

Democratic Fair Allocation of Indivisible Goods*

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We study the problem of fairly allocating indivisible goods to groups of agents. Agents in the same group share the same set of goods even though they may have different preferences. Previous work has focused on *unanimous fairness*, in which all agents in each group must agree that their group’s share is fair. Under this strict requirement, fair allocations exist only for small groups. We introduce the concept of *democratic fairness*, which aims to satisfy a certain fraction of the agents in each group. This concept is better suited to large groups such as cities or countries. We present protocols for democratic fair allocation among two or more arbitrarily large groups of agents with monotonic, additive, or binary valuations. Our protocols approximate both envy-freeness and maximin-share fairness. As an example, for two groups of agents with additive valuations, our protocol yields an allocation that is envy-free up to one good and gives at least half of the maximin share to at least half of the agents in each group.

1. Introduction

Fair division is the study of how to allocate resources among agents with different preferences so that agents perceive the resulting allocation as fair. This problem occurs in a wide range of situations, from negotiating over international interests and reaching divorce settlements [Brams and Taylor, 1996] to dividing household tasks and sharing apartment rent [Goldman and Procaccia, 2014].

Two kinds of fairness criteria are common in the literature. The first, *envy-freeness* (*EF*), means that each agent finds her share at least as good as the share of any other agent. When allocating indivisible goods, envy-freeness is sometimes unattainable (consider two agents quarreling over a single good), so it is often relaxed to *envy-freeness up to one good* (*EF1*), which is always attainable [Lipton et al., 2004, Budish, 2011]. The second kind, *maximin share fairness*, means that each agent finds his share at least as

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good as his *maximin share* (MMS), which is the best share he can secure by dividing the goods into n parts and getting the worst part, where n is the number of agents. Again, with indivisible goods MMS fairness cannot be guaranteed, but at least a constant fraction of the MMS can [Procaccia and Wang, 2014].

Most works on fair division involve individual agents, each of whom has individual preferences. But in reality, resources often have to be allocated among *groups of agents*, such as families or countries. A good allocated to a group is shared among the group members and all of them derive full utility from the good. For example, when dividing real estate among families, all members of a family enjoy their allocated house and backyard. In international negotiations, the divided rights and settled outcomes are enjoyed by all citizens of a country. When resources are allocated between different buildings of a university, all occupants of a building benefit from the whiteboards and open space allocated to their building. However, different group members may have different preferences. The same share can be perceived as fair by one member and unfair by another member of the same group. Ideally, we would like to find an allocation that is considered fair by *all* agents in *all* groups. However, two recent works show that this “unanimous fairness” might be too strong to be practical.

(a) Suksompong [2018a] shows that when allocating indivisible goods among groups, there might be no allocation that is unanimously EF1. Moreover, there might be no division that gives all agents a positive fraction of their MMS. This impossibility occurs even for two groups of three agents each.

(b) Segal-Halevi and Nitzan [2015] show that when allocating a divisible good (“cake”) among groups, there might be no division that is unanimously envy-free and gives each group a single connected piece, or even a constant number of connected pieces. In contrast, with individual agents a connected envy-free division always exists [Stromquist, 1980].

What do groups do when they cannot attain unanimity? In democratic societies, they use some kind of voting. The premise of voting is that it is impossible to satisfy everyone, so we should try to satisfy as many members as possible. Based on this observation, we say that a division is *h-democratic fair*, for some fairness notion and for some $h \in [0, 1]$, if at least a fraction h of the agents in each group believe it is fair. In this paper we focus on allocating indivisible goods. We would like h , the fraction of happy agents, to be as large as possible. We thus pose the following question:

Given a fairness notion, what is the largest h such that an h -democratic fair allocation of indivisible goods can always be found?

We study democratic fairness under three different assumptions on the agents’ valuations. In the most general case, the agents can have arbitrary *monotonic* valuations on bundles of goods. A more common assumption in the literature is that agents’ valuations are *additive* (the value of a bundle is the sum of the values of the goods in the bundle). We also study a special case of additive valuations in which agents’ valuations are *binary* (each agent has a set of desired goods and her utility equals the number of desired goods allocated to her group).

1.1. Overview of our results

Initially (**Section 3**) we consider two groups with binary agents. We study a relaxation of envy-freeness that we call *envy-freeness up to c goods* (EF c), a generalization of EF1. One might expect to have a trade-off curve where a larger c corresponds to a larger h . However, we find that the actual trade-off curve is degenerate: for every constant c , it is possible to guarantee $1/2$ -democratic EF c , and the factor $1/2$ is tight. The same holds for MMS fairness. To get a more flexible trade-off curve, we study a generalization of MMS called *1-out-of- c MMS*, which is the best share an agent can secure by dividing the goods into c subsets and receiving the worst one. We prove that 1-out-of- c MMS can be guaranteed to at least $1 - 1/2^{c-1}$ and at most $1 - 1/2^c$ of the agents in both groups.

Our positive results are attained by an efficient round-robin protocol where each group in turn picks a good using *weighted approval voting* with carefully calculated weights. The weights of agents who lose in early votes are increased in later votes. We believe this weighted voting scheme can be interesting in its own right as a way to make fair group decisions.

Next (**Section 4**) we consider two groups whose agents have arbitrary monotonic valuations. We present an efficient protocol that guarantees EF1 to at least $1/2$ of the agents in each group (which is tight even for binary agents). When all agents are additive, this protocol guarantees $1/2$ of the MMS to $1/2$ of the agents. This is tight: one cannot guarantee more than $1/2$ of the MMS to more than $1/3$ of the agents. Moreover, a positive fraction of the MMS can be guaranteed to at least $3/5$ and at most $2/3$ of the agents in both groups. If we are instead interested in relaxing envy-freeness, it is possible to guarantee unanimous EF($n - 1$) when agents have additive valuations, where n denotes the total number of agents in the two groups.

Finally (**Section 5**), we present two generalizations of our results to $k \geq 3$ groups. The first generalization has stronger fairness guarantees: when all valuations are monotonic, it guarantees EF2 to $1/k$ of the agents in all groups; when all valuations are binary, it guarantees both EF1 and MMS to $1/k$ of the agents in all groups (the factor $1/k$ is tight for EF c for any constant c , even when valuations are binary). However, the run-time of the protocol might be exponential. The second generalization uses a polynomial-time protocol but has weaker guarantees: when valuations are additive, it guarantees an additive approximation of MMS, and when all valuations are binary, it guarantees MMS to $1/k$ of the agents. We also show that for any number of groups, a generalized version of our round-robin protocol from Section 3 gives a positive fraction of the MMS to at least $(e - 1)/(2e - 1) \approx 39\%$ of the agents in each group.

Some of our results and open questions are summarized in Table 1.

1.2. Related work

The group resource allocation problem is relatively new. We already mentioned the impossibility result of Suksompong [2018a], which is for worst-case agents' utilities. On the other hand, if the agents' utilities are drawn at random from probability distributions, Manurangsi and Suksompong [2017a] showed that a unanimously envy-free allocation

Happy $h \downarrow$ Share $q \rightarrow$	Positive	$(0, 1/2]$	$(1/2, 1]$
$(0, 1/3]$	Yes (Thm. 4.5)	Yes (Cor. 4.3)	Bin: Yes (Cor. 4.3), Add: ?
$(1/3, 1/2]$			Bin: Yes (\uparrow), Add: No (\downarrow)
$(1/2, 3/5]$?	?
$(3/5, 2/3]$			
$(2/3, 1]$	No (Prop. 3.1)		

Happy h	EFc for any constant $c \geq 1$
$(0, 1/2]$	Yes (Thm. 4.1)
$(1/2, 1]$	No (Prop. 3.5)

Table 1: Summary of results for two groups with **Binary** and **Additive** agents. For each range of $h, q \in (0, 1]$, the table shows whether there always exists an allocation that gives at least a fraction h of the agents in each group at least a fraction q of their maximin share. For EFc, both results are valid for binary, additive and monotonic valuations. The arrows refer to the directions pointed to in the table. The table is clearer when viewed in color.

exists with high probability as the number of agents and goods grows. In our terminology, unanimous fairness is called *1-democratic-fairness*.¹

The term democratic fairness appears in the work of Segal-Halevi and Nitzan [2015]; however, they use it in the narrower sense that at least $1/2$ of the agents in each group must be satisfied. In our terminology this is called *1/2-democratic fairness*. Hence, our democratic fairness notion generalizes existing notions of group fairness.

A related model, in which a subset of public goods is allocated to a single group of agents but the rest of the goods remain unallocated, has also been studied [Suksompong, 2016, Manurangsi and Suksompong, 2017b].

MMS fairness was introduced by Budish [2011] based on earlier concepts by Moulin [1990]. Budish also considered its relaxation to 1-out-of- $(n+1)$ MMS. The notion 1-out-of- c MMS is a special case of l -out-of- d MMS, recently defined by Babaioff et al. [2017] and studied by Segal-Halevi [2018].

Our group-fairness notions differ from those defined, e.g., by Berliant et al. [1992], Husseinov [2011], and Todo et al. [2011]. In their setting, goods are divided among *individuals*. The challenge comes from the requirement to eliminate envy, not only between individuals, but also between subsets of agents. In our setting, the challenge is that the goods are divided among *groups*. A share that is desirable for some group members might be undesirable for other members of the same group. This motivates the use of social choice techniques such as having each group vote on which goods to pick.

Group preferences are important in matching markets, too. For example, when matching doctors to hospitals, usually a husband and a wife want to be matched to the same hospital. This issue poses a substantial challenge to stable-matching mechanisms

¹See also [Suksompong, 2018b] for an overview of the group resource allocation problem.

[Klaus and Klijn, 2005, 2007, Kojima et al., 2013].

2. Preliminaries

There is a set $G = \{g_1, \dots, g_m\}$ of goods. A *bundle* is a subset of G . There is a set A of agents. The agents are partitioned into k groups A_1, \dots, A_k with n_1, \dots, n_k agents, respectively. Let a_{ij} denote the j th agent in group A_i . Each agent a_{ij} has a nonnegative utility $u_{ij}(G')$ for each $G' \subseteq G$. For any agent a_{ij} , denote by $u_{ij,\max} := \max_{l=1,\dots,m} u_{ij}(g_l)$ the maximum utility of the agent for any single good. Denote by $\mathbf{u}_{ij} = (u_{ij}(g_1), \dots, u_{ij}(g_m))$ the utility vector of agent a_{ij} for single goods. The agents' utility functions are *monotonic*, i.e., $u_{ij}(G'') \leq u_{ij}(G')$ for every $G'' \subseteq G' \subseteq G$ and every agent a_{ij} . A subclass of monotonic utilities is the class of *additive* utilities, i.e., for every bundle $G' \subseteq G$ and every agent $a_{ij} \in A$, we have $u_{ij}(G') = \sum_{g \in G'} u_{ij}(g)$.

Sometimes we will study a special case of additive utilities in which utilities are *binary*, i.e., each agent either approves or disapproves each good. Since we will not engage in interpersonal comparison of utilities, we may assume without loss of generality that in this case $u_{ij}(g) \in \{0, 1\}$ for each i, j, g .

We allocate a bundle $G_i \subseteq G$ to each group A_i . All goods should be allocated. The goods are treated as public goods within each group, i.e., for every group i , the utility of every agent a_{ij} is $u_{ij}(G_i)$. We refer to a setting with agents partitioned into groups, goods and utility functions as an *instance*.

We now define the fairness notions considered in this paper. We begin by defining what it means for an allocation to be fair for a *specific* agent. We start with envy-freeness.

