Supplementary Information for A finite sufficient set of conditions for catalytic majorization

SUPPLEMENTARY NOTE 1: PROOF OF THEOREM 8

1. Preliminary results

We first introduce the essential ingredients that serve as the foundation for proving Theorem 8.

Fact 1. *[1, Theorem 2.7.1] Consider two sets of positive real numbers $\{a_i\}_{i=1}^N$ and $\{b_i\}_{i=1}^N$. The following inequality holds:*

$$\sum_{i=1}^{N} a_i \log\left(\frac{a_i}{b_i}\right) \geq \left(\sum_{i=1}^{N} a_i\right) \log\left(\frac{\sum_{i=1}^{N} a_i}{\sum_{i=1}^{N} b_i}\right).$$

We now present the useful definitions and lemmas from [2] that connect the scaled l_p norms of vectors to a class of symmetric polynomials used to derive our results. To this end, we start with the following definition:

Definition 2. *Consider an integer $r \geq 1$. Denote the r^{th} degree Taylor polynomial expansion of the exponential function by P_r, where $P_r(\nu) = \sum_{i=0}^{r} \frac{\nu^i}{i!}$. For any vector $x \in \mathbb{R}^N$ and any real number $t \in \mathbb{R}$ define*

$$f_r(x,t) := \prod_{i=1}^{N} P_r(x_i t) = \prod_{i=1}^{N} \left(\sum_{j=0}^{r} \frac{(x_i t)^j}{j!}\right). \tag{S1}$$

Further, for any integer $k \geq 1$ we define $F_{k,r}(x)$ as the coefficient of t^k in the expansion of $f_r(x,t)$. Equivalently, we can express $f_r(x,t)$ as:

$$f_r(x,t) := \sum_{k=1}^{r} F_{k,r}(x) t^k. \tag{S2}$$

Fact 3. *[2, Theorem 1] Consider two probability vectors x, y of support size N and take a fixed integer $r \geq 1$. Then we have the following characterization of the scaled l_p with respect to the polynomial coefficients $F_{k,r}$ of the exponential function, for all integers k such that $r \leq k \leq Nr$.*

If $F_{k,r}(x) \leq F_{k,r}(y)$ then

$$\begin{aligned} \|x\|_p &\leq \|y\|_p \quad \text{for } p \in [0,1], \\ \|x\|_p &\geq \|y\|_p \quad \text{for } p \in [1, r+1]\,. \end{aligned}$$

We also need the following relations of the so-called Mellin transforms [2]:

Fact 4. *Define the integral transform of the logarithm of P_r (from Definition 2) as:*

$$\begin{aligned} I_r(p) &:= \int_0^\infty \log\left(\sum_{i=0}^{r} \frac{\nu^i}{i!}\right) \nu^{-p} \frac{d\nu}{\nu} \quad \text{for } p \in (0,1) \text{ and} \\ J_r(p) &:= \int_0^\infty \left(\nu - \log\left(\sum_{i=0}^{r} \frac{\nu^i}{i!}\right)\right) \nu^{-p} \frac{d\nu}{\nu} \quad \text{for } p \in (1, r+1)\,. \end{aligned}$$

Then, if we let $t = \nu/a$ for $a \geq 0$, the following relations hold:

$$\begin{aligned} \frac{1}{I_r(p)} \int_0^\infty \log\left(\sum_{i=0}^{r} \frac{(at)^i}{i!}\right) t^{-p} \frac{dt}{t} &= a^p \quad \text{for } p \in (0,1) \\ \frac{1}{J_r(p)} \int_0^\infty \left(at - \log\left(\sum_{i=0}^{r} \frac{(at)^i}{i!}\right)\right) t^{-p} \frac{dt}{t} &= a^p \quad \text{for } p \in (1, r+1). \end{aligned}$$

We now prove the statement 1 of Fact 14 to analyze the case when the thermal distribution has irrational entries as the following proposition:

Proposition 5. *Consider the spectral decomposition of the thermal state ρ_g in the energy eigenbasis $\{|e_i\rangle\}_{i=1}^d$ as:*

$$\rho_g = \sum_{i=1}^{d} g_i \,|e_i\rangle\langle e_i|$$

with some of the eigenvalues g_i being irrational. Then there exists a state ρ_{g_ε} with all eigenvalues as rational numbers of form $\{\frac{d_i}{N}\}_{i=1}^N$, for N large enough, ρ_{g_ε} is diagonal in the energy eigenbasis and

$$\|\rho_g - \rho_{g_\varepsilon}\|_1 \leq \varepsilon$$

for any given $\varepsilon > 0$.

Proof. The above proposition is simply an extension of the fact that any irrational number can be approximated by a rational number to any desired accuracy.

More formally, let g_ε denote the vector of eigenvalues of ρ_{g_ε} and N, ε are given. Suppose for some $1 \leq i \leq d$, g_i is irrational. Then by Archimedian property of real numbers, there exists an integer N' such that $N' > \max\left[\max_i [g_i], \frac{N}{\varepsilon}\right]$. Now we define $d_i := \lfloor g_i N' \rfloor$ for all $1 \leq i \leq d$.

Since g is a probability vector, therefore $\sum_{i=1}^{d} d_i = N'$.

Finally, define the rational approximation of g_i as: $(g_\varepsilon)_i := \frac{d_i}{N'}$. Thus, $(g_\varepsilon)_i$ satisfy:

$$|g_i - (g_\varepsilon)_i| = \left| g_i - \frac{d_i}{N'} \right| = \left| \frac{N' g_i - d_i}{N'} \right| \leq \frac{\varepsilon}{N'} \; .$$

Note that if for any $1 \leq i \leq d,$ if g_i is rational then $g_i = (g_\varepsilon)_i = \frac{d_i}{N}$. We thus obtain the resultant density matrix

$$\rho_{g_\varepsilon} := \sum_{i=1}^{d} \frac{d_i}{N'} \,|e_i\rangle\langle e_i| \; .$$

This satisfy the property that:

$$\|\rho_g - \rho_{g_\varepsilon}\|_1 = \sum_{i=1}^{d} \left| g_i - \frac{d_i}{N'} \right| \leq N \frac{\varepsilon}{N'} \leq \varepsilon.$$

□

Below we prove a continuity statement for the p-Rényi divergence.

