

THE ROLE OF HELICITY IN TURBULENT FLUID DYNAMICS.

Trevor Charles Lipscombe

Magdalen College.



A thesis submitted to the University of Oxford in
partial fulfilment of the requirements for the degree of
Doctor of Philosophy.

September 1986

ABSTRACT.

In this thesis we consider turbulent fluid systems. We develop a closure scheme in which the mean velocity field of an incompressible fluid is driven by a turbulent velocity field possessing a non-zero mean helicity. We use this to investigate the formation of large scale vortices and the behaviour of the mean kinetic energy, enstrophy and helicity.

The same technique is then applied to the equations of magneto-hydrodynamics, in order to explain the self-generation of mean magnetic fields, and the joint formation of current and vortex structures.

We then discuss the convection of a passive scalar by the fluid and determine an equation for the mean temperature.

Finally we present a theory to account for the behaviour of a two-dimensional electrically conducting fluid subject to a constant external magnetic field driven by external forces. We explain the peaks in the power spectrum, the saturation of the magnetic and kinetic energies, and the insensitiveness of their equilibrium value on the external field. All of these are observed in numerical experiments.

ACKNOWLEDGEMENTS.

Firstly I should like to thank Dr. D. ter Haar, for allowing me the freedom to think for myself, and for his helpful advice and encouragement when I was unable to do so. It has been a pleasure to be supervised by him. I wish to thank Drs. J. Binney and E. Levich for useful discussions.

There have been many people who have helped to make my stay in Oxford a happy one. In particular I should like to thank Elaine, Jennifer, Andrew, Ian, Peter, Philip and Tim for making the Department a brighter place. I would like also to thank my long-suffering flat-mates David, Paul and Geraldine, as well as Claire, who doesn't fit into any category at all.

Finally I must thank the staff of the John Radcliffe Hospital, without whom this thesis may not have been written.

This work was supported by an S.E.R.C. studentship.

To my parents Joan and Eddy, with love,
and to the memory of Barbara Davis.

CONTENTS.

INTRODUCTION	2
CHAPTER ONE: VORTEX FORMATION IN AN INHOMOGENEOUS FLUID.	18
CHAPTER TWO: THE EFFECT OF HELICITY ON MAGNETIC FIELDS	38
CHAPTER THREE: HELICITY AND THE PASSIVE CONVECTION OF A SCALAR FIELD.	.56
CHAPTER FOUR: FORCED MHD IN THE PRESENCE OF AN EXTERNAL MAGNETIC FIELD	.67
CONCLUSIONS:	.83
TABLE 1: SOME TYPICAL VALUES FOR THE REYNOLDS NUMBER. .	87
REFERENCES:	88

INTRODUCTION

Fluids are fascinating. Most people, for example, have looked at the pattern of waves breaking on the sea shore and even more, especially in Britain, take great delight in talking (and complaining) about the weather.

To some extent, however, the motion of fluids is known. The Navier-Stokes equation, which describes the flow of Newtonian fluids (including the Earth's atmosphere and oceans) was derived a century and a half ago. The problem is that it is difficult to *do* anything with it.

In some cases it is possible to simplify the equation. This is the case, for example, in aerodynamics, where the flow of air around a wing can be separated into flow in a boundary layer where friction is important, and flow in an incompressible, irrotational region immediately surrounding it¹.

It is arguable, however, that the greatest challenge is to predict the general properties of fluid motion, either from the Navier-Stokes equation itself, or from a phenomenological model. It is then sensible to consider turbulent motion, since this is the "natural" state of most liquids and gases. By turbulent motion, we mean a state where the fluid exhibits a sensitive dependence on its initial conditions. The parameter which characterises fluid flow is Reynolds' number Re where:

$$Re = \frac{UL}{\nu} \quad (I.1)$$

in which U and L are "typical" speed and length scales, and ν is the kinematic viscosity.

For R_e greater than some critical value, say $R_e(cr)$, the flow becomes turbulent. This critical value depends usually on the particular geometry of the system considered, and varies from a few thousand for flow through a tube, to 10^6 or larger for astrophysical systems. Some typical values for R_e can be found in Table 1.

