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A finite sufficient set of conditions for catalytic majorization

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Understanding when one physical state can be transformed into another is a central problem in quantum information science and thermodynamics. Majorization provides a mathematical tool for describing such transformations. Yet many transitions that are forbidden by majorization can become possible in the presence of a catalyst, an auxiliary system that enables the process without being consumed or altered. Determining the feasibility of such catalytic transformation typically involves checking an infinite set of inequalities involving generalized entropic quantities. Here, we derive a finite sufficient set of inequalities that imply catalysis. Extending this framework to thermodynamics, we also establish a finite set of sufficient conditions for catalytic state transformations under thermal operations. For further examples, we provide a software toolbox implementing these conditions. Our results rely on the connection between a polynomial representation of ℓ_p norm with Rényi p entropies for any real value of p.

Irreversibility is a phenomenon in the realm of physics that serves as a key indicator for transitions from orderly to disorderly states of the systems. It is typically described by the laws of thermodynamics, yet its relevance extends far beyond thermodynamics, influencing diverse fields across physics and mathematics. Intuitively, a process is considered reversible when no dissipation occurs, allowing the system to remain in equilibrium throughout its evolution. Accurately characterizing irreversibility is thus paramount, with applications in areas such as quantum state transformations, entanglement dynamics, characterization of thermal machines and heat engines.

There is a broader notion of irreversibility which extends beyond thermodynamic irreversibility, treating it as a special case. Central to this approach is the concept of *majorization*, which provides a powerful tool for distinguishing between reversible and irreversible processes.

Majorization and its implications have had a significant impact on several scientific disciplines. In mathematics, it is widely used in matrix theory and inequalities, including eigenvalue relations, doubly stochastic matrices, and the Schur-Horn theorem[1]. In machine learning, it has been applied to optimization and incremental learning algorithms[2,3] while in economics, it appears in the study of inequality measures and decision theory[1]. More recently, majorization has played a crucial role in quantum information theory[4–8], quantum thermodynamics[9–11], and more generally on resource theories[12]. A review of early applications can be found in Marshall et al.[1]. In the latter cases, it gives insights into the entanglement structure of quantum states, thermodynamic resources and provides an elegant operational criterion which determines when it is possible to convert one quantum state into another.

Nevertheless, majorization alone falls short of offering a comprehensive characterization of state transformations. There are instances where transforming one quantum state into another is not possible within a specific class of operations. Yet, the introduction of an ancillary state can sometimes facilitate this transformation under the same operations, with the ancillary state ultimately returning to its original and exact form[5]. This additional system that alleviates restrictions on apparently prohibited transitions serves as a catalyst. Comprehensive reviews of different aspects of catalysis in quantum information theory can be found in[13,14].

To illustrate further, let us analyze this from the perspective of thermodynamics. In the macroscopic domain of thermodynamics, a system existing in state ρ can undergo a transition to state σ if there is a decrease in free energy, where the free energy of a state ρ is defined as $\mathcal{F}(\rho) := \langle E(\rho)\rangle - KTS(\rho)$. Here $\langle E(\rho)\rangle$ is the average energy, K is the Boltzmann constant, and $S(\rho)$ is the entropy of the state ρ, with T being the temperature of the surrounding heat bath. Unfortunately, in the realm of microscopic, quantum, or highly correlated systems, a reduction in free energy alone does not suffice as a condition for state transformation. Instead, state transformation is governed by the concept of *thermo-majorization*[9,15]. Nevertheless, even thermo-majorization proves insufficient for characterizing state transformation rules comprehensively. There exist instances where two states ρ and σ do not satisfy thermo-majorization, yet the

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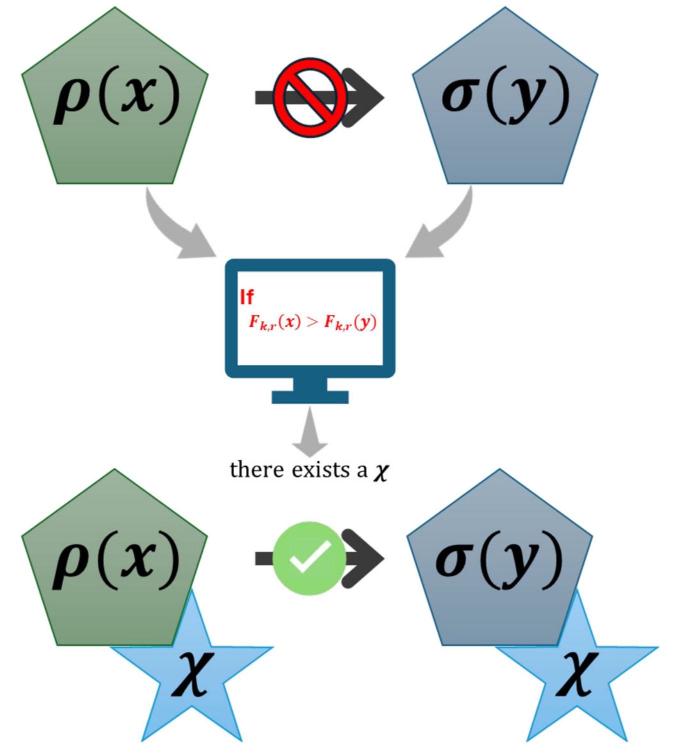

Fig. 1 | Schematic of catalytic majorization and finite sufficient conditions for catalysis. A conceptual illustration of how a finite set of inequalities can certify the existence of a catalyst enabling otherwise forbidden state transformations. Green pentagons represent input states, grey pentagons represent target states, blue stars denote catalysts, a red crossed circle indicates an impossible transformation, and a green checkmark indicates an allowed transformation. A transformation from an initial quantum state ρ (green pentagon) to a target state σ (grey pentagon) is not possible under the chosen free operations, indicated by the red crossed circle between the two states. The eigenvalue vectors of ρ and σ are denoted by x and y, respectively. A finite collection of inequalities derived in this work is evaluated (blue rectangle). When these conditions are satisfied, there exists a catalyst χ (blue star) such that the joint transformation $\rho \otimes \chi \to \sigma \otimes \chi$ becomes possible, indicated by the green checkmark. The catalyst χ is returned unchanged after the transformation, illustrating its catalytic nature.

transition from ρ to σ becomes feasible in the presence of an auxiliary system χ, known as a catalyst, which remains unaltered before and after the transformation[15].

Therefore, when assessing the possibility of transforming ρ into σ, consideration must be given to the existence of another working body or other ancillary systems χ such that the composite system $\rho \otimes \chi$ can be transformed into $\sigma \otimes \chi$. Thus, majorization should be exclusively applied to the total resources, encompassing all potential catalysts and working bodies of arbitrary dimensions, rather than to the system of interest itself. Criteria that investigate transitions between two states in the presence of catalysis are commonly termed as *trumping*. It has been demonstrated that merely decreasing free energy is insufficient to fully encapsulate state transformation rules under thermal operations[9]. Instead (for states diagonal in the energy eigenbasis), the necessary and sufficient conditions for thermodynamic transitions are determined by a family of second laws, which require verifying the decrease of the generalized free energy, i.e., $\mathcal{F}_p(\rho, H) \geq \mathcal{F}_p(\sigma, H)$ for $p \in (-\infty, \infty)$.