Proposition 6. *Let $p > 0$, and any given $\varepsilon > 0$, and two probability distributions x and g. Let $g_{\min} := \min_i g_i$ and g_ε be another probability distributions such that $\|g - g_\varepsilon\|_1 \leq \varepsilon$, then:*

$$|D_p(x||g_\varepsilon) - D_p(x||g)| \;\leq\; \max\left\{1, \frac{p}{|p-1|}\right\} \log\left(1 + \frac{\varepsilon}{g_{\min}}\right), \; \textit{and} \tag{S3}$$

$$|D_1(x||g_\varepsilon) - D_1(x||g)| \;\leq\; \log\left(1 + \frac{\varepsilon}{g_{\min}}\right). \tag{S4}$$

Proof. The above proposition can be proven as follows:

$$
\begin{aligned}
|D_p(x||g_\varepsilon) - D_p(x||g)| &= \frac{1}{|p-1|}\left|\left\{\log\left(\sum_{i=1}^{N} x_i^p (g_\varepsilon)_i^{1-p}\right) - \log\left(\sum_{i=1}^{N} x_i^p g_i^{1-p}\right)\right\}\right| \\
&= \frac{1}{|p-1|\left(\sum_{i=1}^{N} x_i^p {(g_\varepsilon)_i}^{1-p}\right)}\left|\left(\sum_{i=1}^{N} x_i^p {(g_\varepsilon)_i}^{1-p}\right)\log\left(\frac{\sum_{i=1}^{N} x_i^p {(g_\varepsilon)_i}^{1-p}}{\sum_{i=1}^{N} x_i^p {g_i}^{1-p}}\right)\right| \\
&\overset{a}{\leq} \frac{1}{|p-1|\left(\sum_{i=1}^{N} x_i^p {(g_\varepsilon)_i}^{1-p}\right)} \sum_{i=1}^{N} x_i^p {(g_\varepsilon)_i}^{1-p}\left|\log\left(\frac{(g_\varepsilon)_i}{g_i}\right)^{1-p}\right| \\
&\leq \frac{1}{\left(\sum_{i=1}^{N} x_i^p {(g_\varepsilon)_i}^{1-p}\right)}\left(\sum_{i=1}^{N} x_i^p {(g_\varepsilon)_i}^{1-p}\right)\left[\max_i\left|\log\left(\frac{(g_\varepsilon)_i)}{g_i}\right)\right|\right] \\
&= \max_i\left|\log\left(\frac{(g_\varepsilon)_i}{g_i}\right)\right| \\
&\overset{b}{\leq} \log\left(1+\frac{\varepsilon}{g_{\min}}\right),
\end{aligned}
$$

where (a) follows from the log-sum inequality, Fact 1, and (b) follows from the following:

$$
\begin{aligned}
\varepsilon \geq \|g - g_\varepsilon\|_1 &= \sum_{i=1}^{N} g_i\left|\frac{(g_\varepsilon)_i}{g_i} - 1\right| \\
&\geq g_{\min}\sum_{i=1}^{N}\left|\frac{(g_\varepsilon)_i}{g_i} - 1\right| \\
&\geq g_{\min}\max_i\left|\frac{(g_\varepsilon)_i}{g_i} - 1\right| \\
&\Rightarrow \log\left(\frac{(g_\varepsilon)_i}{g_i}\right) \leq \log\left(1+\frac{\varepsilon}{g_{\min}}\right);\ \text{for all } i \in \{1, 2\ldots, N\}.
\end{aligned} \tag{S5}
$$

We now give a continuity statement for the case of $p = 1$ (i.e., for KL-divergence). The simplicity comes from the definition of $D_1(x||g_\varepsilon) := \sum_i x_i \log\frac{x_i}{(g_\varepsilon)_i}$ which leads to:

$$
\begin{aligned}
|D_1(x||g) - D_1(x||g_\varepsilon)| &= \sum_i x_i\left|\log\left(\frac{(g_\varepsilon)_i}{g_i}\right)\right| \\
&\leq \max_i\left|\log\left(\frac{(g_\varepsilon)_i}{g_i}\right)\right|\sum_i x_i \\
&= \max_i\left|\log\left(\frac{(g_\varepsilon)_i}{g_i}\right)\right| \\
&\overset{i}{\leq} \log\left(1+\frac{\varepsilon}{g_{\min}}\right)
\end{aligned} \tag{S6}
$$

where (i) follows from Eq. (S5). This proves Eq. (S4). □

In the following lemmas, we demonstrate how conditions based on symmetric polynomials lead to trumping by explicitly examining the relation between $D_p(\rho||\rho_g)$ and $D_p(\sigma||\rho_g)$ across the full range of $p \in \mathbb{R}$. Throughout, we consider x, y as two probability vectors of support size N, that is, $\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} y_i = 1$ and without loss of generality, the logarithm is taken with base 2.

2. Characterization in terms of relative entropy, for $p = 1$:

For $p = 1$, following two lemma guarantee the conditions of trumping.

Lemma 7. *For any given* $\varepsilon > 0$*, if* $H_1(x) < H_1(y) - 2\log\left(1 + \frac{\varepsilon}{g_{\min}}\right)$*, then*

$$D_1(\rho||\rho_{g_\varepsilon}) > D_1(\sigma||\rho_{g_\varepsilon}) + 2\log\left(1 + \frac{\varepsilon}{g_{\min}}\right) .$$

Proof. The proof follows easily by using the following relation between entropy and the relative entropy:

$$D_1(x||g_\varepsilon) = \log N - H_1(x) .$$

Using the hypothesis of the lemma and above relation completes the proof. □

Lemma 8. *For any given* $\varepsilon > 0$*, if* $D_1(\rho||\rho_{g_\varepsilon}) > D_1(\sigma||\rho_{g_\varepsilon}) + 2\log\left(1 + \frac{\varepsilon}{g_{\min}}\right)$*, then*

$$D_1(\rho||\rho_g) > D_1(\sigma||\rho_g).$$

Proof. The continuity of relative entropy from Eq. (S4) together with the hypothesis of the lemma, gives

$$\begin{aligned} D_1(\rho||\rho_g) &\overset{a}{>} D_1(\rho||\rho_{g_\varepsilon}) - \log\left(1 + \frac{\varepsilon}{g_{\min}}\right) \\ &> D_1(\sigma||\rho_{g_\varepsilon}) + \log\left(1 + \frac{\varepsilon}{g_{\min}}\right) \\ &\overset{b}{>} D_1(\sigma||\rho_g) \end{aligned}$$

where (a) and (b) follows from Eq. (S4). □

3. Characterization in terms of p Rényi divergence, for $p \in (1, \bar{r})$:

For $p \in (1, \bar{r})$, in order to show $F_{k,\bar{r}}(x) < \frac{F_{k,\bar{r}}(y)}{A_1(\varepsilon)}$ implies trumping (for some suitably chosen $A_1(\varepsilon)$), we prove following series of Lemmas.