The onset of turbulence has attracted much attention in recent years. The pioneering work was done by Landau², who visualised the transition to turbulence as being due the instability of periodic flows as Reynolds' number increases. Although the actual scenario proposed by him is no longer thought to be a valid description of the onset of fluid turbulence, his idea is still present in the so-called Feigenbaum and the Ruelle-Takens-Newhouse scenarios³, which describe the transition to chaos as a process of period-doubling bifurcations. This thesis, however, is concerned with fully-developed turbulence, in which all the modes are excited.

The Navier-Stokes equation, which describes the flow of Newtonian fluids (for which the rate of stress is proportional to that of strain) is:

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \nabla) \underline{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \underline{v} \quad (1.2)$$

where $\underline{v}$ is the velocity field, p is the pressure, and ρ is the density.

If we measure velocities, times, lengths, pressures and densities in units U , L/U , L , $\rho_0 U^2$ and ρ_0 (see (I.1); ρ_0 is a "typical" density) the Navier-Stokes equation can also be written in the following dimensionless form:

$$\frac{\partial \underline{v}}{\partial t} - \underline{v} \wedge \underline{\omega} = - \frac{\nabla p}{\rho} - \nabla \left(\frac{v^2}{2} \right) + \frac{1}{Re} \nabla^2 \underline{v} \quad (I.3)$$

where $\underline{\omega} = \text{curl } \underline{v}$ is the vorticity (related to the rotational properties of the flow). In this form, it is easy to see that the non-linear term in (I.2) comprises of a kinematic pressure and a vorticity-producing term.

Equation (I.2) can be obtained by considering the forces acting on an arbitrary fluid element, and is a direct consequence of the conservation of momentum in binary molecular collisions. For derivations from both macroscopic and molecular viewpoints see, for example, chapter five of the book by Huang⁴.

In order to close this set of equations, it is necessary to express the density in terms of the pressure (i.e., to determine the equation of state) and also to find out how the density evolves.

Conservation of mass requires that:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0 \quad (I.4)$$

However, if we consider ^{incompressible} inviscid flow, one consequence of the Navier-Stokes equation, in the case of steady flow, is that the quantity:

$$p/\rho + \frac{1}{2} v^2 \quad (I.5)$$

is constant along a streamline, a curve that is everywhere tangent to the local fluid velocity. The pressure difference δp between the pressure at any point in the fluid and a stagnation point (a point where the fluid velocity is zero) is thus:

$$\delta p = \frac{1}{2} \rho v^2 \quad (I.6)$$

For a barotropic fluid, in which the pressure is a function of the density alone, we have that:

$$\delta p = \frac{\partial p}{\partial \rho} \delta \rho = c_s^2 \delta \rho \quad (I.7)$$

where c_s is the speed of sound. This gives:

$$\frac{\delta \rho}{\rho} \sim \frac{1}{2} \left(\frac{v}{c_s} \right)^2 \quad (I.8)$$

and so, as we shall only consider fluids for which the fluid velocity is small as compared to the speed of sound, the density can be regarded as constant. We thus obtain the incompressibility condition:

$$\nabla \cdot \underline{v} = 0 \quad (I.9)$$

The pressure can then be eliminated from (I.2) by taking the divergence of (I.2) to obtain:

$$\nabla^2 p = -(\partial_i v_j)(\partial_j v_i) \quad (I.10)$$

where ∂_i indicates here and henceforth the i -th component of the gradient operator $\underline{\nabla}$.

To describe the general properties of turbulence, a statistical description needs to be developed. If we consider an ensemble of fluids, which differ only in their initial conditions, and take the average of (I.4) over them, we find that:

$$\frac{\partial \langle \underline{v} \rangle}{\partial t} + \langle (\underline{v} \cdot \underline{\nabla}) \underline{v} \rangle = - \underline{\nabla} \langle p/\rho \rangle + \frac{1}{Re} \nabla^2 \langle \underline{v} \rangle \quad (I.11)$$

and so the equation of motion for the average involves the second moments and so on. This constitutes a closure problem, and has arisen because of the loss of information brought about by the averaging process (specifically, the initial conditions have been "smoothed over"). In order to combat this, many sophisticated closure schemes have been developed. The difficulty with many of them is that, in order to describe the properties of real fluids adequately, they are mathematically complex and difficult to use in practical situations. In spite of this, some have met with partial success, such as the Direct Interaction Approximation, and various cumulant discard methods⁵.