Verifying the existence of catalytic majorization or trumping relationships requires evaluating an infinite number of conditions across the continuous spectrum of p values. Despite having well-defined mathematical expressions for these catalytic transformation conditions, checking an unbounded number of inequalities is computationally infeasible. A fundamental question is whether these infinite conditions can be reduced to a finite, tractable set of conditions that guarantee catalytic majorization.

Prior research has approached this problem from multiple angles. For probability distributions obtained by measuring pure states in a fixed basis or the squared Schmidt coefficients of a pure bipartite state and the vector composed of eigen values of a $d \times d$ mixed states where $d \geq 4$, Gour[16] established that a finite number of conditions cannot be sufficient to determine transformation feasibility. Existing characterizations of necessary and sufficient conditions, such as those developed by Klimesh[17], Turgut[18] and Aubrun and Nechita[19], all require infinite checks and lack computational tractability. These limitations have directed research efforts toward approximate catalytic transformations[20,21], with relatively little progress on exact catalytic transformations. A detailed survey on approximate catalysis and approximate trumping can be found in[21], although these topics are not directly relevant here.

In this work, we derive a finite set of sufficient conditions to determine whether state transformation is possible under a given free operation (as illustrated in Fig. 1). We first derive this finite set of conditions which are sufficient for trumping two state vectors $|x\rangle_{AB}$ and $|y\rangle_{AB}$ shared between two spatially separated parties under local operation and classical communication (LOCC). Further, we also extend our analysis to the second law of thermodynamics and again explicitly derive a finite set of sufficient conditions for trumping states that are block diagonal in the energy eigenbasis via thermal operations. Our analysis also applies to a broader class of general states that are not necessarily block-diagonal in the energy eigenbasis. Finally, we introduce a simulation toolbox to facilitate the practical application of our trumping criteria.

Results

Background: Majorization and Catalytic majorization

We start with the resource theory of entanglement, in which the free operations are local operations and classical communication (LOCC), the free states are separable (unentangled) states, and the resourceful states are entangled ones. A central question in this framework is how to transfer one bipartite state vector to other shared between two distant parties, via LOCC. We first review the conditions governing such LOCC transformations and subsequently examine how these conditions are modified in the presence of catalytic assistance.

Consider a vector in the Hilbert space $|y\rangle_{AB} \in \mathcal{H}_{AB} (= \mathcal{H}_A \otimes \mathcal{H}_B)$ representing some target state that two spatially separated parties, Alice and Bob, wish to share by acting on the available state $|x\rangle_{AB} \in \mathcal{H}_{AB}$ using LOCC[22]. Any such bipartite quantum state in the tensor product space $\mathcal{H}_{AB}$ can be written in the Schmidt form[23]:

$$|x\rangle_{AB} = \sum_{i=1}^{n} \sqrt{x_i} |u_i\rangle_A \otimes |v_i\rangle_B \tag{1}$$

where $\{|u_i\rangle_A\}_i \in \mathcal{H}_A$ and $\{|v\rangle_{iB}\}_i \in \mathcal{H}_B$ are two sets of orthonormal bases and $|\mathcal{H}_{AB}| = |\mathcal{H}_A||\mathcal{H}_B| = n$. The vector $\mathrm{x} = (x_1, \ldots, x_n) \in P_n$, with $P_n = \{(x_1, \ldots, x_n) : x_1 \geq x_2 \geq \ldots x_n \geq 0, \sum_i x = 1\}$, is called the vector of Schmidt coefficients. With a slight abuse of notation for brevity, we will sometimes denote the state by its vector of Schmidt coefficients.

Here we restrict our attention to deterministic transformations. In this case, it was shown by Nielsen[4] that given $\mathrm{x}, \mathrm{y} \in P_n$, $\mathbf{x}$ can be transformed into $\mathbf{y}$ if and only if for all $i \in \{1, \ldots, n\}$

$$\sum_{j=1}^{i} x_j \leq \sum_{j=1}^{i} y_j. \tag{2}$$

This relation is known as majorization. Here we say that y majorizes x or equivalently that x is majorized by y and we compactly denote it by $\mathrm{x} \prec \mathrm{y}$. Majorization is a partial order, i.e. there exist vectors x, y for which neither $\mathrm{x} \prec \mathrm{y}$ nor $\mathrm{y} \prec \mathrm{x}$. We indicate that x is not majorized by y by $\mathrm{x} \nprec \mathrm{y}$.

In some cases, when $x \nprec y$, it is nevertheless possible to convert $|x\rangle_{AB}$ to $|y\rangle_{AB}$: surprisingly, there may exist a special vector that acts as a catalyst, i.e. there exists a set of triples of states with (x, y, z) such that y does not majorize x, but $y \otimes z$ majorizes $x \otimes z$. In other words, by using an auxiliary state $|z\rangle_{AB}$ with vector z one enables the deterministic transformation, and at the end of the latter $|z\rangle_{AB}$ is recovered unscathed. This relation is called catalytic majorization or trumping[5] and henceforth we shall denote it by $x \prec_T y$.

We define by $T(y) = \{x \in P_{<\infty}: x \prec_T y\}$ the set of vectors that are trumped by y, where $P_{<\infty} = \cup_{n=1}^{\infty} P_n$. Note that if dim(x) ≠ dim(y), the definition of trumping and majorization can be extended to probability vectors where the shorter vector is padded with zeros.

The first comprehensive set of conditions characterizing the trumping relation was derived in 2007 independently by Klimesh and Turgut[17,18] (a characterization of a generalized concept of majorization is given in[24]). Although the catalyst may be of arbitrary dimension, these conditions are equivalent to all the following strict inequalities being true[24] (given that x ≠ y and at least one of x and y has no zero components): ero components):

$$x \in T(y) \iff \begin{cases} H_p(x) > H_p(y), & \text{for } p \in \mathbb{R}/\{0\} \\ H_{\text{burg}}(x) > H_{\text{burg}}(y) \end{cases} \tag{3}$$

where for $p \neq \{0, 1\}$:

$$H_p(x) := \frac{\text{sign}(p)}{1-p} \log \sum_{i=1}^{n} (x_i)^p, \tag{4}$$

for $p = 1$:

$$H_1(x) := -\sum_{i=1}^{n} x_i \log x_i, \tag{5}$$

and

$$H_{\text{burg}}(x) := \frac{1}{n} \sum_{i=1}^{n} \log x_i . \tag{6}$$

Note that both H_p for $p < 0$ and H_{burg} evaluate to $-\infty$ if some of the elements of x are zero.

