Stable partitions for games with non-transferable utility and externalities*

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Abstract I propose a model of coalitional bargaining with claims in order to find solutions for games with non-transferable utility and externalities. I show that, for each such game, payoff configurations exist which will not be renegotiated. In the ordinal game derived from these payoff configurations, a core stable partition can be found, i.e. a partition in which no group of players has an incentive to jointly change their coalitions.

Keywords: Games with non-transferable utility in partition function form, Bargaining with claims, Ordinal games, Core stable partitions

JEL Classification: C71, C78

1 Introduction

The most general class of cooperative games are those with non-transferable utility and externalities. I will simply call them *games* throughout the paper. For instance in the context of industrial organizations, they are relevant: not only do they allow the consideration of synergies (and, hence, non-linear payoff functions), they also enable us to take the indirect influence of a cooperation between companies on an outside party's payoff into account. However, the theory of these games is rather meager and even generalizations of well known solutions for games with only one of these two features are far from trivial. The main scope of this paper is to propose an approach that is based on bargaining in every coalition in every partition.

In a bargaining problem (Nash, 1950) players have to agree on a payoff allocation, subject to the allocation being feasible and each player receiving a certain minimum, her so called disagreement point. A game defines for each coalition a set of payoff allocations that are feasible. In particular, each player can receive a payoff when staying alone. This payoff is her natural disagreement point in every coalition. Hence, a game can be interpreted as a collection of

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bargaining problems, one for each coalition. It therefore seems appropriate to use bargaining solutions in order to solve games – an approach that has been proposed for instance by Harsanyi (1963).

In a game players have several coalitions they might join, and thus, do not only have ultimate disagreement points but also certain *claims* in each coalition: a player who can receive a high payoff in one coalition has a good reason to claim a high payoff in another coalition. Claims and disagreement points must not be confused: even if the former is not satisfied, a player might still consider to join a coalition. Hence, claims add another dimension to the classical bargaining problem. Chun and Thomson (1992) were the first who formally defined bargaining problems with claims. A bargaining solution maps every bargaining problem with claims to a (unique) payoff allocation.

The first part of this paper applies the idea of bargaining with claims to games. Suppose that in every coalition every player has a claim. In this case a bargaining solution can be applied to the bargaining problem in each coalition, allocating to every player a payoff. Based on these payoffs players can derive new claims for each coalition. Hence, a negotiation process emerges: claims are used to find payoffs, payoffs are used to derive claims. One result will be that for every game (under very mild conditions) there are payoffs that are robust with respect to such renegotiation. These payoffs are called *consistent.*

Of course, after consistent payoffs have been found, it is still not clear, how players form their coalitions – consistent payoffs are defined for each player in each partition. But based on such payoffs, each player has a preference order over all possible partitions. These preferences form an ordinal game. The question is: Can we find a partition that makes everybody happy? A partition is stable if there is no group of players who have an incentive to leave their coalitions (in this partition) and form a new one. It turns out that if a payoff configuration is consistent with a fair bargaining solution and reasonable claims, then such a partition exists.

In Section 2 the model of coalitional bargaining with claims is formally introduced and the existence of consistent payoff configurations for each game under general conditions is proved. In Section 3 ordinal games are defined and it is shown how an ordinal game can be derived from a payoff configuration. In particular, I give sufficient conditions for a payoff configuration to induce an ordinal game with a stable partition. In Section 4 the results from Section 2 are refined for the proportional bargaining solution. In Section 5 a class of single-valued solutions that are based on opportunity costs and marginal contributions is derived. Section 6 provides a discussion of the related literature and concludes the paper with some possible avenues for further research.

2 Coalitional bargaining with claims

Throughout the paper N shall be a finite set of players. A *coalition* is a nonempty subset $S \subseteq N$, and the set $\mathcal{P} = \mathcal{P}(N)$ is the collection of all coalitions. For $i \in N$ denote by $\mathcal{P}_i$ the collection of all coalitions containing i. A *partition* is a collection $\sigma = \{S^1, \ldots, S^m\}$ of coalitions such that $S^k \cap S^l = \emptyset$ for all $k \neq l$ and $\bigcup_{k=1}^m S^k = N$. The set Σ shall be the collection of all partitions of N, and for a coalition S denote the set of all partitions containing S by Σ_S. For $i \in N$ and a partition σ denote by $\sigma(i)$ the (unique) coalition $S \in \sigma$ with $i \in S$. An *embedded coalition* is a pair $(S, \sigma) \in \mathcal{P} \times \Sigma$ such that $S \in \sigma$. The collection of all embedded coalitions is denoted by $\mathcal{E}$ and the collection of all embedded coalitions (S, σ) with $i \in S$ by $\mathcal{E}_i$.

A *game* is a map V which assigns to each embedded coalition (S, σ) a nonempty, closed, convex subset $V(S, \sigma) \subseteq \mathbb{R}^S$ such that

1. $V(S, \sigma)$ is comprehensive: that is, for $x \in V(S, \sigma)$ and $y \leq x$ it holds that $y \in V(S, \sigma)$,[1]

2. the set $\{x \in V(S, \sigma) : \; x \geq x^*\}$ is bounded for each $x^* \in V(S, \sigma)$.

A game can hence have transferable or non-transferable utility, and be in characteristic function or partition function form. If $x \in V(S, \sigma)$, x is called *feasible* for (S, σ). A game is *non-leveled* if for all (S, σ) and all $x \in \partial V(S, \sigma)$ it holds that $y > x$ only if $y \notin V(S, \sigma)$.

Player i's *disagreement point in* V is $d_i^V = \min_{\tau \in \Sigma_{\{i\}}} \max(V(\{i\}, \tau))$. This is the worst-case-payoff player i can achieve if she stays alone. Throughout this paper I assume that $d_S^V \in V(S, \sigma)$ for all games V and all embedded coalitions (S, σ).[2] This standard condition allows to avoid unnecessary technicalities in what follows; it will become clear later that it is no loss of generality, see Remark 2.7.

A vector $x \in \mathbb{R}^{N \times \Sigma}$ is called a *payoff configuration*. Hence, a payoff configuration specifies in each partition for each player a payoff. For a payoff configuration x player i's *disagreement point in* x is $d_i(x) = \min_{\tau \in \Sigma_{\{i\}}} x_{i,\tau}$. This is the minimal payoff player i can achieve if she stays alone. A payoff configuration x is *individually rational* if $x_{i,\sigma} \geq d_i(x)$ for all $\sigma \in \Sigma$.

For a game V let $\Delta(V) \subseteq \mathbb{R}^{N \times \Sigma}$ be the set of all payoff configurations x with $x_{S,\sigma} \in V(S, \sigma)$ for all embedded coalitions (S, σ) and $x_{i,\sigma} = \max(V(\{i\}, \sigma))$ for all $\sigma \in \Sigma_{\{i\}}$. Hence, for any embedded coalition (S, σ) I require $x_{S,\sigma}$ to be feasible for (S, σ), and if S contains only player i, I require that $x_{i,\sigma}$ is the maximal feasible payoff. This makes sense as there is no good reason to allocate less to i than she could achieve on her own. Note that this implies $d_i(x) = d_i^V$ for all $x \in \Delta(V)$.

Call x *efficient in* V if $y \gg x$ implies $y \notin \Delta(V)$ for all $y \in \mathbb{R}^{N \times \Sigma}$. Hence, $x \in \Delta(V)$ is efficient if and only if $x_{S,\sigma} \in \partial V(S, \sigma)$ for each embedded coalition (S, σ). Call x *individually rational in* V if $x_{i,\sigma} \geq d_i^V$ for all $i \in N$ and all $\sigma \in \Sigma$. In particular, $x \in \Delta(V)$ is individual rational if and only if it is individually rational in V. Denote the subset of all efficient (resp. individually rational) payoff configurations in V by $\Delta_{\text{eff}}(V)$ (resp. $\Delta_{\text{ir}}(V)$).

[1] For $x, y \in \mathbb{R}^S$ I write $x \geq y$ if $x_i \geq y_i$ for all $i \in S$, I write $x > y$ if $x \geq y$ and $x \neq y$, and I write $x \gg y$ if $x_i > y_i$ for all $i \in S$.

[2] For a vector $d = (d_i)_{i \in N} \in \mathbb{R}^N$ I write d_S for $(d_i)_{i \in S} \in \mathbb{R}^S$.

Let $\rho : N \to N$ be a permutation. For a partition $\sigma = \left\{S^1, \ldots, S^k\right\}$ define $\rho(\sigma) = \left\{\rho(S^1), \ldots, \rho(S^k)\right\}$. For a game V define ρV by

$$(\rho V)(S, \sigma) = \left\{x \in \mathbb{R}^S : \left(x_{\rho^{-1}(i)}\right)_{i \in S} \in V\left(\rho^{-1}(S), \rho^{-1}(\sigma)\right)\right\}$$

for each embedded coalition (S, σ). A payoff configuration $x \in \Delta(V)$ is *anonymous* if $x_{\rho(i),\rho(\sigma)} = x_{i,\sigma}$ for each permutation with $\rho V = V$. Denote the set of all anonymous payoff configurations in V by $\Delta_{\text{an}}(V)$.

