_\alpha * \rho^{\varepsilon})\right|(t,r)\, r^{n-1}\,\mathrm{d}r \le C(M, E_0, T)\delta. \tag{2.140}
$$

Therefore, combining (2.136) and (2.138) with (2.140) leads to (2.120) for $\alpha \in (-1, n-1)$.

□

We now have the tools to prove the decay of the energy for the constructed global solution of the CNSREs.

Lemma 2.10.9. *Under the assumptions of* Lemma 2.10.1, *the following holds*:

$$
\begin{aligned}
&\int_0^{\infty} \Big(\frac{1}{2}\Big|\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}\Big|^2 + \rho^\varepsilon e(\rho^\varepsilon) + \frac{\kappa}{2}\rho^\varepsilon(\Phi_\alpha * \rho^\varepsilon)\Big)(t,r)\, r^{n-1}\,\mathrm{d}r \\
&\le \int_0^{\infty} \Big(\frac{1}{2}\Big|\frac{m_0^\varepsilon}{\sqrt{\rho_0^\varepsilon}}\Big|^2 + \rho_0^\varepsilon e(\rho_0^\varepsilon) + \frac{\kappa}{2}\rho_0^\varepsilon(\Phi_\alpha * \rho_0^\varepsilon)\Big)(r)\, r^{n-1}\,\mathrm{d}r.
\end{aligned}
$$

Proof. From Lemma 2.6.1, we obtain

$$
\int_0^{b(t)} \Big(\frac{1}{2}\Big|\frac{m^{\varepsilon,b}}{\sqrt{\rho^{\varepsilon,b}}}\Big|^2 + \rho^{\varepsilon,b} e(\rho^{\varepsilon,b}) + \frac{\kappa}{2}\rho^{\varepsilon,b}\big(\Phi_\alpha * \rho^{\varepsilon,b}\big)\Big)(t,r)\, r^{n-1}\,\mathrm{d}r
$$

$$\leq \int_0^b \Big(\frac{1}{2}\Big|\frac{m_0^{\varepsilon,b}}{\sqrt{\rho_0^{\varepsilon,b}}}\Big|^2 + \rho_0^{\varepsilon,b} e(\rho_0^{\varepsilon,b}) + \frac{\kappa}{2}\rho_0^{\varepsilon,b}\big(\Phi_\alpha * \rho_0^{\varepsilon,b}\big)\Big)(r)\, r^{n-1}\,\mathrm{d}r.$$

Using Lemmas 2.4.4 and 2.4.8, which establishes convergence results for the constructed approximation $(\rho_0^\varepsilon, m_0^\varepsilon)$, together with Lemma 2.7.1, we obtain

$$\begin{aligned}
&\int_0^b \Big(\frac{1}{2}\Big|\frac{m_0^{\varepsilon,b}}{\sqrt{\rho_0^{\varepsilon,b}}}\Big|^2 + \rho_0^{\varepsilon,b} e(\rho_0^{\varepsilon,b}) + \frac{\kappa}{2}\rho_0^{\varepsilon,b}(\Phi_\alpha * \rho_0^{\varepsilon,b})\Big)(t,r)\, r^{n-1}\,\mathrm{d}r \\
&\longrightarrow \int_0^\infty \Big(\frac{1}{2}\Big|\frac{m_0^{\varepsilon}}{\sqrt{\rho_0^{\varepsilon}}}\Big|^2 + \rho_0^{\varepsilon} e(\rho_0^{\varepsilon}) + \frac{\kappa}{2}\rho_0^{\varepsilon}(\Phi_\alpha * \rho_0^{\varepsilon})\Big)(r)\, r^{n-1}\,\mathrm{d}r, \qquad \text{as } b \to \infty.
\end{aligned}$$

By Fatou's Lemma, Lemma 2.7.1, and (2.120), we obtain

$$\begin{aligned}
&\int_0^\infty \Big(\frac{1}{2}\Big|\frac{m^{\varepsilon}}{\sqrt{\rho^{\varepsilon}}}\Big|^2 + \rho^{\varepsilon} e(\rho^{\varepsilon}) + \frac{\kappa}{2}\rho^{\varepsilon}(\Phi_\alpha * \rho^{\varepsilon})\Big)(t,r)\, r^{n-1}\,\mathrm{d}r \\
&\leq \liminf_{b\to\infty} \int_0^{b(t)} \Big(\frac{1}{2}\Big|\frac{m^{\varepsilon,b}}{\sqrt{\rho^{\varepsilon,b}}}\Big|^2 + \rho^{\varepsilon,b} e(\rho^{\varepsilon,b}) + \frac{\kappa}{2}\rho^{\varepsilon,b}\big(\Phi_\alpha * \rho^{\varepsilon,b}\big)\Big)(t,r)\, r^{n-1}\,\mathrm{d}r.
\end{aligned}$$

This completes the proof. □

Thus, we now have enough information to pass the limit in the momentum equation, to obtain existence of global weak solutions of the CNSREs.

Lemma 2.10.10. *Under the assumptions of* Lemma 2.10.1, *let* $\boldsymbol{\psi}(t,\boldsymbol{x}) \in (C_c^2([0,T]\times\mathbb{R}^n))^n$ *be any smooth function with compact support so that* $\boldsymbol{\psi}(T,\boldsymbol{x}) = 0$. *Then*

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+} \Big\{\boldsymbol{\mathcal{M}}^\varepsilon \cdot \partial_t\boldsymbol{\psi} + \frac{\boldsymbol{\mathcal{M}}^\varepsilon}{\sqrt{\rho^\varepsilon}}\cdot\Big(\frac{\boldsymbol{\mathcal{M}}^\varepsilon}{\sqrt{\rho^\varepsilon}}\cdot\nabla\Big)\boldsymbol{\psi} + p(\rho^\varepsilon)\nabla\cdot\boldsymbol{\psi} - \kappa\rho^\varepsilon\nabla(\Phi_\alpha * \rho^\varepsilon)\cdot\boldsymbol{\psi}\Big\}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&\quad + \int_{\mathbb{R}^n} \boldsymbol{\mathcal{M}}_0^\varepsilon\cdot\boldsymbol{\psi}(0,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
&= -\varepsilon\int_{\mathbb{R}^{n+1}_+}\Big\{\frac{1}{2}\boldsymbol{\mathcal{M}}^\varepsilon\cdot\big(\Delta\boldsymbol{\psi} + \nabla(\nabla\cdot\boldsymbol{\psi})\big) + \frac{\boldsymbol{\mathcal{M}}^\varepsilon}{\sqrt{\rho^\varepsilon}}\cdot\big(\nabla\sqrt{\rho^\varepsilon}\cdot\nabla\big)\boldsymbol{\psi} \\
&\qquad\qquad + \nabla\sqrt{\rho^\varepsilon}\cdot\Big(\frac{\boldsymbol{\mathcal{M}}^\varepsilon}{\sqrt{\rho^\varepsilon}}\cdot\nabla\Big)\boldsymbol{\psi}\Big\}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&= \sqrt{\varepsilon}\int_{\mathbb{R}^{n+1}_+}\sqrt{\rho^\varepsilon}\Big\{V^\varepsilon\frac{\boldsymbol{x}\otimes\boldsymbol{x}}{r^2} + \frac{\sqrt{\varepsilon}}{r}\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}}\big(I_{n\times n} - \frac{\boldsymbol{x}\otimes\boldsymbol{x}}{r^2}\big)\Big\} : \nabla\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,
\end{aligned}$$

where $V^\varepsilon(t,r) \in L^2(0,T;L^2(\mathbb{R}^n))$ *is a function such that*

$$\int_0^T\int_{\mathbb{R}^n} |V^\varepsilon(t,\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \leq C(E_0,M),$$

for some $C(E_0,M) > 0$ *independent of* $T > 0$.

The proof of this lemma is the same as the argument for that of [36].

2.11 Compensated compactness methods for the vanishing viscosity

Having established the existence of global weak solutions to the CNSREs system, we now employ a compensated compactness argument to pass the vanishing viscosity limit and thereby obtain solutions to the CEREs system. To this end, we first require the following compactness result for general weak entropy pairs.

Lemma 2.11.1 (H^{-1}_{loc}–Compactness). *Under the assumptions of* Lemma 2.10.1, *let* (η, q) *be a weak entropy pair defined in* (2.94) *for any compactly supported smooth function* $\psi(s)$ *in* $\mathbb{R}$. *Then*

$$\partial_t \eta(\rho^\varepsilon, m^\varepsilon) + \partial_r q(\rho^\varepsilon, m^\varepsilon), \qquad \text{is compact in } H^{-1}_{\mathrm{loc}}(\mathbb{R}^2_+).$$

Proof. We multiply $(2.8)_1$ by $\eta^{\varepsilon,b}_\rho := \eta_\rho(\rho^{\varepsilon,b}, m^{\varepsilon,b})$ and $(2.8)_2$ by $\eta^{\varepsilon,b}_m := \eta_m(\rho^{\varepsilon,b}, m^{\varepsilon,b})$. Then, by the construction of the entropy pair, we obtain

$$\begin{aligned}\partial_t \eta^{\varepsilon,b} + \partial_r q^{\varepsilon,b} =& -\frac{n-1}{r} m^{\varepsilon,b}(\eta^{\varepsilon,b}_\rho + u^{\varepsilon,b}\eta^{\varepsilon,b}_m) - \kappa \eta^{\varepsilon,b}_m \rho^{\varepsilon,b}(\Phi_\alpha * \rho^{\varepsilon,b})_r \\ &+ \varepsilon \eta^{\varepsilon,b}_m \Big\{ \Big(\rho^{\varepsilon,b}(u^{\varepsilon,b}_r + \frac{n-1}{r}u^{\varepsilon,b})\Big)_r - \frac{n-1}{r}\rho^{\varepsilon,b}_r u^{\varepsilon,b}\Big\}. \end{aligned} \tag{2.141}$$

Multiplying (2.141) by $\phi(t,r) \in C^\infty_{\mathrm{c}}(\mathbb{R}^2_+)$ and integrating it over $\mathbb{R}^2_+$, and then integrating by parts, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^2_+} \big(\partial_t \eta^{\varepsilon,b} + \partial_r q^{\varepsilon,b}\big)\phi \,\mathrm{d}r\,\mathrm{d}t \\ &= -\int_{\mathbb{R}^2_+} \frac{n-1}{r} m^{\varepsilon,b}\big(\eta^{\varepsilon,b}_\rho + u^{\varepsilon,b}\eta^{\varepsilon,b}_m\big)\phi \,\mathrm{d}r\,\mathrm{d}t - \varepsilon \int_{\mathbb{R}^2_+} \big(\eta^{\varepsilon,b}_m\big)_r \rho^{\varepsilon,b}\big(u^{\varepsilon,b}_r + \frac{n-1}{r}u^{\varepsilon,b}\big)\phi \,\mathrm{d}r\,\mathrm{d}t \\ &\quad - \varepsilon \int_{\mathbb{R}^2_+} \eta^{\varepsilon,b}_m \rho^{\varepsilon,b}\big(u^{\varepsilon,b}_r + \frac{n-1}{r}u^{\varepsilon,b}\big)\phi_r \,\mathrm{d}r\,\mathrm{d}t - \varepsilon \int_{\mathbb{R}^2_+} \eta^{\varepsilon,b}_m \frac{n-1}{r}\rho^{\varepsilon,b}_r u^{\varepsilon,b}\phi \,\mathrm{d}r\,\mathrm{d}t \\ &\quad - \kappa \int_{\mathbb{R}^2_+} \eta^{\varepsilon,b}_m \rho^{\varepsilon,b}\big(\Phi_\alpha * \rho^{\varepsilon,b}\big)_r \phi \,\mathrm{d}r\,\mathrm{d}t := \sum_{i=1}^5 I^{\varepsilon,b}_i. \end{aligned}$$

Note that the terms $I^{\varepsilon,b}_i$ for $i = 1, \dots, 4$ coincide with those in [36, Lemma 4.12], and can therefore be treated in the same manner. However, the term $I^{\varepsilon,b}_5$ differs and requires a separate argument.

Since ϕ is compactly supported in $\mathbb{R}^2_+$, there exist $(d, D) \Subset (0, \infty)$ and $T > 0$ such that

$$\mathrm{supp}(\phi) \subset (0, T) \times (d, D),$$

so that

$$|I^{\varepsilon,b}_5| \le C \int_0^T \int_d^D \Big|\eta^{\varepsilon,b}_m \rho^{\varepsilon,b}(\Phi_\alpha * \rho^{\varepsilon,b})_r\Big| \,\mathrm{d}r\,\mathrm{d}t$$

$$\leq C\int_0^T\int_d^D \rho^{\varepsilon,b}(r)\int_a^{b(t)} \big|\omega(r,\eta)\big|\rho^{\varepsilon,b}(\eta)\eta^{n-1}\,\mathrm{d}\eta\,\mathrm{d}r\,\mathrm{d}t. \tag{2.142}$$

We need to prove the following estimate:

$$\int_0^T\int_d^D \left|\eta_m^{\varepsilon,b}\rho^{\varepsilon,b}(\Phi_\alpha * \rho^{\varepsilon,b})_r\right|\mathrm{d}r\,\mathrm{d}t \leq C(d,D,M,E_0,T). \tag{2.143}$$

When $\alpha \in (n-2, n-1)$, combining (2.142) with (2.33) and applying Fubini's Theorem yields

$$\begin{aligned}
&\int_0^T\int_d^D \left|\eta_m^{\varepsilon,b}\rho^{\varepsilon,b}(\Phi_\alpha * \rho^{\varepsilon,b})_r\right|\,\mathrm{d}r\,\mathrm{d}t\\
&\leq C\int_0^T\int_d^D \rho^{\varepsilon,b}(r)\int_{(a,b(t))\cap(r-\frac{d}{2},r+\frac{d}{2})} |\omega(r,\eta)|\rho^{\varepsilon,b}(\eta)\eta^{n-1}\,\mathrm{d}\eta\,\mathrm{d}r\,\mathrm{d}t\\
&\quad + C\int_0^T\int_d^D \rho^{\varepsilon,b}(r)\int_{(a,b(t))\cap(r-\frac{d}{2},r+\frac{d}{2})^c} |\omega(r,\eta)|\rho^{\varepsilon,b}(\eta)\eta^{n-1}\,\mathrm{d}\eta\,\mathrm{d}r\,\mathrm{d}t\\
&\leq C\int_0^T\int_d^D \rho^{\varepsilon,b}(r)\int_{(a,b(t))\cap(r-\frac{d}{2},r+\frac{d}{2})} \frac{|r-\eta|^{n-2-\alpha}}{(r\eta)^{\frac{n-1}{2}}}\rho^{\varepsilon,b}(\eta)\eta^{n-1}\,\mathrm{d}\eta\,\mathrm{d}r\,\mathrm{d}t\\
&\quad + C(d)\int_0^T\int_d^D \rho^{\varepsilon,b}(r)\int_{(a,b(t))\cap(r-\frac{d}{2},r+\frac{d}{2})^c} \rho^{\varepsilon,b}(\eta)\eta^{n-1}\,\mathrm{d}\eta\,\mathrm{d}r\,\mathrm{d}t\\
&\leq C(d,M,T) + C\int_0^T\int_d^D \rho^{\varepsilon,b}(r)\int_{(a,b(t))\cap(r-\frac{d}{2},r+\frac{d}{2})} \frac{|r-\eta|^{n-2-\alpha}}{d^{n-1}}\rho^{\varepsilon,b}(\eta)\eta^{n-1}\,\mathrm{d}\eta\,\mathrm{d}r\,\mathrm{d}t\\
&\leq C(d,M,T) + C(d)\int_0^T\int_a^{b(t)} \rho^{\varepsilon,b}(\eta)\eta^{n-1}\int_d^D |r-\eta|^{n-2-\alpha}\rho^{\varepsilon,b}(r)\,\mathrm{d}r\,\mathrm{d}\eta\,\mathrm{d}t.
\end{aligned} \tag{2.144}$$

Thus, for $\gamma > \dfrac{1}{(n-1)-\alpha}$, the combination of Lemma A.0.1 and (2.144) implies

$$\begin{aligned}
&\int_0^T\int_d^D \left|\eta_m^{\varepsilon,b}\rho^{\varepsilon,b}(\Phi_\alpha * \rho^{\varepsilon,b})_r\right|\mathrm{d}r\,\mathrm{d}t\\
&\leq C\int_0^T\int_a^{b(t)} \rho^{\varepsilon,b}(\eta)\eta^{n-1}\sup_{z\in\mathbb{R}_+}\left\{\int_d^D |r-z|^{n-2-\alpha}\rho^{\varepsilon,b}(r)\,\mathrm{d}r\right\}\mathrm{d}\eta\,\mathrm{d}t\\
&= C\int_0^T\int_a^{b(t)} \rho^{\varepsilon,b}(\eta)\eta^{n-1}\sup_{z\in[d,D]}\left\{\int_d^D |r-z|^{n-2-\alpha}\rho^{\varepsilon,b}(r)\,\mathrm{d}r\right\}\mathrm{d}\eta\,\mathrm{d}t\\
&\leq C(K)\|\rho^{\varepsilon,b}\|_{L^\gamma(d,D)}\int_0^T\int_a^{b(t)} \rho^{\varepsilon,b}(\eta)\eta^{n-1}\,\mathrm{d}\eta\,\mathrm{d}t\\
&\leq C(d,D,M,E_0,T).
\end{aligned}$$

When $\alpha \in (-1, n-2]$, following a similar argument to that of (2.144), but using the alternative estimate for ω given in (2.33) for $\alpha \in (-1, n-2]$, we obtain (2.143). □

Now, applying the compensated compactness framework of [37], one can show that there exist radial functions $(\rho, m)(t,r)$ such that

$$(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon) \longrightarrow (\rho, m), \qquad a.e.\ \text{in } \mathbb{R}^2_+ \text{ as } \varepsilon \to 0.$$

The underlying idea of the argument is as follows:

1. For the half plane $\mathbb{H} \coloneqq \{(\rho, u) \in \mathbb{R}^2_+ \,|\, \rho > 0\}$ we define the subset of continuous functions

$$\overline{C}(\mathbb{H}) \coloneqq \left\{ \phi \in C(\overline{\mathbb{H}}) \;\middle|\; \begin{array}{l} \phi(0,u) \text{ is constant and the map} \\ (\rho,u) \mapsto \lim\limits_{s\to\infty} \phi(s\rho, su) \in C(\mathbb{S}^1 \cap \mathbb{H}) \end{array} \right\}.$$

   It can be shown that there exists a compactification $\overline{\mathcal{H}}$ of $\mathbb{H}$, such that the weak-$*$ topology is induced by $C(\overline{\mathcal{H}})$. Moreover, there exists an isometric isomorphism $\iota : \overline{C}(\mathbb{H}) \to C(\overline{\mathcal{H}})$. Let V denote the weak-$*$ closure of the vacuum set $\{(0,u) \,|\, u \in \mathbb{R}_+\}$.

2. The fundamental Theorem of Young measures [2, Theorem 2.4] and [119, Proposition 4.1] implies the existence of a Young measure $\nu_{(t,r)} \in \mathcal{P}(\overline{\mathcal{H}})$ such that, for every $\phi \in C(\overline{\mathbb{H}})$ vanishing on $\partial\mathbb{H}$, the function ϕ is integrable with respect to $\nu_{(t,r)}$ for *a.e.* $(t,r) \in \mathbb{R}^2_+$, and

$$\phi(\rho^\varepsilon(t,r), u^\varepsilon(t,r)) \rightharpoonup \int_{\mathbb{H}} \phi(\rho, \mathrm{u})\, \mathrm{d}\nu_{(\mathrm{t,r})}(\rho, \mathrm{u}), \qquad \text{in } \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^2_+).$$

   Moreover, by choosing $\phi(\rho, u) = (\rho, \rho u)$, we observe that the sequence $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)$ converges almost everywhere to (ρ, m) if and only if

$$\nu_{(t,r)}(\rho, u) = \delta_{(\rho(t,r), m(t,r))}, \qquad \text{for } a.e.\ (t,r) \in \{(t,r) \in \mathbb{R}^2_+ \,|\, \rho(t,r) > 0\},$$

   and $\operatorname{supp}(\nu_{(t,r)}) \subset V$ for *a.e.* $(t,r) \in \{(t,r) \in \mathbb{R}^2_+ \,|\, \rho(t,r) = 0\}$. In other words, it suffices to show that, for *a.e.* $(t,r) \in \mathbb{R}^2_+$, the measure $\nu_{(t,r)}$ is either supported on V or concentrated at a single point in $\mathbb{H}$.

3. One can use the div-curl Lemma (see Murat [115] and Tartar [128]), Lemma 2.11.1 and the higher integrability results in Lemmas 2.9.1 and 2.9.2 to prove the commutator relation

$$\theta(s_2 - s_1)\Big(\overline{\chi(s_1)\chi(s_2)} - \overline{\chi(s_1)}\,\overline{\chi(s_2)}\Big) = (1-\theta)\Big(\overline{\chi(s_1)}\,\overline{u\chi(s_2)} - \overline{\chi(s_2)}\,\overline{u\chi(s_1)}\Big),$$

   for all $s_1, s_2 \in \mathbb{R}$, where $\chi(s) \coloneqq \chi(\cdot,\cdot,s)$ and $\overline{f} = \int_{\overline{\mathcal{H}}} f \,\mathrm{d}\nu_{(t,r)}$.

4. The commutator relation can be used to show that any connected component of the support of the Young measure is bounded. Furthermore, the approach of [100] for $\gamma \in [3,\infty)$ and [31, 101, 51, 53] for $\gamma \in (1,3)$ can be employed to prove that the support of $\nu_{(t,r)}$ is either a subset of V or a single point in $\mathbb{H}$.

Since, $m(t,r) = 0$ *a.e.* on $\{(t,r) \,|\, \rho(t,r) = 0\}$, we define the limit velocity $u(t,r)$ by setting

$$\begin{aligned} u(t,r) &:= \frac{m(t,r)}{\rho(t,r)}, && \text{a.e. on} \{(t,r) \,|\, \rho(t,r) \neq 0\}, \\ u(t,r) &:= 0, && \text{a.e. on} \{(t,r) \,|\, \rho(t,r) = 0\}. \end{aligned}$$

Then

$$m(t,r) = \rho(t,r)u(t,r).$$

We can also define

$$\Big(\frac{m}{\sqrt{\rho}}\Big)(t,r) := \sqrt{\rho(t,r)}u(t,r).$$

By Young's inequality, we have

$$|m|^{\frac{3(\gamma+1)}{\gamma+3}} \leq C(\rho|u|^3 + \rho^{\gamma+1}).$$

Thus, owing to the higher integrability results Lemmas 2.9.1 and 2.9.2, we obtain that, as $\varepsilon \to 0$,

$$\begin{aligned} &\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}} \equiv \sqrt{\rho^\varepsilon}u^\varepsilon \longrightarrow \frac{m}{\sqrt{\rho}} \equiv \sqrt{\rho}u, && \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^2_+), \\ &(\rho^\varepsilon, m^\varepsilon) \longrightarrow (\rho, m), && \text{in } L^{p_1}_{\mathrm{loc}}(\mathbb{R}^2_+) \times L^{p_2}_{\mathrm{loc}}(\mathbb{R}^2_+), \end{aligned}$$

for $p_1 \in [1, \gamma+1)$ and $p_2 \in [1, \frac{3(\gamma+1)}{\gamma+3})$, where $L^p_{\mathrm{loc}}(\mathbb{R}^2_+)$ represents $L^p([0,T] \times K)$ for all $T > 0$ and $K \Subset (0,\infty)$. Tying this together, we have the following lemma.

Lemma 2.11.2. *Under the assumptions of* Theorem 2.1.2*, the following statements hold as* $\varepsilon \to 0$ *(up to a subsequence)*:

$$\begin{aligned} &\frac{m^\varepsilon}{\sqrt{\rho^\varepsilon}} \equiv \sqrt{\rho^\varepsilon}u^\varepsilon \longrightarrow \frac{m}{\sqrt{\rho}} \equiv \sqrt{\rho}u, && \text{a.e. and in } L^2_{\mathrm{loc}}(\mathbb{R}^2_+) \\ &(\rho^\varepsilon, m^\varepsilon) \longrightarrow (\rho, m), && \text{a.e. and in } L^{p_1}_{\mathrm{loc}}(\mathbb{R}^2_+) \times L^{p_2}_{\mathrm{loc}}(\mathbb{R}^2_+). \end{aligned} \tag{2.145}$$

Integrating (2.107) over $[0,T]$, letting $\varepsilon \to 0$ (up to a subsequence), and using (2.145), we obtain

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} \rho(t,\boldsymbol{x})\zeta(t,\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_0^T \int_{\mathbb{R}^n} \rho_0(\boldsymbol{x})\zeta(0,\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ + \int_0^T \int_0^t \int_{\mathbb{R}^n} \big(\rho\zeta_t + \boldsymbol{\mathcal{M}} \cdot \nabla\zeta\big) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\,\mathrm{d}t. \end{aligned} \tag{2.146}$$

Thus, in the same way as Lemma 2.10.5, using (2.146), we obtain

$$\int_0^T \int_{\mathbb{R}^n} \rho(t,\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = MT.$$

Then, arguing as in the proof of Lemma 2.10.8, we obtain

Lemma 2.11.3. *Under the assumptions of* Theorem 2.1.2*, the following statements hold as* $\varepsilon \to 0$ *(up to a subsequence)*:

$$\Phi_\alpha * \rho^\varepsilon \rightharpoonup \Phi_\alpha * \rho, \quad \text{weak-}* \text{ in } L^\infty(0,T; W^{1,\frac{2n}{\alpha+2}}_{\mathrm{loc}}(\mathbb{R}^n)),$$
$$\text{and weakly in } L^2(0,T; W^{1,\frac{2n}{\alpha+2}}_{\mathrm{loc}}(\mathbb{R}^n)),$$
$$\int_0^T \int_0^\infty \left|\rho^\varepsilon(\Phi_\alpha * \rho^\varepsilon) - \rho(\Phi_\alpha * \rho)\right|(t,r)\, r^{n-1}\, \mathrm{d}r\, \mathrm{d}t \longrightarrow 0.$$

In addition, for $\alpha \in (0, n-1)$,

$$\|\Phi_\alpha * \rho(t)\|_{L^{\frac{2n}{\alpha}}(\mathbb{R}^n)} + \|\nabla\Phi_\alpha * \rho(t)\|_{L^{\frac{2n}{\alpha+2}}(\mathbb{R}^n)} \le C(M, E_0), \qquad \text{for } t \ge 0;$$

and, for $\alpha \in (-1, 0]$, *for any* $K \Subset \mathbb{R}^n$,

$$\|\Phi_\alpha * \rho(t)\|_{L^\infty(K)} + \|\nabla\Phi_\alpha * \rho(t)\|_{L^{\frac{2n}{\alpha+2}}(K)} \le C(M, E_0, T, K), \qquad \text{for } t \ge 0.$$

Then, as in [36], we conclude that $(\rho, m)(t, r)$ defines a global finite-energy weak solution of the CEREs system (1.1) in the sense of Definition 2.1.1, thereby yielding Theorem 2.1.2.

2.12 A generalised pressure

Here, we formulate and outline the approach used to extend the previous results to the case of a general pressure law. For brevity, we omit the detailed arguments. This formulation is motivated by applications such as the modelling of white dwarf stars discussed in the introduction. In this setting, we consider a general pressure of the form:

(a) The pressure function $p \in C^4([0,\infty))$ with

$$p'(\rho) > 0, \quad 2p'(\rho) + \rho p''(\rho) > 0. \tag{2.147}$$

(b) There exists a constant $\rho_* > 0$ such that

$$p(\rho) = \kappa_1 \rho^{\gamma_1}(1 + \mathcal{P}_1(\rho)), \quad \text{for } \rho \in [0, \rho_*), \tag{2.148}$$

with $\gamma_1 \in (\frac{2n}{2n-\alpha}, 3)$, $\kappa_1 > 0$ and a function $\mathcal{P}_1 \in C^4([0,\infty))$ satisfying

$$|\mathcal{P}_1^{(j)}(\rho)| \le C_* \rho^{\gamma_1 - 1 - j}, \tag{2.149}$$

for $\rho \in (0, \rho_*)$ and $j \in \{0, \dots, 4\}$, where $C_* > 0$ is a constant depending only on ρ_*.

(c) There exists a constant $\rho^* > \rho_* > 0$ such that

$$p(\rho) = \kappa_2 \rho^{\gamma_2}(1 + \mathcal{P}_2(\rho)), \qquad \text{for } \rho \in [\rho^*, \infty), \tag{2.150}$$

for $\gamma_2 \in (\frac{2n}{2n-\alpha}, \gamma_1]$, $\kappa_2 > 0$ and a function $\mathcal{P}_2 \in C^4([0,\infty))$ satisfying

$$|\mathcal{P}_2^{(j)}(\rho)| \le C^* \rho^{-\varepsilon - j}, \tag{2.151}$$

for $\rho \in [\rho_*, \infty)$, $\varepsilon > 0$ (we can take $\varepsilon < \gamma_2 - 1$ for simplicity) and $j \in \{0, \dots, 4\}$, where $C^* > 0$ is a constant depending only on ρ^*.

The purpose of this formulation is to capture the behaviour of a pressure law that exhibits two distinct power-law regimes as $\rho \to 0$ and $\rho \to \infty$. A representative example is given by (1.3), where the exponents are $\gamma_1 = \frac{5}{3}$ in the low-density regime and $\gamma_2 = \frac{4}{3}$ in the high-density regime.

In the polytropic case, the energy estimates involving the internal energy naturally yield control of the L^γ-norm of the density. In the present setting, however, the pressure is no longer homogeneous, and we only obtain that $\rho e(\rho) \in L^1$, which does not directly provide control of the density. To recover such control, we exploit the condition $\gamma_1 \ge \gamma_2$. Combining this with (2.147)–(2.151), we obtain

Lemma 2.12.1. *Suppose $p : [0,\infty) \to \mathbb{R}$ satisfies (2.147)–(2.151), then for $\beta > 0$ there exists $C = C(\beta) > 0$ such that the following is true*

$$\|\rho\|_{L^{\gamma_2}} \le C\Big(\int_0^\infty \rho(\beta + e(\rho)) r^{n-1} \, \mathrm{d}r \Big)^{\frac{1}{\gamma_2}}.$$

Since the pressure, and hence the internal energy density, is no longer homogeneous, controlling the potential energy by the internal energy for $\gamma \in \big(\frac{2n}{2n-\alpha}, \frac{n+\alpha}{n}\big]$ becomes more delicate. To address this, we first compare $\beta + e(\rho)$ with $\rho^{\gamma_2 - 1}$ and define

$$C_{\max}(\beta) := \max_{\rho \ge 0} \left\{ \frac{\rho^{\gamma_2 - 1}}{\beta + e(\rho)} \right\} < \infty.$$

Without the condition $\beta > 0$, such a constant would, in general, fail to be bounded when $\gamma_1 > \gamma_2$. We therefore introduce a β-dependent critical mass $M_c^\alpha(\beta)$. To this end, we first define a constant B_β, analogous to $B_{n,\gamma,\alpha}$ in the polytropic case, by

$$B_\beta := \frac{C_{\alpha,n}}{2\alpha} \omega_n^{\frac{\alpha - n(\gamma_2 - 1)}{n(\gamma_2 - 1)}} (C_{\max}(\beta))^{\frac{\alpha}{n(\gamma_2 - 1)}}.$$

Furthermore, we define $M_c^\alpha(\beta) > 0$ by the unique solution of

$$\begin{cases} M_c^\alpha(\beta) = B_\beta^{-\frac{n}{n-\alpha}}, & \gamma_2 = \frac{n+\alpha}{n}, \\ \dfrac{\alpha - n(\gamma_2 - 1)}{\alpha} \left(\dfrac{\alpha B_\beta}{n(\gamma_2 - 1)} \right)^{-\frac{n(\gamma_2 - 1)}{\alpha - n(\gamma_2 - 1)}} (M_c^\alpha(\beta))^{-\frac{n(\gamma_2 - 1)}{\alpha - n(\gamma_2 - 1)}} & \gamma_2 \in \left(\frac{2n}{2n-\alpha}, \frac{n+\alpha}{n} \right), \\ \qquad - \omega_n^{-1} \beta M_c^\alpha(\beta) = E_0, \end{cases}$$

Then the critical mass M_c^α is defined by taking the supremum over all such $\beta > 0$,

$$M_c^\alpha = \sup_{\beta>0} M_c^\alpha(\beta).$$

In essence, by carrying out a careful analysis of the entropy and entropy–flux pairs, as in [34], we can extend the results obtained up to the end of §2.10 by replacing γ with γ_2. This entropy framework allows us to exploit the compensated compactness method of [119] in order to pass to the vanishing viscosity limit. Consequently, the existence results established for the polytropic CEREs system can be extended to the case of general pressure laws of the form (2.147)–(2.151).

Theorem 2.12.2 (Existence of radially symmetric solutions of the CEREs)**.** *Consider problem* (1.1) *and* (1.5) *with the pressure* p *satisfying* (2.147)–(2.151) *and* $\alpha \in (-1, n-1)$ *and the initial data of radial symmetry as defined in* (2.1)–(2.2)*. Assume that* $(\rho_0, \boldsymbol{\mathcal{M}}_0)$ *satisfy* (2.4)–(2.5) *and additionally* (2.6) *for* $\alpha \in (-1, 0]$, $\gamma_2 > \frac{1}{n-1-\alpha}$, *and*

- *When* $\kappa = -1$, $\rho_0 \in L^{\frac{2n}{2n-\alpha}}(\mathbb{R}^n)$ *for* $\alpha \in (0, n-1)$ *and*
  - (a) $\gamma_2 > 1$ *for* $\alpha \in (-1, n-2]$,
  - (b) $\gamma_2 \geq \frac{3n}{3n-2(1+\alpha)}$ *for* $\alpha \in (n-2, n-1)$;
- *When* $\kappa = 1$, $\gamma_2 \geq \frac{3n}{3n-2(1+\alpha)}$ *when* $\alpha \neq n-2$ *and*
  - (c) $\gamma_2 > 1$ *for* $\alpha \in (-1, 0]$,
  - (d) $\gamma_2 > \frac{n+\alpha}{n}$ *for* $\alpha \in (0, n-1)$,
  - (e) $M < M_c^\alpha$ *for* $\gamma_2 \in (\frac{2n}{2n-\alpha}, \frac{n+\alpha}{n}]$ *and* $\alpha \in (0, n-1)$.

Then there exists a global radially symmetric finite-energy weak solution $(\rho, \boldsymbol{\mathcal{M}})$ *of problem* (1.1)–(1.5) *in the sense of* Definition 2.1.1.

2.13 Open problems

There are two open problems arising from this work:

1. While local well-posedness of the one-dimensional polytropic CEREs for the full range of Riesz potentials was established in [41], and global well-posedness was subsequently proved in [43] for sufficiently small initial density and initial velocity sufficiently close to a suitable reference state u_0 in an appropriate Sobolev space, the existence of global weak solutions for general one-dimensional initial data remains open. There are several obstacles to extending our approach to the one-dimensional setting. First, our existence theory requires $\alpha < n-1$, since, for

$\alpha \geq n-1$, the derivative of the Riesz potential is defined through the cancellation formula

$$\nabla \mathcal{W}_\alpha(\boldsymbol{x}) = \int_{\mathbb{R}^n} \nabla \Phi_\alpha(\boldsymbol{x}-\boldsymbol{y})\big(\rho(\boldsymbol{y})-\rho(\boldsymbol{x})\big)\,\mathrm{d}\boldsymbol{y},$$

which is not compatible with the HLS estimates used in our analysis. Consequently, in one dimension our method applies only to $\alpha \in (-1,0)$ and does not include the Coulomb/Newtonian case $\alpha = -1$. A second difficulty is that the boundary term

$$\varepsilon(n-1)\int_0^t (\rho u^2)(s,b(s))\,b(s)^{n-2}\,\mathrm{d}s$$

appearing in the energy inequality (2.37) vanishes when $n = 1$. This term is essential for obtaining a lower bound on $b(t)$ and proving expansion of the free-boundary domain. Since analogous boundary terms disappear in one dimension even for other viscosity coefficients satisfying the BD relation, a different approximation scheme will likely be required to establish global existence in the one-dimensional setting. For instance, an approach inspired by [19], where finite-energy entropy solutions are constructed for the one-dimensional compressible Euler equations with a non-local attraction-repulsion potential, may prove useful. Nevertheless, substantial changes to the present framework would be necessary, especially in the proofs of the energy estimates.

2. While the result for general pressures in §2.12 requires the assumption that $\gamma_1 < 3$, it remains unclear to the author, despite some effort, how one might extend the argument to the case $\gamma_1 \geq 3$. The main difficulty lies in the structure of the entropy and entropy flux kernels. In the polytropic case, these kernels admit explicit formulas, which allows one to establish the crucial lower bound on the mechanical entropy flux,

$$q^{\#}(\rho,\rho u) \geq C_\gamma^{-1}\big(\rho|u|^3 + \rho^{\gamma+\theta}\big),$$

as in (2.96). In particular, this estimate yields the higher integrability properties needed for the compensated compactness framework for general $\gamma > 1$. By contrast, in the case of general pressure laws, the kernels are not available in explicit form, and the best estimates currently available for the entropy and entropy flux kernels are not sufficiently sharp. As a result, we are unable to obtain an analogous lower bound. Bridging this gap therefore appears to require a new and sharper approach.

Chapter 3

Existence, stability and instability of stationary states of the compressible Euler-Riesz equations

3.1 Introduction

The aim of this chapter is to establish when stationary states of the attractive CEREs (1.1) are either stable or unstable. For the whole chapter, we assume that we are in the attractive Riesz potential setting $\kappa = 1$. Here, what we mean by a stationary state is that of the following definition.

Definition 3.1.1. $\bar{\rho} \in L^1_+(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is called a stationary state of (1.1) if there exists a constant $\mu > 0$ such that

$$\begin{cases} (\bar{\rho}e(\bar{\rho}))_{\bar{\rho}} + \Phi_\alpha * \bar{\rho} = -\mu, & \boldsymbol{x} \in \Gamma, \\ \Phi_\alpha * \bar{\rho}(\boldsymbol{x}) \geq -\mu, & \boldsymbol{x} \in \mathbb{R}^n \setminus \Gamma, \end{cases} \tag{3.1}$$

where

$$\Gamma := \{\boldsymbol{x} \in \mathbb{R}^n \,|\, \bar{\rho}(\boldsymbol{x}) > 0\}.$$

In particular, letting $\Pi : [0, \infty) \to [0, \infty)$, be given by $\Pi(\tau) = (\tau e(\tau))_\tau$, we see that $\Pi'(\tau) = \frac{p'(\tau)}{\tau} > 0$ for $\tau > 0$, hence Π is invertible. Extending Π^{-1} to $\Pi^{-1}_+ : \mathbb{R} \to [0, \infty)$ by 0 on the rest of the real line, we see that (3.1) is equivalent to

$$\bar{\rho} = \Pi^{-1}_+(-\Phi_\alpha * \bar{\rho} - \mu). \tag{3.2}$$

Thus, in the case the pressure is given by a power-law ($p(\rho) = a_0\rho^\gamma$), (3.2) becomes

$$\bar{\rho} = \left(\frac{\gamma - 1}{a_0\gamma}\right)^{\frac{1}{\gamma-1}} (-\Phi_\alpha * \bar{\rho} - \mu)_+^{\frac{1}{\gamma-1}}.$$

Here, (3.2) is derived by dividing the stationary form of (1.1), in the attractive case ($\kappa = 1$), by the density to obtain

$$\nabla(\bar{\rho}e(\bar{\rho}))_{\bar{\rho}}(\boldsymbol{x}) = -\nabla\bar{\mathcal{W}}_\alpha(\boldsymbol{x}),$$

where we have used that $(\bar{\rho}e(\bar{\rho}))_{\bar{\rho}\bar{\rho}} = \frac{p'(\bar{\rho})}{\bar{\rho}}$. Then, in the case that the density is regular enough, there must exist a constant μ such that

$$(\bar{\rho}e(\bar{\rho}))_{\bar{\rho}}(\boldsymbol{x}) = -(\Phi_\alpha * \bar{\rho})(\boldsymbol{x}) - \mu = \big(- (\Phi_\alpha * \bar{\rho})(\boldsymbol{x}) - \mu \big)_+.$$

We employ arguments to prove that stationary solutions must be translations of radially decreasing stationary states. Thus, by the regularity of $\bar{\rho}$, the support of the stationary states must be compact. To prove existence of such stationary states, we elaborate on what was mentioned in §1.6. In the mass-subcritical regime, such stationary states, of mass $0 < M < \infty$, are expected to be given by minimisers of the free-energy functional

$$\mathcal{G}(\varrho) = \int_{\mathbb{R}^n} \Big(\varrho(\boldsymbol{x}) e(\varrho(\boldsymbol{x})) + \frac{1}{2} \varrho(\boldsymbol{x}) (\Phi_\alpha * \varrho)(\boldsymbol{x}) \Big) \, \mathrm{d}\boldsymbol{x},$$

over the admissible set

$$X_M := \Big\{ \varrho \in L^1_+(\mathbb{R}^n) \cap L^{\frac{n+\alpha}{n}}(\mathbb{R}^n) \,\Big|\, \int_{\mathbb{R}^n} \varrho(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = M \Big\}.$$

In order for a minimiser of $\mathcal{G}$ over X_M to exist, we employ a concentration compactness argument as in [103]. The idea of the concentration compactness argument is to prove that given a minimising sequence of $\mathcal{G}$, that the sequence of distributions must be tight up to translation, conserving the potential energy in the limit. In order to do this, one must rule out behaviour such as the vanishing or dichotomy of minimising sequences; see §4.3 for more information. Such a framework allows us to prove nonlinear Lyapunov stability of stationary states under a contradiction argument. Moreover, we establish quantitative stability using a refined relative entropy method under the assumption of finite-energy entropy solutions. This approach goes beyond purely variational notions of stability. A key difficulty lies in controlling the difference between the density $\rho(t)$ and the steady state $\bar{\rho}$ by the relative entropy and entropy flux terms, which requires a careful analysis. We impose more natural higher-integrability assumptions on the density and momentum, which allow us to control both the relative entropy and the associated entropy flux terms.

While we prove Lyapunov stability of stationary states in the mass-subcritical regime, it is of great interest to understand whether this is a local or global phenomenon. That is, do we expect solutions whose configuration are not suitably close to stationary states to demonstrate behaviour such as the growth of the solutions away from the stationary states or not? By proving the strong convexity of the second moment of the density with respect to time for solutions with non-negative energy, we prove the linear growth of the support of solutions. Thus, owing to the compact support of stationary

states, which have negative energy, this indeed contributes to proving the locality of the stability result.

However, in the polytropic setting, the free-energy functional is not bounded below on the admissible set for $\gamma < \frac{n+\alpha}{n}$, this comes from the competition between the potential energy and the internal energy. Thus, in order to consider stationary states for $\gamma < \frac{n+\alpha}{n}$, an alternative variational framework is required. To overcome this problem, we instead shift our thought to a free-energy functional of the form

$$S_\mu(\varrho) \coloneqq \frac{a_0}{\gamma - 1} \int_{\mathbb{R}^n} \varrho^\gamma \, \mathrm{d}\boldsymbol{x} + \frac{1}{2} \int_{\mathbb{R}^n} \varrho \Phi_\alpha * \varrho \, \mathrm{d}\boldsymbol{x} + \mu \int_{\mathbb{R}^n} \varrho \, \mathrm{d}\boldsymbol{x},$$

for $\mu > 0$. Instead of minimising over an admissible set of fixed mass distributions, one can instead study the behaviour of the functional over the mass-preserving scaling. This instead, leads us to considering an admissible set of the form

$$\mathcal{K} \coloneqq \{\varrho \in L^1_+(\mathbb{R}^n) \cap L^\gamma(\mathbb{R}^n) \,|\, Q(\varrho) = 0,\ \varrho \not\equiv 0\},$$

where

$$Q(\varrho) \coloneqq n a_0 \int_{\mathbb{R}^n} \varrho^\gamma \, \mathrm{d}\boldsymbol{x} + \frac{\alpha}{2} \int_{\mathbb{R}^n} \varrho \Phi_\alpha * \varrho \, \mathrm{d}\boldsymbol{x},$$

which is related to the derivative of $S_\mu(\varrho)$ in the scaling parameter defining mass-preserving dilations given by

$$\lambda \mapsto \varrho_\lambda(\boldsymbol{x}) \coloneqq \lambda \varrho(\lambda^{\frac{1}{n}} \boldsymbol{x}). \tag{3.3}$$

In fact, we will see that S_μ is bounded below on the admissible set $\mathcal{K}$, with $\ell_\mu = \inf_{\varrho \in \mathcal{K}} S_\mu(\varrho) > 0$ and that the Euler-Lagrange equations corresponding to minimisers of the minimisation problem coincide with the equations governing stationary states of (1.1). Moreover, the mass of such stationary states are continuously dependent on the parameter $\mu > 0$. By employing the concavity of the free-energy functional S_μ under the mass-preserving scaling parameter, we prove strong convexity of the second moment of the density with respect to time, proving growth of solutions. Unlike the mass-subcritical regime, such a result is true for solutions with initial conditions arbitrarily close to the stationary state.

3.2 Preliminaries and results

In order to employ the concentration compactness argument of [103], we will of course require that $\mathcal{G}$ is bounded below on X_M. Moreover, we require that the infimum of $\mathcal{G}$ over X_M is negative, that is $g_M \coloneqq \inf_{\varrho \in X_M} \mathcal{G}(\varrho) < 0$. So that both these properties are satisfied, we require the following conditions:

(a)

$$M < \left(\frac{2n \liminf_{\rho \to \infty} p(\rho) \rho^{-\frac{n+\alpha}{n}}}{C_*} \right)^{\frac{n}{n-\alpha}}, \tag{3.4}$$

where $C_* > 0$ is the optimal constant given by the HLS inequality as in Lemma A.0.3. This is required so that the internal energy can control the potential energy for large densities, thus guaranteeing $g_M > -\infty$.

(b)

$$M > \left(\frac{2n \limsup_{\rho \to 0} p(\rho) \rho^{-\frac{n+\alpha}{n}}}{C_*} \right)^{\frac{n}{n-\alpha}}. \tag{3.5}$$

Such a condition is required so that the potential energy dominates the internal energy for small densities, thus guaranteeing $g_M < 0$.

Remark 3.2.1. Note that (3.4) is analogous to that of the Chandrasekhar limit mass. In the polytropic case for $\alpha = n - 2$ and $\gamma = \frac{2(n-1)}{n}$, $M_{\text{ch}} := \left(\frac{2na_0}{C_*}\right)^{\frac{n}{2}}$ for stationary gaseous stars [29].

In order to consider the nonlinear stability of stationary states, we need to define some distance under which we can observe stability. Given a solution $(\rho, \boldsymbol{\mathcal{M}})$ of the CEREs with mass M and minimiser $\bar{\rho}$ of $\mathcal{G}$ over X_M, the way in which we measure the distance of the solution to the stationary state is derived by considering the relative energy given by

$$E(\rho, \boldsymbol{\mathcal{M}} \,|\, \bar{\rho}, \mathbf{0})(t) \coloneqq E(\rho, \boldsymbol{\mathcal{M}})(t) - E(\bar{\rho}, \mathbf{0}) = \mathcal{G}(\rho(t)) - \mathcal{G}(\bar{\rho}) + \frac{1}{2} \int_{\mathbb{R}^n} \left| \frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}} \right|^2 (t, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}. \tag{3.6}$$

We can rewrite the relative free-energy as follows:

$$\mathcal{G}(\rho(t)) - \mathcal{G}(\bar{\rho}) = d(\rho(t), \bar{\rho}) + \frac{1}{2} \int_{\mathbb{R}^n} (\rho(t) - \bar{\rho}) \, \Phi_\alpha * (\rho(t) - \bar{\rho}) \, \mathrm{d}\boldsymbol{x}, \tag{3.7}$$

where the relative free-energy functional $d(\varrho, \bar{\rho})$ for $\varrho, \bar{\rho} \in X_M$, is given by

$$d(\varrho, \bar{\rho}) \coloneqq \int_{\mathbb{R}^n} \left((\varrho e(\varrho) - \bar{\rho} e(\bar{\rho})) + (\varrho - \bar{\rho})(\Phi_\alpha * \bar{\rho}) \right) \mathrm{d}\boldsymbol{x}.$$

In the case that $\bar{\rho}$ is a stationary state of (1.1), by (3.1), there exists $\mu > 0$ such that

$$\begin{aligned} d(\varrho, \bar{\rho}) &= \int_{\mathbb{R}^n} \left((\varrho e(\varrho) - \bar{\rho} e(\bar{\rho})) + (\varrho - \bar{\rho})(\Phi_\alpha * \bar{\rho} + \mu) \right) \mathrm{d}\boldsymbol{x} \\ &\geq \int_{\mathbb{R}^n} \left((\varrho e(\varrho) - \bar{\rho} e(\bar{\rho})) + (\varrho - \bar{\rho})(\bar{\rho} e(\bar{\rho}))_{\bar{\rho}} \right) \mathrm{d}\boldsymbol{x}. \end{aligned}$$

Applying (1.4), we see that $\varrho \mapsto \varrho e(\varrho)$ is convex in ϱ, therefore $d(\varrho, \bar{\rho}) \geq 0$. Notice that we can estimate the second term on the right-hand side of (3.7), which can be obtained by the Hölder inequality and the HLS inequality from Lemma A.0.2, as such

$$\left| \int_{\mathbb{R}^n} (\rho(t) - \bar{\rho}) \, \Phi_\alpha * (\rho(t) - \bar{\rho}) \, \mathrm{d}\boldsymbol{x} \right| \leq C \|\rho(t) - \bar{\rho}\|^2_{L^{\frac{2n}{2n-\alpha}}}.$$

This leads us to the following formulation and result for the nonlinear stability.

Theorem 3.2.2 (Nonlinear Stability of Stationary Solutions)**.** *Suppose $\bar{\rho}$ is a unique minimiser (up to translation) of the functional $\mathcal{G}$ in X_M. Suppose $\alpha \in (0, n)$ with p satisfying (1.4), (3.4) and (3.5). Let $(\rho, \boldsymbol{\mathcal{M}})$ be a global weak solution of the CEREs in the sense of* Definition 2.1.1 *satisfying*

$$\int_{\mathbb{R}^n} \rho_0(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^n} \bar{\rho}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = M.$$

Then, for any $\varepsilon > 0$, there exists $\delta > 0$, such that if

$$d(\rho_0, \bar{\rho}) + \|\rho_0 - \bar{\rho}\|^2_{L^{\frac{2n}{2n-\alpha}}} + \frac{1}{2}\int_{\mathbb{R}^n} \left|\frac{\boldsymbol{\mathcal{M}}_0}{\sqrt{\rho_0}}\right|^2 \mathrm{d}\boldsymbol{x} < \delta, \tag{3.8}$$

there exists a translation function $\boldsymbol{y} \in \mathbb{R}^n$, such that

$$d(\rho(t), T^{\boldsymbol{y}}\bar{\rho}) + \|\rho(t) - T^{\boldsymbol{y}}\bar{\rho}\|^2_{L^{\frac{2n}{2n-\alpha}}} + \frac{1}{2}\int_{\mathbb{R}^n} \left|\frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}}\right|^2 \mathrm{d}\boldsymbol{x} < \varepsilon, \tag{3.9}$$

for almost every $t > 0$, where $T^{\boldsymbol{y}}\bar{\rho}(\boldsymbol{x}) = \bar{\rho}(\boldsymbol{x} + \boldsymbol{y})$.

Remark 3.2.3. For Theorem 3.2.2, the proof follows the same strategy of [117] for the CEPEs. Concerning the existence proof of the minimisers of the free-energy with homogeneous interaction potentials, different from Luo-Smoller [107, 108], we develop a simplified approach by employing the work of [103], as it was done in [25] and extend to the general pressure law case.

In order to consider a relative entropy method to quantify the stability of stationary states, we first define what it means for a finite-energy weak solution to be a weak entropy solution.

Definition 3.2.4. A finite-energy weak solution $(\rho, \boldsymbol{\mathcal{M}})$ as defined in Definition 2.1.1 is called a weak entropy solution if

$$\partial_t \eta(\rho, \boldsymbol{\mathcal{M}}) + \nabla \cdot q(\rho, \boldsymbol{\mathcal{M}}) \leq -\kappa \boldsymbol{\mathcal{M}} \cdot \nabla \mathcal{W}_\alpha,$$

in the sense of distributions for the mechanical entropy

$$\eta(\rho, \boldsymbol{\mathcal{M}}) = \frac{|\boldsymbol{\mathcal{M}}|^2}{2\rho} + h(\rho), \qquad q(\rho, \boldsymbol{\mathcal{M}}) = \frac{1}{2}\boldsymbol{\mathcal{M}}\frac{|\boldsymbol{\mathcal{M}}|^2}{\rho^2} + h'(\rho)\boldsymbol{\mathcal{M}}, \tag{3.10}$$

where $h(\rho) = \rho e(\rho)$.

For the mechanical entropy/entropy flux, we define the corresponding relative entropy/entropy flux with respect to a stationary state $(\bar{\rho}, \mathbf{0})$, given by

$$\eta(\rho, \boldsymbol{\mathcal{M}} \,|\, \bar{\rho}) := \eta(\rho, \boldsymbol{\mathcal{M}}) - \eta(\bar{\rho}, \mathbf{0}) - \partial_\rho \eta(\bar{\rho}, \mathbf{0})(\rho - \bar{\rho}) - \nabla_{\boldsymbol{\mathcal{M}}} \eta(\bar{\rho}, \mathbf{0}) \cdot \boldsymbol{\mathcal{M}}, \tag{3.11}$$

and

$$q(\rho, \boldsymbol{\mathcal{M}} \,|\, \bar{\rho}) := q(\rho, \boldsymbol{\mathcal{M}}) - h'(\bar{\rho})\boldsymbol{\mathcal{M}}, \tag{3.12}$$

respectively. We will show a growth estimate on the relative entropy

$$\mathcal{E}(t) := \int_{\mathbb{R}^n} \eta(\rho, \boldsymbol{\mathcal{M}} \,|\, \bar{\rho}) \, \mathrm{d}\boldsymbol{x}. \tag{3.13}$$

Proposition 3.2.5 (Finite-time stability around steady states)**.** *Let $\alpha \in (0, n-1)$, $\bar{\rho}$ be a compactly supported steady state in the sense of* Definition 3.1.1*, $(\rho, \boldsymbol{\mathcal{M}})$ be a global weak entropy solution of the attractive polytropic CEREs* (1.1) *as in* Definition 3.2.4 *such that $\gamma > \max\{\frac{n+\alpha}{n}, \frac{3n}{3n-2(\alpha+1)}\}$, the relative entropy $\mathcal{E}(t)$ is initially finite and $\rho^{\gamma+\theta}, \frac{|\boldsymbol{\mathcal{M}}|^3}{\rho^2} \in L^1(0,T; L^1(\mathbb{R}^n))$, where $\theta = \frac{\gamma-1}{2}$. Then, there exists $T^* \in (0, \infty]$ (with $T^* < \infty$ for $\frac{3n}{3n-2(\alpha+1)} < \gamma < \frac{n+2(\alpha+1)}{n}$ and $T^* = \infty$ for $\gamma \geq \frac{n+2(\alpha+1)}{n}$) such that for all $0 < T < T^*$, there exists a constant $C_T > 0$, depending only on T, the parameters (γ, α, n), the steady state $(\bar{\rho}, \mathbf{0})$ and the behaviour of $\mathcal{E}(0)$, such that*

$$\operatorname{ess\,sup}_{0 \leq t \leq T} \mathcal{E}(t) \leq \mathcal{E}(0) + C_T.$$

Moreover, C_T is continuous in T and satisfies that $C_T \to 0$ as $T \to 0$. This yields a finite-time relative entropy bound around the steady state.

While we have stability of stationary states in the mass-subcritical regime, we will see that solutions with positive initial energy show growth of the support. In fact, in some cases we derive a lower bound for the linear growth of the support. We suppose that the fluid is confined to a spatially bounded domain $\Omega(t) \subset \mathbb{R}^n$ at time t, that is $\rho|_{\partial\Omega(t)} = 0$, and $\rho > 0$ in $\Omega(t)$. For a stationary solution being considered, $\Omega(t) = \Omega$ is a fixed domain. We define the radius $R(t)$ of the domain $\Omega(t)$ by

$$R(t) = \max_{\boldsymbol{x} \in \Omega(t)} \{|\boldsymbol{x}|\}.$$

When generalising the behaviour of polytropic pressures to general pressures, the following exponent is useful

$$\gamma_{\inf} := 1 + \inf_{\varrho \in \mathbb{R}_+} \left(\frac{p(\varrho)}{\varrho e(\varrho)} \right).$$

The results are summarized in the following theorem.

Theorem 3.2.6. *Suppose that $(\rho, \boldsymbol{\mathcal{M}})$ is a global classical solution of* (1.1) *with p satisfying* (1.4) *and the energy $E = E(\rho, \boldsymbol{\mathcal{M}}) > 0$. We have the following.*

(a) *If $\alpha \in (0, 2] \cap (0, n-1)$, $\gamma_{\inf} \geq \frac{n+\alpha}{n}$, then*

$$\liminf_{t \to \infty} \left(\frac{R(t)}{t} \right) \geq \left(\frac{\alpha E}{M} \right)^{\frac{1}{2}},$$

(b) *If* $\alpha \in (2, n-1)$, $\gamma_{\inf} > \frac{n+2}{n}$ *and* $\lim_{\rho\to\infty} p(\rho)\rho^{-\frac{n+\alpha}{n}} = \infty$ *is satisfied, then there exists* $\tilde{M} > 0$ *such that if* $M < \tilde{M}$, *there exists* $\Lambda_{\min}(M) > 0$ *such that*

$$\liminf_{t\to\infty}\left(\frac{R(t)}{t}\right) \geq \left(\frac{\Lambda_{\min}(M)}{M}\right)^{\frac{1}{2}}.$$

Moreover, if the pressure is polytropic with $\gamma > \frac{n+\alpha}{n}$, $\alpha \in (0,2] \cap (0, n-1)$ *and* $E = E(\rho, \boldsymbol{\mathcal{M}}) = 0$, *then there exists a sequence* $(t_m)_m \subset [0,\infty)$ *such that* $R(t_m) \to \infty$ *as* $m \to \infty$.