The closure of $\overline{T(y)}$ in the l_1-norm was introduced in[19] as a proxy for the study of $T(y)$ (with related developments in[25]). It turns out that by including the limit points of $T(y)$, one can simplify the set of conditions for trumping from Eq. (3)

$$x \in \overline{T(y)} \iff H_p(x) > H_p(y), p \in (1, \infty). \tag{7}$$

$\overline{T(y)}$ can be interpreted as the set of vectors that are approximately trumped by y. In the following, we rephrase the characterization of $\overline{T(y)}$ and $T(y)$ in a more convenient form for the purpose of this paper. Let $x \in \mathbb{R}_+^n$ and $p \in \mathbb{R}$, with slight abuse of notation, we define

$$\| x\|_p := \left(\frac{1}{n} \sum_{i=1}^{n} (x_i)^p \right)^{1/p}. \tag{8}$$

Note that we retain the standard Schatten p-norm notation $\| \cdot \|_p$, although it differs by a factor of $n^{-1/p}$, from the conventional definition for $p \geq 1$. This scaling is adopted purely for mathematical convenience and to simplify subsequent calculations. We call the weight of a vector the number of non-zero entries. We set $\| x\|_0 = \left(\prod_{i=1}^{n} x_i\right)^{1/n}$ and for $p < 0$ and x with weight smaller than n, $\|x\|_p = 0$ such that $\| \cdot \|_p$ is continuous. With this notation we can replace Eq. (3) and Eq. (7) with:

$$x \in T(y) \iff \begin{cases} \| x\|_p < \| y\|_p, & p \in (1, \infty) \\ \| x\|_p > \| y\|_p, & p \in (-\infty, 1) \\ H_1(x) > H_1(y), \end{cases} \tag{9}$$

and

$$x \in \overline{T(y)} \iff \{ \| x\|_p < \| y\|_p, \quad p \in (1, \infty). \tag{10}$$

Despite Eq. (9) (equivalently (3)) and Eq. (10) (equivalently (7)) respectively form a complete set of the necessary and sufficient conditions for trumping and approximate trumping, the number of conditions to check remains infinite. Here, we aim to outline our primary mathematical findings that lead to a finite set of conditions, which guarantee the trumping of bipartite state vectors when the free operation is LOCC.

For the concrete case of four-dimensional vectors and two-dimensional catalysts, necessary and sufficient conditions are known[26,27]. The general problem of characterizing the existence of a k dimensional catalyst for n dimensional vectors has been answered with algorithms that have running time polynomial in n and k[26,28]. However, the algorithm only provides an answer for a given value of k. Some general conditions necessary for trumping were investigated in[29]. Here, we are interested in a different approach to the problem of catalytic majorization. Instead of bounding the dimension of the catalyst, we want to find a discrete set of inequalities that implies the sets in Eqs. (3) and (7).

Partial progress in this direction was made by Klemes[30] by considering a family of symmetric polynomials $F_{k,r}: \mathbb{R}^n \mapsto \mathbb{R}$ indexed by $k, r \in \mathbb{N}$:

$$F_{k,r}(x) = \sum_{\substack{(k_1, \ldots, k_n) \text{ s.t.} \\ \sum_i k_i = k, \max k_i \leq r}} \prod_{i=1}^{n} \frac{(x_i)^{k_i}}{k_i!}. \tag{11}$$

Intuitively, this can be thought of as a polynomial generated from a truncated exponential function. See Supplementary Note 1 for a formal definition. Using this polynomial[30] showed that there is a *finite* set of inequalities that in turn imply the inequality of p norms for a continuous range of values of p. We state it as the following fact:

Fact 1. (Klemes) Let $x, y \in \mathbb{R}_+^n$ and fix $r \in \mathbb{N}$. If

$$F_{k,r}(x) \geq F_{k,r}(y), \; k \in \{r, r+1, \ldots, nr\}$$

Then

$$\| x\|_p \geq \| y\|_p, \; p \in (0, 1)$$

and further if $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ then

$$\| x\|_p \leq \| y\|_p \, p \in (1, r+1).$$

Note, while the condition $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ may seem restrictive, it is trivially satisfied when considering physical systems where x_i, y_i indicate the probability of finding the requisite system in state i.

Remark 2. Henceforth, the vectors $x, y \in P_n$ (equivalently the vector of squared Schmidt coefficients or of the eigen values of the states under consideration) admit the relation $x \neq y$, unless mentioned otherwise. For fixed k, r we note that $F_{k,r}(x)$ is so-called Schur concave in x. This property is used to construct a so-called resource monotone (e.g. relative entropy) in resource theory[15,31] which can quantify the conditions necessary for resource interconversion. However, here our goal is not to construct such a monotone for quantifying trumping but to reduce the infinite set of conditions (which in turn are quantified by a monotone, which the generalized Rényi p-entropy) to finite and hence we don't discuss this aspect in this work any further.

Preliminary definitions and earlier results

We define the generalized Rényi divergence and generalized free energy.

Definition 3. The generalized Rényi divergence of order p (for all $p \in \mathbb{R}$) between any two states ρ and σ, such that $supp(\rho) \subseteq supp(\sigma)$, is defined as:

$$D_p(\rho||\sigma) := \frac{\text{sign}(p)}{p-1} \log \text{Tr}\left[\rho^p \sigma^{1-p}\right]$$

where for the limiting cases

- $p = 1$: $D_1(\rho||\sigma) := \text{Tr}\left[\rho \log \rho - \rho \log \sigma\right]$ is the quantum Kullback-Liebler divergence also known as the von Neumann relative entropy; and
- $p = \infty$: $D_\infty(\rho||\sigma) := D_{\max}(\rho||\sigma) := \inf\{\lambda : \rho \leq 2^\lambda \sigma\}$ is known as the max-Rényi divergence.

Another quantity that will be extensively used in our subsequent analysis is the generalized free energy, defined as follows:

Definition 4. [9] The generalized free energy of order p of a system with Hamiltonian H of a state ρ, is defined as

$$\begin{aligned} \mathcal{F}_p(\rho, H) : &= KT[D_p(\rho||\rho_g) - \log Z] \\ &= \mathcal{F}(\rho_g, H) + KTD_p(\rho||\rho_g) \end{aligned} \quad (12)$$

where $\rho_g := \frac{e^{-\beta H}}{\text{Tr}(e^{-\beta H})}$ is known as the thermal state associated with the system Hamiltonian H and $\beta = 1/KT$ is the inverse temperature of the surrounding bath.

Let us now briefly introduce the resource theory of thermodynamics that provides an operational framework to study state transformations under thermodynamic constraints. In this framework, the allowed operations are energy-conserving unitaries, the free states are thermal (Gibbs) states at a fixed temperature, and any deviation from thermal equilibrium constitutes a resourceful state—that is, a state not in thermal equilibrium with respect to its Hamiltonian. Similar to the resource theory of entanglement, in the absence of any auxiliary system, the criteria for state transformations are fully characterized by thermo-majorization. However, these criteria are insufficient to describe state transformation laws in the presence of catalysis. Based on Definitions 3 and 4, it was shown in[9], Theorem 18 that the necessary and sufficient conditions for catalytic thermo-majorization are obtained by comparing the generalized free energies of order p of ρ and σ for infinite values of p. In particular, if

$$\mathcal{F}_p(\rho, H) > \mathcal{F}_p(\sigma, H) \text{ for all } p \in (-\infty, \infty), \quad (13)$$

catalytic thermo-majorization is possible. Note that this is equivalent (from Definition 4) to checking $D_p(\rho||\rho_g) > D_p(\sigma||\rho_g)$ for all $p \in (-\infty, \infty)$. We also emphasize that in the thermodynamic setting, the direction of majorization is reversed compared to the entanglement-based condition in (2), which is defined in terms of the eigenvalues of the reduced states.