Lemma 9. *If* $F_{k,\bar{r}}(x) < \frac{F_{k,\bar{r}}(y)}{A_1(\varepsilon)}$ *for some real valued positive function* $A_1(\varepsilon)$ *which is independent of* k *(with* $\varepsilon > 0$*) then*

$$f_{\bar{r}}(x,t) < \frac{f_{\bar{r}}(y,t)}{A_1(\varepsilon)}.$$

Proof. By definition 2 of $f_{\bar{r}}(x,t)$ and $F_{k,\bar{r}}(x)$ [2]:

$$\begin{aligned} f_{\bar{r}}(x,t) &= \sum_k F_{k,\bar{r}}(x) t^k \\ &\overset{a}{<} \frac{1}{A_1(\varepsilon)} \sum_k F_{k,\bar{r}}(y) t^k \\ &= \frac{f_{\bar{r}}(y,t)}{A_1(\varepsilon)}. \end{aligned}$$

Here (a) follows from the hypothesis of the lemma. □

Lemma 10. *If* $f_{\bar{r}}(x,t) < \frac{f_{\bar{r}}(y,t)}{A_1(\varepsilon)}$ *for any* $\varepsilon > 0$ *and* $A_1(\varepsilon)$ *is as defined in Lemma 9 and in addition satisfies* $A_1(\varepsilon) \geq 2^{\left[\frac{1}{N}\left\{\left(1+\frac{\varepsilon}{g_{\min}}\right)^{2\bar{r}}\right\}\right]}$*, then*

$$\|x\|_p > \|y\|_p \left(1 + \frac{\varepsilon}{g_{\min}}\right)^2 .$$

Proof. Using Fact 4, we have the following:

$$\sum_{i=1}^{N}\left[y_i t - \log\left(\sum_{j=1}^{\bar{r}} \frac{(y_i t)^j}{j!}\right)\right] = \sum_{i=1}^{N} y_i t - \log f_{\bar{r}}(y,t)$$
$$\overset{a}{<} \sum_{i=1}^{N} y_i t - \log f_{\bar{r}}(x,t) - \log A_1(\varepsilon).$$

Here (a) follows from the hypothesis of the lemma. Now multiplying both sides of the above Eq. by $\frac{t^{-p-1}}{J_{\bar{r}}(p)}$, rearranging the terms, integrating with respect to $t \in (0,\infty)$ and using Fact 4, we get:

$$\begin{aligned}
N\|x\|_p^p &> N\|y\|_p^p + \frac{\log A_1(\varepsilon)}{J_{\bar{r}}(p)} \int_0^\infty t^{-p-1} dt \\
&\overset{a}{>} N\|y\|_p^p + \frac{\log A_1(\varepsilon)}{J_{\bar{r}}(p)} \int_k^\infty t^{-p-1} dt \\
&= N\|y\|_p^p + N \log A_1(\varepsilon) \\
\Rightarrow \|x\|_p^p &> \|y\|_p^p \times \left[1 + \frac{\log A_1(\varepsilon)}{\|y\|_p^p}\right] \\
\Rightarrow \|x\|_p &\overset{b}{>} \|y\|_p \left[1 + \frac{\varepsilon}{g_{\min}}\right]^2,
\end{aligned}$$

where in (a) we set the lower limit of the integral as $k = (NpJ_{\bar{r}}(p))^{-\frac{1}{p}} > 0$ ensuring that the integral evaluates to $NJ_{\bar{r}}(p)$ and (b) holds for the choice of $A_1(\varepsilon)$ as:

$$\begin{aligned}
A_1(\varepsilon) &\geq 2^{\left[\frac{1}{N}\left\{\left(1+\frac{\varepsilon}{g_{\min}}\right)^{2\bar{r}}\right\}\right]} \\
&\overset{a}{\geq} 2^{\|y\|_p^p\left\{\left(1+\frac{\varepsilon}{g_{\min}}\right)^{2\bar{r})}\right\}} \\
&\overset{b}{\geq} 2^{\|y\|_p^p\left\{\left(1+\frac{\varepsilon}{g_{\min}}\right)^{2p}-1\right\}}.
\end{aligned}$$

In the above analysis (a) holds because $\|y\|_p^p = \frac{\Sigma_{i=1}^n y_i^p}{N} \leq \frac{\Sigma_{i=1}^N y_i}{N} = \frac{1}{N}$, since y is a probability vector and (b) follows as $p \in (1, \bar{r})$.
This completes the proof of the lemma. □

Lemma 11. *If $\|x\|_p > \|y\|_p\left(1 + \frac{\varepsilon}{g_{\min}}\right)^2$ for any given $\varepsilon > 0$, then:*

$$H_p(x) < H_p(y) - \frac{2p}{p-1} \log\left(1 + \frac{\varepsilon}{g_{\min}}\right).$$

Proof. From the definition of Renyi entropy (Eq. 3, 4, 5) we have:

$$\begin{aligned}
H_p(x) &:= -\frac{1}{p-1} \log\left(N \times \frac{\sum_{i=1}^{N} x_i^p}{N}\right) \\
&\overset{a}{<} -\frac{\log N}{p-1} - \frac{p}{p-1} \log\|y\|_p - \frac{2p}{p-1} \log\left(1 + \frac{\varepsilon}{g_{\min}}\right) \\
&= H_p(y) - \frac{2p}{p-1} \log\left(1 + \frac{\varepsilon}{g_{\min}}\right),
\end{aligned}$$

where (a) follows from the hypothesis of the lemma. □

Lemma 12. *For any given $\varepsilon > 0$, if $H_p(x) < H_p(y) - \frac{2p}{p-1}\log\left(1 + \frac{\varepsilon}{g_{\min}}\right)$ then:*

$$D_p(\rho||\rho_{g_\varepsilon}) \geq D_p(\sigma||\rho_{g_\varepsilon}) + \frac{2p}{p-1} \log\left(1 + \frac{\varepsilon}{g_{\min}}\right). \tag{S7}$$

Proof. The proof essentially follows from the relation between the relative entropy of a given distribution with uniform distribution and the Shannon entropy of a given state, i.e., $D_p(x||u_N) = \log N - H_p(x)$ and similarly $D_p(y||u_N) = \log N - H_p(y)$. Since,

$$\begin{aligned} D_p(\rho||\rho_{g_\varepsilon}) &= D_p(x||u_N) = \log N - H_p(x) \\ &\overset{a}{>} \log N - H_p(y) + \frac{2p}{p-1}\log\left(1+\frac{\varepsilon}{g_{\min}}\right) \\ &= D_p(y||u_N) + \frac{2p}{p-1}\log\left(1+\frac{\varepsilon}{g_{\min}}\right) \\ &= D_p(\sigma||\rho_{g_\varepsilon}) + \frac{2p}{p-1}\log\left(1+\frac{\varepsilon}{g_{\min}}\right). \end{aligned}$$