The cumulant discard hypothesis is a very simple closure scheme. For a flow of zero mean, the n -th order moment of the velocity can be expanded as:

$$\begin{aligned} \langle \underline{v}(\underline{x}_1) \dots \underline{v}(\underline{x}_N) \rangle &= \sum_{i=1}^N \langle \underline{v}(\underline{x}_1) \dots \underline{v}(\underline{x}_i) \rangle \langle \underline{v}(\underline{x}_{i+1}) \dots \underline{v}(\underline{x}_N) \rangle \\ &\quad + \langle \langle \underline{v}(\underline{x}_1) \dots \underline{v}(\underline{x}_N) \rangle \rangle \end{aligned} \quad (I.12)$$

where $\langle\langle \underline{v}(\underline{x}_1) \dots \underline{v}(\underline{x}_n) \rangle\rangle$ is the n -th order cumulant. If the velocity field obeyed Gaussian statistics, then the third and higher cumulants would all be zero. Moreover, the central limit theorem states that the probability distribution for r independent stochastic variables approaches the Gaussian distribution in the limit $r \rightarrow \infty$. It is to be hoped, therefore, that for real fluid flows, the higher order cumulants may be negligible, allowing the equations of motion to be closed.

The Direct Interaction Approximation is a much more sophisticated and mathematically complex technique. It attempts to obtain expressions for the correlation function $\langle v_i(\underline{x}, t) v_j(\underline{y}, t + \tau) \rangle$ and the mean response function of the velocity to an external driving field $\underline{F}$ in the limit as $|\underline{F}| \rightarrow 0$, that is to say, the quantity:

$$\lim_{|\underline{F}| \rightarrow 0} \left\langle \frac{\delta v_i(\underline{x}, t + \tau)}{\delta F_j(\underline{x}', t)} \right\rangle \quad (I.13)$$

In order to close the set of equations, the direct interaction approximation employs the weak dependence principle and the assumption of maximal randomness.

If we are concerned with the Fourier components $\underline{v}(\underline{k}, t)$ of the velocity field $\underline{v}(\underline{x}, t)$, then the weak dependence principle asserts that the interaction between modes corresponding to different values of $\underline{k}$ is weak. The maximal randomness condition states that the statistical interdependence amongst the Fourier components is brought about entirely by the non-linear coupling term in the Navier-Stokes equation, and not by either an external

force or the initial conditions.

In spite of the success of the Direct Interaction Approximation, it is still very difficult to employ it as a predictive tool. It is perhaps because of this difficulty that there has been a notable change of emphasis in recent years in favour of the development of phenomenological models. It could be argued, for example, that the Kolmogorov spectrum⁴ is a good example of this type of approach.

Kolmogorov, in an attempt to determine how the energy is distributed amongst the various length scales in the fluid, assumed that the non-linear term $(\underline{v} \cdot \nabla) \underline{v}$ was local, in wavenumber space. If energy is put into the system mechanically at some large scale, this non-linear term transfers it down to the smallest scales, where it is dissipated as heat. However, as the non-linear transfer is assumed to be only local, there is a sufficiently large number of scales which "feel" neither the input nor removal of energy. This is the so-called inertial range.

He further assumed that the turbulent fluid would exhibit universality: for "large enough" Reynolds' numbers, the physical properties of the flow would not be altered should the Reynolds number be increased by a factor of ten. In such a situation, the dynamical quantities should be independent of the Reynolds number.

The flow pattern is then characterised by only two quantities: the energy input rate and the viscosity. In equilibrium, the energy transfer rate ϵ , that is, the energy per unit mass transferred per unit time, will be

constant (and equals the rate at which energy is put into the system, for example by stirring.) From dimensional analysis, we have that:

$$[\epsilon] = v^3 l^{-1} \quad (I.14)$$

where v is a velocity and l is a length scale.