We now state an earlier result for trumping the states that are not simultaneously diagonal in the energy eigenbasis of the system's Hamiltonian. These states are referred to as the states having coherence (e.g. [32]). Bu et al.[33] derived the necessary and sufficient conditions for catalytic transformations of pure states under incoherent operations with pure catalysts. More succinctly, for any two incomparable pure states $|\psi\rangle, |\phi\rangle \in \mathcal{H}_d$, the necessary and sufficient condition for the existence of a catalyst $|\chi\rangle$ such that $|\psi\rangle \otimes |\chi\rangle \to |\phi\rangle \otimes |\chi\rangle$ under incoherent operations is

$$\begin{aligned} H_p(\psi) > H_p(\phi) \ \forall p \in (-\infty, \infty)/0 \\ \text{and}, \ \ \text{Tr}(\log[\psi]) > \text{Tr}(\log[\phi]) \end{aligned} \quad (14)$$

where H_p is the Renyi entropy defined earlier and ψ (respectively ϕ) represent the states after removing all off-diagonal coefficients in the relevant basis[13], i.e. $\psi = \sum_i \langle i|\psi\rangle\langle\psi|i\rangle |i\rangle\langle i|$ (resp. $\phi = \sum_i \langle i|\phi\rangle\langle\phi|i\rangle |i\rangle\langle i|$).

Remark 5. It is known that one can uniquely identify a vector from a finite number of Rényi entropies[34], Theorem 42, but to the best of our knowledge there is no algorithm that builds on this information to conclude that trumping is possible and hence one still needs additional tools.

A finite sufficient set of conditions for trumping of state vectors

We present our main result as the following Theorem 6:

Theorem 6. Let $n > 1$, x, y $\in P_n$, x have weight n and choose

$$r = \frac{\log n}{\log y_1 - \log x_1} \text{ and } \bar{r} := \lfloor r + 1 \rfloor.$$

If $r > 0$ and

$$F_{k,\bar{r}}(\text{x}) > F_{k,\bar{r}}(\text{y}), \ k \in \{\bar{r}, \bar{r}+1, \ldots, n\bar{r}\},$$

then x $\in \overline{T(\text{y})}$.

If additionally

$$H_1(\text{x}) > H_1(\text{y})$$

and either

- the weight of y is smaller than n, or,
- the weight of y is n, $s > 0$ with

$$s = \frac{\log n}{\log x_{\min} - \log y_{\min}} \text{ and } \bar{s} := \lfloor s + 1 \rfloor$$

and

$$F_{k,1}\left(\frac{1}{\text{x}^{\bar{s}}}\right) < F_{k,1}\left(\frac{1}{\text{y}^{\bar{s}}}\right), \ k \in \{1, 2, \ldots, n\},$$

then x $\in T(\text{y})$.

The proof of this theorem is provided in the Methods section. Theorem 6 demonstrates a finite set of inequalities that imply the trumping relation for transforming a general state vector x to y when the allowed free operations are LOCC. We note that even if y_1, x_1 and/or $x_{\min}, y_{min}$ are close, the number of conditions increases but are still finite.

Trumping for states diagonal in energy eigenbasis

Here we consider a thermal system evolving according to the Hamiltonian H. We examine two states, ρ and σ, which are simultaneously diagonal in the energy eigenbasis with q_ρ and q_σ as their respective vector of eigenvalues arranged in descending order of magnitude. Our goal is to provide a finite set of sufficient conditions for transforming ρ to σ using a catalyst. We shall use the nomenclature thermo-majorization as an equivalence to the second law of thermodynamics.

A technical hurdle in this endeavor (that is, reducing infinitely many entropic comparisons of Eq. (13) to finitely many) is to address the case where the vector of eigenvalues of the thermal state ρ_g — denoted as g —

contains irrational entries. We will first consider in Corollary 7 a g that contains only rational entries before considering a general g in Theorem 8.

Corollary 7. The following set of finite conditions are sufficient to guarantee the existence of a transformation of a state ρ with the vector of eigenvalues denoted by q_ρ and is block diagonal in the energy eigenbasis into another block diagonal state σ with the vector of eigenvalues denoted by q_σ using catalytic thermal operations, given that the vector of eigenvalues of the thermal state ρ_g, denoted by g, contain only the rational entries:

- $F_{k,\bar{r}}(\mathrm{x}) < F_{k,\bar{r}}(\mathrm{y}),\ k \in \{\bar{r}, \bar{r}+1, \ldots, N\bar{r}\}$,
- $H_1(\mathrm{x}) < H_1(\mathrm{y})$, and either,
- the weight of y is smaller than n, or,the weight of y is n, and $F_{k,1}\left(\frac{1}{\mathrm{x}^{\bar{s}}}\right) > F_{k,1}\left(\frac{1}{\mathrm{y}^{\bar{s}}}\right),\ k \in \{1, 2, \ldots, n\}$,where x andy are the probability vectors obtained from $\mathcal{E}(q_\rho)$ and $\mathcal{E}(q_\sigma)$ and the parameters r and s are chosen to be:

$$r := \frac{\log N}{\log x_1 - \log y_1}\ s := \frac{\log N}{\log y_{\min} - \log x_{\min}}.$$

For a formal proof of Corollary 7 we refer to the Methods section.

Now, we consider the case where the vector g (vector of eigenvalues of the thermal state of the underlying system) has irrational entries. For this we first approximate g by a vector g_ε that is arranged in descending order of coordinates and is close to g in ℓ_1 norm. This is given by Fact 14. Addressing the above technicality, a formal catalytic majorization theorem for a general thermal state is as follows:

Theorem 8. Let $\varepsilon > 0$ be given and ρ_{g_ε} be a density matrix with rational entries that commutes with the thermal state ρ_g and satisfies $\| \rho_g - \rho_{g_\varepsilon} \|_1 \le \varepsilon$. The following set of finite conditions are sufficient to guarantee the existence of a transformation of a state ρ block diagonal in the energy eigenbasis into another block diagonal state σ using catalytic thermal operations:

- $F_{k,\bar{r}}(\mathrm{x}) < \frac{F_{k,\bar{r}}(\mathrm{y})}{A_r(\varepsilon)},\ k \in \{\bar{r}+1, \bar{r}+2, \ldots, N\bar{r}\}$
- $H_1(\mathrm{x}) < H_1(\mathrm{y}) - 2\log(1 + \frac{\varepsilon}{g_{\min}})$, and either,
- the weight of y is smaller than n, or,the weight of y is n, and $F_{k,1}\left(\frac{1}{\mathrm{x}^{\bar{s}}}\right)/A_s(\varepsilon) > F_{k,1}\left(\frac{1}{\mathrm{y}^{\bar{s}}}\right),\ k \in \{1, 2, \ldots, n\}$.Here x andy are the probability vectors obtained from $\mathcal{E}(q_\rho)$ and $\mathcal{E}(q_\sigma)$, where q_ρ, q_σ are the eigenvectors of ρ, σ respectively. $\mathcal{E}$ is defined in Eq. (25), $g_{\min} := \min_i g_i$, and $\|g - g_\varepsilon\| \le \varepsilon$, for any arbitrary $\varepsilon > 0$ with g_ε being a vector with rational entries. The functions $A_r(\varepsilon)$ and $A_s(\varepsilon)$ are such that

$$\frac{1}{A_r(\varepsilon)} = \max\left\{ 2^{\left[-\frac{1}{N}\left\{\left(1+\frac{\varepsilon}{g_{\min}}\right)^{2\bar{r}}\right\}\right]}, 2^{-\frac{2\varepsilon}{N g_{\min}}} \right\},$$

$$\frac{1}{A_s(\varepsilon)} = 2^{-\frac{1}{N}\left\{\left(1+\frac{\varepsilon}{g_{\min}}\right)^{2(1+\bar{s})} - 1\right\}},$$

and the parameters r and s are chosen as:

$$r := \frac{\log N}{\log x_1 - \log\left\{y_1\left(1+\frac{\varepsilon}{g_{\min}}\right)^2\right\}},$$

$$s := \frac{\log N}{\log y_{\min} - \log\left\{x_{\min}\left(1+\frac{\varepsilon}{g_{\min}}\right)^2\right\}}.$$

We give further details of the proof of the above theorem in the Methods section.