Note that (3.1) was derived under the assumptions that (3.4) and (3.5) hold, conditions equivalent to $\gamma > \frac{n+\alpha}{n}$ in the polytropic case. For the mass-supercritical regime $\frac{2n}{2n-\alpha} < \gamma < \frac{n+\alpha}{n}$, it was shown in [28] and [58] that for $a > 0$, $\mu > 0$ and $\alpha \in (0,n) \cap [n-2, n)$, there exists a unique radially symmetric $v \in L^\infty(\mathbb{R}^n)$ such that

$$(-\Delta)^{\frac{n-\alpha}{2}} v = a(v-\mu)_+^{\frac{1}{\gamma-1}}.$$

We note that for $\rho \in L^1(\mathbb{R}^n) \cap L^\gamma(\mathbb{R}^n)$, that the fractional Laplacian $(-\Delta)^{\frac{n-\alpha}{2}}$ is invertible, with the inverse given by a multiple of the Riesz potential Φ_α. As such, if

$$(-\Delta)^{\frac{n-\alpha}{2}} v = c_{n,\alpha}\rho,$$

in the sense of distributions, where

$$c_{n,\alpha} = \frac{2^{n-\alpha}\pi^{\frac{n}{2}}\Gamma\left(\frac{n-\alpha}{2}\right)}{\alpha\Gamma\left(\frac{\alpha}{2}\right)},$$

then $v = -\Phi_\alpha * \rho$. As such, for $a = c_{n,\alpha}\left(\frac{\gamma-1}{a_0\gamma}\right)^{\frac{1}{\gamma-1}}$, we have that there exist unique radially symmetric $\bar{\rho}_\mu \in L^1(\mathbb{R}^n) \cap L^\gamma(\mathbb{R}^n)$ with $\bar{\rho}_\mu \geq 0$ *a.e.* and $\bar{\rho}_\mu \not\equiv 0$ such that $v = -\Phi_\alpha * \bar{\rho}_\mu$ and

$$\bar{\rho}_\mu = \left(\frac{\gamma-1}{a_0\gamma}\right)^{\frac{1}{\gamma-1}}(-\Phi_\alpha * \bar{\rho}_\mu - \mu)_+^{\frac{1}{\gamma-1}}.$$

Hence, there is a unique stationary state (up to translation) for $\mu > 0$ as in (3.1). Moreover, Chan-González-Huang-Mainini-Volzone showed that $\bar{\rho}_\mu$ has compact support and is radially decreasing. We will see later that both of these facts follow from our calculations. We consider the energy-critical scaling

$$^{\xi}\varrho(\boldsymbol{x}) := \xi^{n-\alpha}\rho\left(\xi^{2-\gamma}\boldsymbol{x}\right), \tag{3.14}$$

for $\xi > 0$. For $\alpha \in (0,n)$, if $\bar{\rho}_1$ is a radially symmetric solution of (3.1) for $\mu = 1$, we obtain for $\rho_\mu := \mu^{\frac{1}{(\gamma-1)(n-\alpha)}}\bar{\rho}_1$ that

$$\begin{aligned}\rho_\mu(\boldsymbol{x}) &= \left(\frac{\gamma-1}{a_0\gamma}\right)^{\frac{1}{\gamma-1}}(-\mu\Phi_\alpha * \bar{\rho}_1 - \mu)_+^{\frac{1}{\gamma-1}}\left(\mu^{\frac{2-\gamma}{(\gamma-1)(n-\alpha)}}\boldsymbol{x}\right)\\ &= \left(\frac{\gamma-1}{a_0\gamma}\right)^{\frac{1}{\gamma-1}}(-\Phi_\alpha * \rho_\mu - \mu)_+^{\frac{1}{\gamma-1}}(\boldsymbol{x}).\end{aligned} \tag{3.15}$$

Thus, ρ_μ is a solution of (3.1) for $\mu > 0$. Since, in the regime $\alpha \in [n-2, n)$, radially symmetric solutions of (3.1) are unique, we must have that

$$\bar{\rho}_\mu(\boldsymbol{x}) = \rho_\mu(\boldsymbol{x}) = \mu^{\frac{1}{\gamma-1}} \bar{\rho}_1\Big(\mu^{\frac{2-\gamma}{(\gamma-1)(n-\alpha)}} \boldsymbol{x}\Big).$$

Thus, for any $\alpha \in (0, n)$, if there exists a unique stationary state $\bar{\rho}_{\mu_0}$ satisfying (3.1) for a given $\mu_0 > 0$, then by the previous scaling argument, there exists a unique stationary state $\bar{\rho}_\mu$ satisfying (3.1) for each $\mu > 0$ and

$$\bar{\rho}_\mu(\boldsymbol{x}) = \Big(\frac{\mu}{\mu_0}\Big)^{\frac{1}{\gamma-1}} \bar{\rho}_{\mu_0}\Big(\Big(\frac{\mu}{\mu_0}\Big)^{\frac{2-\gamma}{(\gamma-1)(n-\alpha)}} \boldsymbol{x}\Big).$$

Thus, letting

$$M_1 \coloneqq \int_{\mathbb{R}^n} \bar{\rho}_1 \, \mathrm{d}\boldsymbol{x},$$

be the mass of the unique stationary state for $\mu = 1$. Then the mass of $\bar{\rho}_\mu$ for any $\mu > 0$ is given by

$$\int_{\mathbb{R}^n} \bar{\rho}_\mu \, \mathrm{d}\boldsymbol{x} = \mu^{\frac{n\gamma-(n+\alpha)}{(\gamma-1)(n-\alpha)}} M_1.$$

Hence, given any $M > 0$, taking

$$\mu_M \coloneqq \Big(\frac{M}{M_1}\Big)^{\frac{(\gamma-1)(n-\alpha)}{n\gamma-(n+\alpha)}} > 0,$$

we have that $\bar{\rho}_{\mu_M}$ is the unique stationary state with mass $M > 0$. Hence, we see that the mapping between stationary states indexed by μ and stationary states with prescribed mass $M > 0$ is a bijection. However, in §3.7 we will see that the uniqueness is not required in our approach, just the scaling. Thus, while the admissible sets and free-energy functionals considered in §3.3 and §3.7 are different, their respective minimisation both yield stationary states of the system. This gives the following theorem.

Theorem 3.2.7. *Suppose there exists a minimiser $\bar{\rho}$ of the free-energy functional S_μ over $\mathcal{K}$, then $\bar{\rho}$ is a stationary state as given by Definition 3.1.1 with compact support. Moreover, with the restriction $\alpha \in (0, n) \cap [n-2, n)$, $\bar{\rho}$ is a translation of $\bar{\rho}_\mu$.*

We will see that for $\frac{2n}{2n-\alpha} < \gamma < \frac{n+\alpha}{n}$, that these stationary states are unstable. This is characterised by the following theorem.

Theorem 3.2.8. *Suppose $(\rho, \boldsymbol{\mathcal{M}})$ is a classical solution of the attractive ($\kappa = 1$) polytropic CEREs (1.1) for $\alpha \in (0, n)$, $\frac{2n}{2n-\alpha} < \gamma < \frac{n+\alpha}{n}$ with initial conditions $(\rho_0, \boldsymbol{\mathcal{M}}_0)$ satisfying $Q(\rho_0) > 0$ and*

$$\int_{\mathbb{R}^n} \rho_0 \, \mathrm{d}\boldsymbol{x} < \Big(\ell_1 \frac{(2n-\alpha)\gamma - 2n}{(n-\alpha)(\gamma-1)}\Big)^{\frac{(n-\alpha)(\gamma-1)}{(2n-\alpha)\gamma-2n}} \Big(\frac{(2n-\alpha)\gamma-2n}{n+\alpha-n\gamma} E(\rho_0, \boldsymbol{\mathcal{M}}_0)\Big)^{\frac{n\gamma-n-\alpha}{(2n-\alpha)\gamma-2n}}. \tag{3.16}$$

Then for $R(t) = \max_{\boldsymbol{x}\in\Omega(t)}\{|\boldsymbol{x}|\}$, we have that $R(t) \to \infty$ as $t \to \infty$.

Remark 3.2.9. From the calculation in Lemma 3.8.3, we see that for any $\mu > 0$ and minimiser $\bar{\rho}$ of S_μ over $\mathcal{K}$, that $Q(\bar{\rho}) = 0$ and

$$\int_{\mathbb{R}^n} \bar{\rho}\,\mathrm{d}\boldsymbol{x} \leq \Big(\frac{(2n-\alpha)\gamma - 2n}{n+\alpha-n\gamma} E(\bar{\rho},\mathbf{0})\Big)^{\frac{n\gamma-n-\alpha}{(2n-\alpha)\gamma-2n}} \Big(\ell_1 \frac{(2n-\alpha)\gamma-2n}{(n-\alpha)(\gamma-1)}\Big)^{\frac{(n-\alpha)(\gamma-1)}{(2n-\alpha)\gamma-2n}}.$$

Taking $0 < \lambda < 1$, by Lemma 3.7.1 (d), (e) and (f), we see that $Q(\bar{\rho}_\lambda) > 0$ and

$$E(\bar{\rho}_\lambda, \mathbf{0}) = S_{\mu,\lambda}(\bar{\rho}) - \mu \int_{\mathbb{R}^n} \bar{\rho}_\lambda\,\mathrm{d}\boldsymbol{x} < S_\mu(\bar{\rho}) - \mu \int_{\mathbb{R}^n} \bar{\rho}\,\mathrm{d}\boldsymbol{x} = E(\bar{\rho},\mathbf{0}).$$

Since, for $\frac{2n}{2n-\alpha} < \gamma < \frac{n+\alpha}{n}$ we see that $\frac{2n-(2n-\alpha)\gamma}{n+\alpha-n\gamma} < 0$, we have that for $0 < \lambda < 1$, $(\bar{\rho}_\lambda, \mathbf{0})$ satisfies (3.16). One can see that there exists arbitrarily small perturbations $(\rho_0, \boldsymbol{\mathcal{M}}_0)$ of $(\bar{\rho}, 0)$ that satisfy $Q(\rho_0) > 0$ and (3.16). Thus, since $\bar{\rho}$ is a stationary state with compact support as proved in Theorem 3.2.7, this proves instability.

3.3 Existence of stationary states in the mass-subcritical regime

In this section, we consider the existence of stationary solutions of the CEREs (1.1) for a general pressure, with sufficiently nice asymptotic behaviour at the origin and infinity, which are inclusive of the mass-subcritical regime $\gamma > \frac{n+\alpha}{n}$ in the polytropic case. We recall the definition of the admissible set X_M:

$$X_M \coloneqq \Big\{\varrho \in L^1_+(\mathbb{R}^n) \cap L^{\frac{n+\alpha}{n}}(\mathbb{R}^n) \;\Big|\; \int_{\mathbb{R}^n} \varrho(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = M\Big\}.$$

For $\varrho \in X_M$, we recall the definition of the free-energy functional $\mathcal{G}$:

$$\mathcal{G}(\varrho) \coloneqq \int_{\mathbb{R}^n} \Big((\varrho e(\varrho))(\boldsymbol{x}) + \frac{1}{2}\varrho(\boldsymbol{x})(\Phi_\alpha * \varrho)(\boldsymbol{x})\Big)\,\mathrm{d}\boldsymbol{x}.$$

We shall employ the classical concentration compactness principle of Lions in [103, Theorem II.2] to prove existence of minimisers to the problem

$$g_M = \inf_{\varrho \in X_M} \mathcal{G}(\varrho).$$

We first prove some facts about minimisers of general free-energies. To do this, we consider a general non-local interaction kernel of the form

$$\Psi \in L^p_{\mathrm{w}}(\mathbb{R}^n)\,(\text{weak } L^p \text{ space}),\ \text{for } p \in (1,\infty),\ \text{with } \Psi \geq 0\ a.e., \tag{3.17}$$

such that there exists $\beta \in (0, n)$ satisfying

$$\Psi(t\xi) \geq t^{-\beta}\Psi(\xi), \tag{3.18}$$

for all $t \geq 1$ and *a.e.* $\xi \in \mathbb{R}^n$. We take a convex function Π satisfying

$$\begin{cases} \Pi : [0,\infty) \to [0,\infty) \text{ strictly convex}, \\ \lim_{t\to 0^+} \Pi(t)t^{-1} = 0, \\ \liminf_{t\to\infty} \Pi(t)t^{-q} > 0, \end{cases} \tag{3.19}$$

where $q = \frac{p+1}{p}$. We define the associated free-energy

$$\mathcal{F}(\varrho) := \int_{\mathbb{R}^n} \left(\Pi(\varrho(\boldsymbol{x})) - \frac{1}{2}\varrho(\boldsymbol{x})(\Psi * \varrho)(\boldsymbol{x}) \right) \mathrm{d}\boldsymbol{x},$$

with the corresponding admissible set

$$\mathcal{A}_M := \left\{ \varrho \in L^1_+(\mathbb{R}^n) \cap L^q(\mathbb{R}^n) \,\middle|\, \int_{\mathbb{R}^n} \varrho \, \mathrm{d}\boldsymbol{x} = M \right\}.$$

We define the minimum of $\mathcal{F}$ over $\mathcal{A}_M$ by

$$F_M := \inf_{\varrho\in\mathcal{A}_M} \mathcal{F}(\varrho).$$

Under analogous conditions on the mass M to that of (3.4) and (3.5), one can prove useful facts about the sign of the free-energy.

Lemma 3.3.1. *Suppose that Ψ and Π satisfy (3.17) and (3.19) respectively.*

(a) *Suppose*

$$M < \left(\frac{2 \liminf_{t\to\infty} \Pi(t)t^{-q}}{C^\Psi_*} \right)^{\frac{1}{2-q}}, \tag{3.20}$$

where $C^\Psi_* > 0$ is the sharp constant from the weak Young's convolution inequality Lemma A.0.4*, then we have that $F_M > -\infty$.*

(b) *Suppose that Ψ satisfies (3.18) for $\beta \leq \frac{n}{p}$ and $\limsup_{t\to 0} \Pi(t)t^{-q} < \infty$. If $\beta = \frac{n}{p}$, suppose*

$$M > \left(\frac{2 \limsup_{t\to 0} \Pi(t)t^{-q}}{C^\Psi_*} \right)^{\frac{1}{2-q}}, \tag{3.21}$$

then we have that $F_M < 0$.

Proof. (a) We can take ρ^* large enough such that

$$\Pi(\varrho)\varrho^{-q} > \frac{C^\Psi_*}{2} M^{2-q},$$

for all $\varrho \geq \rho^*$. Then, for any $\varrho \in \mathcal{A}_M$, applying the weak Young's convolution inequality, we have that

$$\mathcal{F}(\varrho) \geq \int_{\{\varrho\geq\rho^*\}} \Pi(\varrho(\boldsymbol{x})) \, \mathrm{d}\boldsymbol{x} - \frac{C^\Psi_*}{2} \|\varrho\|^{2-q}_{L^1} \|\varrho\|^q_{L^q}$$

$$
\begin{aligned}
&\geq \int_{\{\varrho\geq\rho^*\}} \varrho^q\Big(\Pi(\varrho(\boldsymbol{x}))\varrho^{-q} - \frac{C_*^\Psi}{2}M^{2-q}\Big)\,\mathrm{d}\boldsymbol{x} - \frac{C_*^\Psi}{2}M^{2-q}\int_{\{\varrho<\rho^*\}}\varrho^q\,\mathrm{d}\boldsymbol{x}\\
&\geq -\frac{C_*^\Psi}{2}M^{2-q}(\varrho^*)^{q-1}\int_{\{\varrho<\rho^*\}}\varrho\,\mathrm{d}\boldsymbol{x}\\
&\geq -\frac{C_*^\Psi}{2}M^{3-q}(\rho^*)^{q-1},
\end{aligned}
$$

where $\Pi \geq 0$ was used in the first inequality. Thus, $F_M > -\infty$.

(b) There exists $\theta \in (0,1)$ such that

$$
\delta_{\beta,\frac{n}{p}}\limsup_{t\to 0}\Pi(t)t^{-q} < \frac{\theta C_*^\Psi}{2}M^{2-q},
$$

where

$$
\delta_{\beta,\frac{n}{p}} = \begin{cases} 1, & \beta = \frac{n}{p},\\ 0, & \beta < \frac{n}{p}.\end{cases}
$$

By Lemma A.0.4 and the sharpness of $C_*^\Psi > 0$, there exists $\varrho \in L^1_+(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)$ such that

$$
\int_{\mathbb{R}^n}\varrho\Psi * \varrho\,\mathrm{d}\boldsymbol{x} > \theta C_*^\Psi\|\varrho\|_{L^q}^q\|\varrho\|_{L^1}^{2-q}. \tag{3.22}
$$

By density of $C_\mathrm{c}(\mathbb{R}^n)$ in $L^1_+(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)$, we may assume that $\varrho \in C_\mathrm{c}(\mathbb{R}^n)$ and is thus a bounded function. Moreover, by employing the rescaling

$$
\tilde{\varrho}(\boldsymbol{x}) := \varrho\bigg(\boldsymbol{x}\Big(\frac{\|\varrho\|_{L^1}}{M}\Big)^{\frac{1}{n}}\bigg),
$$

we have that $\tilde{\varrho} \in \mathcal{A}_M$. Now, employing the mass-preserving scaling given in (3.3), we have that $\tilde{\varrho}_\lambda \in \mathcal{A}_M$ for all $\lambda > 0$. Since Ψ satisfies (3.18) for $\beta \leq \frac{n}{p}$, we combine this with (3.22) to see for $\lambda < \frac{M}{\|\varrho\|_{L^1}}$ that

$$
\begin{aligned}
\int_{\mathbb{R}^n}\tilde{\varrho}_\lambda\Psi * \tilde{\varrho}_\lambda\,\mathrm{d}\boldsymbol{x} &= \Big(\frac{M}{\|\varrho\|_{L^1}}\Big)^2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\varrho(\boldsymbol{x})\varrho(\boldsymbol{y})\Psi\bigg(\Big(\frac{M}{\lambda\|\varrho\|_{L^1}}\Big)^{\frac{1}{n}}(\boldsymbol{x}-\boldsymbol{y})\bigg)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\\
&\geq \lambda^{\frac{\beta}{n}}\Big(\frac{M}{\|\varrho\|_{L^1}}\Big)^{2-\frac{\beta}{n}}\int_{\mathbb{R}^n}\varrho\Psi * \varrho\,\mathrm{d}\boldsymbol{x}\\
&\geq \theta C_*^\Psi M^{2-\frac{\beta}{n}}\lambda^{\frac{\beta}{n}}\|\varrho\|_{L^q}^q\|\varrho\|_{L^1}^{\frac{\beta}{n}-q}.
\end{aligned}
$$

Thus, taking $\lambda < \frac{M}{\|\varrho\|_{L^1}}$ we obtain

$$
\begin{aligned}
\mathcal{F}(\tilde{\varrho}_\lambda)\lambda^{-\frac{\beta}{n}}\Big(\frac{M}{\|\varrho\|_{L^1}}\Big)^{\frac{\beta-nq}{n}} &\leq \Big(\frac{M}{\lambda\|\varrho\|_{L^1}}\Big)^{\frac{\beta}{n}-\frac{1}{p}}\int_{\mathbb{R}^n}\frac{\Pi(\lambda\varrho)}{(\lambda\varrho)^q}\varrho^q\,\mathrm{d}\boldsymbol{x} - \frac{\theta C_*^\Psi}{2}\|\varrho\|_{L^q}^q M^{2-q}\\
&= \int_{\{\varrho>0\}}\bigg(\Big(\frac{M}{\lambda\|\varrho\|_{L^1}}\Big)^{\frac{\beta}{n}-\frac{1}{p}}\frac{\Pi(\lambda\varrho)}{(\lambda\varrho)^q} - \frac{\theta C_*^\Psi}{2}M^{2-q}\bigg)\varrho^q\,\mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

Since ϱ is a bounded function and

$$\limsup_{\lambda\to 0}\lambda^{\frac{1}{p}-\frac{\beta}{n}}\frac{\Pi(\lambda\varrho)}{(\lambda\varrho)^q}\leq\frac{C_*^\Psi}{2}M^{2-q},$$

for $\varrho>0$, the term $\lambda^{\frac{1}{p}-\frac{\beta}{n}}\frac{\Pi(\lambda\varrho)}{(\lambda\varrho)^q}$ is bounded for $\lambda<\frac{M}{\|\varrho\|_{L^1}}$. Hence, there exists $C>0$ such that $\lambda^{\frac{1}{p}-\frac{\beta}{n}}\frac{\Pi(\lambda\varrho)}{(\lambda\varrho)^q}\leq C$ for $\lambda<\frac{M}{\|\varrho\|_{L^1}}$, and so we can apply the Reverse Fatou Lemma to obtain that

$$\limsup_{\lambda\to 0}\mathcal{F}(\tilde{\varrho}_\lambda)\lambda^{-\frac{\beta}{n}}\left(\frac{M}{\|\varrho\|_{L^1}}\right)^{\frac{\beta-nq}{n}}\leq\|\varrho\|_{L^q}^q\left(\delta_{\beta,\frac{n}{p}}\limsup_{t\to 0}\Pi(t)t^{-q}-\frac{\theta C_*^\Psi}{2}M^{2-q}\right)<0.$$

Hence, we have that $F_M<0$.

□

For the sake of completeness, we remind the reader of the main result in [103].

Theorem 3.3.2 ([103, Theorem II.2]). *Suppose that Ψ, Π satisfy (3.17), (3.19) respectively and that (3.20) is satisfied. Then for any minimising sequence $\{\varrho_m\}_m\subset\mathcal{A}_M$, there exists $\boldsymbol{y}_m\in\mathbb{R}^n$ and a minimiser $\bar{\rho}\in\mathcal{A}_M$ to $\mathcal{F}$ (i.e. $\mathcal{F}(\bar{\rho})=F_M$) with $T^{\boldsymbol{y}_m}\varrho_m$ compact in $L^1(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)$, with (up to a subsequence) $T^{\boldsymbol{y}_m}\varrho_m\to\bar{\rho}$ in $L^1(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)$ if and only if*

$$F_M<F_m+F_{M-m},\qquad \text{for all } m\in(0,M). \tag{3.23}$$

In order to prove existence of minimisers for our problem, it suffices to prove (3.23) in order to apply Theorem 3.3.2. In order to verify condition (3.23), we also employ a corollary from [103].

Corollary 3.3.3 ([103, Corollary II.1]). *Suppose (3.17) and (3.19) are satisfied, if for some $\beta\in(0,n)$,*

$$\Psi(t\xi)\geq t^{-\beta}\Psi(\xi),$$

for all $t\geq 1$ and $\xi\in\mathbb{R}^n$ a.e., then (3.23) is satisfied if and only if $F_M<0$.

Under the assumptions of Theorem 3.3.2, (3.18) and (3.21), Corollary 3.3.3 implies that (3.23) is satisfied. Thus, Theorem 3.3.2 implies there exists a minimiser $\bar{\rho}\in\mathcal{A}_M$, that is, $\mathcal{F}(\bar{\rho})=F_M$. We now show under radial conditions on the interaction kernel Ψ, that minimisers indeed are radially symmetric (up to translation). First of all, we prove a generalisation of Cavalieri's formula. To do this, we use the corollary from [123].

Corollary 3.3.4 ([123, Corollary 9]). *Suppose $I\subset\mathbb{R}$ is a subinterval and $g:I\to\mathbb{R}$ is injective, locally absolutely continuous, and of local bounded variation. Letting $J=g(I)$, if $f:J\to\mathbb{R}$ is measurable and locally bounded on J. Then we have that*

$$\int_J f(t)\,\mathrm{d}t=\int_I f(g(t))g'(t)\,\mathrm{d}t.$$

Lemma 3.3.5. *Let (X, μ) be a topological measure space and suppose Π satisfies (3.19), then for all measurable functions $\varrho : (X, \mu) \to \mathbb{R}$ such that*

$$\int_X \Pi(|\varrho|)\,\mathrm{d}\mu < \infty,$$

we have that the Cavalieri's formula holds, i.e.,

$$\int_X \Pi(|\varrho|)\,\mathrm{d}\mu = \int_0^\infty \Pi'(\lambda)\mu(\{|\varrho| > \lambda\})\,\mathrm{d}\lambda. \tag{3.24}$$

Proof. Note that since Π is convex on $[0, \infty)$, it is differentiable *a.e.*, and therefore (3.24) makes sense. By $(3.19)_2$, we have that $\lim_{t\to 0} \Pi(t)t^{-1} = 0$, which implies that $\Pi(0) = \Pi'_+(0) = 0$ and so since Π is strictly convex, we have that Π is strictly increasing on $[0, \infty)$ and is a bijection from $[0, \infty)$ to $[0, \infty)$. Thus, by applying the standard Cavalieri's formula to the identity function, we get

$$\begin{aligned}\int_X \Pi(|\varrho|)\,\mathrm{d}\mu &= \int_0^\infty \mu(\{\Pi(|\varrho|) > \tau\})\,\mathrm{d}\tau \\ &= \int_0^\infty \mu(\{|\varrho| > \Pi^{-1}(\tau)\})\,\mathrm{d}\tau \\ &= \int_0^\infty \Pi'(\tau)\mu(\{|\varrho| > \lambda\})\,\mathrm{d}\lambda,\end{aligned}$$

where the last line follows from Corollary 3.3.4 with $I = (0, \infty)$, $g = \Pi : (0, \infty) \to (0, \infty)$, $f = \mu(\{|\varrho| > \cdot\}) : (0, \infty) \to (0, \infty)$, which satisfy the properties, as Π being strictly convex, implies local absolute continuity and local bounded variation. □

Lemma 3.3.6. *Suppose the conditions of Lemma 3.3.1 are satisfied, with Ψ strictly radially decreasing. Then there exists a radial minimiser $\bar{\varrho} \in \mathcal{A}_M$ of $\mathcal{F}$ over $\mathcal{A}_M$. Moreover, any minimiser of $\mathcal{F}$ over $\mathcal{A}_M$ is the translation of a radially symmetric minimiser.*

Proof. We can now apply Cavalieri's formula to obtain that for a minimiser $\bar{\rho} \in \mathcal{A}_M$, taking the radially decreasing rearrangement $\rho^\#$ of $\bar{\rho}$, we have that $\rho^\# \in \mathcal{A}_M$ and

$$\int_{\mathbb{R}^n} \Pi(\rho^\#)\,\mathrm{d}\boldsymbol{x} = \int_0^\infty \Pi'(\tau)|\{\rho^\# > \lambda\}|\,\mathrm{d}\lambda = \int_0^\infty \Pi'(\tau)|\{\bar{\rho} > \lambda\}|\,\mathrm{d}\lambda = \int_{\mathbb{R}^n} \Pi(\bar{\rho})\,\mathrm{d}\boldsymbol{x}.$$

We have that if Ψ is radially symmetric and strictly decreasing in r, then $\Psi^* = \Psi$ and by Lemma A.0.5, we have

$$-\int_{\mathbb{R}^n} \rho^\#(\boldsymbol{x})(\Psi * \rho^\#)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \le -\int_{\mathbb{R}^n} \bar{\rho}(\boldsymbol{x})(\Psi * \bar{\rho})(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},$$

with equality only if $\bar{\rho} = \rho^\#(\cdot - \boldsymbol{y})$ for some $\boldsymbol{y} \in \mathbb{R}^n$. Thus, we have $\mathcal{F}(\rho^\#) \le \mathcal{F}(\bar{\rho})$. Since $\bar{\rho}$ is a minimiser of $\mathcal{F}$ over $\mathcal{A}_M$, we have equality and that $\bar{\rho}$ is the translation of a radial minimiser. □

We now apply the general framework to our particular minimisation problem. We take $\Psi = -\Phi_\alpha$, which satisfies (3.17) for $p = \frac{n}{\alpha}$, $q = \frac{p+1}{p} = \frac{n+\alpha}{n}$ and satisfies (3.18), in fact with an equality for $\beta = \alpha$, and for all $\xi \in \mathbb{R}^n \setminus \{0\}$. For the internal energy function $\Pi(\varrho) := \varrho e(\varrho)$, we have that taking the second derivative, that $\Pi''(\varrho) = \frac{p'(\varrho)}{\varrho} > 0$ for $\varrho > 0$ from the assumption (1.4) on p. This implies that Π is strictly convex on $[0, \infty)$ and we can clearly check from the formula for $e(\varrho)$, (1.4), (3.4) and (3.5), that it is non-negative and satisfies (3.19). If

$$M > \left(\frac{2n \limsup_{\rho \to 0} p(\rho) \rho^{-\frac{n+\alpha}{n}}}{C_*} \right)^{\frac{n}{n-\alpha}},$$

we have $g_M < 0$ by Lemma 3.3.1. If

$$M < \left(\frac{2n \liminf_{\rho \to \infty} p(\rho) \rho^{-\frac{n+\alpha}{n}}}{C_*} \right)^{\frac{n}{n-\alpha}},$$

we have $g_M > -\infty$ by Lemma 3.3.1. By Theorem 3.3.2 and Corollary 3.3.3 we deduce that there exists a minimiser $\bar{\rho} \in X_M$ to $\mathcal{G}$ over X_M. Since $\Psi = -\Phi_\alpha$ is radially symmetric and strictly radially decreasing, we apply Lemma 3.3.6 to prove the existence of a radially symmetric minimiser of the functional $\mathcal{G}$ over X_M. Thus, we obtain the following corollary.

Corollary 3.3.7. *Suppose $\alpha \in (0, n)$, $M > 0$ with $p \in C^1([0, \infty))$ satisfying (1.4), (3.4) and (3.5). Then there exists a radially symmetric minimiser $\bar{\rho} \in X_M$ to $\mathcal{G}$.*

Combining this with the approach of [6, 117], one obtains the following corollary.

Corollary 3.3.8. *Under the conditions of Corollary 3.3.7, if $\bar{\rho} \in X_M$ is a minimiser of $\mathcal{G}$ over X_M, then $\bar{\rho}$ is a stationary state as given by Definition 3.1.1 with compact support.*

3.4 Stability of stationary states in the mass-subcritical regime

For this section, we assume uniqueness of the global minimiser (up to translation) as required in Theorem 3.2.2; see [23, Theorem 5.5] for a summary of known results on the uniqueness of minimisers. We now turn to the question of stability of the global minimisers found in the previous section, assumed to be unique (up to translation). We have the following proof for stability of stationary states.

Proof of Theorem 3.2.2. We follow a similar approach to Rein [117], however, we take advantage of having uniqueness of global minimisers modulo translations. Suppose the statement in Theorem 3.2.2 is not true, then there exist $\varepsilon > 0$ and a sequence of

global weak solutions $(\rho^j, \boldsymbol{\mathcal{M}}^j)$ of the attractive CEREs associated to the initial data $(\rho_0^j, \boldsymbol{\mathcal{M}}_0^j)$ and a sequence of times $t_j > 0$ such that

$$d(\rho_0^j, \bar{\rho}) + \|\rho_0^j - \bar{\rho}\|^2_{L^{\frac{2n}{2n-\alpha}}} + \frac{1}{2}\int_{\mathbb{R}^n} \left|\frac{\boldsymbol{\mathcal{M}}_0^j}{\sqrt{\rho_0^j}}\right|^2 \mathrm{d}\boldsymbol{x} < \frac{1}{j}, \tag{3.25}$$

and

$$d(\rho^j(t_j), T^{\boldsymbol{y}}\bar{\rho}) + \|\rho^j(t_j) - T^{\boldsymbol{y}}\bar{\rho}\|^2_{L^{\frac{2n}{2n-\alpha}}} + \frac{1}{2}\int_{\mathbb{R}^n} \left|\frac{\boldsymbol{\mathcal{M}}^j}{\sqrt{\rho^j}}\right|^2(t_j) \mathrm{d}\boldsymbol{x} \geq \varepsilon, \tag{3.26}$$

for all $\boldsymbol{y} \in \mathbb{R}^n$. Notice that the sequence of times t_j can be chosen outside a zero measure set, where the terms in (3.26) are well-defined according to Definition 2.1.1. Then, by (3.25) and (3.7), we have

$$\lim_{j\to\infty} E(\rho_0^j, \boldsymbol{\mathcal{M}}_0^j) = E(\bar{\rho}, 0) = \mathcal{G}(\bar{\rho}).$$

Since the energy E for the weak solutions is non-increasing from the initial energy by definition, we have

$$\limsup_{j\to\infty} \mathcal{G}(\rho^j(t_j)) \leq \limsup_{j\to\infty} E(\rho^j(t_j), \boldsymbol{\mathcal{M}}^j(t_j)) \leq \lim_{j\to\infty} E(\rho_0^j, \boldsymbol{\mathcal{M}}_0^j) = \mathcal{G}(\bar{\rho}).$$

Thus, $\rho^j(t_j) \in X_M$ is a minimising sequence of $\mathcal{G}$ in X_M and, by Theorem 3.3.2, there exists $\boldsymbol{y}_j \in \mathbb{R}^n$ such that (up to a subsequence) $T^{\boldsymbol{y}_j}\rho^j(t_j) \to \bar{\rho}$ in $L^1(\mathbb{R}^n) \cap L^{\frac{n+\alpha}{n}}(\mathbb{R}^n)$, due to the uniqueness of $\bar{\rho}$ up to translation. Then

$$\|\rho^j(t_j) - T^{-\boldsymbol{y}_j}\bar{\rho}\|_{L^{\frac{2n}{2n-\alpha}}} = \|T^{\boldsymbol{y}_j}\rho^j(t_j) - \bar{\rho}\|_{L^{\frac{2n}{2n-\alpha}}} \longrightarrow 0, \qquad \text{as } j \to \infty,$$

since $\frac{2n}{2n-\alpha} \in (1, \frac{n+\alpha}{n})$. We see that, for any i,

$$\lim_{j\to\infty} E(\rho^j(t_j), \boldsymbol{\mathcal{M}}^j(t_j)) = \mathcal{G}(\bar{\rho}) = \mathcal{G}(T^{-\boldsymbol{y}_i}\bar{\rho}). \tag{3.27}$$

Therefore, by (3.6),

$$d(\rho^j(t_j), T^{-\boldsymbol{y}_j}\bar{\rho}) + \frac{1}{2}\int_{\mathbb{R}^n} \left|\frac{\boldsymbol{\mathcal{M}}^j}{\sqrt{\rho^j}}\right|^2(t_j) \mathrm{d}\boldsymbol{x} \longrightarrow 0, \qquad \text{as } j \to \infty. \tag{3.28}$$

This is a contradiction to (3.26). □

Remark 3.4.1. Carrillo-Castorina-Volzone [26] proved that, for $n = 2$, $\alpha = 0$, and $\gamma > 1$, there exists a unique global minimiser of $\mathcal{G}$ in X_M. This is the unique radially decreasing, compactly supported, and smooth in its support stationary state of (1.1) in the sense of distributions; however, in [26], only the weak compactness of the minimising sequence was obtained. In this thesis, we require the strong compactness of the minimising sequence, allowing us to control the relative potential energy. For the

Riesz potential (i.e. $\alpha \in (0, n)$), we are able to control the relative potential energy by the relative $L^{\frac{2n}{2n-\alpha}}$–norm of the densities. However, in the logarithmic potential case, it is no longer apparent how to find the natural controlling relative entropy or how we might obtain such a bound, so that we cannot control the relative potential energy and use (3.6) and (3.27) to obtain (3.28), since

$$\frac{1}{2}\int_{\mathbb{R}^n} (\rho(t) - \bar{\rho})\, \Phi_0 * (\rho(t) - \bar{\rho})\, \mathrm{d}\boldsymbol{x},$$

does not have a sign. Thus, we are unable to use a contradiction argument as structured in the previous argument for this case.

Remark 3.4.2. Suppose that $\bar{\rho}$ is a unique minimiser of $\mathcal{G}$ in X_M. Without loss of generality, suppose that $0 < \delta = \delta(\varepsilon) < \varepsilon$ satisfies Theorem 3.2.2. Define

$$A_\varepsilon \coloneqq \left\{ \boldsymbol{y} \in \mathbb{R}^n \,\middle|\, \begin{array}{l} (3.8)\text{–}(3.9) \text{ are satisfied with } \varepsilon \text{ and } \delta(\varepsilon) \\ \text{for a global weak solution of the CEREs} \end{array} \right\}.$$

Then, for $\boldsymbol{y} \in A_\varepsilon$, there exists a weak solution $(\rho, \boldsymbol{\mathcal{M}})$ of the CEREs such that (3.8)–(3.9) are satisfied for ε and $\delta(\varepsilon)$:

$$\|\bar{\rho} - T^{\boldsymbol{y}}\bar{\rho}\|^2_{L^{\frac{2n}{2n-\alpha}}} \leq \|\bar{\rho} - \rho_0\|^2_{L^{\frac{2n}{2n-\alpha}}} + \|\rho_0 - T^{\boldsymbol{y}}\bar{\rho}\|^2_{L^{\frac{2n}{2n-\alpha}}} \leq \varepsilon + \delta(\varepsilon).$$

Thus, as $\varepsilon \to 0$, we have

$$\sup_{\boldsymbol{y} \in A_\varepsilon} \|\bar{\rho} - T^{\boldsymbol{y}}\bar{\rho}\|^2_{L^{\frac{2n}{2n-\alpha}}} \longrightarrow 0.$$

Since $\bar{\rho} \in L^{\frac{2n}{2n-\alpha}}(\mathbb{R}^n)$, we must have

$$\sup_{\boldsymbol{y} \in A_\varepsilon} |\boldsymbol{y}| \longrightarrow 0, \qquad \text{as } \varepsilon \to 0.$$

That is, the closer we start to $\bar{\rho}$, the closer we remain to $\bar{\rho}$, not just a translation of $\bar{\rho}$.

Remark 3.4.3. In case we no longer have the uniqueness of the minimisers, as in [117], we can define the set of minimisers of $\mathcal{G}$ in X_M by

$$\mathfrak{M}_M \coloneqq \{\bar{\rho} \in X_M \,|\, \mathcal{G}(\bar{\rho}) = g_M\},$$

and prove the following corollary.

Corollary 3.4.4 (Nonlinear stability of the CEREs)**.** *Suppose $\alpha \in (0, n)$ with $p \in C^1([0, \infty))$ satisfying (1.4), (3.4) and (3.5). Let $(\rho, \boldsymbol{\mathcal{M}})$ be a global weak solution of the attractive CEREs ($\kappa = 1$) in the sense of* Definition 2.1.1 *satisfying*

$$\int_{\mathbb{R}^n} \rho_0(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^n} \bar{\rho}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = M.$$

Then, for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, if

$$\inf_{\bar\rho\in\mathfrak{M}_M}\left\{d(\rho_0,\bar\rho) + \|\rho_0-\bar\rho\|^2_{L^{\frac{2n}{2n-\alpha}}} + \frac12\int_{\mathbb{R}^n}\left|\frac{\boldsymbol{\mathcal{M}}_0}{\sqrt{\rho_0}}\right|^2 \mathrm{d}\boldsymbol{x}\right\} < \delta,$$

then

$$\inf_{\bar\rho\in\mathfrak{M}_M}\left\{d(\rho(t),\bar\rho) + \|\rho(t)-\bar\rho\|^2_{L^{\frac{2n}{2n-\alpha}}} + \frac12\int_{\mathbb{R}^n}\left|\frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}}\right|^2(t)\,\mathrm{d}\boldsymbol{x}\right\} < \varepsilon, \qquad \text{for a.e. } t>0.$$

Furthermore, let $\bar\rho \in \mathfrak{M}_M$ be an isolated minimiser (up to translation) of the functional $\mathcal{G}$ in X_M, that is,

$$\delta_0 := \inf\left\{\|\rho-\bar\rho\|^2_{L^{\frac{2n}{2n-\alpha}}} \,\middle|\, \rho\in\mathfrak{M}_M\setminus\{T^{\boldsymbol{y}}\bar\rho\,|\,\boldsymbol{y}\in\mathbb{R}^n\}\right\} > 0,$$

and let

$$(t,\boldsymbol{y}) \longmapsto d(\rho(t),T^{\boldsymbol{y}}\bar\rho) + \|\rho(t)-T^{\boldsymbol{y}}\bar\rho\|^2_{L^{\frac{2n}{2n-\alpha}}} + \frac12\int_{\mathbb{R}^n}\left|\frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}}\right|^2(t)\,\mathrm{d}\boldsymbol{x}, \tag{3.29}$$

be continuous. Then, for any $0<\varepsilon<\frac{\delta_0}{4}$, there exists $0<\delta=\delta(\varepsilon)<\varepsilon$ such that, if

$$d(\rho_0,\bar\rho) + \|\rho_0-\bar\rho\|^2_{L^{\frac{2n}{2n-\alpha}}} + \frac12\int_{\mathbb{R}^n}\left|\frac{\boldsymbol{\mathcal{M}}_0}{\sqrt{\rho_0}}\right|^2 \mathrm{d}\boldsymbol{x} < \delta,$$

there exists a translation $\boldsymbol{y}\in\mathbb{R}^n$ such that

$$d(\rho(t),T^{\boldsymbol{y}}\bar\rho) + \|\rho(t)-T^{\boldsymbol{y}}\bar\rho\|^2_{L^{\frac{2n}{2n-\alpha}}} + \frac12\int_{\mathbb{R}^n}\left|\frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}}\right|^2(t)\,\mathrm{d}\boldsymbol{x} < \varepsilon, \qquad \text{for all } t>0.$$

Notice that this result needs the additional continuity assumption of (3.29); see [117] for details.

3.5 Relative entropy estimates in the mass-subcritical regime

We now turn to a more precise quantification of the stability of steady states in the polytropic case and mass-subcritical regime, that is, when $p(\rho)=a_0\rho^\gamma$ with $\gamma > \frac{n+\alpha}{n}$. The analysis is based on the relative entropy method. We begin by defining the density

$$h(\rho) = \frac{a_0}{\gamma-1}\,\rho^\gamma,$$

and the relative entropy

$$h(\rho|\bar\rho) := h(\rho) - h(\bar\rho) - h'(\bar\rho)(\rho-\bar\rho), \qquad H(t) := \int_{\mathbb{R}^n} h(\rho(t,\boldsymbol{x})|\bar\rho(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}. \tag{3.30}$$

To analyse the relative potential energy, we first establish the following lemmas, which allow us to control the L^γ norm of the density ρ and the difference $\rho-\bar\rho$ in terms of the relative entropy H defined in (3.30).

Lemma 3.5.1 (Control of $\|\rho\|_{L^\gamma}$)**.** *There exists a constant C depending on γ and $\|\bar\rho\|_{L^\gamma}$, such that for all $\rho, \bar\rho \in L^1_+(\mathbb{R}^n) \cap L^\gamma(\mathbb{R}^n)$,*

$$\|\rho\|_{L^\gamma} \le C\big(1 + H(t)^{1/\gamma}\big), \text{ with } H(t) = \int_{\mathbb{R}^n} h(\rho(t,\boldsymbol{x})|\bar\rho(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}.$$

Proof. It suffices to show that there exist constants $c_1, c_2 > 0$, depending only on γ, such that, for all $\rho, \bar\rho \ge 0$,

$$\int_{\mathbb{R}^n} \rho^\gamma \,\mathrm{d}\boldsymbol{x} \le c_1\, H(t) + c_2 \int_{\mathbb{R}^n} \bar\rho^\gamma \,\mathrm{d}\boldsymbol{x}. \tag{3.31}$$

We have

$$h(\rho|\bar\rho) = \frac{a_0}{\gamma-1}\big(\rho^\gamma - \bar\rho^\gamma - \gamma\bar\rho^{\gamma-1}(\rho-\bar\rho)\big) = \frac{a_0}{\gamma-1}\big(\rho^\gamma - \gamma\bar\rho^{\gamma-1}\rho + (\gamma-1)\bar\rho^\gamma\big). \tag{3.32}$$

Combining this with Young's inequality, we obtain a pointwise lower bound. In particular, for any $\varepsilon > 0$, we write

$$\gamma\bar\rho^{\gamma-1}\rho \le \varepsilon\rho^\gamma + C_\varepsilon \bar\rho^{(\gamma-1)\frac{\gamma}{\gamma-1}} = \varepsilon\rho^\gamma + C_\varepsilon\bar\rho^\gamma,$$

where $C_\varepsilon > 0$ depends only on ε and γ. Substituting this into (3.32),

$$h(\rho|\bar\rho) \ge \frac{a_0}{\gamma-1}\big(\rho^\gamma - \big(\varepsilon\rho^\gamma + C_\varepsilon\bar\rho^\gamma\big) + (\gamma-1)\bar\rho^\gamma\big) = \frac{a_0}{\gamma-1}\big((1-\varepsilon)\rho^\gamma + \big(\gamma - 1 - C_\varepsilon\big)\bar\rho^\gamma\big).$$

We choose $\varepsilon = \frac{1}{2}$, so that $1-\varepsilon = \frac{1}{2} > 0$, and retain the $\bar\rho^\gamma$ term with a (possibly negative) coefficient. Hence, there exist constants $c_1, c_2 > 0$ such that

$$h(\rho|\bar\rho) \ge c_1\rho^\gamma - c_2\bar\rho^\gamma.$$

Integrating over $\mathbb{R}^n$ yields

$$H(t) = \int_{\mathbb{R}^n} h(\rho|\bar\rho)\,\mathrm{d}\boldsymbol{x} \ge c_1\int_{\mathbb{R}^n}\rho^\gamma\,\mathrm{d}\boldsymbol{x} - c_2\int_{\mathbb{R}^n}\bar\rho^\gamma\,\mathrm{d}\boldsymbol{x},$$

which is equivalent to (3.31). The bound on $\|\rho\|_{L^\gamma}$ then follows by taking the $1/\gamma$-th power and using the fact that $\int_{\mathbb{R}^n}\bar\rho^\gamma\,\mathrm{d}\boldsymbol{x}$ is finite and independent of t. □

Lemma 3.5.2 (Control of $\|\rho - \bar\rho\|_{L^\gamma}$)**.** *There exists a constant $C > 0$, depending only on γ and $\|\bar\rho\|_{L^\gamma}$, such that for all $\rho, \bar\rho \in L^1_+(\mathbb{R}^n) \cap L^\gamma(\mathbb{R}^n)$,*

$$\|\rho - \bar\rho\|_{L^\gamma} \le C\big(1 + H(t)^{1/\gamma}\big).$$

Proof. It suffices to show that there exists a constant $C > 0$, depending only on γ and $\|\bar\rho\|_{L^\gamma}$, such that, for all $\rho, \bar\rho \ge 0$,

$$|\rho - \bar\rho|^\gamma \le C\big(h(\rho|\bar\rho) + \bar\rho^\gamma\big). \tag{3.33}$$

The proof is based on a scaling argument. Fix $b \geq 0$ and consider, for $a \geq 0$ with $a \neq b$,

$$F(a,b) := \frac{h(a|b) + b^\gamma}{|a-b|^\gamma}.$$

We claim that

$$m := \inf_{b\geq 0} \inf_{a\geq 0,\ a\neq b} F(a,b) > 0.$$

Assuming this for the moment, we obtain that, for all $a, b \geq 0$ with $a \neq b$,

$$|a-b|^\gamma \leq m^{-1}\big(h(a|b) + b^\gamma\big),$$

and the inequality trivially holds when $a = b$. This yields (3.33) with $C = m^{-1}$. It remains to show that $m > 0$. To this end, we exploit the homogeneity of h. A direct computation shows that $h(\lambda a|\lambda b) = \lambda^\gamma h(a|b)$ for all $\lambda > 0$, while $(\lambda b)^\gamma = \lambda^\gamma b^\gamma$ and $|\lambda a - \lambda b|^\gamma = \lambda^\gamma |a-b|^\gamma$. Thus,

$$F(\lambda a, \lambda b) = F(a,b),$$

and we may, without loss of generality, fix $b = 1$. Define

$$f(a) := F(a,1) = \frac{h(a|1)+1}{|a-1|^\gamma}, \qquad \text{for } a \geq 0,\ a \neq 1.$$

We analyse f on $(0,\infty)\setminus\{1\}$.

Case 1. Behaviour as $a \to \infty$. Using the explicit formula

$$h(a|1) = \frac{a_0}{\gamma-1}\big(a^\gamma - 1 - \gamma(a-1)\big) \sim \frac{a_0}{\gamma-1}a^\gamma, \qquad \text{as } a \to \infty,$$

we obtain

$$f(a) \sim \frac{a_0\, a^\gamma + \gamma - 1}{(\gamma-1)(a-1)^\gamma} \longrightarrow \frac{a_0}{\gamma-1} > 0, \qquad \text{as } a \to \infty.$$

Case 2. Behaviour as $a \to 0$. We have

$$h(0|1) = \frac{a_0}{\gamma-1}(0 - 1 - \gamma(0-1)) = a_0\frac{\gamma-1}{\gamma-1} = a_0,$$

so $h(a|1) \to a_0$ as $a \to 0$, and hence

$$f(a) \longrightarrow \frac{a_0+1}{1} = a_0 + 1 > 0, \qquad \text{as } a \to 0.$$

Case 3. Behaviour as $a \to 1$. Writing $a = 1+\varepsilon$ with $|\varepsilon| \ll 1$, a Taylor expansion yields

$$h(1+\varepsilon|1) = \frac{a_0}{\gamma-1}\Big((1+\varepsilon)^\gamma - 1 - \gamma\varepsilon\Big) = \frac{a_0}{\gamma-1}\Big(\gamma(\gamma-1)\frac{\varepsilon^2}{2} + O(|\varepsilon|^3)\Big) = \frac{a_0\gamma}{2}\varepsilon^2 + O(|\varepsilon|^3).$$

Hence $h(1+\varepsilon|1) + 1 = 1 + O(\varepsilon^2)$, and

$$f(1+\varepsilon) = \frac{1+O(\varepsilon^2)}{|\varepsilon|^\gamma} \longrightarrow \infty, \qquad \text{as } \varepsilon \to 0.$$

Thus, f is continuous on $(0,\infty)\setminus\{1\}$, has positive finite limits as $a\to 0$ and $a\to\infty$, and diverges to $+\infty$ as $a\to 1$. Hence, its infimum on $(0,\infty)\setminus\{1\}$ is attained and strictly positive. We denote this infimum by $m_1>0$. By the scaling argument, $m_1=m$, and the claim follows. □

We are now in a position to estimate the relative potential energy.

Lemma 3.5.3. *Under the assumptions of* Proposition 3.2.5*, there exists a constant C depending on n,α,γ,a_0 and $\|\bar\rho\|_{L^1\cap L^\gamma}$ such that*

$$\int_{\mathbb{R}^n}\rho\,|\nabla\mathcal{W}_\alpha|^2\,\mathrm{d}\boldsymbol{x},\quad \int_{\mathbb{R}^n}\rho\,|\nabla\mathcal{W}_\alpha-\nabla\bar{\mathcal{W}}_\alpha|^2\,\mathrm{d}\boldsymbol{x}\;\le\;C\big(1+H(t)^{\sigma_*}\big),\tag{3.34}$$

with

$$\sigma_*=\frac{2(\alpha+1)}{n(\gamma-1)}.$$

Proof. We prove only the second estimate,

$$\int_{\mathbb{R}^n}\rho\,|\nabla\mathcal{W}_\alpha-\nabla\bar{\mathcal{W}}_\alpha|^2\,\mathrm{d}\boldsymbol{x}\;\le\;C\big(1+H(t)^{\sigma_*}\big),$$

since the first estimate follows analogously from Lemma 3.5.1. We divide the proof into two steps.

Step 1.

$$\begin{aligned}\int_{\mathbb{R}^n}\rho\,|\nabla\mathcal{W}_\alpha-\nabla\bar{\mathcal{W}}_\alpha|^2\;\mathrm{d}\boldsymbol{x}&\le\int_{\mathbb{R}^n}\rho\,|\,|\cdot|^{-(\alpha+1)}*(\rho-\bar\rho)|^2\,\mathrm{d}\boldsymbol{x}\\&=\|\rho^{1/2}\,|\cdot|^{-(\alpha+1)}*(\rho-\bar\rho)\|_{L^2}^2.\end{aligned}\tag{3.35}$$

Fix $\gamma>1$ and denote its Hölder conjugate by γ', that is,

$$\frac{1}{\gamma}+\frac{1}{\gamma'}=1.$$

Since $\gamma>\frac{3n}{3n-2(\alpha+1)}$, we can apply Hölder's inequality together with the HLS inequality to obtain

$$\begin{aligned}\|\rho^{1/2}\,|\cdot|^{-(\alpha+1)}*(\rho-\bar\rho)\|_{L^2}^2&\le\|\rho\|_{L^{\frac{3n}{3n-2(\alpha+1)}}}\|\,|\cdot|^{-(\alpha+1)}*(\rho-\bar\rho)\|^2_{L^{\frac{3n}{\alpha+1}}(\mathbb{R}^n)}\\&\le C\|\rho\|_{L^1}^{(1-\phi)}\|\rho\|_{L^\gamma}^{\phi}\|\rho-\bar\rho\|_{L^1}^{2(1-\phi)}\|\rho-\bar\rho\|_{L^\gamma}^{2\phi},\end{aligned}$$

where

$$\phi=\frac{2\gamma(\alpha+1)}{3n(\gamma-1)}\in(0,1).$$

Combining this with (3.35), using conservation of mass and finiteness of $\|\rho\|_{L^1}$ and $\|\rho-\bar\rho\|_{L^1}$, we obtain

$$\int_{\mathbb{R}^n}\rho\,|\nabla\mathcal{W}_\alpha-\nabla\bar{\mathcal{W}}_\alpha|^2\,\mathrm{d}\boldsymbol{x}\le C\,\|\rho\|_{L^\gamma}^{\phi}\,\|\rho-\bar\rho\|_{L^\gamma}^{2\phi}.\tag{3.36}$$

Step 2. Applying Lemmas 3.5.1–3.5.2 to (3.36) yields

$$\int_{\mathbb{R}^n} \rho\,|\nabla \mathcal{W}_\alpha - \nabla \bar{\mathcal{W}}_\alpha|^2 \,\mathrm{d}\boldsymbol{x} \le C\big(1 + H(t)^{1/\gamma}\big)^{3\phi}.$$

Using that, for $a, b \ge 0$ and $H \ge 0$,

$$(1 + H^a)^b \le C\big(1 + H^{ab}\big),$$

we obtain

$$\int_{\mathbb{R}^n} \rho\,|\nabla \mathcal{W}_\alpha - \nabla \bar{\mathcal{W}}_\alpha|^2 \,\mathrm{d}\boldsymbol{x} \le C\Big(1 + H(t)^{\frac{3\phi}{\gamma}}\Big).$$

Thus, the exponent becomes

$$\sigma_* = \frac{3\phi}{\gamma} = \frac{2(\alpha+1)}{n(\gamma-1)}.$$

Therefore, we obtain the claimed bound (3.34).

□

We now employ the form of the mechanical entropy/entropy flux to derive an entropy inequality. Recall the mechanical entropy/entropy flux and relative entropy/entropy flux as given by (3.10), (3.11) and (3.12).

Lemma 3.5.4. *Under the assumptions of* Proposition 3.2.5*, for $\bar{\rho} > 0$, we have*

$$\partial_t \eta(\rho, \boldsymbol{\mathcal{M}} \,|\, \bar{\rho}) + \nabla \cdot q(\rho, \boldsymbol{\mathcal{M}} \,|\, \bar{\rho}) \le -\boldsymbol{\mathcal{M}} \cdot (\nabla \mathcal{W}_\alpha - \nabla \bar{\mathcal{W}}_\alpha), \tag{3.37}$$

in the sense of distributions. For $\bar{\rho} = 0$, we have

$$\partial_t \eta(\rho, \boldsymbol{\mathcal{M}} \,|\, \bar{\rho}) + \nabla \cdot q(\rho, \boldsymbol{\mathcal{M}} \,|\, \bar{\rho}) = \partial_t \eta(\rho, \boldsymbol{\mathcal{M}}) + \nabla \cdot q(\rho, \boldsymbol{\mathcal{M}}) \le -\boldsymbol{\mathcal{M}} \cdot \nabla \mathcal{W}_\alpha, \tag{3.38}$$

in the sense of distributions.

Proof. We divide the proof into three steps.

Step 1. Since $\eta(\bar{\rho}, \mathbf{0}) = h(\bar{\rho})$, $\partial_\rho \eta(\bar{\rho}, \mathbf{0}) = h'(\bar{\rho})$ and $\nabla_{\boldsymbol{\mathcal{M}}} \eta(\bar{\rho}, \mathbf{0}) = \mathbf{0}$, we obtain the explicit form

$$\eta(\rho, \boldsymbol{\mathcal{M}} \,|\, \bar{\rho}) = \frac{|\boldsymbol{\mathcal{M}}|^2}{2\rho} + h(\rho|\bar{\rho}),$$

Step 2. The steady state $(\bar{\rho}, \mathbf{0})$ satisfies

$$\nabla p(\bar{\rho}) + \bar{\rho}\,\nabla \bar{\mathcal{W}}_\alpha = \mathbf{0},$$

with $\bar{\mathcal{W}}_\alpha = \Phi_\alpha * \bar{\rho}$ in the sense of distributions. Using $p'(\rho) = h''(\rho)\rho$, it follows that, for $\bar{\rho} > 0$,

$$\nabla h'(\bar{\rho}) + \nabla \bar{\mathcal{W}}_\alpha = \mathbf{0}, \tag{3.39}$$

in the sense of distributions. Since $\bar{\rho} \in L^\infty(\mathbb{R}^n)$ has compact support, it follows that $\bar{\rho} \in L^p(\mathbb{R}^n)$ for all $1 \le p \le \infty$. Hence, by the HLS inequality (Lemma A.0.2), for $\alpha < n - 1$, we have $\nabla h'(\bar{\rho}) = -\nabla \bar{\mathcal{W}}_\alpha \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ for all $1 \le q < \infty$.