Definition 2.1. Given an agent a_{ij} and an integer $c \geq 0$, an allocation is called *envy-free up to c goods (EF c)* for a_{ij} if for every i' there exists a set $C_{i'} \subseteq G_{i'}$ with $|C_{i'}| \leq c$ such that:

$$u_{ij}(G_i) \geq u_{ij}(G_{i'} \setminus C_{i'}).$$

In other words, one can remove the envy of a_{ij} toward group i' by removing at most c goods from the group's bundle.

An EF0 allocation is also known as *envy-free*.

Next, we define the maximin share concepts.

Definition 2.2. Given an agent a_{ij} and an integer $c \geq 2$, the *1-out-of- c maximin share (MMS)* of a_{ij} is defined as the maximum, over all partitions of G into c sets, of the minimum of the agent's utilities for the sets in the partition:

$$\text{MMS}_{ij}^c(G) := \max_{G'_1, \dots, G'_c} \min(u_{ij}(G'_1), \dots, u_{ij}(G'_c)),$$

where (G'_1, \dots, G'_c) is a partition of G . When $c = k$ (the number of groups), the 1-out-of- k MMS of an agent is simply called his *MMS* and denoted by $\text{MMS}_{ij}(G)$. An allocation (G_1, \dots, G_k) is said to be:

- 1-out-of- c MMS-fair for a_{ij} , if $u_{ij}(G_i) \geq \text{MMS}_{ij}^c(G)$.
- MMS-fair for a_{ij} , if $u_{ij}(G_i) \geq \text{MMS}_{ij}(G)$.
- q -MMS-fair for a_{ij} , for some fraction $q \in (0, 1)$, if $u_{ij}(G_i) \geq q \cdot \text{MMS}_{ij}(G)$.
- positive-MMS-fair for a_{ij} , if $\text{MMS}_{ij}(G) > 0$ implies $u_{ij}(G_i) > 0$.

Note that MMS-fairness implies q -MMS-fairness (for any q), which in turn implies positive-MMS-fairness. The next lemma shows an interesting link between EF1 and MMS-fairness. Part (a) was proved concurrently and independently by Amanatidis et al. [2018, Prop. 3.6].

Lemma 2.1. *If an allocation among k groups is EF1 for an agent with an additive utility function, then:*

- (a) *it is also $\frac{1}{k}$ -MMS-fair for that agent— $\frac{1}{k}$ is tight;*
- (b) *it is also 1-out-of- $(2k - 1)$ -MMS-fair for that agent— $2k - 1$ is tight;*
- (c) *if the agent's utility function is also binary, then the allocation is also MMS-fair for that agent.*

Proof. Denote by u the utility function of the agent and assume without loss of generality that the agent is in group A_1 .

(a) EF1 implies that in each bundle G_i (for $i \in \{2, \dots, k\}$) there exists a subset C_i with $|C_i| \leq 1$ such that $u(G_1) \geq u(G_i \setminus C_i)$. Summing over groups $2, \dots, k$ and adding $u(G_1)$ to both sides gives $k \cdot u(G_1) \geq u(G \setminus (C_2 \cup \dots \cup C_k))$. Now, in any partition of G into k bundles, there is at least one bundle that does not contain any good in $C_2 \cup \dots \cup C_k$. This bundle is contained in $G \setminus (C_2 \cup \dots \cup C_k)$. Therefore, the MMS is at most $u(G \setminus (C_2 \cup \dots \cup C_k))$ which is at most $k \cdot u(G_1)$. Therefore, $u(G_1)$ is at least $1/k$ of the MMS.

To show that the factor $1/k$ is tight, assume that there are $2k - 1$ goods with $u(g_1) = \dots = u(g_k) = 1$ and $u(g_{k+1}) = \dots = u(g_{2k-1}) = k$. If the agent's group gets g_1 and group $i \geq 2$ gets $\{g_i, g_{k+i-1}\}$, the agent gets utility 1 and finds the allocation EF1. However, the MMS is k , as can be seen from the partition $(\{g_1, \dots, g_k\}, \{g_{k+1}\}, \dots, \{g_{2k-1}\})$.

(b) As above EF1 implies $k \cdot u(G_1) \geq u(G \setminus (C_2 \cup \dots \cup C_k))$. In any partition of G into $2k - 1$ bundles, there are at least k bundles that do not contain any good in $C_2 \cup \dots \cup C_k$. The union of these bundles is contained in $G \setminus (C_2 \cup \dots \cup C_k)$. Therefore, at least one of these bundles has utility at most $u(G \setminus (C_2 \cup \dots \cup C_k))/k$ which is at most $u(G_1)$. Therefore, $u(G_1)$ is at least the 1-out-of- $(2k - 1)$ -MMS.

To show that $2k - 1$ is tight, consider the following allocation:

- Group A_1 gets a single good worth $k - 1$;
- Each group A_2, \dots, A_n gets one good worth k plus $k - 1$ goods worth 1 each.

The allocation is EF1 for the agent. However, the 1-out-of- $(2k - 2)$ MMS of the agent is k due to the following partition (note that in total, there are 1 good worth $k - 1$, $k - 1$ goods worth k , and $(k - 1)(k - 1) = k^2 - 2k + 1$ goods worth 1):

- Bundle 1 has the good worth $k - 1$ plus one good worth 1;
- Each of the $k - 1$ bundles $2, \dots, k$ has one good worth k ;
- Each of the $k - 2$ bundles $k + 1, \dots, 2k - 1$ has k goods worth 1.²

(c) Suppose the agent's group wins l of the agent's desired goods. EF1 implies that each of the other $k - 1$ groups wins at most $l + 1$ of the agent's desired goods. Hence the agent has at most $kl + k - 1$ desired goods. Therefore the agent's MMS is at most l , so the allocation is MMS-fair for her. \square

Now we are ready to define our main group fairness notion:

Definition 2.3. For any given fairness notion, an allocation (G_1, \dots, G_k) is said to be *h-democratic fair* if it is fair for at least $h \cdot n_i$ agents in group A_i , for all $i \in \{1, \dots, k\}$.

We also refer to 1-democratic fairness as *unanimous fairness*.

3. Two Groups with Binary Valuations

This section considers the setting where there are two groups, the agents have additive valuations, and each agent either *desires* a good (in which case her utility for the good is 1) or does not desire it (in which case her utility is 0).

3.1. Negative results

Even in the special case of binary valuations, some fairness guarantees are unattainable.

Proposition 3.1. *For any $h > 2/3$, there is a binary instance in which no allocation is h-democratic positive-MMS-fair.*

Proof. Suppose that there are three goods. Each group consists of three members, each of whom has utility 0 for a unique good and utility 1 for each of the other two goods. Each agent has a positive MMS (1), but no allocation gives all agents a positive utility. \square

Moreover, deciding whether an instance admits an allocation that gives all agents a positive utility is a computationally hard problem.

Proposition 3.2. *Deciding whether a binary instance with two groups admits an allocation that gives every agent a positive utility is NP-complete.*

²A similar proof shows that for every $c \geq 1$, EF c implies 1-out-of- $(c(k - 1) + k)$ -MMS fairness.

Proof. For any allocation, we can clearly verify in polynomial time whether it yields a positive utility to every agent. To show that the problem is NP-hard, we reduce from MONOTONE SAT, a variant of the classical satisfiability problem where each clause contains either only positive literals or only negative literals. MONOTONE SAT is known to be NP-hard [Garey and Johnson, 1979, p. 259].

Given a MONOTONE SAT formula ϕ with variables x_1, \dots, x_m , let there be m items corresponding to the m variables. For each clause that contains only positive literals, we construct an agent in the first group who values exactly the items contained in this clause. Similarly, for each clause that contains only negative literals, we construct an agent in the second group who values exactly the items contained in this clause. Any assignment that satisfies ϕ gives rise to an allocation where the items corresponding to true variables in the assignment are allocated to the first group and those corresponding to false variables in the assignment are allocated to the second group; this allocation gives every agent nonzero utility. Likewise, any allocation that gives every agent nonzero utility yields a satisfying assignment of ϕ . Hence the reduction is valid. \square

We now proceed to prove more general negative results. In binary instances, each agent can be represented by two integers, which we denote here by r and s :

- r is the number of goods that the agent finds desirable;
- s is the number of goods that the agent needs in order to consider the allocation fair.

Specific fairness requirements define s as a function of r . For example:

- EFC means that $s = \lfloor (r - c + 1)/2 \rfloor$. In particular, EF1 means that $s = \lfloor r/2 \rfloor$.
- 1-out-of- c MMS means that $s = \lfloor r/c \rfloor$.
- Positive-MMS means that $s = 1$ whenever $r \geq 2$.

We prove a negative result for general r and s , and then use it to derive negative results for specific fairness requirements.

Proposition 3.3. *Let r, s be integers such that $r \geq s \geq 1$. Consider a binary instance with two groups in which each agent desires exactly r goods and needs s goods in order to consider the allocation fair. Then, it is impossible to attain more than $\text{MAXH}(r, s)$ -democratic fairness, where:*

$$\text{MAXH}(r, s) = \begin{cases} 0 & \text{when } r \leq 2s - 1; \\ \frac{1}{2^r} \sum_{i=s}^r \binom{r}{i} = 1 - \frac{1}{2^r} \sum_{i=0}^{s-1} \binom{r}{i} & \text{when } r \geq 2s. \end{cases}$$

Proof. First, suppose that $r \leq 2s - 1$, the total number of goods is r , and all agents in both groups desire all goods. At least one group will get at most $r/2 < s$ goods, so all of its members will be unhappy.

$r \downarrow s \rightarrow$	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0
2	1	<i>0.75</i>	0	0	0	0	0	0	0	0
3	1	0.875	0	0	0	0	0	0	0	0
4	1	0.938	0.688	0	0	0	0	0	0	0
5	1	0.969	0.813	0	0	0	0	0	0	0
6	1	0.985	0.891	0.657	0	0	0	0	0	0
7	1	0.993	0.938	0.774	0	0	0	0	0	0
8	1	0.997	0.966	0.856	0.637	0	0	0	0	0
9	1	0.999	0.982	0.911	0.747	0	0	0	0	0
10	1	1	0.991	0.947	0.829	0.624	0	0	0	0

Table 2: Some values of $\text{MAXH}(r, s)$. The italics at $(r, s) = (2, 1)$ denotes that this upper bound is not tight—Proposition 3.1 shows an upper bound of $2/3$ in this case.

Next, suppose that there are $2m$ goods, with $m \gg r$. In each group there are $\binom{2m}{r}$ members, each of whom wants a distinct subset of r goods. At least one group will get at most m goods. In this group, the fraction of happy agents will be at most:

$$\frac{\sum_{i=s}^r \binom{m}{i} \cdot \binom{m}{r-i}}{\binom{2m}{r}}.$$

When $m \gg r$, the numerator is approximately $\sum_{i=s}^r \frac{m^i}{i!} \cdot \frac{m^{r-i}}{(r-i)!} = \sum_{i=s}^r \frac{m^r}{i!(r-i)!}$ and the denominator is approximately $\frac{(2m)^r}{r!} = \frac{2^r m^r}{r!}$. Therefore when $m \rightarrow \infty$ the expression approaches $\frac{1}{2^r} \sum_{i=s}^r \binom{r}{i}$, which is equal to $1 - \frac{1}{2^r} \sum_{i=0}^{s-1} \binom{r}{i}$. \square

For illustration, some values of $\text{MAXH}(r, s)$ are shown in Table 2.