Trumping for states having coherence

In the previous subsection, we describe a set of sufficient conditions for the catalytic transformation of states that are diagonal in the energy eigenbasis. However, in general the states under consideration may contain coherence, i.e. off-diagonal elements in the energy eigenbasis of the Hamiltonian. The resource theory of coherence and definitions of transformations are reviewed in[32].

The conditions of Eq. (14) for the states having coherence are equivalent to the conditions derived for trumping and hence we can get the following corollary.

Corollary 9. The following set of finite conditions are sufficient to transform a pure state $|\psi\rangle$ into another pure state $|\phi\rangle$ using a pure catalyst and incoherent operations:

- $F_{k,\bar{r}}(\mathrm{x}) > F_{k,\bar{r}}(\mathrm{y}),\ k \in \{\bar{r}, \bar{r}+1, \ldots, n\bar{r}\}$
- Additionally $H_1(\mathrm{x}) > H_1(\mathrm{y})$ and either
- (a) the weight of y is smaller than n, or, (b) the weight of y is n, with

$$F_{k,1}\left(\frac{1}{\mathrm{x}^{\bar{s}}}\right) < F_{k,1}\left(\frac{1}{\mathrm{y}^{\bar{s}}}\right),\ k \in \{1, 2, \ldots, n\}$$

where x and y are the eigenvalues of ψ and ϕ respectively and r and s are real positive number as defined in Theorem 6.

The proof of the above result can be argued analogously from Theorem 6 where x and y are the eigenvalues of ψ and ϕ respectively. Determining a finite set of sufficient conditions for general (mixed) states and mixed catalysis remains an open challenge. Nevertheless, for such general scenario, a finite set of necessary conditions can be derived as outlined in Supplementary Note 2. If any of these conditions are not satisfied, it indicates that catalysis is not possible, however the converse is not true.

Outline of the main proof technique

Before moving to examples of the above results we give a high level description of the proof technique. Our aim is to reduce infinite comparisons of Rényi p-entropies to finitely many. For this, we propose the following steps:

- From Eq. (4) we observe that $H_p(\mathbf{x}) \propto \log ||\mathbf{x}||_p$.
- We then invoke Fact 1 we find the connection between the normalized $||\cdot||_p$ norm and the symmetric polynomial $F_{k,r}(\cdot)$.
- Finally the finite comparisons of $F_{k,r}(\mathbf{x})$ and $F_{k,r}(\mathbf{y})$ leads to infinite many comparisons of the ℓ_p norm of $\mathbf{x}$ and $\mathbf{y}$.
- The above comparison of ℓ_p norm is then mapped onto the infinite comparisons of Rényi p-entropies of $\mathbf{x}$ and $\mathbf{y}$, leading to our Theorem 6 and Corollary 7.Theorem 8 needs some more work. For this, recall the thermo-majorization calls for comparison of free energies which are related to Rényi p-divergences, see Eq. (12). This leads to the comparison of Rényi p divergences (between any state and the thermal state of the system) for infinitely many values of p to ascertain the occurrence of thermo-majorization. These Rényi p divergences can be then expressed in terms of Rényi p entropy only if the entries of the thermal state are rational (via a stochastic embedding map defined in Eq. (25). We can then apply the steps $1-4$ from above. Further, if the entries (some or all of them) of the thermal state of the system are irrational, then the aforementioned embedding map does not suffice to relate the Rényi p divergence to Rényi p entropy, adding a technical constraint. We resolve this issue by proving a continuity property of the Rényi p divergence between a state and the thermal state vs the state and an approximation of the thermal state containing only the rational entries, in Supplementary Note 1. Given this property, we can then apply the embedding map on the Rényi divergence between the state and the rational approximation of the thermal state and subsequently the steps $1-4$ mentioned above.

Examples

We now provide some examples of trumping guaranteed by the sets of sufficient conditions derived in the previous sections.

- *Trumping bipartite state vectors under LOCC*. First, we present examples of two state vectors $|x\rangle_{AB}$ and $|y\rangle_{AB}$, shared between two spatially-separated parties, Alice and Bob. These state vectors are not interconvertible using LOCC since x $\nprec$ y where x and y represent their respective vectors of Schmidt coefficient. However, conversion from $|x\rangle_{AB}$ to $|y\rangle_{AB}$ becomes feasible when catalysis is allowed. Consider the vectors, x = (0.610, 0.305, 0.043, 0.042) and y = (0.732, 0.121, 0.137, 0.010). Here x $\nprec$ y. One can also easily reconcile this with the necessary and sufficient conditions derived by Bosyk et al. which resolve the question when the catalyst is 2-dimensional[27]. However, our findings (presented in Theorem 6), confirms the existence of a catalyst indicating that trumping is indeed possible (detailed checks are provided in[35]). In fact, a suitable catalyst for this example can be found in dimension four, c = (0.48, 0.24, 0.16, 0.12) implying x ⊗ c ≺ y ⊗ c.
- *Trumping states diagonal in the energy eigenbasis under thermal operation*. We now present an example from thermodynamics where two states, initially not interconvertible under thermal operations, the states do not meet thermo-majorization conditions, become interconvertible when catalysis is allowed. Consider two states ρ and σ having vector of eigenvalues q_ρ = (0.936, 0.047, 0.016, 0.001), q_σ = (0.863, 0.130, 0.005, 0.002) respectively. Here $q_\sigma \nprec q_\rho$ under thermal operations, where the thermal state g is defined as $g = \frac{1}{Z}(e^{-\beta E_0}, e^{-\beta E_1}, e^{-\beta E_2}, e^{-\beta E_3})$ with $Z = e^{-\beta E_0} + e^{-\beta E_1} + e^{-\beta E_2} + e^{-\beta E_3}$, $\beta = 1.2$ and $E_0 = 0$, $E_1 = 1$, $E_2 = 2$, $E_3 = 3$. Although q_σ and q_ρ are not interconvertible under thermal operations, they satisfy the conditions from Theorem 8 when choosing ρ_ε as the maximally mixed state. In fact, in the presence of a catalyst c = (0.48, 0.24, 0.16, 0.12), q_ρ thermo-majorizes q_σ.
- *Trumping general states having coherence*. For states having coherence, consider $|\psi\rangle = \sqrt{0.4}|0\rangle + \sqrt{0.4}|1\rangle + \sqrt{0.1}|2\rangle + \sqrt{0.1}|3\rangle$ and $|\phi\rangle = \sqrt{0.5}|0\rangle + \sqrt{0.25}|1\rangle + \sqrt{0.25}|2\rangle$ as presented in[33]. Although the vector of eigenvalues of $|\psi\rangle$ and $|\phi\rangle$ do not majorize each other initially, the transformation becomes feasible in the presence of the catalyst, $|\chi\rangle = \sqrt{0.6}|0\rangle + \sqrt{0.4}|1\rangle$. This transformation is also assured by the finite set of sufficient conditions presented in Corollary 9.The open source software for this numerical exploration is publicly available[35].