Step 3. Let $(\rho, \boldsymbol{\mathcal{M}})$ be a finite-energy weak solution satisfying the entropy inequality

$$\partial_t \eta(\rho, \boldsymbol{\mathcal{M}}) + \nabla \cdot q(\rho, \boldsymbol{\mathcal{M}}) \leq -\boldsymbol{\mathcal{M}} \cdot \nabla \mathcal{W}_\alpha, \tag{3.40}$$

in the sense of distributions. For $\bar{\rho} > 0$, a direct computation using the conservation of mass $\partial_t \rho + \nabla \cdot \boldsymbol{\mathcal{M}} = 0$ gives

$$\partial_t \eta(\rho, \boldsymbol{\mathcal{M}} \,|\, \bar{\rho}) = \partial_t \eta(\rho, \boldsymbol{\mathcal{M}}) + h'(\bar{\rho}) \nabla \cdot \boldsymbol{\mathcal{M}},$$

in the sense of distributions. This expression is well-defined since $\bar{\rho} \in L^q(\mathbb{R}^n)$ for all $1 \leq q \leq \infty$, which allows us to apply a generalised product rule for distributional derivatives. We compute

$$\nabla \cdot q(\rho, \boldsymbol{\mathcal{M}} \,|\, \bar{\rho}) = \nabla \cdot q(\rho, \boldsymbol{\mathcal{M}}) - h'(\bar{\rho}) \nabla \cdot \boldsymbol{\mathcal{M}} - \boldsymbol{\mathcal{M}} \cdot \nabla h'(\bar{\rho}),$$

in the sense of distributions. This is well-defined since $\nabla h'(\bar{\rho}) \mathbb{1}_{\{\bar{\rho}>0\}} \in L^q(\mathbb{R}^n)$ for all $1 \leq q < \infty$. Combining these two identities and substituting into (3.39) yields

$$\partial_t \eta(\rho, \boldsymbol{\mathcal{M}} \,|\, \bar{\rho}) + \nabla \cdot q(\rho, \boldsymbol{\mathcal{M}} \,|\, \bar{\rho}) = \partial_t \eta(\rho, \boldsymbol{\mathcal{M}}) + \nabla \cdot q(\rho, \boldsymbol{\mathcal{M}}) + \boldsymbol{\mathcal{M}} \cdot \nabla \bar{\mathcal{W}}_\alpha,$$

in the sense of distributions. For $\bar{\rho} > 0$, using the entropy inequality (3.40), we obtain

$$\partial_t \eta(\rho, \boldsymbol{\mathcal{M}} \,|\, \bar{\rho}) + \nabla \cdot q(\rho, \boldsymbol{\mathcal{M}} \,|\, \bar{\rho}) \leq -\boldsymbol{\mathcal{M}} \cdot (\nabla \mathcal{W}_\alpha - \nabla \bar{\mathcal{W}}_\alpha),$$

in the sense of distributions. For $\bar{\rho} = 0$, by (3.40) we have

$$\partial_t \eta(\rho, \boldsymbol{\mathcal{M}} \,|\, \bar{\rho}) + \nabla \cdot q(\rho, \boldsymbol{\mathcal{M}} \,|\, \bar{\rho}) = \partial_t \eta(\rho, \boldsymbol{\mathcal{M}}) + \nabla \cdot q(\rho, \boldsymbol{\mathcal{M}}) \leq -\boldsymbol{\mathcal{M}} \cdot \nabla \mathcal{W}_\alpha,$$

in the sense of distributions.

□

Testing (3.37)–(3.38) against ψ, a non-negative Lipschitz function with compact support in $[0, T) \times \mathbb{R}^n$, and integrating by parts gives the standard relative entropy inequality:

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^n} \Big(\big(\partial_\tau \psi\big)\, \eta(\rho, \boldsymbol{\mathcal{M}} \mid \bar{\rho}) + \nabla \psi \cdot q(\rho, \boldsymbol{\mathcal{M}} \mid \bar{\rho}) \Big) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\tau + \int_{\mathbb{R}^n} (\psi\, \eta(\rho, \boldsymbol{\mathcal{M}} \mid \bar{\rho}))(0, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \\ &\qquad \geq \int_0^T \int_{\mathbb{R}^n} \psi\, \boldsymbol{\mathcal{M}} \cdot \Big(\mathbb{1}_{\{\bar{\rho}>0\}} (\nabla \mathcal{W}_\alpha - \nabla \bar{\mathcal{W}}_\alpha) + \mathbb{1}_{\{\bar{\rho}=0\}} \nabla \mathcal{W}_\alpha \Big) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\tau. \end{aligned} \tag{3.41}$$

Now, we can employ Lemmas 3.5.3 and 3.5.4 to prove Proposition 3.2.5. We recall the definition of $\mathcal{E}$ given by (3.13).

Proof of Proposition 3.2.5. Here we develop a new approach to establish a finite-time stability estimate, inspired by Luo–Smoller [107]. By employing a suitable Lipschitz test function, we derive a finite-time stability estimate without requiring the assumption $(\rho, \boldsymbol{\mathcal{M}}) \in L^\infty$. We divide the proof into six steps.

Step 1. Fix $t \in (0,T)$, $L > 0$ and $\varepsilon > 0$ small. Let $\vartheta_\varepsilon = \vartheta_\varepsilon(\tau)$ and $\varsigma_\varepsilon = \varsigma_\varepsilon(\tau, \boldsymbol{x})$ be the following cutoff functions:

$$\vartheta_\varepsilon(\tau) = \begin{cases} 1, & 0 \le \tau < t, \\ \frac{1}{\varepsilon}(t-\tau)+1, & t \le \tau < t+\varepsilon, \\ 0, & t+\varepsilon \le \tau \le T, \end{cases}$$

and

$$\varsigma_\varepsilon(\boldsymbol{x}) = \begin{cases} 1, & |\boldsymbol{x}| < L, \\ \frac{1}{\varepsilon}\big(L - |\boldsymbol{x}|\big)+1, & L \le |\boldsymbol{x}| < L+\varepsilon, \\ 0, & |\boldsymbol{x}| \ge L+\varepsilon. \end{cases} \tag{3.42}$$

We employ $\psi_\varepsilon(\tau, \boldsymbol{x}) = \varsigma_\varepsilon(\boldsymbol{x})\vartheta_\varepsilon(\tau)$, a non-negative Lipschitz function compactly supported in $[0,T) \times \mathbb{R}^n$, as a test function in (3.41).

Step 2. We now estimate the right-hand side of (3.41). By the Cauchy–Schwarz inequality,

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} \psi_\varepsilon \boldsymbol{\mathcal{M}} \cdot \Big(\mathbb{1}_{\{\bar\rho>0\}} (\nabla \mathcal{W}_\alpha - \nabla \bar{\mathcal{W}}_\alpha) + \mathbb{1}_{\{\bar\rho=0\}} \nabla \mathcal{W}_\alpha \Big) \,\mathrm{d}\boldsymbol{x} \right| \\ &\le \int_{\mathbb{R}^n} |\boldsymbol{\mathcal{M}}| \big(|\nabla \mathcal{W}_\alpha - \nabla \bar{\mathcal{W}}_\alpha| + |\nabla \mathcal{W}_\alpha| \big) \,\mathrm{d}\boldsymbol{x} \\ &\le \int_{\mathbb{R}^n} \frac{|\boldsymbol{\mathcal{M}}|^2}{\rho} \,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_{\mathbb{R}^n} \rho |\nabla \mathcal{W}_\alpha|^2 \,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_{\mathbb{R}^n} \rho |\nabla \mathcal{W}_\alpha - \nabla \bar{\mathcal{W}}_\alpha|^2 \,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{3.43}$$

Integrating in time, and using Lemma 3.5.3, the bound $H \le \mathcal{E}$, and Young's inequality, we obtain

$$\begin{aligned} &\left| \int_0^T \int_{\mathbb{R}^n} \psi_\varepsilon \boldsymbol{\mathcal{M}} \cdot \Big(\mathbb{1}_{\{\bar\rho>0\}} (\nabla \mathcal{W}_\alpha - \nabla \bar{\mathcal{W}}_\alpha) + \mathbb{1}_{\{\bar\rho=0\}} \nabla \mathcal{W}_\alpha \Big) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau \right| \\ &\le C \int_0^T \big(1 + \mathcal{E}(\tau) + \mathcal{E}(\tau)^{\sigma_*} \big) \,\mathrm{d}\tau \\ &\le C \int_0^T \big(1 + \mathcal{E}(\tau)^{\max\{1,\sigma_*\}} \big) \,\mathrm{d}\tau. \end{aligned}$$

Step 3. We consider the entropy flux term in (3.41), given by

$$I_\varepsilon := \int_0^T \int_{\mathbb{R}^n} \nabla \psi_\varepsilon(\tau, \boldsymbol{x}) \cdot q(\rho, \boldsymbol{\mathcal{M}} \mid \bar\rho)(\tau, \boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau.$$

Note that $\nabla \psi_\varepsilon = \vartheta_\varepsilon \nabla \varsigma_\varepsilon$, since ϑ_ε depends only on τ. Define the shell

$$\mathcal{S}_\varepsilon(L) := \{ \boldsymbol{x} \in \mathbb{R}^n \,|\, L \le |\boldsymbol{x}| < L+\varepsilon \}.$$

By construction in (3.42), $\varsigma_\varepsilon(\tau, \cdot)$ is constant outside $\mathcal{S}_\varepsilon(L)$ and linear in it. Hence,

$$\nabla \varsigma_\varepsilon(\tau, \boldsymbol{x}) = -\frac{1}{\varepsilon}\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\, \mathbb{1}_{\mathcal{S}_\varepsilon(L)}(\boldsymbol{x}), \quad \text{for } a.e.\ (\tau, \boldsymbol{x}), \qquad |\nabla \varsigma_\varepsilon| = \frac{1}{\varepsilon}\mathbb{1}_{\mathcal{S}_\varepsilon(L)}.$$

Since

$$q(\rho, \boldsymbol{\mathcal{M}} \,|\, \bar{\rho}) = \frac{1}{2}\boldsymbol{\mathcal{M}}\frac{|\boldsymbol{\mathcal{M}}|^2}{\rho^2} + (h'(\rho) - h'(\bar{\rho}))\boldsymbol{\mathcal{M}},$$

we have

$$|I_\varepsilon| \leq \frac{C}{\varepsilon}\int_0^T \int_{\mathcal{S}_\varepsilon(L)} \left(\frac{|\boldsymbol{\mathcal{M}}|^3}{\rho^2} + |\boldsymbol{\mathcal{M}}|\ |h'(\rho) - h'(\bar{\rho})| \right) \mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau =: C\big(J_{1,\varepsilon} + J_{2,\varepsilon}\big). \tag{3.44}$$

We will show later, in (3.52), that $J_{1,\varepsilon}$ can be controlled as $\varepsilon \to 0$, owing to the integrability property $\frac{|\boldsymbol{\mathcal{M}}|^3}{\rho^2} \in L^1(0,T;L^1(\mathbb{R}^n))$. To analyse $J_{2,\varepsilon}$, we use the fact that $\bar{\rho} \in L^\infty(\mathbb{R}^n)$ is compactly supported. Consequently, there exists $R_0 > 0$ such that

$$\mathrm{supp}(\bar{\rho}) \subset B_{R_0}(\mathbf{0}).$$

We choose $L > R_0$ such that $\mathrm{supp}(\nabla \varsigma_\varepsilon) \subset \{\bar{\rho} = 0\}$. Therefore,

$$J_{2,\varepsilon} \leq \frac{C}{\varepsilon}\int_0^T \int_{\mathcal{S}_\varepsilon(L)} \rho^{\gamma-1}|\boldsymbol{\mathcal{M}}|\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}\tau.$$

Applying Young's inequality, we obtain

$$\frac{1}{\varepsilon}\int_{\mathcal{S}_\varepsilon(L)} \rho^{\gamma-1}|\boldsymbol{\mathcal{M}}|\ \mathrm{d}\boldsymbol{x} \leq C\Big(\int_{\mathbb{R}^n} |\nabla\varsigma_\varepsilon| \frac{|\boldsymbol{\mathcal{M}}|^3}{\rho^2}\ \mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^n} |\nabla\varsigma_\varepsilon|\rho^{\gamma+\theta}\ \mathrm{d}\boldsymbol{x}\Big), \tag{3.45}$$

where $\theta = \frac{\gamma-1}{2}$. Thus, combining (3.41), (3.44) and (3.45), we obtain

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^n} \partial_\tau\psi_\varepsilon\, \eta(\rho, \boldsymbol{\mathcal{M}} \mid \bar{\rho})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau + \int_{\mathbb{R}^n} \psi_\varepsilon(\boldsymbol{x},0)\eta(\rho,\boldsymbol{\mathcal{M}}\mid\bar{\rho})(0,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ + C\int_0^T \big(1 + \mathcal{E}(\tau)^{\max\{1,\sigma_*\}}\big)\ \mathrm{d}\tau \\ \geq -C\int_0^T \int_{\mathbb{R}^n} |\nabla\varsigma_\varepsilon| \left(\rho + \rho^{\gamma+\theta} + \frac{|\boldsymbol{\mathcal{M}}|^3}{\rho^2}\right)\mathrm{d}\boldsymbol{x}\ \mathrm{d}\tau.\end{aligned} \tag{3.46}$$

Step 4. We consider the first term on the left-hand side of (3.46). Since $\psi_\varepsilon = \varsigma_\varepsilon\vartheta_\varepsilon$,

$$\partial_\tau\psi_\varepsilon = \varsigma_\varepsilon\,\vartheta_\varepsilon',$$

since ς_ε depends only on $\boldsymbol{x}$. By definition, $\vartheta_\varepsilon'(\tau) = 0$ for $\tau \in (0,t)$ and $\tau \in (t+\varepsilon, T)$, while

$$\vartheta_\varepsilon'(\tau) = -\frac{1}{\varepsilon}, \qquad \text{for } a.e.\ \tau \in (t, t+\varepsilon).$$

Therefore,

$$\int_0^T \int_{\mathbb{R}^n} \varsigma_\varepsilon\,\vartheta_\varepsilon'\,\eta(\rho,\boldsymbol{\mathcal{M}}\mid\bar{\rho})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau = -\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_{\mathbb{R}^n} \varsigma_\varepsilon(\boldsymbol{x})\,\eta(\rho,\boldsymbol{\mathcal{M}}\mid\bar{\rho})(\tau,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau. \tag{3.47}$$

We have

$$\varsigma_\varepsilon(\boldsymbol{x}) = 1 \text{ on } \{|\boldsymbol{x}| < L\},$$

and ς_ε is supported in $\{|\boldsymbol{x}| \le L + \varepsilon\}$. Moreover $0 \le \varsigma_\varepsilon \le 1$. Since $\eta(\rho, \boldsymbol{\mathcal{M}} \mid \bar{\rho}) \in L^\infty(0,T; L^1(\mathbb{R}^n))$, the map

$$\tau \mapsto \int_{\mathbb{R}^n} \mathbb{1}_{B_L}(\boldsymbol{x})\, \eta(\rho, \boldsymbol{\mathcal{M}} \mid \bar{\rho})(\tau, \boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$$

belongs to $L^\infty(0,T)$, and therefore admits Lebesgue points for almost every t. We fix such a Lebesgue point t for the function $\tau \mapsto \int_{B_L} \eta(\tau, \boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$. Then, by the dominated convergence Theorem and

$$\mathbb{1}_{B_L} \le \varsigma_\varepsilon \le \mathbb{1}_{B_{L+\varepsilon}},$$

almost every t,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \int_{\mathbb{R}^n} \varsigma_\varepsilon(\boldsymbol{x})\, \eta(\rho, \boldsymbol{\mathcal{M}} \mid \bar{\rho})(\tau, \boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\tau = \int_{|\boldsymbol{x}|<L} \eta(\rho, \boldsymbol{\mathcal{M}} \mid \bar{\rho})(t, \boldsymbol{x})\, \mathrm{d}\boldsymbol{x}. \tag{3.48}$$

Combining (3.47) and (3.48) yields

$$\lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{R}^n} \varsigma_\varepsilon\, \vartheta_\varepsilon'\, \eta(\rho, \boldsymbol{\mathcal{M}} \mid \bar{\rho})\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\tau = -\int_{|\boldsymbol{x}|<L} \eta(\rho, \boldsymbol{\mathcal{M}} \mid \bar{\rho})(t, \boldsymbol{x})\, \mathrm{d}\boldsymbol{x}, \qquad \text{for } a.e.\ t. \tag{3.49}$$

Step 5. We consider the second term on the left-hand side of (3.46). At $\tau = 0$, we have $\vartheta_\varepsilon(0) = 1$, hence

$$\psi_\varepsilon(0, \boldsymbol{x}) = \varsigma_\varepsilon(\boldsymbol{x}) = \begin{cases} 1, & |\boldsymbol{x}| < L, \\ \in [0,1], & L \le |\boldsymbol{x}| < L + \varepsilon, \\ 0, & |\boldsymbol{x}| \ge L + \varepsilon. \end{cases}$$

Thus, $\psi_\varepsilon(0, \cdot) \to \mathbb{1}_{B_L}$ pointwise almost everywhere and monotonically as $\varepsilon \to 0$. Since $\eta(\rho, \boldsymbol{\mathcal{M}} \mid \bar{\rho})(0, \cdot) \in L^1(\mathbb{R}^n)$, the dominated convergence Theorem yields

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \psi_\varepsilon(0, \boldsymbol{x}) \eta(\rho, \boldsymbol{\mathcal{M}} \mid \bar{\rho})(0, \boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \int_{|\boldsymbol{x}|<L} \eta(\rho, \boldsymbol{\mathcal{M}} \mid \bar{\rho})(0, \boldsymbol{x})\, \mathrm{d}\boldsymbol{x}. \tag{3.50}$$

Step 6. Thus, combining (3.46), (3.49) and (3.50), we obtain that

$$\begin{aligned} \int_{|\boldsymbol{x}|<L} \eta(\rho, \boldsymbol{\mathcal{M}} \mid \bar{\rho})(t, \boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \le \int_{|\boldsymbol{x}|<L} \eta(\rho, \boldsymbol{\mathcal{M}} \mid \bar{\rho})(0, \boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \\ + C \int_0^T \left(1 + \mathcal{E}(\tau)^{\max\{1, \sigma_*\}}\right) \mathrm{d}\tau + C\, \mathcal{R}(L), \end{aligned} \tag{3.51}$$

for almost every $t \in (0,T)$, where

$$\mathcal{R}(L) := \limsup_{\varepsilon \to 0} \int_0^T \int_{\mathbb{R}^n} |\nabla \varsigma_\varepsilon| \left(\rho + \rho^{\gamma+\theta} + \frac{|\boldsymbol{\mathcal{M}}|^3}{\rho^2}\right) \mathrm{d}\boldsymbol{x}\, \mathrm{d}\tau. \tag{3.52}$$

Since $\eta(\rho, \boldsymbol{\mathcal{M}} \mid \bar{\rho}) \geq 0$, the left-hand side of (3.51) increases monotonically to $\mathcal{E}(t)$ as $L \to \infty$, and similarly the initial term increases to $\mathcal{E}(0)$. By the monotone convergence Theorem,

$$\lim_{L\to\infty} \int_{|\boldsymbol{x}|<L} \eta(t, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^n} \eta(t, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = \mathcal{E}(t), \qquad \lim_{L\to\infty} \int_{|\boldsymbol{x}|<L} \eta(0, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = \mathcal{E}(0).$$

It remains to control the remainder $\mathcal{R}(L)$. Let $f := \left(\rho + \rho^{\gamma+\theta} + \frac{|\boldsymbol{\mathcal{M}}|^3}{\rho^2}\right) \in L^1(0, T; L^1(\mathbb{R}^n))$ by assumption. Then the co-area formula yields

$$\int_0^T \int_{\mathbb{R}^n} f \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\tau = \int_0^\infty \left(\int_0^T \int_{\{|\boldsymbol{x}|=s\}} f \, \mathrm{d}S(\boldsymbol{x}) \, \mathrm{d}t \right) \mathrm{d}s.$$

In particular, the function

$$G(s) := \int_0^T \int_{\{|\boldsymbol{x}|=s\}} f \, \mathrm{d}S(\boldsymbol{x}) \, \mathrm{d}t,$$

belongs to $L^1(\mathbb{R}_+)$. Therefore, there exists a sequence $s_j \to +\infty$ of Lebesgue points of G such that $G(s_j) \to 0$. Fix $t \in (0, T)$ and set $L_j := s_j$. For this sequence, we have $L_j \to +\infty$, and by the definition of ς_ε, the support of $|\nabla \varsigma_\varepsilon|$ is contained in the slab $\{\, l \in [L_j,\, L_j + \varepsilon] \,\}$. Therefore, by the co-area formula,

$$\int_0^T \int_{\mathbb{R}^n} |\nabla \varsigma_\varepsilon| \, f \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\tau = \frac{1}{\varepsilon} \int_{L_j}^{L_j+\varepsilon} G(s) \, \mathrm{d}s,$$

and letting $\varepsilon \to 0$ gives $\mathcal{R}(L_j) \leq G(s_j) \to 0$. Passing to the limit $j \to \infty$ along this subsequence in (3.51) yields

$$\mathcal{E}(t) \leq \mathcal{E}(0) + C \int_0^T \left(1 + \mathcal{E}(\tau)^{\max\{1,\sigma_*\}}\right) \mathrm{d}\tau.$$

Note that $\sigma_* > 1$ for $\frac{3n}{3n-2(\alpha+1)} < \gamma < \frac{n+2(\alpha+1)}{n}$ and $0 < \sigma_* \leq 1$ for $\gamma \geq \frac{n+2(\alpha+1)}{n}$. Thus, using a Grönwall type inequality, we obtain the result.

□

3.6 Growth of solutions in the mass-subcritical regime

Throughout this section, we assume that $(\rho, \boldsymbol{\mathcal{M}})$ is a classical solution of the attractive CEREs with compactly supported density and sufficiently regular free boundary, so that all integrations by parts are justified. In the mass-subcritical regime, as seen in §3.3 and §3.4, while we have, in some sense, the local Lyapunov stability of stationary states, the question still remains open whether stationary states are globally stable.

In fact, we see that the radius of the support of solutions with positive initial energy grow linearly with respect to time.

However, for $\gamma = \frac{n+\alpha}{n}$, where there is a unique stationary state (up to translation) for a critical mass $M = M_c$ (see [16, Theorem 3]), the total energy of the stationary state is 0, as such, one can show that any small perturbation $(\rho_0, \boldsymbol{\mathcal{M}}_0)$ of the stationary state $(\bar{\rho}, \mathbf{0})$ will have positive energy, unless it is a translation of the stationary solution. Since the support of the stationary state is compact, the linear growth in the radius of the support proves instability of the stationary state. In order to prove this for a more general pressure, we define the quantity

$$\gamma_{\text{inf}} \coloneqq 1 + \inf_{\varrho \in \mathbb{R}_+} \left(\frac{p(\varrho)}{\varrho e(\varrho)} \right). \tag{3.53}$$

Note that in the case of a polytropic pressure, if $\gamma \geq \frac{n+\alpha}{n}$ we have $\gamma_{\text{inf}} = \gamma \geq \frac{n+\alpha}{n}$. When considering global classical solutions $(\rho, \boldsymbol{\mathcal{M}})$ of (1.1) with compact support, one can see that the total energy $E(\rho(t), \boldsymbol{\mathcal{M}}(t))$ is conserved for all $t \geq 0$. Hence, for the rest of this section, we denote $E \coloneqq E(\rho_0, \boldsymbol{\mathcal{M}}_0)$. Different from [49], where they consider the polytropic CEPEs, we can extend their approach to considering general pressures and the non-local Riesz potential. We suppose that the gas is confined to a spatially bounded domain $\Omega(t) \subset \mathbb{R}^n$ at time t, that is $\rho|_{\partial\Omega(t)} = 0$, and $\rho > 0$ in $\Omega(t)$. For a stationary solution being considered, $\Omega(t) = \Omega$ is a fixed domain.

Lemma 3.6.1. *Suppose that $\alpha \in (0, 2] \cap (0, n-1)$, $\gamma_{\text{inf}} \geq \frac{n+\alpha}{n}$ and that $(\rho, \boldsymbol{\mathcal{M}})$ is a global classical solution of the attractive CEREs ($\kappa = 1$) (1.1) with p satisfying (1.4). If the energy $E > 0$, then*

$$\liminf_{t \to \infty} \left(\frac{R(t)}{t} \right) \geq \left(\frac{\alpha E}{M} \right)^{\frac{1}{2}},$$

with $R(t) = \max_{\boldsymbol{x} \in \Omega(t)} \{|\boldsymbol{x}|\}$.

Proof. Taking derivative of the second inertia

$$H(t) = \int_{\Omega(t)} \rho |\boldsymbol{x}|^2 \, \mathrm{d}\boldsymbol{x}.$$

Then, using $(1.1)_1$, integration by parts and $\rho|_{\partial\Omega(t)} \equiv 0$, we see that

$$H'(t) = \int_{\partial\Omega(t)} |\boldsymbol{x}|^2 \boldsymbol{\mathcal{M}} \cdot \nu \, \mathrm{d}S(\boldsymbol{x}) + \int_{\Omega(t)} \rho_t |\boldsymbol{x}|^2 \, \mathrm{d}\boldsymbol{x} = 2 \int_{\Omega(t)} \boldsymbol{\mathcal{M}} \cdot \boldsymbol{x} \, \mathrm{d}\boldsymbol{x},$$

where $\nu(\boldsymbol{x})$ is the normal to the surface $\partial\Omega(t)$ at $\boldsymbol{x} \in \partial\Omega(t)$. Note, that similarly, using $(1.1)_2$, that

$$\begin{aligned} H''(t) &= 2 \int_{\partial\Omega(t)} \left(\frac{\boldsymbol{\mathcal{M}} \otimes \boldsymbol{\mathcal{M}}}{\rho} : \boldsymbol{x} \otimes \nu \right) \mathrm{d}S(\boldsymbol{x}) + 2 \int_{\Omega(t)} \boldsymbol{\mathcal{M}}_t \cdot \boldsymbol{x} \, \mathrm{d}\boldsymbol{x} \\ &= 2 \Big(\int_{\Omega(t)} \left| \frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}} \right|^2 \mathrm{d}\boldsymbol{x} + n \int_{\Omega(t)} p \, \mathrm{d}\boldsymbol{x} - \int_{\Omega(t)} \rho \nabla \Phi_\alpha * \rho \cdot \boldsymbol{x} \, \mathrm{d}\boldsymbol{x} \Big). \end{aligned} \tag{3.54}$$

Simple calculation shows that

$$\begin{aligned}
I &:= \int_{\Omega(t)} \rho \nabla \Phi_\alpha * \rho \cdot \boldsymbol{x} \, \mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega(t)} \rho(\boldsymbol{x}) \int_{\Omega(t)} \frac{\rho(\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot \boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^{\alpha+2}} \, \mathrm{d}\boldsymbol{y} \, \mathrm{d}\boldsymbol{x} \\
&= -\int_{\Omega(t)} \rho(\boldsymbol{x}) \int_{\Omega(t)} \frac{\rho(\boldsymbol{y})|\boldsymbol{x}-\boldsymbol{y}|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{\alpha+2}} \, \mathrm{d}\boldsymbol{y} \, \mathrm{d}\boldsymbol{x} + \int_{\Omega(t)} \rho(\boldsymbol{x}) \int_{\Omega(t)} \frac{\rho(\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot \boldsymbol{x}}{|\boldsymbol{x}-\boldsymbol{y}|^{\alpha+2}} \, \mathrm{d}\boldsymbol{y} \, \mathrm{d}\boldsymbol{x} \\
&= -\int_{\Omega(t)} \rho(\boldsymbol{x}) \int_{\Omega(t)} \frac{\rho(\boldsymbol{y})|\boldsymbol{x}-\boldsymbol{y}|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{\alpha+2}} \, \mathrm{d}\boldsymbol{y} \, \mathrm{d}\boldsymbol{x} - \int_{\Omega(t)} \rho(\boldsymbol{y}) \int_{\Omega(t)} \frac{\rho(\boldsymbol{x})(\boldsymbol{y}-\boldsymbol{x})\cdot \boldsymbol{x}}{|\boldsymbol{x}-\boldsymbol{y}|^{\alpha+2}} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{y} \\
&= -\alpha \int_{\Omega(t)} \rho \Phi_\alpha * \rho \, \mathrm{d}\boldsymbol{x} - I.
\end{aligned}$$

Then, we have

$$I = -\frac{\alpha}{2} \int_{\Omega(t)} \rho \Phi_\alpha * \rho \, \mathrm{d}\boldsymbol{x}. \tag{3.55}$$

We also note that

$$E = \int_{\mathbb{R}^n} \left(\frac{1}{2} \left| \frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}} \right|^2 (t,\boldsymbol{x}) + (\rho e(\rho))(t,\boldsymbol{x}) + \frac{1}{2} \rho(t,\boldsymbol{x}) (\Phi_\alpha * \rho)(t,\boldsymbol{x}) \right) \mathrm{d}\boldsymbol{x}.$$

Thus, we have for (3.54),

$$\begin{aligned}
H''(t) &= 2 \int_{\Omega(t)} \left| \frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}} \right|^2 \mathrm{d}\boldsymbol{x} + 2n \int_{\Omega(t)} p \, \mathrm{d}\boldsymbol{x} + 2\alpha \left(E - \int_{\Omega(t)} \frac{1}{2} \left| \frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}} \right|^2 + \rho e(\rho) \, \mathrm{d}\boldsymbol{x} \right) \\
&= 2\alpha E + (2-\alpha) \int_{\Omega(t)} \left| \frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}} \right|^2 \mathrm{d}\boldsymbol{x} + 2 \int_{\Omega(t)} (np - \alpha \rho e(\rho)) \, \mathrm{d}\boldsymbol{x} \\
&\geq 2\alpha E + (2-\alpha) \int_{\Omega(t)} \left| \frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}} \right|^2 \mathrm{d}\boldsymbol{x} + 2 \left(n - \frac{\alpha}{\gamma_{\mathrm{inf}} - 1} \right) \int_{\Omega(t)} p \, \mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Then, we need to assume $\alpha \in (0,2]$, and $\gamma_{\mathrm{inf}} \geq \frac{n+\alpha}{n}$, such that $H''(t) \geq 2\alpha E$. Hence,

$$H(t) \geq H(0) + H'(0)t + \alpha E t^2.$$

On the other hand, it holds trivially that

$$H(t) = \int_{\Omega(t)} \rho(t,\boldsymbol{x}) |\boldsymbol{x}|^2 \, \mathrm{d}\boldsymbol{x} \leq R(t)^2 M.$$

Thus,

$$H(0) + H'(0)t + \alpha E t^2 \leq H(t) \leq R(t)^2 M.$$

This implies that

$$O(t^{-1}) + \alpha E \leq \frac{R(t)^2 M}{t^2}.$$

Therefore, we have

$$\liminf_{t \to \infty} \left(\frac{R(t)}{t} \right) \geq \left(\frac{\alpha E}{M} \right)^{\frac{1}{2}}.$$

□

For $\alpha > 2$, we are no longer able to use the previous method to obtain a positive lower bound for the second derivative of $H(t)$. In order to combat this problem, we have to consider how the internal energy compares to the gravitational potential energy. In fact, we find that for small enough initial mass, the internal energy can control the gravitational potential energy found in the second derivative of $H(t)$.

Lemma 3.6.2. *Suppose that $\alpha \in (2, n-1)$, $\gamma_{\text{inf}} > \frac{n+2}{n}$, $\lim_{\rho\to\infty} p(\rho)\rho^{-\frac{n+\alpha}{n}} = \infty$, the energy $E > 0$ and that $(\rho, \boldsymbol{\mathcal{M}})$ is a global classical solution of the attractive CEREs ($\kappa = 1$) (1.1) with p satisfying (1.4). Then, there exists an $\tilde{M} \in (0, \infty]$ such that if $M < \tilde{M}$, then*

$$\Lambda_{\min}(M) := 2E + \inf_{R>0} \frac{M}{R}\left(np(R) - 2Re(R) - \frac{(\alpha-2)}{2\alpha} C_* M^{\frac{n-\alpha}{n}} R^{\frac{n+\alpha}{n}}\right) > 0, \tag{3.56}$$

and

$$\liminf_{t\to\infty} \left(\frac{R(t)}{t}\right) \geq \left(\frac{\Lambda_{\min}(M)}{M}\right)^{\frac{1}{2}},$$

where $R(t)$ is defined in Lemma 3.6.1.

Remark 3.6.3. In the polytropic case, (3.56) is equivalent to $\gamma \geq \frac{n+\alpha}{n}$ with the critical mass given by $\tilde{M} = M_*$, defined by

$$M_* := \begin{cases} \left(\left(\frac{4n\alpha(\gamma-1)E}{(\alpha-2)(n(\gamma-1)-\alpha)C_*}\right)^{n(\gamma-1)} \left(\frac{(n(\gamma-1)-2)(n(\gamma-1)-\alpha)}{2\alpha E(\gamma-1)}\right)^{\alpha}\right)^{\frac{1}{(2n-\alpha)\gamma-2n}}, & \gamma > \frac{n+\alpha}{n}, \\ \left(\frac{2\alpha(n(\gamma-1)-2)}{(\alpha-2)(\gamma-1)C_*}\right)^{\frac{n(\gamma-1)}{(2n-\alpha)\gamma-2n}}, & \gamma = \frac{n+\alpha}{n}, \end{cases}$$

where $C_* > 0$ is the sharp constant from the variant of the HLS inequality.

Proof. As in Lemma 3.6.1, we have that for

$$H(t) = \int_{\Omega(t)} \rho|\boldsymbol{x}|^2 \,\mathrm{d}\boldsymbol{x},$$

that

$$\begin{aligned} H''(t) &= 2\left(\int_{\Omega(t)} \left|\frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}}\right|^2 \mathrm{d}\boldsymbol{x} + n\int_{\Omega(t)} p \,\mathrm{d}\boldsymbol{x} + \frac{\alpha}{2}\int_{\Omega(t)} \rho\Phi_\alpha * \rho \,\mathrm{d}\boldsymbol{x}\right) \\ &= 4E + 2\int_{\Omega(t)} (np - 2\rho e(\rho)) \,\mathrm{d}\boldsymbol{x} + (\alpha-2)\int_{\Omega(t)} \rho\Phi_\alpha * \rho \,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{3.57}$$

Applying the variant of the HLS inequality in Lemma A.0.3 to (3.57), we obtain

$$H''(t) \geq 4E + 2\int_{\Omega(t)} (np - 2\rho e(\rho)) \,\mathrm{d}\boldsymbol{x} - \frac{(\alpha-2)}{\alpha} C_* \|\rho\|_{L^1}^{\frac{n-\alpha}{n}} \|\rho\|_{L^{\frac{n+\alpha}{n}}}^{\frac{n+\alpha}{n}}. \tag{3.58}$$

We would like to find a lower bound for the right-hand side of (3.58). To do this, we will show that it is minimised by a function of the form $\sigma_R = R\mathbb{1}_{B_{(\frac{nM}{R\omega_n})^{1/n}}} \in X_M$. As such, we need to show that the following function $f_M : X_M \to \mathbb{R}$ for fixed M is minimised by σ_R for some R, where

$$f_M(\varrho) := \int_{\Omega(t)} (np - 2\varrho e(\varrho))\,\mathrm{d}\boldsymbol{x} - \frac{(\alpha-2)}{2\alpha} C_* M^{\frac{n-\alpha}{n}} \int_{\Omega(t)} \varrho^{\frac{n+\alpha}{n}}\,\mathrm{d}\boldsymbol{x}.$$

Plugging in σ_R, one obtains

$$f_M(\sigma_R) = \frac{M}{R}\Big(np(R) - 2Re(R) - \frac{(\alpha-2)}{2\alpha} C_* M^{\frac{n-\alpha}{n}} R^{\frac{n+\alpha}{n}}\Big). \tag{3.59}$$

Thus, we consider the infimum $\Lambda_{\min}(M)$ of this plus $2E$

$$\Lambda_{\min}(M) = 2E + \inf_{R>0} f_M(\sigma_R).$$

From (3.59) and the fact that $\gamma_{\inf} > \frac{n+2}{n}$ where $\gamma_{\inf}$ is defined in (3.53), we deduce that

$$f_M(\sigma_R) \geq \frac{M}{R}\Big(\Big(n - \frac{2}{\gamma_{\inf}-1}\Big)p(R) - \frac{(\alpha-2)}{2\alpha} C_* M^{\frac{n-\alpha}{n}} R^{\frac{n+\alpha}{n}}\Big). \tag{3.60}$$

Since $\lim_{\varrho\to\infty} p(\varrho)\varrho^{-\frac{n+\alpha}{n}} = \infty$, we have that,

$$\lim_{R\to\infty} f_M(\sigma_R) = \infty.$$

Therefore, this means that $\Lambda_{\min}(M) > -\infty$. One can see that $M \mapsto \Lambda_{\min}(M)$ is continuous, as $p(R)$ and $e(R)$ are continuous. Combining (3.60) and $\lim_{\varrho\to\infty} p(\varrho)\varrho^{-\frac{n+\alpha}{n}} = \infty$, we have that there exists $C \in \mathbb{R}$ such that for $0 < M < 1$,

$$\frac{R}{M} f_M(\sigma_R) \geq \Big(\Big(n - \frac{2}{\gamma_{\inf}-1}\Big)p(R) - \frac{(\alpha-2)}{2\alpha} C_* R^{\frac{n+\alpha}{n}}\Big) \geq C.$$

Thus, we see that $\liminf_{M\to 0} \Lambda_{\min}(M) \geq 2E > 0$.

Case 1. Suppose $\Lambda_{\min}(M) > 0$ for all $M > 0$. Then, let $\tilde{M} := \infty$.

Case 2. Suppose there exists $M > 0$ such that $\Lambda_{\min}(M) \leq 0$. We can see that $M \mapsto f_M(\sigma_R)$ is strictly decreasing in the set $\{M > 0 \,|\, f_M(\sigma_R) \leq 0\}$. Hence $M \mapsto \Lambda_{\min}(M)$ is decreasing in the set $\{M > 0 \,|\, \Lambda_{\min}(M) \leq 0\}$. Thus, since $\liminf_{M\to 0} \Lambda_{\min}(M) > 0$, there exists $\tilde{M} \in (0, \infty)$ (the first mass $M > 0$ such that $\Lambda_{\min}(M) = 0$) such that $\Lambda_{\min}(M) \leq 0$ for all $M \geq \tilde{M}$ and $\Lambda_{\min}(M) > 0$ for all $M < \tilde{M}$.

Now, we show that $\Lambda_{\min}(M) - 2E$ is the infimum of the function f_M. Consider $\varrho \in X_M$, we have that by the definition of $\Lambda_{\min}(M)$, that

$$f_M(\varrho) = \frac{1}{M}\int_{\{\varrho>0\}} f_M(\sigma_{\varrho(\boldsymbol{x})})\varrho(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \geq \frac{\Lambda_{\min}(M) - 2E}{M}\int_{\mathbb{R}^n} \varrho\,\mathrm{d}\boldsymbol{x} = \Lambda_{\min}(M) - 2E.$$

Thus, $\Lambda_{\min}(M) = 2E + \inf_{\varrho \in X_M} f_M(\varrho)$. As such, for $M < \tilde{M}$, employing (3.58), we have

$$H''(t) \geq 2\Lambda_{\min}(M) > 0.$$

Hence,

$$H(t) \geq H(0) + H'(0)t + \Lambda_{\min}(M)t^2.$$

On the other hand, it holds trivially that

$$H(t) = \int_{\Omega(t)} \rho(t, \boldsymbol{x})|\boldsymbol{x}|^2 \, \mathrm{d}\boldsymbol{x} \leq R(t)^2 M.$$

Thus,

$$H(0) + H'(0)t + \Lambda_{\min}(M)t^2 \leq H(t) \leq R(t)^2 M.$$

It implies that

$$O(t^{-1}) + \Lambda_{\min}(M) \leq \frac{R(t)^2 M}{t^2}.$$

Therefore, we have

$$\liminf_{t \to \infty} \left(\frac{R(t)}{t} \right) \geq \left(\frac{\Lambda_{\min}(M)}{M} \right)^{\frac{1}{2}}.$$

□

Through developing a different approach to [49], in the polytropic case and the case that $\alpha \in (0, 2]$, we are able to show growth of solutions with energy $E = 0$ as follows.

Lemma 3.6.4. *Suppose that $\gamma > \frac{n+\alpha}{n}$, $\alpha \in (0, 2] \cap (0, n-1)$ and that $(\rho, \boldsymbol{\mathcal{M}})$ is a global classical solution of the attractive polytropic CEREs ($\kappa = 1$) (1.1). If the energy $E = 0$, then there exists a sequence $(t_m)_m \subset [0, \infty)$ such that $R(t_m) \to \infty$ as $m \to \infty$.*

Proof. As in Lemma 3.6.1, we have that for

$$H(t) = \int_{\Omega(t)} \rho|\boldsymbol{x}|^2 \, \mathrm{d}\boldsymbol{x},$$

that

$$H''(t) = 2\int_{\Omega(t)} \left| \frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}} \right|^2 \mathrm{d}\boldsymbol{x} + 2n \int_{\Omega(t)} p \, \mathrm{d}\boldsymbol{x} - 2\alpha \int_{\Omega(t)} \frac{1}{2} \left| \frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}} \right|^2 + \rho e(\rho) \, \mathrm{d}\boldsymbol{x}.$$

Then, we need to assume $\alpha \in (0, 2]$, and $\gamma \geq \frac{n+\alpha}{n}$, such that

$$H''(t) \geq 2a_0 \frac{n(\gamma - 1) - \alpha}{\gamma - 1} \int_{\Omega(t)} \rho^\gamma \, \mathrm{d}\boldsymbol{x}.$$

Note, we have that by the Hölder inequality

$$M = \int_{\Omega(t)} \rho \, \mathrm{d}\boldsymbol{x} \leq \left(\int_{\Omega(t)} \rho^\gamma \, \mathrm{d}\boldsymbol{x} \right)^{\frac{1}{\gamma}} |\Omega(t)|^{\frac{\gamma-1}{\gamma}}.$$

As such, we obtain the lower bound for the second derivative of H given by

$$H''(t) \geq 2a_0 \frac{n(\gamma-1)-\alpha}{\gamma-1} M^\gamma |\Omega(t)|^{1-\gamma} \geq 2a_0 \frac{n(\gamma-1)-\alpha}{\gamma-1} M^\gamma \left(\frac{\omega_n}{n} R(t)^n\right)^{1-\gamma}, \quad (3.61)$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$. Note that by Taylor's Theorem, for each $t \in [0,\infty)$, there exists $\xi_t \in [0,t)$ such that

$$H(t) = H(0) + tH'(0) + \frac{t^2}{2} H''(\xi_t). \quad (3.62)$$

As such, combining (3.61) and (3.62), we obtain

$$H(t) \geq H(0) + tH'(0) + t^2 a_0 \frac{n(\gamma-1)-\alpha}{\gamma-1} M^\gamma \left(\frac{\omega_n}{n} R(\xi_t)^n\right)^{1-\gamma}.$$

From the same approach as in Lemma 3.6.1, we see that

$$\liminf_{t\to\infty} \left(\frac{R(t)R(\xi_t)^{\frac{n(\gamma-1)}{2}}}{t}\right) \geq \left(a_0 \frac{n(\gamma-1)-\alpha}{(\gamma-1)} \left(\frac{nM}{\omega_n}\right)^{\gamma-1}\right)^{\frac{1}{2}}.$$

As such, taking any sequence $(t_m)_m \subset [0,\infty)$ such that $t_m \to \infty$ as $m \to \infty$, we have that $R(t_m)R(\xi_{t_m})^{\frac{n(\gamma-1)}{2}} \to \infty$ as $m \to \infty$. Thus, there exists a subsequence of $(t_m)_m$ (which we just relabel as $(t_m)_m$) such that either $R(t_m) \to \infty$ or $R(\xi_{t_m}) \to \infty$ as $m \to \infty$. In the latter case, we relabel ξ_{t_m} to t_m. □

In the previous lemmas, we require that our solutions are strong solutions. However, we would now like to consider finite-energy weak solutions of the attractive CEREs (1.1) with finite second moment.

Corollary 3.6.5. *Suppose $(\rho, \boldsymbol{\mathcal{M}})$ is a finite-energy weak solution as in Definition 2.1.1 for all $T > 0$ with p satisfying (1.4) and finite second-moment $\rho(t,\boldsymbol{x})|\boldsymbol{x}|^2 \in L^1(0,T;L^1(\mathbb{R}^n))$ for all $T > 0$. If there exists $\Lambda > 0$ such that*

$$\Lambda \leq \int_{\mathbb{R}^n} \left|\frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}}\right|^2 \mathrm{d}\boldsymbol{x} + n\int_{\mathbb{R}^n} p\,\mathrm{d}\boldsymbol{x} + \frac{\alpha}{2}\int_{\mathbb{R}^n} \rho\Phi_\alpha * \rho\,\mathrm{d}\boldsymbol{x},$$

for a.e. $t \geq 0$, then either there exists a set $A \subset (0,\infty)$ with non-zero measure such that $R(t) = \infty$ for all $t \in A$ or

$$\lim_{t\to\infty} \operatorname*{ess\,inf}_{s\geq t} \left(\frac{R(s)}{s}\right) \geq \left(\frac{\Lambda}{M}\right)^{\frac{1}{2}},$$

where $R(t) \coloneqq \operatorname{ess\,sup}_{\boldsymbol{x}\in\Omega(t)} |\boldsymbol{x}|$.

Proof.

Case 1. Suppose $\boldsymbol{x} \mapsto \rho(t,\boldsymbol{x})$ does not have compact support on a non-zero measure set of $t \geq 0$, then we have that on a non-zero measure set of $t \geq 0$, that $R(t) = \infty$.

Case 2. Suppose $\boldsymbol{x} \mapsto \rho(t, \boldsymbol{x})$ has compact support for *a.e.* $t \geq 0$. Let $\varphi \in C_c^\infty(\mathbb{R})$ and $\psi \in C_c^\infty(\mathbb{R}^n)$ be standard radially symmetric mollifiers with φ_ε and ψ_δ defined in (2.19). Then, we define the following notation for taking the mollification of a function $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^m$ by

$$\begin{cases} f_\varepsilon(t, \boldsymbol{x}) = (f(\cdot, \boldsymbol{x}) * \varphi_\varepsilon)(t), & \text{for } (t, \boldsymbol{x}) \in [\varepsilon, \infty) \times \mathbb{R}^n, \\ f_\delta(t, \boldsymbol{x}) = (f(t, \cdot) * \psi_\delta)(\boldsymbol{x}), & \text{for } (t, \boldsymbol{x}) \in [0, \infty) \times \mathbb{R}^n, \\ f_{\varepsilon,\delta} = (f_\varepsilon)_\delta = (f_\delta)_\varepsilon. \end{cases}$$

Note, for $f(t, \cdot) \in L^1(\mathbb{R}^n)$, we have that

$$\int_{\mathbb{R}^n} f_\delta(t, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^n} f(t, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}. \tag{3.63}$$

We define $H_{\varepsilon,\delta} : [\varepsilon, \infty) \to \mathbb{R}$ by

$$H_{\varepsilon,\delta}(t) := \int_{\mathbb{R}^n} \rho_{\varepsilon,\delta}(t, \boldsymbol{x}) |\boldsymbol{x}|^2 \, \mathrm{d}\boldsymbol{x}.$$

Note that since $(\rho, \boldsymbol{\mathcal{M}})$ is a weak solution, we have that for $(t, \boldsymbol{x}) \in (\varepsilon, \infty) \times \mathbb{R}^n$, the mollification satisfies the following:

$$\begin{aligned} &(\rho_{\varepsilon,\delta})_t + \nabla \cdot (\boldsymbol{\mathcal{M}}_{\varepsilon,\delta}) \\ &= \int_0^\infty \int_{\mathbb{R}^n} \Big(\rho(s, \boldsymbol{y})(\varphi_\varepsilon)_t(t - s)\psi_\delta(\boldsymbol{x} - \boldsymbol{y}) + \boldsymbol{\mathcal{M}}(s, \boldsymbol{y}) \cdot \varphi_\varepsilon(t - s) \nabla_{\boldsymbol{x}} \psi_\delta(\boldsymbol{x} - \boldsymbol{y}) \Big) \, \mathrm{d}\boldsymbol{y} \, \mathrm{d}s \\ &= - \int_0^\infty \int_{\mathbb{R}^n} \Big(\rho(s, \boldsymbol{y}) \big[\varphi_\varepsilon(t - \cdot)\psi_\delta(\boldsymbol{x} - \cdot) \big]_s (s, \boldsymbol{y}) \\ &\qquad\qquad + \boldsymbol{\mathcal{M}}(s, \boldsymbol{y}) \cdot \nabla_{\boldsymbol{y}} \big[\varphi_\varepsilon(t - \cdot)\psi_\delta(\boldsymbol{x} - \cdot) \big](s, \boldsymbol{y}) \Big) \, \mathrm{d}\boldsymbol{y} \, \mathrm{d}s \\ &= 0, \end{aligned} \tag{3.64}$$

where the last line follows from the fact that since $t > \varepsilon$, we have that $\big[\varphi_\varepsilon(t - \cdot)\psi_\delta(\boldsymbol{x} - \cdot)\big] \in C_c^\infty((0, \infty) \times \mathbb{R}^n)$ and so can be used as a test function in the definition of weak solutions of the CEREs. Similarly, one can show that for $(t, \boldsymbol{x}) \in (\varepsilon, \infty) \times \mathbb{R}^n$,

$$\partial_t \boldsymbol{\mathcal{M}}_{\varepsilon,\delta} + \nabla \cdot \left(\frac{\boldsymbol{\mathcal{M}} \otimes \boldsymbol{\mathcal{M}}}{\rho} \right)_{\varepsilon,\delta} + \nabla p_{\varepsilon,\delta} + (\rho \nabla \Phi_\alpha * \rho)_{\varepsilon,\delta} = \mathbf{0}. \tag{3.65}$$

Note that

$$\int_{\mathbb{R}^n} \boldsymbol{\mathcal{M}} \cdot \boldsymbol{x} \, \mathrm{d}\boldsymbol{x} \in L^1(0, T),$$

as $\rho(t, \boldsymbol{x})|\boldsymbol{x}|^2 \in L^1(0, T; L^1(\mathbb{R}^n))$ and $\frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}} \in L^\infty(0, T; L^2(\mathbb{R}^n))$. Now, when differentiating $H_{\varepsilon,\delta}$ in t, applying (3.64), Lebesgue's differentiation Theorem, Fubini's Theorem and integration by parts, we get

$$H'_{\varepsilon,\delta}(t) = 2 \int_{\mathbb{R}^n} \boldsymbol{\mathcal{M}}_{\varepsilon,\delta}(t) \cdot \boldsymbol{x} \, \mathrm{d}\boldsymbol{x} = 2 \int_{\mathbb{R}^n} \boldsymbol{\mathcal{M}}_\varepsilon(t) \cdot \boldsymbol{x}_\delta \, \mathrm{d}\boldsymbol{x} = 2 \Big(\int_{\mathbb{R}^n} \boldsymbol{\mathcal{M}} \cdot \boldsymbol{x} \, \mathrm{d}\boldsymbol{x} \Big)_\varepsilon (t).$$

Now, differentiating again, applying (3.65) and using the fact that $\frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}} \in L^\infty(0,T;L^2(\mathbb{R}^n))$, $p(\rho) \in L^\infty(0,T;L^1(\mathbb{R}^n))$, $\rho\Phi_\alpha * \rho \in L^\infty(0,T;L^1(\mathbb{R}^n))$, $\boldsymbol{x} \mapsto \rho(t,\boldsymbol{x})$ has compact support for *a.e.* $t \geq 0$, to apply integration by parts and obtain that the boundary terms disappear, we get

$$H''_{\varepsilon,\delta}(t) = 2\Big(\int_{\mathbb{R}^n} \left(\left| \frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}} \right|^2 \right)_{\varepsilon,\delta} \mathrm{d}\boldsymbol{x} + n \int_{\mathbb{R}^n} p_{\varepsilon,\delta} \,\mathrm{d}\boldsymbol{x} - \int_{\mathbb{R}^n} (\rho \nabla \Phi_\alpha * \rho)_{\varepsilon,\delta} \cdot \boldsymbol{x} \,\mathrm{d}\boldsymbol{x} \Big). \tag{3.66}$$

We have that $\boldsymbol{x} \mapsto (\rho\nabla\Phi_\alpha * \rho)_\delta(s,\boldsymbol{x}) \cdot \boldsymbol{x}$ has compact support for *a.e.* $s \geq 0$. Now, for $s \geq 0$, $\boldsymbol{x} \mapsto (\rho\nabla\Phi_\alpha * \rho)_\delta(s,\boldsymbol{x}) \cdot \boldsymbol{x}$ is continuous. Hence, we can apply Fubini's Theorem to obtain

$$\int_{\mathbb{R}^n} (\rho\nabla\Phi_\alpha * \rho)_\delta \cdot \boldsymbol{x} \,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^n} (\rho\nabla\Phi_\alpha * \rho) \cdot \boldsymbol{x}_\delta \,\mathrm{d}\boldsymbol{x}.$$

But, $\boldsymbol{x}_\delta = \boldsymbol{x}$ and so this implies that $(\rho\nabla\Phi_\alpha * \rho) \cdot \boldsymbol{x}$ is integrable for *a.e.* $s \geq 0$, and we get

$$\begin{aligned}
I &:= \int_{\mathbb{R}^n} \rho\nabla\Phi_\alpha * \rho \cdot \boldsymbol{x} \,\mathrm{d}\boldsymbol{x} \\
&= \int_{\mathbb{R}^n} \rho(\boldsymbol{x}) \int_{\mathbb{R}^n} \frac{\rho(\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^{\alpha+2}} \,\mathrm{d}y \,\mathrm{d}\boldsymbol{x} \\
&= -\int_{\mathbb{R}^n} \rho(\boldsymbol{x}) \int_{\mathbb{R}^n} \frac{\rho(\boldsymbol{y})|\boldsymbol{x}-\boldsymbol{y}|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{\alpha+2}} \,\mathrm{d}y \,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^n} \rho(\boldsymbol{x}) \int_{\mathbb{R}^n} \frac{\rho(\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{x}}{|\boldsymbol{x}-\boldsymbol{y}|^{\alpha+2}} \,\mathrm{d}y \,\mathrm{d}\boldsymbol{x} \\
&= -\int_{\mathbb{R}^n} \rho(\boldsymbol{x}) \int_{\mathbb{R}^n} \frac{\rho(\boldsymbol{y})|\boldsymbol{x}-\boldsymbol{y}|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{\alpha+2}} \,\mathrm{d}y \,\mathrm{d}\boldsymbol{x} - \int_{\mathbb{R}^n} \rho(\boldsymbol{y}) \int_{\mathbb{R}^n} \frac{\rho(\boldsymbol{x})(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{x}}{|\boldsymbol{x}-\boldsymbol{y}|^{\alpha+2}} \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}y \\
&= -\alpha \int_{\mathbb{R}^n} \rho\Phi_\alpha * \rho \,\mathrm{d}\boldsymbol{x} - I.
\end{aligned}$$

Then, we have

$$\int_{\mathbb{R}^n} (\rho\nabla\Phi_\alpha * \rho)_\delta \cdot \boldsymbol{x} \,\mathrm{d}\boldsymbol{x} = I = -\frac{\alpha}{2} \int_{\mathbb{R}^n} \rho\Phi_\alpha * \rho \,\mathrm{d}\boldsymbol{x}.$$

Now, for $t > \varepsilon$, multiplying by $\varphi_\varepsilon(t-s)$ and integrating over s, we obtain

$$\int_{\mathbb{R}^n} (\rho\nabla\Phi_\alpha * \rho)_{\varepsilon,\delta} \cdot \boldsymbol{x} \,\mathrm{d}\boldsymbol{x} = -\frac{\alpha}{2} \int_{\mathbb{R}^n} (\rho\Phi_\alpha * \rho)_\varepsilon \,\mathrm{d}\boldsymbol{x}.$$

As such, applying (3.63) and (3.66), one obtains

$$\begin{aligned}
H''_{\varepsilon,\delta}(t) &= 2\Big(\int_{\mathbb{R}^n} \left(\left| \frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}} \right|^2 \right)_{\varepsilon} \mathrm{d}\boldsymbol{x} + n \int_{\mathbb{R}^n} p_\varepsilon \,\mathrm{d}\boldsymbol{x} + \frac{\alpha}{2} \int_{\mathbb{R}^n} (\rho\Phi_\alpha * \rho)_\varepsilon \,\mathrm{d}\boldsymbol{x} \Big) \\
&= 2\Big(\int_{\mathbb{R}^n} \left| \frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}} \right|^2 \mathrm{d}\boldsymbol{x} + n \int_{\mathbb{R}^n} p \,\mathrm{d}\boldsymbol{x} + \frac{\alpha}{2} \int_{\mathbb{R}^n} (\rho\Phi_\alpha * \rho) \,\mathrm{d}\boldsymbol{x} \Big)_\varepsilon
\end{aligned}$$

and so is independent of δ. As such, if there exists $\Lambda > 0$ such that

$$\Lambda \leq \int_{\mathbb{R}^n} \left| \frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}} \right|^2 \mathrm{d}\boldsymbol{x} + n \int_{\mathbb{R}^n} p \,\mathrm{d}\boldsymbol{x} + \frac{\alpha}{2} \int_{\mathbb{R}^n} \rho\Phi_\alpha * \rho \,\mathrm{d}\boldsymbol{x},$$

for *a.e.* $t > 0$, then since φ is non-negative, we see that

$$H''_{\varepsilon,\delta}(t) \geq 2\Lambda.$$

Then we have that

$$H_{\varepsilon,\delta}(t) \geq H_{\varepsilon,\delta}(t_0) + (t-s)H'_{\varepsilon,\delta}(t_0) + (t-t_0)^2\Lambda, \tag{3.67}$$

for *a.e.* $\varepsilon < t_0 < t$. Since $\int_{\mathbb{R}^n} \rho(\boldsymbol{x})|\boldsymbol{x}|^2 \,\mathrm{d}\boldsymbol{x}$, $\int_{\mathbb{R}^n} \boldsymbol{\mathcal{M}} \cdot \boldsymbol{x} \,\mathrm{d}\boldsymbol{x} \in L^1(0,T)$ for all $T > 0$, we see that for *a.e.* $0 < t_0 < t$, letting $\varepsilon \to 0$ in (3.67), that

$$H(t) \geq H(t_0) + (t-t_0)\int_{\mathbb{R}^n} \boldsymbol{\mathcal{M}}(t_0) \cdot \boldsymbol{x} \,\mathrm{d}\boldsymbol{x} + (t-t_0)^2\Lambda.$$

This is true for all $T > 0$. Thus, it can be extended to all $t > 0$. Hence, applying the arguments used in Lemma 3.6.1, we obtain that

$$\lim_{t\to\infty} \operatorname*{ess\,inf}_{s\geq t}\left(\frac{R(s)}{s}\right) \geq \left(\frac{\Lambda}{M}\right)^{\frac{1}{2}}.$$

In either case, we obtain growth of solutions. □

3.7 Existence of stationary states in the mass-supercritical regime

In this section we prove existence of stationary solutions in the polytropic mass-supercritical case ($\frac{2n}{2n-\alpha} < \gamma < \frac{n+\alpha}{n}$). Given $\varrho \in L^1_+(\mathbb{R}^n) \cap L^\gamma(\mathbb{R}^n)$, and $p(\varrho) = a_0\varrho^\gamma$, with $\frac{2n}{2n-\alpha} < \gamma < \frac{n+\alpha}{n}$, we recall the definition of the free-energy functional

$$Q(\varrho) = na_0 \int_{\mathbb{R}^n} \varrho^\gamma \,\mathrm{d}\boldsymbol{x} + \frac{\alpha}{2}\int_{\mathbb{R}^n} \varrho\Phi_\alpha * \varrho \,\mathrm{d}\boldsymbol{x}.$$

This quantity will play a crucial role in this section. More precisely, let us write the energy (1.6) as

$$E(\rho, \boldsymbol{\mathcal{M}}) = \int_{\mathbb{R}^n} \frac{1}{2}\left|\frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}}\right|^2 \mathrm{d}\boldsymbol{x} + S_\mu(\rho) - \mu\int_{\mathbb{R}^n} \rho \,\mathrm{d}\boldsymbol{x}, \tag{3.68}$$

where the free-energy $S_\mu(\varrho)$ is defined similarly to [39] as

$$S_\mu(\varrho) := \frac{a_0}{\gamma-1}\int_{\mathbb{R}^n} \varrho^\gamma \,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_{\mathbb{R}^n} \varrho\Phi_\alpha * \varrho \,\mathrm{d}\boldsymbol{x} + \mu\int_{\mathbb{R}^n} \varrho \,\mathrm{d}\boldsymbol{x}.$$

As we shall see next, the identity (3.68) is convenient for the rest of the section and $Q(\varrho)$ is related to the derivative of $S_\mu(\varrho)$ in the scaling parameter defining mass-preserving dilations. Note, that $S_\mu(\varrho)$ is chosen to obtain (3.1) as an Euler-Lagrange

equation under some restriction on ϱ. We will see later that any unique stationary state $\bar{\rho}_\mu$ satisfies $Q(\bar{\rho}_\mu) = 0$, as such we would like to show that minimisers of S_μ over the admissible set

$$\mathcal{K} = \{\varrho \in L^1_+(\mathbb{R}^n) \cap L^\gamma(\mathbb{R}^n) \,|\, Q(\varrho) = 0,\ \varrho \not\equiv 0\},$$

are perturbations of $\bar{\rho}_\mu$. We define $\ell_\mu := \inf_{\varrho\in\mathcal{K}} S_\mu(\varrho)$, the minimisation of S_μ over $\mathcal{K}$. Suppose $\varrho \in \mathcal{K}$, then, owing to the fact that $\gamma < \frac{n+\alpha}{n}$, we have that

$$S_\mu(\varrho) = a_0 \frac{\alpha - n(\gamma-1)}{\alpha(\gamma-1)} \int_{\mathbb{R}^n} \varrho^\gamma \,\mathrm{d}\boldsymbol{x} + \mu \int_{\mathbb{R}^n} \varrho \,\mathrm{d}\boldsymbol{x} \geq 0. \tag{3.69}$$

Thus, this implies that $\ell_\mu \geq 0$, later we will show that in fact $\ell_\mu > 0$.