Proposition 3.4. *For any integer $c \geq 2$ and $h > 1 - 1/2^c$, there is a binary instance with two groups in which no allocation is h -democratic 1-out-of- c MMS-fair.*

Proof. Apply Proposition 3.3 with $r = c$ and $s = 1$. Then $\text{MAXH}(r, s) = 1 - \frac{1}{2^c} \cdot 1$. \square

Note that for $c = 2$, Proposition 3.1 gives a tighter upper bound of $2/3$.

Proposition 3.5. *For any constant integer $c \geq 1$ and $h > 1/2$, there is a binary instance with two groups in which no allocation is h -democratic EF c .*

Proof. Let l be a large positive integer. Apply Proposition 3.3 with $r = 2l$ and $s = \lfloor (r - c + 1)/2 \rfloor = l - \lfloor c/2 \rfloor$. Then $\text{MAXH}(r, s) = \frac{1}{2^{2l}} \sum_{i=l-\lfloor c/2 \rfloor}^{2l} \binom{2l}{i}$. When $l \rightarrow \infty$, this expression approaches $1/2$. \square

3.2. Positive results

Our positive results are attained with a protocol we call *Round-robin with Weighted Approval Voting (RWAV)*. In this protocol, each group picks a single good in turn, until all goods are taken. Each group picks its good using a weighted approval voting scheme. The members' weights are determined using fiat money. Initially, each group has an account that starts at a balance of 0, and each agent also starts with zero money. Each agent can pay money to the group account or receive money from the group account. RWAV proceeds as follows.

- (a) Initially, each member pays to his group account some amount of fiat money to be calculated later.
- (b) Whenever it is the group's turn to pick, each member is assigned a *weight* to be calculated later.
- (c) For each good, the *total weight* is calculated as the sum of the weights of the members who desire this good. The group picks a good with a maximal total weight (breaking ties arbitrarily).
- (d) Every member whose desired good was picked by his group pays to his group account his weight plus a certain extra amount to be calculated later.
- (e) When it is the other group's turn to pick, each member whose desired good was picked by the other group receives his weight from his own group account.

We now to calculate the agents' weights. An agent's weight will be a function of the number r of his desired goods that remain untaken, and the number s of desired goods that he should still receive. Note that both r and s of an agent weakly decrease as the protocol runs. An agent becomes happy when $s = 0$ and unhappy when $r - s < 0$. Let $w(r, s)$ be the weight of such an agent and $B(r, s)$ the net amount paid by such an agent to his group account.

In step (e), each losing agent receives $w(r, s)$, so $\forall r \geq s > 0$:

$$(*) \quad B(r, s) - w(r, s) = B(r - 1, s)$$

In step (d), we would like the winning agents to pay enough money for covering the payments to the losing agents. Therefore, each winning agent should pay at least his weight $w(r, s)$. But, when $r \geq 2$, an agent winning a good in step (d) might then lose a different good in step (e), where his weight will be $w(r - 1, s - 1)$. Therefore the agent must pay at least $w(r - 1, s - 1)$. All in all, a winning agent pays $\max[w(r, s), w(r - 1, s - 1)]$ when $r \geq 2$ and $w(r, s)$ otherwise:³

$$(**) \quad \begin{aligned} B(r, s) + \max[w(r, s), w(r - 1, s - 1)] &= B(r - 1, s - 1) & r \geq 2 \\ B(r, s) + w(r, s) &= B(r - 1, s - 1) & r = 1 \end{aligned}$$

³In the conference version of this paper [Segal-Halevi and Suksompong, 2018], we erroneously wrote that a winning agent should pay $w(r, s)$. This is a mistake since it neglects the possibility of an agent winning a good and immediately losing another good afterwards.

Note that in the case $r = 1$, the recurrence relation is relevant only when $s = 1$.

After some rearrangements we get the following recurrence relation for $B(r, s)$:⁴

$$(***) \quad \forall r \geq s > 0, r \geq 2: \quad B(r, s) = \min \left[\frac{1}{2}[B(r-1, s) + B(r-1, s-1)], B(r-2, s-1) \right]$$

$$\text{for } r = 1, s = 1: \quad B(r, s) = \frac{1}{2}[B(r-1, s) + B(r-1, s-1)]$$

We set the initial conditions of the recurrence relation so that, when the protocol ends, the net payment paid by each happy agent is 1 and by each unhappy agent is 0 (i.e., each unhappy agent got all his money back):

$$\begin{aligned} \forall r \geq 0: & \quad B(r, 0) = 1 \quad (\text{happy agents}) \\ \forall r < s: & \quad B(r, s) = 0 \quad (\text{unhappy agents}) \end{aligned}$$

Using this recurrence relation, $B(r, s)$ and $w(r, s)$ can be calculated for any r, s . Some values are shown in Table 3.

To complete the specification of the protocol, we have to calculate the initial payment of each agent in step (a). Consider an agent with r desired goods. Given a fairness criterion, we can calculate the number $s(r)$ of goods that such an agent should receive in order to satisfy the fairness criterion. So in step (a), each such agent should pay $B(r, s(r))$ to the group account.

We are now ready to state the main lemma.

Lemma 3.6. *Given a fairness criterion represented by an integer function $s(r)$, for every group $i \in \{1, 2\}$, the RWAV protocol with initial payments $B(r, s(r))$ yields an allocation that is fair for at least a fraction h_i of the agents in group i , where:*

$$\begin{aligned} h_1 &= \inf_{r=1,2,\dots} B(r, s(r)); \\ h_2 &= \inf_{r=1,2,\dots} B(r-1, s(r)). \end{aligned}$$

Proof. We have to prove that, when RWAV ends, in each group i there are at least $h_i \cdot n_i$ happy agents. By the boundary conditions on B , when RWAV ends, each unhappy agent has paid to the group account a net amount of 0, and each happy agent has paid a net

⁴ The case $r = 1$ is easy: just sum (*) and (**) and divide by 2.

Proof of the case $r \geq 2$: (*) implies: $w(r, s) = B(r, s) - B(r-1, s)$ and $w(r-1, s-1) = B(r-1, s-1) - B(r-2, s-1)$.

Substituting these into (**) and re-arranging gives:

$$\max[2B(r, s) - B(r-1, s) - B(r-1, s-1), B(r, s) - B(r-2, s-1)] = 0,$$

which implies that $B(r, s)$ satisfies the following inequalities

$$B(r, s) \leq [B(r-1, s) + B(r-1, s-1)]/2 \quad \text{and} \quad B(r, s) \leq B(r-2, s-1),$$

and one of them must be tight. Hence $B(r, s)$ is the minimum of the two right-hand side expressions.

$r \downarrow \mid s \rightarrow$	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	0	0
1	1	0.5	0	0	0	0	0	0	0	0
2	1	0.75	0	0	0	0	0	0	0	0
3	1	0.875	0.375	0	0	0	0	0	0	0
4	1	0.938	0.625	0	0	0	0	0	0	0
5	1	0.969	0.782	0.313	0	0	0	0	0	0
6	1	0.985	0.876	0.548	0	0	0	0	0	0
7	1	0.993	0.931	0.712	0.274	0	0	0	0	0
8	1	0.997	0.962	0.822	0.493	0	0	0	0	0
9	1	0.999	0.98	0.892	0.658	0.247	0	0	0	0
10	1	1	0.99	0.936	0.775	0.453	0	0	0	0

$B(r, s)$. The boldfaced cells are the cells of $B(r - 1, s(r))$ where $s(r) = \lfloor \frac{r}{3} \rfloor$ = the function corresponding to 1-out-of-3-MMS-fairness.

$r \downarrow \mid s \rightarrow$	0	1	2	3	4	5	6
0	0	0	0				
1	0	.500	0				
2	0	.250	0 [.500]	0			
3	0	.125	.375	0			
4	0	.063	.250	0 [.375]	0		
5	0	.031	.156	.313	0		
6	0	.016	.094	.234	0 [.313]	0	
7	0	.008	.055	.164	.273	0	
8	0	.004	.031	.109	.219	0 [.273]	0
9	0	.002	.018	.070	.164	.246	0
10	0	.001	.010	.044	.117	.205	0 [.246]

$w(r, s)$: In most cases, the payment of a winner is $w(r, s)$ too. In the few cases in which a winner should pay more than $w(r, s)$, this payment is shown in brackets.

Table 3: Some values of $B(r, s)$ and $w(r, s)$. Compare to $\text{MAXH}(r, s)$ in Table 2.

amount of 1. Therefore, it is sufficient to prove that the final balance of group i is at least $h_i \cdot n_i$. We prove this in two parts: (1) The initial balance of group i at its first turn to pick a good is at least $h_i \cdot n_i$; (2) The balance of each group weakly increases from its first turn until the protocol ends.

Part (1). The claim is obvious for group 1 since, by definition of the initial payments, each agent with r desirable goods initially pays the group $B(r, s(r))$, which is at least h_1 . As for group 2, before its first turn it might have to pay to members whose desired good was picked by group 1. To each member with r desired goods, it has to pay either 0 or $w(r, s(r))$, so the new net payment of each member is at least $B(r, s(r)) - w(r, s(r)) = B(r - 1, s(r))$, which is at least h_2 .

Part (2). We calculate the change in the balance of group i in a pair of turns in which group i picks a good g_i and then the other group picks a good g_{-i} . The change in balance is determined by the weights of three groups of agents, which we denote by:

- D_i — agents who desire g_i and do not care about g_{-i} . Each agent in this group (with r remaining desired goods and s goods still needed) pays $\max[w(r, s), w(r - 1, s - 1)] \geq w(r, s)$.
- D_{-i} — agents who do not care about g_i and desire g_{-i} . Each agent in this group receives $w(r, s)$.
- D_0 — agents who desire both g_i and g_{-i} . Each agent in this group pays $\max[w(r, s), w(r - 1, s - 1)]$ and then receives $w(r - 1, s - 1)$, so his net payment is positive.

Therefore the total change in the group balance after the two turns is at least:

$$\Delta[Balance] = \sum_{j \in D_i} w(r_j, s_j) - \sum_{j \in D_{-i}} w(r_j, s_j)$$

Now, the group chose g_i while g_{-i} was still available, which means that the total weight of g_i is weakly larger:

$$\sum_{j \in D_i} w(r_j, s_j) + \sum_{j \in D_0} w(r_j, s_j) \geq \sum_{j \in D_{-i}} w(r_j, s_j) + \sum_{j \in D_0} w(r_j, s_j)$$

which implies that $\Delta[Balance] \geq 0$. □

Remark 3.7. For simplicity we assume that both groups have the same fairness criterion. However, in general, each group i can use a different function $s_i(r)$ and get the corresponding guarantee regardless of the function s_{-i} used by the other group.

Remark 3.8. One can prove that, when s is fixed, $B(r, s)$ is an increasing function of r (see Lemma A.2). Therefore, in Lemma 3.6, $h_1 \geq h_2$, so Lemma 3.6 guarantees h_2 -democratic fairness to both groups.

Remark 3.9. It is possible to obtain improved guarantees if we allow randomization; the details can be found in Appendix B.

We now use Lemma 3.6 to get some more specific fairness guarantees. To this end we establish some properties of $B(r, s)$.