Here (a) follows from the hypothesis of the lemma. □

Lemma 13. *For any given $\varepsilon > 0$, if $D_p(\rho||\rho_{g_\varepsilon}) \geq D_p(\sigma||\rho_{g_\varepsilon}) + \frac{2p}{p-1}\log\left(1+\frac{\varepsilon}{g_{\min}}\right)$ then:*

$$D_p(\rho||\rho_g) > D_p(\sigma||\rho_g).$$

Proof.

$$\begin{aligned} D_p(\rho||\rho_g) &\overset{a}{>} D_p(\rho||\rho_{g_\varepsilon}) - \frac{p}{p-1}\log\left(1+\frac{\varepsilon}{g_{\min}}\right) \\ &\overset{b}{>} D_p(\sigma||\rho_{g_\varepsilon}) + \frac{2p}{p-1}\log\left(1+\frac{\varepsilon}{g_{\min}}\right) - \frac{p}{p-1}\log\left(1+\frac{\varepsilon}{g_{\min}}\right) \\ &\overset{c}{>} D_p(\sigma||\rho_g) - \frac{p}{p-1}\log\left(1+\frac{\varepsilon}{g_{\min}}\right) + \frac{2p}{p-1}\log\left(1+\frac{\varepsilon}{g_{\min}}\right) - \frac{p}{p-1}\log\left(1+\frac{\varepsilon}{g_{\min}}\right) \\ &= D_p(\sigma||\rho_g) \end{aligned}$$

where (a) follows from Proposition 6, (b) follows from the hypothesis of the Lemma and (c) follows from Proposition 6. □

Thus, $F_{k,\bar{r}}(x) < \frac{F_{k,\bar{r}}(y)}{A_1(\varepsilon)}$ along with the choice of $A_1(\varepsilon) = 2^{\left[\frac{1}{N}\left\{\left(1+\frac{\varepsilon}{g_{\min}}\right)^{2\bar{r}}\right\}\right]}$ is sufficient to guarantee trumping.

4. Characterization in terms of p Rényi divergence, for $p \in (0,1)$:

Since the relation between scaled l_p norms of two vectors differ in the regime where $p \in (0,1)$ compared to when $p \in (1,\bar{r})$, some of the lemmas presented above have different bounds and quantifiers. Here we prove the lemmas for the region $p \in (0,1)$ for clarity and completeness.

Lemma 14. *If $f_{\bar{r}}(x,t) < \frac{f_{\bar{r}}(y,t)}{A_2(\varepsilon)}$ for any $\varepsilon > 0$ and $A_2(\varepsilon)$ is as defined in Lemma 9 and in addition satisfies $A_2(\varepsilon) \geq 2^{\frac{2\varepsilon}{N g_{\min}}}$, then*

$$\|x\|_p < \begin{cases} \|y\|_p\left(1+\frac{\varepsilon}{g_{\min}}\right)^{-2\left(\frac{1-p}{p}\right)} & \text{for } p \in [0,\frac{1}{2}] \\ \|y\|_p\left(1+\frac{\varepsilon}{g_{\min}}\right)^{-2} & \text{for } p \in (\frac{1}{2},1] \end{cases}$$

Proof. Using Fact 3, we have the following:

$$\begin{aligned} N\|x\|_p^p &= \frac{1}{I_{\bar{r}}(p)}\int_0^\infty t^{-p-1}\log f_{\bar{r}}(x,t)dt \\ &\overset{a}{<} \frac{1}{I_{\bar{r}}(p)}\int_0^\infty t^{-p-1}\log f_{\bar{r}}(y,t)dt - \log A_2(\varepsilon)\int_0^\infty \frac{t^{-p-1}}{I_{\bar{r}}(p)}dt \\ &\overset{b}{<} N\|y\|_p^p - \log A_2(\varepsilon)\int_k^\infty \frac{t^{-p-1}}{I_{\bar{r}}(p)}dt \\ &= N\|y\|_p^p\left\{1 - N\eta(p)\log A_2(\varepsilon)\right\}. \end{aligned} \tag{S8}$$

where (a) follows from the hypothesis of the lemma, while inequality (b) arises from restricting the integration range of the second term to a modified lower bound $k = (\eta(p)pN^2\|y\|_p^p I_{\bar{r}}(p))^{-1/p}$ instead of 0. This ensures that the definite integral in the second term evaluates to $N^2\|y\|_p^p\eta(p)$ with the chosen function $\eta(p) = \max\{p, (1-p)\}$ depending on the range of p.
Now, we choose $A_2(\varepsilon) \geq 2^{\frac{2\varepsilon}{N g_{\min}}}$ (implying that $\log A_2(\varepsilon) > 0$), which further leads to:

$$\begin{aligned}\eta(p)\log A_2(\varepsilon) &\geq \frac{2\eta(p)\varepsilon}{N g_{\min}} \\ &= \frac{1}{N}\left[1 - \left(1 - \frac{2\eta(p)\varepsilon}{g_{\min}}\right)\right] \\ &\overset{i}{\geq} \frac{1}{N}\left[1 - \left(1 + \frac{\varepsilon}{g_{\min}}\right)^{-2\eta(p)}\right] ,\end{aligned} \tag{S9}$$

where (i) holds since $(1+x)^{-n} \geq 1 - nx$, for $x \in [0,1]$.

Case 1: For $p \in (1/2, 1]$, we have $\eta(p) = p$ and then substituting Eq. (S9) in Eq. (S8) gives:

$$\|x\|_p < \|y\|_p \left(1 + \frac{\varepsilon}{g_{\min}}\right)^{-2}, \text{ for } p \in (1/2, 1] .$$

Case 2: For $p \in [0, 1/2)$, we have $\eta(p) = 1 - p$ and again :

$$\log A_2(\varepsilon) \geq \frac{1}{N}\left[1 - \left(1 + \frac{\varepsilon}{g_{\min}}\right)^{-2(1-p)}\right].$$

Substituting Eq. (S9) in Eq. (S8) gives

$$\|x\|_p < \|y\|_p \left(1 + \frac{\varepsilon}{g_{\min}}\right)^{-2\left(\frac{1-p}{p}\right)} \quad \text{for } p \in [0, 1/2).$$

This completes the proof of the lemma. ☐

Lemma 15. *If* $\|x\|_p < \|y\|_p\left(1 + \frac{\varepsilon}{g_{\min}}\right)^{-2\gamma}$ *where* $\gamma := \max\left\{\left(\frac{1-p}{p}\right), 1\right\}$ *for any given* $\varepsilon > 0$*, then:*

$$H_p(x) < H_p(y) - 2\frac{\gamma\, p}{1-p}\log\left(1 + \frac{\varepsilon}{g_{\min}}\right).$$