From the Navier-Stokes equation:

$$\epsilon = \left\langle \frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^2 \right) \right\rangle = \left\langle \nu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)^2 \right\rangle \quad (I.15)$$

or, in terms of Fourier transforms:

$$\epsilon = 2\nu \int k^2 E(k) dk \quad (I.16)$$

where:

$$E(k) = 4\pi k^2 \int \langle v_i(x) v_i(x+r) \rangle e^{-ik \cdot r} d^3 r \quad (I.17)$$

i.e. $E(k)$ is the contribution to the kinetic energy from modes in the wavenumber range k to $k+\delta k$, and we have assumed the turbulence to be isotropic.

In terms of dimensions:

$$[\epsilon] = \nu k^3 E(k) \quad (I.18)$$

Finally, in a steady state, the Navier-Stokes equation is written as:

$$(\underline{v} \cdot \nabla) \underline{v} \simeq \nu_t \nabla^2 \underline{v} \quad (I.19)$$

where ν_t is an "eddy viscosity" introduced to account for the pressure term. In many cases, such as in atmospheric physics, this eddy viscosity will be much greater than the kinematic viscosity. Dimensional analysis gives:

$$[\nu] \sim \nu K^{-1} \quad (I.20)$$

and so, by combining (I.14) to (I.20) we obtain the famous Kolmogorov spectrum:

$$E(K) \propto \epsilon^{2/3} K^{-5/3} \quad (I.21)$$

This is, of course, a far from rigorous treatment. An important objection, for example, was raised by Landau⁷, who argued that since ϵ itself is a fluctuating quantity, the assumption of universality is violated. Perhaps the most surprising aspect of the Kolmogorov spectrum is that it is in very good agreement with experiment.

The concept itself is capable of adaptation, and Fleck⁸, for example, has developed a similar method to describe the velocity-length-scale correlations of molecular clouds.

The Kolmogorov description does not, however, explain all of the observed features of turbulence. The most outstanding of these are, perhaps, the phenomenon of intermittency and the persistence of large scale structures.

By intermittent, it is meant that the turbulence is restricted to "patches" of space and time. This feature, together with the idea of a cascade of energy from large to small scales formed the basis for the Novikov-Stewart⁸ model. They considered a cube of fluid (of side length L) to be constituted of n cubes, only m of which are "active" and contain turbulence.

The m cubes are then similarly sub-divided, in the same ratio m/n , until a length scale is reached on which viscosity acts, and on which the turbulence is considered to be uniformly distributed over the remaining cubes. They obtained the expression:

$$\langle \epsilon^2 \rangle = \langle \epsilon \rangle^2 \left(\frac{n}{m} \right)^5 \quad (1.22)$$

for the mean-square dissipation rate, where s is the number of sub-divisions required to reach the viscosity scale l_0 , and hence:

$$\langle \epsilon^2 \rangle = \langle \epsilon \rangle^2 \left(\frac{L}{l_0} \right)^{3q} \quad q = 1 - \frac{\ln m}{\ln n} \quad (1.23)$$

They then proceed to make some further assumptions about the spatial distribution of the turbulence-containing cubes in order to obtain an expression for the energy spectrum. Further details of this, and refinements of it (such as the β model) can be found in the monograph by Monin and Yaglom⁹, and a good qualitative discussion can be found in the book by Mandelbrot¹⁰.

The Novikov-Stewart model is capable of great improvement. Thus far, the eddies have been regarded as being of undetermined shape. It can be postulated, however, that the vorticity is restricted either to vortex tubes, sheets or elements. It would then be possible to make use of the features of the inviscid Navier-Stokes equation to construct a more realistic model, since this corresponds to the $R_e \rightarrow \infty$ limit. For example, the energy:

$$H = \int \frac{1}{2} \rho v^2 d^3r \quad (I.24)$$

the circulation:

$$K = \oint_C \underline{v} \cdot d\underline{L} \quad (I.25)$$

where $d\underline{L}$ is a line element of the closed material curve C , and the helicity:

$$\Gamma = \int \underline{v} \cdot \underline{\omega} d^3r \quad (I.26)$$

are all conserved.

One model, proposed by Childress¹¹, visualises the vorticity as being restricted to vortex tubes. The circulation is then just the (constant) strength of the tubes, whilst the helicity measures their knottedness. The tubes are then permitted to split and reform, subject to the three constraints above, and allows an expression for the energy spectrum to be determined. This model is very similar to one of the earliest proposed descriptions of turbulence, namely that suggested by Townsend¹², where

the vorticity was restricted to randomly oriented vortex sheets.

