

Edge conditions under a bearing load; use of asymptotic solutions

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Abstract

This short paper focuses on the nature of the tractions at edge of a complete contact in the form of a sharp wedge on a flat surface. The possible states of stress, slip and separation are dependent on the wedge angle and the coefficient of friction. The requirement of remote loading is only that it holds the parts together. The choice of this feature may thus have consequences for the integrity of some components.

Keywords: Complete contact, Asymptotic analysis, Friction, Separation, Slip.

1 Problem

The object of this note is to show how we may infer the contact conditions at the edge of some flared out pedestal, such as the one shown schematically in Figure 1. The main load carried is assumed to be approximately as shown but it does not need to be applied strictly normal to the bearing surface. Provided that the two components pressed together are made from the same material, the only independent variable in the problem are the contact angle, ϕ , and the coefficient of friction between the bodies, denoted by f .

First, assume that the interface is in intimate contact right up to the very edge. If this is the case the two bodies may be taken together, as a monolith, of internal

angle $(\pi + \phi)$. Thus, the state of stress near to the edge may be determined, to within a constant (which must be found by taking into account the exact geometry of the problem, if additional detail is required) by using the Williams solution¹ for a wedge of that angle. The Williams solution shows that, if r is a coordinate measured inwards from the (left) contact edge the state of stress varies in the form

$$\sigma_{ij}(r, \theta) = K_I r^{\lambda_I - 1} g_{ij}^I(\theta) + K_{II} r^{\lambda_{II} - 1} g_{ij}^{II}(\theta)$$

where the state of stress is referred to the bisector, as shown in Figure 1.. As $\lambda_I < \lambda_{II}$ the first term will always dominate the second, and the value of λ_I is given by the smallest non-zero root of the equation

$$\lambda \sin \phi - \sin \lambda (\pi + \phi) = 0.$$

From the eigenvalue we follow the method implied by Williams and find the eigenvector and thus the stresses so, that for a wedge on a flat, the edge will be adhered providing that $K_I < 0$ and that, in addition $f > g_{r\theta}^I / g_{\theta\theta}^I$ evaluated at the interface, thus

$$\frac{g_{r\theta}^I(\chi)}{g_{\theta\theta}^I(\chi)} = \frac{\cos(\lambda_I - 1)\phi - \cos(\lambda_I + 1)\phi}{\sin(\lambda_I - 1)\phi - \frac{\lambda_I + 1}{\lambda_I - 1} \sin(\lambda_I - 1)\phi} \times \frac{\sin(\lambda_I - 1)\phi \sin(\lambda_I + 1)\chi - \sin(\lambda_I + 1)\phi \sin(\lambda_I - 1)\chi}{\cos(\lambda_I - 1)\phi \cos(\lambda_I + 1)\chi - \cos(\lambda_I + 1)\phi \cos(\lambda_I - 1)\chi},$$

where $\phi = \pi + \phi$ and $\chi = (\pi - \phi) / 2$. This limit is included in Figure 2 (solid), and shows that, for example, if $\phi = 65^\circ$, a value of $f \gtrsim 0.870$ will ensure that the corner will certainly stick; lower coefficients of friction will be accompanied by a slipping region at the contact edge. For typical metal on metal contacts, it is therefore very likely that the outer edges of the contact will slip if a value of $\phi \lesssim 70^\circ$ is chosen ($f < 0.789$).

Suppose that the coefficient of friction is, indeed, too low to ensure that stick persists to the edge. We may now fit in a new asymptotic form, viz. the one derived by

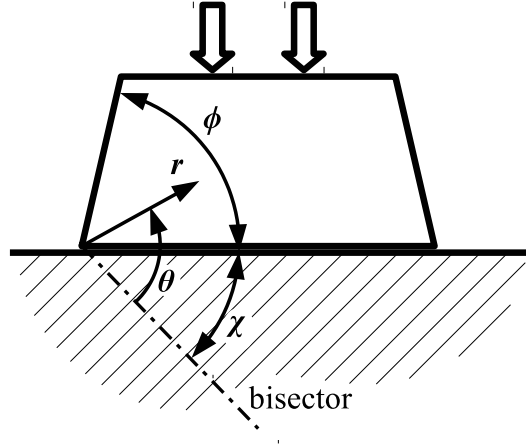


Figure 1: Geometry of the problem. The angle between the bisector and interface $\chi = (\pi - \phi) / 2$