Conclusions

In this work, we derive a finite set of inequalities that imply trumping. Specifically, we establish a finite set of conditions for trumping of state vectors under LOCC and general state transformation under thermal operations. Our results are also applicable to other restricted subsets of thermal operations known as *elementary thermal operations* and *Markovian thermal operations*, as the hierarchy of all such thermal processes collapses when catalysis is taken into consideration[36]. Moreover, the analysis presented here is general and we believe that it can be non-trivially extended to other concepts, such as Matrix-majorization[37]. Our derivation builds on a set of inequalities relating ℓ_p norms to a specific polynomial[30]. We note that, the set of inequalities introduced in[38] might be amenable to a similar result. The latter have a special significance in quantum information theory[39], as many of the basic entropic quantities have a much more elegant representation when defined over symmetric polynomials.

We leave for future work a comparison between the sufficient conditions derived here and other known conditions for the existence of a catalyst. One such condition has been proposed[27]. There, the authors derive necessary and sufficient conditions for 4-dimensional vectors to admit a 2-dimensional catalyst. It turns out that the set of conditions derived in our work (which apply to vectors and catalysts of arbitrary dimension) is incomparable to their condition. For instance, the example of a pair of 4-dimensional vectors presented previously in Example 2 satisfies our computable criterion, but fails to satisfy these conditions, thus implying that the catalyst needs to be at least 3-dimensional[27]. On the other hand, we present the following two vectors which satisfy the criterion and thus admit a 2-dimensional catalyst[27], but at the same time do not satisfy our criterion: x = (0.465, 0.273, 0.204, 0.058), y = (0.468, 0.269, 0.207, 0.056). With this observation, we note that our derived conditions are sufficient but not necessary.

We note that the conditions do not shed light on how to find the catalyst. In particular, we do not relate our conditions with a bound on the catalyst dimension. In fact, recent work hints at the computational *intractability* of this problem[40]. We conclude by mentioning that coming up with a set of *finite* necessary and sufficient conditions for trumping would still require a different tool set, which we leave as an open problem.

Methods

Notation. We now introduce some notation used throughout the manuscript. Given $x \in \mathbb{R}^n$, we denote by $x_{\min}$ the minimum non-zero element of **x**: $x_{\min} = \min\{x_i, 1 \le i \le n : x_i > 0\}$. We denote by x^p the point-wise exponentiation of the vector x. That is: $x^p = (x_1^p, \ldots, x_n^p)$ where we follow the convention that $0^p = 0$ for $p \in \mathbb{R}$. Abusing notation, we denote $x^{-1} = \frac{1}{x}$. Finally, let $x \in \mathbb{R}$, we denote the greatest integer smaller than or equal to $x+1$ by $\bar{x} = \lfloor x+1 \rfloor$.

We make use of Fact 1 to construct a *finite* set of sufficient conditions for checking whether an element belongs to the set T(y) (and also for $\overline{T(y)}$) in Theorem 6. However, before proceeding we state the following corollaries and lemma derived from Fact 1 that will be pivotal in the proof of our main Theorem 6. Corollary 10 is a slight modification of the Fact 1 replacing inequalities with strict inequalities, and some basic properties of the function $\|\cdot\|_p$.

Corollary 10. Let $n > 1$, x, y ∈ P_n and fix $r \in \mathbb{N}$. If

$$F_{k,r}(x) > F_{k,r}(y),\ k \in \{r, r+1, \ldots, nr\}$$

Then

$$\|x\|_p > \|y\|_p,\ p \in (0,1)$$

and

$$\|x\|_p < \|y\|_p,\ p \in (1, r+1).$$

The point-wise notation introduced in the previous section allows us to make a series of identities between different values of p. In particular, let x, y ∈ P_n. We have that for $p, m \in \mathbb{R}_+$ and $p, m \neq 0$:

$$\|x\|_{pm} = \left(\frac{1}{n}\sum_{i=1}^{n}(x_i)^{pm}\right)^{1/(pm)} = \left(\|x^m\|_p\right)^{1/m}, \tag{15}$$

and if x has weight n

$$\|x\|_p = \frac{1}{\left(\frac{1}{n}\sum_{i=1}^{n}(x_i)^p\right)^{-1/p}} = \frac{1}{\|\frac{1}{x}\|_{-p}}. \tag{16}$$

From Eq. (15) and Eq. (16) it follows:

$$\|x\|_{pm} > \|y\|_{pm} \iff \|x^m\|_p > \|y^m\|_p \tag{17}$$

and if x, y have both weight n

$$\| \mathrm{x}\|_p > \| \mathrm{y}\|_p \iff \left\|\frac{1}{\mathrm{x}}\right\|_{-p} < \left\|\frac{1}{\mathrm{y}}\right\|_{-p}. \tag{18}$$

Mitra and Ok[41] observed that in order to compare inequalities between p-norms in the range $(1, \infty)$ it is enough to check the range $(1, r)$ where r depends only on the ratio of x_1/y_1 and the dimension of the vectors. The following lemma is a slight generalization of this statement:

Lemma 11. Let $n > 1$, $\mathrm{x}, \mathrm{y} \in \mathbb{R}_+^n$, $y_1 > x_1 > 0$ and let

$$r = \frac{\log n}{\log y_1 - \log x_1}$$

Then, $\|\mathrm{x}\|_p < \|\mathrm{y}\|_p$ for $\forall\, p > r$.

Proof. Consider the following chain of inequalities:

$$\begin{aligned} \| \mathrm{x}\|_p^p &\le (x_1)^p \\ &\le \frac{1}{n}(y_1)^p \\ &\le \frac{1}{n}\sum_{i=1}^{n}(y_i)^p \\ &= \| \mathrm{y}\|_p^p. \end{aligned} \tag{19}$$

The first inequality holds by our modified definition of ℓ_p norm in Eq. (8) and the third inequality holds as all the components of vector y are non-negative. In addition, the first and third inequality hold independently of the value of p. We rewrite the second inequality as:

$$\left(\frac{y_1}{x_1}\right)^p > n. \tag{20}$$

Thus, the second inequality holds if $p > \log n/\log(y_1/x_1)$. □

A similar statement holds for the negative range of p and is as follows:

Corollary 12. Let $n > 1$, $\mathrm{x}, \mathrm{y} \in \mathbb{R}_+^n$ with $x_{\min} > y_{\min} > 0$ and let

$$s = \frac{\log n}{\log x_{\min} - \log y_{\min}}$$

Then, $\|\mathrm{x}\|_p > \|\mathrm{y}\|_p$ for $p < -s$.

Proof. Consider the vectors 1/x and 1/y. We have $(1/x)_1 = 1/x_{\min}$, $(1/y)_1 = 1/y_{\min}$ and, as a consequence, $(1/y)_1 > (1/x)_1 > 0$. From Lemma 11, we have that for $p > s$:

$$\left\|\frac{1}{\mathrm{x}}\right\|_p < \left\|\frac{1}{\mathrm{y}}\right\|_p.$$

Finally, from (16), we have that

$$\left\|\frac{1}{\mathrm{x}}\right\|_p < \left\|\frac{1}{\mathrm{y}}\right\|_p \iff \| \mathrm{x}\|_{-p} > \| \mathrm{y}\|_{-p}$$

□

Proof of Theorem 6: The proof builds on Lemma 11, Corollary 12 and Corollary 10 mentioned above.