A *bargaining problem (with claims)* is a quadruple (S, X, d, c) of a coalition S with $|S| \geq 2$, a closed, convex, and comprehensive subset $X \subseteq \mathbb{R}^S$ such that $\{x \in X : x \geq x^*\}$ is bounded for each $x^* \in X$, a *disagreement point* $d \in X$, and a *claim point* $c \geq d$.[3] A *bargaining solution* is a map F which maps each bargaining problem (S, X, d, c) to a vector $x \in X$. The first three of the following four properties are standard and do not need much discussion.

Individual rationality. A bargaining solution F is *individually rational* if $F(S, X, d, c) \geq d$ for each bargaining problem (S, X, d, c).

Efficiency. A bargaining solution F is *efficient* if $y \gg F(S, X, d, c)$ implies $y \notin X$ for all bargaining problems (S, X, d, c).

Anonymity. A bargaining solution F is *anonymous* if

$$F_i(S, X, d, c) = F_{\rho(i)}\left(\rho(S), X', d_{\rho(S)}, c_{\rho(S)}\right)$$

for all bargaining problems (S, X, d, c) and all permutations ρ, where $x \in X'$ if and only if $x_{\rho^{-1}(S)} \in X$.

Continuity. Let (S, X, d, c) be a bargaining problem. A bargaining solution F is *continuous in* c if for all sequences c^n with $c^n \geq d$ and $\lim_{n\to\infty} c^n = c$ it holds that $F(S, X, d, c^n) \to F(S, X, d, c)$. F is *continuous* if it is continuous in c for all bargaining problems (S, X, d, c).

Continuity of F ensures that the bargaining solution does not jump after a small change in c. I do not consider continuity of F in any other variable than c throughout this paper; therefore, as this cannot lead to confusion, by continuity I always mean continuity in c.

2.1 Claim forms

Let V be a game and let $x \in \Delta(V)$ be a payoff configuration. Suppose that x builds the starting point for negotiations. Then players can and will derive claims from x: if player i knows that she could receive a high payoff in a coalition T, she might use this potential payoff as a bargaining chip when negotiating payoffs in another coalition S. This motivates for the following definitions.

[3] Note that in the original definition of Chun and Thomson (1992) it is required that $c \notin X$. I do not impose this condition.

A *claim form* maps each game V to a function $C^V : \Delta(V) \to \mathbb{R}^{N\times\Sigma}$ with $C^V_{i,\sigma}(x) \geq d_i(x)$ for all $i \in N$, all $\sigma \in \Sigma$ and all $x \in \Delta(V)$. The function C^V is called *claim function*. Hence, a claim form specifies in each partition a claim for each player that depends both on the game V and a previous payoff configuration $x \in \Delta(V)$. Recall that $d(x) = d^V$ for such x, hence, the condition $C^V_{i,\sigma}(x) \geq d_i(x)$ simply states that a player claims at least her guaranteed payoff in the game V. A claim form C is *continuous* if C^V is continuous for all games V.

Anonymity. A claim form C is *anonymous* if

$$C^{\rho V}_{\rho(i),\rho(\sigma)}\left(\left(x_{\rho(i),\rho(\sigma)}\right)_{i\in N,\sigma\in\Sigma}\right) = C^V_{i,\sigma}(x) \tag{1}$$

for each permutation ρ, each game V, and each $x \in \Delta(V)$.

Anonymity of a claim form can be interpreted as the condition that player i would derive the same claims as player j does, if i were in j's shoes. Although this seems like a strong condition at first sight, Equation (1) not only ensures that i and j change their roles with respect to the payoff configuration x, but also with respect to the game V. This means, that i would make the same claims as j if she were to face x from j's perspective and had j's utility function. Thus, anonymity is a very natural property of a claim function.

Given a payoff configuration x, a player's claim in an embedded coalition should not be entirely arbitrary. Let $\sigma \in \Sigma$ and let $i \in N$. A coalition $T \in \mathcal{P}_i$ is an *outside option* of i in σ if for all $j \in T \setminus \{i\}$ it holds that $x_{j,\tau} \geq x_{j,\sigma}$ for all $\tau \in \Sigma_T$ and $x_{j,\tau} > x_{j,\sigma}$ for some $\tau \in \Sigma_T$. That means an outside option is a coalition in which all players except i can, according to x, only gain compared to their payoffs in σ. (In particular, staying alone is an outside option.) Hence, it is rather easy for i to convince the other members of T (if any) to collaborate. An outside option T of i is *positive* if $x_{i,\tau} \geq x_{i,\sigma}$ for all $\tau \in \Sigma_T$ and $x_{i,\tau} > x_{i,\sigma}$ for some $\tau \in \Sigma_T$.

α-reasonableness. A claim function C^V is *α-reasonable in x* if

$$C^V_{i,\sigma}(x) \leq \max_{\tau\in\Sigma} x_{i,\tau} \text{ and} \tag{2a}$$

$$C^V_{i,\sigma}(x) > x_{i,\sigma} \text{ if and only if } i \text{ has a positive outside option in } \sigma \tag{2b}$$

for all $i \in N$ and all $\sigma \in \Sigma$. A claim function C^V is *α-reasonable* if C^V is α-reasonable in all $x \in \Delta(V)$. A claim form C is *α-reasonable* if C^V is α-reasonable for each game V.

β-reasonableness. A claim function C^V is *β-reasonable in x* if

$$C^V_{i,\sigma}(x) \leq \max_{(S,\sigma)\in\mathcal{E}_i} \left\{y_i : y \in V(S,\sigma), y_j \geq d^V_j \text{ for all } j \in S \setminus \{i\}\right\} \tag{3}$$

for all $i \in N$ and all $\sigma \in \Sigma$. For a claim function and a claim form β-reasonableness is defined as before.

Claim functions specify how players derive their claims in each coalition in each partition depending on all payoffs of all players in all partitions. Inequality (3) guarantees that a player claims no more than she would receive in any other coalition where payoffs are shared individually rationally. β-reasonableness is a very general condition and related to the concepts of the core and the bargaining set: a player i can claim a payoff even though there is no partition in which this payoff would actually be allocated to her. If C is α-reasonable this is not possible anymore. Condition (2a) guarantees that a player claims no more than she would receive in any other partition according to x. Condition (2b) states that a player can (and does) claim strictly more than she already gets, only if she actually can receive a higher payoff when forming another coalition.

Both axioms should be considered as minimal requirements on a claim function depending on the context. β-reasonableness seems appropriate if players can easily make proposals how to share a surplus. This might be the case in very simple structures, where developing and negotiating proposals can be done at (almost) no costs. However, if negotiations and potential contracts are very complicated, a claim can be justified by an existing proposal of a third party rather than by the mere possibility of an unspecified collaboration with such a party.

Remark 2.1. In a positive outside option every player is at least as well off as in the partition they deviated from. Hence, the definition of a positive outside option reflects a rather pessimistic view of players on the outcome of deviating. I am well aware of criticism of this point of view (for instance in Ray and Vohra, 1997). Although many results (not all) can be derived for different notions of outside options, I use this approach as it reflects the idea that the payoff player i receives in a positive outside option T of σ might serve as a bargaining chip when negotiating payoffs in $\sigma(i)$. But this bargaining chip is useful only in so far as it cannot be reduced by some *bloodthirsty behavior* (Ray and Vohra, 1997) of players in $N \setminus T$.

Note that, in general, α-reasonableness does not imply β-reasonableness or vice versa (see Example 2.2). However, if C^V is α-reasonable on $\Delta_{\mathrm{ir}}(V)$ then C^V is also β-reasonable on $\Delta_{\mathrm{ir}}(V)$.

Example 2.2.

1. The function C with $C_{i,\sigma}(x) = d_i(x)$ represents a claim form if one defines C^V as the restriction $C_{|\Delta(V)}$ for all games V. In particular, C is continuous, anonymous, and β-reasonable but not α-reasonable.

2. Consider the claim form C which is defined by

$$C^V_{i,\sigma}(x) = \max_{T \in \mathcal{P}_i} \min_{\tau \in \Sigma_T} \max_{y \in V(T,\tau)} \{y_i : \, y_j \geq x_{j,\sigma} \text{ for all } j \in T \setminus \{i\}\}.$$

   This claim function is related to the idea of *deviations* in the definition of the core[4]. In any partition σ player i claims the maximal worst-case-payoff

[4]The precise relation of this claim function to the core (Gillies, 1959) and the bargaining set (Davis and Maschler, 1963) is investigated in Karos (2015).

she can obtain in any coalition T provided that each member of T receives at least the payoff she would receive in σ. Note that C is continuous and anonymous but neither α- nor β-reasonable: if x is not individual rational, Inequality (3) can easily be violated; and $C^V_{i,\sigma}(x) > x_{i,\sigma}$ is possible without i having an outside option.

3. For a payoff configuration x, a partition $\sigma \in \Sigma$, and a player $i \in N$ let $\mathcal{T}^*_{i,\sigma}(x)$ be the set of outside options of i in σ. The function

$$C_{i,\sigma}(x) = \max_{T \in \mathcal{T}^*_{i,\sigma}(x)} \min_{\tau \in \Sigma_T} x_{i,\tau} \tag{4}$$

is an anonymous and α-reasonable claim form, but neither β-reasonable (x might not be individually rational) nor continuous. The latter observation is further explored in Example 2.9.