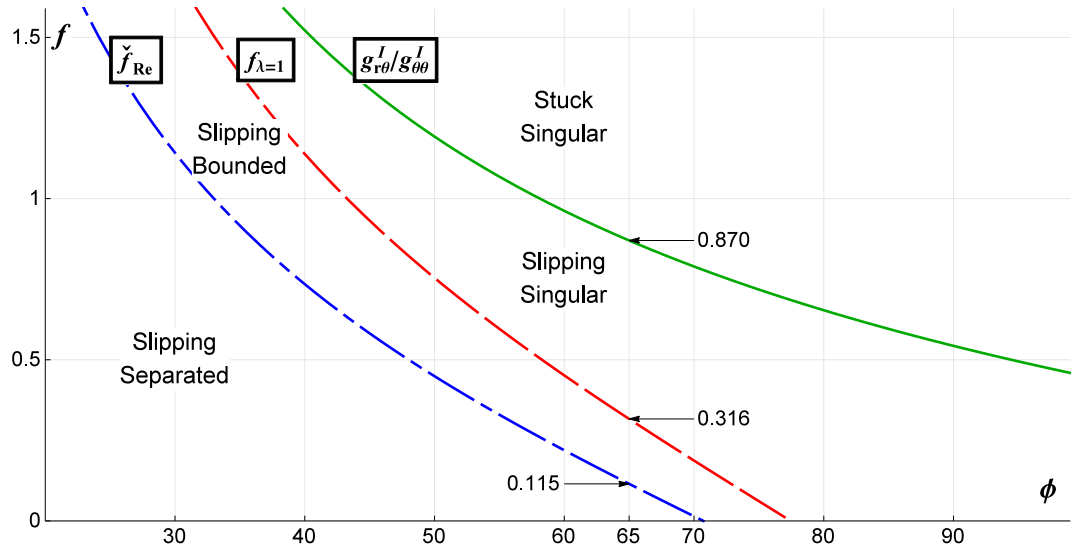


Figure 2: Friction limits v. wedge angle, ϕ . The zones of contact behaviour are indicated and the values for the example case of 65° are shown.

Gdoutos and Theocaris² for a semi-infinite wedge sliding along another wedge, which, in general may be elastically dissimilar. We now specialise that solution to the case where the bodies are made from the same material. The eigenvector component for shear, $g_{r\theta}^I$, is positive for positive θ thus when $K_I < 0$, as with indentation by a finite body ($0 < \phi < \pi$), the shear stress at the interface associated with the first mode will be negative and thus *always* tend to ‘spread’ the contact so that both corners are ‘leading edge’ in character. Gdoutos and Theocaris used a sign convention appropriate to analysis of ‘trailing edge’ contacts, we have modified our formulae so that a positive number is appropriate. The state of stress in the neighbourhood of the contact edge is now of the form

$$\sigma_{ij}(r, \theta) = K_s r^{\lambda_s - 1} g_{ij}^s(\theta)$$

and the value of λ_s which is relevant is the smallest non-zero root of the equation

$$\begin{aligned} & \cos \lambda_s \pi (\sin^2 \lambda_s \phi - \lambda_s^2 \sin^2 \phi) \\ & + \frac{1}{2} \sin \lambda_s \pi (\sin 2\lambda_s \phi + \lambda_s \sin 2\phi) \\ & - f \lambda_s (1 + \lambda_s) \sin \lambda_s \pi \sin^2 \phi = 0. \end{aligned}$$

The root of this characteristic equation must be found numerically, and a typical variation of the lowest root with coefficient of friction is shown in Figure 3 for the example case of $\phi = 65^\circ$. We distinguish between three kinds of dominant root;

1. If the coefficient of friction is slightly less than that needed to maintain full adhesion it is found that $\lambda_s < 1$ and the contact pressure remains singular as the contact corner is approached. From Figure 2, we see that for $\phi = 65^\circ$ this behaviour occurs when the coefficient of friction is in the range $0.316 \lesssim f \lesssim 0.870$. Note that if $f \gtrsim 0.870$ the contact, as already shown above, sticks everywhere.
2. If a lower coefficient of friction is present $\lambda_s > 1$ and, as the contact pressure varies as $r^{\lambda_s - 1}$ it is bounded as the corner is approached. For the example case of the 65° contact angle the order of the bounding may be read from the figure, and a contour showing the lower value of behaviour at which this kind of characteristic