Lemma 3.10. *For every $r \geq 0$, $B(r, 1) = 1 - 1/2^r$.*

Proof. We proceed by induction on r . For $r = 0$, $B(0, 1) = 0$ by the boundary condition of B . Now assume that $r > 0$ and that the claim is true for $r - 1$. Then, by the recurrence (***):

$$\begin{aligned} B(r, 1) &= \min \left[\frac{1}{2}(B(r-1, 1) + B(r-1, 0)), B(r-2, 0) \right] \\ &= \min \left[\frac{1}{2}(1 - 1/2^{r-1} + 1), 1 \right] \quad (\text{using the induction assumption}) \\ &= 1 - 1/2^r. \end{aligned} \quad \square$$

This lemma is useful for groups who follow the egalitarian philosophy and want to ensure that as many members as possible receive a positive utility. By letting $s(r) \equiv 1$ they guarantee that, if each member has at least r desirable goods, then at least a fraction $1 - 1/2^{r-1}$ of their members will receive a positive utility.

Corollary 3.11. *If each agent has at least r desirable goods, then RWAV can guarantee a positive utility to at least a fraction $1 - 1/2^{r-1}$ of the members in both groups (and at least a fraction $1 - 1/2^r$ of the members in the first group).*

It is interesting to follow the execution of RWAV in this case. The initial voting weight of a member with r desired goods is $w(r, 1) = B(r, 1) - B(r-1, 1) = 1/2^r$. As the protocol progresses, the weight of a member whose desired good is taken by her own group drops to zero, and the weight of a member whose desired good was taken by the other group is multiplied by 2. Thus the interests of poorer agents are prioritized.

Asymptotically, the lower bound of Corollary 3.11 is almost tight, since the upper bound implied by Proposition 3.3 is $1 - 1/2^r$. However, for small values of r it can be substantially improved.

Theorem 3.12. *If each agent has at least r desirable goods, then it is possible to guarantee a positive utility to at least a fraction $\frac{2^r-1}{2^{r+1}} = 1 - \frac{2}{2^{r+1}}$ of the members in both groups.*

Proof. If, in one of the groups, at least $\frac{2^r-1}{2^{r+1}}$ of the agents desire the same good g , then give g to that group and give all other goods to the other group.

Otherwise, run RWAV as usual. As in the proof of Lemma 3.6, we have to prove that, for each group i , its balance when it first picks an item is at least $\frac{2^r-1}{2^{r+1}} \cdot n_i$.

We have $s(r) \equiv 1$ for all r , so in step (a), the initial payment of each agent is $B(r, 1) = 1 - 1/2^r$. Therefore the initial balance of group 1 is $\frac{2^r-1}{2^r} \cdot n_1 > \frac{2^r-1}{2^{r+1}} \cdot n_1$.

As for group 2, before its first turn it might have to pay to members whose good was picked by group 1. There are less than $\frac{2^r-1}{2^{r+1}} \cdot n_2$ such members, and the weight of each is $w(r, 1) = 1/2^r$. Therefore the initial balance of group 2 is above $(1 - 1/2^r)n_2 - (1/2^r) \cdot \frac{2^r-1}{2^{r+1}} \cdot n_2 = \frac{2^r-1}{2^{r+1}} \cdot n_2$. \square

The lower bound of Theorem 3.12 still does not completely close the gap. For example, for $r = 2$ the lower bound is $3/5$ while the upper bound is $2/3$ (Proposition 3.1); for $r = 3$ the lower bound is $7/9$ while the upper bound is $7/8$ (Proposition 3.3).

What lower bounds on h can we obtain for envy-freeness and MMS-fairness?

For EF1, it is apparent from Table 3 that RWAV does not give any meaningful lower bounds. This is because the relevant cells in the table, $B(r-1, \lfloor r/2 \rfloor)$, form a decreasing sequence that approaches 0. However, we can prove an existential lower bound that matches the upper bound of Proposition 3.5.^{5,6}

Theorem 3.13. *When a round-robin protocol is used for dividing goods between two groups, each group can pick goods in a way that guarantees EF1 to at least $1/2$ of its members, regardless of what the other group does.*

Proof. It is sufficient to prove the claim for group 2, since group 1 can always assume the role of group 2 by passing its first turn (or picking an arbitrary good). Therefore, throughout this proof we refer only to agents of group 2; we treat group 1 as an adversary whose only goal is to prevent group 2 from attaining its fairness goals.

Each agent i with $2s_i$ or $2s_i + 1$ desired goods needs to get at least s_i of these goods in order for EF1 to be satisfied. After group 1 picks a good, each such agent has at least $2s_i - 1$ remaining desired goods. For simplicity we assume that, at this point, agent i has exactly $2s_i - 1$ remaining goods (if the agent has more remaining goods, we just ignore them).

We claim that, for every sequence of goods picked by group 1, group 2 can pick goods so that at least half of its agents i get at least s_i of their desired goods.

Suppose by contradiction that the claim is not true. This means that, for *every* sequence of goods picked by group 2, group 1 can pick goods (from the second step onwards) such that more than half the members of group 2 get less than s_i goods. This means that, for more than half the members i of group 2, group 1 holds at least s_i of their desired goods.

But if group 1 had such a strategy, group 2 could copy this strategy and play it against group 1 (from the first step onwards). This would guarantee that more than half the members of group 2 receive at least s_i of their desired goods. \square

Theorem 3.13 provides a lower bound of $1/2$ for both EF1 and (equivalently) 1-out-of-2-MMS-fairness. We can get better lower bounds for 1-out-of- c MMS-fairness when $c > 2$.

Theorem 3.14. *For every $c \geq 3$ and $s \geq 2$:*

$$B(cs - 1, s) \geq 1 - 1/2^{c-1}.$$

Therefore, RWAV attains $(1 - 1/2^{c-1})$ -democratic 1-out-of- c MMS-fairness.

⁵ In fact, in Section 4 we present a protocol for two groups that guarantees $1/2$ -democratic EF1 even for additive agents (Theorem 4.1). However, Theorem 3.13 is interesting since it shows that this guarantee can also be attained by a round-robin protocol.

⁶ We are grateful to J. Kreft for the proof idea.

The proof requires various technical lemmas on binomial coefficients. These can be found in Appendix A. The lemma itself is proved as Lemma A.11.⁷

Theorem 3.14 implies the following unanimous-fairness guarantee:

Corollary 3.15. *For two groups each containing at most n agents with binary valuations, there exists a unanimous 1-out-of- $(\lceil \log_2(n+1) \rceil + 1)$ MMS-fair allocation.*

Proof. Let $c = \lceil \log_2(n+1) \rceil + 1$, so $n \leq 2^{c-1} - 1$. By Theorem 3.14, we can attain $(1 - 1/2^{c-1})$ -democratic 1-out-of- c MMS-fairness. However, $(1 - 1/2^{c-1})$ -democratic fairness and unanimous fairness are equivalent when the number of agents in each group is at most $2^{c-1} - 1$. \square

Proposition 3.4 implies that the $O(\log n)$ rate cannot be improved.

4. Two Groups with General Valuations

In this section we assume that there are two groups and each agent can have an arbitrary monotonic utility function.

We start with a positive result: it is always possible to efficiently allocate goods so that at least half of the agents in each group believe the division is EF1. The protocol mirrors the well-known “cut-and-choose” protocol for dividing a cake between two agents. Despite the simplicity of the protocol, we find the result important since, unlike previous results in this setting [Manurangsi and Suksompong, 2017a, Suksompong, 2018a], our result holds for worst-case instances with any number of agents in the groups and very general utility functions.

Theorem 4.1. *For two groups of agents with monotonic valuations, 1/2-democratic EF1 is attainable.*

Proof. We arrange the goods in a line and process them from left to right. Starting from an empty block, we add one good at a time until the current block is EF1 for at least half of the agents in at least one group. We allocate the current block to one such group, and the remaining goods to the other group.

Since the whole set of goods is EF1 for both groups, the protocol terminates. Assume without loss of generality that the left block G_1 is allocated to the first group A_1 , and the right block G_2 to the second group A_2 . By the description of the protocol, the allocation is EF1 for at least half of the agents in A_1 , so it remains to show that the same holds for A_2 . Let g be the last good added to the left block. More than half of the agents in A_2 think that $G_1 \setminus \{g\}$ is not EF1, so for these agents, $G_1 \setminus \{g\}$ is worth less than $G_2 \cup \{g\} \setminus \{g'\}$ for any $g' \in G_2 \cup \{g\}$. Taking $g' = g$, we find that these agents value $G_1 \setminus \{g\}$ less than G_2 . But this implies that the agents find G_2 to be EF1, completing the proof. \square

⁷ This claim appeared in the conference version [Segal-Halevi and Suksompong, 2018]. Due to the error explained in Footnote 3, the proof there was much shorter. Happily, the theorem still holds after correcting the error.

Theorem 4.1 shows that if the goods lie on a line, we can find a $1/2$ -democratic EF1 allocation that moreover gives each group a contiguous block on the line. This may be important, for example, if the goods are houses on a street and each group wants to have all its houses in a contiguous block [Bouveret et al., 2017, Suksompong, 2017, Barrera et al., 2015].

Proposition 3.5 shows that the factor $1/2$ in Theorem 4.1 cannot be improved even for binary agents and even if we relax EF1 to EF c for any constant c . Nevertheless, if we let the relaxation of envy-freeness depend on the number of agents, it is possible to obtain a unanimous fairness guarantee.

Theorem 4.2. *For any two groups of agents with additive valuations, there exists an allocation that is EF($n - 1$) for all agents, where $n = n_1 + n_2$ is the total number of agents in both groups.*

Proof. Choose an arbitrary agent in one of the groups. We will partition the goods into two parts and let the agent choose the part that she prefers. The resulting allocation is envy-free and therefore EF($n - 1$) for this agent. It therefore suffices to show that there exists a partition in which each bundle is EF($n - 1$) (with respect to the other bundle) for all of the remaining $n - 1$ agents.

To this end, assume that there is a divisible good (“cake”) represented by the half-open interval $(0, m]$. The value-density functions of the agents over the cake are piecewise-constant: for every $l \in \{1, \dots, m\}$, the value-density v_{ij} in the half-open interval $(l - 1, l]$ equals $u_{ij}(g_l)$.

It is known that there exists a partition of the cake into two parts, using at most $n - 1$ cuts, in which every agent has equal value for both parts [Alon, 1987]. Starting with two empty bundles, for each $l \in \{1, \dots, m\}$, we add good g_l to the bundle corresponding to the part that contains at least half of the interval $(l - 1, l]$. (If both parts contain exactly half of the interval, we add g_l to an arbitrary bundle.)

We claim that every agent finds either bundle to be EF($n - 1$). Fix an agent a_{ij} and a bundle G' . From our partitioning choice, we have that $u_{ij}(G \setminus G') - u_{ij}(G') \leq u_{ij}(G'')$ for some set $G'' \subseteq G \setminus G'$ of size at most $n - 1$. This implies that the agent finds G' to be EF($n - 1$) with respect to $G \setminus G'$, as claimed. \square

We can combine Theorem 4.1 with Lemma 2.1 to get:

Corollary 4.3. *For two groups with additive agents:*

- (a) *$1/2$ -democratic $1/2$ -MMS-fairness is attainable;*
- (b) *$1/2$ -democratic 1-out-of-3 MMS-fairness is attainable;*
- (c) *If the valuations are binary, then $1/2$ -democratic MMS-fairness is attainable.*

For $1/2$ -MMS-fairness, the factor $1/2$ (in $1/2$ -democratic) in Corollary 4.3 is “almost” tight:

Proposition 4.4. *For any $h > 1/3$ and $q > 1/2$, there is an additive instance with two groups in which no allocation is h -democratic q -MMS-fair.*

Proof. Consider an instance with $m = 3$ goods and $n_1 = n_2 = 3$ agents in each group, with utility vectors: $\mathbf{u}_{i1} = (2, 1, 1)$, $\mathbf{u}_{i2} = (1, 2, 1)$, and $\mathbf{u}_{i3} = (1, 1, 2)$ for $i = 1, 2$. The MMS of every agent is 2. In any allocation, one group receives at most one good, so at most one of its three agents receives utility more than 1. In that group, at most $1/3$ of the agents receive more than $1/2$ of their MMS. \square

A corollary of Proposition 4.4 is that, for every $h \in (1/3, 1/2]$, the maximum fraction q such that there always exists an h -democratic q -MMS-fair allocation is $q = 1/2$.