4. For a payoff configuration x, a partition $\sigma \in \Sigma$, and a player $i \in N$ define

$$\mathcal{T}_{i,\sigma}(x) = \left\{ T \in \mathcal{P}_i : \min_{j \in T} \min_{\tau \in \Sigma_T} x_{j,\tau} \geq x_{j,\sigma} \right\},$$

and for all $T \in \mathcal{P}_i$ let

$$\tilde{\gamma}_{i,\sigma}(T,x) = \begin{cases} \min_{j \in T} \max_{\tau \in \Sigma_T} x_{j,\tau} - x_{j,\sigma}, & \text{if } T \in \mathcal{T}_{i,\sigma}(x), \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $\tilde{\gamma}_{i,\sigma}(T,.)$ is continuous for all x, and so is its normalized version

$$\gamma_{i,\sigma}(T,x) = \frac{\tilde{\gamma}_{i,\sigma}(T,x)}{1 + \sum_{T' \in \mathcal{P}_i} \tilde{\gamma}_{i,\sigma}(T,x)}.$$

Moreover, $\gamma_{i,\sigma}(T,x) > 0$ if and only if T is a positive outside option of i in σ. Hence, the claim form $C_{i,\sigma}(x) = \max \left\{ d_i(x), \tilde{C}_{i,\sigma}(x) \right\}$ where

$$\tilde{C}_{i,\sigma}(x) = \left(1 - \sum_{T \in \mathcal{P}_i} \gamma_{i,\sigma}(T,x) \right) x_{i,\sigma} + \sum_{T \in \mathcal{P}_i} \gamma_{i,\sigma}(T,x) \min_{\tau \in \Sigma_T} x_{i,\tau} \tag{5}$$

is continuous, α-reasonable, anonymous, and satisfies $C_{i,\sigma}(x) \geq x_{i,\sigma}$ for all $i \in N$ and all $\sigma \in \Sigma$. For a deviation T one can interpret $\gamma_{i,\sigma}(T,x)$ as an indicator for the likelihood that T actually forms, given the maximum each player can obtain from T. In particular, if there is some player who cannot gain from joining T, this likelihood is 0.

2.2 Consistent payoff configurations

Let C be a claim form, let V be a game, and let $x \in \Delta(V)$. Then $d(x) = d^V$ and, by definition, $C^V_{S,\sigma}(x) \geq d_S(x) = d^V_S$ for all embedded coalitions (S,σ). Hence,

$\left(S, V(S,\sigma), d_S^V, C_{S,\sigma}^V(x)\right)$ is a bargaining problem for each embedded coalition (S,σ) with $|S| \geq 2$. This means that, given a claim form C, any game V can be interpreted as a collection of bargaining problems with claims.

As claim forms can be used in order to renegotiate payoffs, the interesting payoff configurations are those that are invariant with respect to such renegotiations. This idea is captured by the following definition.

Definition 2.3. Let V be a game, let C be a claim form, and let F be a bargaining solution. A payoff configuration $x \in \Delta(V)$ is *consistent (with F under C)* if

$$x_{i,\sigma} = F_i\left(\sigma(i), V(\sigma(i),\sigma), d_{\sigma(i)}^V, C_{\sigma(i),\sigma}^V(x)\right) \tag{6}$$

for all $i \in N$ and all partitions $\sigma \in \Sigma$ with $|\sigma(i)| \geq 2$. Denote by $\mathcal{K}_F^C(V)$ the collection of all payoff configurations that are consistent with F under C.

Note that for $x \in \mathcal{K}_F^C(V)$ the payoffs $x_{i,\sigma}$ are already uniquely determined for partitions with $|\sigma(i)| = 1$ as $x \in \Delta(V)$. Applying the bargaining rule F to the claims derived from x according to C will result in x again; and hence, players would not renegotiate consistent payoff configurations.

Proposition 2.4. *Let V be a game, let C be a continuous claim form, and let F be a continuous bargaining solution.*

1. *If F is individually rational then $\mathcal{K}_F^C(V)$ is a nonempty, closed subset of $\Delta_{\mathrm{ir}}(V)$.*

2. *If C is β-reasonable then $\mathcal{K}_F^C(V)$ is a nonempty, compact set.*

Proof. Let $\hat{F}^V : \Delta(V) \to \Delta(V)$ be defined as

$$\hat{F}_{i,\sigma}^V(x) = \begin{cases} F_i\left(\sigma(i), V(\sigma(i),\sigma), d_{\sigma(i)}^V, C_{\sigma(i),\sigma}^V(x)\right), & \text{if } |\sigma(i)| \geq 2, \\ x_{i,\sigma}, & \text{otherwise.} \end{cases}$$

A payoff configuration $x \in \Delta(V)$ is consistent with F if and only if $\hat{F}^V(x) = x$. It is clear that $\hat{F}^V$ is continuous since F and C are continuous.

1. Let F be individually rational. As $\hat{F}^V$ is a map from $\Delta_{\mathrm{ir}}(V)$ into itself and as $\Delta_{\mathrm{ir}}(V)$ is compact and convex, $\hat{F}^V$ must have a fixed point in $\Delta_{\mathrm{ir}}(V)$ by Brouwer's fixed point theorem, that is $\mathcal{K}_F^C(V) \neq \emptyset$. Let (x^n) be a sequence of consistent payoff configurations with $x = \lim_{n\to\infty} x^n$. Then $\hat{F}^V(x) = \hat{F}^V(\lim_{n\to\infty} x^n) = \lim_{n\to\infty} \hat{F}^V(x^n) = \lim_{n\to\infty} x^n = x$ by continuity of $\hat{F}^V$. Hence, $\mathcal{K}_F^C(V)$ is closed.

2. Let C be β-reasonable. Then $C^V(\Delta(V))$ is a compact set. Define the map $\tilde{F}^V$ as

$$\tilde{F}_{i,\sigma}^V(c) = \begin{cases} F_i\left(\sigma(i), V(\sigma(i),\sigma), d^V, c_{\sigma(i),\sigma}\right), & \text{if } |\sigma(i)| \geq 2 \\ \max\left(V(\{i\},\sigma)\right), & \text{if } |\sigma(i)| = 1. \end{cases}$$

Then $Q = \tilde{F}^V \left(C^V \left(\Delta(V) \right) \right)$ is a compact subset of $\Delta(V)$. In particular, $\hat{F}$ is a continuous map from the convex hull of Q into itself, so it must have a fixed point. The set of fixed points must be closed for the same reasons as in the previous part.

□

An example of a continuous and individually rational bargaining solution is the *constrained egalitarian solution* (Aumann and Maschler, 1985; Curiel et al., 1987): $\tilde{E}_i \left(S, X, d, c \right) = \max \left(c_i - \lambda, d_i \right)$, where $\lambda \in \mathbb{R}$ is uniquely determined by the efficiency of $\tilde{E}$. Note that, unlike in the original definition, λ might take negative values in the model presented here.

Without any further restrictions it might happen that a bargaining solution assigns to one player more and to another player less than they claim. Such a bargaining solutions would usually be perceived as unfair. The next two axioms account for this.

Strong fairness. A bargaining solution F is called *strongly fair* if for all bargaining problems (S, X, d, c) there is $i \in S$ with $F_i \left(S, X, d, c \right) \geq c_i$ only if $F_j \left(S, X, d, c \right) \geq c_j$ for all $j \in S$.

Weak fairness. A bargaining solution F is called *weakly fair*, if for all bargaining problems (S, X, d, c) there is $i \in S$ with $F_i(S, X, d, c) > c_i$ only if $F_j \left(S, X, d, c \right) \geq c_j$ for all $j \in S$.

Strong fairness guarantees that nobody's claim can be satisfied while someone else's is not; weak fairness guarantees that nobody's claim can be exceeded while someone else's is not satisfied. Both axioms are (trivially) satisfied by all solutions in classical bankruptcy theory as nobody's claim is satisfied there. The *proportional solution* is defined as

$$P \left(S, X, d, c \right) = \begin{cases} d + (c - d) \max \left\{ t \in \mathbb{R} : \ d + t \left(c - d \right) \in X \right\}, & \text{if } c > d, \\ d + \mathbb{1}_S \max \left\{ t \in \mathbb{R} : \ d + t \mathbb{1}_S \in X \right\}, & \text{otherwise.} \end{cases}$$

It is weakly but not strongly fair. This bargaining solution will be investigated in detail in Section 4. The *egalitarian bargaining solution*, defined as

$$E \left(S, X, d, c \right) = c + \mathbb{1}_S \cdot \max \left\{ t \in \mathbb{R} : \ c + t \mathbb{1}_S \in X \right\},$$

is a strongly fair and continuous bargaining solution. This solution is the natural extension of the classical egalitarian solution for cases in which the reference point (usually the disagreement point, here c) lies outside X. But E is not individually rational. In fact, individual rationality and strong fairness are incompatible.