Proof. From the definition of Renyi entropy (Eq. 3, 4, 5) we have:

$$\begin{aligned}H_p(x) &:= \frac{1}{1-p}\log\left(N \times \frac{\sum_{i=1}^{N} x_i^p}{N}\right) \\ &\overset{a}{<} \frac{\log N}{1-p} + \frac{p}{1-p}\log\|y\|_p - 2\frac{\gamma\, p}{1-p}\log\left(1 + \frac{\varepsilon}{g_{\min}}\right) \\ &= H_p(y) - 2\frac{\gamma\, p}{1-p}\log\left(1 + \frac{\varepsilon}{g_{\min}}\right)\end{aligned}$$

where (a) follows from the hypothesis of the lemma. The lemma thus implies:

$$H_p(x) < \begin{cases} H_p(y) - 2\log\left(1 + \frac{\varepsilon}{g_{\min}}\right) & \text{for } p \in [0, 1/2) \\ H_p(y) - \frac{2p}{1-p}\log\left(1 + \frac{\varepsilon}{g_{\min}}\right) & \text{for } p \in [1/2, 1) \end{cases}$$

This completes the proof of the lemma. ☐

Lemma 16. *For any given $\varepsilon > 0$, if $H_p(x) < H_p(y) - 2\frac{\gamma\, p}{1-p}\log\left(1+\frac{\varepsilon}{g_{\min}}\right)$, for $\gamma := \max\left\{\left(\frac{1-p}{p}\right),1\right\}$ then:*

$$D_p(\rho||\rho_{g_\varepsilon}) \geq D_p(\sigma||\rho_{g_\varepsilon}) + 2\frac{\gamma\, p}{1-p}\log\left(1+\frac{\varepsilon}{g_{\min}}\right).$$

Proof. The proof essentially follows from the relation between the relative entropy of a given state with maximally mixed state and the von Neumann entropy of a given state, very similar to that of Lemma 12 (with the factor $p-1$ being replaced by $1-p$ in the denominator of the second term on the right hand side of the desired inequality). □

Lemma 17. *For any given $\varepsilon > 0$, if $D_p(\rho||\rho_{g_\varepsilon}) \geq D_p(\sigma||\rho_{g_\varepsilon} + 2\frac{\gamma\, p}{1-p}\log\left(1+\frac{\varepsilon}{g_{\min}}\right)$ for $\gamma := \max\left\{\left(\frac{1-p}{p}\right),1\right\}$ then:*

$$D_p(\rho||\rho_g) > D_p(\sigma||\rho_g).$$

Proof. The essential idea is to use the continuity of $D_p(\rho||\rho_g)$ in the second argument, which follows from Proposition 6. We do the analysis under the following two cases:

Case 1: For $p \in [0,1/2)$ range, we have $\frac{\gamma\, p}{1-p} = 1$. This leads to:

$$\begin{aligned} D_p(\rho||\rho_g) &\overset{i}{>} D_p(\rho||\rho_{g_\varepsilon}) - \log\left(1+\frac{\varepsilon}{g_{\min}}\right) \\ &\overset{ii}{>} D_p(\sigma||\rho_{g_\varepsilon}) + 2\log\left(1+\frac{\varepsilon}{g_{\min}}\right) - \log\left(1+\frac{\varepsilon}{g_{\min}}\right) \\ &\overset{iii}{>} D_p(\sigma||\rho_g) - \log\left(1+\frac{\varepsilon}{g_{\min}}\right) + 2\log\left(1+\frac{\varepsilon}{g_{\min}}\right) - \log\left(1+\frac{\varepsilon}{g_{\min}}\right) \\ &= D_p(\sigma||\rho_g). \end{aligned}$$

Case 2: For the range $p \in [1/2,1)$, we have $\frac{\gamma\, p}{1-p} = \frac{p}{1-p}$. Thus:

$$\begin{aligned} D_p(\rho||\rho_g) &\overset{i}{>} D_p(\rho||\rho_{g_\varepsilon}) - \frac{p}{1-p}\log\left(1+\frac{\varepsilon}{g_{\min}}\right) \\ &\overset{ii}{>} D_p(\sigma||\rho_{g_\varepsilon}) + \frac{2p}{1-p}\log\left(1+\frac{\varepsilon}{g_{\min}}\right) - \frac{p}{1-p}\log\left(1+\frac{\varepsilon}{g_{\min}}\right) \\ &\overset{iii}{>} D_p(\sigma||\rho_g) - \frac{p}{1-p}\log\left(1+\frac{\varepsilon}{g_{\min}}\right) + \frac{2p}{1-p}\log\left(1+\frac{\varepsilon}{g_{\min}}\right) - \frac{p}{1-p}\log\left(1+\frac{\varepsilon}{g_{\min}}\right) \\ &= D_p(\sigma||\rho_g). \end{aligned}$$

where for both the above cases, (i) and (iii) follows from Proposition 6; (ii) follows from hypothesis of the lemma.

From analysis of both the cases above, we get that $D_p(\rho||\rho_g) > D_p(\sigma||\rho_g)$ which completes the proof of the lemma. □

For our purpose, it is thus sufficient to choose $\frac{1}{A_2(\varepsilon)} = 2^{-\frac{2\varepsilon}{N g_{\min}}}$ which guarantee trumping.

5. Characterization in terms of p Rényi divergence, for $p \in (-\bar{s}, 0)$:

In the regime $p \in (-\bar{s}, 0)$, we need to show

$$\frac{F_{k,1}\left(\frac{1}{\mathrm{x}^{\bar{s}}}\right)}{A_s(\varepsilon)} > F_{k,1}\left(\frac{1}{\mathrm{y}^{\bar{s}}}\right), \qquad 1 \leq k \leq N\ ,$$

implies trumping for some real valued positive function $A_s(\varepsilon)$. To demonstrate this, we need the following lemmas.