Suppose we are only interested in positive-MMS-fairness. What fraction of the agents can be made happy? The upper bound of $2/3$ in Proposition 3.1 is proved for binary agents, so it is certainly true for the more general case of additive agents. The lower bound of Theorem 3.12 is also valid for additive agents:

Theorem 4.5. *For any two groups with additive agents, there exists a $3/5$ -democratic positive-MMS allocation.*

Proof. An agent's maximin share is positive only if the agent has at least two goods with a positive utility. Therefore, for positive-MMS it is sufficient to give an agent at least one of his two most valuable goods.

This can be done by converting the additive instance to a binary instance: change the utility of each agent to 1 for his two most valuable goods and 0 for all other goods. Then apply Theorem 3.12 with $r = 2$. This guarantees that at least $3/5$ of the agents in each group will receive at least one of their two most valuable goods, so their utility will be positive. \square

5. Three or More Groups

In this section we study the most general setting where we allocate goods among any number of groups. Similarly to the previous sections, we start with negative results and move on to positive results.

5.1. Negative results

The following proposition generalizes Proposition 3.1.

Proposition 5.1. *For any $k \geq 2$ and any $h > k/(2k - 1)$, there is a binary instance in which no allocation is h -democratic positive-MMS-fair.*

Proof. There are $2k - 1$ goods placed in a circle. In each group there are $2k - 1$ members. Each member of a group values a unique block of k consecutive goods on the circle; the member has utility 1 for each of the k goods and utility 0 for the remaining $k - 1$ goods. Each agent has a positive MMS (1). However, in any allocation some group gets at most one good, and only k members of the group get positive utility. \square

In particular, when the number of groups is large, it is not possible to satisfy more than about half of the agents.

Next, we generalize Proposition 3.3 to an arbitrary number of groups.

Proposition 5.2. *Let r, s be integers such that $r \geq s \geq 1$. Consider a binary instance with k groups in which each agent desires exactly r goods and needs s goods in order to consider the allocation fair. Then, it is impossible to attain more than $\text{MAXH}(r, s)$ -democratic fairness, where:*

$$\text{MAXH}(r, s) = \begin{cases} 0 & \text{when } r \leq ks - 1; \\ \frac{1}{k^r} \sum_{i=s}^r (k-1)^{r-i} \binom{r}{i} & \text{when } r \geq ks. \end{cases}$$

Proof. First, suppose that $r \leq ks - 1$, the total number of goods is r , and all agents in all groups desire all goods. At least one group will get at most $r/k < s$ goods, so all of its members will be unhappy.

Next, suppose that there are km goods, with $m \gg r$. In each group there are $\binom{km}{r}$ members, each of whom wants a distinct subset of r goods. At least one group will get at most m goods. In this group, the fraction of happy agents will be at most:

$$\frac{\sum_{i=s}^r \binom{m}{i} \cdot \binom{km-m}{r-i}}{\binom{km}{r}}.$$

When $m \gg r$, the numerator is approximately $\sum_{i=s}^r \frac{m^i}{i!} \cdot \frac{(k-1)^{r-i} m^{r-i}}{(r-i)!} = \sum_{i=s}^r \frac{(k-1)^{r-i} m^r}{i!(r-i)!}$ and the denominator is approximately $\frac{(km)^r}{r!} = \frac{k^r m^r}{r!}$. Therefore when $m \rightarrow \infty$ the expression approaches $\frac{1}{k^r} \sum_{i=s}^r (k-1)^{r-i} \binom{r}{i}$. \square

In particular, we get the following generalization of Proposition 3.4:

Proposition 5.3. *For any integers $k, c \geq 2$ and any $h > 1 - (\frac{k-1}{k})^c$, there is a binary instance with k groups in which no allocation is h -democratic 1-out-of- c MMS-fair.*

Proof. Apply Proposition 5.2 with $r = c$ and $s = 1$. Then $\text{MAXH}(r, s) \leq \frac{1}{k^c} \sum_{i=1}^c (k-1)^{c-i} \binom{c}{i} = \frac{k^c - (k-1)^c}{k^c} = 1 - (\frac{k-1}{k})^c$. \square

As a generalization of Proposition 3.5, we get:

Proposition 5.4. *For any constant integer $c \geq 1$ and any $h > 1/k$, there is a binary instance with k groups in which no allocation is h -democratic EFe.*

Proof. Assume that there are $m = km'$ goods for some large positive integer m' . Each group consists of 2^m agents, each of whom values a distinct combination of the goods. Consider first an allocation that gives exactly m' goods to each group, and fix a group. We claim that the fraction of the agents in the group whose utilities for some two bundles differ by at most c converges to 0 for large m' . Indeed, this follows from the central limit theorem: Fix two bundles and consider a random agent from the group; let X be the random variable denoting the (possibly negative) difference between the agent's utilities for the two bundles. Then X is a sum of m' independent and identically distributed random variables with mean 0. The central limit theorem implies that for any fixed $\epsilon > 0$, there exists a constant d such that $\Pr[|X| \leq c] \leq \Pr[|X| \leq d\sqrt{m'}] \leq \epsilon$ for any sufficiently large m' .

Taking the union bound over all pairs of bundles, we find that the fraction of agents in the group who value some two bundles within c of each other approaches 0 as m' goes to infinity. This means that all but a negligible fraction of the agents find only one bundle to be $\text{EF}c$. By symmetry, $1/k$ of these agents find the bundle allocated to the group to be $\text{EF}c$. It follows that the fraction of agents in the group for whom the allocation is $\text{EF}c$ converges to $1/k$.

It remains to consider the case where the allocation does not give the same number of goods to all groups. In this case, let \mathcal{G} denote the set of bundles with the smallest number of goods, which must be strictly smaller than m' goods. If we move goods from bundles with more than m' goods to bundles in \mathcal{G} in such a way that the number of goods in each bundle in \mathcal{G} increases by exactly one, the fraction of agents in an arbitrary group that receives a bundle in \mathcal{G} who finds the allocation to be $\text{EF}c$ can only increase. We can repeat this process, at each step possibly adding bundles to \mathcal{G} , until all bundles contain the same number of goods, which is the case we have already handled. Since the fraction of agents for whom the allocation is $\text{EF}c$ is bounded above by $1/k$ for large m' in the latter allocation, and this fraction only increases during our process of moving goods, the same is true for the original allocation. \square

5.2. Positive results

When there are two groups, the protocol in Theorem 4.1 is computationally efficient and yields an allocation that is both approximately envy-free and approximately MMS-fair. We present two ways of generalizing the result to multiple groups: one keeps the approximate envy-freeness guarantee but loses computational efficiency, while the other keeps only the approximate MMS-fairness guarantee but also retains computational efficiency.

5.2.1. Approximate envy-freeness

Our first theorem establishes the existence of a $1/k$ -democratic $\text{EF}2$ allocation for k groups with arbitrary monotonic valuations. Note that by Proposition 5.4, the factor $1/k$ cannot be improved.

Theorem 5.5. *For k groups with agents having arbitrary monotonic valuations, there exists an allocation that is $1/k$ -democratic $\text{EF}2$.*

Proof. Bilò et al. [2018] proved that for any k agents with monotonic valuations, if the goods lie on a line, there exists an $\text{EF}2$ allocation that gives each agent a contiguous block on the line. We present their proof briefly (keeping only the details required for our purposes), and then show how to adapt their proof to the group setting.

Bilò et al.'s protocol considers a k -vertex simplex in \mathbb{R}^k that is triangulated to smaller k -vertex sub-simplices. It identifies each vertex of the triangulation with a *sub-partition*, i.e., a partition of some subset of G into k parts (in particular, some goods might not appear in any part). This identification has the following properties:

1. In each main vertex $i \in \{1, \dots, k\}$ of the main simplex, part i contains all of G while the other $k - 1$ parts are empty.
2. In each face spanned by main vertices i_1, \dots, i_l of the main simplex, only parts i_1, \dots, i_l contain goods, while the other $k - l$ parts are empty.
3. In each small sub-simplex of the triangulation, consider the k sub-partitions attached to its k vertices. For each good $g \in G$, there is a unique $i \in \{1, \dots, k\}$ such that
 - (a) g belongs to part i in at least one of these k sub-partitions;
 - (b) g does not belong to another part in any of these k sub-partitions;
 - (c) for each part i in any sub-partition, there are at most two goods that do not belong to this part i but belong to part i in some other sub-partition.

For instance, if there are $k = 4$ agents and $m = 12$ goods, a possible set of four sub-partitions that satisfy property 3 is the following:

- $(\{g_1, g_2, g_3\}, \{g_4, g_5, g_6\}, \{g_7, g_8, g_9\}, \{g_{11}, g_{12}\})$
- $(\{g_1, g_2, g_3\}, \{g_5, g_6\}, \{g_7, g_8, g_9\}, \{g_{11}, g_{12}\})$
- $(\{g_1, g_2, g_3\}, \{g_5, g_6\}, \{g_8, g_9\}, \{g_{11}, g_{12}\})$
- $(\{g_1, g_2, g_3\}, \{g_5, g_6\}, \{g_8, g_9, g_{10}\}, \{g_{11}, g_{12}\})$

For each vertex of the triangulation, the protocol asks each agent which of the k parts in the attached sub-partition he prefers the most, and label the vertex with the answers. Since the agents' valuations are monotonic, we can assume that an agent never prefers an empty bundle to a non-empty bundle. Hence, by property 1, each main vertex i only has label i . By property 2, all face vertices are labeled only with labels from the endpoints i_1, \dots, i_l of the face. Thus, each agent's labeling satisfies Sperner's boundary condition. Therefore, by Bapat [1989]'s generalization of Sperner's lemma, there exists a sub-simplex and a matching of its vertices to the agents such that, in the vertex matched to agent i , agent i prefers part i . By property 3, if we unite all parts numbered i in all the k sub-partitions of the sub-simplex, we get a partition of G . (In the example above, the united partition is $(\{g_1, g_2, g_3\}, \{g_4, g_5, g_6\}, \{g_7, g_8, g_9, g_{10}\}, \{g_{11}, g_{12}\})$.) In this united partition, each part j is larger than the corresponding parts j in the sub-partitions by at most two goods. Therefore, each agent i finds part i better than any other part j , up to at most two goods. As a result, giving part i of the united partition to agent i yields an EF2 allocation.