Lemma 2.5. *There is no individually rational and strongly fair bargaining solution.*

Proof. Suppose F is a bargaining rule with both properties and let the bargaining problem (S, X, d, c) be defined as $S = \{1, 2\}$, $c = (2, 0)$, $d = (0, 0)$, and $X = \{x \in \mathbb{R}^2 : x_1 + x_2 \le 1\}$. By individual rationality, $F_2(S, X, d, c) \ge 0 = c_2$ and $F_1(S, X, d, c) \le 1 < c_1$, contradicting strong fairness. $\square$

This negative result is a drawback for sure. But it will become clear later (see Remark 3.4) that it might be a good idea to skip individual rationality for strong fairness. In fact for an α-reasonable claim function fairness can replace individual rationality in Proposition 2.4.

Proposition 2.6. *Let V be a game, let C be a continuous and α-reasonable claim form, and let F be a continuous bargaining solution. If F is, additionally, weakly fair then $\mathcal{K}^C_F(V)$ is a nonempty, compact set.*

Proof. Let $\hat{F}^V$ be defined as in the proof of Proposition 2.4 and define

$$x^*_+ = \max_{i \in N} \max_{(S,\sigma) \in \mathcal{E}_i} \left\{ x_i : x \in \partial V(S, \sigma) \text{ and } x_j \ge d^V_j \text{ for all } j \in S \setminus \{i\} \right\},$$

$$x^*_- = \min_{i \in N} \min \left(d_i(x), \min_{(S,\sigma) \in \mathcal{E}_i, |S| \ge 2} \min_{c \in \prod_{j \in S} \left[d_j(x), x^*_+ \right]} \left\{ F_i \left(S, V(S, \sigma), d^V_S, c \right) \right\} \right),$$

Let $Q \subseteq \Delta(V)$ be the set of payoff configurations with $x^*_- \le x_{i,\sigma} \le x^*_+$ for all partitions σ and all $i \in N$. Obviously, Q is compact and convex. Hence, it is sufficient to show that $\hat{F}^V(Q) \subseteq Q$. To do this let $x \in Q$ and let $(S, \sigma) \in \mathcal{E}$, with $|S| \ge 2$, arbitrary but fixed. Clearly, $d^V_i \le C^V_{i,\sigma}(x) \le x^*_+$ for all $i \in N$ and $\sigma \in \Sigma$ as C is α-reasonable. By weak fairness one has to consider only the following two cases.

1. Suppose $\hat{F}^V_{i,\sigma}(x) \ge C^V_{i,\sigma}(x)$ for all $i \in S$. Clearly, $\hat{F}^V_{i,\sigma} \ge C^V_{i,\sigma}(x) \ge d^V_i \ge x^*_-$. This also implies $\hat{F}^V_{i,\sigma} \le x^*_+$, as $\hat{F}^V_{j,\sigma} \ge d^V_j$ for all $j \in S \setminus \{i\}$.

2. Suppose $\hat{F}^V_{i,\sigma}(x) \le C^V_{i,\sigma}(x)$ for all $i \in S$. Then $\hat{F}^V_{i,\sigma}(x) \le x^*_+$. Further, by construction, $\hat{F}^V_{i,(S,\sigma)}(x) \ge x^*_-$.

As (S, σ) was arbitrary, $\hat{F}^V(x) \in Q$; and the non-emptyness of $\mathcal{K}^C_F(V)$ is proved by applying Brouwer's fixed point theorem. The same arguments as in the proof of Proposition 2.4 show that $\mathcal{K}^C_F(V)$ is closed. $\square$

Remark 2.7. Suppose that V is a game such that $d^V_S \notin V(S, \sigma)$ for some embedded coalition (S, σ). Then partition σ will not form under any circumstances as coalition S would break apart. Hence, define $\Sigma^* = \Sigma^*(V)$ to be the collection of all partitions σ with $d^V_S \in V(S, \sigma)$ for all $S \in \sigma$. If one now requires Equation (6) to hold only for all $i \in N$ and all $\sigma \in \Sigma^*$ with $|\sigma(i)| \ge 2$, the results of this paper remain valid. Hence, the condition $d^V_S \in V(S, \sigma)$ is no loss of generality.

Remark 2.8. It is obvious that efficiency of F implies $\mathcal{K}^C_F(V) \subseteq \Delta_{\text{eff}}(V)$, and individual rationality of F implies $\mathcal{K}^C_F(V) \subseteq \Delta_{\text{ir}}(V)$ for any game V. It can also be shown that the set $\Delta_{\mathrm{an}}(V)$ is convex for all V.[5] As the same fixed point arguments as in the proofs of Propositions 2.4 and 2.6 can be used on $\Delta_{\mathrm{an}}(V)$ rather than $\Delta(V)$, this implies that $\mathcal{K}_F^C(V) \cap \Delta_{\mathrm{an}}(V) \neq \emptyset$ for all bargaining solutions F and claim rules C that satisfy the respective conditions and are, additionally, anonymous.

The next example shows that for the claim function in Equation (4) a consistent payoff configuration need not to exist.

Example 2.9. Let v be the proper monotonic simple three player game which is defined by its minimal winning coalitions $\{1,2\}$ and $\{1,3\}$,[6] and let V be the corresponding game.[7] Then there is no payoff configuration $x \in \Delta(V)$ which is consistent with the egalitarian bargaining solution E under the claim function C from Equation (4) in Example 2.2. Assume, on the contrary, that x is such a payoff configuration (which must be efficient) and let $q^1 = x_{1,\{12,3\}}$ and $q^2 = x_{1,\{13,2\}}$.[8] Suppose $q^1, q^2 < 1$. Obviously, $13 \in \mathcal{T}^*_{1,\{12,3\}}(x)$ and therefore $C_{1,\{12,3\}}(x) \geq q^2$. For player 2 it holds that $2 \in \mathcal{T}^*_{2,\{12,3\}}(x)$; $23 \in \mathcal{T}^*_{2,\{12,3\}}(x)$ only if $x_{2,\{1,23\}} < 0$; and $N \in \mathcal{T}^*_{2,\{12,3\}}(x)$ only if $x_{1,\{N\}} > q^1, x_{3,\{N\}} > 0$ and therefore $x_{2,\{N\}} < 1 - q^1$. Hence, $C_{2,\{12,3\}}(x) < 1 - q^1 = x_{2,\{12,3\}}$. The definition of E implies that in this case $q^2 \leq C_{1,\{12,3\}}(x) < x_{1,\{12,3\}} = q^1$ and for the same reasons it must hold as well that $q^1 < q^2$. As this is impossible, either q^1 or q^2 must be at least 1. Without loss of generality let $q^1 \geq 1$. In this case $N \in \mathcal{T}^*_{1,\{12,3\}}$ only if $x_{1,\{N\}} < q^1$; and $13 \in \mathcal{T}^*_{1,\{12,3\}}$ only if $q^2 < 1 \leq q^1$. Therefore, $C_{1,\{12,3\}}(x) < q^1 = x_{1,\{12,3\}}$. Again, by definition of E, it must hold that $x_{2,\{12,3\}} > C_{2,\{12,3\}}(x)$. Hence, $0 \geq x_{2,\{12,3\}} > C_{2,\{12,3\}}(x) \geq 0$ by definition of C. This is impossible as well.

3 Coalition formation

An *ordinal game* is a pair $(N, \succeq)$ where $\succeq$ is a profile of preferences $(\succeq_i)_{i \in N}$ over partitions. A coalition T is a *deviation* of a partition σ if for all $i \in T$ it holds that $\tau \succeq_i \sigma$ for all $\tau \in \Sigma_T$ and $\tau \succ_i \sigma$ for some $\tau \in \Sigma_T$. In this case T *blocks* σ. If $i \in T$, i *has deviation* T in σ. Partition σ is *blocked*, if there is a coalition T that blocks σ, and *core stable* if it is not blocked. The idea behind ordinal games is that players have preferences over partitions. A player is happy to stay in a coalition if she is unable to gather a group of others around her that would leave their coalitions in order to form a new coalition together. Again, players in my model have pessimistic views on the world (recall Remark 2.1): in order to form a new coalition, they must be sure that they won't lose out compared

[5] See Karos (2015) for details.

[6] A proper monotonic simple game is a function $v : \mathcal{P} \to \{0,1\}$ such that $v(N) = 1$, $v(S) \leq v(T)$ if $S \subseteq T$, and $v(S) + v(N \setminus S) \leq 1$ for all nonempty $S \subsetneq N$.

[7] That is $V(S) = \left\{x \in \mathbb{R}^S : \sum_{i \in S} x_i \leq v(S)\right\}$ for all $S \in \mathcal{P}$.

[8] In order to avoid brackets I use the notation jk for coalition $\{j,k\}$ and accordingly $\{i, jk\}$ for the partition $\{\{i\},\{j,k\}\}$ here and in the remainder of the paper.

to the status quo and have a chance of improving. If in a partition no group of players has any interest in forming a new coalition, this partition is core stable.

From a payoff configuration x one canonically derives[9] the ordinal game $(N, \succeq^x)$ where $\succeq_i^x$ is defined for all $i \in N$ as

$$\tau \succeq_i^x \sigma \quad \text{if and only if} \quad x_{i,\tau} \geq x_{i,\sigma}.$$

Let x be a payoff configuration and let σ be a partition. It can easily be checked that T is a deviation of $i \in T$ in σ in the game $(N \succeq^x)$ if and only if T is a positive outside option of i.