Lemma 18. *If $\frac{F_{k,1}\left(\frac{1}{x^{\bar{s}}}\right)}{A_s(\varepsilon)} > F_{k,1}\left(\frac{1}{y^{\bar{s}}}\right)$, for some real-valued positive function $A_s(\varepsilon)$ with $1 \leq k \leq N$ and $\varepsilon > 0$, then*

$$f_1\left(\frac{1}{y^{\bar{s}}}, t\right) < \frac{f_1\left(\frac{1}{x^{\bar{s}}}, t\right)}{A_s(\varepsilon)}.$$

Proof. By Definition 2 of $f_1(x,t)$ and $F_{k,1}(x)$ [2]:

$$\begin{aligned} f_1\left(\frac{1}{\mathrm{y}^{\bar{s}}},t\right) &= \sum_k F_{k,1}\left(\frac{1}{\mathrm{y}^{\bar{s}}}\right)t^k \\ &\overset{a}{<} \frac{1}{A_s(\varepsilon)}\sum_k F_{k,1}\left(\frac{1}{\mathrm{x}^{\bar{s}}}\right)t^k \\ &= \frac{f_1\left(\frac{1}{\mathrm{x}^{\bar{s}}},t\right)}{A_s(\varepsilon)}. \end{aligned}$$

where (a) follows from the hypothesis of the lemma. □

Lemma 19. *If* $f_1\left(\frac{1}{y^{\bar{s}}},t\right) < \frac{f_1\left(\frac{1}{x^{\bar{s}}},t\right)}{A_s(\varepsilon)}$*, then for any* $\varepsilon > 0$*,* $\xi \in (0,1)$ *and the choice of* $A_s(\varepsilon)$ *as*

$$A_s(\varepsilon) \geq 2^{\left\{\frac{\left(1+\frac{\varepsilon}{g_{\min}}\right)^{2(1+\bar{s})}-1}{N}\right\}}$$

the following holds:

$$\left\|\frac{1}{x^{\bar{s}}}\right\|_\xi > \left\|\frac{1}{y^{\bar{s}}}\right\|_\xi \left(1+\frac{\varepsilon}{g_{\min}}\right)^{2^{\frac{1+\xi\bar{s}}{\xi}}}.$$

Proof. From Fact 3, we have the following:

$$\begin{aligned} N\left\|\frac{1}{\mathrm{x}^{\bar{s}}}\right\|_\xi^\xi &= \frac{1}{I_r(\xi)}\int_0^\infty t^{-\xi-1}\log f_r\left(\frac{1}{\mathrm{x}^{\bar{s}}},t\right)dt \\ &\overset{a}{>} \frac{1}{I_r(\xi)}\int_0^\infty t^{-\xi-1}\log f_r\left(\frac{1}{\mathrm{y}^{\bar{s}}},t\right)dt + \log A_s(\varepsilon)\int_0^\infty \frac{t^{-\xi-1}}{I_r(\xi)}dt \\ &\overset{b}{>} N\left\|\frac{1}{\mathrm{y}^{\bar{s}}}\right\|_\xi^\xi + \log A_s(\varepsilon)\int_k^\infty \frac{t^{-\xi-1}}{I_r(\xi)}dt \\ &= N\left\|\frac{1}{\mathrm{y}^{\bar{s}}}\right\|_\xi^\xi \left\{1+N\log A_s(\varepsilon)\right\}. \end{aligned} \tag{S10}$$

Here (a) follows from the hypothesis of the lemma whereas in (b) we restrict the range of the definite integral to $\left[k = \left(\xi N^2 \left\|\frac{1}{\mathrm{y}^{\bar{s}}}\right\|_\xi^\xi I_r(\xi)\right)^{-1/\xi}, \infty\right]$ instead of considering the full domain $[0,\infty]$, ensuring that it evaluates to $N^2\left\|\frac{1}{\mathrm{y}^{\bar{s}}}\right\|_\xi^\xi$.

Now, the choice of $A_s(\varepsilon) \geq 2^{\left\{\frac{\left(1+\frac{\varepsilon}{g_{\min}}\right)^{2(1+\bar{s})}-1}{N}\right\}}$ implies:

$$1+N\log A_s(\varepsilon) \geq \left(1+\frac{\varepsilon}{g_{\min}}\right)^{2(1+\bar{s})} \overset{i}{>} \left(1+\frac{\varepsilon}{g_{\min}}\right)^{2(1+\xi\bar{s})} \tag{S11}$$

where (i) holds since $\xi \in (0,1)$. Substituting Eq. (S11) in Eq. (S10) and taking the ξ^{th} root both the sides, completes the proof. □

We now state a lemma that serves as a bridge between $\|x\|_{-p}$ and $\left\|\frac{1}{\mathrm{x}}\right\|_{k\bar{s}}$.

Lemma 20. $\|x\|_p < \|y\|_p\left(1+\frac{\varepsilon}{g_{\min}}\right)^{2^{\frac{1-p}{p}}}$ *for* $p \in (-\bar{s},0) \iff \left\|\frac{1}{x^{\bar{s}}}\right\|_\xi > \left\|\frac{1}{y^{\bar{s}}}\right\|_\xi\left(1+\frac{\varepsilon}{g_{\min}}\right)^{2^{\frac{1+\xi\bar{s}}{\xi}}}$ *for* $\xi \in (0,1)$.

Proof. The equivalence in the lemma can be shown by the following analysis for any $\xi \in (0,1)$:

$$\left\|\frac{1}{\mathrm{x}^{\bar{s}}}\right\|_{\xi} > \left\|\frac{1}{\mathrm{y}^{\bar{s}}}\right\|_{\xi}\left(1+\frac{\varepsilon}{g_{\min}}\right)^{2\frac{1+\xi\bar{s}}{\xi}} \iff \left\|\frac{1}{\mathrm{x}}\right\|_{\xi\bar{s}} > \left\|\frac{1}{\mathrm{y}}\right\|_{\xi\bar{s}}\left(1+\frac{\varepsilon}{g_{\min}}\right)^{2\frac{1+\xi\bar{s}}{\xi\bar{s}}} \tag{S12}$$

$$\iff \frac{1}{\|\mathrm{x}\|_{-\xi\bar{s}}} > \frac{1}{\|\mathrm{y}\|_{-\xi\bar{s}}}\left(1+\frac{\varepsilon}{g_{\min}}\right)^{2\frac{1+\xi\bar{s}}{\xi\bar{s}}}$$

$$\iff \|\mathrm{x}\|_p < \|\mathrm{y}\|_p\left(1+\frac{\varepsilon}{g_{\min}}\right)^{-2\frac{1+|p|}{|p|}}, \text{ where } p := -\xi\bar{s} \text{ and hence} p \in (-\bar{s}, 0) \tag{S13}$$

$$\iff \|\mathrm{x}\|_p < \|\mathrm{y}\|_p\left(1+\frac{\varepsilon}{g_{\min}}\right)^{2\frac{1-p}{p}}, \quad \text{for all } p \in (-\bar{s}, 0)\ .$$