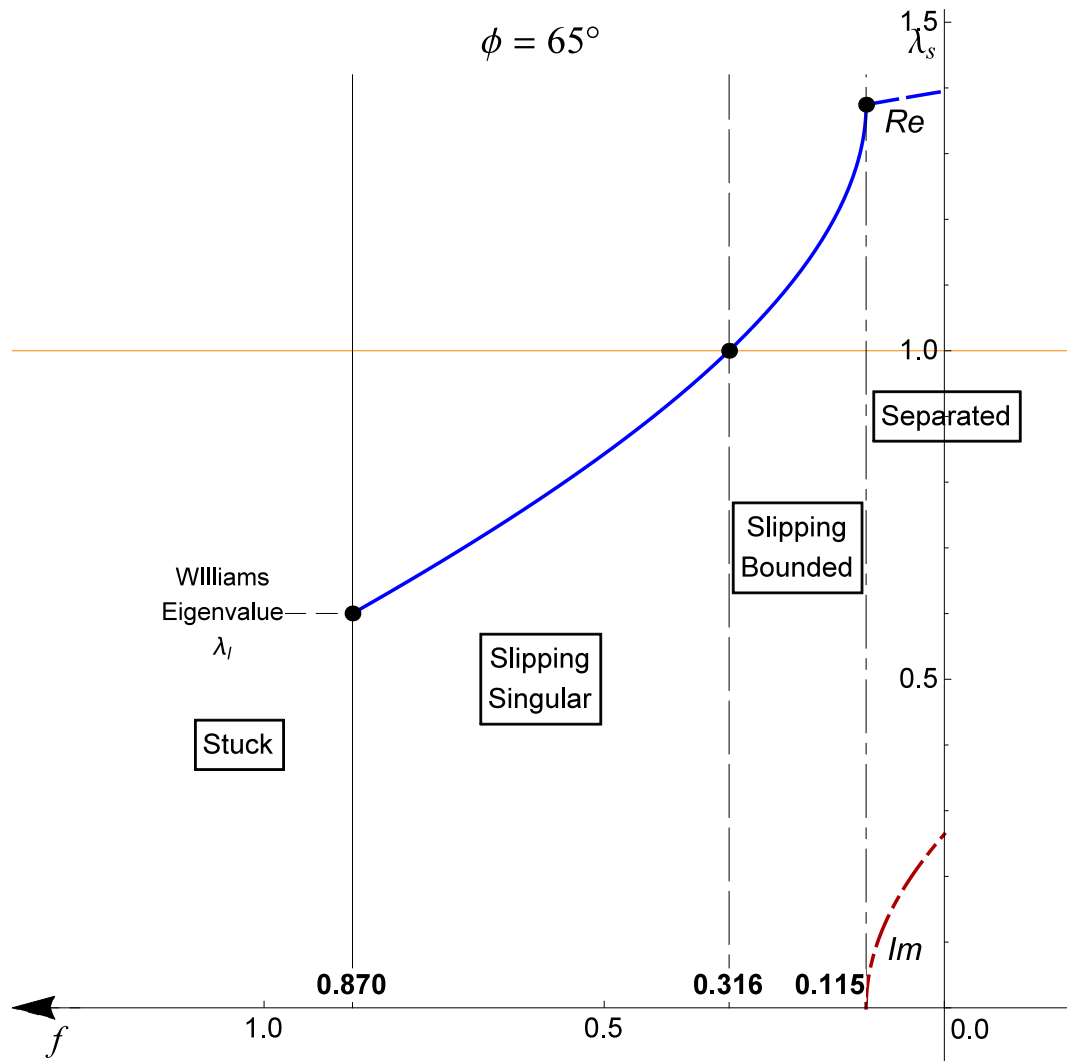


Figure 3: Eigenvalue v Friction, for $\phi = 65^\circ$. continuous = real, dashed and chained = real and imaginary parts when complex. Transitions: Stuck to Slipping, Singular to Bounded, Closed to Separated, occur as friction reduces at $f = 0.870, 0.316, 0.115$, respectively

may be expected is included in Figure 2 (dashed), labelled $f_{\lambda=1}$. From Figure 3, we see that for $\phi = 65^\circ$ this behaviour occurs when the coefficient of friction is in the range $0.115 \lesssim f \lesssim 0.316$.

3. Lastly, if the coefficient is very low (for 65° , $f \lesssim 0.115$) the root becomes complex, and, as discovered and analysed in detail by Karuppanan³ this means that there will be local separation. At the new contact edge, which will be in-board from the edge of the pedestal, contact pressure will varies in a square root bounded manner as this is now an ‘incomplete’ contact. Again, the boundary of this region is shown in Figure 2 (chained), labelled \check{f}_{Re} , being the minimum with a real eigenvalue .

Note that at a lower angle of ϕ , where the contact is flared out more, the relevant coefficients of friction denoting the boundaries between different kinds of behaviour are higher, as shown in the plots of Fig. 2.

Also to be noted from Figure 2 is that even for zero friction when $\phi > 70.72^\circ$ there will not be separation and, for $\phi > 77.45^\circ$ the stress at the edge will always be singular even if there is slip. For such cases the plot equivalent to that shown in Figure 3 will have corresponding less regions of differing behaviour. The ultimate limiting case is when $\phi = 180^\circ$ (extending the axis in Figure 2 to the right) which is where the limit for the stick-slip transition meets the axis ($f = 0$).

2 Summary

Williams and related asymptotic methods are very powerful ways that allow us to predict what the behaviour near corners is going to be. They not only enable us to say whether the local pressure will be singular (implying, in practice, some plasticity and probably a very well bedded-in contact) or bounded in character, but they also enable us to say whether a corner will stick or slip. And this part of the solution, obtained by careful consideration of the characteristic equation, applies wholly independently of the multiplier on the solution, that is, the calibration for the generalised stress intensity factor.

Most manufactured items are ‘square’ (i.e. $\phi = 90^\circ$) and will be fully stuck if $f > 0.543$ and only have slipping edges if the coefficient of friction is lower; the contact tractions between foot and substrate will always be singular at the corners. The picture is rather different if the edge is ‘flared out’ with an angle of say 65° (our example) or less. Now, it requires a coefficient of friction of almost unity to make the corners stick, so that, in practice, there are likely to be edge zones of slip and, if the coefficient of friction is low enough, the contact pressure will fall to zero at the edges. These basic points would be of interest in design for structures that include parts clamped together and then subject to fatigue loads.

References

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