Proof. Consider the first implication, that is that $\mathrm{x} \in \overline{T(\mathrm{y})}$ or equivalently from (10) that $\|\mathrm{x}\|_p < \|\mathrm{y}\|_p$ for $p \in (1, \infty)$. By hypothesis, the following condition holds:

$$F_{k,\bar{r}}(\mathrm{x}) > F_{k,\bar{r}}(\mathrm{y})\; k \in \{\bar{r}, \bar{r}+1, \ldots, n\bar{r}\}. \tag{21}$$

Then, by Corollary 10 we have that $\|\mathrm{x}\|_p < \|\mathrm{y}\|_p$ in the range $p \in (1, \bar{r})$. Moreover, since $r > 0$, by Lemma 11 we conclude that $\|\mathrm{x}\|_p < \|\mathrm{y}\|_p$ also in the range $p \in (r, \infty)$.

Let us now investigate the second implication. To verify that x is in $T(\mathrm{y})$, it needs to hold additionally that $H_1(\mathrm{x}) > H_1(\mathrm{y})$ and that $\|\mathrm{x}\|_p > \|\mathrm{y}\|_p$ for $p \in (-\infty, 1)/\{0\}$ (see (9)). The first condition holds by hypothesis, let us verify the second one.

From (21) we have that $\|\mathrm{x}\|_p > \|\mathrm{y}\|_p$ in the range (0, 1). It remains to check the range $(-\infty, 0)$. We divide the range into two partially overlapping ranges: $(-\infty, -s)$ and $(-\bar{s}, 0)$.

Since $s > 0$, we have by Corollary 12 that: $\|\mathrm{x}\|_p > \|\mathrm{y}\|_p$ for $p < -s$.

To check the range $(-\bar{s}, 0)$, we have by assumption that:

$$F_{k,1}\left(\frac{1}{\mathrm{x}^{\bar{s}}}\right) < F_{k,1}\left(\frac{1}{\mathrm{y}^{\bar{s}}}\right)\; k \in \{1, 2, \ldots, n\}, \tag{22}$$

which implies from Corollary 10 that

$$\left\|\frac{1}{\mathrm{x}^{\bar{s}}}\right\|_t < \left\|\frac{1}{\mathrm{y}^{\bar{s}}}\right\|_t \; t \in (0,1) \tag{23}$$

We conclude by showing the equivalence between (23) and the desired inequality in the range $(-\bar{s}, 0)$:

$$\begin{aligned} &\left\|\frac{1}{\mathrm{x}^{\bar{s}}}\right\|_t < \left\|\frac{1}{\mathrm{y}^{\bar{s}}}\right\|_t \; t \in (0,1) \\ \iff &\left\|\frac{1}{\mathrm{x}}\right\|_{t\bar{s}} < \left\|\frac{1}{\mathrm{y}}\right\|_{t\bar{s}} \; t \in (0,1) \\ \iff &\left\|\frac{1}{\mathrm{x}}\right\|_{p} < \left\|\frac{1}{\mathrm{y}}\right\|_{p} \; p \in (0,\bar{s}) \\ \iff &\| \mathrm{x}\|_p > \| \mathrm{y}\|_p \, p \in (-\bar{s}, 0) \end{aligned} \tag{24}$$

□

Proof of Corollary 7: An important ingredient in the proof is the following *extension* map:

Definition 13. Let $d > 0$, $v \in \mathbb{N}^d$ with $v_i > 0$ and $\sum_{i=1}^d v_i = N$; the embedding map, denoted by $\mathcal{E} : P_d \to P_N$, $(N \ge d)$ is defined as:

$$\mathcal{E}(q) := \Bigg\{ \underbrace{\frac{q_1}{v_1}, \ldots, \frac{q_1}{v_1}}_{v_1}, \underbrace{\frac{q_2}{v_2}, \ldots, \frac{q_2}{v_2}}_{v_2}, \ldots, \underbrace{\frac{q_d}{v_d}, \ldots, \frac{q_d}{v_d}}_{v_d} \Bigg\}. \tag{25}$$

The embedding map $\mathcal{E}$ is also a valid classical channel (a stochastic matrix) and possesses the following properties:

- $\mathcal{E}$ maps the probability distribution $g := \{\frac{v_1}{N}, \frac{v_2}{N}, \ldots, \frac{v_d}{N}\}$ to the uniform distribution on support size N, that is:

$$\mathcal{E}(g) = \underbrace{\left\{\frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N}\right\}}_{N} := u_N. \tag{26}$$

- It preserves equality of the p-Rényi divergence between any probability distribution (q) and g which becomes:

$$D_p(q||g) = D_p(\mathcal{E}(q)||\mathcal{E}(g)) = D_p(\mathcal{E}(q)||u_N)\,. \tag{27}$$

Using this map we now prove Corollary 7:

Proof. This corollary follows directly from Theorem 6. For $p \in (1, \bar{r})$ and $p \in (\bar{r}, \infty)$ (from Corollary 10 and Lemma 11 respectively) we get (by interchanging the roles of x and y):

$$\| x\|_p > \| y\|_p \text{ and thus } H_p(x) < H_p(y). \tag{28}$$

Now,

$$\begin{aligned} D_p(\rho||\rho_g) &\overset{a}{=} D_p(q_\rho||g) \overset{b}{=} D_p(\mathcal{E}(q_\rho)||\mathcal{E}(g)) = D_p(x||u_N) \\ &\overset{c}{=} logN - H_p(x) \\ &\overset{a}{>} \log N - H_p(y) \\ &\overset{e}{=} D_p(y||u_N) = D_p(\mathcal{E}(q_\sigma)||\mathcal{E}(g)) \\ &\overset{f}{=} D_p(q_\sigma||g) = gD_p(\sigma||\rho_g). \end{aligned}$$

Here (a) and (g) hold since q_ρ, q_σ, g are the vector of eigenvalues of ρ, σ, ρ_g respectively and in (b) and (f), we use the properties of the embedding map (27), in (c) and (e) we use the relation between $D_p(.\,||u_N)$ and $H_p(.)$, and in (d) we use Eq. (28).