□

Lemma 21. *If* $\|x\|_p < \|y\|_p\left(1+\frac{\varepsilon}{g_{\min}}\right)^{2\frac{1-p}{p}}$ *for any given* $\varepsilon > 0$*, then:*

$$H_p(x) < H_p(y) - 2\log\left(1+\frac{\varepsilon}{g_{\min}}\right).$$

Proof. From the definition of Rényi p-entropy (Eq. 4) and the notation $p = -|p|$ (since p is negative) we have:

$$\begin{aligned}
H_p(x) &= -\frac{1}{1-p}\log\left(N \times \frac{\sum_{i=1}^{N} x_i^p}{N}\right) \\
&= -\frac{\log N}{1+|p|} + \frac{|p|}{1+|p|}\log\|x\|_p \\
&\overset{a}{<} -\frac{\log N}{1+|p|} + \frac{|p|}{1+|p|}\log\|y\|_p + \frac{|p|}{1+|p|} \times \frac{2(1-p)}{p}\log\left(1+\frac{\varepsilon}{g_{\min}}\right) \\
&= -\frac{\log N}{1+|p|} + \frac{|p|}{1+|p|}\log\|y\|_p - \frac{|p|}{1+|p|} \times \frac{2(1+|p|)}{|p|}\log\left(1+\frac{\varepsilon}{g_{\min}}\right) \\
&= H_p(y) - 2\log\left(1+\frac{\varepsilon}{g_{\min}}\right),
\end{aligned}$$

where (a) follows from the hypothesis of the lemma. □

Lemma 22. *For any given* $\varepsilon > 0$*, if* $H_p(x) < H_p(y) - 2\log\left(1+\frac{\varepsilon}{g_{\min}}\right)$ *then:*

$$D_p(\rho||\rho_{g_\varepsilon}) \ \geq\ D_p(\sigma||\rho_{g_\varepsilon}) + 2\log\left(1+\frac{\varepsilon}{g_{\min}}\right). \tag{S14}$$

Proof. The proof essentially follows from the relation between the relative entropy of a given distribution with uniform distribution and the Shannon entropy of a given state, that is,

$$\begin{aligned}
D_p(\rho||\rho_{g_\varepsilon}) &= D_p(x||u_N) = -\log N - H_p(x) \\
&\overset{a}{>} -\log N - H_p(y) + 2\log\left(1+\frac{\varepsilon}{g_{\min}}\right) \\
&= D_p(y||u_N) + 2\log\left(1+\frac{\varepsilon}{g_{\min}}\right) \\
&= D_p(\sigma||\rho_{g_\varepsilon}) + 2p\log\left(1+\frac{\varepsilon}{g_{\min}}\right)\ .
\end{aligned}$$

Here (a) follows from the hypothesis of the lemma. □

Lemma 23. *For any given $\varepsilon > 0$, if $D_p(\rho||\rho_{g_\varepsilon}) \geq D_p(\sigma||\rho_{g_\varepsilon}) + 2\log\left(1+\frac{\varepsilon}{g_{\min}}\right)$ then:*

$$D_p(\rho||\rho_g) > D_p(\sigma||\rho_g).$$

Proof. The proof follows from the following analysis:

$$\begin{aligned}
D_p(\rho||\rho_g) &\overset{a}{>} D_p(\rho||\rho_{g_\varepsilon}) - \log\left(1+\frac{\varepsilon}{g_{\min}}\right) \\
&\overset{b}{>} D_p(\sigma||\rho_{g_\varepsilon}) + 2\log\left(1+\frac{\varepsilon}{g_{\min}}\right) - \log\left(1+\frac{\varepsilon}{g_{\min}}\right) \\
&\overset{c}{>} D_p(\sigma||\rho_g) - \log\left(1+\frac{\varepsilon}{g_{\min}}\right) + 2\log\left(1+\frac{\varepsilon}{g_{\min}}\right) - \log\left(1+\frac{\varepsilon}{g_{\min}}\right) \\
&= D_p(\sigma||\rho_g).
\end{aligned}$$

where (a) and (c) follow from Eq. (S4) of Proposition 6, (b) follows from the hypothesis of the Lemma and (c) follows from Proposition 6. Above expression implies $D_p(\rho||\rho_g) > D_p(\sigma||\rho_g)$. □

6. Characterization in terms of p Rényi divergence, for $p \in (\bar{r}, \infty)$ and $p \in (-\infty, -\bar{s}]$:

Following Lemma explores the ranges $p \in (\bar{r}, \infty)$ and $p \in (-\infty, -\bar{s}]$ to guarantee the conditions for trumping.

Lemma 24. *For a fixed $\varepsilon > 0$, let $N > 1$, $x, y \in \mathbb{R}_+^N$, $x_1 > y_1\left(1+\frac{\varepsilon}{g_{\min}}\right)^2 > 0$ and $y_{\min} > x_{\min}\left(1+\frac{\varepsilon}{g_{\min}}\right)^2 > 0$. Let r and s be chosen as:*

$$r = \frac{\log N}{\log x_1 - \log y_1\left(1+\frac{\varepsilon}{g_{\min}}\right)^2} \quad \text{and} \quad s = \frac{\log N}{\log y_{\min} - \log\left\{x_{\min}\left(1+\frac{\varepsilon}{g_{\min}}\right)^2\right\}}.$$

Then, $D_p(\rho||\rho_g) > D_p(\sigma||\rho_g)$ for $\forall p \geq \bar{r}$ and $\forall p < -\bar{s}$.

Proof. Using the choice of r and following the steps outlined in Lemma 11 we get:

$$\|\mathrm{x}\|_p \geq \|\mathrm{y}\|_p\left(1+\frac{\varepsilon}{g_{\min}}\right)^2 \quad \text{for } p \geq \bar{r}\,.$$

Now using the following relation between the p-Rényi divergence and the our scaled ℓ_p norm we get:

$$D_p(\mathrm{x}||u_N) = \log N + \frac{p}{p-1}\log\|\mathrm{x}\|_p \quad \Rightarrow \quad D_p(\mathrm{x}||u_N) > D_p(\mathrm{y}||u_N) + \frac{2p}{p-1}\log\left(1+\frac{\varepsilon}{g_{\min}}\right). \tag{S15}$$

The following analysis leads us to the proof of the lemma for $p > \bar{r}$:

$$\begin{aligned}
D_p(\rho||\rho_g) &= D_p(\mathrm{x}||g) \\
&\overset{a}{\geq} D_p(\mathrm{x}||g_\varepsilon) - \log\left(1+\frac{\varepsilon}{g_{\min}}\right) \\
&\overset{b}{>} D_p(\mathrm{y}||g_\varepsilon) + \frac{p}{p-1}\log\left(1+\frac{\varepsilon}{g_{\min}}\right) \\
&\overset{c}{>} D_p(\mathrm{y}||g) = D_p(\sigma||\rho_g)\,.
\end{aligned}$$

where (a) and (c) follows from the continuity of p-Rényi divergence mentioned in Eq. (S3) of Proposition 6; (b) follows from Eq. (S15).