For general ordinal games it is not clear in which cases core stable partitions exist. I will show that the following two (technical) conditions are sufficient. It will become clear later that they are closely related to consistent payoff configurations.

Definition 3.1. A payoff configuration x is *balanced* if for each embedded coalition (S, σ) there is $i \in S$ with a positive outside option in σ if and only if each $j \in S$ has a positive outside option in σ. A payoff configuration x is *constrained balanced* if for all embedded coalitions (S, σ) and all $i \in S$ one of the following holds.

1. $x_{S,\sigma} \ll d_S(x)$, or
2. $x_{S,\sigma} = d_S(x)$, or
3. $x_{S,\sigma} > d_S(x)$ and no $j \in S$ with $x_{j,\sigma} = d_j(x)$ has a positive outside option, and there is $i \in S$ with a positive outside option if and only if each $j \in S$ with $x_{j,\sigma} > d_j(x)$ has a positive outside option.

The somewhat complicated formulation of constrained balancedness ensures that the first priority is to satisfy all disagreement points, and only after that any further potential outside options are considered. The promised result about the existence of core stable partitions is the following lemma.

Lemma 3.2. *Let x be (constrained) balanced. Then there is a core stable partition in the ordinal game $(N, \succeq^x)$.*

Proof. I prove the claim for constrained balanced x and sketch how it would be done for balanced x afterwards. Let $j^1 \in N$, define

$$\tilde{S}^1 = \arg\max_{S \in \mathcal{P}_{j^1}} \min_{\sigma \in \Sigma_S} x_{j^1,\sigma},$$

and let $\sigma^1 = \arg\min_{\sigma' \in \Sigma_{\tilde{S}^1}} x_{j^1,\sigma'}$. Then for all partitions $\sigma' \in \Sigma_{\tilde{S}^1}$

$$x_{j^1,\sigma'} \geq x_{j^1,\sigma^1} \geq \max_{T \in \mathcal{P}_{j^1}} \min_{\tau \in \Sigma_T} x_{j^1,\tau}.$$

[9] This idea is based on Shenoy (1979).

Hence, if σ' is a partition containing $\tilde{S}^1$, j^1 has no positive outside option in σ'. In particular, $x_{j^1,\sigma'} \geq d_{j^1}$. Let

$$S^1 = \begin{cases} \{j^1\}, & \text{if } x_{j^1,\sigma^1} = d_{j^1}, \\ \tilde{S}^1, & \text{if } x_{j^1,\sigma^1} > d_{j^1}. \end{cases}$$

If $S^1 = N$, let $\sigma^* = \{N\}$. Otherwise, recursively construct the following partition. Define $N^k = N \setminus (\bigcup_{l=1,\ldots,k-1} S^l)$ for $k \geq 2$ and, if $N^k \neq \emptyset$, choose $j^k \in N^k$. Let

$$\tilde{S}^k = \underset{S \in \mathcal{P}_{j^k}(N^k)}{\arg\max} \min_{\sigma \in \Sigma, \{S_1,\ldots,S_{k-1},S\} \subseteq \sigma} x_{j^k,\sigma},$$

let $\sigma^k = \arg\min_{\sigma \in \Sigma, S^1,\ldots S^{k-1}, \tilde{S}^k \in \sigma} x_{j^k,\sigma}$, and define

$$S^k = \begin{cases} \{j^k\}, & \text{if } x_{j^k,\sigma^k} = d_{j^k}, \\ \tilde{S}^k, & \text{if } x_{j^k,\sigma^k} > d_{j^k}. \end{cases}$$

As N is finite, there is k^* such that $N^{k^*+1} = \emptyset$, that is $\sigma^* = \{S^1, \ldots, S^{k^*}\}$ is a partition. I show that no player has a positive outside option in σ^*. Note that there is no positive outside option in σ^* for any $j \in S^1$. Indeed, if $S^1 = \{j^1\}$, this is clear as $\max_{T \in \mathcal{P}_{j^1}} \min_{\tau \in \Sigma_T} x_{j^1,\tau} \leq x_{j^1,\sigma^1} \leq x_{j^1,\sigma^*}$. If $S^1 = \tilde{S}^1$, then j^1 cannot have a positive outside option. As x is constrained balanced and $x_{j^1,\sigma^*} \geq x_{j^1,\sigma^1} > d_{j^1}$, this means that no $j \in S^1$ has a positive outside option.

Now let $k \geq 2$ and suppose that there is no positive outside option in σ^* for any $j \in \bigcup_{l=1}^{k-1} S^l$. From the construction of S^k it is clear that j^k cannot have a positive outside option T in σ^* with $T \subseteq N^{k-1}$. Together with the hypothesis, this implies that there is no positive outside option of j^k in σ^* at all. If $S^k = \{j^k\}$, there is no positive outside option for any $j \in S^k$. So, let $|S^k| \geq 2$. In this case $x_{j^k,\sigma^*} \geq x_{j^k,\sigma^k} > d_{j^k}$. By the constrained balancedness of x, there cannot be a positive outside option in σ^* for any $j \in S^k$. Hence, σ^* must be core stable.

If x is balanced, simply define $S^k = \tilde{S}^k$ and keep the rest of the proof identical. □

The close relation between fair bargaining solutions and balanced payoff configurations seems obvious. The following theorem makes this relation more explicit.

Theorem 3.3. *Let V be a game, let C be an α-reasonable claim form, and let F be a strongly fair bargaining solution. Then for each $x \in \mathcal{K}_F^C(V)$ there is a core stable partition in the ordinal game $(N, \succeq^x)$.*

Proof. Let $x \in \mathcal{K}_F^C(V)$. By Lemma 3.2 it is sufficient to show that x is balanced. Let (S, σ) be an embedded coalition and let $i \in S$. If i has a positive outside option in σ then $C_{i,\sigma}^V(x) > x_{i,\sigma} = \hat{F}_{i,\sigma}^V(x)$ (where $\hat{F}$ is defined as in the proof of Proposition 2.4) by the α-reasonableness of C; and by the strong fairness of F the same is true for all $j \in S$. Hence, each $j \in S$ has a positive outside option. □

Remark 3.4. If a payoff configuration is consistent with a strongly fair bargaining solution under an α-reasonable claim function, there is a partition of the player set such that in every coalition every player's claim is satisfied. The fact that the bargaining solution is not individually rational will not affect the stability of this partition: in any embedded coalition where at least one player's disagreement point is not met, nobody's claim can be met and the respective coalition will not form.

Remark 3.5. Suppose that x is a payoff configuration without externalities, i.e. with $x_{i,\sigma} = x_{i,\pi}$ whenever $\sigma(i) = \pi(i)$. Then the ordinal game $(N, \succeq^x)$ is a *hedonic game* (Drèze and Greenberg, 1980). It can easily be checked that for such games the definitions of deviations and core stability in this paper coincide with those in the literature. In this case the (constrained) balancedness of x implies that $(N, \succeq^x)$ exhibits the *weak top coalition property* introduced by Banerjee et al. (2001). The authors have shown that this property is a sufficient (but not necessary) condition for the existence of a core stable partition.

4 The proportional bargaining solution

Recall the proportional bargaining solution, defined as

$$P(S, X, d, c) = \begin{cases} d + (c - d) \max\{t :\ d + t(c - d) \in X\}, & \text{if } c > d, \\ d + \mathbb{1}_S \max\{t :\ d + t\mathbb{1}_S \in X\}, & \text{otherwise.} \end{cases}$$

It is easy to see that P is weakly fair, individually rational, efficient, and anonymous.

Example 4.1. Let the claim form C be defined by

$$C^V_{i,\sigma}(x) = \max_{y \in V(\sigma(i),\sigma)} \{y_i :\ y \geq d^V_{\sigma(i)}\}$$

for each game V. Then for each game V there is a unique payoff configuration $x \in \mathcal{K}^C_P(V)$, and for each embedded coalition (S, σ) the payoff vector $x_{S,\sigma}$ coincides with the Kalai-Smorodinsky bargaining solution (Kalai and Smorodinsky, 1975) of the bargaining problem (without claims) $\left(S, V(S,\sigma), d^V_S\right)$.

Propositions 2.4 and 2.6 do not apply for P as P is, in general, not continuous in $c = d$. But for a bargaining problem (S, X, d, c) it is continuous everywhere if $d \in \partial X$. The conditions on the claim function in the following proposition are satisfied for instance by the claim function in Equation (5).