To adapt Bilò et al.'s proof to the group setting, we define a *representative* for each group, and run their protocol on the k representatives. Whenever a representative is asked to select a best part, he uses *plurality voting* among the group members, and answers by specifying the index of the part preferred by the largest number of members (breaking ties arbitrarily). Since each agent prefers a non-empty bundle to an empty bundle, the representative also prefers a non-empty bundle to an empty bundle. Hence

the representative's answers satisfy Sperner's boundary conditions. This means that there exists a sub-simplex in which, in each vertex i , the representative of group i prefers part i . By the pigeonhole principle, in the sub-partition attached to this vertex i , at least $1/k$ of the members in group i prefer part i . In the united partition, these members find part i better than any other part up to at most 2 goods. Therefore, giving part i to group i yields a $1/k$ -democratic EF2 allocation. \square

If agents have binary valuations, the fairness guarantee can be improved to EF1 by adapting the proof for the individual setting [Barrera et al., 2015, Suksompong, 2017] to the group setting. Since the techniques are similar to those used in Theorem 5.5, we only state the result here and defer the proof to Appendix C.

Theorem 5.6. *For k groups in which all agents have binary valuations, there exists an allocation that is $1/k$ -democratic EF1 and $1/k$ -democratic MMS-fair.*

By Proposition 5.4, the factor $1/k$ is again tight.

Remark 5.7. Bilò et al. [2018] proved that, for $k \leq 4$ agents, the fairness guarantee can be improved from EF2 to EF1 even when the agents have arbitrary monotonic valuations. It is possible that this result can be adapted to the group setting using plurality voting in a similar manner as in the previous two theorems. This would mean that for $k \leq 4$ groups of agents with arbitrary monotonic valuations, there exists a $1/k$ -democratic EF1 allocation. However, their proof is rather involved and we have not been able to verify that our reduction works for it.

Remark 5.8. Barrera et al. [2015], Suksompong [2017] and Bilò et al. [2018] did not provide efficient algorithms for computing the corresponding approximately fair allocations. It is an interesting question whether such allocations can be found in polynomial time, both for the individual setting and the group setting.

5.2.2. Approximate MMS

In this subsection, we show that if we weaken our fairness requirement to approximate MMS, it is possible to compute a fair allocation in time polynomial in the input size.

Lemma 5.9. *When agents have additive valuations, there always exists an allocation such that at least $1/k$ of the agents a_{ij} in each group A_i receive utility at least $\frac{1}{k} \cdot u_{ij}(G) - \frac{k-1}{k} \cdot u_{ij,\max}$, and such an allocation can be computed efficiently.*

This lemma generalizes the corresponding result for the setting with one agent per group by Suksompong [2017]. The factor $(k-1)/k$ is tight even for individual agents.

Proof. We arrange the goods in a line and process them from left to right. Starting from an empty block, we add one good at a time until the current block yields utility at least $\frac{1}{k} \cdot u_{ij}(G) - \frac{k-1}{k} \cdot u_{ij,\max}$ for at least $1/k$ of the agents in at least one group. We allocate the current block to one such group and repeat the process with the remaining $k-1$ groups. It is clear that this algorithm can be implemented efficiently. Any group

that receives a block from the algorithm meets the requirement, so it suffices to show that the algorithm allocates a block to every group. We claim that if l groups are yet to receive a block, at least l/k of the agents a_{ij} in each of these groups have utility at least $\frac{l}{k} \cdot u_{ij}(G) - \frac{k-l}{k} \cdot u_{ij,\max}$ for the remaining goods. This would imply the desired result because for the last group, at least $1/k$ of the agents have utility $\frac{1}{k} \cdot u_{ij}(G) - \frac{k-1}{k} \cdot u_{ij,\max}$, which is exactly our requirement.

To show the claim, we proceed by backward induction on l . The claim trivially holds when $l = k$. Suppose that the statement holds when there are $l + 1$ groups left, and consider a group j that is not the next one to receive a block. At least $(l + 1)/k$ of the agents a_{ij} in the group have utility at least $\frac{l+1}{k} \cdot u_{ij}(G) - \frac{k-l-1}{k} \cdot u_{ij,\max}$ for the remaining goods. Since the group does not receive the next block, less than $1/k$ of the agents in the group have utility at least $\frac{1}{k} \cdot u_{ij}(G) - \frac{k-1}{k} \cdot u_{ij,\max}$ for the block excluding the last good. Hence, less than $1/k$ of the agents have utility at least $\frac{1}{k} \cdot u_{ij}(G) + \frac{1}{k} \cdot u_{ij,\max}$ for the whole block. This means that at least l/k of the agents have utility at least $(\frac{l+1}{k} \cdot u_{ij}(G) - \frac{k-l-1}{k} \cdot u_{ij,\max}) - (\frac{1}{k} \cdot u_{ij}(G) + \frac{1}{k} \cdot u_{ij,\max}) = \frac{l}{k} \cdot u_{ij}(G) - \frac{k-l}{k} \cdot u_{ij,\max}$, completing the induction. \square

It is clear by definition that the MMS of any agent a_{ij} is at most $\frac{1}{k} \cdot u_{ij}(G)$. Lemma 5.9 therefore implies the following:

Theorem 5.10. *When agents have additive valuations, there always exists an allocation such that at least $1/k$ of the agents a_{ij} in each group A_i receive utility at least $\text{MMS}_{ij}(G) - \frac{k-1}{k} \cdot u_{ij,\max}$, and such an allocation can be computed efficiently.*

For binary valuations, if we change the stopping condition in Lemma 5.9 to be when the current block yields the MMS for at least $1/k$ of the agents in some group, we get:

Theorem 5.11. *When agents have binary valuations, there always exists a $1/k$ -democratic MMS-fair allocation, and such an allocation can be computed efficiently.*

5.3. Generalizing the RWAV protocol

The RWAV protocol of Section 3 can be generalized to k groups. The main change is that each member whose desired good was picked by its group should pay $k - 1$ times his weight to the group account, since there are $k - 1$ rounds in which the group might have to pay to losing agents.

Currently, we do not know how to handle agents that win a good and immediately lose another good. Therefore, we can write and solve the recurrence relation only for the case $s = 1$ (i.e., an agent who wins a single good is already considered happy):

$$\begin{aligned}
\forall r \geq 0 : \quad & B(r, 0) = 1 && \text{(happy agents)} \\
& B(0, 1) = 0 && \text{(unhappy agents)} \\
& B(r, 1) + (k - 1) \cdot w(r, 1) = B(r - 1, 0) = 1 && \text{by step (d)} \\
& B(r, 1) - w(r, 1) = B(r - 1, 1) && \text{by step (e)}
\end{aligned}$$

This implies the following recurrence relation for $B(r, s)$:

$$\forall r > 0 : \quad B(r, 1) = \frac{(k-1) \cdot B(r-1, 1) + 1}{k}$$

Its solution is:

$$\forall r \geq 0 : \quad B(r, 1) = \frac{1}{k^r} \sum_{i=1}^r \binom{r}{i} (k-1)^{r-i} = 1 - \left(\frac{k-1}{k} \right)^r$$

Note that for $k = 2$ we get $B(r, 1) = 1 - 1/2^r$ as in Lemma 3.10.

This allows us to prove the following positive result.

Theorem 5.12. *For any k groups with additive agents, there exists a h_0 -democratic positive-MMS allocation, where $h_0 = (e-1)/(2e-1) \approx 0.3873$.*

Proof. An agent's maximin share is positive only if the agent has at least k goods with a positive utility. Therefore, for positive-MMS it is sufficient to give an agent at least one of his best k goods. Moreover, we can convert all valuations to binary by assuming each agent desires only his k most valuable goods (breaking ties arbitrarily). Now, we prove that it is possible to give to at least h_0 of the agents in each group at least one desired good.

The proof is by induction on k . For $k = 2$ we already proved in Theorem 4.5 that it is possible to satisfy $3/5$ of the agents in each group, which is larger than h_0 .

Assume the claim is true up until $k-1$; we will prove it for k .

If in some group i at least $h_0 \cdot n_i$ members desire the same good g , give them good g and divide the remaining goods among the remaining groups recursively. Note that in each remaining group, each agent now desires at least $k-1$ goods, so by the inductive hypothesis, it is possible to satisfy at least an h_0 fraction of each group.

Otherwise, run RWAV modified for k groups as explained above. As in the proof of Lemma 3.6, it is sufficient to prove that, for each group i , its balance when it first picks an item is at least $h_0 \cdot n_i$.

We have $f(d) \equiv 1$ for all d , so the initial payment of each member is $B(k, 1) = 1 - ((k-1)/k)^k > 1 - 1/e$, and the initial amount paid to each group i is at least $(1 - 1/e) \cdot n_i$. This is also the balance of group 1 when it first picks an item.

The balance of groups $2, 3, \dots, k$ is smaller since they have to pay to their members whose desired goods were picked. Obviously group k is in the worst situation since it has to pay $k-1$ times, so we focus on this group. Each time a good is picked, the group has to pay to at most $h_0 \cdot n_k$ members. It has to pay $w(r, 1)$ to each member with r remaining goods. Note that $w(r, 1) = (k-1)^{r-1}/k^r$, which is larger when r is smaller. Therefore, the worst case for group k is when it has to pay again and again to the same $h_0 \cdot n_k$ members. In this case it has to pay $h_0 \cdot n_k \cdot \sum_{r=2}^k w(r, 1) = h_0 \cdot n_k \cdot [B(k, 1) - B(1, 1)]$. The total balance remaining in group k 's account when it first picks an item is thus at least: $B(k, 1) \cdot n_k - h_0 \cdot n_k \cdot [B(k, 1) - B(1, 1)] > B(k, 1) \cdot n_k (1 - h_0) > (1 - 1/e) \cdot n_k (1 - h_0) = n_k \cdot h_0$. \square

Note that, by Proposition 5.1, the asymptotic upper bound on h when $k \rightarrow \infty$ is 0.5.

6. Conclusion and Future Work

For two groups, we have a comprehensive understanding of possible democratic fairness guarantees. We have a complete characterization of possible envy-freeness approximations, and upper and lower bounds for maximin-share-fairness approximations. Some remaining gaps are shown in Table 1; closing them raises interesting combinatorial challenges.

For $k \geq 3$ groups, the challenges are much greater. Currently all our fairness guarantees are to no more than 39% of the agents in each group. From a practical perspective, it may be important in some settings to give fairness guarantees to at least half of the agents in all groups. Finding protocols that provide such guarantees is an avenue for future work. From an algorithmic perspective, it is interesting whether there exists a polynomial-time algorithm that guarantees EF1 to any positive fraction of the agents.

A possible concern about democratic fairness is that it completely leaves aside a fraction of the agents in each group. As Proposition 3.1 shows, it might be inevitable to leave some agents with zero utility. In these cases, the goal of an egalitarianist is to minimize the fraction of such poor agents. While the weighting scheme used by our RWAV protocol indeed prioritizes the interests of poor agents (see the remarks after Corollary 3.11), it may be interesting to develop an algorithm that directly minimizes the maximum fraction of poor agents across all groups.

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A. Properties of the Function B

The function $B(r, s)$, defined in Section 3.2, represents a lower bound on the fraction of agents that can be given at least s out of r desired goods. It is defined using the following recurrence relation:

$$\begin{aligned}
 (***) \quad \forall r \geq 2, r \geq s > 0 : \quad B(r, s) &= \min \left[\frac{1}{2} [B(r-1, s) + B(r-1, s-1)], \right. \\
 &\quad \left. B(r-2, s-1) \right] \\
 \forall r \geq 0 : \quad B(r, 0) &= 1 \\
 \forall r < s : \quad B(r, s) &= 0 \\
 B(1, 1) &= 1/2
 \end{aligned}$$

Some values are shown in Table 3. In this section we prove some properties of B .