We use the standard scaling that preserves L^1 mass, which for this section is defined by

$$\varrho_\lambda(\boldsymbol{x}) := \lambda \varrho(\lambda^{\frac{1}{n}}\boldsymbol{x}),$$

for $\lambda > 0$, *i.e.*

$$\int_{\mathbb{R}^n} \varrho_\lambda \,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^n} \varrho \,\mathrm{d}\boldsymbol{x}.$$

We define $S_{\mu,\lambda}(\varrho) \coloneqq S_\mu(\varrho_\lambda)$ and state some important facts that will be useful later.

Lemma 3.7.1. *Suppose $\alpha \in (0,n)$, $\frac{2n}{2n-\alpha} < \gamma < \frac{n+\alpha}{n}$, $\varrho \in L^1_+(\mathbb{R}^n) \cap L^\gamma(\mathbb{R}^n)$, $\varrho \not\equiv 0$, then the following statements are true*:

(a) *If $\varrho_m \rightharpoonup \tilde{\varrho}$ in $L^\gamma(\mathbb{R}^n)$ as $m \to \infty$, then $(\varrho_m)_\lambda \rightharpoonup \tilde{\varrho}_\lambda$ in $L^\gamma(\mathbb{R}^n)$ as $m \to \infty$.*

(b) *$(\varrho_\lambda)^\# = (\varrho^\#)_\lambda$ for all $\lambda > 0$, where $(\cdot)^\#$ represents the radially decreasing rearrangement.*

(c) *There exists a unique $\lambda^*(\varrho) > 0$ such that $\varrho_{\lambda^*(\varrho)} \in \mathcal{K}$.*

(d) $\lambda^*(\varrho) > 1 \iff Q(\varrho) > 0$.

(e) $\lambda^*(\varrho) = 1 \iff Q(\varrho) = 0$.

(f) *$\lambda \mapsto S_{\mu,\lambda}(\varrho)$ is strictly increasing (decreasing) for $\lambda \leq \lambda^*(\varrho)$ ($\lambda \geq \lambda^*(\varrho)$). Moreover, $S_{\mu,\lambda}(\varrho) < S_{\mu,\lambda^*(\varrho)}(\varrho)$, for all $\lambda \neq \lambda^*(\varrho)$ and $\lambda > 0$.*

Proof. Both (a) and (b) follow trivially. For (c) we have that

$$Q(\varrho_\lambda) = n a_0 \lambda^{\gamma-1} \int_{\mathbb{R}^n} \varrho^\gamma \,\mathrm{d}\boldsymbol{x} + \frac{\alpha \lambda^{\frac{\alpha}{n}}}{2} \int_{\mathbb{R}^n} \varrho \Phi_\alpha * \varrho \,\mathrm{d}\boldsymbol{x}.$$

Then, there exists a unique $\lambda^*(\varrho) > 0$ such that $\varrho_{\lambda^*(\varrho)} \in \mathcal{K}$. We have

$$\lambda^*(\varrho) := \left(\frac{2n a_0 \int_{\mathbb{R}^n} \varrho^\gamma \,\mathrm{d}\boldsymbol{x}}{-\alpha \int_{\mathbb{R}^n} \varrho \Phi_\alpha * \varrho \,\mathrm{d}\boldsymbol{x}} \right)^{\frac{n}{\alpha - n(\gamma-1)}}. \tag{3.70}$$

Both (d) and (e) follow from (3.70). For (f), we calculate by a change of variables that

$$S_{\mu,\lambda}(\varrho) = \frac{a_0\lambda^{\gamma-1}}{\gamma-1}\int_{\mathbb{R}^n}\varrho^\gamma\,\mathrm{d}\boldsymbol{x} + \frac{\lambda^{\frac{\alpha}{n}}}{2}\int_{\mathbb{R}^n}\varrho\Phi_\alpha * \varrho\,\mathrm{d}\boldsymbol{x} + \mu\int_{\mathbb{R}^n}\varrho\,\mathrm{d}\boldsymbol{x}.$$

Then, differentiating with respect to λ, we obtain

$$\frac{\mathrm{d}S_{\mu,\lambda}(\varrho)}{\mathrm{d}\lambda} = a_0\lambda^{\gamma-2}\int_{\mathbb{R}^n}\varrho^\gamma\,\mathrm{d}\boldsymbol{x} + \frac{\alpha\lambda^{\frac{\alpha}{n}-1}}{2n}\int_{\mathbb{R}^n}\varrho\Phi_\alpha * \varrho\,\mathrm{d}\boldsymbol{x} = \frac{Q(\varrho_\lambda)}{n\lambda}. \tag{3.71}$$

From part (d) and (e), we see that if $\lambda < \lambda^*(\varrho)$, then $Q(\varrho_\lambda) > 0$ and if $\lambda > \lambda^*(\varrho)$, then $Q(\varrho_\lambda) < 0$. As such, the map $\lambda \mapsto S_{\mu,\lambda}(\varrho)$ is strictly increasing for $0 < \lambda < \lambda^*(\varrho)$ and strictly decreasing for $\lambda > \lambda^*(\varrho)$, part (f) follows. □

We will first prove existence of minimisers to the problem $\ell_\mu = \inf_{\varrho\in\mathcal{K}} S_\mu(\varrho)$ in order to write $\bigcup_{\mu>0}\mathcal{I}_\mu$ in a more explicit form. We need to prove that given a minimising sequence of S_μ over $\mathcal{K}$, that we obtain convergence of the potential term. Firstly, we can employ the following lemma to obtain weak convergence of minimising sequences.

Lemma 3.7.2. *Suppose that $q > 1$, $\varrho_k \in L^1_+(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)$ is a bounded sequence. Then, there exists $\tilde{\varrho}\in L^1_+(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)$ such that $\varrho_k \rightharpoonup \tilde{\varrho}$ in $L^q(\mathbb{R}^n)$ (up to a subsequence) and*

$$\|\tilde{\varrho}\|_{L^1} \le \liminf_{k\to\infty}\|\varrho_k\|_{L^1}.$$

The proof is the same as in [39].

Unlike the previous work of [39], for $\alpha \neq n-2$, the potential does not correspond to the extremely well studied Poisson equation. Hence, in order to obtain convergence, we obtain compactness by considering the potential in some sufficiently well chosen Sobolev space, or in the case that $\alpha \ge n-1$, a fractional Sobolev space. Here we are able to make use of the compact embeddings for such spaces and decay estimates for radial functions to obtain convergence. For the next lemma, we note that $L^p_{\mathrm{rad}}(\mathbb{R}^n)$ is the Banach space made up of all radial functions in $L^p(\mathbb{R}^n)$ endowed with the subspace norm.

Lemma 3.7.3. *Suppose that $n \ge 2$, $\alpha\in(0,n)$, $\frac{2n}{2n-\alpha} < \gamma < \frac{n}{n-\alpha}$, $\varrho_m \in L^1_{rad}(\mathbb{R}^n)\cap L^\gamma_{rad}(\mathbb{R}^n)$ is a bounded sequence and $\varrho_m \rightharpoonup \tilde{\varrho}$ as $m\to\infty$ in $L^\gamma(\mathbb{R}^n)$. Then, we have (up to a subsequence)*

$$\int_{\mathbb{R}^n}\varrho_m\Phi_\alpha * \varrho_m\,\mathrm{d}\boldsymbol{x} \longrightarrow \int_{\mathbb{R}^n}\tilde{\varrho}\Phi_\alpha * \tilde{\varrho}\,\mathrm{d}\boldsymbol{x}, \qquad \text{as } m\to\infty.$$

Proof. We divide the proof into five steps.

Step 1.

Case 1 ($\alpha \in (0, n-1)$)**.** Take $\frac{n\gamma}{n(\gamma-1)+\gamma} < p < \frac{n\gamma}{n-\gamma(n-\alpha-1)}$, then $\frac{np}{n+p(n-\alpha)} < \frac{np}{n+p(n-\alpha-1)} < \gamma$ and for $R, R' > 0$, using the HLS inequality and Hölder's inequality, we have that

$$\begin{aligned}\|\Phi_\alpha * (\mathbb{1}_{B_R}\varrho)\|_{L^p(B_{R'})} &\leq C\||\cdot|^{-\alpha} * (\mathbb{1}_{B_R}\varrho)\|_{L^p(B_{R'})} \\ &\leq C\|\varrho\|_{L^{\frac{np}{n+p(n-\alpha)}}(B_R)} \\ &\leq CR^{\frac{\gamma(n+p(n-\alpha))-np}{p\gamma}}\|\varrho\|_{L^\gamma},\end{aligned}$$

and

$$\begin{aligned}\|\nabla\Phi_\alpha * (\mathbb{1}_{B_R}\varrho)\|_{L^p(B_{R'})} &\leq \||\cdot|^{-(\alpha+1)} * (\mathbb{1}_{B_R}\varrho)\|_{L^p(B_{R'})} \\ &\leq C\|\varrho\|_{L^{\frac{np}{n+p(n-\alpha-1)}}(B_R)} \\ &\leq CR^{\frac{\gamma(n+p(n-\alpha-1))-np}{p\gamma}}\|\varrho\|_{L^\gamma}.\end{aligned}$$

As such, since ϱ_m is a bounded sequence in $L^\gamma(\mathbb{R}^n)$, we have that $\Phi_\alpha * (\mathbb{1}_{B_R}\varrho_m) \in W^{1,p}(B_{R'})$ is a bounded sequence. The Rellich-Kondrachov Theorem tell us that for $n \geq 2$, since $\gamma < \frac{n}{n-\alpha}$, that $1 < p < \frac{n\gamma}{n-\gamma(n-\alpha-1)} < n$ and

$$W^{1,p}(B_{R'}) \Subset L^q(B_{R'}),$$

for any $1 \leq q < \frac{np}{n-p}$. Note, that since $\frac{n\gamma}{n(\gamma-1)+\gamma} < p$, we have

$$\frac{np}{n-p} > \frac{\gamma}{\gamma-1} = \gamma',$$

where $\gamma' > 1$ is the Hölder conjugate of γ. So we have that

$$W^{1,p}(B_{R'}) \Subset L^{\gamma'}(B_{R'}).$$

Thus, by compactness, there exists $g \in L^{\gamma'}(B_{R'})$ such that (up to a subsequence)

$$\Phi_\alpha * (\mathbb{1}_{B_R}\varrho_m) \longrightarrow g, \qquad \text{in } L^{\gamma'}(B_{R'}) \text{ as } m \to \infty.$$

Case 2 ($\alpha \in [n-1, n)$)**.** In order to prove local convergence of the potential in the more singular case, we need to consider fractional Sobolev spaces, we take $0 < s < \mu < n - \alpha \leq 1$ and $\frac{n\gamma}{n(\gamma-1)+s\gamma} < p < \frac{n\gamma}{n-\gamma(n-\alpha-s)}$. Then, for $R, R' > 0$ and $\varrho \in L^\gamma(\mathbb{R}^n)$, we have that by the HLS inequality, that

$$\|\Phi_\alpha * (\mathbb{1}_{B_R}\varrho)\|_{L^p(B_{R'})} \leq C\|\varrho\|_{L^{\frac{np}{n+p(n-\alpha)}}(B_R)}. \tag{3.72}$$

Since, $p < \frac{n\gamma}{n-\gamma(n-\alpha-s)}$, we have that $\frac{np}{n+p(n-\alpha)} < \gamma$. As such, combining this with (3.72) and using Hölder's inequality, we obtain

$$\|\Phi_\alpha * (\mathbb{1}_{B_R}\varrho)\|_{L^p(B_{R'})} \leq CR^{\frac{\gamma(n+p(n-\alpha))-np}{p\gamma}}\|\varrho\|_{L^\gamma}.$$

Now, we consider the semi-norm for the space $W^{s,p}(B_{R'})$, as defined in Definition B.0.1, given by

$$\begin{aligned}[\Phi_\alpha * (\mathbb{1}_{B_R}\varrho)]^p_{W^{s,p}(B_{R'})} &= \int_{B_{R'}}\int_{B_{R'}} \frac{|\Phi_\alpha * (\mathbb{1}_{B_R}\varrho)(\boldsymbol{x}) - \Phi_\alpha * (\mathbb{1}_{B_R}\varrho)(\boldsymbol{y})|^p}{|\boldsymbol{x}-\boldsymbol{y}|^{n+sp}}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} \\ &= \int_{B_{R'}}\int_{B_{R'}} \left| \int_{B_R} \frac{|\boldsymbol{x}-\boldsymbol{z}|^{-\alpha} - |\boldsymbol{y}-\boldsymbol{z}|^{-\alpha}}{\alpha|\boldsymbol{x}-\boldsymbol{y}|^{\frac{n}{p}+s}}\varrho(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\right|^p \mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{3.73}$$

Now, considering the difference of the two Riesz potentials in (3.73), we get

$$\begin{aligned}\left|\frac{|\boldsymbol{x}-\boldsymbol{z}|^{-\alpha} - |\boldsymbol{y}-\boldsymbol{z}|^{-\alpha}}{\alpha}\right| &= \left|\frac{1}{\alpha}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\big(t|\boldsymbol{x}-\boldsymbol{z}| + (1-t)|\boldsymbol{y}-\boldsymbol{z}|\big)^{-\alpha}\,\mathrm{d}t\right| \\ &= \left|\int_0^1 \frac{|\boldsymbol{x}-\boldsymbol{z}| - |\boldsymbol{y}-\boldsymbol{z}|}{\big(t|\boldsymbol{x}-\boldsymbol{z}| + (1-t)|\boldsymbol{y}-\boldsymbol{z}|\big)^{\alpha+1}}\,\mathrm{d}t\right| \\ &\le \int_0^1 \frac{|\boldsymbol{x}-\boldsymbol{y}|}{\big(t|\boldsymbol{x}-\boldsymbol{z}| + (1-t)|\boldsymbol{y}-\boldsymbol{z}|\big)^{\alpha+1}}\,\mathrm{d}t.\end{aligned} \tag{3.74}$$

Note, that given $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in\mathbb{R}^n$, then we must have that $|\boldsymbol{x}-\boldsymbol{y}| \le 2|\boldsymbol{x}-\boldsymbol{z}|$ or $|\boldsymbol{x}-\boldsymbol{y}| \le 2|\boldsymbol{y}-\boldsymbol{z}|$. As suppose the former is not true, then $|\boldsymbol{x}-\boldsymbol{y}| > 2|\boldsymbol{x}-\boldsymbol{z}|$, but then

$$|\boldsymbol{x}-\boldsymbol{y}| \le |\boldsymbol{x}-\boldsymbol{z}| + |\boldsymbol{y}-\boldsymbol{z}| < \frac{1}{2}|\boldsymbol{x}-\boldsymbol{y}| + |\boldsymbol{y}-\boldsymbol{z}|.$$

Rearranging, we obtain the latter is true and vice versa. As such, if $|\boldsymbol{x}-\boldsymbol{z}| \le |\boldsymbol{y}-\boldsymbol{z}|$, then $|\boldsymbol{x}-\boldsymbol{y}| \le 2|\boldsymbol{y}-\boldsymbol{z}|$ is true and vice versa. Thus, suppose that $|\boldsymbol{x}-\boldsymbol{z}| \le |\boldsymbol{y}-\boldsymbol{z}|$, then combining with (3.74), we obtain

$$\begin{aligned}\left|\frac{|\boldsymbol{x}-\boldsymbol{z}|^{-\alpha} - |\boldsymbol{y}-\boldsymbol{z}|^{-\alpha}}{\alpha}\right| &\le |\boldsymbol{x}-\boldsymbol{y}|^{\mu}|\boldsymbol{x}-\boldsymbol{z}|^{\mu-\alpha}\int_0^1 \left(\frac{1-t}{2}\right)^{\mu-1}\mathrm{d}t \\ &= \frac{2^{1-\mu}}{\mu}|\boldsymbol{x}-\boldsymbol{y}|^{\mu}|\boldsymbol{x}-\boldsymbol{z}|^{-\mu-\alpha}.\end{aligned} \tag{3.75}$$

A similar inequality holds for $|\boldsymbol{y}-\boldsymbol{z}| \le |\boldsymbol{x}-\boldsymbol{z}|$. We define the set

$$B_R^{(\boldsymbol{x},\boldsymbol{y})} \coloneqq B_R \cap \{\boldsymbol{z}\in\mathbb{R}^n \,|\, |\boldsymbol{x}-\boldsymbol{z}| \le |\boldsymbol{y}-\boldsymbol{z}|\}.$$

Thus, combining (3.73) with (3.75), we get

$$\begin{aligned}&[\Phi_\alpha * (\mathbb{1}_{B_R}\varrho)]^p_{W^{s,p}(B_{R'})} \\ &\le \int_{B_{R'}}\int_{B_{R'}} |\boldsymbol{x}-\boldsymbol{y}|^{(\mu-s)p-n}\Bigg(\Bigg(\int_{B_R^{(\boldsymbol{x},\boldsymbol{y})}} + \int_{B_R^{(\boldsymbol{y},\boldsymbol{x})}}\Bigg)\left|\frac{|\boldsymbol{x}-\boldsymbol{z}|^{-\alpha} - |\boldsymbol{y}-\boldsymbol{z}|^{-\alpha}}{\alpha|\boldsymbol{x}-\boldsymbol{y}|^{\mu}}\right| \\ &\qquad\qquad \times \varrho(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\Bigg)^p \mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} \\ &\le C\int_{B_{R'}}\int_{B_{R'}} |\boldsymbol{x}-\boldsymbol{y}|^{(\mu-s)p-n}\left(\int_{B_R}\big(|\boldsymbol{x}-\boldsymbol{z}|^{-\mu-\alpha} + |\boldsymbol{y}-\boldsymbol{z}|^{-\mu-\alpha}\big)\varrho(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\right)^p \mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} \\ &= C\int_{B_{R'}}\int_{B_{R'}} |\boldsymbol{x}-\boldsymbol{y}|^{(\mu-s)p-n}\big(|\cdot|^{-\mu-\alpha} * (\mathbb{1}_{B_R}\varrho)(\boldsymbol{x}) + |\cdot|^{-\mu-\alpha} * (\mathbb{1}_{B_R}\varrho)(\boldsymbol{y})\big)^p \mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{3.76}$$

Applying Minkowski's inequality and Tonelli's Theorem to (3.76), we get that

$$\begin{aligned}
&[\Phi_\alpha * (\mathbb{1}_{B_R}\varrho)]^p_{W^{s,p}(B_{R'})} \\
&\leq C\bigg(\Big(\int_{B_{R'}}\int_{B_{R'}} |\boldsymbol{x}-\boldsymbol{y}|^{(\mu-s)p-n}\big(|\cdot|^{-\mu-\alpha} * (\mathbb{1}_{B_R}\varrho)(\boldsymbol{x})\big)^p \,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}\Big)^{\frac{1}{p}} \\
&\qquad + \Big(\int_{B_{R'}}\int_{B_{R'}} |\boldsymbol{x}-\boldsymbol{y}|^{(\mu-s)p-n}\big(|\cdot|^{-\mu-\alpha} * (\mathbb{1}_{B_R}\varrho)(\boldsymbol{y})\big)^p \,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}\Big)^{\frac{1}{p}}\bigg)^p \qquad (3.77)\\
&= C\int_{B_{R'}}\int_{B_{R'}} |\boldsymbol{x}-\boldsymbol{y}|^{(\mu-s)p-n}\big(|\cdot|^{-\mu-\alpha} * (\mathbb{1}_{B_R}\varrho)(\boldsymbol{y})\big)^p \,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} \\
&\leq C\int_{B_{R'}} |\cdot|^{(\mu-s)p-n} * \big(|\cdot|^{-\mu-\alpha} * (\mathbb{1}_{B_R}\varrho)\big)^p \,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Now, taking $1 < r \leq \frac{n\gamma}{p(n-\gamma(n-\alpha-s))}$, we get $\frac{n\gamma}{n(\gamma-1)+s\gamma} < pr \leq \frac{n\gamma}{n-\gamma(n-\alpha-s)}$. Applying Hölder's inequality to the last integral in (3.77) and applying the HLS inequality twice, we get that

$$\begin{aligned}
[\Phi_\alpha * (\mathbb{1}_{B_R}\varrho)]^p_{W^{s,p}(B_{R'})} &\leq C(R')^{\frac{n(r-1)}{r}} \big\| |\cdot|^{(\mu-s)p-n} * \big(|\cdot|^{-\mu-\alpha} * (\mathbb{1}_{B_R}\varrho)\big)^p \big\|_{L^r(\mathbb{R}^n)} \\
&\leq C(R')^{\frac{n(r-1)}{r}} \big\| |\cdot|^{-\mu-\alpha} * (\mathbb{1}_{B_R}\varrho) \big\|^p_{L^{\frac{npr}{n+pr(\mu-s)}}(\mathbb{R}^n)} \qquad (3.78)\\
&\leq C(R')^{\frac{n(r-1)}{r}} \|\varrho\|^p_{L^{\frac{npr}{n+pr(n-\alpha-s)}}(B_R)}.
\end{aligned}$$

Note that both the applications of the HLS inequality are well-defined, as both $0 < n-(\mu-s)p, \mu+\alpha < n$ and, thanks to the fact that $\frac{n\gamma}{n(\gamma-1)+s\gamma} < pr$ and $\frac{2n}{2n-\alpha} < \gamma < \frac{n}{n-\alpha}$, we get

$$1 < \frac{n\gamma}{(2n-\alpha)\gamma - n} < \frac{npr}{n+pr(n-\alpha-s)} < \frac{npr}{n+pr(\mu-s)}.$$

Since $pr \leq \frac{n\gamma}{n-\gamma(n-\alpha-s)}$, we get $\frac{npr}{n+pr(n-\alpha-s)} \leq \gamma$. Thus, combining with (3.78) and applying Hölder's inequality, we obtain

$$[\Phi_\alpha * (\mathbb{1}_{B_R}\varrho)]^p_{W^{s,p}(B_{R'})} \leq C(R')^{\frac{n(r-1)}{r}} R^{\frac{n+pr(n-\alpha-s)}{r}} \|\varrho\|^p_{L^\gamma}.$$

As such, since ϱ_m is a bounded sequence in $L^\gamma(\mathbb{R}^n)$, we have that $\Phi_\alpha * (\mathbb{1}_{B_R}\varrho_m) \in W^{s,p}(B_{R'})$ is a bounded sequence. For fractional Sobolev spaces, since $sp < n$, by Lemma B.0.2, we have that

$$W^{s,p}(B_{R'}) \Subset L^q(B_{R'}),$$

for any $1 \leq q < \frac{np}{n-sp}$. Note, that since $\frac{n\gamma}{n(\gamma-1)+s\gamma} < p$, we have

$$\frac{np}{n-sp} > \frac{\gamma}{\gamma-1} = \gamma'.$$

So we have that

$$W^{s,p}(B_{R'}) \Subset L^{\gamma'}(B_{R'}).$$

Thus, by compactness, there exists $g \in L^{\gamma'}(B_{R'})$ such that (up to a subsequence)

$$\Phi_\alpha * (\mathbb{1}_{B_R}\varrho_m) \longrightarrow g, \qquad \text{in } L^{\gamma'}(B_{R'}) \text{ as } m \to \infty.$$

Step 2. Now, since $\varrho_m \rightharpoonup \tilde{\varrho}$ as $m \to \infty$ in $L^\gamma(\mathbb{R}^n)$ and $\gamma > \frac{2n}{2n-\alpha}$, for $f \in L^\gamma(B_{R'})$, that by the HLS inequality and that B_R has finite measure, that $\Phi_\alpha * f \in L^{\frac{n\gamma}{n-\gamma(n-\alpha)}}(B_R) \subset L^{\gamma'}(B_R)$ (where we extend f by considering it to be 0 outside $B_{R'}$) and by Fubini's Theorem

$$\int_{B_{R'}} f\Phi_\alpha * (\mathbb{1}_{B_R}\varrho_m)\,\mathrm{d}\boldsymbol{x} = \int_{B_R} \varrho_m \Phi_\alpha * f\,\mathrm{d}\boldsymbol{x} \longrightarrow \int_{B_R} \tilde{\varrho}\Phi_\alpha * f\,\mathrm{d}\boldsymbol{x} = \int_{B_{R'}} f\Phi_\alpha * (\mathbb{1}_{B_R}\tilde{\varrho})\,\mathrm{d}\boldsymbol{x},$$

as $m \to \infty$. Hence, we have that $\Phi_\alpha * (\mathbb{1}_{B_R}\varrho_m) \rightharpoonup \Phi_\alpha * (\mathbb{1}_{B_R}\tilde{\varrho})$ as $m \to \infty$ in $L^{\gamma'}(B_{R'})$. Thus, by uniqueness of limits, we must have that $g \equiv \Phi_\alpha * (\mathbb{1}_{B_R}\tilde{\varrho})$ in $L^{\gamma'}(\mathbb{R}^n)$. Hence (up to a subsequence),

$$\Phi_\alpha * (\mathbb{1}_{B_R}\varrho_m) \longrightarrow \Phi_\alpha * (\mathbb{1}_{B_R}\tilde{\varrho}), \qquad \text{in } L^{\gamma'}(B_{R'}) \text{ as } m \to \infty.$$

Which implies that (up to a subsequence)

$$\int_{B_{R'}} \varrho_m \Phi_\alpha * (\mathbb{1}_{B_R}\varrho_m)\,\mathrm{d}\boldsymbol{x} \longrightarrow \int_{B_{R'}} \tilde{\varrho}\Phi_\alpha * (\mathbb{1}_{B_R}\tilde{\varrho})\,\mathrm{d}\boldsymbol{x}, \qquad \text{as } m \to \infty. \tag{3.79}$$

Step 3. Using that ϱ_m are bounded in $L^1(\mathbb{R}^n)$ and taking $R' > R > 0$, we get that

$$\begin{aligned}\left|\int_{B^c_{R'}} \varrho_m \Phi_\alpha * (\mathbb{1}_{B_R}\varrho_m)\,\mathrm{d}\boldsymbol{x}\right| &\leq C\frac{1}{(R'-R)^\alpha}\int_{B^c_{R'}} |\varrho_m(\boldsymbol{x})| \int_{B_R} |\varrho_m(\boldsymbol{y})|\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} \\ &\leq C\frac{1}{(R'-R)^\alpha}.\end{aligned} \tag{3.80}$$

Hence, taking $R' > 0$ sufficiently larger than $R > 0$ in (3.80) and combining with (3.79), we get that

$$\int_{\mathbb{R}^n} \varrho_m \Phi_\alpha * (\mathbb{1}_{B_R}\varrho_m)\,\mathrm{d}\boldsymbol{x} \longrightarrow \int_{\mathbb{R}^n} \tilde{\varrho}\Phi_\alpha * (\mathbb{1}_{B_R}\tilde{\varrho})\,\mathrm{d}\boldsymbol{x}, \qquad \text{as } m \to \infty. \tag{3.81}$$

Step 4. We have that since ϱ_m is radial, we can use the radial representation formula with its bounds as in Lemma 2.5.1, to get

$$(\Phi_\alpha * \varrho_m)(\boldsymbol{x}) = \int_0^\infty K(|\boldsymbol{x}|,\eta)\varrho_m(\eta)\eta^{n-1}\,\mathrm{d}\eta,$$

where

$$K(r,\eta) \coloneqq \int_{\mathbb{S}^{n-1}} \frac{|r\boldsymbol{e_1} - \eta\boldsymbol{y}|^{-\alpha}}{-\alpha}\,\mathrm{d}S(\boldsymbol{y})$$

satisfies, for $\alpha \neq n-1$,

$$|K(r,\eta)| \leq C\begin{cases}(r\eta)^{-\frac{\alpha}{2}} & \text{for } \alpha \in (0, n-1), \\ (r\eta)^{-\frac{n-1}{2}}|r-\eta|^{n-1-\alpha} & \text{for } \alpha \in (n-1, n).\end{cases}$$

Case 1. For $\alpha \in (0, n-1)$, taking $R, R' > 0$ and using that ϱ_m are bounded in $L^1(\mathbb{R}^n)$, we get

$$\begin{aligned}\left| \int_{B^c_{R'}} \varrho_m \Phi_\alpha * (\mathbb{1}_{B^c_R} \varrho_m) \, \mathrm{d}\boldsymbol{x} \right| &= \left| \int_{R'}^\infty \varrho_m(r) r^{n-1} \int_R^\infty K(r,\eta) \varrho_m(\eta) \eta^{n-1} \, \mathrm{d}\eta \, \mathrm{d}r \right| \\ &\leq C \frac{1}{(R'R)^{\frac{\alpha}{2}}} \int_{R'}^\infty |\varrho_m(r)| r^{n-1} \int_R^\infty |\varrho_m(\eta)| \eta^{n-1} \, \mathrm{d}\eta \, \mathrm{d}r \\ &\leq C \frac{1}{(R'R)^{\frac{\alpha}{2}}}.\end{aligned} \tag{3.82}$$

Case 2. For $\alpha \in (n-1, n)$, taking $R, R' > 0$, using that ϱ_m are bounded in $L^1(\mathbb{R}^n) \cap L^\gamma(\mathbb{R}^n)$, the HLS inequality and that $\frac{2}{n+1-\alpha} < \frac{2n}{2n-\alpha} < \gamma$, we get

$$\begin{aligned}\left| \int_{B^c_{R'}} \varrho_m \Phi_\alpha * (\mathbb{1}_{B^c_R} \varrho_m) \, \mathrm{d}\boldsymbol{x} \right| &= \left| \int_{R'}^\infty \varrho_m(r) r^{n-1} \int_R^\infty K(r,\eta) \varrho_m(\eta) \eta^{n-1} \, \mathrm{d}\eta \, \mathrm{d}r \right| \\ &\leq C \int_{R'}^\infty |\varrho_m(r)| r^{\frac{n-1}{2}} \int_R^\infty |r-\eta|^{n-1-\alpha} |\varrho_m(\eta)| \eta^{\frac{n-1}{2}} \, \mathrm{d}\eta \, \mathrm{d}r \\ &\leq C \Big(\int_{R'}^\infty |\varrho_m(r)|^{\frac{2}{n+1-\alpha}} r^{\frac{n-1}{n+1-\alpha}} \\ &\qquad \times \int_R^\infty |\varrho_m(\eta)|^{\frac{2}{n+1-\alpha}} \eta^{\frac{n-1}{n+1-\alpha}} \, \mathrm{d}\eta \, \mathrm{d}r \Big)^{\frac{n+1-\alpha}{2}} \\ &\leq C \frac{1}{(R'R)^{\frac{(n-1)(n-\alpha)}{n+1-\alpha}}} \|\varrho_m\|^2_{L^{\frac{2}{n+1-\alpha}}(\mathbb{R}^n)} \\ &\leq C \frac{1}{(R'R)^{\frac{(n-1)(n-\alpha)}{n+1-\alpha}}} \|\varrho_m\|_{L^1}^{\frac{\gamma(n+1-\alpha)-2}{\gamma-1}} \|\varrho_m\|_{L^\gamma}^{\frac{\gamma(\alpha-(n-1))}{\gamma-1}} \\ &\leq C \frac{1}{(R'R)^{\frac{(n-1)(n-\alpha)}{n+1-\alpha}}}.\end{aligned} \tag{3.83}$$

Case 3. For $\alpha = n-1$, taking $R, R' > 0$, we note that by considering the behaviour of Φ_α around and away from the origin separately, we can obtain control of the potential energy, by the use of less and more singular Riesz potentials. That is, given a $0 < \delta < 1$, we have that

$$\begin{aligned}\left| \int_{B^c_{R'}} \varrho_m \Phi_{n-1} * (\mathbb{1}_{B^c_R} \varrho_m) \, \mathrm{d}\boldsymbol{x} \right| \leq C \Big(&\left| \int_{B^c_{R'}} \varrho_m \Phi_{n-1-\delta} * (\mathbb{1}_{B^c_R} \varrho_m) \, \mathrm{d}\boldsymbol{x} \right| \\ &+ \left| \int_{B^c_{R'}} \varrho_m \Phi_{n-1+\delta} * (\mathbb{1}_{B^c_R} \varrho_m) \, \mathrm{d}\boldsymbol{x} \right| \Big).\end{aligned} \tag{3.84}$$

Thus, taking $0 < \delta < \frac{n-1}{n} = \frac{\alpha}{n}$ we see that $\frac{2}{2-\delta} < \frac{2n}{n+1} = \frac{2n}{2n-\alpha}$. Thus, combining with (3.82)-(3.84), we see that

$$\left| \int_{B^c_{R'}} \varrho_m \Phi_{n-1} * (\mathbb{1}_{B^c_R} \varrho_m) \, \mathrm{d}\boldsymbol{x} \right| \leq C \Big(\frac{1}{(R'R)^{\frac{n-1-\delta}{2}}} + \frac{1}{(R'R)^{\frac{(n-1)(1-\delta)}{2-\delta}}} \Big). \tag{3.85}$$

Step 5. Note, that for $R > R' > 0$, as in (3.80), using that ϱ_m are bounded in $L^1(\mathbb{R}^n)$ and Fubini's Theorem, we get by symmetry that

$$\left| \int_{B_{R'}} \varrho_m \Phi_\alpha * (\mathbb{1}_{B_R^c} \varrho_m) \, \mathrm{d}\boldsymbol{x} \right| \leq \frac{C}{(R - R')^\alpha}. \tag{3.86}$$

Combining (3.81)-(3.83), (3.85)-(3.86) while taking $R > R' > 0$ large with $R > 0$ sufficiently larger than $R' > 0$, we obtain the desired result.

□

Lemma 3.7.4. *For $n \geq 2$, $\alpha \in (0, n)$, $\frac{2n}{2n-\alpha} < \gamma < \frac{n+\alpha}{n}$, we have that $\ell_\mu > 0$ and that there exists a minimiser of S_μ over $\mathcal{K}$.*

Proof. We divide the proof into two steps.

Step 1. Suppose that $\varrho_m \in \mathcal{K}$ is a minimising sequence of S_μ, i.e. that

$$Q(\varrho_m) = 0,$$

and

$$\lim_{m \to \infty} S_\mu(\varrho_m) = \ell_\mu.$$

We would like to take a minimising sequence that is radially symmetric, in order to do this, we first take the radially decreasing rearrangement to define $w_m = \varrho_m^\#$. However, it is now unclear whether w_m exists in $\mathcal{K}$, so we take $\tilde{w}_m = (w_m)_{\lambda^*(w_m)}$. Note, that by Lemma 3.7.1 (b) and (c), we have that $\tilde{w}_m = ((\varrho_m)_{\lambda^*(w_m)})^\#$ and so $\tilde{w}_m \in \mathcal{K}$ are radially symmetric. Using Lemma 3.7.1 (f) and the same method as in Lemma 3.3.6, we can see that $\tilde{w}_m$ is a minimising sequence as

$$\ell_\mu \leq S_\mu(\tilde{w}_m) \leq S_\mu((\varrho_m)_{\lambda^*(w_m)}) \leq S_\mu(\varrho_m).$$

As such, we can assume that $\{\varrho_m\}_m$ is a sequence of radially symmetric, radially decreasing and non-negative functions. Using (3.69), we see that

$$\|\varrho_m\|_{L^\gamma} + \|\varrho_m\|_{L^1} \leq C. \tag{3.87}$$

We have that since $\varrho_m \in \mathcal{K}$, that

$$\|\varrho_m\|_{L^\gamma}^\gamma = -\frac{\alpha}{2na_0} \int_{\mathbb{R}^n} \varrho_m \Phi_\alpha * \varrho_m \, \mathrm{d}\boldsymbol{x} \leq C \|\varrho_m\|_{L^1}^{\frac{\gamma(2n-\alpha)-2n}{n(\gamma-1)}} \|\varrho_m\|_{L^\gamma}^{\frac{\gamma\alpha}{n(\gamma-1)}}. \tag{3.88}$$

Thus, combining this with (3.87), since $\frac{2n}{2n-\alpha} < \gamma < \frac{n+\alpha}{n}$, we get

$$\|\varrho_m\|_{L^\gamma}^{\frac{\gamma(n(\gamma-1)-\alpha)}{n(\gamma-1)}} \leq C \|\varrho_m\|_{L^1}^{\frac{\gamma(2n-\alpha)-2n}{n(\gamma-1)}} \leq C,$$

and thus

$$\|\varrho_m\|_{L^\gamma} \geq C. \tag{3.89}$$

Thus, using (3.69), we see that $\ell_\mu > 0$. By Lemma 3.7.2, we have that (3.87) implies there exists $\tilde{\varrho} \in L^\gamma_{\mathrm{rad}}(\mathbb{R}^n)$ such that (up to a subsequence) $\varrho_m \rightharpoonup \tilde{\varrho}$ as $m \to \infty$ in $L^\gamma(\mathbb{R}^n)$.

Step 2. However, $\tilde{\varrho}$ may not satisfy that $Q(\tilde{\varrho}) = 0$. As such, we take $\bar{\rho} := \tilde{\varrho}_{\lambda^*(\tilde{\varrho})} \in \mathcal{K}$. Then, Lemma 3.7.1 (a) tells us that (up to a subsequence) $\varrho_m^* := (\varrho_m)_{\lambda^*(\tilde{\varrho})} \rightharpoonup \bar{\rho}$ as $m \to \infty$ in $L^\gamma(\mathbb{R}^n)$. From Lemma 3.7.2, we have that $\bar{\varrho} \geq 0$ *a.e.* and

$$\|\bar{\rho}\|_{L^\gamma} + \|\bar{\rho}\|_{L^1} \leq C.$$

Since $\frac{n+\alpha}{n} < \frac{n}{n-\alpha}$, Lemma 3.7.3 tells us that (up to a subsequence)

$$\int_{\mathbb{R}^n} \varrho_m^* \Phi_\alpha * \varrho_m^* \,\mathrm{d}\boldsymbol{x} \longrightarrow \int_{\mathbb{R}^n} \bar{\rho}\Phi_\alpha * \bar{\rho} \,\mathrm{d}\boldsymbol{x}, \qquad \text{as } m \to \infty. \tag{3.90}$$

Combining (3.88) and (3.89), we have that

$$-\int_{\mathbb{R}^n} \varrho_m^* \Phi_\alpha * \varrho_m^* \,\mathrm{d}\boldsymbol{x} \geq C.$$

Combining with (3.90), we have that

$$-\int_{\mathbb{R}^n} \bar{\rho}\Phi_\alpha * \bar{\rho} \,\mathrm{d}\boldsymbol{x} \geq C.$$

Hence, we see that $\bar{\rho} \not\equiv 0$. Now, by Lemma 3.7.2 and weak lower semi-continuity of $\int_{\mathbb{R}^n} \varrho^\gamma \,\mathrm{d}\boldsymbol{x}$, we get that

$$\int_{\mathbb{R}^n} \bar{\rho} \,\mathrm{d}\boldsymbol{x} \leq \liminf_{m\to\infty} \int_{\mathbb{R}^n} \varrho_m^* \,\mathrm{d}\boldsymbol{x}, \qquad \text{and} \qquad \int_{\mathbb{R}^n} (\bar{\rho})^\gamma \,\mathrm{d}\boldsymbol{x} \leq \liminf_{m\to\infty} \int_{\mathbb{R}^n} (\varrho_m^*)^\gamma \,\mathrm{d}\boldsymbol{x}.$$

Hence, using Lemma 3.7.1 (f), we have that

$$S_\mu(\bar{\rho}) \leq \liminf_{m\to\infty} S_\mu(\varrho_m^*) \leq \liminf_{m\to\infty} S_\mu(\varrho_m) = \ell_\mu.$$

Thus $\bar{\rho}$ is a radially symmetric minimiser.

□

Thus, we can now prove existence of stationary states.

Proof of Theorem 3.2.7. We divide the proof into two steps.

Step 1. We consider the Euler-Lagrange equations for the minimisation problem. To do this, for $\varepsilon > 0$, we consider the set

$$\Gamma_\varepsilon := \Big\{ \boldsymbol{x} \in \mathbb{R}^n \,|\, \varepsilon < \bar{\rho}(\boldsymbol{x}) < \frac{1}{\varepsilon} \Big\}.$$

Now, take any $w \in L^\infty(\mathbb{R}^n)$ with compact support, that is non-negative on $\mathbb{R}^n \setminus \Gamma_\varepsilon$. Now, for $\tau > 0$ small enough, we have that

$$\bar{\rho}_\tau := \bar{\rho} + \tau w,$$

satisfies that $\bar{\rho}_\tau \geq 0$ *a.e.* and $\bar{\rho}_\tau \not\equiv 0$. In particular, we have that $\bar{\rho}_\tau \in L^1(\mathbb{R}^n) \cap L^\gamma(\mathbb{R}^n)$ and so by Lemma 3.7.1 (c), there exists unique

$$\lambda^*(\bar{\rho}_\tau) = \left(\frac{2na_0 \int_{\mathbb{R}^n} \bar{\rho}_\tau^\gamma \,\mathrm{d}\boldsymbol{x}}{-\alpha \int_{\mathbb{R}^n} \bar{\rho}_\tau \Phi_\alpha * \bar{\rho}_\tau \,\mathrm{d}\boldsymbol{x}} \right)^{\frac{n}{\alpha - n(\gamma-1)}},$$

such that $(\bar{\rho}_\tau)_{\lambda^*(\bar{\rho}_\tau)} \in \mathcal{K}$. Letting $f(\tau) = S_\mu((\bar{\rho}_\tau)_{\lambda^*(\bar{\rho}_\tau)})$ and using (3.71), we obtain that

$$\begin{aligned} f'(\tau) &= \frac{\mathrm{d}\lambda^*(\bar{\rho}_\tau)}{\mathrm{d}\tau} \frac{\partial S_{\mu,\lambda}(\bar{\rho}_\tau)}{\partial \lambda}\bigg|_{\lambda=\lambda^*(\bar{\rho}_\tau)} + \lambda^*(\bar{\rho}_\tau)^{\gamma-1} \frac{a_0\gamma}{\gamma-1} \int_{\mathbb{R}^n} w \bar{\rho}_\tau^{\gamma-1} \,\mathrm{d}\boldsymbol{x} \\ &\qquad + \lambda^*(\bar{\rho}_\tau)^{\frac{\alpha}{n}} \int_{\mathbb{R}^n} w \Phi_\alpha * \bar{\rho}_\tau \,\mathrm{d}\boldsymbol{x} + \mu \int_{\mathbb{R}^n} w \,\mathrm{d}\boldsymbol{x} \qquad (3.91) \\ &= \int_{\mathbb{R}^n} \Big(\lambda^*(\bar{\rho}_\tau)^{\gamma-1} \frac{a_0\gamma}{\gamma-1} \bar{\rho}_\tau^{\gamma-1} + \lambda^*(\bar{\rho}_\tau)^{\frac{\alpha}{n}} \Phi_\alpha * \bar{\rho}_\tau + \mu \Big) w \,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

Note that the right-hand derivative $f'_+(\tau)$ exists at $\tau = 0$ and satisfies $f'_+(0) = \lim_{\tau \to 0^+} f'(\tau) \geq 0$ as $\bar{\rho}$ is a minimiser of S_μ over $\mathcal{K}$. Thus, since $\lambda^*(\bar{\rho}_0) = 1$, employing (3.91), we obtain

$$0 \leq f'_+(0) = \int_{\mathbb{R}^n} \Big(\frac{a_0\gamma}{\gamma-1} \bar{\rho}^{\gamma-1} + \Phi_\alpha * \bar{\rho} + \mu \Big) w \,\mathrm{d}\boldsymbol{x}. \qquad (3.92)$$

Since (3.92) is true for all such w, we obtain that

$$\begin{aligned} &\frac{a_0\gamma}{\gamma-1} \bar{\rho}^{\gamma-1} + \Phi_\alpha * \bar{\rho} + \mu = 0, && \text{a.e. on } \Gamma_\varepsilon, \\ &\frac{a_0\gamma}{\gamma-1} \bar{\rho}^{\gamma-1} + \Phi_\alpha * \bar{\rho} + \mu \geq 0, && \text{a.e. on } \mathbb{R}^n \setminus \Gamma_\varepsilon. \end{aligned}$$

Thus, letting $\varepsilon \to 0$, we obtain that

$$\begin{aligned} &\frac{a_0\gamma}{\gamma-1} \bar{\rho}^{\gamma-1} + \Phi_\alpha * \bar{\rho} + \mu = 0, && \text{a.e. on } \Gamma, \\ &\Phi_\alpha * \bar{\rho} + \mu \geq 0, && \text{a.e. on } \mathbb{R}^n \setminus \Gamma. \end{aligned} \qquad (3.93)$$

Note, that since $\bar{\rho} \in L^1(\mathbb{R}^n) \cap L^\gamma(\mathbb{R}^n)$, we have that by the HLS inequality, that $\Phi_\alpha * \bar{\rho} \in L^p(\mathbb{R}^n)$ for $\frac{n}{\alpha} < p \leq \frac{n\gamma}{n-\gamma(n-\alpha)}$. Since $\frac{2n}{2n-\alpha} < \gamma < \frac{n+\alpha}{n}$, we have that

$\frac{n}{\alpha} < \frac{\gamma}{\gamma-1} \leq \frac{n\gamma}{n-\gamma(n-\alpha)}$. Thus, we have that $\bar{\rho}^{\gamma-1}, \Phi_\alpha * \bar{\rho} \in L^{\frac{\gamma}{\gamma-1}}(\mathbb{R}^n)$. So, combining this with (3.93) and the fact that $\bar{\rho}$ is radially decreasing, we must have that the support of $\bar{\rho}$ is compact. Similar to [6, Theorem A], we have that $\bar{\rho} \in L^\infty(\mathbb{R}^n)$. Moreover, manipulating (3.93), we have that

$$\bar{\rho} = \left(\frac{\gamma-1}{a_0\gamma}\right)^{\frac{1}{\gamma-1}} (-\Phi_\alpha * \bar{\rho} - \mu)_+^{\frac{1}{\gamma-1}}.$$

Step 2. For $\alpha \in [n-2, n)$, as mentioned at the start of the section, $\bar{\rho}_\mu$ is the unique radially symmetric solution to this equation. We will show that $\bar{\rho}$ must be a translation of $\bar{\rho}_\mu$. We have, by Lemma 3.7.1 (c), that $(\bar{\rho}^\#)_{\lambda^*(\bar{\rho}^\#)} \in \mathcal{K}$. By Lemma 3.7.1 (b) and (f),

$$\ell_\mu \leq S_\mu\Big((\bar{\rho}^\#)_{\lambda^*(\bar{\rho}^\#)}\Big) = S_\mu\Big(\big((\bar{\rho})_{\lambda^*(\bar{\rho}^\#)}\big)^\#\Big) \leq S_\mu\Big((\bar{\rho})_{\lambda^*(\bar{\rho}^\#)}\Big) \leq S_\mu(\bar{\rho}) = \ell_\mu.$$

Hence, we have that all inequalities must be equalities in the previous statement. As such, we conclude $\lambda^*(\bar{\rho}^\#) = \lambda^*(\bar{\rho}) = 1$ and $S_\mu(\bar{\rho}^\#) = S_\mu(\bar{\rho})$. This implies that $\bar{\rho}^\# = \bar{\rho}_\mu$. Moreover, we must have

$$\int_{\mathbb{R}^n} \bar{\rho}^\# \Phi_\alpha * \bar{\rho}^\# \,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^n} \bar{\rho}\Phi_\alpha * \bar{\rho}\,\mathrm{d}\boldsymbol{x}.$$

By Lemma 3.3.6, we have that $\bar{\rho}$ is a translation of $\bar{\rho}^\# = \bar{\rho}_\mu$. This concludes the proof.

□

3.8 Instability of stationary states in the mass-supercritical regime

We aim to show that if we have a solution $(\rho, \boldsymbol{\mathcal{M}})$ of (1.1) with initial conditions $(\rho_0, \boldsymbol{\mathcal{M}}_0)$ satisfying

$$\int_{\mathbb{R}^n} \rho_0 \,\mathrm{d}\boldsymbol{x} < \left(\ell_1 \frac{(2n-\alpha)\gamma - 2n}{(n-\alpha)(\gamma-1)}\right)^{\frac{(n-\alpha)(\gamma-1)}{(2n-\alpha)\gamma-2n}} \left(\frac{(2n-\alpha)\gamma-2n}{n+\alpha-n\gamma} E(\rho_0, \boldsymbol{\mathcal{M}}_0)\right)^{\frac{n\gamma-n-\alpha}{(2n-\alpha)\gamma-2n}},$$

then either $R(t) \to \infty$ as $t \to \infty$ or there exists some $\Lambda > 0$ such that $Q(\rho(t)) \geq \Lambda$ for all $t \geq 0$. The latter can be used to show, as in the previous section, $H''(t) \geq 2\Lambda$ and hence we obtain instability of stationary states (3.1). To do this, we require our initial conditions to be in some suitable set of the form $\mathcal{I}_\mu$.

Definition 3.8.1. We define

$$\mathcal{I}_\mu := \{(\rho, \boldsymbol{\mathcal{M}}) \,|\, Q(\rho) > 0,\ I_\mu(\rho, \boldsymbol{\mathcal{M}}) < \ell_\mu\},$$

where

$$I_\mu(\rho, \boldsymbol{\mathcal{M}}) := E(\rho, \boldsymbol{\mathcal{M}}) + \mu \int_{\mathbb{R}^n} \rho \,\mathrm{d}\boldsymbol{x}.$$

We show that employing the minimisation problem of §3.7, that the set $\mathcal{I}_\mu$ must remain invariant under the semi-flow of (1.1).

Lemma 3.8.2 (Invariance of $\mathcal{I}_\mu$ under the solution semi-flow)**.** *Let $\alpha \in (0,n)$, $\frac{2n}{2n-\alpha} < \gamma < \frac{n+\alpha}{n}$ and $(\rho, \boldsymbol{\mathcal{M}})$ be a classical solution of the attractive polytropic CEREs with initial data $(\rho_0, \boldsymbol{\mathcal{M}}_0) \in \mathcal{I}_\mu$. Then*

$$(\rho(t), \boldsymbol{\mathcal{M}}(t)) \in \mathcal{I}_\mu, \qquad \text{for all } t > 0.$$

Proof. By conservation of mass and energy,

$$I_\mu(\rho(t), \boldsymbol{\mathcal{M}}(t)) = I_\mu(\rho_0, \boldsymbol{\mathcal{M}}_0) < \ell_\mu, \quad \text{for all } t > 0.$$

Suppose by contradiction that there exists $t_0 > 0$ such that $Q(\rho(t_0)) \leq 0$. Since $Q(\rho_0) > 0$ and $t \mapsto Q(\rho(t))$ is continuous, there exists a first time $t^* > 0$ such that $Q(\rho(t^*)) = 0$. By definition of ℓ_μ, we have

$$S_\mu(\rho(t^*)) \geq \ell_\mu.$$

On the other hand,

$$S_\mu(\rho(t^*)) \leq I_\mu(\rho(t^*), \boldsymbol{\mathcal{M}}(t^*)) = I_\mu(\rho_0, \boldsymbol{\mathcal{M}}_0) < \ell_\mu,$$

which is a contradiction. □

We now give a more explicit description of $\bigcup_{\mu>0} \mathcal{I}_\mu$.