Proposition 4.2. *Let V be a non-leveled game and let C be a continuous claim form with $C^V(x) \geq x$. Then $\mathcal{K}^C_P(V)$ is a nonempty, closed subset of $\Delta_{\text{ir,eff}}(V)$. If C is, additionally, anonymous, then $\mathcal{K}^C_P(V) \cap \Delta_{\text{an}}(V) \neq \emptyset$.*

Proof. Define $\hat{P}^V$ on $\text{convh}(\Delta_{\text{ir,eff}}(V))$, the convex hull of $\Delta_{\text{ir}}(V) \cap \Delta_{\text{eff}}(V)$, by

$$\hat{P}^V_{i,\sigma}(x) = \begin{cases} P_i\left(\sigma(i), V(\sigma(i),\sigma), d^V_{\sigma(i)}, C^V_{\sigma(i),\sigma}(x)\right), & \text{if } |\sigma(i)| \geq 2, \\ x_{i,\sigma}, & \text{otherwise.} \end{cases}$$

I show that $\hat{P}^V$ is a continuous map from $\operatorname{convh}(\Delta_{\mathrm{ir,eff}}(V))$ into itself. It is clear that $\hat{P}^V(x) \in \Delta_{\mathrm{ir,eff}}(V)$ for all x, as P is individually rational and efficient. I show that $\hat{P}^V$ is, indeed, continuous. Let $x \in \operatorname{convh}(\Delta_{\mathrm{ir,eff}}(V))$ and let $(S,\sigma) \in \mathcal{E}$ with $|S| \geq 2$. If $x_{S,\sigma} > d_S^V$ then $C_{S,\sigma}^V(x) \geq x_{S,\sigma} > d_S^V$, so that P is continuous in $C_{S,\sigma}^V(x)$. In this case, $\hat{P}_{i,\sigma}^V$ is continuous in x for all $i \in S$. Suppose that $x_{S,\sigma} = d_S^V$. As $x_{S,\sigma}$ is a convex combination of points $y \in \partial V(S,\sigma)$ with $y \geq d_S^V$, this is possible only if $x_{S,\sigma}$ is an extreme point of the convex hull. This means $d_S^V = x_{S,\sigma} \in \partial V(S,\sigma)$. As V is non-leveled, d_S^V is the only efficient and individually rational element of $V(S,\sigma)$. In particular, for any sequence (x^n) in $\operatorname{convh}(\Delta_{\mathrm{ir,eff}}(V))$ approaching x it holds that $x_{S,\sigma}^n = d_S^V$. Hence, $\lim_{n\to\infty} \hat{P}_{S,\sigma}^V(x^n) = d_S^V = \hat{P}_{S,\sigma}^V(x) = \hat{P}_{S,\sigma}^V(\lim_{n\to\infty} x^n)$, so that $\hat{P}_{i,\sigma}^V$ is continuous in x for all $i \in S$. Therefore, $\hat{P}^V$ is a continuous map from $\operatorname{convh}(\Delta_{\mathrm{ir,eff}}(V))$ into itself and, by Brouwer's fixed point theorem, must have a fixed point. Clearly, this fixed point is individually rational, efficient in V, and consistent with P under C. Closeness of $\mathcal{K}_P^C(V)$ is shown in the same way as in the proof of Propositions 2.4 and 2.6.

If C is, additionally, anonymous, $\hat{P}^V$ also maps $\operatorname{convh}(\Delta_{\mathrm{an,ir,eff}}(V))$ into itself. Hence, $\hat{P}^V$ has a fixed point in $\Delta_{\mathrm{an,ir,eff}}(V)$ which is consistent with P under C. □

In the context of bargaining the condition $C(x) \geq x$ is easily justified. Assuming that a payoff configuration is the result of previous negotiations, there is no reason for players to claim less than they had agreed to before. This does not necessarily mean that these claims are satisfied in the end.

Theorem 4.3. *Let V be a game and let C be an α-reasonable claim function. Then for each $x \in \mathcal{K}_P^C(V)$ the ordinal game $(N, \succeq^x)$ has a core stable partition.*

Proof. Let $x \in \mathcal{K}_P^C(V)$. By Lemma 3.2 it is sufficient to show that x is constrained balanced. Let $(S,\sigma) \in \mathcal{E}$ with $|S| \geq 2$ and let $i \in S$. Clearly, $x_{S,\sigma} \geq d_S^V$. If $x_{S,\sigma} = d_S^V$, there is nothing to show, so let $x_{S,\sigma} > d_S^V$. Suppose that there is $i \in S$ with $x_{i,\sigma} = d_i^V$ such that i has a positive outside option. Then $C_{i,\sigma}^V(x) > x_{i,\sigma}$ by the α-reasonableness of C. Also, $d_i^V = x_{i,\sigma} = \hat{P}_{i,\sigma}(x)$. Hence,

$$d_i^V = P_i\left(S, V(S,\sigma), d_S^V, C_{S,\sigma}^V(x)\right) = d_i^V + t_{S,\sigma}\left(C_{i,\sigma}^V(x) - d_i^V\right)$$

for some $t_{S,\sigma} \geq 0$. As $C_{i,\sigma}^V(x) > x_{i,\sigma} \geq d_i^V$, it is necessary that $t_{S,\sigma} = 0$. Hence, $x_{S,\sigma} = P\left(S, V(S,\sigma), d_S^V, C_{S,\sigma}^V(x)\right) = d_S^V$, in contradiction to $x_{S,\sigma} > d_S^V$. Therefore, no i with $x_{i,\sigma} = d_i^V$ can have a positive outside option.

Suppose now that there is $i \in S$ with a positive outside option. Then $x_{i,\sigma} > d_i^V$, so that

$$C_{i,\sigma}^V(x) > x_{i,\sigma} = d_i^V + t_{S,\sigma}\left(C_{i,\sigma}^V(x) - d_i^V\right) > d_i^V$$

with $0 < t_{S,\sigma} < 1$ as C is α-reasonable. In particular, for any $j \in S$ with $x_{j,\sigma} > d_j^V$ it holds that $C_{j,\sigma}^V(x) > d_j^V$ and, hence,

$$x_{j,\sigma} = d_j + t_{S,\sigma}\left(C_{j,\sigma}^V(x) - d_i\right) < C_{j,\sigma}^V(x).$$

By the α-reasonableness of C, j must have a positive outside option. □

Proposition 4.2 and Theorem 4.3 together show that for each non-leveled game an efficient, individually rational, and anonymous payoff configuration can be found that is stable with respect to renegotiation and induces an ordinal game for which a core stable partition exists.

5 Claim forms and opportunity costs

So far, I have imposed properties on claim forms and investigated what the corresponding sets of payoff configurations look like. Another way to motivate claims is to ask: What drives players' expectations when bargaining? In this section, I will focus on two quantities, namely opportunity costs and marginal contributions. Let x be a payoff configuration, let $\sigma \in \Sigma$, and let $i \in N$. Player i's *opportunity costs* of joining $\sigma(i)$ rather than any of the other coalitions are given by

$$O_{i,\sigma}(x) = \max_{T \in \{\emptyset\} \cup \sigma \setminus \{\sigma(i)\}} x_i^{\sigma_{T+i}} - d_i(x),$$

where $\sigma_{T+i} = (\sigma \cup \{\sigma(i) \setminus \{i\}, T \cup \{i\}\}) \setminus \{\emptyset, \sigma(i), T\}$. The opportunity costs of i in σ are the surplus (net of her disagreement point) she would have received if she had joined her best coalition in σ apart from $\sigma(i)$.

The definition of opportunity costs is straightforward, but things become complicated when dealing with marginal contributions in the presence of non-transferable utility or externalities (or both). A suggestion how to deal with externalities is given in de Clippel and Serrano (2008), where vectors of marginal contributions are defined. Otten et al. (1998) propose a definition of marginal contributions for games with non-transferable utility and without externalities. A discussion how marginal contributions should be defined in either case lies beyond the scope of this paper, and I will only assume that $M_{i,\sigma}^V$ is a suitable measure of player i's marginal contribution to coalition $\sigma(i)$ in partition σ in the game V.

Opportunity costs are independent of the game V and only depend on a specified payoff configuration x, whereas marginal contributions are independent of x and depend only on the game V. Marginal contributions are well established in cooperative game theory, but opportunity costs appear only implicitly, for instance as a motivation for the bargaining set. Nonetheless, both seem to be important for players' expectations of and agreement on their payoffs. From a rational point of view, marginal contributions should be irrelevant for a player's decision to accept or reject a payoff allocation. And yet, experimental studies (for instance of the ultimatum game, see Oosterbeek et al., 2004) show that outside options alone cannot explain people's behavior. I, therefore, consider a "hybrid" claim form which mixes opportunity costs and marginal contributions – more specifically, claim forms of type

$$C_{i,\sigma}^{\mu,V}(x) = d_i(x) + \mu \max\left\{0, M_{i,\sigma(i)}^V\right\} + (1-\mu) O_{i,\sigma}(x) \tag{7}$$

for $\mu \in [0,1]$. A few comments on this claim functions are in order. First, it is normalized in such a way that $C^{\mu,V}_{i,\sigma}(x) \geq d_i(x)$. Even without any marginal contributions, there is no good argument for a claim to lie below the payoff a player could realize by simply staying alone. Second, marginal contributions might be negative. But it seems unreasonable to include a negative marginal contribution in one's claim. I therefore assume that marginal contributions are relevant only if they are positive. Third, while opportunity costs are an easy-to-interpret absolute value, the scale of $M^V_{i,\sigma(i)}$ is not clear. However, this shall not cause too much trouble if we choose $M^V_{i,\sigma(i)}$ as an absolute measure of the damage player i can cause when leaving $\sigma(i)$. Fourth, the weight μ can be interpreted as a measure of cooperation: while opportunity costs are a rational and justified claim (they can be realized when leaving a coalition), marginal contributions are a rather destructive bargaining chip. By leaving a coalition, player i cannot gain $M^V_{i,\sigma(i)}$; she can just cause a damage of $M^V_{i,\sigma(i)}$ for the remaining players.