A similar model is employed by Moffatt¹³, in an attempt to account for the observed increase in the enstrophy Ω , where:

$$\Omega = \int \frac{1}{2} \omega^2 d^3r \quad (I.27)$$

He considers two forms for the distribution of vorticity to explain this: the first is a collection of vortex filaments, for which the helicity obeys:

$$\Gamma \sim K^2 \left(J + \frac{1}{2\pi} \int \tau(s) ds \right) \quad (I.28)$$

where $\tau(s)$ is the torsion of the filament, defined by the equation:

$$\frac{d}{ds} (\underline{t} \wedge \underline{n}) = \tau(s) \underline{n} \quad (I.29)$$

where $\underline{t}$ and $\underline{n}$ are the unit vectors tangent and normal to the vortex filament, and s is the arc length, and J is the Gauss integral:

$$J = \frac{1}{4\pi} \int d^3x d^3x' \frac{(\underline{x} - \underline{x}') \cdot (d\underline{x} \wedge d\underline{x}')}{|\underline{x} - \underline{x}'|^3} \quad (I.30)$$

It can be seen from equation (I.28) that the constancy of helicity places a constraint on the torsion of the filament.

The second case dealt with by Moffatt is when the

vorticity is confined to randomly convoluted vortex sheets. These he shows to be unstable to Kelvin-Helmholtz instabilities, and lead to the formation of spiral singularities. From this stage it is possible to obtain an expression for the energy spectrum.

The models described above, whilst phenomenological, do draw on some of the conservation laws for ideal flow, and hence some connexion with the mathematical formulation of the problem. Unfortunately, they use the fact that turbulence is intermittent in order to obtain an expression for the energy spectrum, but do not provide even a qualitative picture of why the phenomenon occurs. Ideally we should seek an explanation both for this and for the persistence of large-scale, coherent vortex structures (those vortices that are recognisable after a few eddy turnover times $\tau = (\nu k^2)^{-1}$).

In a region of high helicity, where $\underline{v}$ is mainly parallel to $\underline{\omega}$, the non-linear interaction term $\underline{v} \times \underline{\omega}$ will be small, and so energy transfer will be restricted. If a vortex was generated in this region, it may well be long-lived. Moreover, between regions of opposite sign helicity, a large shear would be generated, leading to dissipation. This would explain intermittency (dissipation being restricted within the helical region). It remains to be shown that, in a reflexionally symmetric fluid, helical regions will occur, and also that large vortices could be formed in such a region.

Levich and Tsinober¹⁴ have recently shown that in a homogeneous fluid, the quantity:

$$I = \int \langle \gamma(\underline{x}) \gamma(\underline{x} + \underline{r}) \rangle d^3 \underline{r} \quad (I.31)$$

where the helicity density:

$$\gamma(\underline{x}) = \underline{v}(\underline{x}) \cdot \underline{\omega}(\underline{x}) \quad (I.32)$$

is a constant, up to $O(R_e^{-1})$. Physically, for a fluid in which $\langle \Gamma \rangle = 0$, I represents the helicity fluctuations. It remains to be shown that $I \neq 0$.

Intuitively, it seems implausible that helicity fluctuations do not exist. Also, the form of I is of the same form as the Saffmann¹⁵ integral:

$$M_{ij} = \frac{1}{(2\pi)^3} \int \langle f_i(\underline{x}) f_j(\underline{x} + \underline{r}) \rangle d^3 \underline{r} \quad (I.33)$$

where $\underline{f}$ is a force per unit mass acting on the fluid. This can be shown, assuming ergodic behaviour, to be non-zero in a fluid if, in a volume V , $f \propto V^{1/2}$. More conclusively, perhaps, when the velocity field obeys Gaussian statistics, we have:

$$I = \int E^2(k) dk \quad (I.34)$$

This equation, as we shall show later, introduces the concept of an inverse cascade.