In order to prove the lemma for $p < -\bar{s}$, we make the following observations (for $p < 0$):

$$\left\|\frac{1}{\mathrm{x}}\right\|_p = \frac{1}{\|\mathrm{x}\|_{-p}}; \quad H_p(x) = \frac{|p|}{1+|p|}\log\|\mathrm{x}\|_{-|p|} \quad \Rightarrow \quad D_p(\mathrm{x}||u_N) = -\log N - H_p(x)\,.$$

From Corollary 12 and the choice of s, we get:

$$\begin{aligned}
&\left\|\frac{1}{\mathrm{x}}\right\|_{-|p|} > \left\|\frac{1}{\mathrm{y}}\right\|_{-|p|}\left(1+\frac{\varepsilon}{g_{\min}}\right)^2 \\
&\Rightarrow \frac{1}{\|\mathrm{x}\|_{-|p|}} > \frac{1}{\|\mathrm{y}\|_{-|p|}}\left(1+\frac{\varepsilon}{g_{\min}}\right)^2 \\
&\Rightarrow \|\mathrm{x}\|_{-|p|} < \|\mathrm{y}\|_{-|p|}\left(1+\frac{\varepsilon}{g_{\min}}\right)^{-2} \\
&\Rightarrow H_p(\mathrm{x}) < H_p(\mathrm{y}) - \frac{2|p|}{|p|+1}\left(1+\frac{\varepsilon}{g_{\min}}\right) \\
&\Rightarrow D_p(\mathrm{x}||u_N) > D_p(\mathrm{y}||u_N) + \frac{2|p|}{|p|+1}\left(1+\frac{\varepsilon}{g_{\min}}\right)
\end{aligned}$$

Now doing similar analysis for the case $p > \overline{r}$, gives:

$$D_p(\rho||\rho_g) > D_p(\sigma||\rho_g) \text{ for } p \in (-\infty, -\bar{s}]\ .$$

This completes the proof the lemma. □

SUPPLEMENTARY NOTE 2: DISCUSSION ON TRUMPING CONDITIONS FOR GENERAL STATES HAVING COHERENCE

At first, it may be noted that no necessary and sufficient set of conditions are currently known for catalytic state transformations involving states having coherence under the corresponding free operations, rather only necessary conditions have been established [3]. Here, we outline a method to derive a finite set of necessary conditions for catalytic majorization in quantum states having coherence.

Before discussing the catalytic majorization conditions for quantum states having coherence, let us first explore the formal aspects of the concept known as *Symmetry* [3, 4]. This concept has significant applications in physics, and recent insights indicate that any deviation from *Symmetry*, termed as *asymmetry* serves as a resource for information processing tasks [4].

Symmetry group and symmetry operation: For a symmetry group G, the symmetry transformation of a density matrix ρ corresponding to the group element g can be represented as follows:

$$g \in G \implies \rho \rightarrow \mathcal{U}_g(\rho) = U_g(\rho)U_g^\dagger$$

where U_g is unitary.

An evolution $\mathcal{M}$ is said to be symmetric with respect to G if it commutes with the action of the symmetry group i.e., if $[\mathcal{M}, G] = 0$ for all $g \in G$. This implies that the order in which the symmetry transformation and the dynamics occur does not affect the final state. In other words, for every density matrix ρ and each element g in the group G, the Equation $\mathcal{M}[\mathcal{U}_g(\rho)] = \mathcal{U}_g[\mathcal{M}(\rho)]$ holds. Similarly, a state is called symmetric if it is invariant under symmetry transformations; otherwise, it is called asymmetric.

Indeed, a resource theory can be formulated where the set of free operations comprises those demonstrating symmetry with respect to the group G. Specifically, this theory pertains to quantum coherence between eigenspaces of the observables generating G. If the generator is the Hamiltonian H_s, the corresponding channel $\mathcal{M}$ is time-translation symmetric.

Definition 25. *For any $p \geq 0$, the free coherence of a state ρ with respect to the Hamiltonian H is*

$$A_p(\rho) := D_p(\rho||\mathcal{N}_H(\rho))$$

where $\mathcal{N}_H$ is the operation that removes all coherence between energy eigenspaces and D_p is the Réyni divergence as defined earlier.

Similar to how free energies determine the extent to which a state deviates from being thermal, free coherence serves as a measure of how far a state strays from being incoherent in energy. In order to check state transformation between two states in presence of a catalyst, the free energy relations are no longer sufficient rather it is only necessary to go beyond free energy relations to capture the role of quantum coherence in thermodynamical state transformation.

Let us now discuss the conditions for the transformation of states having coherence. For that we borrow and state the theorems from [3, 5].

Fact 26. *[3, Theorem 1] The set of Thermal operations on a quantum state is a strict subset of the set of symmetric quantum operations with respect to the time-translation.*

Fact 27. *[3, Theorem 2] For all $p \geq 0$, we necessarily have $\Delta A_p \leq 0$ for any thermal operation.*

It is easy to argue that the above fact is true in general for any thermal operation. The Rényi divergence follows the relation $D_p(\mathcal{M}(\rho)||\mathcal{M}(\sigma)) \leq D_p(\rho||\sigma)$ for all p. Since from Fact 26, $[\mathcal{M}, \mathcal{N}_H] = 0$, we derive the condition that the above fact is indeed true.

A state ρ in $\mathcal{H}$ can be transformed into a state σ (i.e., $\rho \to \sigma$) through a catalytic thermal operation, if there is another quantum state ρ_c in Hilbert space $\mathcal{H}_c$ with Hamiltonian H_c and a thermal operation $\mathcal{M}$ on $\mathcal{H} \otimes \mathcal{H}_c$ i.e.,

$$\mathcal{M}(\rho \otimes \rho_c) = \sigma \otimes \rho_c.$$

Fact 28. *[3, Theorem 4] Catalytic thermal operations with a block-diagonal catalyst are symmetric operations, i.e., if H is the system's Hamiltonian and $\mathcal{C}$ is a catalytic thermal operation then*

$$\mathcal{C}(e^{-iHt}\rho e^{iHt}) = e^{-iHt}\mathcal{C}(\rho)e^{iHt}.$$

Fact 29. *[3, Theorem 5] If $[\rho_c, H_c] = 0$, for $\rho \otimes \rho_c \to \sigma \otimes \rho_c$, we necessarily have*

$$A_p(\sigma) \leq A_p(\rho) \quad \forall p \geq 0.$$

This implies the necessary conditions for catalytically transforming states having coherence, involve comparing the generalized coherence-free energies (of order p) of ρ and σ across an infinite range of p values.

One can now get a finite number of necessary conditions for catalytic majorization by replacing ρ_g with a state $\rho_f =: \mathcal{N}_H(\rho)$ that removes all the off diagonal elements of the state into the discussion of Corollary 7 and Theorem 8.

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