Lemma A.1. *For every fixed r , $B(r, s)$ is a weakly decreasing function of s .*

Proof. By induction on r . For $r = 0, 1, 2, 3$ this is apparent from Table 3 (there are finitely many values to check). Now let $r \geq 4$. We assume the claim is true for $r-2$ and $r-1$ and prove it is true for r .

$B(r, s)$ is a minimum of two expressions. In each of these expressions, the first operand is less than r . Therefore, by the induction assumption, each of these expressions is decreasing with s . Therefore the same is true for $B(r, s)$. \square

Lemma A.2. *For every fixed s , $B(r, s)$ is a weakly increasing function of r .*

Proof. We have to prove that, for every s and every $r \geq 1$, $B(r, s) \geq B(r-1, s)$. We prove this by induction on r . For $r = 1, 2, 3$ this is apparent from Table 3 (there are finitely many values to check). Now let $r \geq 4$. We assume the claim is true for $r-2$ and $r-1$ and prove it is true for r . By the induction assumption, each term in the formula of $B(r, s)$ in (***) is no less than the corresponding term in the formula of $B(r-1, s)$. Hence it follows that $B(r, s) \geq B(r-1, s)$. \square

Lemma A.3. *For every r, s such that $0 \leq s \leq r$, $0 \leq B(r, s) \leq 1$.*

Proof. The boundary conditions on B imply that, for every r , $B(r, 0) = 1$ and $B(r, r+1) = 0$. Lemma A.1 implies that, for every fixed r , $B(r, s)$ decreases from 1 to 0. \square

Lemma A.4. *For every r, s such that $0 \leq s \leq r$, $0 \leq w(r, s) \leq 1$.*

Proof. By (**), $w(r, s) = B(r, s) - B(r-1, s)$. By Lemma A.2, this difference is at least 0. By Lemma A.3, the difference is at most 1. \square

Lemma A.5. *For every $s \geq 1$ and every $r \leq 2s-2$:*

$$B(r, s) = 0.$$

Proof. By induction on s . When $s = 1$, the claim should be verified only for $r = 0$; indeed it is true by the boundary condition $B(0, 1) = 0$.

Assume that $s \geq 2$ and that the claim is true for $s - 1$. Let r be an integer such that $r \leq 2s - 2$. By the recurrence relation defining B :

$$B(r, s) \leq B(r - 2, s - 1).$$

Since $r \leq 2s - 2$, $r - 2 \leq 2(s - 1) - 2$. Hence by the induction assumption on $s - 1$: $B(r - 2, s - 1) = 0$, so $B(r, s) \leq 0$. But by Lemma A.3, $B(r, s) \geq 0$, so we must have $B(r, s) = 0$. \square

Lemma A.6. *For every $s \geq 1$ and every $r \geq 2s - 1$:*

$$B(r, s) = \frac{1}{2}[B(r - 1, s) + B(r - 1, s - 1)].$$

Proof. By the recurrence relation (***) , it is sufficient to prove that whenever $r \geq 2s - 1$:

$$B(r - 1, s) + B(r - 1, s - 1) - 2B(r - 2, s - 1) \leq 0.$$

We prove this by induction on r . For $r = 1$ and $r = 2$ we only have to check the case $s = 1$; indeed the claim can be verified in Table 3. Assume that $r > 2$ and that the claim holds for $r - 1$ and $r - 2$. We prove that it holds for r by considering two cases.

Case A: $r = 2s - 1$. Then, by Lemma A.5, $B(r - 1, s) = 0$. However, $B(r - 1, s - 1)$ and $B(r - 2, s - 1)$ are subject to the induction assumption, since $r - 1 \geq 2(s - 1) - 1$ and $r - 2 \geq 2(s - 1) - 1$. Hence:

$$\begin{aligned} & B(r - 1, s) + B(r - 1, s - 1) - 2B(r - 2, s - 1) \\ &= 0 + \frac{1}{2}[B(r - 2, s - 1) + B(r - 2, s - 2)] - [B(r - 3, s - 1) + B(r - 3, s - 2)] \\ &= \frac{1}{2}[B(r - 2, s - 1) + B(r - 2, s - 2) - 2B(r - 3, s - 2)] \quad (\text{since } B(r - 3, s - 1) = 0) \\ &\leq 0 \quad (\text{by the induction assumption on } r - 1, \text{ since } r - 1 \geq 2(s - 1) - 1). \end{aligned}$$

Case B: $r \geq 2s$. Then, all three terms in the inequality are subject to the induction assumption. Hence:

$$\begin{aligned} & B(r - 1, s) + B(r - 1, s - 1) - 2B(r - 2, s - 1) \\ &= \frac{1}{2}[B(r - 2, s) + B(r - 2, s - 1) + B(r - 2, s - 1) + B(r - 2, s - 2)] \\ &\quad - [B(r - 3, s - 1) + B(r - 3, s - 2)] \\ &= \frac{1}{2}[B(r - 2, s) + B(r - 2, s - 1) - 2B(r - 3, s - 1)] \\ &\quad + \frac{1}{2}[B(r - 2, s - 1) + B(r - 2, s - 2) - 2B(r - 3, s - 2)] \\ &\leq 0 \quad (\text{by the induction assumption on } r - 1, \text{ since } r - 1 \geq 2s - 1). \quad \square \end{aligned}$$

In light of Lemma A.6, the recurrence relation of B can be simplified to:

$$\begin{aligned}
 (\text{****}) \quad & \forall s \geq 1, r \geq 2s - 1 : & B(r, s) &= \frac{1}{2}[B(r-1, s) + B(r-1, s-1)] \\
 & \forall s \geq 1, r \leq 2s - 2 : & B(r, s) &= 0 \\
 & \forall r \geq 0 : & B(r, 0) &= 1
 \end{aligned}$$

In the next lemma we find a closed-form solution to the function B in (****):

Lemma A.7. *The following function satisfies the recurrence relation (****):*

$$B(r, s) = \frac{1}{2^r} \sum_{i=s-1}^{r-s} \binom{r}{i} = \frac{1}{2^r} \sum_{i=s}^{r-s+1} \binom{r}{i},$$

where we assume that $\binom{a}{b} = 0$ if $b < 0$ or $b > a$.

Proof. When $s = 0$, the sum goes from -1 to r , however, for $i = -1$ the summand is zero so $B(r, s) = \frac{1}{2^r} \sum_{i=0}^r \binom{r}{i} = 1$.

When $r \leq 2s - 2$, the sum starts at $s - 1$ and ends at (at most) $s - 2$ so it is 0.

When $r \geq 2s - 1$, it is sufficient to prove that $2^{r-1}B(r-1, s) + 2^{r-1}B(r-1, s-1) = 2^r B(r, s)$. We have

$$\begin{aligned}
 2^{r-1}B(r-1, s) &= \sum_{i=s-1}^{r-s-1} \binom{r-1}{i}, \text{ and} \\
 2^{r-1}B(r-1, s-1) &= \sum_{i=s-2}^{r-s} \binom{r-1}{i} = \sum_{i=s-1}^{r-s+1} \binom{r-1}{i-1} \\
 &= \sum_{i=s-1}^{r-s-1} \binom{r-1}{i-1} + \left[\binom{r-1}{r-s-1} + \binom{r-1}{r-s} \right],
 \end{aligned}$$

since when $r \geq 2s - 1$, the sum $\sum_{i=s-1}^{r-s+1} \binom{r-1}{i-1}$ contains at least two elements. Summing the above two equations gives:

$$\begin{aligned}
 & 2^{r-1}[B(r-1, s) + B(r-1, s-1)] \\
 &= \sum_{i=s-1}^{r-s-1} \left[\binom{r-1}{i} + \binom{r-1}{i-1} \right] + \left[\binom{r-1}{r-s-1} + \binom{r-1}{r-s} \right]
 \end{aligned}$$

By two applications of Pascal's identity:

$$\begin{aligned}
 2^{r-1}[B(r-1, s) + B(r-1, s-1)] &= \sum_{i=s-1}^{r-s-1} \binom{r}{i} + \binom{r}{r-s} \\
 &= \sum_{i=s-1}^{r-s} \binom{r}{i} = 2^r B(r, s). \quad \square
 \end{aligned}$$

Next, we prove several technical lemmas about binomial coefficients and their sums.

Lemma A.8. *For every $s \geq 1$:*

$$\binom{3s-1}{s-1} \frac{3s}{s+2} \leq 2^{3s-3}$$

Proof. By induction on s . For $s = 1, 2, 3, 4$ the claim can be verified manually. We assume the claim for some $s \geq 4$ and prove it for $s + 1$. When s grows to $s + 1$, the right-hand side is multiplied by 8. The left-hand side is multiplied by:

$$\begin{aligned} \left[\binom{3s+2}{s} \frac{3s+3}{s+3} \right] / \left[\binom{3s-1}{s-1} \frac{3s}{s+2} \right] &= \frac{(3s+2)!(3s+3)(s-1)!(2s)!(s+2)}{(s)!(2s+2)!(s+3)(3s-1)!(3s)} \\ &= \frac{(3s+2)(3s+1)(3s)(3s+3)(s+2)}{(s)(2s+2)(2s+1)(s+3)(3s)} \\ &= \frac{3(3s+2)(3s+1)(s+2)}{2(s)(2s+1)(s+3)} \\ &\leq \frac{3 \cdot 3 \cdot (3s+2)}{2 \cdot 2 \cdot (s)} = 2.25 \cdot (3 + 2/s). \end{aligned}$$

When $s \geq 4$ this expression is less than 8, so the left-hand side remains smaller than the right-hand side. \square

Lemma A.9. *For every $s \geq 1$:*

$$\sum_{i=0}^{s-1} \binom{3s-1}{i} + \sum_{i=0}^{s-2} \binom{3s-1}{i} \leq 2^{3s-3}.$$

Proof. For $N \geq 2k$, denote by $f(N, k)$ the sum of the first k binomial coefficients: $f(N, k) := \sum_{i=0}^k \binom{N}{i}$. Michael Lugo⁸ proved the following upper bound on this sum:

$$f(N, k) \leq \binom{N}{k} \frac{N - k + 1}{N - 2k + 1}.$$

Therefore:

$$\begin{aligned} f(N, k+1) + f(N, k) &= 2f(N, k+1) - \binom{N}{k+1} \\ &\leq \binom{N}{k+1} \left[2 \cdot \frac{N - k}{N - 2k - 1} - 1 \right] \\ &= \binom{N}{k+1} \frac{N + 1}{N - 2k - 1}. \end{aligned}$$

The left-hand side of the claim is this expression with $N = 3s - 1$ and $k = s - 2$, so it is no more than:

$$\binom{3s-1}{s-1} \frac{3s}{3s-1-2s+4-1} = \binom{3s-1}{s-1} \frac{3s}{s+2},$$

which by Lemma A.8 is at most 2^{3s-3} . \square

⁸ Here: <https://mathoverflow.net/a/17236/34461>

Our next lemma is a generalization of Lemma A.9.⁹

Lemma A.10. *For all integers $c \geq 3$ and $s \geq 1$:*

$$\sum_{i=0}^{s-1} \binom{cs-1}{i} + \sum_{i=0}^{s-2} \binom{cs-1}{i} \leq 2^{cs-c}.$$

Proof. We prove the claim by induction on c for every fixed s . For $c = 3$, the inequality follows from Lemma A.9. We now assume the claim is true for some $c \geq 3$. When c grows to $c + 1$, the left-hand side still has the same number of summands ($2s - 3$ summands), where in each summand, the $cs - 1$ at the top becomes $cs + s - 1$. Meanwhile, the right-hand side is multiplied by 2^{s-1} . Therefore, it is sufficient to show that in the left-hand side, each summand grows by a factor of at most 2^{s-1} . Indeed, for every $i \leq s - 1$:

$$\begin{aligned} \frac{\binom{cs+s-1}{i}}{\binom{cs-1}{i}} &= \frac{(cs+s-1)!/(cs+s-1-i)!}{(cs-1)!/(cs-1-i)!} \\ &= \frac{(cs+s-1) \cdots (cs+s-i)}{(cs-1) \cdots (cs-i)} \\ &= \left(1 + \frac{s}{cs-1}\right) \cdots \left(1 + \frac{s}{cs-i}\right) \\ &\leq \left(1 + \frac{s}{cs-i}\right)^i && \text{(the rightmost term is the largest)} \\ &\leq \left(1 + \frac{s}{cs-(s-1)}\right)^{s-1} && (i \leq s-1) \\ &< \left(1 + \frac{s}{2s-s+1}\right)^{s-1} && (c > 2) \\ &< 2^{s-1}. \end{aligned}$$

This completes the proof. □

We now use this combinatorial lemma to prove a useful lower bound on $B(r, s)$, which implies a democratic fairness guarantee.