The aim of this section is to prove the following theorem which shows that unique consistent payoff configurations can actually be obtained for some claim functions of the form (7).

Theorem 5.1. *Let $\mu > \frac{1}{2}$. Then for every game V the set $\mathcal{K}^{C^\mu}_E(V)$ is nonempty and single-valued, and $x \in \mathcal{K}^{C^\mu}_E(V)$ is efficient.*

The following lemma plays a key role in the proof of Theorem 5.1. I use the maximum norm $\|.\|_\infty$ throughout this section, that is $\|x\|_\infty = \max_{i\in S}|x_i|$ for all $x \in \mathbb{R}^S$.

Lemma 5.2. *Let (S,X,d,b) and (S,X,d,c) be two bargaining problems. Then*

$$\|E(S,X,d,b) - E(S,X,d,c)\|_\infty \leq 2\,\|b-c\|_\infty\,. \tag{8}$$

Proof. Let $b \neq c$ and assume without loss of generality $b \in \partial X$, that is $b = E(S,X,d,b)$. Further assume that c minimizes $\|b-c'\|_\infty$ subject to $c' = c + t\mathbb{1}_S$ for some $t \in \mathbb{R}$. (Note that $E(S,X,d,c') = E(S,X,d,c)$ for all such c'.) Define

$$\begin{aligned} a^1 &= \inf\{c - t'\mathbb{1}_S : t' \in \mathbb{R} \text{ and } c - t'\mathbb{1}_S \not\ll b\}\,, \\ a^2 &= \sup\{c + t'\mathbb{1}_S : t' \in \mathbb{R} \text{ and } c + t'\mathbb{1}_S \not\gg b\}\,. \end{aligned}$$

Then $a^1 = c - t^1\mathbb{1}_S$ where $t^1 \leq \|b-c\|_\infty$, and $a^2 = c + t^2\mathbb{1}_S$ where $t^2 \leq \|b-c\|_\infty$. As E is strongly fair, either $E(S,X,d,c) \leq c$ or $E(S,X,d,c) \geq c$. Suppose $E(S,X,d,c) \leq c$. By definition of E and comprehensiveness of X it must hold that $E(S,X,d,c) = \lambda c + (1-\lambda)a^1$ for some $\lambda \in [0,1]$. Hence,

$$\begin{aligned} \|E(S,X,d,b) - E(S,X,d,c)\|_\infty &= \left\|b - \lambda c - (1-\lambda)a^1\right\|_\infty \\ &= \left\|b - \lambda c - (1-\lambda)(c - t^1\mathbb{1}_S)\right\|_\infty \\ &\leq \|b-c\|_\infty + (1-\lambda)\,t^1 \\ &\leq 2\,\|b-c\|_\infty\,. \end{aligned}$$

If $E(S,X,d,c) \geq c$ then $E(S,X,d,c) = \lambda c + (1-\lambda)a^2$ for some $\lambda \in [0,1]$ and the same equations hold for a^2 instead of a^1. □

Proof of Theorem 5.1. Let V be a game, let $x, y \in \Delta_{\text{eff}}(V)$, let $(S, \sigma) \in \mathcal{E}$, and let $i \in S$. Let $T, T' \in \{\emptyset\} \cup \sigma \setminus \{S\}$ be such that $O_{i,\sigma}(x) = x_{i,\sigma_{T+i}} - d_i(x)$ and $O_{i,\sigma}(y) = y_{i,\sigma_{T'+i}} - d_i(y)$. Without loss of generality let $x_{i,\sigma_{T+i}} \geq y_{i,\sigma_{T'+i}}$. Clearly, $y_{i,\sigma_{T'+i}} \geq y_{i,\sigma_{T+i}}$ by definition of $O_{i,\sigma}(y)$. Hence,

$$\begin{aligned}\left|C^{\mu,V}_{i,\sigma}(x) - C^{\mu,V}_{i,\sigma}(y)\right| &= (1-\mu)\left|O_{i,\sigma}(x) - O_{i,\sigma}(y)\right| \\ &= (1-\mu)\left|x_{i,\sigma_{T+i}} - y_{i,\sigma_{T'+i}}\right| \\ &\leq (1-\mu)\left|x_{i,\sigma_{T+i}} - y_{i,\sigma_{T+i}}\right| \\ &\leq (1-\mu)\left\|x-y\right\|_\infty .\end{aligned}$$

This implies $\left\|C^{\mu,V}(x) - C^{\mu,V}(y)\right\|_\infty \leq (1-\mu)\left\|x-y\right\|_\infty$. Let $\hat{E}^V$ be defined as in the proofs of Subsection 2.2. By Lemma 5.2

$$\begin{aligned}\left|\hat{E}^V_{i,\sigma}(x) - \hat{E}^V_{i,\sigma}(y)\right| &= \left|E_i\left(S, V(S,\sigma), d^V_S, C^{\mu,V}_{S,\sigma}(x)\right) - E_i\left(S, V(S,\sigma), d^V_S, C^{\mu,V}_{S,\sigma}(y)\right)\right| \\ &\leq 2\left\|C^{\mu,V}(x) - C^{\mu,V}(y)\right\|_\infty \\ &\leq 2(1-\mu)\left\|x-y\right\|_\infty .\end{aligned}$$

As $\mu > \frac{1}{2}$, $\hat{E}^V$ is a contraction from $\Delta_{\text{eff}}(V)$ into itself. Hence, by Banach's fixed point theorem, $\hat{E}^V$ has a unique fixed point in $\Delta_{\text{eff}}(V)$. □

The power of Lemma 5.2 lies in the fact that Inequality (8) does not depend on the set X. However, it can be considerably sharpened if the structure of X is known. In particular, if $X = \left\{x \in \mathbb{R}^S : \sum_{i \in S} x_i \leq y\right\}$ for some $y \in \mathbb{R}$ then E is simply an orthogonal projection of c on the surface of X. In this case it holds that $\left\|E(S, X, d, b) - E(S, X, d, c)\right\|_\infty \leq \left\|b-c\right\|_\infty$. Hence, for a game V with transferable utility the set $\mathcal{K}^{C^\mu}_E(V)$ is single-valued actually for all $\mu > 0$.[10]

Remark 5.3. For $\mu > \frac{1}{2}$ the payoff configuration $x \in \mathcal{K}^{C^\mu}_E(V)$ is not only stable with respect to renegotiation: it can be interpreted as the result of (infinite) bargaining. Start with an arbitrary payoff configuration x^0 of a game V and define $x^t = \hat{E}^V\left(x^{t-1}\right)$ for all $t \geq 1$. That is x^t is the payoff configuration after the payoff configuration x^{t-1} has been renegotiated. This delivers a sequence that converges towards the (unique) payoff configuration that is consistent with E under C^μ.

Instead of the egalitarian solution one might use the proportional solution in order to obtain similar results. If attention is restricted to those embedded coalitions in which at least one player has a strictly positive marginal contribution (similar to Remark 2.7), there are payoff configurations that are consistent with P under C^μ for any $\mu > 0$. Attempts to find uniqueness results were undertaken in Karos (2013) for proper monotonic simple games, but they were not fruitful for general games. The following example shall illustrate the results of this paper and closes this section.

[10] A deeper investigation of this solution in the context of proper monotonic simple games, including a example where uniqueness fails for $\mu = 0$, can be found in Karos (2013).

Example 5.4. In spring 2014 General Electrics (GE) fought a bidding war against a consortium of Siemens and Mitsubishi (SM) about taking over (parts of) the French company Alstom: on April 24 the news portal Bloomberg reported that GE planned to acquire Alstom and that negotiations were already taking place.[11] Although this report was neither confirmed nor denied by any of the parties, three days later Siemens released a statement to the press that "a letter has been submitted to the Board of Alstom to signal [Siemens'] willingness to discuss future strategic opportunities".[12] On April 29 Siemens announced their intention to make an offer for Alstom provided that access to Alstom's data room is granted for four weeks.[13] Alstom announced on May 6 that a binding offer had been made by GE,[14] and a joint offer of SM was made on June 16.[15] GE revised their offer on June 19;[16] SM revised theirs on June 20.[17] On June 26 Alstom's Board of Directors recommended the acceptance of the offer of GE.[18]

The negotiation process in Remark 5.3 nicely captures these events: assume that only coalitions {Alstom, GE} and {Alstom, SM} could make profit. (Alstom staying alone was not an option as they were close to bankruptcy.) In the spirit of Remark 2.7, attention can be restricted to payoffs in the two relevant partitions only. The first round of bargaining lasted from May 6 to June 16, the second round from June 19 to June 20. The result of every round depended on Alstom's outside options in the previous round and the derived (maybe implicit) claims.