Lemma 3.8.3. *For $n \geq 2$, $\alpha \in (0,n)$, $\frac{2n}{2n-\alpha} < \gamma < \frac{n+\alpha}{n}$, we have that*

$$\bigcup_{\mu>0} \mathcal{I}_\mu = \mathcal{I} := \Big\{ (\rho, \boldsymbol{\mathcal{M}}) \,\Big|\, Q(\rho) > 0,\ \int_{\mathbb{R}^n} \rho \,\mathrm{d}\boldsymbol{x} < \Big(\ell_1 \frac{(2n-\alpha)\gamma - 2n}{(n-\alpha)(\gamma-1)} \Big)^{\frac{(n-\alpha)(\gamma-1)}{(2n-\alpha)\gamma-2n}} \times \Big(\frac{(2n-\alpha)\gamma - 2n}{n+\alpha-n\gamma} E(\rho, \boldsymbol{\mathcal{M}}) \Big)^{\frac{n\gamma-n-\alpha}{(2n-\alpha)\gamma-2n}} \Big\}.$$

Proof. By the definition of $\mathcal{I}_\mu$, we have that if $(\rho, \boldsymbol{\mathcal{M}}) \in \mathcal{I}_\mu$, then $E(\rho, \boldsymbol{\mathcal{M}}) < \ell_\mu - \mu \int_{\mathbb{R}^n} \rho \,\mathrm{d}\boldsymbol{x}$. Thus, defining

$$f(\mu) \coloneqq \ell_\mu - \mu \int_{\mathbb{R}^n} \rho \,\mathrm{d}\boldsymbol{x},$$

we have that if $(\rho, \boldsymbol{\mathcal{M}}) \in \bigcup_{\mu>0} \mathcal{I}_\mu$, then

$$E(\rho, \boldsymbol{\mathcal{M}}) < \max_{\mu>0} f(\mu). \tag{3.94}$$

By Lemma 3.7.4, there exists $\bar{\rho}_1 \in \mathcal{K}$ that is a minimiser of S_1 over $\mathcal{K}$ and by Theorem 3.2.7, $\bar{\rho}_1$ is a solution of (3.1) for $\mu = 1$. As mentioned, $\bar{\rho}_\mu(\boldsymbol{x}) = \mu^{\frac{1}{\gamma-1}} \bar{\rho}_1 \Big(\mu^{\frac{2-\gamma}{(\gamma-1)(n-\alpha)}} \boldsymbol{x} \Big)$ is a solution of (3.1) for $\mu > 0$. Note that due to the scaling, $\bar{\rho}_\mu \in \mathcal{K}$. Suppose $\bar{\rho}_\mu$

is not a minimiser of S_μ over $\mathcal{K}$, then we have that there exists $\varrho_\mu^* \in \mathcal{K}$ such that $S_\mu(\varrho_\mu^*) < S_\mu(\bar{\rho}_\mu)$. But then defining $\varrho_1^*(\boldsymbol{x}) \coloneqq \mu^{\frac{-1}{\gamma-1}} \varrho_\mu^*\Big(\mu^{\frac{\gamma-2}{(\gamma-1)(n-\alpha)}}\boldsymbol{x}\Big)$, $\varrho_1^* \in \mathcal{K}$ and

$$S_1(\varrho_1^*) = \mu^{\frac{2n-\gamma(2n-\alpha)}{(\gamma-1)(n-\alpha)}} S_\mu(\varrho_\mu^*) < \mu^{\frac{2n-\gamma(2n-\alpha)}{(\gamma-1)(n-\alpha)}} S_\mu(\bar{\rho}_\mu) = S_1(\bar{\rho}_1),$$

a contradiction. Thus, $\bar{\rho}_\mu$ is a minimiser of S_μ over $\mathcal{K}$. Hence, we have that

$$\ell_\mu = \mu^{\frac{(2n-\alpha)\gamma-2n}{(\gamma-1)(n-\alpha)}} \ell_1.$$

Thus,

$$f(\mu) = \mu^{\frac{(2n-\alpha)\gamma-2n}{(\gamma-1)(n-\alpha)}} \ell_1 - \mu \int_{\mathbb{R}^n} \rho \,\mathrm{d}\boldsymbol{x}.$$

We find the value μ_0 where f attains its maximum. As such, μ_0 satisfies

$$f'(\mu_0) = \frac{(2n-\alpha)\gamma-2n}{(n-\alpha)(\gamma-1)} \mu_0^{\frac{n\gamma-n-\alpha}{(\gamma-1)(n-\alpha)}} \ell_1 - \int_{\mathbb{R}^n} \rho \,\mathrm{d}\boldsymbol{x} = 0.$$

Hence, we see that

$$\mu_0 = \left(\frac{(n-\alpha)(\gamma-1) \int_{\mathbb{R}^n} \rho \,\mathrm{d}\boldsymbol{x}}{((2n-\alpha)\gamma-2n)\ell_1} \right)^{\frac{(\gamma-1)(n-\alpha)}{n\gamma-n-\alpha}}. \tag{3.95}$$

Since $\frac{2n}{2n-\alpha} < \gamma < \frac{n+\alpha}{n}$, we see that $\mu_0 > 0$ and that $f''(\mu_0) < 0$. Hence, f attains its maximum at μ_0. Plugging this in, we obtain

$$\begin{aligned}
\max_{\mu>0} f(\mu) &= f(\mu_0) \\
&= \left(\frac{(n-\alpha)(\gamma-1)}{((2n-\alpha)\gamma-2n)\ell_1} \right)^{\frac{(n-\alpha)(\gamma-1)}{n\gamma-n-\alpha}} \left(\frac{n+\alpha-n\gamma}{(2n-\alpha)\gamma-2n} \right) \left(\int_{\mathbb{R}^n} \rho \,\mathrm{d}\boldsymbol{x} \right)^{\frac{(2n-\alpha)\gamma-2n}{n\gamma-n-\alpha}}.
\end{aligned}$$

hence, rearranging and applying (3.94), since $n(\gamma-1)-\alpha < 0$ we get that

$$\int_{\mathbb{R}^n} \rho \,\mathrm{d}\boldsymbol{x} < \left(\ell_1 \frac{(2n-\alpha)\gamma-2n}{(n-\alpha)(\gamma-1)} \right)^{\frac{(n-\alpha)(\gamma-1)}{(2n-\alpha)\gamma-2n}} \left(\frac{(2n-\alpha)\gamma-2n}{n+\alpha-n\gamma} E(\rho, \boldsymbol{\mathcal{M}}) \right)^{\frac{n\gamma-n-\alpha}{(2n-\alpha)\gamma-2n}}.$$

Thus, $(\rho, \boldsymbol{\mathcal{M}}) \in \mathcal{I}$. Thus $\bigcup_{\mu>0} \mathcal{I}_\mu \subset \mathcal{I}$.

Now, suppose $(\rho, \boldsymbol{\mathcal{M}}) \in \mathcal{I}$. Then, from the previous calculations and $\mu_0 > 0$ defined as in (3.95), we have that $(\rho, \boldsymbol{\mathcal{M}}) \in \mathcal{I}_{\mu_0} \subset \bigcup_{\mu>0} \mathcal{I}_\mu$. Thus, $\bigcup_{\mu>0} \mathcal{I}_\mu \supseteq \mathcal{I}$. Hence, we conclude the proof. □

We now study the concavity of the map $\lambda \mapsto S_{\mu,\lambda}(\varrho)$. As mentioned previously, the concavity of the free-energy with respect to the mass-preserving scaling λ is fundamental in proving the convexity of the second moment of the density, allowing us to prove growth of the support of the density.

Lemma 3.8.4. *Suppose $\alpha \in (0,n)$, $\frac{2n}{2n-\alpha} < \gamma < \frac{n+\alpha}{n}$, $\varrho \in L^1_+(\mathbb{R}^n) \cap L^\gamma(\mathbb{R}^n)$, $\varrho \not\equiv 0$, then there exists a neighbourhood of $\lambda^*(\varrho)$ such that $\lambda \mapsto S_{\mu,\lambda}(\varrho)$ is a concave mapping. In particular, the map $\lambda \mapsto S_{\mu,\lambda}(\varrho)$ is concave for $0 < \lambda \leq \lambda^*(\varrho)$.*

Proof. Differentiating (3.71) again, we see

$$\frac{\mathrm{d}^2 S_{\mu,\lambda}(\varrho)}{\mathrm{d}\lambda^2} = \lambda^{\gamma-3}\Big(a_0(\gamma-2)\int_{\mathbb{R}^n}\varrho^\gamma\,\mathrm{d}\boldsymbol{x} + \frac{\alpha(\alpha-n)\lambda^{\frac{\alpha-n(\gamma-1)}{n}}}{2n^2}\int_{\mathbb{R}^n}\varrho\Phi_\alpha * \varrho\,\mathrm{d}\boldsymbol{x}\Big). \tag{3.96}$$

Thus, substituting in $\lambda^*(\varrho)$ and using the fact that $\gamma < \frac{n+\alpha}{n}$, we obtain

$$\frac{\mathrm{d}^2 S_{\mu,\lambda}(\varrho)}{\mathrm{d}\lambda^2}\bigg|_{\lambda^*(\varrho)} = \frac{a_0(n(\gamma-1)-\alpha)\lambda^*(\varrho)^{\gamma-3}}{n}\int_{\mathbb{R}^n}\varrho^\gamma\,\mathrm{d}\boldsymbol{x} < 0. \tag{3.97}$$

Since, we can see that $\lambda \mapsto \frac{\mathrm{d}^2 S_{\mu,\lambda}(\varrho)}{\mathrm{d}\lambda^2}$ is continuous from (3.96), we obtain concavity in a neighbourhood of $\lambda^*(\varrho)$. For $\alpha \in (0,n)$, we have the map

$$\lambda \mapsto a_0(\gamma-2)\int_{\mathbb{R}^n}\varrho^\gamma\,\mathrm{d}\boldsymbol{x} + \frac{\alpha(\alpha-n)\lambda^{\frac{\alpha-n(\gamma-1)}{n}}}{2n^2}\int_{\mathbb{R}^n}\varrho\Phi_\alpha * \varrho\,\mathrm{d}\boldsymbol{x},$$

is increasing in λ. Hence, applying similar calculations to that of (3.97) and using (3.96), we obtain that for $0 < \lambda \leq \lambda^*(\varrho)$

$$\frac{\mathrm{d}^2 S_{\mu,\lambda}(\varrho)}{\mathrm{d}\lambda^2} \leq \frac{a_0(n(\gamma-1)-\alpha)\lambda^{\gamma-3}}{n}\int_{\mathbb{R}^n}\varrho^\gamma\,\mathrm{d}\boldsymbol{x} < 0.$$

Thus, $\lambda \mapsto S_{\mu,\lambda}(\varrho)$ is concave in $(0,\lambda^*(\varrho)]$. □

Now we show that if $(\rho_0, \boldsymbol{\mathcal{M}}_0) \in \mathcal{I}_\mu$, $R(t) \to \infty$ as $t \to \infty$. Moreover, the following lemma allows us to characterise the asymptotic behaviour of the radius of the support $R(t)$.

Lemma 3.8.5. *Suppose $(\rho, \boldsymbol{\mathcal{M}})$ is a classical solution of the attractive polytropic CEREs ($\kappa = 1$) (1.1) for $\alpha \in (0,n)$, $\frac{2n}{2n-\alpha} < \gamma < \frac{n+\alpha}{n}$ with initial conditions $(\rho_0, \boldsymbol{\mathcal{M}}_0) \in \mathcal{I}_\mu$. Then, $R(t) \to \infty$ as $t \to \infty$.*

Remark 3.8.6. Note that unlike [39], we cannot make the assumption in the previous lemma that $\lim_{t\to\infty} Q(\rho(t)) = 0$ implies that $\lim_{t\to\infty}\lambda^*(\rho(t)) = 1$. Hence, we need to consider the cases where $\lim_{t\to\infty}\lambda^*(\rho(t)) = 1$ is and is not true and what we can conclude in each case.

Proof. We divide the proof into two steps.

Step 1. Suppose that there does not exist such $\Lambda > 0$ such that $Q(\rho(t)) \geq \Lambda$ for all $t > 0$. Then there exists a time $t_0 \in (0,\infty]$ such that $Q(\rho(t)) > 0$ for all $t \in [0,t_0)$ and $\lim_{t\to t_0} Q(\rho(t)) = 0$. Note that $t_0 = \infty$, as if $t_0 < \infty$, then $Q(\rho(t_0)) = 0$, which is impossible due to the invariance of $\mathcal{I}_\mu$ as proved in Lemma 3.8.2. Thus, from the scaling properties in Lemma 3.7.1, we have that $\lambda^*(\rho(t)) > 1$ for $t \in [0,\infty)$. From Lemma 3.8.4, we have that $\lambda \mapsto S_{\mu,\lambda}(\rho(t))$ is concave on $[1,\lambda^*(\rho(t))]$. Due to the

concavity and the relation of the derivative of $S_\mu(\rho)$ with respect to the scaling and $Q(\rho)$ as seen in (3.71), we have for all $t > 0$

$$\begin{aligned}\ell_\mu \leq S_{\mu,\lambda^*(\rho(t))}(\rho(t)) &\leq S_\mu(\rho(t)) + (\lambda^*(\rho(t)) - 1)\frac{\mathrm{d}S_{\mu,\lambda}(\rho(t))}{\mathrm{d}\lambda}\bigg|_{\lambda=1} \\ &= S_\mu(\rho(t)) + (\lambda^*(\rho(t)) - 1)\frac{Q(\rho(t))}{n}.\end{aligned}$$

Rearranging and using conservation of energy, we obtain

$$Q(\rho(t)) \geq \frac{n(\ell_\mu - S_\mu(\rho(t)))}{\lambda^*(\rho(t)) - 1} \geq \frac{n(\ell_\mu - I_\mu(\rho_0))}{\lambda^*(\rho(t)) - 1}, \tag{3.98}$$

where $\ell_\mu - I_\mu(\rho_0) > 0$.

Case 1. Suppose that

$$\liminf_{t\to\infty} \frac{Q(\rho(t))}{-\int_{\mathbb{R}^n} \rho(t)\Phi_\alpha * \rho(t)\,\mathrm{d}\boldsymbol{x}} > 0.$$

Since $\lim_{t\to\infty} Q(\rho(t)) = 0$, this implies that $\lim_{t\to\infty} \int_{\mathbb{R}^n} \rho(t)\Phi_\alpha * \rho(t)\,\mathrm{d}\boldsymbol{x} = 0$ and equally that $\lim_{t\to\infty} \int_{\mathbb{R}^n} \rho(t)^\gamma\,\mathrm{d}\boldsymbol{x} = 0$. Suppose that $R(t) \not\to \infty$ as $t \to \infty$, then there exists a sequence $t_m \to \infty$ as $m \to \infty$ and $C > 0$ such that $R(t_m) \leq C$. But then we have

$$M = \int_{\Omega(t_m)} \rho(t_m)\,\mathrm{d}\boldsymbol{x} = \int_{B_C} \rho(t_m)\,\mathrm{d}\boldsymbol{x} \leq \left(\frac{\omega_n C^n}{n}\right)^{\frac{\gamma-1}{\gamma}} \|\rho(t_m)\|_{L^\gamma} \longrightarrow 0, \qquad \text{as } m \to \infty.$$

This is a contradiction, as $M > 0$, so we must have that $R(t) \to \infty$ as $t \to \infty$.

Case 2. Otherwise, suppose that

$$\liminf_{t\to\infty} \frac{Q(\rho(t))}{-\int_{\mathbb{R}^n} \rho(t)\Phi_\alpha * \rho(t)\,\mathrm{d}\boldsymbol{x}} = 0.$$

Then, there exists a sequence $t_m \to \infty$ as $m \to \infty$, such that

$$\lim_{m\to\infty} \frac{Q(\rho(t_m))}{-\int_{\mathbb{R}^n} \rho(t_m)\Phi_\alpha * \rho(t_m)\,\mathrm{d}\boldsymbol{x}} = 0. \tag{3.99}$$

Due to formula (3.70), we see that

$$\lambda^*(\rho) = \left(1 - \frac{2Q(\rho)}{\alpha\int_{\mathbb{R}^n} \rho\Phi_\alpha * \rho\,\mathrm{d}\boldsymbol{x}}\right)^{\frac{1}{\alpha - n(\gamma-1)}}.$$

Combining this with (3.99), we have that $\lim_{m\to\infty} \lambda^*(\rho(t_m)) = 1$. Hence, from (3.98), we obtain $\lim_{m\to\infty} Q(\rho(t_m)) = \infty$, which is a contradiction. Thus, we must actually be in Case 1.

Step 2. Now, we suppose that there exists $\Lambda > 0$ such that $Q(\rho(t)) \geq \Lambda$ for all $t > 0$. As in the proof of Lemma 3.6.1, we have that

$$\begin{aligned} H''(t) &= 2\Big(\int_{\Omega(t)} \left| \frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}} \right|^2 (t)\, \mathrm{d}\boldsymbol{x} + na_0 \int_{\Omega(t)} \rho^\gamma(t)\, \mathrm{d}\boldsymbol{x} + \frac{\alpha}{2} \int_{\Omega(t)} \rho(t) \Phi_\alpha * \rho(t)\, \mathrm{d}\boldsymbol{x} \Big) \\ &\geq 2Q(\rho(t)) \\ &\geq 2\Lambda. \end{aligned}$$

Thus, as in Lemma 3.6.1, we have that

$$H(0) + H'(0)t + \Lambda t^2 \leq H(t) \leq R(t)^2 M.$$

Thus,

$$\liminf_{t\to\infty} \left(\frac{R(t)}{t} \right) \geq \left(\frac{\Lambda}{M} \right)^{\frac{1}{2}}.$$

The result follows.

□

Proof of Theorem 3.2.8. Lemma 3.8.3 proves that the condition $(\rho_0, \boldsymbol{\mathcal{M}}_0) \in \bigcup_{\mu>0} \mathcal{I}_\mu$, is equivalent to $Q(\rho_0) > 0$ and (3.16). By Lemma 3.8.5, since $(\rho_0, \boldsymbol{\mathcal{M}}_0) \in \mathcal{I}_\mu$ for some $\mu > 0$, then $R(t) \to \infty$ as $t \to \infty$. □

3.9 Open problems

In the mass-supercritical regime, the homogeneity of the polytropic pressure plays a crucial role in analysing the behaviour of the free-energy, enabling us not only to establish the existence of stationary solutions but also to prove their instability. However, it remains unclear whether a comparable analysis can be extended to more general pressure laws. In the author's view, such an extension is unlikely to be straightforward and would likely require a fundamentally different approach.

Chapter 4

Existence, stability and instability of rotating Riesz star solutions of the compressible Euler-Riesz equations

4.1 Introduction

Another important part of this thesis is the existence and stability of rotating Riesz star solutions. These are the analogue of rotating star solutions for the attractive CEPEs. Here, what we mean by a rotating Riesz star solution, is that of a steady state of the attractive CEREs with rotation about an axis, in particular, the solution is axisymmetric. For the whole chapter, we assume that we are in the attractive Riesz potential setting $\kappa = 1$. We say a solution $(\rho, \boldsymbol{\mathcal{M}})$ of (1.1) for $n = 3$ is axisymmetric if, given the basis

$$\boldsymbol{e}_r = \Big(\frac{x_1}{r}, \frac{x_2}{r}, 0\Big), \qquad \boldsymbol{e}_\theta = \Big(-\frac{x_2}{r}, \frac{x_1}{r}, 0\Big), \qquad \boldsymbol{e}_3 = (0, 0, 1),$$

the density ρ and momentum $\boldsymbol{\mathcal{M}}$ satisfy

$$\begin{cases} \rho(t, \boldsymbol{x}) = \rho(t, r, z), \\ \boldsymbol{\mathcal{M}}(t, \boldsymbol{x}) = \mathcal{M}_r(t, r, z)\boldsymbol{e}_r + \mathcal{M}_\theta(t, r, z)\boldsymbol{e}_\theta + \mathcal{M}_3(t, r, z)\boldsymbol{e}_3, \end{cases} \tag{4.1}$$

where $r = r(\boldsymbol{x}) = \sqrt{x_1^2 + x_2^2}$ and $z = z(\boldsymbol{x}) = x_3$. Suppose the angular momentum of the solution is given by $J(m_\rho(r))$ with

$$m_\rho(r) \coloneqq \int_{\sqrt{x_1^2+x_2^2} \le r} \rho(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = 2\pi \int_0^r \int_{-\infty}^{\infty} \rho(\eta, z)\eta \, \mathrm{d}z \, \mathrm{d}\eta,$$

then, the rotational momentum is given by

$$\mathcal{M}_\theta(t, r) = \frac{\rho(t, r, z) J(m_{\rho(t)}(r))}{r}.$$

Thus, more specifically, $(\bar{\rho}, \bar{\boldsymbol{\mathcal{M}}})$ is a rotating Riesz star solution if the velocity field satisfies

$$\bar{\boldsymbol{u}}(\boldsymbol{x}) \coloneqq \Big(\frac{\bar{\boldsymbol{\mathcal{M}}}}{\bar{\rho}}\Big)(\boldsymbol{x}) = \frac{J(m_{\bar{\rho}}(r))}{r}\boldsymbol{e}_\theta.$$

Plugging this into (1.1), we obtain

$$\begin{cases} \partial_r p(\bar\rho) = \bar\rho \dfrac{L(m_{\bar\rho}(r))}{r^3} - \bar\rho \partial_r(\Phi_\alpha * \bar\rho), \\ \partial_z p(\bar\rho) = -\bar\rho \partial_z(\Phi_\alpha * \bar\rho), \end{cases} \tag{4.2}$$

where $L = J^2$. Thus, dividing (4.2) by $\bar\rho$ and integrating against r and z, respectively, as in Chapter 3, one obtains that there exists $\mu > 0$ such that

$$\begin{aligned} (\bar\rho e(\bar\rho))_{\bar\rho} &= -\int_{r(\boldsymbol{x})}^{\infty} L(m_{\bar\rho}(s)) s^{-3} \,\mathrm{d}s - \Phi_\alpha * \bar\rho - \mu \\ &= \left(-\int_{r(\boldsymbol{x})}^{\infty} L(m_{\bar\rho}(s)) s^{-3} \,\mathrm{d}s - \Phi_\alpha * \bar\rho - \mu \right)_+ . \end{aligned}$$

Thus, we consider $L \in C^1([0, M])$ such that

$$\begin{cases} L(0) = 0, \\ L(m) \geq 0 \text{ for all } m \in [0, M], \end{cases} \tag{4.3}$$

and have the following definition for rotating Riesz star solutions.

Definition 4.1.1. $\bar\rho \in L^1_+(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ is called a rotating Riesz star solution of (1.1) if $\bar\rho$ is axisymmetric, there exists an associated angular momentum squared L and a constant $\mu > 0$ such that

$$\begin{cases} (\bar\rho e(\bar\rho))_{\bar\rho} + \int_{r(\boldsymbol{x})}^{\infty} L(m_{\bar\rho}(s)) s^{-3} \,\mathrm{d}s + \Phi_\alpha * \bar\rho = -\mu, & \boldsymbol{x} \in \Gamma, \\ \int_{r(\boldsymbol{x})}^{\infty} L(m_{\bar\rho}(s)) s^{-3} \,\mathrm{d}s + \Phi_\alpha * \bar\rho(\boldsymbol{x}) \geq -\mu, & \boldsymbol{x} \in \mathbb{R}^3 \setminus \Gamma, \end{cases} \tag{4.4}$$

where

$$\Gamma := \{\boldsymbol{x} \in \mathbb{R}^3 \,|\, \bar\rho(\boldsymbol{x}) > 0\}.$$

Then, the associated velocity field $\bar{\boldsymbol{u}}$, is given by

$$\bar{\boldsymbol{u}}(\boldsymbol{x}) = \left(\frac{\bar{\boldsymbol{\mathcal{M}}}}{\bar\rho}\right)(\boldsymbol{x}) = \frac{J(m_{\bar\rho}(r))}{r} \boldsymbol{e}_\theta.$$

In the mass-subcritical case, in order to prove existence and stability of rotating Riesz star solutions for a general pressure, we employ a similar variational approach to §3.3. While the free-energy functional $\mathcal{G}$, did not contain a kinetic energy term, we now have to incorporate the kinetic energy term corresponding to the velocity field $\bar{\boldsymbol{u}}$. Thus, we define the free-energy functional $\mathcal{G}^a$:

$$\mathcal{G}^a(\varrho) = \int_{\mathbb{R}^3} \varrho(\boldsymbol{x}) \Big(e(\varrho(\boldsymbol{x})) + \frac{1}{2} \frac{L(m_\varrho(r(\boldsymbol{x})))}{r(\boldsymbol{x})^2} + \frac{1}{2} (\Phi_\alpha * \varrho)(\boldsymbol{x}) \Big) \,\mathrm{d}\boldsymbol{x}.$$

This is paired with the set of admissible functions X^a_M:

$$X^a_M := \left\{ \varrho \in L^1_+(\mathbb{R}^3) \cap L^{\frac{3+\alpha}{3}}_{\mathrm{axi}}(\mathbb{R}^3) \,\middle|\, \int_{\mathbb{R}^3} \varrho \,\mathrm{d}\boldsymbol{x} = M, \int_{\mathbb{R}^3} \frac{\varrho(\boldsymbol{x}) L(m_\varrho(r(\boldsymbol{x})))}{r(\boldsymbol{x})^2} \,\mathrm{d}\boldsymbol{x} < \infty \right\},$$

where

$$L^p_{\text{axi}}(\mathbb{R}^3) \coloneqq \{\varrho \in L^p(\mathbb{R}^3) \,|\, \varrho \text{ is axisymmetric}\}.$$

We adapt the original concentration compactness framework of [103] to include the rotational kinetic term. Moreover, the axisymmetric requirement on the density creates added difficulties, for instance, in the case of dichotomy of minimising sequences, one must prove that the two diverging distributions must be axisymmetric themselves. Furthermore, owing to the form of the kinetic term, it is no longer clear whether radially decreasing rearrangements of minimisers of $\mathcal{G}^a$ over X^a_M are indeed minimisers themselves.

In the polytropic mass-supercritical range ($\frac{6}{6-\alpha} < \gamma < \frac{3+\alpha}{3}$), as in Chapter 3, the free-energy functional $\mathcal{G}^a$ is no longer bounded below on X^a_M. Thus, as before, we introduce a different free-energy functional of the form

$$S^a_\mu(\varrho) \coloneqq \frac{1}{2}\int_{\mathbb{R}^3} \frac{\varrho L(m_\varrho(r))}{r^2}\,\mathrm{d}\boldsymbol{x} + \frac{a_0}{\gamma-1}\int_{\mathbb{R}^3}\varrho^\gamma\,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_{\mathbb{R}^3}\varrho\Phi_\alpha * \varrho\,\mathrm{d}\boldsymbol{x} + \mu\int_{\mathbb{R}^3}\varrho\,\mathrm{d}\boldsymbol{x},$$

for $\mu > 0$. If, as in the previous chapter, we proceed with considering the behaviour of S^a_μ with respect to the mass-preserving scaling, we observe different behaviour to that of §3.7. By considering the derivative of S^a_μ with respect to the mass-preserving scaling parameter, one obtains the functional

$$Q^a(\varrho) \coloneqq \int_{\mathbb{R}^3} \frac{\varrho L(m_\varrho(r))}{r^2}\,\mathrm{d}\boldsymbol{x} + 3a_0\int_{\mathbb{R}^3}\varrho^\gamma\,\mathrm{d}\boldsymbol{x} + \frac{\alpha}{2}\int_{\mathbb{R}^3}\varrho\Phi_\alpha * \varrho\,\mathrm{d}\boldsymbol{x}.$$

While in the non-rotating case, for any $\varrho \in L^1_+(\mathbb{R}^3) \cap L^\gamma(\mathbb{R}^3)$, there exists a unique scaling $\lambda^*(\varrho)$ such that $Q(\varrho_{\lambda^*(\varrho)}) = 0$, we have that due to the scaling of the kinetic term, that this is not true in general in the rotating case. While for $\alpha \in (2,3)$, one can prove that there is a unique $\lambda^*(\varrho)$ such that $Q^a(\varrho_{\lambda^*(\varrho)}) = 0$, when $\alpha \in (0,2]$, one can no longer guarantee that there exists $\lambda > 0$ such that $Q^a(\varrho_\lambda) = 0$. In fact, if $\alpha \in (0,2)$, if there exists such a λ, then it is not guaranteed to be unique, that is, there can exist at most two $0 < \lambda^*_1(\varrho) \leq \lambda^*_2(\varrho)$ such that $Q^a(\varrho_{\lambda^*_i(\varrho)}) = 0$ for $i = 1,2$. While, in the non-rotational case, the scaling $\lambda^*(\varrho)$ represents the global maximum of $\lambda \mapsto S_{\mu,\lambda}(\varrho)$, in the rotational case when $\alpha \in (0,2)$, if $\lambda^*_1(\varrho)$ and $\lambda^*_2(\varrho)$ exist and are distinct, then $\lambda^*_1(\varrho)$ represents the local maximum of $\lambda \mapsto S^a_{\mu,\lambda}(\varrho)$ and $\lambda^*_2(\varrho)$ the local minimum, while if $\lambda^*_1(\varrho) = \lambda^*_2(\varrho)$, it is an inflection point. Thus, simply minimising the functional S^a_μ over the set of ϱ such that $Q^a(\varrho) = 0$ may not produce a rotating Riesz star solution, in fact, S^a_μ is not even bounded below on such a set. Thus, we require a different admissible set in order to prove existence of rotating Riesz star solutions. In fact, since $\lambda^*_1(\varrho)$ corresponds to a local maximum, one expects that, as in the non-rotating case, minimising over ϱ with $\lambda^*_1(\varrho) = 1$ should result in a rotating Riesz star

solution. Furthermore, since we will observe that this is equivalent to $Q^a(\varrho) = 0$ and $Q^a(\varrho_\lambda) > 0$, for all $0 < \lambda < 1$, we will consider the admissible set

$$\mathcal{K}^a := \{\varrho \in L^1_+(\mathbb{R}^3) \cap L^\gamma_{\text{axi}}(\mathbb{R}^3) \,|\, Q^a(\varrho) = 0,\, Q^a(\varrho_\lambda) > 0 \text{ for all } 0 < \lambda < 1,\, \varrho \not\equiv 0\}.$$

Note that if $L \equiv 0$, without the axisymmetric assumption, the admissible set $\mathcal{K}^a$ reduces to $\mathcal{K}$. Note that the admissible set does not impose conditions on the mass of a distribution ϱ. Thus, it no longer makes sense to consider L of the form (4.3). Instead, we require, $L \in C^1([0,\infty))$ such that

$$\begin{cases} L(0) = 0, \\ L(m) \geq 0 \text{ for all } m \in [0,\infty). \end{cases} \tag{4.5}$$

Note that given a function L of the form (4.3), this can be extended to a function $\bar{L}$ satisfying (4.5).

Thus, in the mass-subcritical regime, by employing a concentration compactness argument, we prove that for $\alpha \in (0,2)$, there exists a rotating Riesz star solution corresponding to a given angular momentum L satisfying suitable sub-homogeneity conditions. Moreover, if such a solution is unique, we obtain strong convergence of minimising sequences to this solution. Combined with conservation of mass, this yields the nonlinear stability of the rotating Riesz star solution. In contrast, the lack of analogous existence and stability results in the range $\alpha \in [2,3)$ suggests that the rotational term has a destabilising effect in the more singular regime. Furthermore, in the mass-supercritical regime, by an approach inspired by §3.7 and §3.8, we establish the existence of rotating Riesz star solutions and their instability for a given L satisfying suitable super-homogeneity conditions. Notably, the conditions imposed on L to guarantee instability of the corresponding steady states become more restrictive as α decreases, contributing to the previous idea that the rotational term has a stronger destabilising effect for larger α, or equivalently, a stronger stabilising effect for smaller α. Finally, since our results do not impose any restrictions on the magnitude of the angular momentum, they appear to be new, to the best of the author's knowledge, even in the context of the more extensively studied CEPEs.

4.2 Preliminaries and results

As in §3.3, for the mass-subcritical regime, we will employ a concentration compactness argument to prove existence of rotating Riesz star solutions of the CEREs (1.1). As before, we require that both (3.4) and (3.5) are satisfied for $n = 3$ so that for

$$g^a_M := \inf_{\varrho \in X^a_M} \mathcal{G}^a(\varrho),$$

both $g_M^a > -\infty$ and $g_M^a < 0$ are guaranteed respectively. Furthermore, to guarantee that we cannot have dichotomy behaviour of the minimising sequences, as defined in Lemma 4.3.8, we require sub-homogeneity of order $\frac{5-\alpha}{3}$ for the angular momentum squared, that is

$$L(am) \geq a^{\frac{5-\alpha}{3}} L(m), \tag{4.6}$$

for all $0 < a < 1$ and $0 \leq m \leq M$. Given a global finite-energy weak solution $(\rho, \boldsymbol{\mathcal{M}})$ of the CEREs as in Definition 2.1.1 with mass $M > 0$, the momentum field can be written in the form

$$\boldsymbol{\mathcal{M}}(t, \boldsymbol{x}) = \mathcal{M}_r(t, \boldsymbol{x})\boldsymbol{e}_r + \mathcal{M}_\theta(t, \boldsymbol{x})\boldsymbol{e}_\theta + \mathcal{M}_3(t, \boldsymbol{x})\boldsymbol{e}_3.$$

Thus, given a rotating Riesz star solution $\bar{\rho} \in X_M^a$ with associated angular momentum squared L, as in Chapter 3, in order to discover the natural distance that should be used in the nonlinear stability result, we consider the relative energy. We obtain

$$E(\rho, \boldsymbol{\mathcal{M}} \,|\, \bar{\rho}, \bar{\boldsymbol{\mathcal{M}}})(t) = \mathcal{G}(\rho(t)) - \mathcal{G}^a(\bar{\rho}) + \frac{1}{2}\int_{\mathbb{R}^3} \left(\left| \frac{\boldsymbol{\mathcal{M}}^r}{\sqrt{\rho}} \right|^2 + \left| \frac{\boldsymbol{\mathcal{M}}^\theta}{\sqrt{\rho}} \right|^2 + \left| \frac{\boldsymbol{\mathcal{M}}^3}{\sqrt{\rho}} \right|^2 \right) \mathrm{d}\boldsymbol{x}. \tag{4.7}$$

In particular, to take advantage of the variational framework, we require the rotational velocity to be in the same format as the rotating Riesz star solution $(\bar{\rho}, \bar{\boldsymbol{\mathcal{M}}})$. Thus, we require that

$$u^\theta(t, \boldsymbol{x}) = \left(\frac{\mathcal{M}^\theta}{\rho} \right)(t, \boldsymbol{x}) = \frac{J(m_{\rho(t)}(r))}{r}, \tag{4.8}$$

with $J^2 = L$. In order for this to be true, we impose the following assumptions on the initial data:

(I_1) The mass of the solution is the same as the stationary state

$$\int_{\mathbb{R}^3} \rho_0 \,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^3} \bar{\rho} \,\mathrm{d}\boldsymbol{x} = M.$$

(I_2) The initial angular momentum $ru_0^\theta(\boldsymbol{x}) = ru_0^\theta(r, z)$ depends only on the mass in the cylinder $\{\boldsymbol{y} \in \mathbb{R}^3 \,|\, r(\boldsymbol{y}) \leq r(\boldsymbol{x})\}$, that is, there exists J such that

$$u_0^\theta(\boldsymbol{x}) = \left(\frac{\mathcal{M}_0^\theta}{\rho_0} \right)(\boldsymbol{x}) = \frac{J(m_{\rho_0}(r))}{r}.$$

(I_3) The angular momentum squared is L,

$$J^2(m) = L(m), \qquad 0 \leq m \leq M.$$

As in [107], we have the following assumptions on the weak solutions:

(A$_1$) For $D_t = \{\boldsymbol{x} \in \mathbb{R}^3 \,|\, \rho(t,\boldsymbol{x}) > 0\}$, there exists a measurable set $G_t \subset D_t$ such that $|D_t \setminus G_t| = 0$ and a unique backwards particle path $\xi(\cdot,\cdot;t) : [0,t] \times G_t \to \mathbb{R}^3$ such that

$$\partial_\tau \xi(\tau,\boldsymbol{x};t) = \boldsymbol{u}(\tau,\xi(\tau,\boldsymbol{x};t)), \qquad \xi(t,\boldsymbol{x};t) = \boldsymbol{x},$$

where $\boldsymbol{u} = \frac{\boldsymbol{\mathcal{M}}}{\rho}$. We define $\xi_{-t}(\boldsymbol{x}) \coloneqq \xi(0,\boldsymbol{x};t)$.

(A$_2$) Conservation of the angular momentum along the particle path $j(t,\boldsymbol{x}) = j_0(\xi_{-t}(\boldsymbol{x}))$ for all $\boldsymbol{x} \in G_t$.

(A$_3$) Conservation of the mass along the particle path $m_{\rho(t)}(r(\boldsymbol{x})) = m_{\rho_0}(r(\xi_{-t}(\boldsymbol{x})))$ for all $\boldsymbol{x} \in G_t$.

Thus, as in [107], under the assumptions (I$_1$)–(I$_3$) and (A$_1$)–(A$_3$) we have that (4.8) is satisfied. Condition (A$_1$) is a reasonable assumption with (A$_2$) and (A$_3$) shown to be true in [129] for regular enough solutions. Thus, under these assumptions, (4.7) becomes

$$E(\rho,\boldsymbol{\mathcal{M}} \,|\, \bar{\rho},\bar{\boldsymbol{\mathcal{M}}})(t) = \mathcal{G}^a(\rho(t)) - \mathcal{G}^a(\bar{\rho}) + \frac{1}{2}\int_{\mathbb{R}^3} \left(\left| \frac{\boldsymbol{\mathcal{M}}^r}{\sqrt{\rho}} \right|^2 + \left| \frac{\boldsymbol{\mathcal{M}}^3}{\sqrt{\rho}} \right|^2 \right) \mathrm{d}\boldsymbol{x}.$$

Thus, considering the difference of the free-energy, we have that

$$\mathcal{G}^a(\rho(t)) - \mathcal{G}^a(\bar{\rho}) = d_1^a(\rho(t),\bar{\rho}) + d_2^a(\rho(t),\bar{\rho}) + \frac{1}{2}\int_{\mathbb{R}^3} (\rho(t) - \bar{\rho})\, \Phi_\alpha * (\rho(t) - \bar{\rho})\, \mathrm{d}\boldsymbol{x}, \tag{4.9}$$

where the relative free-energy functionals $d_1^a(\varrho,\bar{\rho})$ and $d_2^a(\varrho,\bar{\rho})$ for $\varrho,\bar{\rho} \in X_M^a$, are given by

$$\begin{cases} d_1^a(\varrho,\bar{\rho}) \coloneqq \displaystyle\int_{\mathbb{R}^3} \left((\varrho e(\varrho) - \bar{\rho} e(\bar{\rho})) + (\varrho - \bar{\rho}) \left(\Phi_\alpha * \bar{\rho} + \int_{r(\boldsymbol{x})}^\infty \frac{L(m_{\bar{\rho}}(s))}{s^3}\, \mathrm{d}s \right) \right) \mathrm{d}\boldsymbol{x}, \\ d_2^a(\varrho,\bar{\rho}) \coloneqq \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^3} \frac{\varrho L(m_\varrho(r(\boldsymbol{x}))) - \bar{\rho} L(m_{\bar{\rho}}(r(\boldsymbol{x})))}{r(\boldsymbol{x})^2}\, \mathrm{d}\boldsymbol{x} \\ \qquad\qquad - \displaystyle\int_{\mathbb{R}^3} (\varrho - \bar{\rho}) \int_{r(\boldsymbol{x})}^\infty \frac{L(m_{\bar{\rho}}(s))}{s^3}\, \mathrm{d}s\, \mathrm{d}\boldsymbol{x}. \end{cases}$$

In the case that $\bar{\rho}$ is a rotating Riesz star solution of (1.1), by (4.4), there exists $\mu > 0$ such that

$$\begin{aligned} d_1^a(\varrho,\bar{\rho}) &= \int_{\mathbb{R}^3} \left((\varrho e(\varrho) - \bar{\rho} e(\bar{\rho})) + (\varrho - \bar{\rho}) \left(\Phi_\alpha * \bar{\rho} + \int_{r(\boldsymbol{x})}^\infty \frac{L(m_{\bar{\rho}}(s))}{s^3}\, \mathrm{d}s + \mu \right) \right) \mathrm{d}\boldsymbol{x} \\ &\geq \int_{\mathbb{R}^3} \left((\varrho e(\varrho) - \bar{\rho} e(\bar{\rho})) + (\varrho - \bar{\rho})(\bar{\rho} e(\bar{\rho}))_{\bar{\rho}} \right) \mathrm{d}\boldsymbol{x}. \end{aligned}$$

Applying (1.4), we see that $\varrho \mapsto \varrho e(\varrho)$ is convex in ϱ, therefore $d_1^a(\varrho,\bar{\rho}) \geq 0$. On the other hand, in order to have positivity of $d_2^a(\rho,\bar{\rho})$, we require the technical assumption

(A$_4$) $\lim_{r\to 0^+} L(m_\rho(r)+m_{\bar\rho}(r))m_{\rho-\bar\rho}(r)r^{-2}=0.$

Lemma 4.2.1 ([107, Lemma 3.8]). *Suppose L satisfies (4.3) and*

$$L'(m)\geq 0, \tag{4.10}$$

for all $0\leq m\leq M$, then for any $\varrho\in X_M^a$ satisfying

$$\lim_{r\to 0^+} L(m_\varrho(r)+m_{\bar\rho}(r))m_{\varrho-\bar\rho}(r)r^{-2}=0,$$

we have that

$$d_2^a(\varrho,\bar\rho)\geq 0.$$

As in Chapter 3, we have that the term on the right-hand side of (4.9) can be controlled by Hölder's inequality and the HLS inequality from Lemma A.0.2 to obtain

$$\left|\int_{\mathbb{R}^3}(\rho(t)-\bar\rho)\,\Phi_\alpha * (\rho(t)-\bar\rho)\,\mathrm{d}\boldsymbol{x}\right|\leq C\|\rho(t)-\bar\rho\|^2_{L^{\frac{6}{6-\alpha}}}.$$

This leads us to the following stability theorem.

Theorem 4.2.2 (Nonlinear Stability of Rotating Riesz Star Solutions). *Suppose $\bar\rho$ is a unique minimiser (up to translation in the z-axis) of the functional $\mathcal{G}^a$ in X_M^a. Suppose $\alpha\in(0,2)$ with $p\in C^1([0,\infty))$ satisfying (1.4), (3.4)–(3.5) for $n=3$ and L satisfies (4.3), (4.6) and (4.10). Let $(\rho,\boldsymbol{\mathcal{M}})$ be a global finite-energy weak solution of the CEREs in the sense of* Definition 2.1.1 *satisfying* (I$_1$)–(I$_3$) *and* (A$_1$)–(A$_4$) *with the square of the angular momentum given by $L(m_\rho(r(\boldsymbol{x})))$. Then, for any $\varepsilon>0$, there exists $\delta>0$, such that if*

$$d_1^a(\rho_0,\bar\rho)+d_2^a(\rho_0,\bar\rho)+\|\rho_0-\bar\rho\|^2_{L^{\frac{6}{6-\alpha}}}+\frac{1}{2}\int_{\mathbb{R}^3}\left(\left|\frac{\boldsymbol{\mathcal{M}}_0^r}{\sqrt{\rho_0}}\right|^2+\left|\frac{\boldsymbol{\mathcal{M}}_0^3}{\sqrt{\rho_0}}\right|^2\right)\mathrm{d}\boldsymbol{x}<\delta,$$

there exists a translation function $\boldsymbol{y}\in\{\mathbf{0}\}\times\mathbb{R}$, such that

$$d_1^a(\rho(t),T^{\boldsymbol{y}}\bar\rho)+d_2^a(\rho(t),T^{\boldsymbol{y}}\bar\rho)+\|\rho(t)-T^{\boldsymbol{y}}\bar\rho\|^2_{L^{\frac{6}{6-\alpha}}}+\frac{1}{2}\int_{\mathbb{R}^3}\left(\left|\frac{\boldsymbol{\mathcal{M}}^r}{\sqrt{\rho}}\right|^2+\left|\frac{\boldsymbol{\mathcal{M}}^3}{\sqrt{\rho}}\right|^2\right)(t)\,\mathrm{d}\boldsymbol{x}<\varepsilon,$$

for almost every $t>0$.

In the polytropic setting, similar to the non-rotating case (3.15), if $L(m)=m^{\omega^*}$, where

$$\omega_*:=\frac{7-\alpha-(5-\alpha)\gamma}{3+\alpha-3\gamma}, \tag{4.11}$$

with $\bar\rho_1$ a rotating Riesz star solution, as in Definition 4.1.1 for $\mu=1$, then $\bar\rho_\mu=\mu^{\frac{1}{(\gamma-1)(3-\alpha)}}\bar\rho_1$ satisfies Definition 4.1.1 for any $\mu>0$. Thus, there exists a scaling

between rotating Riesz star solutions, this highlights the criticality of ω_*. However, in the mass-supercritical regime, for $\alpha \in (0,2]$ and L satisfying (4.5), in order to prove existence of minimisers of S_μ^a over $\mathcal{K}^a$, we will require super-homogeneity of order $\omega > \omega_*$ for the angular momentum squared, that is

$$L(am) \leq a^\omega L(m), \tag{4.12}$$

for all $0 < a < 1$ and $m \geq 0$. Such an assumption is required to prove that not only is the admissible set $\mathcal{K}^a$ non-empty, but with the further restrictions that $\omega > \omega_* + \bar{\omega}$, that minimisers of the variational problem, indeed correspond to rotating Riesz star solutions. Here,

$$\bar{\omega} = \frac{(2-\alpha)((6-\alpha)\gamma - 6)(5-3\gamma)}{3\alpha(\gamma-1)(3+\alpha-3\gamma)} \geq 0.$$

Remark 4.2.3. The addition of $\bar{\omega}$ represents the stabilising property of the addition of rotation to steady states. In particular, as γ approaches the mass-critical exponent $\frac{3+\alpha}{3}$, we require stronger conditions on the super-homogeneity of L. Such stabilising properties are only apparent for $\alpha \in (0,2)$, as we observe that as α approaches 2, the conditions on the super-homogeneity of L are weaker. In particular, when $\alpha \to 2$, $\bar{\omega} \to 0$.

In order to prove existence of minimisers of S_μ^a over $\mathcal{K}^a$, we also require the weak lower semi-continuity of the rotational kinetic term

$$\varrho \mapsto \int_{\mathbb{R}^3} \frac{\varrho L(m_\varrho(r))}{r^2}\, \mathrm{d}\boldsymbol{x}.$$

In order to guarantee this, we require the condition that

$$L'(m) \geq 0, \tag{4.13}$$

for all $m \geq 0$. Thus, we have the following theorem.

Theorem 4.2.4. *Suppose $\alpha \in (0,3)$, $\frac{6}{6-\alpha} < \gamma < \frac{3+\alpha}{3}$, L satisfies (4.5), (4.13) and, for $\alpha \in (0,2]$, (4.12) for $\omega > \omega_* + \bar{\omega}$. If $\bar{\rho}$ is a minimiser of S_μ^a over $\mathcal{K}^a$, then $\bar{\rho}$ is a rotating Riesz star solution as given by Definition 4.1.1 with compact support.*

Moreover, as in Chapter 3, we will prove under an extra condition that such rotating Riesz star solutions are unstable.

4.3 Existence of rotating Riesz star solutions in the mass-subcritical regime

In this section, we consider the existence of rotating Riesz star solutions of the CEREs (1.1) for a general pressure, with sufficiently nice asymptotic behaviour at the origin and infinity, which are inclusive of the mass-subcritical regime $\gamma > \frac{3+\alpha}{3}$ in the polytropic case. We recall the definition for the admissible set of functions

$$X_M^a := \left\{ \varrho \in L^1_+(\mathbb{R}^3) \cap L^{\frac{3+\alpha}{3}}_{\text{axi}}(\mathbb{R}^3) \,\middle|\, \int_{\mathbb{R}^3} \varrho \, \mathrm{d}\boldsymbol{x} = M, \int_{\mathbb{R}^3} \frac{\varrho(\boldsymbol{x}) L(m_\varrho(r(\boldsymbol{x})))}{r(\boldsymbol{x})^2} \, \mathrm{d}\boldsymbol{x} < \infty \right\}.$$

For $\varrho \in X_M^a$, we recall the definition of the free-energy functional

$$\mathcal{G}^a(\varrho) = \int_{\mathbb{R}^3} \varrho(\boldsymbol{x}) \Big(e(\varrho(\boldsymbol{x})) + \frac{1}{2} \frac{L(m_\varrho(r(\boldsymbol{x})))}{r(\boldsymbol{x})^2} + \frac{1}{2} (\Phi_\alpha * \varrho)(\boldsymbol{x}) \Big) \, \mathrm{d}\boldsymbol{x}.$$

We substantially adapt the concentration compactness argument of [103] to prove existence of minimisers to the problem:

$$g_M^a = \inf_{\varrho \in X_M^a} \mathcal{G}^a(\varrho).$$

We first prove some facts about minimisers of general free-energies. To do this, we consider a general non-local interaction kernel of the form

$$\Psi \in L^p_{\text{w}}(\mathbb{R}^3) \, (\text{weak } L^p \text{ space}), \text{ for } p \in (1, \infty), \text{ with } \Psi \geq 0 \text{ } a.e., \tag{4.14}$$

such that there exists $\beta \in (0, 3)$ satisfying

$$\Psi(t\xi) \geq t^{-\beta} \Psi(\xi), \tag{4.15}$$

for all $t \geq 1$ and $a.e.$ $\xi \in \mathbb{R}^3$. We take a convex function Π satisfying (3.19), that is

$$\begin{cases} \Pi : [0, \infty) \to [0, \infty) \text{ strictly convex}, \\ \lim_{t \to 0^+} \Pi(t) t^{-1} = 0, \\ \liminf_{t \to \infty} \Pi(t) t^{-q} > 0, \end{cases}$$

where $q = \frac{p+1}{p}$. We consider a function I such that

$$\begin{cases} I \in C^1([0, M]), \\ I(0) = 0, \\ I'(m) \geq 0 \text{ for all } m \in [0, M], \end{cases} \tag{4.16}$$

we also require sub-homogeneity of order $\frac{5-\beta}{3}$, that is

$$I(am) \geq a^{\frac{5-\beta}{3}} I(m), \tag{4.17}$$

for all $0 < a < 1$ and $0 \leq m \leq M$. We define the associated free-energy

$$\mathcal{F}^a(\varrho) = \int_{\mathbb{R}^3} \left(\Pi(\varrho(\boldsymbol{x})) + \frac{\varrho(\boldsymbol{x}) I(m_\varrho(r(\boldsymbol{x})))}{r(\boldsymbol{x})^2} - \frac{1}{2}\varrho(\boldsymbol{x})(\Psi * \varrho)(\boldsymbol{x}) \right) \mathrm{d}\boldsymbol{x},$$

with the corresponding admissible set

$$\mathcal{A}_M^a := \left\{ \varrho \in L^1_+(\mathbb{R}^3) \cap L^q_{\mathrm{axi}}(\mathbb{R}^3) \,\middle|\, \int_{\mathbb{R}^3} \varrho \,\mathrm{d}\boldsymbol{x} = M, \int_{\mathbb{R}^3} \frac{\varrho(\boldsymbol{x}) I(m_\varrho(r(\boldsymbol{x})))}{r(\boldsymbol{x})^2} \,\mathrm{d}\boldsymbol{x} < \infty \right\}.$$

We define the minimum of $\mathcal{F}^a$ over $\mathcal{A}_M^a$ by

$$F_M^a := \inf_{\varrho \in \mathcal{A}_M^a} \mathcal{F}^a(\varrho).$$

As in §3.3, we require the conditions (3.20) and (3.21), which are analogous to (3.4) and (3.5), in order to prove useful facts about the sign and bounds of F_M^a.

Theorem 4.3.1. *Let $p \in (1, \infty)$, with $q = \frac{p+1}{p}$. Suppose $F_M^a < 0$, (3.20) is satisfied, Ψ satisfies (4.14)–(4.15), Π satisfies (3.19) and I satisfies (4.16)–(4.17). Then for any minimising sequence $\{\varrho_k\}_k \subset \mathcal{A}_M^a$, there exists $\boldsymbol{y}_k \in \{\mathbf{0}\} \times \mathbb{R}$ and a minimiser $\bar{\rho} \in \mathcal{A}_M^a$ to $\mathcal{F}^a$ (*i.e.* $\mathcal{F}^a(\bar{\rho}) = F_M^a$) with $T^{\boldsymbol{y}_k}\varrho_k$ compact in $L^1(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$, with (up to a subsequence) $T^{\boldsymbol{y}_k}\varrho_k \to \bar{\rho}$ in $L^1(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$.*

As in §3.3, we now characterise how the conditions (3.20) and (3.21) on the mass affect the sign and finiteness of the minimised energy.

Lemma 4.3.2. *Let $p \in (1, \infty)$, with $q = \frac{p+1}{p}$. Suppose that Ψ satisfies (4.14), Π satisfies (3.19) and I satisfies (4.16).*

(a) *Suppose (3.20) is satisfied, then we have that $F_M^a > -\infty$.*

(b) *Suppose that $p > \frac{3}{2}$, Ψ satisfies (4.15) for $\beta \leq \frac{3}{p}$ and is axisymmetric with $r \mapsto \Psi(r, z)$ a radially decreasing function for a.e. z. If $\beta = \frac{3}{p}$, suppose (3.21) is satisfied, then we have that $F_M^a < 0$.*

Proof. (a) Follows the same as the proof of (a) in Lemma 3.3.1.

(b) There exists $\theta \in (0, 1)$ such that

$$\delta_{\beta, \frac{3}{p}} \limsup_{t \to 0} \Pi(t) t^{-q} < \frac{\theta C_*^\Psi}{2} M^{2-q},$$

where

$$\delta_{\beta, \frac{3}{p}} = \begin{cases} 1, & \beta = \frac{3}{p}, \\ 0, & \beta < \frac{3}{p}. \end{cases}$$

By Lemma A.0.4 and the sharpness of $C_*^\Psi > 0$, there exists $\varrho \in L^1_+(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \varrho \Psi * \varrho \, \mathrm{d}\boldsymbol{x} > \theta C_*^\Psi \|\varrho\|_{L^q}^q \|\varrho\|_{L^1}^{2-q}. \tag{4.18}$$

We take the radial decreasing rearrangement $\varrho^\#$ in the first two variables (x_1, x_2) of ϱ and, since Ψ is axisymmetric with $r \mapsto \Psi(r, z)$ a radially decreasing function for *a.e.* z, we obtain that the radial decreasing rearrangement $\Psi^\#$ in the first two variables (x_1, x_2) of Ψ satisfies $\Psi^\# = \Psi$. Thus, applying Lemma A.0.5, we obtain that $\varrho^\#$ satisfies (4.18) and is axisymmetric. Moreover, taking $\varepsilon > 0$, we can then define $\varrho_\varepsilon^* = \varrho^\# \mathbb{1}_{B_\varepsilon^c \times \mathbb{R}}$, where B_ε represents the open ball of radius ε centred at $\mathbf{0} \in \mathbb{R}^2$, that satisfies

$$\int_{\mathbb{R}^3} \frac{\varrho_\varepsilon^*(\boldsymbol{x}) I(m_{\varrho_\varepsilon^*}(r(\boldsymbol{x})))}{r(\boldsymbol{x})^2} \, \mathrm{d}\boldsymbol{x} < \infty.$$

Taking $\varepsilon > 0$ sufficiently small, we have that ϱ_ε^* satisfies (4.18). So without loss of generality, we may assume that ϱ is axisymmetric and satisfies

$$\int_{\mathbb{R}^3} \frac{\varrho(\boldsymbol{x}) I(m_\varrho(r(\boldsymbol{x})))}{r(\boldsymbol{x})^2} \, \mathrm{d}\boldsymbol{x} < \infty.$$

By density of $C_\mathrm{c}(\mathbb{R}^3)$ in $L^1(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$, we may assume that $\varrho \in C_\mathrm{c}(\mathbb{R}^3)$ and thus is a bounded function. Moreover, by employing the rescaling

$$\tilde{\varrho}(\boldsymbol{x}) := \varrho\bigg(\boldsymbol{x}\bigg(\frac{\|\varrho\|_{L^1}}{M}\bigg)^{\frac{1}{3}}\bigg),$$

we have that $\tilde{\varrho} \in \mathcal{A}_M^a$. Now, employing the mass-preserving scaling given in (3.3) for $n = 3$, we have that $\tilde{\varrho}_\lambda \in \mathcal{A}_M^a$ for all $\lambda > 0$. Considering the kinetic energy term, by integration by parts, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \frac{\tilde{\varrho}_\lambda I(m_{\tilde{\varrho}_\lambda}(r(\boldsymbol{x})))}{r(\boldsymbol{x})^2} \, \mathrm{d}\boldsymbol{x} \\
&= \int_{\mathbb{R}^3} \frac{\tilde{\varrho}_\lambda I\Big(M\|\varrho\|_{L^1(\mathbb{R}^3)}^{-1} m_\varrho\big(r\big(\lambda^{\frac{1}{3}}\|\varrho\|_{L^1(\mathbb{R}^3)}^{\frac{1}{3}} M^{-\frac{1}{3}}\boldsymbol{x}\big)\big)\Big)}{r(\boldsymbol{x})^2} \, \mathrm{d}\boldsymbol{x} \\
&= \lambda^{\frac{2}{3}}\bigg(\frac{M}{\|\varrho\|_{L^1(\mathbb{R}^3)}}\bigg)^{\frac{1}{3}} \int_{\mathbb{R}^3} \frac{\varrho I\big(M\|\varrho\|_{L^1}^{-1} m_\varrho(r(\boldsymbol{x}))\big)}{r(\boldsymbol{x})^2} \, \mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Since Ψ satisfies (4.15) for $\beta \leq \frac{3}{p}$, we combine this with (4.18) to see for $\lambda < \frac{M}{\|\varrho\|_{L^1}}$ that

$$\begin{aligned}
\int_{\mathbb{R}^3} \tilde{\varrho}_\lambda \Psi * \tilde{\varrho}_\lambda \, \mathrm{d}\boldsymbol{x} &= \bigg(\frac{M}{\|\varrho\|_{L^1}}\bigg)^2 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \varrho(\boldsymbol{x})\varrho(\boldsymbol{y}) \Psi\bigg(\bigg(\frac{M}{\lambda\|\varrho\|_{L^1}}\bigg)^{\frac{1}{3}}(\boldsymbol{x}-\boldsymbol{y})\bigg) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{y} \\
&\geq \lambda^{\frac{\beta}{3}}\bigg(\frac{M}{\|\varrho\|_{L^1}}\bigg)^{2-\frac{\beta}{3}} \int_{\mathbb{R}^3} \varrho\Psi * \varrho \, \mathrm{d}\boldsymbol{x}
\end{aligned}$$

$$\geq \theta C_*^\Psi M^{2-\frac{\beta}{3}} \lambda^{\frac{\beta}{3}} \|\varrho\|_{L^q}^q \|\varrho\|_{L^1}^{\frac{\beta}{3}-q}.$$

In order for the internal energy and potential energy terms to dominate as $\lambda \to 0$, we require that $\beta < 2$. Since $p > \frac{3}{2}$, we have that $\beta \leq \frac{3}{p} < 2$. Thus, taking $\lambda < \frac{M}{\|\varrho\|_{L^1}}$, we obtain

$$\begin{aligned}
\mathcal{F}^a(\tilde{\varrho}_\lambda)\lambda^{-\frac{\beta}{3}}\left(\frac{M}{\|\varrho\|_{L^1}}\right)^{\frac{\beta-3q}{3}} &\leq \left(\frac{M}{\lambda\|\varrho\|_{L^1}}\right)^{\frac{\beta}{3}-\frac{1}{p}} \int_{\mathbb{R}^3} \frac{\Pi(\lambda\varrho)}{(\lambda\varrho)^q}\varrho^q \,\mathrm{d}\boldsymbol{x} - \frac{\theta C_*^\Psi}{2}\|\varrho\|_{L^q}^q M^{2-q} \\
&\quad + \frac{\lambda^{\frac{2-\beta}{3}}}{2}\left(\frac{M}{\|\varrho\|_{L^1}}\right)^{\frac{\beta-3q+1}{3}} \int_{\mathbb{R}^3} \frac{\varrho I\big(M\|\varrho\|_{L^1}^{-1} m_\varrho(r(\boldsymbol{x}))\big)}{r(\boldsymbol{x})^2}\,\mathrm{d}\boldsymbol{x} \\
&= \int_{\{\varrho>0\}} \left(\left(\frac{M}{\lambda\|\varrho\|_{L^1}}\right)^{\frac{\beta}{3}-\frac{1}{p}} \frac{\Pi(\lambda\varrho)}{(\lambda\varrho)^q} - \frac{\theta C_*^\Psi}{2}M^{2-q}\right)\varrho^q \,\mathrm{d}\boldsymbol{x} \\
&\quad + \frac{\lambda^{\frac{2-\beta}{3}}}{2}\left(\frac{M}{\|\varrho\|_{L^1}}\right)^{\frac{\beta-3q+1}{3}} \int_{\mathbb{R}^3} \frac{\varrho I\big(M\|\varrho\|_{L^1}^{-1} m_\varrho(r(\boldsymbol{x}))\big)}{r(\boldsymbol{x})^2}\,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Since ϱ is a bounded function and

$$\limsup_{\lambda\to 0} \lambda^{\frac{1}{p}-\frac{\beta}{3}} \frac{\Pi(\lambda\varrho)}{(\lambda\varrho)^q} \leq \frac{C_*^\Psi}{2} M^{2-q},$$

for $\varrho > 0$, the term $\lambda^{\frac{1}{p}-\frac{\beta}{3}} \frac{\Pi(\lambda\varrho)}{(\lambda\varrho)^q}$ is bounded for $\lambda < \frac{M}{\|\varrho\|_{L^1}}$. Hence, there exists $C > 0$ such that $\lambda^{\frac{1}{p}-\frac{\beta}{3}} \frac{\Pi(\lambda\varrho)}{(\lambda\varrho)^q} \leq C$ for $\lambda < \frac{M}{\|\varrho\|_{L^1}}$, and so we can apply the Reverse Fatou Lemma to obtain that

$$\limsup_{\lambda\to 0} \mathcal{F}^a(\tilde{\varrho}_\lambda)\lambda^{-\frac{\beta}{3}}\left(\frac{M}{\|\varrho\|_{L^1}}\right)^{\frac{\beta-3q}{3}} \leq \|\varrho\|_{L^q}^q \left(\delta_{\beta,\frac{3}{p}} \limsup_{t\to 0} \Pi(t)t^{-q} - \frac{\theta C_*^\Psi}{2}M^{2-q}\right) < 0.$$

Hence, we have that $F_M^a < 0$.

