

On the topology of algebraic surfaces and reduction modulo p

Dosang Joe and Minhyong Kim

Abstract

We show that the topology of a simply-connected smooth projective surface is determined by its algebraic structure modulo p .

1 Observation

The goal of this note is to make the following simple

Observation: Let X and Y be simply-connected smooth projective surfaces. Suppose they are isomorphic modulo p for some prime $p \neq 2$ of good reduction. Then X and Y are homeomorphic by a map preserving the complex orientations.

Here, isomorphic modulo p means the following: We can find a common finitely-generated ring of definition R for X and Y (that is, such that X and Y can both be defined by equations with coefficients in R), and a prime ideal with residue field k of characteristic p such that

$$X \otimes_R \bar{k} \simeq Y \otimes_R \bar{k}$$

That is, X and Y are isomorphic when the coefficients are regarded as lying in the algebraic closure of k . ‘Good reduction’ means the two varieties appearing in this isomorphism are smooth over \bar{k} . For example, they might both be defined over \mathbf{Z} , in which case the meaning of everything is clear.

Here is the proof: By Freedman’s theorem ([7], theorem 1.5) the oriented homeomorphism-type of a simply-connected oriented smooth 4-manifold X is determined by the unimodular intersection pairing

$$B : H^2(X, \mathbf{Z}) \otimes H^2(X, \mathbf{Z}) \rightarrow H^4(X, \mathbf{Z}) \simeq \mathbf{Z}$$

on the free abelian group $H^2(X, \mathbf{Z})$. We examine this for algebraic surfaces. By the smooth and proper base-change theorems ([8], chapters 12-14),

$$B_l : [H^2(X, \mathbf{Z}) \otimes \mathbf{Z}_l] \otimes [H^2(X, \mathbf{Z}) \otimes \mathbf{Z}_l] \rightarrow \mathbf{Z}_l,$$

the \mathbf{Z}_l -linear extension of B , is determined by $X \bmod p$ for any $l \neq p$. In particular, the rank and type (even or odd, by taking $l = 2$) of B is determined by reduction mod p . On the other hand, the signature of X is determined by Hirzebruch's signature theorem ([9], theorem 8.2.2) to be

$$(c_1^2(X) - 2c_2(X))/3$$

The numerical invariants in this formula are preserved under specialization. Therefore, we conclude that the rank, type, and signature of B are determined by reduction mod p . This concludes the proof when B is indefinite ([11], chapter 5, theorem 6). But if B is definite, Donaldson's theorem ([5],[6] theorem 1.3.1) says that

$$B \cong \pm(x_1^2 + x_2^2 + \cdots x_r^2)$$

where r is the rank. But whether or not B is definite (as well as the sign) is also determined by the rank and signature, and hence, by the reduction modulo p . So we are done.

2 Comments

Determination of topological invariants of varieties by modulo p arithmetic is of course a well-known side-effect of modern arithmetic geometry.

For a smooth and proper variety, the Betti numbers, for example, are determined by the reduction of the variety modulo a (good) prime p .

More intricate invariants can also be brought in (where we always assume that the variety is smooth and proper and the prime is good):

-The Hodge numbers, for example, are determined by reduction modulo an *ordinary* prime p ([2] Theorem (0.7)).

-The integral cohomology groups are determined by reduction modulo *two* primes: this is because two primes are sufficient to determine all $H^i(X, \mathbf{Z}_p)$, and then, $H^i(X, \mathbf{Z})$ by the universal coefficient theorem.

-For simply-connected varieties, the rational homotopy groups are determined by reduction modulo p ([1], [4], [10]).

-For simply-connected varieties, the integral higher homotopy groups are determined by reduction modulo two primes [1].

But the case of simply-connected surfaces is the only one we know of where something as refined as the homeomorphism-type is actually determined by reduction modulo p . Our proof is of course a consequence of the very powerful classification theorems for four-manifolds. As such, it appears essentially to be an accident. On the other hand, it would be interesting to seek out other non-trivial examples of such theorems, if only to probe their accidental nature.

We point out also that there are parallel results where topological invariants are preserved by *conjugation*, that is, hitting the coefficients of some defining equations with an automorphism of the complex numbers. As is obvious from our proof, this would be true, for example, for the homeomorphism type of simply-connected smooth projective surfaces.

One final remark: As is well-known, as a consequence of the Weil conjectures [3], the Betti numbers of a smooth proper variety are determined just by the zeta-function of its reduction mod p . So in this very rigid case of simply-connected surfaces, it is natural to ask if the homeomorphism type is determined by the zeta function. However, the simplest possible case of $\mathbf{P}^1 \times \mathbf{P}^1$ and \mathbf{P}^2 blown up at a point provides a counter-example. In fact, this shows that even the rational homotopy type cannot be determined by the zeta function.

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M.K.: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ARIZONA, TUCSON,
AZ 85721 and KOREA INSTITUTE FOR ADVANCED STUDY, 207-43 CHEONGRYANGRI-
DONG DONGDAEMUN-GU, SEOUL, KOREA, 130-012. EMAIL: kim@math.arizona.edu

D.J.: DEPARTMENT OF MATHEMATICS, POHANG UNIVERSITY OF SCIENCE
AND TECHNOLOGY, POHANG, KYOUNGBUK, KOREA 790-784. EMAIL: joe@euclid.postech.ac.kr