Further, conditions (2) and (3) which are the extension to other values of p (i.e., $p = 1$, $p \in (0, 1)$, $p \in (-\bar{s}, 0)$, and $p \in (-\infty, -\bar{s})$) is a direct consequence, leveraging Corollary 10, Corollary 12, and thus fulfilling the conditions of Theorem 6 (albeit in a reverse direction to ensure the relative Rényi entropy aligns with the desired direction for thermo-majorization). Thus, ρ can undergo transformation to σ via thermal operation if the conditions outlined in Corollary 7 are met. □

Proof of Theorem 8: One may note that the definition of $\mathcal{E}$ in (25) relies on integer multiplicities v_i, and thus g must have rational entries. If g contains irrational components, one instead need to use a rational approximation g_ε satisfying $\|g - g_\varepsilon\|_1 \leq \varepsilon$, ensuring continuity of all relevant quantities and smooth extension of the proof to the general case. We therefore use the following fact as an essential ingredient in the proof:

Fact 14. ([9], Lemma 15) Let $g = \{g_i\}_{i=1}^d$ be a probability distribution of support size d, in descending order, and containing irrational entries. Then for any $\varepsilon > 0$, there exists a probability distribution g_ε such that:

- $\|g - g_\varepsilon\|_1 \leq \varepsilon$;
- Each entry of g_ε is a rational number, that is there exists a set of natural numbers $\{v_i\}_{i=1}^d$ such that $g_\varepsilon = \{\frac{v_i}{N}\}_{i=1}^d$, with $\Sigma^d_{i=1} v_i = N$; and
- There exists a valid classical channel $\mathcal{N}$ such that $\mathcal{N}(g) = g_\varepsilon$ and for any other distribution x, $\| x - \mathcal{N}(x)\|_1 \leq O(\sqrt{\varepsilon})$.

We give a simple proof of the statement 1 of the above Fact 14 in Supplementary Note 1. Subsequently, given any fixed ε, the approximation parameter, we define a density matrix ρ_{g_ε} diagonal in the same basis as ρ_g, with the vector of eigenvalues given by g_ε. This is now a rational approximation of the actual thermal state and will be used to show catalytic majorization. This approximation parameter, will influence the number of conditions (comparisons of the Rényi p-entropies) to be checked for catalytic majorization.

Using this Fact 14 we now prove Theorem 8.

Proof. We prove that the conditions stated in the theorem imply $D_p(\rho||\rho_g) > D_p(\sigma||\rho_g)$ for $p \in \mathbb{R}$ in a piece-wise fashion. First note that for any given $\varepsilon > 0$, the existence of ρ_{g_ε} is shown by Fact 14. This approximation calls for an additional analysis to relate $D_p(\rho||\rho_g)$ with $D_p(\rho||\rho_{g_\varepsilon})$ by proving a continuity argument in Supplementary Note 1 Proposition 20. We now indicate the lemmas and main ideas for each regime, by taking into account this approximation, essentially leading to $A_r(\varepsilon)$ and $A_s(\varepsilon)$ in the following analysis:

For $p = 1$, the proof is provided in Lemma 21 in Supplementary Note 1.

For the regime $1 < p < \bar{r}$, a detailed proof is presented in Supplementary Note 1. Here, we provide a brief outline for completeness. For a sufficiently small positive real-valued function $A_1(\varepsilon)$ we have:

$$\begin{aligned} & F_{k,\bar{r}}(\mathrm{x}) < \frac{F_{k,\bar{r}}(\mathrm{y})}{A_1(\varepsilon)} \text{ for } k \in \{\bar{r}, \bar{r}+1, \ldots, N\bar{r}\} \\ & \overset{a}{\Rightarrow} f_{\bar{r}}(x,t) < \frac{f_{\bar{r}}(y,t)}{A_1(\varepsilon)} \\ & \overset{b}{\Rightarrow} \| \mathrm{x}\|_p > \| \mathrm{y}\|_p \left(1 + \frac{\varepsilon}{g_{\min}}\right)^2 \\ & \overset{c}{\Rightarrow} H_p(\mathrm{x}) < H_p(\mathrm{y}) - \frac{2p}{p-1}\log\left(1 + \frac{\varepsilon}{g_{\min}}\right) \\ & \overset{d}{\Rightarrow} D_p(\rho||\rho_g) > D_p(\sigma||\rho_g). \end{aligned}$$

where (a), (b), (c), and (d) follows from a series of Lemmas (Lemma 23, 24, 25, 26 and 27 of Supplementary Note 1) with $\frac{1}{A_1(\varepsilon)} \leq 2^{\left[-\frac{1}{N}\left\{\left(1+\frac{\varepsilon}{g_{\min}}\right)^{2\bar{r}}\right\}\right]}$.

For the range $p \in (0, 1)$, a similar proof is outlined in Supplementary Note 1, building on Lemmas 28, 29, 30 and 31 and considering $\frac{1}{A_2(\varepsilon)} \leq 2^{-\frac{2\varepsilon}{Ng_{\min}}}$.

Since condition 1 is a consequence of the range, $p \in (1, \bar{r})$ and $p \in (0, 1)$, $\frac{1}{A_r(\varepsilon)}$ can be chosen to be as small as $\max\left\{\frac{1}{A_1(\varepsilon)}, \frac{1}{A_2(\varepsilon)}\right\}$.

We now consider the regime $p \in (-\bar{s}, 0)$. In this range, we have

$$\begin{aligned} & F_{k,1}\left(\frac{1}{\mathrm{x}^{\bar{s}}}\right)/A_s(\varepsilon) > F_{k,1}\left(\frac{1}{\mathrm{y}^{\bar{s}}}\right), \text{ for } k \in \{1, 2, \ldots, N\} \\ & \overset{a}{\Rightarrow} f_1\left(\frac{1}{\mathrm{y}^{\bar{s}}}, t\right) < \frac{f_1(\frac{1}{\mathrm{x}^{\bar{s}}}, t)}{A_s(\varepsilon)}, \\ & \overset{b}{\Rightarrow} \left\|\frac{1}{\mathrm{x}^{\bar{s}}}\right\|_\xi > \left\|\frac{1}{\mathrm{y}^{\bar{s}}}\right\|_\xi \left(1 + \frac{\varepsilon}{g_{\min}}\right)^{2\frac{1+\xi\bar{s}}{\xi}}, \text{ for any } \xi \in (0,1) \\ & \overset{c}{\Rightarrow} \| \mathrm{x}\|_p < \| \mathrm{y}\|_p \left(1 + \frac{\varepsilon}{g_{\min}}\right)^{2\frac{1-p}{p}} \\ & \overset{d}{\Rightarrow} H_p(\mathrm{x}) < H_p(\mathrm{y}) - 2\log\left(1 + \frac{\varepsilon}{g_{\min}}\right) \\ & \overset{e}{\Rightarrow} D_p(\rho||\rho_g) > D_p(\sigma||\rho_g). \end{aligned}$$

Here (a) Follows from Lemma 32 with $A_s(\varepsilon) \geq 2^{\frac{1}{N}\left\{\left(1+\frac{\varepsilon}{g_{\min}}\right)^{2(1+\bar{s})} - 1\right\}}$, (b) follows from Lemma 33, (c) follows from Lemma 34, (d) follows from Lemma 35, and (e) Lemma 36, and Lemma 37 (see Supplementary Note 1).

Note that the direction of inequality (b) is the reverse of that appearing in trumping without the thermal operations. This is because in order to prove thermo-majorization one needs to use the relation between $\|\mathrm{x}\|_p$, $H_p(\mathrm{x})$ and $D_p(\rho||\rho_g)$, the definition of the extension map $\mathcal{E}$, and Eq. (27).

Finally, for the regime $p > \bar{r}$ and $p < -\bar{s}$, the proof is provided in Lemma 38 of Supplementary Note 1.

This completes the proof. □

Data availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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Author contributions

A.M., A.N. worked out the technical derivations and wrote the first draft. D.E., S.S. contributed equally to the supervision of the project. All authors revised the text and prepared the final manuscript.

Competing interests

The authors declare no competing interests.

Additional information

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