6 Discussion

Since the seminal book of von Neumann and Morgenstern (1944) the theory of cooperative games has made significant developments. The characteristic function was extended by partition functions (Thrall and Lucas, 1963), games with non-transferable utility were introduced and studied (Luce and Raiffa, 1957;

[11]Kirchfeld, Campbell, and McCracken, *General Electric Said in Talks to Buy France's Alstom*, April 24, 2014, Bloomberg

[12]Press Release AXX201404.31, *Siemens signals Alstom willingness to discuss*, April 27, 2014, Siemens

[13]Press Release AXX201404.33, *Siemens will make an offer to Alstom*, April 29, 2014, Siemens

[14]Press Release, *Alstom is considering the proposed acquisition of its Energy activities by GE and the creation of a strong standalone market leader in the rail industry*, May 6, 2014, Alstom

[15]Joint Press Release AXX201406.46, *Mitsubishi Heavy Industries and Siemens provide a compelling proposal for Alstom*, June 16, 2014, Mitsubishi and Siemens; see also: Ad-hoc Announcement according to 15 WpHG (Securities Trading Act) *Siemens provides a proposal for Alstom together with Mitsubishi Heavy Industries*, June 16, 2014

[16]Press Release, *GE Announces Energy and Transport Alliance with Alstom*, June 19, 2014, General Electric

[17]Joint Press Release AXX201406.50, *Mitsubishi Heavy Industries and Siemens specify proposal to Alstom*, June 20, 2014, Mitsubishi and Siemens

[18]Press Release, *Alstom Board of Directors recommends General Electrics offer*, June 26, 2014

Aumann, 1967), and also the special case of bargaining problems received a lot of attention since Nash (1950). Today, many solutions can be found for many games in many contexts. The purpose of the present paper is to provide a solution that applies to the most general games, and yet can be refined when appropriate. Two things for such a solution seem necessary: players have to partition into coalitions, and in each coalition an allocation must be chosen. These two aspects are not independent but crucially depend on each other. A coalition will form only if no player expects to find a higher payoff in another coalition, and the payoff a player receives depends on her bargaining strength and therefore, ultimately, on the options she has in other coalitions. The solutions proposed in this article focus on both aspects: in a consistent payoff configuration, no renegotiation will take place, and in a core stable partition in the derived ordinal game players don't have any incentive to leave their coalitions.

6.1 Further related literature

The idea of considering a game with non-transferable utility as a collection of bargaining problems dates back at least to Harsanyi (1963) (see also Kalai, 1977). The model provided here is very much in this spirit, except that the advancements of the theory of bargaining with claims is used.

Before Chun and Thomson (1992) claims have appeared mainly in the context of bankruptcy games (see for instance Aumann and Maschler, 1985; Curiel et al., 1987). Related to this work is Moulin (2000) where rationing rules are investigated. Although only demands are explicitly defined there, the assumption of non-negative payoffs is equivalent to the idea of ultimate disagreement points in Chun and Thomson (1992). Models in the same spirit have also been proposed more recently by Hougaard et al. (2012, 2013).

The consideration of all players' (potential) payoffs in all coalitions when looking for outcomes is very much in the spirit of classical solutions such as the core (Gillies, 1959) or the bargaining set (Davis and Maschler, 1963; Peleg, 1963, for a generalization to games with non-transferable utility). Interestingly, these solutions can be obtained when choosing claim functions appropriately, for details see Karos (2015).

Ordinal games with externalities have not received a lot of attention so far. Without externalities they are better known under the name hedonic games, but not much is known about the existence of core stable partitions. Some sufficient conditions can be found for instance in Banerjee et al. (2001) or Bogomolnaia and Jackson (2002), and a necessary and sufficient condition is presented in Iehlé (2007).

The special case of ordinal games that are derived from payoff configurations has first been considered by Shenoy (1979) and more recently by Dimitrov and Haake (2008) and Karos (2014). But neither of the authors presented a necessary and sufficient condition for the existence of core stable partitions.

6.2 Further research

It will be important to investigate what bargaining rules and claim forms are the right choice, either to depict reality or to derive normative results. The research on axiomatic bargaining, which would be used for the latter, is already very advanced (see for instance Peters, 1992). In the context of bargaining with claims (or rationing with constraints) it seems that the proportional solution has many useful properties, both in my model and in previous articles (Chun and Thomson, 1992; Moulin, 2000). But it remains an open question what other efficient, individually rational, and weakly fair bargaining solutions may look like.

From a descriptive perspective, the results of Nydegger and Owen (1974) also suggest that the proportional bargaining rule is appropriate (see for instance Kalai, 1977). Findings from experiments in the context of the ultimatum game (see Oosterbeek et al., 2004, for a meta-study) suggest that fairness is an important property of division rules. Responders often reject proposals they consider too low, and this observation can be explained in a model of bargaining with claims where a bargaining rule is accepted only if it is fair. The fairness property of the proportional solution makes it therefore very appealing.

With the research into claim functions one enters unknown territory. In the context of bankruptcy problems, claims were assumed to be externally given, for instance in the form of unpaid debts. In my framework many ideas come together and it will probably be impossible to find the optimal claim function for every context. Both marginal contributions and opportunity costs have their justification, as have α and β-reasonableness.

References

Aumann, R., 1967. A survey of cooperative games without side payments, in: Shubik, M. (Ed.), Essays in Mathematical Economics in Honor of Oskar Morgenstern. Princeton University Press, pp. 3–27.

Aumann, R., Maschler, M., 1985. Game theoretic analysis of a bankruptcy problem from the talmud. Journal of Economic Theory 36, 195–213.

Banerjee, S., Konishi, H., Sönmez, T., 2001. Core in a simple coalition formation game. Social Choice and Welfare 19, 135–153.

Bogomolnaia, A., Jackson, M., 2002. The stability of hedonic coalition structures. Games and Economic Behavior 38, 201–230.

Chun, Y., Thomson, W., 1992. Bargaining problems with claims. Mathematical Social Science 24, 19–33.

de Clippel, G., Serrano, S., 2008. Marginal contributions and externalities in the value. Econometrica 76, 1413–1436.

Curiel, I., Maschler, M., Tijs, S., 1987. Bankruptcy games. Zeitschrift für Operations Research 31, 143–159.

Davis, M., Maschler, M., 1963. Existence of stable payoff configurations for cooperative games. Bulletin of the American Mathematical Society 69, 106–108.

Dimitrov, D., Haake, C., 2008. Stable governments and the semistrict core. Games and Economic Behavior 62, 460–475.

Drèze, J., Greenberg, J., 1980. Hedonic coalitions: Optimalilty and stability. Econometrica 48, 987–1003.

Gillies, D., 1959. Solutions to general non-zero-sum games, in: Kuhn, A., Luce, R. (Eds.), Contributions to the Theory of Games. Princeton University Press, pp. 47–85.

Harsanyi, J., 1963. A simplified bargaining model for the n-person cooperative game. International Economic Review 4, 194–220.

Hougaard, J., Moreno-Ternero, J., Østerdaal, 2012. A unifying framework for the problem of adjudicating conflicting claims. Journal of Mathematical Economics 48, 107–114.

Hougaard, J., Moreno-Ternero, J., Østerdaal, 2013. Rationing in the presence of baseline. Social Choice and Welfare 40, 1047–1066.

Iehlé, V., 2007. The core-partition of hedonic games. Mathematical Social Sciences 54, 176–185.

Kalai, E., 1977. Proportional solutions to bargaining situations: Interpersonal utility comparisons. Econometrica 45, 1623–1630.

Kalai, E., Smorodinsky, M., 1975. Other solutions to nash's bargaining problem. Econometrica 43, 513–518.

Karos, D., 2013. Bargaining and power. FEEM Working Paper No. 63.

Karos, D., 2014. Coalition formation in general apex games under monotonic power indices. Games and Economic Behavior 87, 239–252.

Karos, D., 2015. Stable partitions for games with non-tranferable utilities and externalities. Oxford Working Paper Series No. 741.

Luce, R., Raiffa, H., 1957. Games and Decisions. Wiley.

Moulin, H., 2000. Priority rules and other assymmetric rationing methods. Econometrica 68, 643–684.

Nash, J., 1950. The bargaining problem. Econometrica 18, 155–162.

von Neumann, J., Morgenstern, O., 1944. Theory of Games and Economic Behaviour. Princeton University Press.

Nydegger, R., Owen, G., 1974. Two-person bargaining: An experimental test of the nash axioms. International Journal of Game Theory 3, 239–249.

Oosterbeek, H., Sloof, R., van de Kuilen, G., 2004. Cultural differences in ultimatum game experiments: Evidence from a meta-analysis. Experimental Economics 6, 171–188.

Otten, G., Borm, P., Peleg, B., Tijs, S., 1998. The mc-value for monotonic ntu-games. International Journal of Game Theory 27, 37–47.

Peleg, B., 1963. Bargaining sets for cooperative games without side payments. Israel Journal of Mathematics 1, 197–200.

Peters, H., 1992. Axiomatic Bargaining Theory. Kluwer Academic Publishers.

Ray, D., Vohra, R., 1997. Equilibrium binding agreements. Journal of Economic Theory 73, 30–78.

Shenoy, P., 1979. On coalition formation: A game-theoretical approach. International Journal of Game Theory 8, 133–164.

Thrall, R., Lucas, W., 1963. n-person games in partition function form. Naval Research Logistics Quarterly 10, 281–298.