□

The proof of Theorem 4.3.1 is inspired by [103, Theorem II.1], although the introduction of the rotational kinetic energy adds difficulty that requires new analysis. We first show that minimising sequences are indeed bounded in L^q.

Lemma 4.3.3. *Under the assumptions of Theorem 4.3.1, let $m \in (0, M]$ and $\varrho_k \in \mathcal{A}_m^a$ be a minimising sequence (i.e. $\lim_{k\to\infty} \mathcal{F}^a(\varrho_k) = F_m^a$). Then, $\|\varrho_k\|_{L^q(\mathbb{R}^3)}$ and $\int_{\mathbb{R}^3} \varrho_k \Psi * \varrho_k \,\mathrm{d}\boldsymbol{x}$ are bounded.*

Proof. By (3.20), there exists $\delta > 0$ and $\varrho^* > 0$ such that

$$\Pi(\varrho)\varrho^{-q} > \frac{C_*^\Psi M^{2-q}}{2} + \delta,$$

for all $\varrho \geq \varrho^*$. By Lemma A.0.4, we have that

$$\begin{aligned}
\mathcal{F}^a(\varrho_k) &\geq \int_{\{\varrho_k \geq \varrho^*\}} \Pi(\varrho_k)\,\mathrm{d}\boldsymbol{x} - \frac{1}{2}\int_{\mathbb{R}^3} \varrho_k \Psi * \varrho_k\,\mathrm{d}\boldsymbol{x} \\
&\geq \int_{\{\varrho_k \geq \varrho^*\}} \Pi(\varrho_k)\,\mathrm{d}\boldsymbol{x} - \frac{C_*^\Psi m^{2-q}}{2}\int_{\mathbb{R}^3} \varrho_k^q\,\mathrm{d}\boldsymbol{x} \\
&= \int_{\{\varrho_k \geq \varrho^*\}} \varrho_k^q\Big(\Pi(\varrho_k(\boldsymbol{x}))\varrho_k^{-q} - \frac{C_*^\Psi m^{2-q}}{2}\Big)\,\mathrm{d}\boldsymbol{x} - \frac{C_*^\Psi m^{2-q}}{2}\int_{\{\varrho_k < \varrho^*\}} \varrho_k^q\,\mathrm{d}\boldsymbol{x} \\
&\geq \delta\|\varrho_k\|_{L^q(\mathbb{R}^3)}^q - \frac{C_*^\Psi m^{3-q}\varrho_*^{q-1}}{2}.
\end{aligned}$$

Hence, we have that

$$\|\varrho_k\|_{L^q(\mathbb{R}^3)}^q \leq C\big(m^{3-q} + \mathcal{F}^a(\varrho_k)\big), \tag{4.19}$$

where $C > 0$ only depends on Ψ, ϱ^*, Π. Equally, since by Lemma A.0.4,

$$\int_{\mathbb{R}^3} \varrho\Psi * \varrho\,\mathrm{d}\boldsymbol{x} \leq C_*^\Psi \|\varrho\|_{L^q(\mathbb{R}^3)}^q \|\varrho\|_{L^1(\mathbb{R}^3)}^{2-q},$$

we have that

$$\int_{\mathbb{R}^3} \varrho_k\Psi * \varrho_k\,\mathrm{d}\boldsymbol{x} \leq Cm^{2-q}\big(m^{3-q} + \mathcal{F}^a(\varrho_k)\big). \tag{4.20}$$

The result follows. □

In order to rule out behaviour such as dichotomy of minimising sequences, as detailed in Lemma 4.3.8, we will require the continuity of the minimised energy and that

$$F_M^a < F_m^a + F_{M-m}^a,$$

for all $m \in (0, M)$. In order to do this, we use the following lemma that compares the minimised energy for different masses.

Lemma 4.3.4. *Under the assumptions of Theorem 4.3.1, we have that* $F_l^a \geq \left(\frac{l}{m}\right)^{\frac{6-\beta}{3}} F_m^a$ *for all* $0 < l \leq m \leq M$.

Proof. The proof follows a similar argument to that of [103] and [107]. Let $\varrho_m \in \mathcal{A}_m^a$, we take $b = \left(\frac{m}{l}\right)^{\frac{1}{3}}$ and define $\varrho_l = \varrho_m(b\,\cdot) \in \mathcal{A}_l^a$. Then by integration by parts and (4.17), we have that

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{\varrho_l I(m_{\varrho_l}(r(\boldsymbol{x})))}{r(\boldsymbol{x})^2}\,\mathrm{d}\boldsymbol{x} &= \left(\frac{l}{m}\right)^{\frac{1}{3}} \int_{\mathbb{R}^3} \frac{\varrho_m I\left(\frac{l}{m} m_{\varrho_m}(r(\boldsymbol{x}))\right)}{r(\boldsymbol{x})^2}\,\mathrm{d}\boldsymbol{x} \\
&\geq \left(\frac{l}{m}\right)^{\frac{6-\beta}{3}} \int_{\mathbb{R}^3} \frac{\varrho_m I(m_{\varrho_m}(r(\boldsymbol{x})))}{r(\boldsymbol{x})^2}\,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Employing (4.15), we have that

$$\begin{aligned}
\int_{\mathbb{R}^3} \varrho_l(\boldsymbol{x})\varrho_l(\boldsymbol{y})\Psi(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} &= \left(\frac{l}{m}\right)^2 \int_{\mathbb{R}^3} \varrho_m(\boldsymbol{x})\varrho_m(\boldsymbol{y})\Psi\Big(\Big(\frac{l}{m}\Big)^{\frac{1}{3}}(\boldsymbol{x}-\boldsymbol{y})\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \\
&\leq \left(\frac{l}{m}\right)^{\frac{6-\beta}{3}} \int_{\mathbb{R}^3} \varrho_l(\boldsymbol{x})\varrho_l(\boldsymbol{y})\Psi(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}.
\end{aligned}$$

Thus, combining, we see that

$$\begin{aligned}\mathcal{F}^a(\varrho_l) &\geq \frac{l}{m}\int_{\mathbb{R}^3}\Pi(\varrho_m)\,\mathrm{d}\boldsymbol{x} + \left(\frac{l}{m}\right)^{\frac{6-\beta}{3}}\int_{\mathbb{R}^3}\frac{\varrho_m I(m_{\varrho_m}(r(\boldsymbol{x})))}{r(\boldsymbol{x})^2}\,\mathrm{d}\boldsymbol{x}\\ &\quad - \left(\frac{l}{m}\right)^{\frac{6-\beta}{3}}\int_{\mathbb{R}^3}\varrho_l(\boldsymbol{x})\varrho_l(\boldsymbol{y})\Psi(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\\ &\geq \left(\frac{l}{m}\right)^{\frac{6-\beta}{3}}\mathcal{F}^a(\varrho_m).\end{aligned}$$

Since $\varrho_m \mapsto \varrho_l$ is a one-to-one mapping between $\mathcal{A}^a_m$ and $\mathcal{A}^a_l$, the result follows. □

Thus, we can now prove that the minimised energy satisfies the desired inequality.

Lemma 4.3.5. *Under the assumptions of Theorem 4.3.1,*

$$F^a_M < F^a_m + F^a_{M-m}, \tag{4.21}$$

for all $m \in (0, M)$.

Proof. Suppose that $m \in (0, M)$ and let $\varrho \in \mathcal{A}^a_m$, such that $\mathcal{F}^a(\varrho) < \infty$, $\varrho \in C_{\mathrm{c}}(\mathbb{R}^3)$ and thus a bounded function. Using $\varrho_\lambda = \lambda\varrho(\lambda^{\frac{1}{3}}\cdot) \in \mathcal{A}^a_m$ as defined in §3.7, by $(3.19)_2$ ($\lim_{t\to 0^+}\Pi(t)t^{-1} = 0$) and the dominated convergence Theorem, we obtain

$$\begin{aligned}\int_{\mathbb{R}^3}\Pi(\varrho_\lambda)\,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^3}\frac{\varrho_\lambda I(m_{\varrho_\lambda}(r(\boldsymbol{x})))}{r(\boldsymbol{x})^2}\,\mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}^3}\frac{\Pi(\lambda\varrho)}{\lambda}\,\mathrm{d}\boldsymbol{x} + \lambda^{\frac{2}{3}}\int_{\mathbb{R}^3}\frac{\varrho I(m_\varrho(r(\boldsymbol{x})))}{r(\boldsymbol{x})^2}\,\mathrm{d}\boldsymbol{x}\\ &\longrightarrow 0, \qquad \text{as } \lambda \to 0.\end{aligned}$$

Hence, we have that $F^a_m \leq 0$. Suppose that (4.21) does not hold for some $m \in (0, M)$, that is

$$0 > F^a_M \geq F^a_m + F^a_{M-m}.$$

Then without loss of generality, we have that $F^a_m < 0$. Thus, by Lemma 4.3.4 and the fact that $\beta \in (0, 3)$, we have that

$$\frac{m}{M}F^a_M \leq \left(\frac{M}{m}\right)^{\frac{\beta-3}{3}}F^a_m < F^a_m.$$

Equally, since $F^a_{M-m} \leq 0$, we get

$$\frac{M-m}{M}F^a_M \leq F^a_{M-m}.$$

Combining, we see that

$$F^a_M = \frac{m}{M}F^a_M + \frac{M-m}{M}F^a_M < F^a_m + F^a_{M-m}.$$

Thus, (4.21) must hold for all $m \in (0, M)$. □

The following lemma proves the continuity of $m \mapsto F_m^a$.

Lemma 4.3.6. *Under the assumptions of Theorem 4.3.1, we have that $m \mapsto F_m^a$ is continuous for all $m \in (0, M]$.*

Proof. We split the proof into two steps.

Step 1. We first prove that $\limsup_{l\to m}(F_l^a - F_m^a) \le 0$.

Case 1. Suppose that $l > m$, applying Lemma 4.3.4, we have that

$$\limsup_{l\to m^+}(F_l^a - F_m^a) \le \limsup_{l\to m^+}\left(\left(\frac{l}{m}\right)^{\frac{6-\beta}{3}} - 1\right)F_m^a = 0.$$

Case 2. Suppose that $l < m$, we have that for all $\varepsilon > 0$, there exists $\varrho_m \in \mathcal{A}_m^a$ such that $F_m^a \ge \mathcal{F}^a(\varrho_m) - \varepsilon$. Then, we see that $F_l^a - F_m^a \le \mathcal{F}^a\left(\frac{l\varrho_m}{m}\right) - \mathcal{F}^a(\varrho_m) + \varepsilon$. Since $I'(m) \ge 0$ for all $m \in [0, M]$ and $\Pi : [0,\infty) \to [0,\infty)$ is strictly convex with $\Pi(0) = 0$, Π must be increasing, we have that

$$\mathcal{F}^a\left(\frac{l\varrho_m}{m}\right) - \mathcal{F}^a(\varrho_m) \le \left(1 - \left(\frac{l}{m}\right)^2\right)\int_{\mathbb{R}^3} \varrho_m \Psi * \varrho_m \,\mathrm{d}\boldsymbol{x}.$$

Thus,

$$\limsup_{l\to m^-}(F_l^a - F_m^a) \le \varepsilon + \limsup_{l\to m^-}\left(1 - \left(\frac{l}{m}\right)^2\right)\int_{\mathbb{R}^3} \varrho_m \Psi * \varrho_m \,\mathrm{d}\boldsymbol{x} = \varepsilon.$$

Hence, since $\varepsilon > 0$ is arbitrary, we see that $\limsup_{l\to m^-}(F_l^a - F_m^a) \le 0$.

Step 2. We now prove that $\liminf_{l\to m}(F_l^a - F_m^a) \ge 0$.

Case 1. Suppose that $l < m$, applying Lemma 4.3.4, we have that

$$\liminf_{l\to m^-}(F_l^a - F_m^a) \ge \liminf_{l\to m^-}\left(\left(\frac{l}{m}\right)^{\frac{6-\beta}{3}} - 1\right)F_m^a = 0.$$

Case 2. Suppose that $l > m$, we have that for all $\varepsilon > 0$, there exists $\varrho_l \in \mathcal{A}_l^a$ such that $F_l^a \ge \mathcal{F}^a(\varrho_l) - \varepsilon$. Then, we see that $F_l^a - F_m^a \ge \mathcal{F}^a(\varrho_l) - \mathcal{F}^a\left(\frac{m\varrho_l}{l}\right) - \varepsilon$. Since I and Π are increasing, we have that

$$\mathcal{F}^a(\varrho_m) - \mathcal{F}^a\left(\frac{m\varrho_l}{l}\right) \ge \left(\left(\frac{m}{l}\right)^2 - 1\right)\int_{\mathbb{R}^3} \varrho_l \Psi * \varrho_l \,\mathrm{d}\boldsymbol{x}.$$

We need to show that $\int_{\mathbb{R}^3} \varrho_l \Psi * \varrho_l \,\mathrm{d}\boldsymbol{x}$ remains bounded as $l \to m$ in order to pass the limit. By employing Lemma 4.3.4 and a similar argument to that used to obtain (4.19) and (4.20) in Lemma 4.3.3, we have that

$$\|\varrho_l\|_{L^q(\mathbb{R}^3)}^q \le C\left(l^{3-q} + \mathcal{F}^a(\varrho_l)\right) \le C\left(\varepsilon + l^{3-q} + \left(\frac{l}{m}\right)^{\frac{6-\beta}{3}} F_m^a\right),$$

and

$$\int_{\mathbb{R}^3} \varrho_l \Psi * \varrho_l \,\mathrm{d}\boldsymbol{x} \leq C l^{2-q}\big(l^{3-q} + \mathcal{F}^a(\varrho_l)\big) \leq C l^{2-q}\Big(\varepsilon + l^{3-q} + \Big(\frac{l}{m}\Big)^{\frac{6-\beta}{3}} F_m^a\Big).$$

Thus,

$$\liminf_{l\to m^+}(F_l^a - F_m^a) \geq -\varepsilon + \liminf_{l\to m^+}\left(\Big(\frac{m}{l}\Big)^2 - 1\right)\int_{\mathbb{R}^3} \varrho_m \Psi * \varrho_m \,\mathrm{d}\boldsymbol{x} = -\varepsilon.$$

Hence, since $\varepsilon > 0$ is arbitrary, we see that $\liminf_{l\to m^+}(F_l^a - F_m^a) \geq 0$.

□

In order to prove that the weak limit of a minimising sequence is indeed a minimiser of $\mathcal{F}^a$ over $\mathcal{A}_M^a$, we require that the kinetic energy is weakly lower semi-continuous. This will also come in useful when proving the existence of minimisers in §4.5. This is proven in the following lemma. For this, it will be useful to define

$$\mathbb{R}^3_* \coloneqq \mathbb{R}^3 \setminus (\{\mathbf{0}\} \times \mathbb{R}). \tag{4.22}$$

Lemma 4.3.7. *Suppose that I satisfies (4.16), $q > 1$, $\varrho_k \in \mathcal{A}_M^a$ and there exists $\tilde{\varrho} \in L^1_+(\mathbb{R}^3) \cap L^q_{\mathrm{axi}}(\mathbb{R}^3)$ such that $\varrho_k \rightharpoonup \tilde{\varrho}$ as $k \to \infty$ in $L^q(\mathbb{R}^3)$. Then, we have that*

$$\int_{\mathbb{R}^3} \frac{\tilde{\varrho} I(m_{\tilde{\varrho}}(r))}{r^2}\,\mathrm{d}\boldsymbol{x} \leq \liminf_{k\to\infty} \int_{\mathbb{R}^3} \frac{\varrho_k I(m_{\varrho_k}(r))}{r^2}\,\mathrm{d}\boldsymbol{x}.$$

Proof. If $\liminf_{k\to\infty}\int_{\mathbb{R}^3} \frac{\varrho_k I(m_{\varrho_k}(r))}{r^2}\,\mathrm{d}\boldsymbol{x} = \infty$, the result follows trivially. So, without loss of generality, we assume that $\int_{\mathbb{R}^3} \frac{\varrho_k I(m_{\varrho_k}(r))}{r^2}\,\mathrm{d}\boldsymbol{x}$ is a bounded sequence. Let $S > 0$ and define

$$m^S_{\varrho_k}(r) \coloneqq \int_{-S}^{S}\int_{B_r} \varrho_k(\boldsymbol{y}, z)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}z.$$

Since $\varrho_k \rightharpoonup \tilde{\varrho}$ in $L^q(\mathbb{R}^3)$ as $k \to \infty$,

$$m^S_{\varrho_k}(r) \longrightarrow m^S_{\tilde{\varrho}}(r), \qquad \text{as } k \to \infty.$$

As in [107], since $m^S_{\tilde{\varrho}}(r)$ is continuous by the Lebesgue dominated convergence Theorem, we use the variation of Dini's Theorem in [118, p.167] to obtain that for any $R > 0$, that

$$\|m^S_{\varrho_k}(r) - m^S_{\tilde{\varrho}}(r)\|_{L^\infty([0,R])} \longrightarrow 0, \qquad \text{as } k \to \infty.$$

Since $I \in C^1([0,\infty))$, we have that

$$\|I(m^S_{\varrho_k}(r)) - I(m^S_{\tilde{\varrho}}(r))\|_{L^\infty([0,R])} \longrightarrow 0, \qquad \text{as } k \to \infty. \tag{4.23}$$

Thus, since $I'(m) \geq 0$ for all $m \geq 0$, we have that

$$\int_{\mathbb{R}^3} \frac{\varrho_k I(m^S_{\varrho_k}(r))}{r^2}\,\mathrm{d}\boldsymbol{x} \leq \int_{\mathbb{R}^3} \frac{\varrho_k I(m_{\varrho_k}(r))}{r^2}\,\mathrm{d}\boldsymbol{x},$$

is a bounded sequence. Hence, by weak compactness in the space of signed Radon measures $\mathcal{M}(\mathbb{R}^3)$, there exists $\nu^S \in \mathcal{M}(\mathbb{R}^3)$ such that

$$\frac{\varrho_k I(m^S_{\varrho_k}(r))}{r^2} \rightharpoonup \nu^S, \qquad \text{in } \mathcal{M}(\mathbb{R}^3) \text{ as } k \to \infty.$$

By properties of weak convergence of signed Radon measures, as detailed in [57, Theorem 1.1.3], we have that

$$\nu^S(\{\mathbf{0}\} \times \mathbb{R}) \geq \limsup_{k\to\infty} \int_{\{\mathbf{0}\}\times\mathbb{R}} \frac{\varrho_k I(m^S_{\varrho_k}(r))}{r^2} \,\mathrm{d}\boldsymbol{x} = 0, \tag{4.24}$$

and

$$\nu^S(\mathbb{R}^3) \leq \liminf_{k\to\infty} \int_{\mathbb{R}^3} \frac{\varrho_k I(m^S_{\varrho_k}(r))}{r^2} \,\mathrm{d}\boldsymbol{x} \leq \liminf_{k\to\infty} \int_{\mathbb{R}^3} \frac{\varrho_k I(m_{\varrho_k}(r))}{r^2} \,\mathrm{d}\boldsymbol{x} < \infty. \tag{4.25}$$

Let $g \in C_c(\mathbb{R}^3_*)$, due to (4.23), we have that since $\varrho_k \rightharpoonup \tilde{\varrho}$ in $L^q(\mathbb{R}^3)$ as $k \to \infty$, that

$$\int_{\mathbb{R}^3} \frac{\varrho_k(\boldsymbol{x}) I(m^S_{\varrho_k}(r))}{r^2} g(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \longrightarrow \int_{\mathbb{R}^3} \frac{\tilde{\varrho}(\boldsymbol{x}) I(m^S_{\tilde{\varrho}}(r))}{r^2} g(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}.$$

This implies that on $\mathbb{R}^3_*$, $\nu^S \equiv \frac{\tilde{\varrho} I(m^S_{\tilde{\varrho}}(r))}{r^2}$. Thus, combining this with (4.24) and (4.25), we have that

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{\tilde{\varrho}(\boldsymbol{x}) I(m^S_{\tilde{\varrho}}(r))}{r^2} \,\mathrm{d}\boldsymbol{x} &= \nu^S(\mathbb{R}^3_*) \\ &= \nu^S(\mathbb{R}^3) - \nu^S(\{\mathbf{0}\} \times \mathbb{R}) \\ &\leq \liminf_{k\to\infty} \int_{\mathbb{R}^3} \frac{\varrho_k I(m_{\varrho_k}(r))}{r^2} \,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Thus, applying the Monotone convergence Theorem and the fact that $I'(m) \geq 0$ for all $m \geq 0$, we have that

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{\tilde{\varrho}(\boldsymbol{x}) I(m_{\tilde{\varrho}}(r))}{r^2} \,\mathrm{d}\boldsymbol{x} &= \lim_{S\to\infty} \int_{\mathbb{R}^3} \frac{\tilde{\varrho}(\boldsymbol{x}) I(m^S_{\tilde{\varrho}}(r))}{r^2} \,\mathrm{d}\boldsymbol{x} \\ &\leq \liminf_{k\to\infty} \int_{\mathbb{R}^3} \frac{\varrho_k I(m_{\varrho_k}(r))}{r^2} \,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

□

Now, we require the following lemma that characterises the behaviour of minimising sequences. With the framework we have built up, we will prove that behaviour such as vanishing and dichotomy cannot be observed for minimising sequences.

Lemma 4.3.8 ([103, Lemma I.1]). *Let $\varrho_k \in L^1(\mathbb{R}^n)$ be a sequence satisfying:*

$$\varrho_k \geq 0 \text{ in } \mathbb{R}^n, \qquad \int_\Omega \varrho_k \,\mathrm{d}\boldsymbol{x} = M,$$

for fixed $M > 0$. Then there exists a subsequence ϱ_{k_l} satisfying one of the three possibilities:

(a) (*compactness*) *There exists* $\boldsymbol{y}_l \in \mathbb{R}^n$ *such that* $\varrho_{k_l}(\cdot + \boldsymbol{y}_l)$ *is tight, that is, for all* $\varepsilon > 0$*, there exists* $R > 0$ *such that*

$$\int_{\boldsymbol{y}_l + B_R} \varrho_{k_l} \, \mathrm{d}\boldsymbol{x} \geq M - \varepsilon;$$

(b) (*vanishing*) *For all* $R < \infty$*,*

$$\lim_{l \to \infty} \sup_{\boldsymbol{y} \in \mathbb{R}^n} \int_{\boldsymbol{y} + B_R} \varrho_{k_l} \, \mathrm{d}\boldsymbol{x} = 0;$$

(c) (*dichotomy*) *There exists* $m \in (0, M)$ *such that for all* $\varepsilon > 0$*, there exist* $l_0 \geq 1$ *and* $\varrho_l^1, \varrho_l^2 \in L^1_+(\mathbb{R}^n)$ *satisfying for* $l \geq l_0$

$$\begin{cases} \|\varrho_{k_l} - (\varrho_l^1 + \varrho_l^2)\|_{L^1(\mathbb{R}^n)} \leq \varepsilon, \\ \left| \displaystyle\int_{\mathbb{R}^n} \varrho_l^1 \, \mathrm{d}\boldsymbol{x} - m \right| \leq \varepsilon, \\ \left| \displaystyle\int_{\mathbb{R}^n} \varrho_l^2 \, \mathrm{d}\boldsymbol{x} - (M - m) \right| \leq \varepsilon, \\ \operatorname{dist}(\operatorname{supp}(\varrho_l^1), \operatorname{supp}(\varrho_l^2)) \longrightarrow \infty, \ \ \textit{as } l \to \infty. \end{cases}$$

Now, unlike in [103], where they can consider a radially symmetric sequence of minimisers, due to the nature of the problem involving the angular momentum, it is no longer true that the radially decreasing rearrangement of the minimising sequence is still a minimising sequence. As such, we apply Lemma 4.3.8 in a different way that means we have to consider a new case for potential behaviour of the minimising sequence in L^1.

Proof of Theorem 4.3.1. We split the proof into five steps.

Step 1. From Lemma 4.3.2 (a), we have that $F_M^a > -\infty$, as such, there exists a minimising sequence $\varrho_k \in \mathcal{A}_M^a$ such that $\mathcal{F}^a(\varrho_k) \to F_M^a$ as $k \to \infty$. From Lemma 4.3.3, we have that the sequence ϱ_k is bounded in $L^1_+(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$. We can apply Lemma 4.3.8 to our minimising sequence, our aim is to show that vanishing and dichotomy cannot occur. For $\overline{r}$, the extension of $r > 0$ to $\mathbb{R}$, we define

$$f_k(\overline{r}, z) \coloneqq \begin{cases} 2\pi \varrho_k(\overline{r}, z)\overline{r}, & \overline{r} > 0, \\ 0, & \overline{r} \leq 0. \end{cases}$$

We must have that $f_k \in L^1(\mathbb{R}^2)$. We apply Lemma 4.3.8 to f_k.

Step 2. Suppose that (c) (dichotomy) in Lemma 4.3.8 occurs, then (up to a subsequence), for each $\varepsilon > 0$, there exists $f_k^1, f_k^2, g_k \in L^1_+(\mathbb{R}^2)$ such that $0 \leq f_k^1, f_k^2, g_k \leq f_k$

and $f_k^1 f_k^2 = f_k^1 g_k = f_k^2 g_k = 0$ almost everywhere. We let $d_k = \operatorname{dist}(\operatorname{supp}(f_k^1), \operatorname{supp}(f_k^2))$ and

$$m_k^1 = \int_{\mathbb{R}^2} f_k^1 \,\mathrm{d}\overline{r}\,\mathrm{d}z, \qquad m_k^2 = \int_{\mathbb{R}^2} f_k^2 \,\mathrm{d}\overline{r}\,\mathrm{d}z.$$

Then, without loss of generality, we may assume that $d_k \to \infty$ as $k \to \infty$, $m_k^1 \to m^1(\varepsilon)$, $m_k^2 \to m^2(\varepsilon)$ for $m^1(\varepsilon) \in (0, M)$ and $m^2(\varepsilon) \in (0, M - m^1(\varepsilon)]$ with $|m^1(\varepsilon) - m| \le \varepsilon$ and $|m^2(\varepsilon) - (M - m)| \le \varepsilon$ for some $m \in (0, M)$. Defining

$$\begin{aligned}
\varrho_k^1(r,z) &:= \frac{f_k^1(r,z)}{2\pi r},\\
\varrho_k^2(r,z) &:= \frac{f_k^2(r,z)}{2\pi r},\\
\sigma_k(r,z) &:= \frac{g_k(r,z)}{2\pi r},
\end{aligned}$$

$\varrho_k^1, \varrho_k^2, \sigma_k \in L^1_+(\mathbb{R}^3)$ are axisymmetric such that $0 \le \varrho_k^1, \varrho_k^2, \sigma_k \le \varrho_k$ and $\varrho_k^1 \varrho_k^2 = \varrho_k^1 \sigma_k = \varrho_k^2 \sigma_k = 0$ almost everywhere. They also satisfy

$$m_k^1 = \int_{\mathbb{R}^3} \varrho_k^1 \,\mathrm{d}\boldsymbol{x}, \qquad m_k^2 = \int_{\mathbb{R}^3} \varrho_k^2 \,\mathrm{d}\boldsymbol{x},$$

$\operatorname{dist}(\operatorname{supp}(\varrho_k^1), \operatorname{supp}(\varrho_k^2)) = d_k \to \infty$ as $k \to \infty$. As in the proof of [103, Theorem II.1], one can show that

$$\left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \varrho_k^1(\boldsymbol{x}) \varrho_k^2(\boldsymbol{y}) \Psi(\boldsymbol{x} - \boldsymbol{y}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \right| \longrightarrow 0, \qquad \text{as } k \to \infty,$$

and there exists a $C > 0$ such that for $i = 1, 2$,

$$\left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \varrho_k^i(\boldsymbol{x}) \sigma_k(\boldsymbol{y}) \Psi(\boldsymbol{x} - \boldsymbol{y}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \right|^2 + \left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sigma_k(\boldsymbol{x}) \sigma_k(\boldsymbol{y}) \Psi(\boldsymbol{x} - \boldsymbol{y}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \right| \le C\varepsilon^{2-q}.$$

Moreover, since $I'(m) \ge 0$ for all $m \in [0, M]$ and $\Pi : [0, \infty) \to [0, \infty)$ is strictly convex with $\Pi(0) = 0$, Π must be increasing, so it can be seen that

$$\begin{aligned}
\int_{\mathbb{R}^3} \left(\Pi(\varrho_k(\boldsymbol{x})) + \frac{\varrho_k(\boldsymbol{x}) I(m_{\varrho_k}(r(\boldsymbol{x})))}{r(\boldsymbol{x})^2} \right) \mathrm{d}\boldsymbol{x} &\ge \int_{\mathbb{R}^3} \left(\Pi(\varrho_k^1(\boldsymbol{x})) + \frac{\varrho_k^1(\boldsymbol{x}) I(m_{\varrho_k^1}(r(\boldsymbol{x})))}{r(\boldsymbol{x})^2} \right) \mathrm{d}\boldsymbol{x}\\
&\quad + \int_{\mathbb{R}^3} \left(\Pi(\varrho_k^2(\boldsymbol{x})) + \frac{\varrho_k^2(\boldsymbol{x}) I(m_{\varrho_k^2}(r(\boldsymbol{x})))}{r(\boldsymbol{x})^2} \right) \mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Thus, we obtain that

$$\begin{aligned}
F_M^a &\ge \liminf_{k\to\infty} \left(\int_{\mathbb{R}^3} \left(\Pi(\varrho_k^1(\boldsymbol{x})) + \frac{\varrho_k^1(\boldsymbol{x}) I(m_{\varrho_k^1}(r(\boldsymbol{x})))}{r(\boldsymbol{x})^2} - \frac{1}{2} \varrho_k^1 \Psi * \varrho_k^1 \right) \mathrm{d}\boldsymbol{x} \right)\\
&\quad + \liminf_{k\to\infty} \left(\int_{\mathbb{R}^3} \left(\Pi(\varrho_k^2(\boldsymbol{x})) + \frac{\varrho_k^2(\boldsymbol{x}) I(m_{\varrho_k^2}(r(\boldsymbol{x})))}{r(\boldsymbol{x})^2} - \frac{1}{2} \varrho_k^2 \Psi * \varrho_k^2 \right) \mathrm{d}\boldsymbol{x} \right) + O(\varepsilon^{1-\frac{q}{2}})\\
&\ge F_{m^1(\varepsilon)}^a + F_{m^2(\varepsilon)}^a + O(\varepsilon^{1-\frac{q}{2}}).
\end{aligned}$$

Letting $\varepsilon \to 0$, by Lemma 4.3.6, we obtain that

$$F_M^a \geq F_m^a + F_{M-m}^a,$$

a contradiction to Lemma 4.3.5.

Step 3. Suppose that (b) (vanishing) in Lemma 4.3.8 occurs, then as in the proof of [103, Theorem II.1] (up to a subsequence), we have that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \varrho_k(\boldsymbol{x})\varrho_k(\boldsymbol{y})\Psi(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \longrightarrow 0, \qquad \text{as } k \to \infty.$$

But, this would imply that $F_M^a \geq 0$, a contradiction.

Step 4. Thus, we must have that (a) (compactness) in Lemma 4.3.8 occurs, then (up to a subsequence) there exists $(\overline{r}_k, z_k) \in \mathbb{R}^2$ such that for any $\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{B_R(\overline{r}_k, z_k)^c} f_k \,\mathrm{d}\overline{r}\,\mathrm{d}z \leq \varepsilon.$$

In particular, due to the form of f_k, we can assume that $\overline{r}_k \geq 0$ and thus, use the notation $r_k = \overline{r}_k$. We can assume $z_k = 0$ by taking $\tilde{f}_k = f_k(\cdot, \cdot - z_k)$ and $\tilde{\varrho}_k = \varrho_k(\cdot, \cdot - z_k)$. Suppose that r_k does not have a bounded subsequence, in particular, we have that $r_k \to \infty$ as $k \to \infty$. Then, we define $\tilde{f}_k^1(r, z; R) = \mathbb{1}_{B_R(r_k,0)}\tilde{f}_k$ and $\tilde{f}_k^2(r, z; R) = \mathbb{1}_{B_R(r_k,0)^c}\tilde{f}_k$. Then, as before, we define

$$\tilde{\varrho}_k^1(r, z; R) := \frac{\tilde{f}_k^1(r, z; R)}{2\pi r},$$
$$\tilde{\varrho}_k^2(r, z; R) := \frac{\tilde{f}_k^2(r, z; R)}{2\pi r},$$

we have that $\tilde{\varrho}_k = \tilde{\varrho}_k^1 + \tilde{\varrho}_k^2$, $0 \leq \tilde{\varrho}_k^1, \tilde{\varrho}_k^2, \leq \tilde{\varrho}_k$ and $\tilde{\varrho}_k^1\tilde{\varrho}_k^2 = 0$ almost everywhere. For convenience, we suppress the R in the notation so that $\tilde{\varrho}_k^1(\boldsymbol{x}; R) = \tilde{\varrho}_k^1(\boldsymbol{x})$. We claim that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \tilde{\varrho}_k(\boldsymbol{x})\tilde{\varrho}_k(\boldsymbol{y})\Psi(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \longrightarrow 0, \qquad \text{as } k \to \infty.$$

This comes from the dissipation of the density, meaning that as $k \to \infty$, the amount of local interactions decrease. By Lemma A.0.4, we have that

$$\begin{aligned}
\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \tilde{\varrho}_k^2(\boldsymbol{x})\tilde{\varrho}_k^2(\boldsymbol{y})\Psi(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} &\leq C_*^\Psi \varepsilon^{2-q}\|\tilde{\varrho}_k\|_{L^q(\mathbb{R}^3)}^q, \\
\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \tilde{\varrho}_k^2(\boldsymbol{x})\tilde{\varrho}_k^1(\boldsymbol{y})\Psi(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} &\leq C_*^\Psi (\varepsilon M)^{\frac{2-q}{2}}\|\tilde{\varrho}_k\|_{L^q(\mathbb{R}^3)}^q.
\end{aligned} \tag{4.26}$$

For $\delta > 0$, we have that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \tilde{\varrho}_k^1(\boldsymbol{x})\tilde{\varrho}_k^1(\boldsymbol{y})\mathbb{1}_{\{\Psi \leq \delta\}}\Psi(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \leq \delta M^2. \tag{4.27}$$

Defining, $\Psi_\delta \coloneqq \mathbb{1}_{\{\Psi\geq\delta\}}\Psi \in L^r(\mathbb{R}^3)$ for all $1 \leq r < p$, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \tilde{\varrho}_k^1(\boldsymbol{x})\tilde{\varrho}_k^1(\boldsymbol{y})\Psi_\delta(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\\
&= \int_{\mathbb{R}^3}\int_{\{|\boldsymbol{x}-\boldsymbol{y}|<\sqrt{r_k}\}} \tilde{\varrho}_k^1(\boldsymbol{x})\tilde{\varrho}_k^1(\boldsymbol{y})\Psi_\delta(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\\
&\quad + \int_{\mathbb{R}^3}\int_{\{|\boldsymbol{x}-\boldsymbol{y}|\geq\sqrt{r_k}\}} \tilde{\varrho}_k^1(\boldsymbol{x})\tilde{\varrho}_k^1(\boldsymbol{y})\Psi_\delta(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}.
\end{aligned}$$

By Young's convolution inequality, taking $\frac{q}{2(q-1)} = \frac{pq}{2} \leq r < p$, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\{|\boldsymbol{x}-\boldsymbol{y}|\geq\sqrt{r_k}\}} \tilde{\varrho}_k^1(\boldsymbol{x})\tilde{\varrho}_k^1(\boldsymbol{y})\Psi_\delta(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\\
&\leq C\|\Psi_\delta\|_{L^r((B_{\sqrt{r_k}})^c)} M^{\frac{q(2r-1)-2r}{r(q-1)}} \|\tilde{\varrho}_k\|_{L^q(\mathbb{R}^3)}^{\frac{q}{r(q-1)}}.
\end{aligned}$$

Note, we have that $\|\Psi_\delta\|_{L^r((B_{\sqrt{r_k}})^c)} \to 0$ as $k \to \infty$ by the dominated convergence Theorem. Thus,

$$\int_{\mathbb{R}^3}\int_{\{|\boldsymbol{x}-\boldsymbol{y}|\geq\sqrt{r_k}\}} \tilde{\varrho}_k^1(\boldsymbol{x})\tilde{\varrho}_k^1(\boldsymbol{y})\Psi_\delta(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \longrightarrow 0, \qquad \text{as } k \to \infty. \tag{4.28}$$

For $\boldsymbol{x} \in \mathbb{R}^3$, we define $\boldsymbol{x}^2 = (x_1, x_2)$. For fixed $R > 0$, letting $k \to \infty$, there exists $C = C(R) > 0$ such that the amount of open balls $B_{\sqrt{r_k}}(\boldsymbol{x}) = \{\boldsymbol{y} \in \mathbb{R}^3 \,|\, |\boldsymbol{x}-\boldsymbol{y}| \leq \sqrt{r_k}\}$ with $|\boldsymbol{x}^2| = r_k$ and $x_3 = 0$ needed to cover the set $\{\boldsymbol{y} \in \mathbb{R}^3 \,|\, (|\boldsymbol{y}^2|, y_3) \in B_R(r_k, 0)\}$ is

$$\sim C r_k^{\frac{1}{2}},$$

as $k \to \infty$. Hence, for k large enough, there exists a covering set $\mathrm{Cov}_k(R) \subset \{\boldsymbol{x} \in \mathbb{R}^3 \,|\, |\boldsymbol{x}^2| = r_k,\, x_3 = 0\}$ such that $|\mathrm{Cov}_k(R)| \leq C r_k^{\frac{1}{2}}$ and

$$\mathrm{supp}(\tilde{\varrho}_k^1) \subset \{\boldsymbol{y} \in \mathbb{R}^3 \,|\, (|\boldsymbol{y}^2|, y_3) \in B_R(r_k, 0)\} \subset \bigcup_{\boldsymbol{x}\in\mathrm{Cov}_k(R)} B_{\sqrt{r_k}}(\boldsymbol{x}).$$

Due to the axisymmetry of $\tilde{\varrho}_k^1$, we have that there exists $C = C(R) > 0$ such that

$$\begin{aligned}
\|\tilde{\varrho}_k^1\|_{L^1\left(B_{2\sqrt{r_k}}(\boldsymbol{z})\right)} &\leq C r_k^{-\frac{1}{2}} \|\tilde{\varrho}_k^1\|_{L^1(\mathbb{R}^3)},\\
\|\tilde{\varrho}_k^1\|^q_{L^q\left(B_{2\sqrt{r_k}}(\boldsymbol{z})\right)} &\leq C r_k^{-\frac{1}{2}} \|\tilde{\varrho}_k^1\|^q_{L^q(\mathbb{R}^3)}.
\end{aligned} \tag{4.29}$$

Thus, applying Lemma A.0.4 and (4.29), we have that

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\{|\boldsymbol{x}-\boldsymbol{y}|<\sqrt{r_k}\}} \tilde{\varrho}_k^1(\boldsymbol{x})\tilde{\varrho}_k^1(\boldsymbol{y})\Psi_\delta(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\\
&\leq \sum_{\boldsymbol{z}\in \mathrm{Cov}_k(R)}\int_{B_{\sqrt{r_k}}(\boldsymbol{z})}\int_{\{|\boldsymbol{x}-\boldsymbol{y}|<\sqrt{r_k}\}} \tilde{\varrho}_k^1(\boldsymbol{x})\tilde{\varrho}_k^1(\boldsymbol{y})\Psi_\delta(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\\
&\leq \sum_{\boldsymbol{z}\in \mathrm{Cov}_k(R)}\int_{B_{\sqrt{r_k}}(\boldsymbol{z})}\int_{B_{2\sqrt{r_k}}(\boldsymbol{z})} \tilde{\varrho}_k^1(\boldsymbol{x})\tilde{\varrho}_k^1(\boldsymbol{y})\Psi_\delta(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\\
&\leq C_*^\Psi \sum_{\boldsymbol{z}\in \mathrm{Cov}_k(R)} \|\tilde{\varrho}_k^1\|_{L^1(B_{2\sqrt{r_k}}(\boldsymbol{z}))}^{2-q}\|\tilde{\varrho}_k^1\|_{L^q(B_{2\sqrt{r_k}}(\boldsymbol{z}))}^{q}\\
&\leq C r_k^{-\frac{(2-q)}{2}} M^{2-q}\|\tilde{\varrho}_k\|_{L^q(\mathbb{R}^3)}^q.
\end{aligned}$$

Thus, we have that

$$\int_{\mathbb{R}^3}\int_{\{|\boldsymbol{x}-\boldsymbol{y}|<\sqrt{r_k}\}} \tilde{\varrho}_k^1(\boldsymbol{x})\tilde{\varrho}_k^1(\boldsymbol{y})\Psi_\delta(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\longrightarrow 0,\qquad \text{as } k\to\infty. \tag{4.30}$$

Thus, combining (4.26)–(4.30), we have that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \tilde{\varrho}_k(\boldsymbol{x})\tilde{\varrho}_k(\boldsymbol{y})\Psi(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\longrightarrow 0,\qquad \text{as } k\to\infty.$$

This implies that $F_M^a \geq 0$, which is a contradiction. Thus our argument shows that r_k must remain bounded and hence we can assume without loss of generality that $r_k = 0$ for all k.

Step 5. Hence for all $\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{B_R^c} \tilde{\varrho}_k\,\mathrm{d}\boldsymbol{x} \leq \varepsilon. \tag{4.31}$$

Since $\tilde{\varrho}_k \in L^1_+(\mathbb{R}^3)\cap L^q(\mathbb{R}^3)$ is a bounded sequence in $L^1_+(\mathbb{R}^3)\cap L^q(\mathbb{R}^3)$, by Lemma 3.7.2, there exists $\bar{\rho}\in L^1_+(\mathbb{R}^3)\cap L^q(\mathbb{R}^3)$ such that (up to a subsequence) $\tilde{\varrho}_k \rightharpoonup \bar{\rho}$ in $L^q(\mathbb{R}^3)$. Since $\tilde{\varrho}_k$ is axisymmetric, so is $\bar{\rho}$. Due to the weak semi-continuity of the L^1 norm and (4.31), we have that

$$\int_{B_R^c} \bar{\rho}\,\mathrm{d}\boldsymbol{x} \leq \varepsilon. \tag{4.32}$$

By weak convergence in L^q, we see that

$$\int_{B_R} \tilde{\varrho}_k\,\mathrm{d}\boldsymbol{x}\longrightarrow \int_{B_R}\bar{\rho}\,\mathrm{d}\boldsymbol{x},\qquad \text{as } k\to\infty. \tag{4.33}$$

Tying together (4.32) and (4.33), we see that $\int_{\mathbb{R}^3}\bar{\rho}\,\mathrm{d}\boldsymbol{x} = M$. By Lemma 4.3.7, we obtain

$$\int_{\mathbb{R}^3}\frac{\bar{\rho}(\boldsymbol{x})I(m_{\bar{\rho}}(r(\boldsymbol{x})))}{r(\boldsymbol{x})^2}\,\mathrm{d}\boldsymbol{x}\leq \liminf_{k\to\infty}\int_{\mathbb{R}^3}\frac{\tilde{\varrho}_k(\boldsymbol{x})I(m_{\tilde{\varrho}_k}(r(\boldsymbol{x})))}{r(\boldsymbol{x})^2}\,\mathrm{d}\boldsymbol{x}.$$

Hence, $\bar{\rho} \in \mathcal{A}_M^a$. We now aim to show that $\bar{\rho}$ is in fact a minimiser of $\mathcal{F}^a$ over $\mathcal{A}_M^a$. We have that since the functional $\varrho \mapsto \int_{\mathbb{R}^3} \Pi(\varrho)\,\mathrm{d}\boldsymbol{x}$ is convex and strongly lower semi-continuous by Fatou's Lemma, it must be weakly lower semi-continuous. Hence,

$$\int_{\mathbb{R}^3} \Pi(\bar{\rho})\,\mathrm{d}\boldsymbol{x} \leq \liminf_{k\to\infty} \int_{\mathbb{R}^3} \Pi(\tilde{\varrho}_k)\,\mathrm{d}\boldsymbol{x}.$$

As in [103], one can prove that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \tilde{\varrho}_k(\boldsymbol{x})\tilde{\varrho}_k(\boldsymbol{y})\Psi(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \longrightarrow \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \bar{\rho}(\boldsymbol{x})\bar{\rho}(\boldsymbol{y})\Psi(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}, \quad \text{as } k\to\infty.$$

Thus, we see that $\bar{\rho}$ is indeed a minimiser of $\mathcal{F}^a$ over $\mathcal{A}_M^a$ and in fact that

$$\int_{\mathbb{R}^3} \Pi(\tilde{\varrho}_k)\,\mathrm{d}\boldsymbol{x} \longrightarrow \int_{\mathbb{R}^3} \Pi(\bar{\rho})\,\mathrm{d}\boldsymbol{x}, \qquad \text{as } k\to\infty.$$

As in [103], owing to the strict convexity of Π, we have that $\tilde{\varrho}_k \to \bar{\rho}$ in measure. Thus, we can prove that $\tilde{\varrho}_k \to \bar{\rho}$ in $L^1(\mathbb{R}^3)\cap L^q(\mathbb{R}^3)$.

□

We now apply the general framework to our particular minimisation problem. Taking $\Psi = -\Phi_\alpha$, which satisfies (4.14) for $p = \frac{3}{\alpha}$, $q = \frac{p+1}{p} = \frac{3+\alpha}{3}$ and satisfies (4.15) for $\beta = \alpha$. The internal energy function $\Pi(\varrho) := \varrho e(\varrho)$, as in §3.3, satisfies (3.19). Moreover, $I = \frac{1}{2}L$ satisfies (4.16) and (4.17) for $\beta = \alpha$. If $\alpha \in (0,2)$ and

$$M > \left(\frac{6\limsup_{\rho\to 0} p(\rho)\rho^{-\frac{3+\alpha}{3}}}{C_*}\right)^{\frac{3}{3-\alpha}},$$

we have $g_M^a < 0$ by Lemma 4.3.2. If

$$M < \left(\frac{6\liminf_{\rho\to\infty} p(\rho)\rho^{-\frac{3+\alpha}{3}}}{C_*}\right)^{\frac{3}{3-\alpha}},$$

we have $g_M^a > -\infty$ by Lemma 4.3.2. By Theorem 4.3.1 we deduce that there exists a minimiser $\bar{\rho}$ to $\mathcal{G}^a$ over X_M^a. Thus, we obtain the following corollary.

Corollary 4.3.9. *Suppose $\alpha \in (0,2)$, $M > 0$ with $p \in C^1([0,\infty))$ satisfying (1.4), (3.4)–(3.5) for $n = 3$ and L satisfies (4.3), (4.6) and (4.10). Then there exists a minimiser $\bar{\rho} \in X_M^a$ to $\mathcal{G}^a$.*

Combining this with the approach of [6, 107, 108], one obtains the following corollary.

Corollary 4.3.10. *Under the conditions of Corollary 4.3.9, if $\bar{\rho} \in X_M^a$ is a minimiser of $\mathcal{G}^a$ over X_M^a, then $\bar{\rho}$ is a rotating Riesz star solution as given by Definition 4.1.1 with compact support.*

4.4 Stability of rotating Riesz star solutions in the mass-subcritical regime

To show stability, we consider axisymmetric solutions $(\rho, \boldsymbol{\mathcal{M}})$ of (1.1) as defined in (4.1) with initial rotational momentum of the form

$$\mathcal{M}_0^\theta(\boldsymbol{x}) = \frac{\rho_0(\boldsymbol{x}) J(m_{\rho_0}(r))}{r}.$$

Under the conditions (A_1)–(A_4), the rotational momentum field is given by

$$\mathcal{M}^\theta(t, \boldsymbol{x}) = \frac{\rho(t, \boldsymbol{x}) J(m_{\rho(t)}(r))}{r}.$$

Similar to Theorem 3.2.2, we can now prove the stability of rotating Riesz star solutions.

Proof of Theorem 4.2.2. As in §3.4, suppose the statement in Theorem 4.2.2 is not true, then there exist $\varepsilon > 0$ and a sequence of global weak solutions $(\rho^j, \boldsymbol{\mathcal{M}}^j)$ of the CEREs associated to the initial data $(\rho_0^j, \boldsymbol{\mathcal{M}}_0^j)$ and a sequence of times $t_j > 0$ such that

$$d_1^a(\rho_0^j, \bar{\rho}) + d_2^a(\rho_0^j, \bar{\rho}) + \|\rho_0^j - \bar{\rho}\|^2_{L^{\frac{6}{6-\alpha}}} + \frac{1}{2}\int_{\mathbb{R}^3} \left(\left| \frac{\mathcal{M}_{r,0}^j}{\sqrt{\rho_0^j}} \right|^2 + \left| \frac{\mathcal{M}_{3,0}^j}{\sqrt{\rho_0^j}} \right|^2 \right) \mathrm{d}\boldsymbol{x} < \frac{1}{j}, \tag{4.34}$$

and

$$\begin{aligned} d_1^a(\rho^j(t_j), T^{\boldsymbol{y}}\bar{\rho}) + d_2^a(\rho^j(t_j), T^{\boldsymbol{y}}\bar{\rho}) + \|\rho^j(t_j) - T^{\boldsymbol{y}}\bar{\rho}\|^2_{L^{\frac{6}{6-\alpha}}} \\ + \frac{1}{2}\int_{\mathbb{R}^3} \left(\left| \frac{\mathcal{M}_r^j}{\sqrt{\rho^j}} \right|^2 + \left| \frac{\mathcal{M}_3^j}{\sqrt{\rho^j}} \right|^2 \right)(t_j)\, \mathrm{d}\boldsymbol{x} \geq \varepsilon, \end{aligned} \tag{4.35}$$

for all $\boldsymbol{y} \in \{\mathbf{0}\} \times \mathbb{R}$. Notice that the sequence of times t_j can be chosen outside a zero measure set, where the terms in (4.35) are well-defined according to Definition 2.1.1. By Lemma 4.2.1, we have that $d_2^a(\rho(t), \bar{\rho}) \geq 0$ for all $t \geq 0$. Then, by (4.9) and (4.34), we have

$$\lim_{j\to\infty} E(\rho_0^j, \boldsymbol{\mathcal{M}}_0^j) = E(\bar{\rho}, \bar{\boldsymbol{\mathcal{M}}}) = \mathcal{G}^a(\bar{\rho}).$$

Since the energy E for the weak solutions is non-increasing from the initial energy by definition, we have

$$\limsup_{j\to\infty} \mathcal{G}^a(\rho^j(t_j)) \leq \limsup_{j\to\infty} E(\rho^j(t_j), \boldsymbol{\mathcal{M}}^j(t_j)) \leq \lim_{j\to\infty} E(\rho_0^j, \boldsymbol{\mathcal{M}}_0^j) = \mathcal{G}^a(\bar{\rho}).$$

Thus, $\rho^j(t_j) \in X_M^a$ is a minimising sequence of $\mathcal{G}^a$ in X_M^a and, by Theorem 4.3.1, there exists a sequence $\boldsymbol{y}_j \in \{\mathbf{0}\} \times \mathbb{R}$ such that (up to a subsequence) $T^{\boldsymbol{y}_j}\rho^j(t_j) \to \bar{\rho}$ in $L^1(\mathbb{R}^3) \cap L^{\frac{3+\alpha}{3}}(\mathbb{R}^3)$. Then

$$\|\rho^j(t_j) - T^{-\boldsymbol{y}_j}\bar{\rho}\|_{L^{\frac{6}{6-\alpha}}} = \|T^{\boldsymbol{y}_j}\rho^j(t_j) - \bar{\rho}\|_{L^{\frac{6}{6-\alpha}}} \longrightarrow 0,$$

as $j \to \infty$, since $\frac{6}{6-\alpha} \in (1, \frac{3+\alpha}{3})$. We see that, for any i,

$$\lim_{j\to\infty} E(\rho^j(t_j), \boldsymbol{\mathcal{M}}^j(t_j)) = \mathcal{G}^a(\bar{\rho}) = \mathcal{G}^a(T^{-\boldsymbol{y}_i}\bar{\rho}).$$

Therefore, by (4.9),

$$d_1^a(\rho^j(t_j), T^{-\boldsymbol{y}_j}\bar{\rho}) + d_2^a(\rho^j(t_j), T^{-\boldsymbol{y}_j}\bar{\rho}) + \frac{1}{2}\int_{\mathbb{R}^3}\left(\left|\frac{\mathcal{M}_r^j}{\sqrt{\rho^j}}\right|^2 + \left|\frac{\mathcal{M}_3^j}{\sqrt{\rho^j}}\right|^2\right)(t_j)\,\mathrm{d}\boldsymbol{x} \longrightarrow 0,$$

as $j \to \infty$. This is a contradiction to (4.35), which completes the proof. □

Remark 4.4.1. Combining Corollary 4.3.10 with Theorem 3.2.6, we see that Theorem 4.2.2 is indeed a local stability result.

Remark 4.4.2. As in Corollary 3.4.4, one can prove an analogous stability result in the case of non-uniqueness of rotating Riesz star solutions (up to translation), which we do not state for the sake of space.