Lemma A.11. *For every $c \geq 3$ and $s \geq 2$:*

$$B(cs-1, s) \geq 1 - 1/2^{c-1}.$$

Therefore, RWAV attains $(1 - 1/2^{c-1})$ -democratic 1-out-of- c MMS-fairness.

Proof. Using the closed form for $B(r, s)$ from Lemma A.7, we have to prove that:

$$\frac{1}{2^{cs-1}} \sum_{i=s}^{cs-s} \binom{cs-1}{i} \geq 1 - \frac{1}{2^{c-1}}$$

⁹ We are grateful to Alex Francisco and Y. Forman for their help in proving this lemma here: <https://math.stackexchange.com/a/2604279/29780>

$$\begin{aligned}
&\Leftrightarrow \frac{1}{2^{cs-1}} \left[\sum_{i=0}^{s-1} \binom{cs-1}{i} + \sum_{i=cs-s+1}^{cs-1} \binom{cs-1}{i} \right] \leq \frac{1}{2^{c-1}} \\
&\Leftrightarrow \sum_{i=0}^{s-1} \binom{cs-1}{i} + \sum_{i=cs-s+1}^{cs-1} \binom{cs-1}{i} \leq 2^{cs-c} \\
&\Leftrightarrow \sum_{i=0}^{s-1} \binom{cs-1}{i} + \sum_{i=0}^{s-2} \binom{cs-1}{i} \leq 2^{cs-c},
\end{aligned}$$

which we already proved in Lemma A.10. \square

B. A Randomized Algorithm

Although our main focus in this paper is on deterministic algorithms, for completeness we show that it is possible to obtain improved democratic fairness guarantees if we are allowed to use a randomized algorithm.

We define a protocol called Coin-toss with Weighted Approval Voting (CWAV) as follows: each turn, a group is picked at random by a fair coin-toss, and this group can pick a single good.

We assume that each group picks its good using a weighted-approval scheme where the weights are defined as follows:

$$\begin{aligned}
C(r, s) - w(r, s) &= C(r-1, s) \\
C(r, s) + w(r, s) &= C(r-1, s-1) \\
\forall r \geq 0 : C(r, 0) &= 1 \\
\forall r < s : C(r, s) &= 0
\end{aligned}$$

Note that these equations are identical to the equations of B in Section 3.2, except the second one where there is no max operation.

Lemma B.1. *Given a fairness criterion represented by an integer function $s(r)$, if the CWAV protocol is run with initial payments $C(r, s(r))$, then for every group, the expected fraction of happy agents is at least h , where:*

$$h = \inf_{r=1,2,\dots} C(r, s(r))$$

Proof. The proof idea is similar to Lemma 3.6. The initial balance in each group i is at least $h \cdot n_i$. It is sufficient to prove that, when CWAV ends, the expected balance of each group i is at least as large as its initial balance.

Suppose that, if group i wins the coin-toss it picks a good g_i , while if group $-i$ wins it picks a good g_{-i} . The change in balance is determined by the weights of these two groups:

- D_i — agents who desire g_i and do not care about g_{-i} . Each agent in this group (with r remaining goods and s goods that should be received) pays $w(r, s)$.

$r \downarrow s \rightarrow$	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	0	0
1	1	0.5	0	0	0	0	0	0	0	0
2	1	0.75	0.25	0	0	0	0	0	0	0
3	1	0.875	0.5	0.125	0	0	0	0	0	0
4	1	0.938	0.688	0.313	0.063	0	0	0	0	0
5	1	0.969	0.813	0.501	0.188	0.032	0	0	0	0
6	1	0.985	0.891	0.657	0.345	0.11	0.016	0	0	0
7	1	0.993	0.938	0.774	0.501	0.228	0.063	0.008	0	0
8	1	0.997	0.966	0.856	0.638	0.365	0.146	0.036	0.004	0
9	1	0.999	0.982	0.911	0.747	0.502	0.256	0.091	0.02	0.002
10	1	1	0.991	0.947	0.829	0.625	0.379	0.174	0.056	0.011

Table 4: Some values of $C(r, s)$. Compare to $\text{MAXH}(r, s)$ in Table 2 and $B(r, s)$ in Table 3.

- D_{-i} — agents who do not care about g_i and desire g_{-i} . Each agent in this group receives $w(r, s)$.

Therefore the expected change in the group balance after one turn is:

$$\mathbf{E}[\Delta[\text{Balance}]] = \frac{1}{2} \left[\sum_{j \in D_i} w(r_j, s_j) - \sum_{j \in D_{-i}} w(r_j, s_j) \right]$$

Since the group chose g_i over g_{-i} , the total weight of g_i is weakly larger, so the expected change in balance is ≥ 0 .

Since the coin-toss in each turn is independent of the other turns, the expected balance after the last turn is weakly larger than in the first turn. \square

Some values of $C(r, s)$ are shown in Table 4. By solving the recurrence relation we can express $C(r, s)$ as:

$$C(r, s) = \frac{1}{2^r} \sum_{i=s}^r \binom{r}{i}.$$

Note that this is the same as the function MAXH of Proposition 3.3, except the part where $r \leq 2s - 1$, for which $\text{MAXH} = 0$ but CWAV attains a positive fraction in expectation. (Obviously, with a randomized protocol we can always attain an expected fraction of $1/2$ by simply giving all goods to a random group, so the range where $C(r, s) \leq 1/2$ is not interesting).

C. Proof of Theorem 5.6

To establish this theorem, we prove two lemmas that may be of independent interest—one on cake-cutting and the other on group allocation for agents with additive valuations.

The result on cake-cutting generalizes the theorems of Stromquist [1980] and Su [1999], who prove the existence of contiguous envy-free cake allocations for individual agents. Since these results are well-known, we present the model and proof quite briefly, focusing on the changes required to generalize from individuals to groups.

We consider a “cake” modeled as the interval $[0, 1]$. Each agent a_{ij} has a value-density function $v_{ij} : [0, 1] \rightarrow \mathbb{R}_{\geq 0}$. The value of an agent for a piece X is $V_{ij}(X) = \int_{x \in X} v_{ij}(x) dx$. Denoting by X_i the allocation to group i , an allocation is *envy-free* for an agent a_{ij} if $V_{ij}(X_i) \geq V_{ij}(X_{i'})$ for every group i' . A *contiguous allocation* is an allocation of the cake in which each group gets a contiguous interval.

Lemma C.1. *There always exists a contiguous cake allocation that is $1/k$ -democratic envy-free. The factor $1/k$ is tight.*

Proof. The space of all contiguous partitions corresponds to the standard simplex in \mathbb{R}^k . Triangulate that simplex and assign each vertex of the triangulation to one of the groups. In each vertex, ask the group owning that vertex to select one of the k pieces using *plurality voting* among its members, breaking ties arbitrarily. Label that vertex with the group’s selection. The resulting labeling satisfies the conditions of *Sperner’s lemma* (see Su [1999]). Therefore, the triangulation has a *Sperner subsimplex*—a subsimplex all of whose labels are different. We can repeat this process with finer and finer triangulations. This gives an infinite sequence of smaller and smaller Sperner subsimplices. This sequence has a subsequence that converges to a single point. By the continuity of preferences, this limit point corresponds to a partition in which each group selects a different piece. Since the selection is by plurality, at least $1/k$ of the agents in each group prefer their group’s piece over all other pieces.

The tightness of the $1/k$ factor follows from Lemma 6 of Segal-Halevi and Nitzan [2015]. It shows an example with k groups and n' agents in each group with the property that in order to give a positive value to q out of n' agents in each group, we need to cut the cake into at least $k(kq - n')/(k - 1)$ intervals. In a contiguous partition there are exactly k intervals. Therefore, the fraction of agents in each group that can be guaranteed a positive value is $q/n' \leq 1/k + 1/n' - 1/kn'$. Since n' can be arbitrarily large, the largest fraction that can be guaranteed is $1/k$. \square

The next lemma presents a reduction from approximate envy-free allocation of indivisible goods to envy-free cake-cutting. We call this approximation “EF-minus-2”. An allocation is *EF-minus-2* for agent a_{ij} if for every group i' , $u_{ij}(G_i) > u_{ij}(G_{i'}) - 2u_{ij,\max}$. The reduction generalizes Theorem 5 of Suksompong [2017]; a similar reduction was used in Theorem 3 of Barrera et al. [2015].

Lemma C.2. *When agents have additive valuations, there always exists a contiguous allocation of indivisible goods that is $1/k$ -democratic EF-minus-2.*

Proof. We create an instance of the cake-cutting problem in the following way.

- The cake is the half-open interval $(0, m]$.

- The value-density functions are piecewise constant: for every $l \in \{1, \dots, m\}$, the value-density v_{ij} in the half-open interval $(l-1, l]$ equals $u_{ij}(g_l)$.

By Lemma C.1, there exists a contiguous cake allocation that is envy-free for at least $1/k$ of the agents in each group. From this allocation we construct an allocation of goods as follows.

- If point g of the cake is in the interior of a piece, then good g is given to the group owning that piece.
- If point g of the cake is at the boundary between two pieces, then good g is given to the group owning the piece to its left.

A group gets good g only if it owns a positive fraction of the interval $(g-1, g]$. Hence, in the allocation, each group loses strictly less than the value of a good and gains strictly less than the value of a good (relative to its value in the cake division). This means that every agent who believes that the cake allocation is envy-free also believes that the goods allocation is EF-minus-2. \square

We are now ready to prove Theorem 5.6.

Proof of Theorem 5.6. Suppose an allocation is EF-minus-2 for some agent a_{ij} . This means that the agent's envy towards any other group is less than $2u_{ij,\max} \leq 2$. Since the agent has binary valuations, the envy is at most 1, meaning that the allocation is EF1 for that agent. Hence any $1/k$ -democratic EF-minus-2 allocation, which is guaranteed to exist by Lemma C.2, is also $1/k$ -democratic EF1. By Lemma 2.1 it is also $1/k$ -democratic MMS-fair. \square

The cake-cutting protocol of Lemma C.1 might take infinitely many steps to converge. In fact, there is no finite protocol for contiguous envy-free cake-cutting even for individuals [Stromquist, 2008]. However, the division guaranteed by Lemma C.2 and Theorem 5.6 can be found in finite time (exponential in the input size) by checking all possible allocations. An interesting open question is whether a faster algorithm exists.