4.5 Existence of rotating Riesz star solutions in the mass-supercritical regime

In this section we prove existence of rotating Riesz star solutions in the polytropic mass-supercritical case ($\frac{6}{6-\alpha} < \gamma < \frac{3+\alpha}{3}$). Given $\varrho \in L^1_+(\mathbb{R}^3) \cap L^\gamma(\mathbb{R}^3)$, and $p(\varrho) = a_0\varrho^\gamma$, with $\frac{6}{6-\alpha} < \gamma < \frac{3+\alpha}{3}$, we recall the definition of the free-energy functional $Q^a(\varrho)$:

$$Q^a(\varrho) = \int_{\mathbb{R}^3} \frac{\varrho L(m_\varrho(r))}{r^2}\,\mathrm{d}\boldsymbol{x} + 3a_0\int_{\mathbb{R}^3}\varrho^\gamma\,\mathrm{d}\boldsymbol{x} + \frac{\alpha}{2}\int_{\mathbb{R}^3}\varrho\Phi_\alpha * \varrho\,\mathrm{d}\boldsymbol{x}.$$

As in §3.7, this quantity will play a critical role in the analysis. We also reintroduce the associated free-energy functional $S^a_\mu(\varrho)$:

$$S^a_\mu(\varrho) = \frac{1}{2}\int_{\mathbb{R}^3} \frac{\varrho L(m_\varrho(r))}{r^2}\,\mathrm{d}\boldsymbol{x} + \frac{a_0}{\gamma-1}\int_{\mathbb{R}^3}\varrho^\gamma\,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_{\mathbb{R}^3}\varrho\Phi_\alpha * \varrho\,\mathrm{d}\boldsymbol{x} + \mu\int_{\mathbb{R}^3}\varrho\,\mathrm{d}\boldsymbol{x},$$

for $\mu > 0$. In the axisymmetric setting, with only rotational velocity, $S^a_\mu(\varrho)$ represents μ times the mass of ϱ plus the energy of the distribution ϱ with velocity field

$$\boldsymbol{u}(\boldsymbol{x}) = \frac{J(m_\varrho(r))}{r}\boldsymbol{e}_\theta,$$

with $J^2 = L$. While, as in §3.7, we consider the mass-preserving scaling, we use a different power of λ in the scaling. Moreover, in this section, we define the mass-preserving scaling by

$$\varrho_\lambda(\boldsymbol{x}) \coloneqq \lambda^{\frac{3}{2}}\varrho(\lambda^{\frac{1}{2}}\boldsymbol{x}),$$

for $\lambda > 0$. Analogous to §3.7, defining $S^a_{\mu,\lambda}(\varrho) \coloneqq S^a_\mu(\varrho_\lambda)$, we have that

$$S^a_{\mu,\lambda}(\varrho) \coloneqq \frac{\lambda}{2}\int_{\mathbb{R}^3} \frac{\varrho L(m_\varrho(r))}{r^2}\,\mathrm{d}\boldsymbol{x} + \frac{a_0\lambda^{\frac{3(\gamma-1)}{2}}}{\gamma-1}\int_{\mathbb{R}^3}\varrho^\gamma\,\mathrm{d}\boldsymbol{x} + \frac{\lambda^{\frac{\alpha}{2}}}{2}\int_{\mathbb{R}^3}\varrho\Phi_\alpha * \varrho\,\mathrm{d}\boldsymbol{x} + \mu\int_{\mathbb{R}^3}\varrho\,\mathrm{d}\boldsymbol{x},$$

and

$$\begin{aligned}\frac{\mathrm{d}S^a_{\mu,\lambda}(\varrho)}{\mathrm{d}\lambda} &= \frac{1}{2}\int_{\mathbb{R}^3}\frac{\varrho L(m_\varrho(r))}{r^2}\,\mathrm{d}\boldsymbol{x} + \frac{3a_0\lambda^{\frac{3\gamma-5}{2}}}{2}\int_{\mathbb{R}^3}\varrho^\gamma\,\mathrm{d}\boldsymbol{x} + \frac{\alpha\lambda^{\frac{\alpha-2}{2}}}{4}\int_{\mathbb{R}^3}\varrho\Phi_\alpha * \varrho\,\mathrm{d}\boldsymbol{x}\\ &= \frac{Q^a(\varrho_\lambda)}{2\lambda}.\end{aligned}$$

The analysis is more complicated than in the stationary state case of §3.7 owing to the competition of the kinetic energy with the potential energy and internal energy. As mentioned, in §3.7, there exists a unique $\lambda^* = \lambda^*(\varrho) > 0$ such that $Q(\varrho_{\lambda^*}) = 0$ for each $\varrho \in L^1_+ \cap L^\gamma$, this is not necessarily true with the addition of the rotational kinetic energy. This is characterised by the following two lemmas.

Lemma 4.5.1. *Suppose $\alpha \in [2,3)$, $\frac{6}{6-\alpha} < \gamma < \frac{3+\alpha}{3}$, L satisfies (4.5) and $\varrho \in L^1_+(\mathbb{R}^3)\cap L^\gamma(\mathbb{R}^3)$, $\varrho \not\equiv 0$,*

$$\int_{\mathbb{R}^3}\frac{\varrho L(m_\varrho(r))}{r^2}\,\mathrm{d}\boldsymbol{x} < \infty,$$

with, for $\alpha = 2$,

$$\int_{\mathbb{R}^3}\frac{\varrho L(m_\varrho(r))}{r^2}\,\mathrm{d}\boldsymbol{x} < -\int_{\mathbb{R}^3}\varrho\Phi_\alpha * \varrho\,\mathrm{d}\boldsymbol{x},$$

then the following statements are true:

(a) *There exists a unique $\lambda^*(\varrho) > 0$ such that $Q^a(\varrho_{\lambda^*(\varrho)}) = 0$.*

(b) *$\lambda^*(\varrho) > 1 \iff Q^a(\varrho) > 0$.*

(c) *$\lambda \mapsto S^a_{\mu,\lambda}(\varrho)$ is strictly increasing (decreasing) for $\lambda \le \lambda^*(\varrho)$ ($\lambda \ge \lambda^*(\varrho)$). Moreover, $S^a_{\mu,\lambda}(\varrho) < S^a_{\mu,\lambda^*(\varrho)}(\varrho)$, for all $\lambda \ne \lambda^*(\varrho)$ and $\lambda > 0$.*

Proof.

Case 1 ($\alpha \in (2,3)$)**.** Since $\alpha > 2, 3(\gamma-1)$ and $\int_{\mathbb{R}^3}\varrho\Phi_\alpha * \varrho\,\mathrm{d}\boldsymbol{x} < 0$, we have that

$$Q^a(\varrho_\lambda) = \lambda\int_{\mathbb{R}^3}\frac{\varrho L(m_\varrho(r))}{r^2}\,\mathrm{d}\boldsymbol{x} + 3a_0\lambda^{\frac{3(\gamma-1)}{2}}\int_{\mathbb{R}^3}\varrho^\gamma\,\mathrm{d}\boldsymbol{x} + \frac{\alpha\lambda^{\frac{\alpha}{2}}}{2}\int_{\mathbb{R}^3}\varrho\Phi_\alpha * \varrho\,\mathrm{d}\boldsymbol{x} \longrightarrow -\infty,$$

as $\lambda \to \infty$. Similarly, due to the positivity of the kinetic energy and the internal energy, we obtain that

$$\frac{Q^a(\varrho_\lambda)}{\lambda^{\frac{\alpha}{2}}} \longrightarrow \infty, \qquad \text{as } \lambda \to 0.$$

Thus, since $\lambda \mapsto Q^a(\varrho_\lambda)$ is continuous, there exists $\lambda^* = \lambda^*(\varrho) > 0$ such that $Q^a(\varrho_{\lambda^*}) = 0$.

Case 2 ($\alpha = 2$)**.** Similar to the previous case, as $2 > 3(\gamma - 1)$ and

$$\int_{\mathbb{R}^3} \frac{\varrho L(m_\varrho(r))}{r^2}\,\mathrm{d}\boldsymbol{x} < -\int_{\mathbb{R}^3} \varrho\Phi_\alpha * \varrho\,\mathrm{d}\boldsymbol{x},$$

we obtain that

$$Q^a(\varrho_\lambda) \longrightarrow -\infty, \qquad \text{as } \lambda \to \infty,$$

and

$$\frac{Q^a(\varrho_\lambda)}{\lambda} \longrightarrow \infty, \qquad \text{as } \lambda \to 0.$$

Thus, there exists $\lambda^* = \lambda^*(\varrho) > 0$ such that $Q^a(\varrho_{\lambda^*}) = 0$.

Now, for $\alpha \in [2, 3)$, calculating the derivative of $\lambda \mapsto Q^a(\rho_\lambda)$, we see that

$$\begin{aligned}\frac{\mathrm{d}Q^a(\varrho_\lambda)}{\mathrm{d}\lambda} &= \int_{\mathbb{R}^3} \frac{\varrho L(m_\varrho(r))}{r^2}\,\mathrm{d}\boldsymbol{x} + \frac{9a_0(\gamma-1)\lambda^{\frac{3\gamma-5}{2}}}{2}\int_{\mathbb{R}^3}\varrho^\gamma\,\mathrm{d}\boldsymbol{x} \\ &\qquad + \frac{\alpha^2\lambda^{\frac{\alpha-2}{2}}}{4}\int_{\mathbb{R}^3}\varrho\Phi_\alpha * \varrho\,\mathrm{d}\boldsymbol{x} \\ &= \frac{\alpha Q^a(\rho_\lambda)}{2\lambda} + \Big(\frac{2-\alpha}{2}\Big)\int_{\mathbb{R}^3}\frac{\varrho L(m_\varrho(r))}{r^2}\,\mathrm{d}\boldsymbol{x} \\ &\qquad + \frac{3a_0(3\gamma-3-\alpha)}{2}\lambda^{\frac{3\gamma-5}{2}}\int_{\mathbb{R}^3}\varrho^\gamma\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\mathrm{d}Q^a(\varrho_\lambda)}{\mathrm{d}\lambda}\bigg|_{\lambda=\lambda^*} &= \Big(\frac{2-\alpha}{2}\Big)\int_{\mathbb{R}^3}\frac{\varrho L(m_\varrho(r))}{r^2}\,\mathrm{d}\boldsymbol{x} + \frac{3a_0(3\gamma-3-\alpha)}{2}(\lambda^*)^{\frac{3(\gamma-1)-2}{2}}\int_{\mathbb{R}^3}\varrho^\gamma\,\mathrm{d}\boldsymbol{x} \\ &< 0.\end{aligned}$$

Thus, λ^* must be unique with $Q^a(\rho_\lambda) > 0$ for $0 < \lambda < \lambda^*$ and $Q^a(\rho_\lambda) < 0$ for $\lambda > \lambda^*$. Thus, (a), (b), and (c) follow. □

However, when $\alpha \in (0, 2)$, there may no longer be a unique scaling $\lambda^* > 0$ such that $Q^a(\varrho_{\lambda^*}) = 0$.

Lemma 4.5.2. *Suppose $\alpha \in (0, 2)$, $\frac{6}{6-\alpha} < \gamma < \frac{3+\alpha}{3}$, L satisfies (4.5) and $\varrho \in L^1_+(\mathbb{R}^3) \cap L^\gamma(\mathbb{R}^3)$, $\varrho \not\equiv 0$,*

$$\int_{\mathbb{R}^3} \frac{\varrho L(m_\varrho(r))}{r^2}\,\mathrm{d}\boldsymbol{x} < \infty,$$

with $Q^a(\varrho_\lambda) \le 0$ for some $\lambda > 0$, then the following statements are true:

(a) *There exists a $\lambda^*(\varrho) > 0$ such that $Q^a(\varrho_{\lambda^*(\varrho)}) = 0$. Moreover, there are at most two such $\lambda^*(\varrho)$, label these $0 < \lambda_1^*(\varrho) \le \lambda_2^*(\varrho) < \infty$.*

(b) *$\lambda_1^*(\varrho) < 1 < \lambda_2^*(\varrho) \iff Q^a(\varrho) < 0$.*

(c) $\lambda \mapsto S^a_{\mu,\lambda}(\varrho)$ *is strictly increasing (decreasing) for* $\lambda \le \lambda_1^*(\varrho)$ *and* $\lambda \ge \lambda_2^*(\varrho)$ $(\lambda_1^*(\varrho) \le \lambda \le \lambda_2^*(\varrho))$.

(d) $\lambda_1^*(\varrho) \le \kappa(\varrho) \le \lambda_2^*(\varrho)$ *with* $Q^a(\varrho_\kappa) \le 0$, *where*

$$\kappa = \kappa(\varrho) := \left(\frac{6a_0(5-3\gamma)\int_{\mathbb{R}^3} \varrho^\gamma \,\mathrm{d}\boldsymbol{x}}{\alpha(\alpha-2)\int_{\mathbb{R}^3} \varrho \Phi_\alpha * \varrho \,\mathrm{d}\boldsymbol{x}} \right)^{\frac{2}{3+\alpha-3\gamma}}, \tag{4.36}$$

with equality in either of the inequalities if and only if $\lambda_1^*(\varrho) = \lambda_2^*(\varrho)$.

Proof. We split the proof into two steps.

Step 1. Since $3(\gamma-1) < \alpha < 2$, we obtain that

$$Q^a(\varrho_\lambda) \longrightarrow \infty, \qquad \text{as } \lambda \to \infty,$$

and

$$\frac{Q^a(\varrho_\lambda)}{\lambda} \longrightarrow \infty, \qquad \text{as } \lambda \to 0.$$

Thus, since there exists $\lambda > 0$ such that $Q^a(\varrho_\lambda) \le 0$, by continuity of $\lambda \mapsto Q^a(\varrho_\lambda)$, there exists $\lambda^*(\varrho) > 0$ such that $Q^a(\varrho_{\lambda^*(\varrho)}) = 0$. Now, considering the derivative of $\lambda \mapsto Q^a(\varrho_\lambda)$, as before, we see that

$$\frac{\mathrm{d}Q^a(\varrho_\lambda)}{\mathrm{d}\lambda} = \frac{Q^a(\rho_\lambda)}{\lambda} + \frac{3a_0(3\gamma-5)}{2}\lambda^{\frac{3(\gamma-1)-2}{2}} \int_{\mathbb{R}^3} \varrho^\gamma \,\mathrm{d}\boldsymbol{x} + \frac{\alpha(\alpha-2)}{4}\lambda^{\frac{\alpha-2}{2}} \int_{\mathbb{R}^3} \varrho\Phi_\alpha * \varrho \,\mathrm{d}\boldsymbol{x}.$$

Thus, we obtain that

$$\left.\frac{\mathrm{d}Q^a(\varrho_\lambda)}{\mathrm{d}\lambda}\right|_{\lambda=\lambda^*} = \frac{3a_0(3\gamma-5)}{2}(\lambda^*)^{\frac{3(\gamma-1)-2}{2}} \int_{\mathbb{R}^3} \varrho^\gamma \,\mathrm{d}\boldsymbol{x} + \frac{\alpha(\alpha-2)}{4}(\lambda^*)^{\frac{\alpha-2}{2}} \int_{\mathbb{R}^3} \varrho\Phi_\alpha * \varrho \,\mathrm{d}\boldsymbol{x}.$$

Hence, we see that

$$\begin{cases} \left.\frac{\mathrm{d}Q^a(\varrho_\lambda)}{\mathrm{d}\lambda}\right|_{\lambda=\lambda^*} > 0, & \text{for } \lambda^* > \kappa, \\ \left.\frac{\mathrm{d}Q^a(\varrho_\lambda)}{\mathrm{d}\lambda}\right|_{\lambda=\lambda^*} = 0, & \text{for } \lambda^* = \kappa, \\ \left.\frac{\mathrm{d}Q^a(\varrho_\lambda)}{\mathrm{d}\lambda}\right|_{\lambda=\lambda^*} < 0, & \text{for } \lambda^* < \kappa, \end{cases} \tag{4.37}$$

where κ is defined in (4.36). Thus, there can be at most one $\lambda^* > 0$ in each of $\{\lambda > \kappa\}$, $\{\lambda = \kappa\}$ and $\{\lambda < \kappa\}$ such that $Q^a(\varrho_{\lambda^*}) = 0$. Suppose that all three sets contain a vanishing point of $\lambda \mapsto Q^a(\varrho_\lambda)$, label these $0 < \lambda_1^* < \lambda_2^* < \lambda_3^*$, where $\lambda_2^* = \kappa$. Then, due to (4.37), we see that $Q^a(\varrho_\lambda) \le 0$ for all $\lambda_1^* < \lambda < \lambda_3^*$. However, we see that

$$\left.\frac{\mathrm{d}^2Q^a(\varrho_\lambda)}{\mathrm{d}\lambda^2}\right|_{\lambda=\lambda_2^*} = \frac{3a_0(\lambda_2^*)^{\frac{3\gamma-7}{2}}}{2} \int_{\mathbb{R}^3} \varrho^\gamma \,\mathrm{d}\boldsymbol{x} > 0.$$

This implies by Taylor's Theorem that there exists $\lambda_2^* < \lambda < \lambda_3^*$ such that $Q^a(\varrho_\lambda) > 0$. A contradiction. Hence, (a), (b) and (c) follow.

Step 2. Suppose $Q^a(\varrho_{\kappa(\varrho)}) = 0$. Considering, the map $\lambda \mapsto \frac{Q^a(\varrho_\lambda)}{2\lambda} = \frac{\mathrm{d}S^a_{\mu,\lambda}(\varrho)}{\mathrm{d}\lambda}$, looking at the derivative, we see that

$$\frac{\mathrm{d}^2 S^a_{\mu,\lambda}(\varrho)}{\mathrm{d}\lambda^2} = \frac{3a_0(3\gamma-5)}{4}\lambda^{\frac{3\gamma-7}{2}}\int_{\mathbb{R}^3}\varrho^\gamma\,\mathrm{d}\boldsymbol{x} + \frac{\alpha(\alpha-2)}{8}\lambda^{\frac{\alpha-4}{2}}\int_{\mathbb{R}^3}\varrho\Phi_\alpha * \varrho\,\mathrm{d}\boldsymbol{x}. \tag{4.38}$$

Hence, $\lambda = \kappa(\varrho)$ is the only minimum of $\lambda \mapsto \frac{Q^a(\varrho_\lambda)}{2\lambda}$. Thus, we see that $Q^a(\varrho_\lambda) > 0$ for $\lambda \neq \kappa(\varrho)$. Hence, (d) follows.

□

Remark 4.5.3. In the proof of Lemma 4.5.2, it is important that we use the mass-preserving scaling $\varrho_\lambda(\boldsymbol{x}) = \lambda^{\frac{3}{2}}\varrho(\lambda^{\frac{1}{2}}\boldsymbol{x})$ rather than $\lambda\varrho(\lambda^{\frac{1}{3}}\boldsymbol{x})$, due to the disappearance of the kinetic term in the second derivative of $S^a_{\mu,\lambda}$, allowing us to characterise the behaviour of critical points $\lambda_1^*(\varrho)$ and $\lambda_2^*(\varrho)$.

Thus, for $\alpha \in (0,2)$, considering the behaviour of $S^a_{\mu,\lambda}(\varrho)$, we see that if there exists $\lambda > 0$ such that $Q^a(\varrho_\lambda) < 0$, then $S^a_{\mu,\lambda_1^*}(\varrho)$ is a local maximum of $\lambda \mapsto S^a_{\mu,\lambda}(\varrho)$, while $S^a_{\mu,\lambda_2^*}(\varrho)$ is a local minimum of $\lambda \mapsto S^a_{\mu,\lambda}(\varrho)$. Moreover, it is not clear whether $S^a_{\mu,\lambda}(\varrho)$ remains bounded below on the set of ϱ such that $\lambda_2^*(\varrho) = 1$. As in §3.7, we prove existence when $S^a_{\mu,\lambda^*}(\varrho)$ is a local maximum or inflection point of $\lambda \mapsto S^a_{\mu,\lambda}(\varrho)$. As such, for $\alpha \in (0,3)$ we look to minimise S^a_μ on the following admissible set

$$\mathcal{K}^a := \{\varrho \in L^1_+(\mathbb{R}^3) \cap L^\gamma_{\mathrm{axi}}(\mathbb{R}^3) \,|\, Q^a(\varrho) = 0,\, Q^a(\varrho_\lambda) > 0 \text{ for all } 0 < \lambda < 1,\, \varrho \not\equiv 0\}.$$

For $\alpha \in (0,2)$, the admissible set $\mathcal{K}^a$ is equivalent to $\lambda_1^*(\varrho) = 1$. For $\alpha \in (0,2]$, to show that this set is non-empty, we require that L is super-homogeneous of order $\omega > \omega_*$, where ω_* is given by (4.11), that is

$$L(am) \leq a^\omega L(m),$$

for all $0 < a < 1$ and $m \geq 0$.

Lemma 4.5.4. *Suppose $\alpha \in (0,3)$, $\frac{6}{6-\alpha} < \gamma < \frac{3+\alpha}{3}$, L satisfies (4.5) and, for $\alpha \in (0,2]$, (4.12) for $\omega > \omega^*$, then $\mathcal{K}^a \neq \emptyset$.*

Proof. Take $\varrho = \mathbb{1}_{\{1\leq r(\boldsymbol{x})\leq 2\}\times[-1,1]} \in L^1_+(\mathbb{R}^3) \cap L^\gamma(\mathbb{R}^3)$, we have that

$$\int_{\mathbb{R}^3} \frac{\varrho L(m_\varrho(r))}{r^2}\,\mathrm{d}\boldsymbol{x} < \infty.$$

Case 1 ($\alpha \in (2,3)$)**.** By Lemma 4.5.1, there exists $\lambda^* > 0$ such that $\varrho_{\lambda^*} \in \mathcal{K}^a$.

Case 2 ($\alpha \in (0,2)$)**.** By considering $\varrho_{\kappa(\varrho)}$, we may assume $\kappa(\varrho) = 1$. We employ the energy-critical scaling as defined in (3.14) for $n = 3$,

$$ {}^{\xi}\varrho(\boldsymbol{x}) = \xi^{3-\alpha}\rho\big(\xi^{2-\gamma}\boldsymbol{x}\big), $$

for $\xi > 0$. We consider how $Q^a({}^{\xi}\varrho)$ scales with $\xi > 0$. Applying a change of variables, one obtains that $\kappa({}^{\xi}\varrho) = \kappa(\varrho) = 1$ and further that

$$ \begin{aligned} 3a_0 \int_{\mathbb{R}^3} {}^{\xi}\varrho^{\gamma}\,\mathrm{d}\boldsymbol{x} + \frac{\alpha}{2}\int_{\mathbb{R}^3} {}^{\xi}\varrho\Phi_\alpha * {}^{\xi}\varrho\,\mathrm{d}\boldsymbol{x} &= \xi^{(6-\alpha)\gamma-6}\bigg(3a_0\int_{\mathbb{R}^3}\varrho^{\gamma}\,\mathrm{d}\boldsymbol{x} + \frac{\alpha}{2}\int_{\mathbb{R}^3}\varrho\Phi_\alpha * \varrho\,\mathrm{d}\boldsymbol{x}\bigg) \\ &= -3a_0\xi^{(6-\alpha)\gamma-6}\frac{3+\alpha-3\gamma}{2-\alpha}\int_{\mathbb{R}^3}\varrho^{\gamma}\,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{4.39} $$

Employing (4.12) and a change of variables, we see that for $\xi \geq 1$

$$ \begin{aligned} \int_{\mathbb{R}^3}\frac{{}^{\xi}\varrho L(m_{{}^{\xi}\varrho}(r))}{r^2}\,\mathrm{d}\boldsymbol{x} &= \xi^{\gamma+1-\alpha}\int_{\mathbb{R}^3}\frac{\varrho L\big(\xi^{3\gamma-3-\alpha}m_{\varrho}(r)\big)}{r^2}\,\mathrm{d}\boldsymbol{x} \\ &\leq \xi^{\gamma+1-\alpha-\omega(3+\alpha-3\gamma)}\int_{\mathbb{R}^3}\frac{\varrho L(m_{\varrho}(r))}{r^2}\,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{4.40} $$

Thus, combining, we see that for $\xi \geq 1$

$$ Q^a({}^{\xi}\varrho) \leq \xi^{\gamma+1-\alpha-\omega(3+\alpha-3\gamma)}\int_{\mathbb{R}^3}\frac{\varrho L(m_{\varrho}(r))}{r^2}\,\mathrm{d}\boldsymbol{x} - 3a_0\xi^{(6-\alpha)\gamma-6}\frac{3+\alpha-3\gamma}{2-\alpha}\int_{\mathbb{R}^3}\varrho^{\gamma}\,\mathrm{d}\boldsymbol{x}, \tag{4.41} $$

with equality for $\xi = 1$. Then, since $\omega > \omega_*$, we have that

$$ \gamma + 1 - \alpha - \omega(3+\alpha-3\gamma) < (6-\alpha)\gamma - 6. $$

Hence, from (4.41), we see that

$$ Q^a({}^{\xi}\varrho) \longrightarrow -\infty, \qquad \text{as } \xi \to \infty. $$

Thus, there exists $\xi > 0$ such that $Q^a({}^{\xi}\varrho) \leq 0$ and so by Lemma 4.5.2, there exists $\lambda_1^* = \lambda_1^*({}^{\xi}\varrho) > 0$, such that ${}^{\xi}\varrho_{\lambda_1^*} \in \mathcal{K}^a$.

Case 3 ($\alpha = 2$)**.** As in Case 2, we have that

$$ \begin{aligned} \int_{\mathbb{R}^3}\frac{{}^{\xi}\varrho L(m_{{}^{\xi}\varrho}(r))}{r^2}\,\mathrm{d}\boldsymbol{x} + \frac{\alpha}{2}\int_{\mathbb{R}^3}{}^{\xi}\varrho\Phi_\alpha * {}^{\xi}\varrho\,\mathrm{d}\boldsymbol{x} &\leq \xi^{\gamma+1-\alpha-\omega(3+\alpha-3\gamma)}\int_{\mathbb{R}^3}\frac{\varrho L(m_{\varrho}(r))}{r^2}\,\mathrm{d}\boldsymbol{x} \\ &\quad + \xi^{(6-\alpha)\gamma-6}\frac{\alpha}{2}\int_{\mathbb{R}^3}\varrho\Phi_\alpha * \varrho\,\mathrm{d}\boldsymbol{x}. \end{aligned} $$

Hence, we see that

$$ \int_{\mathbb{R}^3}\frac{{}^{\xi}\varrho L(m_{{}^{\xi}\varrho}(r))}{r^2}\,\mathrm{d}\boldsymbol{x} + \frac{\alpha}{2}\int_{\mathbb{R}^3}{}^{\xi}\varrho\Phi_\alpha * {}^{\xi}\varrho\,\mathrm{d}\boldsymbol{x} \longrightarrow -\infty, \qquad \text{as } \xi \to \infty. $$

Thus, there exists $\xi > 0$ such that

$$ \int_{\mathbb{R}^3}\frac{{}^{\xi}\varrho L(m_{{}^{\xi}\varrho}(r))}{r^2}\,\mathrm{d}\boldsymbol{x} < -\frac{\alpha}{2}\int_{\mathbb{R}^3}{}^{\xi}\varrho\Phi_\alpha * {}^{\xi}\varrho\,\mathrm{d}\boldsymbol{x}. $$

Hence by Lemma 4.5.1, there exists $\lambda^* = \lambda^*({}^{\xi}\varrho) > 0$, such that ${}^{\xi}\varrho_{\lambda^*} \in \mathcal{K}^a$.

□

Now that we have proven that the admissible set is non-empty for L satisfying (4.5) and (4.12) for $\omega > \omega_*$, we define

$$\ell_\mu^a \coloneqq \inf_{\varrho \in \mathcal{K}^a} S_\mu^a(\varrho),$$

the minimised free-energy S_μ^a over the admissible set $\mathcal{K}^a$. We show that $\ell_\mu^a \geq 0$.

Lemma 4.5.5. *Suppose $\alpha \in (0,3)$, $\frac{6}{6-\alpha} < \gamma < \frac{3+\alpha}{3}$, L satisfies (4.5) and (4.12) for $\omega > \omega_*$, then $\ell_\mu^a \geq 0$.*

Proof.

Case 1 ($\alpha \in [2,3)$)**.** Let $\varrho \in \mathcal{K}^a$, then

$$S_\mu^a(\varrho) = \frac{Q^a(\varrho)}{\alpha} + \frac{\alpha - 2}{2\alpha} \int_{\mathbb{R}^3} \frac{\varrho L(m_\varrho(r))}{r^2} \,\mathrm{d}\boldsymbol{x} + a_0 \frac{3 + \alpha - 3\gamma}{\alpha(\gamma - 1)} \int_{\mathbb{R}^3} \varrho^\gamma \,\mathrm{d}\boldsymbol{x} + \mu \int_{\mathbb{R}^3} \varrho \,\mathrm{d}\boldsymbol{x}. \tag{4.42}$$

Since $L \geq 0$, $\alpha \geq 2$ and $\gamma < \frac{3+\alpha}{3}$, we have that $S_\mu^a(\varrho) > 0$. Since this is true for all $\varrho \in \mathcal{K}^a$, the result follows.

Case 2 ($\alpha \in (0,2)$)**.** Let $\varrho \in \mathcal{K}^a$, then by Lemma 4.5.2 (d), $\kappa(\varrho) \geq \lambda_1^*(\varrho) = 1$. Thus, since $\gamma < \frac{3+\alpha}{3}$, we see that

$$\begin{aligned} S_\mu^a(\varrho) &= \frac{Q^a(\varrho)}{2} + \frac{2 - \alpha}{4} \int_{\mathbb{R}^3} \varrho \Phi_\alpha * \varrho \,\mathrm{d}\boldsymbol{x} + a_0 \frac{5 - 3\gamma}{2(\gamma - 1)} \int_{\mathbb{R}^3} \varrho^\gamma \,\mathrm{d}\boldsymbol{x} + \mu \int_{\mathbb{R}^3} \varrho \,\mathrm{d}\boldsymbol{x} \\ &= a_0 \frac{5 - 3\gamma}{2\alpha(\gamma - 1)} (\alpha - 3(\gamma - 1)\kappa(\varrho)^{3(\gamma-1)-\alpha}) \int_{\mathbb{R}^3} \varrho^\gamma \,\mathrm{d}\boldsymbol{x} + \mu \int_{\mathbb{R}^3} \varrho \,\mathrm{d}\boldsymbol{x} \\ &\geq a_0 \frac{5 - 3\gamma}{2\alpha(\gamma - 1)} (3 + \alpha - 3\gamma) \int_{\mathbb{R}^3} \varrho^\gamma \,\mathrm{d}\boldsymbol{x} + \mu \int_{\mathbb{R}^3} \varrho \,\mathrm{d}\boldsymbol{x} \\ &> 0. \end{aligned} \tag{4.43}$$

Thus, the result follows.

□

To prove strong convergence of the potential energy for minimising sequences in this section, we require the following lemma that characterises the axisymmetric representation of the potential similar to that of Lemma 2.5.1.

Lemma 4.5.6. *Let $\alpha \in (0,3)$. Given an axisymmetric function $f \in L^{\frac{6}{6-\alpha}}_{\mathrm{axi}}(\mathbb{R}^3)$ then $(\Phi_\alpha * f)(\boldsymbol{x})$ is well-defined, axisymmetric and can be written as*

$$(\Phi_\alpha * f)(r,z) = \int_{\mathbb{R}} \int_0^\infty K(r,\eta,z,w) f(\eta,w)\, \eta \,\mathrm{d}\eta \,\mathrm{d}w,$$

where

$$K(r,\eta,z,w) \coloneqq \int_{\mathbb{S}^1} \Phi_\alpha(r\boldsymbol{e_1} - \eta\boldsymbol{y}, z-w)\,\mathrm{d}S(\boldsymbol{y}),$$

and $\boldsymbol{e}_1 = (1,0)$. *Moreover, there exists* $C = C(\alpha) > 0$ *such that*

$$\begin{cases} |K(r,\eta,z,w)| \le C(r\eta)^{-\frac{\alpha}{2}}, & \text{for } \alpha \in (0,1), \\ |K(r,\eta,z,w)| \le C(r\eta)^{-\frac{1}{2}}|(r,z)-(\eta,w)|^{1-\alpha}, & \text{for } \alpha \in (1,3). \end{cases} \tag{4.44}$$

Proof. Since $f \in L^{\frac{6}{6-\alpha}}_{\mathrm{axi}}(\mathbb{R}^3)$, by the HLS inequality Lemma A.0.2, we have that $\Phi_\alpha * f$ is well-defined. Moreover, since f is axisymmetric, we can apply Fubini's Theorem and the co-area formula to obtain

$$\begin{aligned}(\Phi_\alpha * f)(\boldsymbol{v}) &= \int_{\mathbb{R}^3} \Phi_\alpha(\boldsymbol{v}-\boldsymbol{u}) f(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u} \\ &= \int_{\mathbb{R}}\int_0^\infty \int_{\{|(y_1,y_2)|=\eta\}} \Phi_\alpha(\boldsymbol{v}-(y_1,y_2,w))\,\mathrm{d}S(y_1,y_2) f(\eta,w)\,\mathrm{d}\eta\,\mathrm{d}w \\ &= \int_{\mathbb{R}}\int_0^\infty \int_{\mathbb{S}^1} \Phi_\alpha(\boldsymbol{v}-(\eta\boldsymbol{y},w))\,\mathrm{d}S(\boldsymbol{y}) f(\eta,w)\eta\,\mathrm{d}\eta\,\mathrm{d}w.\end{aligned}$$

Due to the rotational symmetry of Φ_α, we have that for $\boldsymbol{v} = (x_1,x_2,z)$

$$\int_{\mathbb{S}^1} \Phi_\alpha(\boldsymbol{v}-(\eta\boldsymbol{y},w))\,\mathrm{d}S(\boldsymbol{y}) = \int_{\mathbb{S}^1} \Phi_\alpha(r(\boldsymbol{v})\boldsymbol{e_1} - \eta\boldsymbol{y}, z-w)\,\mathrm{d}S(\boldsymbol{y}).$$

Thus, $\Phi_\alpha * f$ is axisymmetric and the representation formula follows. Using polar coordinates, we get

$$\begin{aligned}|K(r,\eta,z,w)| &= \left|\int_{\mathbb{S}^1} \Phi_\alpha(r\boldsymbol{e_1}-\eta\boldsymbol{y}, z-w)\,\mathrm{d}S(\boldsymbol{y})\right| \\ &= \frac{1}{\alpha}\int_0^{2\pi} \left((r-\eta\cos\theta)^2 + (\eta\sin\theta)^2 + (z-w)^2\right)^{-\frac{\alpha}{2}}\mathrm{d}\theta \\ &= \frac{2}{\alpha}\int_0^{\pi} \left((r-\eta)^2 + 2r\eta(1-\cos\theta) + (z-w)^2\right)^{-\frac{\alpha}{2}}\mathrm{d}\theta.\end{aligned}$$

For $\theta \in [0,\pi]$, we have that there exists $c > 0$ such that $2(1-\cos\theta) \ge c\theta^2$. Thus, we obtain that

$$\begin{aligned}|K(r,\eta,z,w)| &= \left|\int_{\mathbb{S}^1} \Phi_\alpha(r\boldsymbol{e_1}-\eta\boldsymbol{y}, z-w)\,\mathrm{d}S(\boldsymbol{y})\right| \\ &\le \frac{2}{\alpha}\int_0^{\pi} \left((r-\eta)^2 + cr\eta\theta^2 + (z-w)^2\right)^{-\frac{\alpha}{2}}\mathrm{d}\theta.\end{aligned}$$

Case 1 ($\alpha \in (0,1)$)**.** Then we obtain that

$$|K(r,\eta,z,w)| \le \frac{2}{\alpha}\int_0^{\pi} \left(cr\eta\theta^2\right)^{-\frac{\alpha}{2}}\mathrm{d}\theta \le C(r\eta)^{-\frac{\alpha}{2}}.$$

Case 2 ($\alpha \in (1,3)$)**.** Then, letting $\zeta > 0$, we obtain that

$$\begin{aligned}|K(r,\eta,z,w)| &\leq \frac{2}{\alpha}\Big(\int_0^{\zeta}\big((r-\eta)^2+(z-w)^2\big)^{-\frac{\alpha}{2}}\,\mathrm{d}\theta + \int_{\zeta}^{\infty}\big(cr\eta\theta^2\big)^{-\frac{\alpha}{2}}\,\mathrm{d}\theta\Big)\\ &\leq C\big(\zeta|(r,z)-(\eta,w)|^{-\alpha} + \zeta^{1-\alpha}(r\eta)^{-\frac{\alpha}{2}}\big).\end{aligned}$$

This upper bound is minimised when $\zeta \propto \frac{|r-\eta|}{\sqrt{r\eta}}$. Thus, we obtain that

$$|K(r,\eta,z,w)| \leq C(r\eta)^{-\frac{1}{2}}|(r,z)-(\eta,w)|^{1-\alpha}.$$

□

Now we can prove strong convergence of the potential energy term.

Lemma 4.5.7. *Suppose that $\alpha \in (0,3)$, $\frac{6}{6-\alpha} < \gamma < \frac{3}{3-\alpha}$, $\varrho_k \in L^1_+(\mathbb{R}^3)\cap L^{\gamma}_{\mathrm{axi}}(\mathbb{R}^3)$ is a bounded sequence, $z \mapsto \varrho_k(r,z)$ is a radially decreasing function for a.e. r and there exists $\tilde{\varrho} \in L^1_+(\mathbb{R}^3)\cap L^{\gamma}_{\mathrm{axi}}(\mathbb{R}^3)$ such that $\varrho_k \rightharpoonup \tilde{\varrho}$ as $k\to\infty$ in $L^{\gamma}(\mathbb{R}^3)$. Then, we have (up to a subsequence)*

$$\int_{\mathbb{R}^3}\varrho_k\Phi_\alpha * \varrho_k\,\mathrm{d}\boldsymbol{x} \longrightarrow \int_{\mathbb{R}^3}\tilde{\varrho}\Phi_\alpha * \tilde{\varrho}\,\mathrm{d}\boldsymbol{x}, \qquad \text{as } k\to\infty.$$

Proof. We split the proof into six steps.

Step 1. We define the cylinder $U(R) \coloneqq B_R\times[-R,R]$, where $B_R = \{\boldsymbol{x}\in\mathbb{R}^2 \,|\, |\boldsymbol{x}|<R\}$ and $R\in\mathbb{N}$. As in Step 1 of the proof of Lemma 3.7.3, we can apply the theory of compact embeddings of Sobolev spaces and fractional Sobolev spaces to obtain for $R, R'\in\mathbb{N}$, that

$$\int_{U(R')}\varrho_k\Phi_\alpha * (\varrho_k\mathbb{1}_{U(R)})\,\mathrm{d}\boldsymbol{x} \longrightarrow \int_{U(R')}\tilde{\varrho}\Phi_\alpha * (\tilde{\varrho}\mathbb{1}_{U(R)})\,\mathrm{d}\boldsymbol{x}, \qquad \text{as } k\to\infty. \tag{4.45}$$

Step 2. Taking $R' > R \geq 1$, we have that since ϱ_k are bounded in $L^1(\mathbb{R}^3)$, that

$$\begin{aligned}\Big|\int_{U(R')^c}\varrho_k\Phi_\alpha * (\varrho_k\mathbb{1}_{U(R)})\,\mathrm{d}\boldsymbol{x}\Big| &\leq \frac{C}{(R'-R)^{\alpha}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|\varrho_k(\boldsymbol{x})||\varrho_k(\boldsymbol{y})|\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\\ &\leq \frac{C}{(R'-R)^{\alpha}}.\end{aligned} \tag{4.46}$$

Thus, taking R' sufficiently larger than R, combining (4.45) and (4.46), we obtain

$$\int_{\mathbb{R}^3}\varrho_k\Phi_\alpha * (\varrho_k\mathbb{1}_{U(R)})\,\mathrm{d}\boldsymbol{x} \longrightarrow \int_{\mathbb{R}^3}\tilde{\varrho}\Phi_\alpha * (\tilde{\varrho}\mathbb{1}_{U(R)})\,\mathrm{d}\boldsymbol{x}, \qquad \text{as } k\to\infty.$$

Step 3. By symmetry, we can apply the previous steps so we have that

$$\int_{U(R)}\varrho_k\Phi_\alpha * (\varrho_k\mathbb{1}_{U(R)^c})\,\mathrm{d}\boldsymbol{x} \longrightarrow \int_{U(R)}\tilde{\varrho}\Phi_\alpha * (\tilde{\varrho}\mathbb{1}_{U(R)^c})\,\mathrm{d}\boldsymbol{x}, \qquad \text{as } k\to\infty.$$

Step 4. Now, we need only show that for R large enough,

$$\int_{U(R)^c} \varrho_k \Phi_\alpha * (\varrho_k \mathbb{1}_{U(R)^c}) \, \mathrm{d}\boldsymbol{x},$$

is sufficiently small for all k. Now, we have that

$$\begin{aligned}\int_{U(R)^c} \varrho_k \Phi_\alpha * (\varrho_k \mathbb{1}_{U(R)^c}) \, \mathrm{d}\boldsymbol{x} &= \int_{B_R^c \times \mathbb{R}} \varrho_k \Phi_\alpha * (\varrho_k \mathbb{1}_{B_R^c \times \mathbb{R}}) \, \mathrm{d}\boldsymbol{x} \\ &\quad + 2 \int_{B_R^c \times \mathbb{R}} \varrho_k \Phi_\alpha * (\varrho_k \mathbb{1}_{B_R \times [-R,R]^c}) \, \mathrm{d}\boldsymbol{x} \\ &\quad + \int_{B_R \times [-R,R]^c} \varrho_k \Phi_\alpha * (\varrho_k \mathbb{1}_{B_R \times [-R,R]^c}) \, \mathrm{d}\boldsymbol{x} \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

Considering Lemma 4.5.6, we have that I_1 can be written as

$$I_1 = 2\pi \int_{\mathbb{R}} \int_R^\infty \varrho_k(r,z)\, r \int_{\mathbb{R}} \int_R^\infty K(r,\eta,z,w) \varrho_k(\eta,w)\, \eta \, \mathrm{d}\eta \, \mathrm{d}w \, \mathrm{d}r \, \mathrm{d}z. \tag{4.47}$$

We have the following cases.

Case 1 ($\alpha \in (0,1)$)**.** From (4.44), (4.47) and that ϱ_k are bounded in $L^1(\mathbb{R}^3)$, we see that

$$|I_1| \leq C R^{-\alpha} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\varrho_k(\boldsymbol{x})||\varrho_k(\boldsymbol{y})| \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{y} \leq C R^{-\alpha}. \tag{4.48}$$

Case 2 ($\alpha \in (1,3)$)**.** We have that $1 < \frac{4}{5-\alpha} < \frac{6}{6-\alpha} < \gamma$. Since ϱ_k are bounded in $L^1(\mathbb{R}^3) \cap L^\gamma(\mathbb{R}^3)$, we have that by interpolation, ϱ_k are bounded in $L^{\frac{4}{5-\alpha}}(\mathbb{R}^3)$. From (4.44), (4.47), Hölder's inequality and the HLS inequality Lemma A.0.2, we see that

$$\begin{aligned}|I_1| &\leq C R^{-\frac{3-\alpha}{2}} \int_{\mathbb{R}} \int_R^\infty \int_{\mathbb{R}} \int_R^\infty |\varrho_k(r,z)| \, |(r,z)-(\eta,w)|^{1-\alpha} |\varrho_k(\eta,w)| \\ &\qquad \times r^{\frac{5-\alpha}{4}} \eta^{\frac{5-\alpha}{4}} \, \mathrm{d}\eta \, \mathrm{d}w \, \mathrm{d}r \, \mathrm{d}z \\ &\leq C R^{-\frac{3-\alpha}{2}} \Big(\int_{\mathbb{R}} \int_R^\infty |\varrho_k(r,z)|^{\frac{4}{5-\alpha}} \, r \, \mathrm{d}r \, \mathrm{d}z \Big)^{\frac{5-\alpha}{2}} \\ &\leq C R^{-\frac{3-\alpha}{2}} \|\varrho_k\|^2_{L^{\frac{4}{5-\alpha}}(\mathbb{R}^3)} \\ &\leq C R^{-\frac{3-\alpha}{2}}.\end{aligned} \tag{4.49}$$

Case 3 ($\alpha = 1$)**.** By the Cauchy-Schwarz inequality, we see that for any $0 < \delta < 1$, that

$$\Phi_1 \leq C(\Phi_{1+\delta} + \Phi_{1-\delta}).$$

Thus, taking δ small enough so that the HLS exponent corresponding to $\Phi_{1+\delta}$ satisfies $\frac{4}{4-\delta} < \frac{6}{5} < \gamma$. Thus, we can apply Case 1 and Case 2 to obtain that

$$|I_1| \leq C\big(R^{\delta-1} + R^{\frac{\delta}{2}-1}\big). \tag{4.50}$$

Combining (4.48), (4.49) and (4.50), we see that $I_1 \to 0$ as $R \to \infty$.

Step 5. We now consider I_3. Since ϱ_k are bounded in $L^1(\mathbb{R}^3) \cap L^\gamma(\mathbb{R}^3)$, we have that by interpolation that there exists $\bar{C} > 0$ such that $\|\varrho_k\|^p_{L^p(\mathbb{R}^3)} \leq 2\bar{C}$ for all $1 \leq p \leq \gamma$ and k. We define $V(R,i) \coloneqq B_R \times [i,(i+1)]$ for $R, i \in \mathbb{N}$. Since $z \mapsto \varrho_k(r,z)$ is radially decreasing for *a.e.* r, we have that

$$\|\varrho_k\|^p_{L^p(V(R,i))} \leq \frac{\bar{C}}{i+1},$$

for all $1 \leq p \leq \gamma$, $i \in \mathbb{N}$ and k. Thus, since $z \mapsto \varrho_k(r,z)$ is a radial function for *a.e.* r, we have that

$$\begin{aligned} |I_3| &\leq 4 \int_{\bigcup_{i=R}^\infty V(R,i)} \int_{\bigcup_{j=R}^\infty V(R,j)} |\varrho_k(\boldsymbol{x})||\Phi_\alpha(\boldsymbol{x}-\boldsymbol{y})||\varrho_k(\boldsymbol{y})| \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \\ &\leq 8 \sum_{i=R}^\infty \sum_{j=i}^\infty \int_{V(R,i)} \int_{V(R,j)} |\varrho_k(\boldsymbol{x})||\Phi_\alpha(\boldsymbol{x}-\boldsymbol{y})||\varrho_k(\boldsymbol{y})| \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}. \end{aligned} \tag{4.51}$$

For $j = i, i+1$, we have that by the HLS inequality Lemma A.0.2,

$$\begin{aligned} \int_{V(R,i)} \int_{V(R,j)} |\varrho_k(\boldsymbol{x})||\Phi_\alpha(\boldsymbol{x}-\boldsymbol{y})||\varrho_k(\boldsymbol{y})| \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} &\leq C\|\varrho_k\|_{L^{\frac{6}{6-\alpha}}(V(R,i))}\|\varrho_k\|_{L^{\frac{6}{6-\alpha}}(V(R,j))} \\ &\leq \frac{C}{(i+1)^{\frac{6-\alpha}{3}}}. \end{aligned} \tag{4.52}$$

For $j \geq i+2$, using the form of Φ_α, we see that

$$\begin{aligned} \int_{V(R,i)} \int_{V(R,j)} |\varrho_k(\boldsymbol{x})||\Phi_\alpha(\boldsymbol{x}-\boldsymbol{y})||\varrho_k(\boldsymbol{y})| \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} &\leq C(j-i-1)^{-\alpha}\|\varrho_k\|_{L^1(V(R,i))}\|\varrho_k\|_{L^1(V(R,j))} \\ &\leq \frac{C(j-i-1)^{-\alpha}}{(i+1)(j+1)}. \end{aligned} \tag{4.53}$$

Here, the last inequality will be used in order to obtain convergence of the sum in (4.51).

Case 1 ($\alpha \in (0,1]$)**.** We have that for $j \geq i+2$, that

$$\frac{(j-i-1)^{-\alpha}}{(i+1)(j+1)} \leq \frac{(j-i-1)^{-\frac{2+\alpha}{2}}}{(i+1)^{\frac{2+\alpha}{2}}}.$$

Combining this with (4.51), (4.52), (4.53), we see that

$$|I_3| \leq C\Big(1 + \sum_{j=1}^\infty j^{-\frac{2+\alpha}{2}}\Big) \sum_{i=R}^\infty (i+1)^{-\frac{2+\alpha}{2}}. \tag{4.54}$$

Since $\frac{2+\alpha}{2} > 1$, the series converge.

Case 2 ($\alpha \in (1,3)$)**.** Combining (4.51), (4.52), (4.53), we see that

$$|I_3| \leq C\Big(1 + \sum_{j=1}^{\infty} j^{-\alpha}\Big) \sum_{i=R}^{\infty} (i+1)^{-\frac{6-\alpha}{3}}. \tag{4.55}$$

Since $\alpha \in (1,3)$, the series converge.

Combining (4.54) and (4.55), we obtain that $I_3 \to 0$ as $R \to \infty$.

Step 6. We split I_2 into two parts

$$\begin{aligned} I_2 &= 2\int_{(B_{2R}\setminus B_R)\times\mathbb{R}} \varrho_k \Phi_\alpha * (\varrho_k \mathbb{1}_{B_R\times[-R,R]^c})\,\mathrm{d}\boldsymbol{x} \\ &\quad + 2\int_{B_{2R}^c\times\mathbb{R}} \varrho_k \Phi_\alpha * (\varrho_k \mathbb{1}_{B_R\times[-R,R]^c})\,\mathrm{d}\boldsymbol{x} \\ &=: I_{2,1} + I_{2,2}. \end{aligned}$$

By the same approach as in Step 5, one can show that $I_{2,1} \to 0$ as $R \to \infty$. For the second integral, we can see that

$$\operatorname{dist}(B_R \times [-R,R]^c, B_{2R}^c \times \mathbb{R}) = R,$$

Hence, we have from the form of Φ_α and the fact that ϱ_k are bounded in $L^1(\mathbb{R}^3)\cap L^\gamma(\mathbb{R}^3)$

$$|I_{2,2}| \leq CR^{-\alpha} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |\varrho_k(\boldsymbol{x})||\varrho_k(\boldsymbol{y})|\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \leq CR^{-\alpha}.$$

Hence $I_{2,2} \to 0$ as $R \to \infty$. This concludes the proof.

□

Now we can prove existence of minimisers of the free-energy functional S_μ^a over $\mathcal{K}^a$.

Lemma 4.5.8. *Suppose $\alpha \in (0,3)$, $\frac{6}{6-\alpha} < \gamma < \frac{3+\alpha}{3}$, L satisfies (4.5), (4.13) and, for $\alpha \in (0,2]$, (4.12) for $\omega > \omega_*$, then there exists a minimiser $\bar\rho$ of S_μ^a over $\mathcal{K}^a$ and $\ell_\mu^a > 0$. Moreover, for any minimiser $\bar\rho$, there exists a translation $\boldsymbol{y} \in \{\mathbf{0}\} \times \mathbb{R}$ such that $z \mapsto T^{\boldsymbol{y}}\bar\rho(r,z)$ is a radially decreasing function for a.e r.*

Proof. We split the proof into three steps.

Step 1. Suppose that $\varrho_k \in \mathcal{K}^a$ is a minimising sequence of S_μ^a, i.e. that

$$Q^a(\varrho_k) = 0,$$

for all $0 < \lambda < 1$ we have that

$$Q^a((\varrho_k)_\lambda) > 0,$$

and

$$\lim_{k\to\infty} S_\mu^a(\varrho_k) = \ell_\mu^a.$$

We would like to show that we can take a minimising sequence that is radially decreasing in the z–coordinate. To do this, we take the radially decreasing rearrangement in Lemma A.0.5 in the z–coordinate. That is, define $\tilde{\varrho}_k$ by $\tilde{\varrho}_k(r,z) = (\varrho_k(r,\cdot))^{\#}(z)$. Note that since the radially decreasing rearrangement preserves L^p norms, we have that

$$\int_{\mathbb{R}^3} \tilde{\varrho}_k \,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}} \tilde{\varrho}_k(\boldsymbol{y},z)\,\mathrm{d}z\,\mathrm{d}\boldsymbol{y} = \int_{\mathbb{R}^2}\int_{\mathbb{R}} \varrho_k(\boldsymbol{y},z)\,\mathrm{d}z\,\mathrm{d}\boldsymbol{y} = \int_{\mathbb{R}^3} \varrho_k \,\mathrm{d}\boldsymbol{x}.$$

Similarly, we see that $\|\tilde{\varrho}_k\|_{L^\gamma(\mathbb{R}^3)} = \|\varrho_k\|_{L^\gamma(\mathbb{R}^3)}$ and

$$\int_{\mathbb{R}^3} \frac{\tilde{\varrho}_k L(m_{\tilde{\varrho}_k}(r))}{r^2}\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^3} \frac{\varrho_k L(m_{\varrho_k}(r))}{r^2}\,\mathrm{d}\boldsymbol{x}.$$

By Lemma (A.0.5), we see that since $z \mapsto -\Phi_\alpha(\boldsymbol{x},z)$ is a radially decreasing function for $z \in \mathbb{R}$ and any $\boldsymbol{x} \in \mathbb{R}^2$, we obtain that

$$\begin{aligned}
\int_{\mathbb{R}^3} \tilde{\varrho}_k \Phi_\alpha * \tilde{\varrho}_k \,\mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \tilde{\varrho}_k(\boldsymbol{x},z)\Phi_\alpha(\boldsymbol{x}-\boldsymbol{y},z-w)\tilde{\varrho}_k(\boldsymbol{y},w)\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \\
&\leq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \varrho_k(\boldsymbol{x},z)\Phi_\alpha(\boldsymbol{x}-\boldsymbol{y},z-w)\varrho_k(\boldsymbol{y},w)\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \\
&= \int_{\mathbb{R}^3} \varrho_k \Phi_\alpha * \varrho_k \,\mathrm{d}\boldsymbol{x}.
\end{aligned} \tag{4.56}$$

Hence, we have that $\tilde{\varrho}_k \in L^1_+(\mathbb{R}^3) \cap L^\gamma(\mathbb{R}^3)$ and satisfies $Q^a(\tilde{\varrho}_k) \leq 0$. Thus, by Lemma 4.5.1 or 4.5.2, there exists $0 < \lambda^*(\tilde{\varrho}_k) \leq 1$ such that $(\tilde{\varrho}_k)_{\lambda^*(\tilde{\varrho}_k)} \in \mathcal{K}^a$. Moreover, it is easy to show that $(\tilde{\varrho}_k)_{\lambda^*(\tilde{\varrho}_k)}(r,z) = ((\varrho_k)_{\lambda^*(\tilde{\varrho}_k)}(r,\cdot))^{\#}(z)$. Hence, by Lemma 4.5.1 (c) or Lemma 4.5.2 (c) we see that

$$S^a_\mu((\tilde{\varrho}_k)_{\lambda^*(\tilde{\varrho}_k)}) \leq S^a_\mu((\varrho_k)_{\lambda^*(\tilde{\varrho}_k)}) \leq S^a_\mu(\varrho_k).$$

This implies that $(\tilde{\varrho}_k)_{\lambda^*(\tilde{\varrho}_k)}$ is a minimising sequence. Hence, we can assume that ϱ_k are radially decreasing in the z–coordinate. Using (4.42) and (4.43), we see that

$$\|\varrho_k\|_{L^\gamma} + \|\varrho_k\|_{L^1} \leq C. \tag{4.57}$$

We have that since $\varrho_k \in \mathcal{K}^a$, that

$$\|\varrho_k\|^\gamma_{L^\gamma} \leq -\frac{\alpha}{6a_0}\int_{\mathbb{R}^3} \varrho_k \Phi_\alpha * \varrho_k \,\mathrm{d}\boldsymbol{x} \leq C\|\varrho_k\|_{L^1}^{\frac{\gamma(6-\alpha)-6}{3(\gamma-1)}}\|\varrho_k\|_{L^\gamma}^{\frac{\gamma\alpha}{3(\gamma-1)}}. \tag{4.58}$$

Thus, combining this with (4.57), since $\frac{6}{6-\alpha} < \gamma < \frac{3+\alpha}{3}$, we get

$$\|\varrho_k\|_{L^\gamma}^{\frac{\gamma(3\gamma-3-\alpha)}{3(\gamma-1)}} \leq C\|\varrho_k\|_{L^1}^{\frac{\gamma(6-\alpha)-6}{3(\gamma-1)}} \leq C,$$

and thus

$$\|\varrho_k\|_{L^\gamma} \geq C. \tag{4.59}$$

Thus, using (4.42) and (4.43), we see that $\ell^a_\mu > 0$. By Lemma 3.7.2, we have that (4.57) implies there exists $\tilde{\varrho} \in L^\gamma(\mathbb{R}^3)$ such that (up to a subsequence) $\varrho_k \rightharpoonup \tilde{\varrho}$ as $k \to \infty$ in $L^\gamma(\mathbb{R}^3)$.

Step 2. However, by weak lower semi-continuity of the L^γ norm, Lemma 4.5.7 and verbatim the same approach as that of the proof of Lemma 4.3.7, we obtain that $Q^a(\tilde{\varrho}) \leq 0$. As such, there exists $0 < \lambda^*(\tilde{\varrho}) \leq 1$ such that $\bar{\rho} := \tilde{\varrho}_{\lambda^*(\tilde{\varrho})} \in \mathcal{K}^a$. Then, we have that (up to a subsequence) $\varrho_k^* := (\varrho_k)_{\lambda^*(\tilde{\varrho})} \rightharpoonup \bar{\rho}$ as $k \to \infty$ in $L^\gamma(\mathbb{R}^3)$. From Lemma 3.7.2, we have that $\bar{\varrho} \geq 0$ *a.e.* and

$$\|\bar{\rho}\|_{L^\gamma} + \|\bar{\rho}\|_{L^1} \leq C.$$

Since $\frac{3+\alpha}{3} < \frac{3}{3-\alpha}$, Lemma 4.5.7 tells us that (up to a subsequence)

$$\int_{\mathbb{R}^3} \varrho_k^* \Phi_\alpha * \varrho_k^* \,\mathrm{d}\boldsymbol{x} \longrightarrow \int_{\mathbb{R}^3} \bar{\rho}\Phi_\alpha * \bar{\rho} \,\mathrm{d}\boldsymbol{x}, \qquad \text{as } k \to \infty. \tag{4.60}$$

Combining (4.58) and (4.59), we have that

$$-\int_{\mathbb{R}^3} \varrho_k^* \Phi_\alpha * \varrho_k^* \,\mathrm{d}\boldsymbol{x} \geq C.$$

Combining with (4.60), we have that

$$-\int_{\mathbb{R}^3} \bar{\rho}\Phi_\alpha * \bar{\rho} \,\mathrm{d}\boldsymbol{x} \geq C.$$

Hence, we see that $\bar{\rho} \not\equiv 0$. Now, by Lemma 3.7.2 and weak lower semi-continuity of $\int_{\mathbb{R}^3} \varrho^\gamma \,\mathrm{d}\boldsymbol{x}$, we get that

$$\int_{\mathbb{R}^3} \bar{\rho} \,\mathrm{d}\boldsymbol{x} \leq \liminf_{k\to\infty} \int_{\mathbb{R}^3} \varrho_k^* \,\mathrm{d}\boldsymbol{x}, \qquad \text{and} \qquad \int_{\mathbb{R}^3} (\bar{\rho})^\gamma \,\mathrm{d}\boldsymbol{x} \leq \liminf_{k\to\infty} \int_{\mathbb{R}^3} (\varrho_k^*)^\gamma \,\mathrm{d}\boldsymbol{x}.$$

By verbatim the same approach as that of the proof of Lemma 4.3.7, since L satisfies (4.5), (4.13), we have that

$$\int_{\mathbb{R}^3} \frac{\bar{\rho} L(m_{\bar{\rho}}(r))}{r^2} \,\mathrm{d}\boldsymbol{x} \leq \liminf_{k\to\infty} \int_{\mathbb{R}^3} \frac{\varrho_k^* L(m_{\varrho_k^*}(r))}{r^2} \,\mathrm{d}\boldsymbol{x}.$$

Hence, using Lemma 4.5.1 or 4.5.2, we have that

$$S_\mu^a(\bar{\rho}) \leq \liminf_{k\to\infty} S_\mu^a(\varrho_k^*) \leq \liminf_{k\to\infty} S_\mu^a(\varrho_k) = \ell_\mu^a.$$

Thus $\bar{\rho}$ is a minimiser.

Step 3. Employing Lemma A.0.5, for the radially decreasing rearrangement in the z–coordinate of $\bar{\rho}$, given by $\rho^*(r,z) = (\bar{\rho}(r,\cdot))^\#(z)$, as in (4.56), we have equality in

$$\begin{aligned}
\int_{\mathbb{R}^3} \rho^*\Phi_\alpha * \rho^* \,\mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \rho^*(\boldsymbol{x},z)\Phi_\alpha(\boldsymbol{x}-\boldsymbol{y},z-w)\rho^*(\boldsymbol{y},w)\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \\
&\leq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \bar{\rho}(\boldsymbol{x},z)\Phi_\alpha(\boldsymbol{x}-\boldsymbol{y},z-w)\bar{\rho}(\boldsymbol{y},w)\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \\
&= \int_{\mathbb{R}^3} \bar{\rho}\Phi_\alpha * \bar{\rho} \,\mathrm{d}\boldsymbol{x},
\end{aligned} \tag{4.61}$$

if and only if there is a translation $\boldsymbol{y} \in \{\mathbf{0}\} \times \mathbb{R}$ such that $z \mapsto T^{\boldsymbol{y}}\bar{\rho}(r,z)$ is a radially decreasing function for *a.e.* r. If there is no equality in (4.61), then as in Step 1, we can obtain a function $\bar{\rho}^* \in \mathcal{K}^a$ such that $S^a_\mu(\bar{\rho}^*) < S^a_\mu(\bar{\rho})$, a contradiction, thus we conclude the lemma.

□

For $\alpha \in (0,2)$, in order to prove that minimisers of S^a_μ over $\mathcal{K}^a$ are indeed rotating Riesz star solutions and that such steady states are unstable, we require slightly stronger assumptions on the super-homogeneity of L. We define

$$\bar{\omega} := \frac{(2-\alpha)((6-\alpha)\gamma - 6)(5-3\gamma)}{3\alpha(\gamma-1)(3+\alpha-3\gamma)} > 0.$$

We will require that $\omega > \omega_* + \bar{\omega}$.

Lemma 4.5.9. *Suppose $\alpha \in (0,2)$, $\frac{6}{6-\alpha} < \gamma < \frac{3+\alpha}{3}$, L satisfies (4.5) and (4.12) for $\omega > \omega_* + \bar{\omega}$. Then, there exists $\bar{Q} \in (0,\infty]$ such that for all $\varrho \in L^1_+(\mathbb{R}^3) \cap L^\gamma(\mathbb{R}^3)$ satisfying*

$$0 < Q^a(\varrho) < \bar{Q} \int_{\mathbb{R}^3} \varrho^\gamma \, \mathrm{d}\boldsymbol{x},$$

and $\kappa(\varrho) = 1$, we have that $S^a_\mu(\varrho) > \ell^a_\mu$. Moreover, if (4.12) is satisfied for

$$\omega \geq \frac{\gamma + 1 - \alpha}{3 + \alpha - 3\gamma}, \tag{4.62}$$

then $\bar{Q} = \infty$.

Remark 4.5.10. Note that the condition (4.62) is required so that

$$\xi \mapsto \int_{\mathbb{R}^3} \frac{{}^\xi\varrho L(m_{{}^\xi\varrho}(r))}{r^2} \, \mathrm{d}\boldsymbol{x},$$

is non-increasing in ξ. This is only a stronger assumption than $\omega > \omega_* + \bar{\omega}$ in the case that $\alpha \in (1,2)$ and $\frac{5-\alpha}{3} < \gamma < \frac{3+\alpha}{3}$, as this is the only range under which

$$\frac{\gamma + 1 - \alpha}{3 + \alpha - 3\gamma} > \omega_* + \bar{\omega}.$$

Proof. Take $\varrho \in L^1_+(\mathbb{R}^3) \cap L^\gamma(\mathbb{R}^3)$ such that $Q^a(\varrho) > 0$ and $\kappa(\varrho) = 1$. Employing (4.41), as in Lemma 4.5.4, we obtain that $\xi \mapsto Q^a({}^\xi\varrho)$ is strictly decreasing and thus there exists $\bar{\xi} > 1$ such that $Q^a({}^{\bar{\xi}}\varrho) = 0$. Since $\kappa({}^\xi\varrho) = \kappa(\varrho) = 1$, by Lemma 4.5.2 (c), we see that $\lambda_1^*({}^{\bar{\xi}}\varrho) = 1$. Hence, by the definition of ℓ^a_μ, we obtain $S^a_\mu({}^{\bar{\xi}}\varrho) \geq \ell^a_\mu$. We aim to show that $\xi \mapsto S^a_\mu({}^\xi\varrho)$ is strictly decreasing. Employing estimates similar to (4.39) and (4.40), we obtain that

$$S^a_\mu({}^\xi\varrho) \leq \xi^{\gamma+1-\alpha-\omega(3+\alpha-3\gamma)} \frac{1}{2} \int_{\mathbb{R}^3} \frac{\varrho L(m_\varrho(r))}{r^2} \, \mathrm{d}\boldsymbol{x}$$

$$
\begin{aligned}
&+ \xi^{(6-\alpha)\gamma-6}\frac{a_0(5-\alpha-3\gamma)(\alpha+3-3\gamma)}{\alpha(2-\alpha)(\gamma-1)}\int_{\mathbb{R}^3}\varrho^\gamma\,\mathrm{d}\boldsymbol{x} + \xi^{3\gamma-3-\alpha}\mu\int_{\mathbb{R}^3}\varrho\,\mathrm{d}\boldsymbol{x}\\
&=: T_\mu(\varrho,\xi) + \xi^{3\gamma-3-\alpha}\mu\int_{\mathbb{R}^3}\varrho\,\mathrm{d}\boldsymbol{x}, \qquad (4.63)
\end{aligned}
$$

with equality for $\xi = 1$. Since $\gamma < \frac{3+\alpha}{3}$, we have that

$$
\xi \mapsto \xi^{3\gamma-3-\alpha}\mu\int_{\mathbb{R}^3}\varrho\,\mathrm{d}\boldsymbol{x},
$$

is strictly decreasing. Thus, from (4.63), $\xi \mapsto S^a_\mu({}^\xi\varrho)$ is strictly decreasing if $\xi \mapsto T_\mu(\varrho,\xi)$ is strictly decreasing. For $\omega > \omega_*$, we have that for $\xi \geq 1$,

$$
\begin{aligned}
\frac{\mathrm{d}T_\mu(\varrho,\xi)}{\mathrm{d}\xi} &= \xi^{3\gamma-3-\alpha}\Bigg(\xi^{7-\alpha-(5-\alpha)\gamma-\omega(3+\alpha-3\gamma)}\frac{\gamma+1-\alpha-\omega(3+\alpha-3\gamma)}{2}\int_{\mathbb{R}^3}\frac{\varrho L(m_\varrho(r))}{r^2}\,\mathrm{d}\boldsymbol{x}\\
&\quad + \frac{a_0(5-\alpha-3\gamma)(\alpha+3-3\gamma)((6-\alpha)\gamma-6)}{\alpha(2-\alpha)(\gamma-1)}\int_{\mathbb{R}^3}\varrho^\gamma\,\mathrm{d}\boldsymbol{x}\Bigg)\\
&\leq \xi^{3\gamma-3-\alpha}\frac{\mathrm{d}T_\mu(\varrho,\xi)}{\mathrm{d}\xi}\bigg|_{\xi=1}.
\end{aligned}
$$

Thus, we just need to show that $\frac{\mathrm{d}T_\mu(\varrho,\xi)}{\mathrm{d}\xi}\big|_{\xi=1} < 0$. Employing (4.41) for $\xi = 1$ and some tedious algebra, we obtain that

$$
\begin{aligned}
\frac{\mathrm{d}T_\mu(\varrho,\xi)}{\mathrm{d}\xi}\bigg|_{\xi=1} &= \frac{\gamma+1-\alpha-\omega(3+\alpha-3\gamma)}{2}Q^a(\varrho)\\
&\quad + (\omega_* + \bar\omega - \omega)\frac{3a_0(\alpha+3-3\gamma)^2}{2(2-\alpha)}\int_{\mathbb{R}^3}\varrho^\gamma\,\mathrm{d}\boldsymbol{x}. \qquad (4.64)
\end{aligned}
$$

Thus, if $\omega > \omega_* + \bar\omega$, then there exists such a $\bar Q$ such that $\frac{\mathrm{d}T_\mu(\varrho,\xi)}{\mathrm{d}\xi}\big|_{\xi=1} < 0$ if $Q^a(\varrho) < \bar Q\int_{\mathbb{R}^3}\varrho^\gamma\,\mathrm{d}\boldsymbol{x}$. Moreover, if ω also satisfies $\omega \geq \frac{\gamma+1-\alpha}{3+\alpha-3\gamma}$. Then the first term on the right-hand side of (4.64) is negative, hence $\bar Q = \infty$. □

Now we can prove that minimisers are indeed rotating Riesz star solutions. We recall the definition of $\mathbb{R}^3_*$ as given by (4.22).

Proof of Theorem 4.2.4. We will show that the Euler-Lagrange equations of the minimisation problem correspond to (4.4). We split the proof into five steps.

Step 1. As such, for $\varepsilon > 0$, we consider the set

$$
\Gamma_\varepsilon := \Big\{\boldsymbol{x} \in \mathbb{R}^3_* \,|\, \varepsilon < \bar\rho(\boldsymbol{x}) < \frac{1}{\varepsilon}\Big\}.
$$

Let $w \in L^\infty(\mathbb{R}^3)$ be such that w has compact support in $\mathbb{R}^3_*$ and is non-negative on $\mathbb{R}^3_* \setminus \Gamma_\varepsilon$. Then, defining

$$
\bar\rho_\tau := \bar\rho + \tau w,
$$

for $\tau \geq 0$ small enough, we have $\bar{\rho}_\tau \in L^1_+(\mathbb{R}^3) \cap L^\gamma(\mathbb{R}^3)$. We show that for such τ, $\tau \mapsto Q^a(\rho_\tau)$ and $\tau \mapsto S^a_\mu(\rho_\tau)$ are both continuous. This follows from the fact that L is continuous, $w \in L^\infty$ and has compact support in $\mathbb{R}^3_*$, thus there exists $\delta > 0$ such that $\operatorname{supp}(w) \subset \mathbb{R}^3 \setminus (B_\delta \times \mathbb{R})$ and for all $0 \leq \tau \leq \Upsilon$,

$$L(m_{\bar{\rho}_\tau}(r)) \leq L(\|\bar{\rho}\|_{L^1} + \Upsilon \|w\|_{L^1}) =: L_\Upsilon,$$

where the inequality follows from the fact that $L'(m) \geq 0$ for all $m \geq 0$. Thus, $\bar{\rho}_\tau L(m_{\bar{\rho}_\tau}(r)) r^{-2}$ can be dominated by

$$\frac{\bar{\rho} L(m_{\bar{\rho}}(r))}{r^2} \mathbb{1}_{\{r<\delta\}} + \frac{(\bar{\rho} + \Upsilon|w|) L_\Upsilon}{\delta^2} \mathbb{1}_{\{r \geq \delta\}},$$

for $0 \leq \tau \leq \Upsilon$, which is integrable. Hence, by the dominated convergence Theorem, we have continuity of

$$\tau \mapsto \int_{\mathbb{R}^3} \frac{\bar{\rho}_\tau L(m_{\bar{\rho}_\tau}(r))}{r^2} \, \mathrm{d}\boldsymbol{x}.$$

The continuity of the internal energy, potential energy and mass are trivial.

Step 2. Suppose $\alpha \in (0, 2)$, there are two cases.

Case 1. There exists $\Upsilon > 0$ such that $Q^a((\bar{\rho}_\tau)_{\kappa(\bar{\rho}_\tau)}) < 0$ for $0 < \tau < \Upsilon$. Thus, by Lemma 4.5.2, we have that $\lambda_1^*(\bar{\rho}_\tau) < \kappa(\bar{\rho}_\tau)$ for $0 < \tau < \Upsilon$. For simplicity, we write $\lambda(\tau) \coloneqq \lambda_1^*(\bar{\rho}_\tau)$. We will show that $\lambda(\tau)$ is differentiable for $0 < \tau < \Upsilon$. To do this, we define the function $f : (0, \Upsilon) \times \mathbb{R}_+ \to \mathbb{R}$ by

$$f(\tau, \lambda) \coloneqq Q^a((\bar{\rho}_\tau)_\lambda).$$

We will apply the Implicit Function Theorem to f to prove differentiability of $\lambda(\tau)$. As in the proof of Lemma 4.5.2, we have that $\frac{\partial f}{\partial \lambda}(\tau, \lambda)$ exists and as in Step 1, it is continuous in (τ, λ). By (4.37), we have that since $\lambda(\tau) < \kappa(\bar{\rho}_\tau)$ for $0 < \tau < \Upsilon$, that

$$\frac{\partial f}{\partial \lambda}(\tau, \lambda(\tau)) < 0.$$

We now show that f is differentiable with respect to τ. We need only show this for the kinetic energy, as the other terms are trivially differentiable. For $0 < \tau < \Upsilon$ and $h \neq 0$ such that $0 < \tau + h < \Upsilon$, since L' is continuous, we have by integration by parts that

$$\left| \frac{L(m_{\bar{\rho}_{\tau+h}}(r)) - L(m_{\bar{\rho}_\tau}(r))}{h} \right| = \left| \int_0^1 L'(m_{\bar{\rho}_{\tau+th}}(r)) m_w(r) \, \mathrm{d}t \right| \leq L'_\Upsilon |m_w(r)|,$$

where

$$L'(m_{\bar{\rho}_\tau}(r)) \leq \max_{m \in [0, \|\bar{\rho}\|_{L^1} + \Upsilon \|w\|_{L^1}]} L'(m) =: L'_\Upsilon.$$

Since $\operatorname{supp}(w) \subset \mathbb{R}^3 \setminus (B_\delta \times \mathbb{R})$, we note that

$$\frac{\bar{\rho}_{\tau+h} L(m_{\bar{\rho}_{\tau+h}}(r)) - \bar{\rho}_\tau L(m_{\bar{\rho}_\tau}(r))}{hr^2} = \frac{wL(m_{\bar{\rho}_{\tau+h}}(r))}{r^2} + \frac{\bar{\rho}_\tau \big(L(m_{\bar{\rho}_{\tau+h}}(r)) - L(m_{\bar{\rho}_\tau}(r))\big)}{hr^2},$$

is dominated by

$$\left(\frac{|w|L_\Upsilon + (\bar{\rho} + \Upsilon|w|)L'_\Upsilon |m_w(r)|}{\delta^2}\right)\mathbb{1}_{\{r \geq \delta\}}.$$

Thus, we can apply the dominated convergence Theorem to obtain that f is differentiable with respect to τ, with

$$\begin{aligned}\frac{\partial f}{\partial \tau}(\tau, \lambda) = \lambda \int_{\mathbb{R}^3} \frac{wL(m_{\bar{\rho}_\tau}(r)) + \bar{\rho}_\tau L'(m_{\bar{\rho}_\tau}(r)) m_w(r)}{r^2} \,\mathrm{d}\boldsymbol{x} \\ + 3a_0\gamma\lambda^{\frac{3(\gamma-1)}{2}} \int_{\mathbb{R}^3} w\bar{\rho}_\tau^{\gamma-1} \,\mathrm{d}\boldsymbol{x} + \alpha\lambda^{\frac{\alpha}{2}} \int_{\mathbb{R}^3} w\Phi_\alpha * \bar{\rho}_\tau \,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

By a similar reasoning to before, this is continuous in (τ, λ). Since $f(\tau, \lambda(\tau)) = 0$ for each $0 < \tau < \Upsilon$, by the Implicit Function Theorem, there exists a neighbourhood interval $I \subset (0, \Upsilon)$ of τ and a continuously differentiable function $g : I \to \mathbb{R}_+$ such that

$$f(s, g(s)) = 0,$$

for all $s \in I$ and $g(\tau) = \lambda(\tau)$. By Lemma 4.5.2, since $Q^a((\bar{\rho}_s)_{\kappa(\bar{\rho}_s)}) < 0$ for $s \in I$, we must have that $g(s) = \lambda(s) < \kappa(\bar{\rho}_s)$. Thus, this implies that $\tau \mapsto \lambda(\tau)$ is continuously differentiable for $0 < \tau < \Upsilon$. Since $\lambda(\tau) < \kappa(\bar{\rho}_\tau)$ and $\tau \mapsto \kappa(\bar{\rho}_\tau)$ is continuous, we have that $\lambda(\tau)$ is bounded for $0 \leq \tau < \Upsilon$. Suppose that $\lambda(\tau) \not\to 1 = \lambda_1^*(\bar{\rho})$ as $\tau \to 0$. Then, there exists $\tau_k \to 0$, as $k \to \infty$, such that $\lambda(\tau_k) \not\to 1$, but since this sequence is bounded, there exists $\iota \neq \lambda_1^*(\bar{\rho})$ and a subsequence (which we relabel) such that

$$\lambda(\tau_k) \to \iota \leq \kappa(\bar{\rho}), \qquad \text{as } k \to \infty.$$

But then, by continuity, we have that

$$0 = Q^a((\bar{\rho}_{\tau_k})_{\lambda(\tau_k)}) \longrightarrow Q^a(\bar{\rho}_\iota), \qquad \text{as } k \to \infty.$$

But, this contradicts the uniqueness of $\lambda_1^*(\bar{\rho})$. Thus, we must have that $\lambda(\tau) \to 1 = \lambda_1^*(\bar{\rho})$ as $\tau \to 0$. Thus, $\tau \mapsto \lambda(\tau)$ is continuous for $0 \leq \tau < \Upsilon$. Hence, similar to the previous part, we have that $\tau \mapsto S_\mu^a((\bar{\rho}_\tau)_{\lambda(\tau)})$ is continuous for $0 \leq \tau < \Upsilon$ and differentiable for $0 < \tau < \Upsilon$. Since $\bar{\rho}$ is the minimiser of S_μ^a over $\mathcal{K}^a$ we can apply the mean value Theorem, to obtain for each $0 < \tau < \Upsilon$, there exists $\zeta_\tau \in (0, \tau)$, such that

$$\begin{aligned}0 &\leq \frac{S_\mu^a((\bar{\rho}_\tau)_{\lambda_1^*(\bar{\rho}_\tau)}) - S_\mu^a(\bar{\rho}_{\lambda_1^*(\bar{\rho})})}{\tau} \\ &= \frac{\mathrm{d}\lambda(\zeta_\tau)}{\mathrm{d}\tau} \frac{Q^a((\bar{\rho}_{\zeta_\tau})_{\lambda(\zeta_\tau)})}{2\lambda(\zeta_\tau)} + \frac{\lambda(\zeta_\tau)}{2} \int_{\mathbb{R}^3} \frac{wL(m_{\bar{\rho}_{\zeta_\tau}}(r)) + \bar{\rho}_\tau L'(m_{\bar{\rho}_{\zeta_\tau}}(r)) m_w(r)}{r^2} \,\mathrm{d}\boldsymbol{x} \\ &\quad + \frac{a_0\gamma\lambda(\zeta_\tau)^{\frac{3(\gamma-1)}{2}}}{\gamma - 1} \int_{\mathbb{R}^3} w\bar{\rho}_{\zeta_\tau}^{\gamma-1} \,\mathrm{d}\boldsymbol{x} + \lambda(\zeta_\tau)^{\frac{\alpha}{2}} \int_{\mathbb{R}^3} w\Phi_\alpha * \bar{\rho}_{\zeta_\tau} \,\mathrm{d}\boldsymbol{x} + \mu \int_{\mathbb{R}^3} w \,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

The first term on the right-hand side is 0, as $Q^a((\bar{\rho}_{\zeta_\tau})_{\lambda(\zeta_\tau)}) = 0$ by definition. Using a similar approach to the previous step, letting $\tau \to 0$, we obtain that

$$0 \leq \frac{1}{2}\int_{\mathbb{R}^3} \frac{wL(m_{\bar{\rho}}(r)) + \bar{\rho}L'(m_{\bar{\rho}}(r))m_w(r)}{r^2}\,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^3} w\left(\frac{a_0\gamma}{\gamma-1}\bar{\rho}^{\gamma-1} + \Phi_\alpha * \bar{\rho} + \mu\right)\mathrm{d}\boldsymbol{x}. \tag{4.65}$$

Case 2. There exists a sequence $\tau_k \to 0$, as $k \to \infty$, such that $Q^a((\bar{\rho}_{\tau_k})_{\kappa(\bar{\rho}_{\tau_k})}) \geq 0$. By continuity, we have that $Q^a((\bar{\rho}_{\tau_k})_{\kappa(\bar{\rho}_{\tau_k})}) \to 0$ as $k \to \infty$. Then, by Lemma 4.5.9 and the fact that $\bar{\rho}$ is a minimiser, we have that for k large enough

$$S^a_\mu((\bar{\rho}_{\tau_k})_{\kappa(\bar{\rho}_{\tau_k})}) - S^a_\mu(\bar{\rho}) \geq 0.$$

Thus, since $\tau \mapsto \kappa(\bar{\rho}_\tau)$ is continuous and differentiable for all $\tau \geq 0$, using a similar approach to the previous case, we can apply L'Hôpital's rule, to obtain

$$\begin{aligned}
0 &\leq \lim_{k\to\infty} \frac{S^a_\mu((\bar{\rho}_{\tau_k})_{\kappa(\bar{\rho}_{\tau_k})}) - S^a_\mu(\bar{\rho})}{\tau_k} \\
&= \frac{\mathrm{d}\kappa(\bar{\rho}_\tau)}{\mathrm{d}\tau}\bigg|_{\tau=0} \frac{Q^a(\bar{\rho})}{2} + \frac{1}{2}\int_{\mathbb{R}^3} \frac{wL(m_{\bar{\rho}}(r)) + \bar{\rho}L'(m_{\bar{\rho}}(r))m_w(r)}{r^2}\,\mathrm{d}\boldsymbol{x} \\
&\quad + \frac{a_0\gamma}{\gamma-1}\int_{\mathbb{R}^3} w\bar{\rho}^{\gamma-1}\,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^3} w\Phi_\alpha * \bar{\rho}\,\mathrm{d}\boldsymbol{x} + \mu\int_{\mathbb{R}^3} w\,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Since $Q^a(\bar{\rho}) = 0$, this reduces to (4.65).

Step 3. Suppose $\alpha \in [2,3)$. Then, by the continuity of $\tau \mapsto Q^a(\rho_\tau)$, there exists $\Upsilon > 0$ such that $\lambda^*(\bar{\rho}_\tau)$ exists for all $0 \leq \tau < \Upsilon$. Thus, by a similar argument to Case 1 of the previous step, (4.65) is satisfied.

Step 4. Since (4.65) is satisfied in all cases, we focus upon the kinetic terms. We have that by the co-area formula and integration by parts

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{\bar{\rho}L'(m_{\bar{\rho}}(r))m_w(r)}{r^2}\,\mathrm{d}\boldsymbol{x} &= \int_0^\infty \frac{(L(m_{\bar{\rho}}(r)))_r m_w(r)}{r^2}\,\mathrm{d}r \\
&= 2\pi\int_0^\infty \int_{\mathbb{R}} w(r,z)r\,\mathrm{d}z \int_r^\infty \frac{(L(m_{\bar{\rho}}(s)))_s}{s^2}\,\mathrm{d}s\,\mathrm{d}r \\
&\quad - \left[m_w(r)\int_r^\infty \frac{(L(m_{\bar{\rho}}(s)))_s}{s^2}\,\mathrm{d}s\right]_{r=0}^\infty \\
&= \int_{\mathbb{R}^3} w\int_r^\infty \frac{(L(m_{\bar{\rho}}(s)))_s}{s^2}\,\mathrm{d}s\,\mathrm{d}\boldsymbol{x}.
\end{aligned} \tag{4.66}$$

Note that applying integration by parts, we obtain

$$\int_r^\infty \frac{(L(m_{\bar{\rho}}(s)))_s}{s^2}\,\mathrm{d}s = \left[\frac{L(m_{\bar{\rho}}(s))}{s^2}\right]_{s=r}^\infty + 2\int_r^\infty \frac{L(m_{\bar{\rho}}(s))}{s^3}\,\mathrm{d}s. \tag{4.67}$$

Thus, combining (4.65), (4.66) and (4.67), we obtain

$$0 \leq \int_{\mathbb{R}^3} w\left(\frac{a_0\gamma}{\gamma-1}\bar{\rho}^{\gamma-1} + \int_r^\infty \frac{L(m_{\bar{\rho}}(s))}{s^3}\,\mathrm{d}s + \Phi_\alpha * \bar{\rho} + \mu\right)\mathrm{d}\boldsymbol{x}.$$

Since this is true for all such w, we have that

$$\begin{aligned}
&\frac{a_0\gamma}{\gamma-1}\bar{\rho}^{\gamma-1} + \int_r^\infty \frac{L(m_{\bar{\rho}}(s))}{s^3}\,\mathrm{d}s + \Phi_\alpha * \bar{\rho} + \mu = 0, && \text{a.e. on } \Gamma_\varepsilon,\\
&\frac{a_0\gamma}{\gamma-1}\bar{\rho}^{\gamma-1} + \int_r^\infty \frac{L(m_{\bar{\rho}}(s))}{s^3}\,\mathrm{d}s + \Phi_\alpha * \bar{\rho} + \mu \geq 0, && \text{a.e. on } \mathbb{R}^3 \setminus \Gamma_\varepsilon.
\end{aligned}$$

Thus, letting $\varepsilon \to 0$, we obtain

$$\begin{aligned}
&\frac{a_0\gamma}{\gamma-1}\bar{\rho}^{\gamma-1} + \int_r^\infty \frac{L(m_{\bar{\rho}}(s))}{s^3}\,\mathrm{d}s + \Phi_\alpha * \bar{\rho} + \mu = 0, && \text{a.e. on } \Gamma,\\
&\int_r^\infty \frac{L(m_{\bar{\rho}}(s))}{s^3}\,\mathrm{d}s + \Phi_\alpha * \bar{\rho} + \mu \geq 0, && \text{a.e. on } \mathbb{R}^3 \setminus \Gamma.
\end{aligned} \tag{4.68}$$

Similar to [6, Theorem A], we have that $\bar{\rho} \in L^\infty(\mathbb{R}^3)$. Thus $\bar{\rho}$ is a rotating Riesz star solution as given by Definition 4.1.1.

Step 5. Since $\bar{\rho} \in \mathcal{K}^a \cap L^\infty(\mathbb{R}^3)$, we have that $\Phi_\alpha * \bar{\rho} \in C(\mathbb{R}^3)$ and that $(\Phi_\alpha * \bar{\rho})(\boldsymbol{x}) \to 0$ uniformly as $|\boldsymbol{x}| \to \infty$. Rearranging (4.68) and using the fact that L is non-negative, we obtain that

$$\begin{aligned}
\bar{\rho} &= \left(\frac{\gamma-1}{a_0\gamma}\right)^{\frac{1}{\gamma-1}} \left(- \int_r^\infty L(m_{\bar{\rho}}(s)) s^{-3}\,\mathrm{d}s - \Phi_\alpha * \bar{\rho} - \mu \right)_+^{\frac{1}{\gamma-1}}\\
&\leq \left(\frac{\gamma-1}{a_0\gamma}\right)^{\frac{1}{\gamma-1}} (-\Phi_\alpha * \bar{\rho} - \mu)_+^{\frac{1}{\gamma-1}}.
\end{aligned}$$

Combining this with the fact that $(\Phi_\alpha * \bar{\rho})(\boldsymbol{x}) \to 0$ uniformly as $|\boldsymbol{x}| \to \infty$, we obtain that there exists $R > 0$ such that $\Gamma \subset B_R$. Hence, $\bar{\rho}$ has compact support.

□

4.6 Instability of rotating Riesz star solutions in the mass-supercritical regime

We now show that, under suitable conditions on L, there exist solutions of the CEREs (1.1) that start arbitrarily close to a rotating Riesz star solution and exhibit growth of their support. Since it was established in Theorem 4.2.4 that rotating Riesz star solutions are compactly supported, this behaviour constitutes an instability result.

For $\alpha \in (0, 2)$, we define the set

$$\mathcal{I}_\mu^a := \{(\varrho, \boldsymbol{\mathcal{M}}) \,|\, Q^a(\varrho) > 0,\ \kappa(\varrho) > 1,\ I_\mu(\varrho, \boldsymbol{\mathcal{M}}) < \ell_\mu^a\},$$

while for $\alpha \in [2, 3)$, we define

$$\mathcal{I}_\mu^a := \{(\varrho, \boldsymbol{\mathcal{M}}) \,|\, Q^a(\varrho) > 0,\ I_\mu(\varrho, \boldsymbol{\mathcal{M}}) < \ell_\mu^a\},$$

where

$$I_\mu(\varrho, \boldsymbol{\mathcal{M}}) := E(\varrho, \boldsymbol{\mathcal{M}}) + \mu \int_{\mathbb{R}^3} \varrho \, \mathrm{d}\boldsymbol{x}.$$

We will show that $\mathcal{I}_\mu^a$ is invariant under the semi-flow of the CEREs (1.1).

Remark 4.6.1. In the case that $\alpha \in (0,2)$, it would no longer be true that the set is invariant without the assumption that $\kappa(\varrho) > 1$. Otherwise, it would be possible for $Q^a(\rho(t)) = 0$ with $\lambda_2^*(\rho(t)) = 1$.

Lemma 4.6.2 (Invariance of $\mathcal{I}_\mu^a$ under the solution semi-flow)**.** *Suppose $\alpha \in (0,3)$, $\frac{6}{6-\alpha} < \gamma < \frac{3+\alpha}{3}$, L satisfies (4.5), (4.13) and, for $\alpha \in (0,2)$, (4.12) for $\omega > \omega_* + \bar{\omega}$ and $\omega \geq \frac{\gamma+1-\alpha}{3+\alpha-3\gamma}$. Then, if $(\rho, \boldsymbol{\mathcal{M}})$ is a classical solution of the attractive polytropic CEREs satisfying* (I_1)*–*(I_3) *and* (A_1)*–*(A_4) *with the square of the angular momentum given by $L(m_\rho(r(\boldsymbol{x})))$ and initial data $(\rho_0, \boldsymbol{\mathcal{M}}_0) \in \mathcal{I}_\mu^a$. Then*

$$(\rho(t), \boldsymbol{\mathcal{M}}(t)) \in \mathcal{I}_\mu^a, \qquad \text{for all } t > 0.$$

Proof. For $\alpha \in [2,3)$, the proof is the same as Lemma 3.8.2. So we now assume that $\alpha \in (0,2)$. By conservation of mass and energy,

$$S_\mu^a(\rho(t)) \leq I_\mu(\rho(t), \boldsymbol{\mathcal{M}}(t)) = I_\mu(\rho_0, \boldsymbol{\mathcal{M}}_0) < \ell_\mu^a, \quad \text{for all } t > 0.$$

Suppose for a contradiction the statement is false, then by continuity, there exists a first time $t_0 > 0$ such that $(\rho(t_0), \boldsymbol{\mathcal{M}}(t_0)) \notin \mathcal{I}_\mu^a$.

Case 1. Suppose $Q^a(\rho(t_0)) = 0$ and $\kappa(\rho(t_0)) \geq 1$. Then, by Lemma 4.5.2, $\rho(t_0) \in \mathcal{K}^a$. By definition of ℓ_μ^a, we have

$$S_\mu^a(\rho(t_0)) \geq \ell_\mu^a,$$

which is a contradiction.

Case 2. Suppose $Q^a(\rho(t_0)) > 0$ and $\kappa(\rho(t_0)) = 1$. By Lemma 4.5.9, we have

$$S_\mu^a(\rho(t_0)) > \ell_\mu^a,$$

which is a contradiction.

□

We next look at the concavity of the mapping $\lambda \mapsto S_{\mu,\lambda}^a(\varrho)$, fundamental in proving the instability of rotating Riesz star solutions. While $\kappa(\varrho)$ was defined for $\alpha \in (0,2)$, we note that the formula is still well-defined for $\alpha \in (2,3)$ with $\gamma > \frac{5}{3}$.

Lemma 4.6.3. *Suppose* $\alpha \in (0,3)$, $\frac{6}{6-\alpha} < \gamma < \frac{3+\alpha}{3}$, *$L$ satisfies (4.5) and* $\varrho \in L^1_+(\mathbb{R}^3) \cap L^\gamma(\mathbb{R}^3)$ *such that* $\varrho \not\equiv 0$ *and*

$$\int_{\mathbb{R}^3} \frac{\varrho L(m_\varrho(r))}{r^2}\,\mathrm{d}\boldsymbol{x} < \infty,$$

then $\lambda \mapsto S^a_{\mu,\lambda}(\varrho)$ *is concave for*

$$\begin{cases} 0 < \lambda \leq \kappa(\varrho), & \text{if } \alpha \in (0,2), \\ \lambda > 0, & \text{if } \alpha \in [2,3) \text{ and } \gamma \leq \frac{5}{3}, \\ \kappa(\varrho) \leq \lambda < \infty, & \text{if } \alpha \in (2,3) \text{ and } \gamma > \frac{5}{3}. \end{cases}$$

Proof. We see that from (4.38)

$$\frac{\mathrm{d}^2 S^a_{\mu,\lambda}(\varrho)}{\mathrm{d}\lambda^2} = \frac{3a_0(3\gamma-5)\lambda^{\frac{3(\gamma-1)-4}{2}}}{4}\int_{\mathbb{R}^3}\varrho^\gamma\,\mathrm{d}\boldsymbol{x} + \frac{\alpha(\alpha-2)\lambda^{\frac{\alpha-4}{2}}}{8}\int_{\mathbb{R}^3}\varrho\Phi_\alpha * \varrho\,\mathrm{d}\boldsymbol{x}.$$

Case 1 ($\alpha \in (0,2)$)**.** Since $\gamma < \frac{3+\alpha}{3}$, we see that $\frac{\mathrm{d}^2 S^a_{\mu,\lambda}(\varrho)}{\mathrm{d}\lambda^2} \leq 0$ if and only if $0 < \lambda \leq \kappa(\varrho)$ with equality for $\lambda = \kappa(\varrho)$.

Case 2 ($\alpha \in [2,3)$ and $\gamma \leq \frac{5}{3}$)**.** Then $\frac{\mathrm{d}^2 S^a_{\mu,\lambda}(\varrho)}{\mathrm{d}\lambda^2} < 0$ for all $\lambda > 0$.

Case 3 ($\alpha \in [2,3)$ and $\gamma > \frac{5}{3}$)**.** Then the signs of the coefficients are flipped as compared to Case 1. Rearranging, we see that $\frac{\mathrm{d}^2 S^a_{\mu,\lambda}(\varrho)}{\mathrm{d}\lambda^2} \leq 0$ if and only if $\kappa(\varrho) \leq \lambda < \infty$ with equality for $\lambda = \kappa(\varrho)$.

□

Theorem 4.6.4. *Suppose* $\alpha \in (0,3)$, $\frac{6}{6-\alpha} < \gamma < \frac{3+\alpha}{3}$, *$L$ satisfies (4.5), (4.13) and, for* $\alpha \in (0,2)$, *(4.12) for* $\omega > \omega_* + \bar{\omega}$ *and* $\omega \geq \frac{\gamma+1-\alpha}{3+\alpha-3\gamma}$. *If* $(\rho, \boldsymbol{\mathcal{M}})$ *is a classical solution of the attractive polytropic CEREs satisfying* (I_1)–(I_3) *and* (A_1)–(A_4) *with the square of the angular momentum given by* $L(m_\rho(r(\boldsymbol{x})))$ *with initial conditions* $(\rho_0, \boldsymbol{\mathcal{M}}_0) \in \mathcal{I}^a_\mu$ *for some* $\mu > 0$. *Then for* $R(t) = \max_{\boldsymbol{x}\in\Omega(t)}\{|\boldsymbol{x}|\}$, *we have that* $R(t) \to \infty$ *as* $t \to \infty$.

Remark 4.6.5. Similar to Remark 3.2.9, for any minimiser $\bar{\rho}$ of S_μ over $\mathcal{K}^a$, taking any $0 < \lambda < 1$, we obtain that $(\bar{\rho}_\lambda, \mathbf{0}) \in \mathcal{I}^a_\mu$. Thus, since $\bar{\rho}$ is a rotating Riesz star solution with compact support as proved in Theorem 4.2.4, then Theorem 4.6.4 proves instability of rotating Riesz star solutions.

Proof. We split the proof into five steps.

Step 1. Suppose there does not exist $\Lambda > 0$ such that $Q^a(\rho(t)) \geq \Lambda$ for all $t > 0$. Thus, there exists $t_0 \in (0,\infty]$ such that $\lim_{t\to t_0} Q^a(\rho(t)) = 0$. By Lemma 4.6.2, we have that $Q^a(\rho(t)) > 0$, $S^a_\mu(\rho(t)) < \ell^a_\mu$ and, for $\alpha \in (0,2)$, $\kappa(\rho(t)) > 1$ for all $t > 0$. Thus, since $t \mapsto Q^a(\rho(t))$ is continuous, $t_0 = \infty$.

Step 2. We first consider when $\alpha \in [2,3)$.

Case 1. Suppose $\alpha \in [2,3)$ with $\gamma \leq \frac{5}{3}$. By Lemma 4.6.3, and a similar proof to that of Lemma 3.8.5, $R(t) \to \infty$ as $t \to \infty$.

Case 2. Suppose $\alpha \in (2,3)$ with $\gamma > \frac{5}{3}$. If

$$\liminf_{t\to\infty} \frac{Q^a(\rho(t))}{-\int_{\mathbb{R}^3} \rho(t)\Phi_\alpha * \rho(t)\,\mathrm{d}\boldsymbol{x}} > 0,$$

owing to the positivity of L, as in Case 1 of the proof of Lemma 3.8.5, we have that $R(t) \to \infty$ as $t \to \infty$. If

$$\liminf_{t\to\infty} \frac{Q^a(\rho(t))}{-\int_{\mathbb{R}^3} \rho(t)\Phi_\alpha * \rho(t)\,\mathrm{d}\boldsymbol{x}} = 0,$$

then there exists a sequence $t_m \to \infty$ as $m \to \infty$, such that

$$\lim_{m\to\infty} \frac{Q^a(\rho(t_m))}{-\int_{\mathbb{R}^3} \rho(t_m)\Phi_\alpha * \rho(t_m)\,\mathrm{d}\boldsymbol{x}} = 0.$$

Using the definition of $Q^a(\rho)$ and the positivity of L, we can rewrite $\kappa(\rho(t))$ as

$$\begin{aligned}\kappa(\rho) &= \left(\frac{2(5-3\gamma)\big(Q^a(\rho) - \int_{\mathbb{R}^3} \rho L(m_\rho(r))r^{-2}\,\mathrm{d}\boldsymbol{x} - \frac{\alpha}{2}\int_{\mathbb{R}^3}\rho\Phi_\alpha * \rho\,\mathrm{d}\boldsymbol{x}\big)}{\alpha(\alpha-2)\int_{\mathbb{R}^3}\rho\Phi_\alpha * \rho\,\mathrm{d}\boldsymbol{x}}\right)^{\frac{2}{3+\alpha-3\gamma}} \\ &\leq \left(\frac{3\gamma-5}{\alpha-2} + \frac{2(5-3\gamma)Q^a(\rho)}{\alpha(\alpha-2)\int_{\mathbb{R}^3}\rho\Phi_\alpha * \rho\,\mathrm{d}\boldsymbol{x}}\right)^{\frac{2}{3+\alpha-3\gamma}}.\end{aligned}$$

Thus, combining this with the fact that $\frac{3\gamma-5}{\alpha-2} < 1$ since $\gamma < \frac{3+\alpha}{3}$, we obtain that $\lim_{m\to\infty}\kappa(\rho(t_m)) < 1$. Thus, for large enough m, employing Lemmas 4.5.1 (b) and Lemma 4.6.3, we have that $\lambda \mapsto S^a_\mu(\rho(t_m))$ is concave on $[1, \lambda^*(\rho(t_m))]$. By a similar proof to that of Case 2 of the proof of Lemma 3.8.5, we obtain a contradiction. Thus, we must have that $R(t) \to \infty$ as $t \to \infty$.

Step 3. Suppose that $\alpha \in (0,2)$. We show that $\kappa(\rho(t)) \to \infty$ as $t \to \infty$. By Lemma 4.6.3, we have that $\lambda \mapsto S^a_{\mu,\lambda}(\rho(t))$ is concave for all $0 < \lambda \leq \kappa(\rho(t))$.

Case 1. Suppose that $Q^a(\rho(t)_\lambda) > 0$ for all $\lambda > 0$. Then, since $Q^a(\rho(t)_{\kappa(\rho(t))}) > 0$ and $\kappa(\rho(t)_{\kappa(\rho(t))}) = 1$, by Lemma 4.5.9 and Taylor's Theorem we have that

$$\ell^a_\mu \leq S^a_\mu(\rho(t)_{\kappa(\rho(t))}) \leq S^a_\mu(\rho(t)) + \frac{Q^a(\rho(t))}{2}(\kappa(\rho(t)) - 1). \tag{4.69}$$

Case 2. Suppose there exists $\lambda > 0$ such that $Q^a(\rho(t)_\lambda) \leq 0$, then by continuity and Lemma 4.5.2 (d), we have that $\lambda_1^*(\rho(t)) \leq \kappa(\rho(t))$. Thus, since $\rho(t)_{\lambda_1^*(\rho(t))} \in \mathcal{K}^a$, by Taylor's Theorem, we have that

$$\begin{aligned}\ell^a_\mu \leq S^a_\mu(\rho(t)_{\lambda_1^*(\rho(t))}) &\leq S^a_\mu(\rho(t)) + \frac{Q^a(\rho(t))}{2}(\lambda_1^*(\rho(t)) - 1) \\ &\leq S^a_\mu(\rho(t)) + \frac{Q^a(\rho(t))}{2}(\kappa(\rho(t)) - 1).\end{aligned} \tag{4.70}$$

Thus, combining and rearranging (4.69) and (4.70), we obtain that

$$\kappa(\rho(t)) \geq \frac{2(\ell_\mu^a - S_\mu^a(\rho(t)))}{Q^a(\rho(t))} \geq \frac{2(\ell_\mu^a - I_\mu(\rho_0, \boldsymbol{\mathcal{M}}_0))}{Q^a(\rho(t))},$$

for all $t > 0$. Thus, since $Q^a(\rho(t)) \to 0$ as $t \to \infty$, we have that $\kappa(\rho(t)) \to \infty$ as $t \to \infty$.

Step 4. We now show that $R(t) \to \infty$ as $t \to \infty$.

Case 1. Suppose there exists $C > 0$ such that $\int_{\mathbb{R}^3} \rho(t)^\gamma \, \mathrm{d}\boldsymbol{x} \leq C$ for all $t > 0$. Then, since

$$\kappa(\rho(t)) = \left(\frac{6a_0(2 - 3(\gamma - 1)) \int_{\mathbb{R}^3} \rho^\gamma(t) \, \mathrm{d}\boldsymbol{x}}{\alpha(\alpha - 2) \int_{\mathbb{R}^3} (\rho \Phi_\alpha * \rho)(t) \, \mathrm{d}\boldsymbol{x}} \right)^{\frac{2}{3+\alpha-3\gamma}} \longrightarrow \infty, \qquad \text{as } t \to \infty,$$

we have that $\int_{\mathbb{R}^3} (\rho \Phi_\alpha * \rho)(t) \, \mathrm{d}\boldsymbol{x} \to 0$ as $t \to \infty$. Thus, since the kinetic energy and internal energy are positive and $Q^a(\rho(t)) \to 0$ as $t \to \infty$, we must have that $\int_{\mathbb{R}^3} \rho^\gamma(t) \, \mathrm{d}\boldsymbol{x} \to 0$ as $t \to \infty$. If there exists a sequence $t_k \to \infty$ as $k \to \infty$ such that $R(t_k)$ is bounded. Then, there exists $D > 0$ such that $R(t_k) \leq D$ for all k. Thus, by Hölder's inequality, we have that

$$M = \|\rho(t_k)\|_{L^1(B_D)} \leq \left(\frac{2\pi D^3}{3} \right)^{\frac{\gamma-1}{\gamma}} \|\rho(t_k)\|_{L^\gamma(\mathbb{R}^3)} \longrightarrow 0, \qquad \text{as } k \to \infty.$$

This contradicts conservation of mass, so we obtain that $R(t) \to \infty$ as $t \to \infty$.

Case 2. Suppose there exists a sequence $t_k \to \infty$ as $k \to \infty$ such that $\int_{\mathbb{R}^3} \rho(t_k)^\gamma \, \mathrm{d}\boldsymbol{x} \to \infty$ as $k \to \infty$. Then, since

$$\kappa(\rho(t_k)) \longrightarrow \infty, \qquad \text{as } k \to \infty,$$

we have that $Q^a(\rho(t_k)) \to \infty$ as $k \to \infty$. This is a contradiction, so we must be in Case 1.

Step 5. We now suppose that there exists $\Lambda > 0$ such that $Q^a(\rho(t)) \geq \Lambda$ for all $t > 0$. Consider the second moment of the density,

$$H(t) = \int_{\Omega(t)} \rho(t) |\boldsymbol{x}|^2 \, \mathrm{d}\boldsymbol{x}.$$

Combining (3.54) and (3.55), we obtain

$$\begin{aligned} H''(t) &= 2\Big(\int_{\Omega(t)} \left| \frac{\boldsymbol{\mathcal{M}}}{\sqrt{\rho}} \right|^2 \mathrm{d}\boldsymbol{x} + 3a_0 \int_{\Omega(t)} \rho^\gamma \, \mathrm{d}\boldsymbol{x} + \frac{\alpha}{2} \int_{\Omega(t)} \rho \Phi_\alpha * \rho \, \mathrm{d}\boldsymbol{x} \Big)(t) \\ &= 2Q^a(\rho(t)) + 2 \int_{\mathbb{R}^3} \left(\left| \frac{\mathcal{M}^r}{\sqrt{\rho}} \right|^2 + \left| \frac{\mathcal{M}^3}{\sqrt{\rho}} \right|^2 \right)(t) \, \mathrm{d}\boldsymbol{x} \\ &\geq 2\Lambda. \end{aligned}$$

Thus, by Taylor's Theorem, we see that

$$H(t) \geq H(0) + H'(0)t + \Lambda t^2. \tag{4.71}$$

Equally, we see that by Hölder's inequality, that

$$H(t) = \int_{\Omega(t)} \rho(t)|\boldsymbol{x}|^2 \,\mathrm{d}\boldsymbol{x} \leq R(t)^2 M. \tag{4.72}$$

Thus, combining (4.71) and (4.72) and rearranging, we obtain that

$$\frac{R(t)}{t} \geq \left(\frac{\Lambda}{M}\right)^{\frac{1}{2}} + O(t^{-\frac{1}{2}}).$$

Thus, we see that $R(t) \to \infty$ as $t \to \infty$.

□

4.7 Open problems

There are two clear open problems arising from this work:

1. While we prove the existence and stability of rotating star solutions for $\alpha \in (0, 2)$ in the mass-subcritical regime in Corollary 4.3.9, Corollary 4.3.10, and Theorem 4.2.2, it remains open what happens in the range $\alpha \in [2, 3)$, where the mass-critical scaling of the potential energy can no longer dominate the scaling of the kinetic energy. In this regime, we expect phenomena such as dichotomy or vanishing for minimising sequences. Accordingly, rotating Riesz star solutions are not expected to be stable in the more singular case.

2. In the mass-supercritical regime, the proof of existence of rotating Riesz star solutions and their instability, as in Theorem 4.2.4 and Theorem 4.6.4 respectively, requires suitable super-homogeneity conditions on L. Without such an assumption, the energy-critical scaling argument no longer applies. It is possible that minimisers of the variational problem remain unstable, but the corresponding instability mechanism may take a different form; for instance, one may instead observe stability around minimisers satisfying the mass-preserving scaling $\lambda_2^*(\varrho)$, when this is well-defined. It is also possible that such minimisers are stable.

Appendix A

Convolution inequalities

We now present several convolution inequalities that have been used in the main text of the thesis.

Lemma A.0.1 ([62, Theorem II.11.2]). *Suppose that $\Omega \subset \mathbb{R}^n$ is bounded, $\alpha \in (0,n)$, and $f \in L^q(\Omega)$ for $q \in (1,\infty)$. If $\alpha < n(1-\frac{1}{q})$, then $|\cdot|^{-\alpha} * (\mathbb{1}_\Omega f) \in C^{0,\mu}(\overline{\Omega})$ for $\mu := \min\left\{1, n(1-\frac{1}{q}) - \alpha\right\}$ and there exists some constant $C = C(\Omega, n, q, \alpha)$ such that*

$$\left\| |\cdot|^{-\alpha} * (\mathbb{1}_\Omega f) \right\|_{C^{0,\mu}(\overline{\Omega})} \leq C \|f\|_{L^q(\Omega)}.$$

Lemma A.0.2 (Hardy-Littlewood-Sobolev (HLS) Inequality). *Let $\alpha \in (0,n)$ and $1 < p, r < \infty$ such that*

$$\frac{1}{p} + \frac{\alpha}{n} = 1 + \frac{1}{r}.$$

Then, there exists a constant $C > 0$ such that for all $f \in L^p(\mathbb{R}^n)$,

$$\left\| |\cdot|^{-\alpha} * f \right\|_{L^r(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}.$$

When $r = \frac{2n}{\alpha}$ and $p = \frac{2n}{2n-\alpha}$, the constant is given by $C_{n,\alpha}$, where

$$C_{n,\alpha} := \pi^{\frac{\alpha}{2}} \frac{\Gamma(\frac{n-\alpha}{2})}{\Gamma(n - \frac{\alpha}{2})} \left(\frac{\Gamma(\frac{n}{2})}{\Gamma(n)} \right)^{\frac{\alpha-n}{n}}.$$

Lemma A.0.3 (Variation of HLS Inequality [15, Theorem 3.1]). *Suppose that $\alpha \in (0,n)$ and $p \geq \frac{2n}{2n-\alpha}$. Then there exists $C_* = C_*(\alpha, p, n) > 0$ such that, for any $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$,*

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\boldsymbol{x}) |\boldsymbol{x} - \boldsymbol{y}|^{-\alpha} f(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{y} \right| \leq C_* \|f\|_{L^1}^{\frac{p(2n-\alpha)-2n}{n(p-1)}} \|f\|_{L^p}^{\frac{\alpha p}{n(p-1)}}. \tag{A.1}$$

In particular, when $p = \frac{n+\alpha}{n} > \frac{2n}{2n-\alpha}$, for any $f \in L^1(\mathbb{R}^n) \cap L^{\frac{n+\alpha}{n}}(\mathbb{R}^n)$,

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\boldsymbol{x}) |\boldsymbol{x} - \boldsymbol{y}|^{-\alpha} f(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{y} \right| \leq C_* \|f\|_{L^1}^{\frac{n-\alpha}{n}} \|f\|_{L^{\frac{n+\alpha}{n}}}^{\frac{n+\alpha}{n}}.$$

Furthermore, C_* is the sharp constant that satisfies (A.1) and is strictly bounded above by the standard HLS constant $C_{n,\alpha}$, i.e., $C_* < C_{n,\alpha}$.

Lemma A.0.4 (Weak Young's Convolution Inequality)**.** *Let $1 < p < \infty$ and $q \geq \frac{2p}{2p-1}$. Then, for $h \in L^p_{\mathrm{w}}(\mathbb{R}^n)$, there exists $C^h_* > 0$ such that for any $f, g \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$,*

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\boldsymbol{x}) h(\boldsymbol{x}-\boldsymbol{y}) g(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \right| \leq C^h_* (\|f\|_{L^1}\|g\|_{L^1})^{\frac{q(2p-1)-2p}{2p(q-1)}} (\|f\|_{L^q}\|g\|_{L^q})^{\frac{q}{2p(q-1)}}.$$

Lemma A.0.5 (Riesz's Rearrangement Inequality [96])**.** *Let $f, g, h : \mathbb{R}^n \to \mathbb{R}_+$ be measurable functions. Define*

$$I(f,g,h) \coloneqq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\boldsymbol{x}) g(\boldsymbol{x}-\boldsymbol{y}) h(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}.$$

For a given measurable function $\phi : \mathbb{R}^n \to \mathbb{R}_+$, define the radially decreasing rearrangement $\phi^\#$ of ϕ as

$$\phi^\#(\boldsymbol{x}) \coloneqq \int_0^\infty \mathbb{1}_{\{\boldsymbol{y} : \phi(\boldsymbol{y}) > t\}^\#}(\boldsymbol{x}) \,\mathrm{d}t,$$

with

$$A^\# = \left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \frac{\omega_n}{n} |\boldsymbol{x}|^n \leq |A| \right\},$$

for any Lebesgue measurable set $A \subset \mathbb{R}^n$. Then

$$I(f,g,h) \leq I(f^\#, g^\#, h^\#). \tag{A.2}$$

If g is a strictly radially decreasing function, then the equality holds in (A.2) if and only if there exists some $\boldsymbol{z} \in \mathbb{R}^n$ such that $f(\boldsymbol{x}) = f^\#(\boldsymbol{x}-\boldsymbol{z})$ and $h(\boldsymbol{x}) = h^\#(\boldsymbol{x}-\boldsymbol{z})$.

Lemma A.0.6 (The Riesz Composition Formula [55, Theorem 3.1])**.** *Suppose that $0 < \alpha < n$, $0 < \beta < n$, and $0 < \alpha + \beta < n$. Then*

$$\int_{\mathbb{R}^n} |\boldsymbol{x}-\boldsymbol{z}|^{\alpha-n} |\boldsymbol{z}-\boldsymbol{y}|^{\beta-n} \,\mathrm{d}\boldsymbol{z} = \kappa_{\alpha,\beta} \, |\boldsymbol{x}-\boldsymbol{y}|^{\alpha+\beta-n},$$

where

$$\kappa_{\alpha,\beta} = \frac{\pi^{\frac{n}{2}} \Gamma(\frac{\alpha}{2}) \Gamma(\frac{\beta}{2}) \Gamma(\frac{n-\alpha-\beta}{2})}{\Gamma(\frac{n-\alpha}{2}) \Gamma(\frac{n-\beta}{2}) \Gamma(\frac{\alpha+\beta}{2})}. \tag{A.3}$$

Appendix B

Fractional Sobolev spaces

We present the definition and a useful compactness result for fractional Sobolev spaces.

Definition B.0.1 (Fractional Sobolev Spaces $W^{s,p}$)**.** Let $\Omega \subset \mathbb{R}^n$ be an open set, $s \in (0,1)$ and $1 \leq p < \infty$, the fractional Sobolev space $W^{s,p}(\Omega)$ is defined by

$$W^{s,p}(\Omega) \coloneqq \left\{ u \in L^p(\Omega) \;\middle|\; \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|}{|\boldsymbol{x} - \boldsymbol{y}|^{\frac{n}{p}+s}} \in L^p(\Omega \times \Omega) \right\}.$$

We define the semi-norm $[\,\cdot\,]_{W^{s,p}(\Omega)}$ by

$$[u]_{W^{s,p}(\Omega)} \coloneqq \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|^p}{|\boldsymbol{x} - \boldsymbol{y}|^{n+sp}} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \right)^{\frac{1}{p}},$$

and the norm of $W^{s,p}(\Omega)$ by

$$\|u\|_{W^{s,p}(\Omega)} \coloneqq \left(\|u\|^p_{L^p(\Omega)} + [u]^p_{W^{s,p}(\Omega)} \right)^{\frac{1}{p}}.$$

Lemma B.0.2 (Compact Embeddings of $W^{s,p}$ [50, Corollary 7.2])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded extension domain for $W^{s,p}$, $s \in (0,1)$ and $1 \leq p < \infty$ such that $sp < n$. Let $q \in [1, p^*)$, where $p^* = \frac{np}{n-sp}$ is the fractional critical exponent, then*

$$W^{s,p}(\Omega) \Subset L^q(\Omega).$$

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