From dimensional analysis, we have:

$$[I] = E^2(k) k \quad (I.35)$$

However, from the Kolmogorov argument, we have that:

$$\epsilon \propto E^{3/2}(k) k^{5/2} = \text{constant} \quad (\text{I.36})$$

Equations (I.35) and (I.36) are clearly incompatible in the same inertial range, which suggests that there are two such ranges: one which cascades energy down to small scales, and the other creating helicity correlations on ever larger scales. This latter cascade would form conditions favourable to the formation of large scale vortex structures. A similar situation occurs in two-dimensional fluid dynamics, where the enstrophy is conserved, and hence plays a role analogous to that of the mean-square helicity in three dimensions.

Recently, in a series of numerical experiments¹⁶, evidence for the existence of helical fluctuations in a reflexionally symmetric fluid has been found. Moreover, the form of the kinetic energy spectrum at the largest scales derived by Levich and Tsinober have been applied to mesospheric phenomena¹⁷, where the theory would suggest that large scale features such as cyclones may occur due to energy transfer from the smaller scale cumulus clouds.

The important concept of helicity fluctuations and the role of helicity in general have only recently been realised. Motivated by the work described above, and based on a model first proposed by Moiseev et al¹⁸, we develop in chapter one a model of an incompressible fluid with a non-zero mean helicity. In the second chapter, a brief discussion is given of magnetic helicity and its use in MHD. The closure scheme developed in chapter one

applied to the MHD equations, and then, in chapter three, to that of a passively convected scalar field.

Finally, in chapter four, some analytical work on forced two-dimensional MHD turbulence of a conducting fluid subject to an external field is presented. This work explains, in part, some of the properties of such flows suggested in numerical studies^{19,20,21}.

CHAPTER ONE

LARGE-SCALE VORTEX FORMATION IN AN INHOMOGENEOUS FLUID.

The work of Levich and Tsinober described in the introduction indicates that even in a reflexionally symmetric ensemble of fluids (those for which $\langle T \rangle = 0$) the fluctuations in the helicity may still be large, and that the fundamental I invariant cascades the energy to large scales, thereby creating conditions favourable to the birth of large scale structures. Should these form in a region of large helicity, i.e. in a region where $\underline{y}$ is mainly parallel to $\underline{\omega}$, the energy transfer term $\underline{y} \times \underline{\omega}$ will be small, and hence the structures may be long-lived.

We now proceed to construct a model of a fluid driven by an underlying Gaussian turbulent flow of a given helicity. The initial theory, due to Moiseev et al¹⁸, dealt specifically with a homogeneous fluid whose equation of state (used for simplicity) was $p = k_0 \rho^2$. The derivation given here is for a more general case, since we consider a weakly non-homogeneous flow of a particular nature of an incompressible fluid. Inhomogeneities are likely to arise in any natural flow.

Assume that for some time $t < t_0$ the (turbulent) velocity is given by $\underline{y} = \underline{y}^t$; $p = p^t$, where $\underline{y}$ is a δ -correlated field with Gaussian statistics, has zero mean and has a definite helicity. This field will obey the Navier-Stokes equation, for an incompressible fluid, so that:

$$\partial_t v_i^t + v_j \partial_j v_i^t = \nu \nabla^2 v_i^t - \partial_i \pi^t \quad (1.1)$$

where $\partial_t \equiv \partial/\partial t$ and $\pi^t \equiv p^t/\rho^t$. Taking the divergence of this equation gives:

$$\nabla^2 \pi^t = -(\partial_i v_j^t)(\partial_j v_i^t) \quad (1.2)$$

The assumption of incompressibility is valid so long as the Mach number M (the ratio of fluid to sound velocity) is small ($M \ll 0.3$).

At time t_0 , "switch on" a small mean velocity $\langle \underline{v} \rangle$ (i.e. such that $|\langle \underline{v} \rangle| \ll |\underline{v}^t|$ everywhere. Realistically, this can only be satisfied "almost" everywhere, as occasionally $\underline{v}^t$ may vanish.) At all subsequent times, the total velocity can be expressed as the sum of these two fields plus a third term. This third term, denoted by $\underline{v}^1$, is a "mixing" velocity, of zero mean, which is a function of the turbulent velocity, so that $\underline{v}^1 = \underline{v}^1(\underline{v}^t, \pi^t)$. The same argument is also given for the total pressure.

We note that $\underline{v}^1$ is well-defined: at any time later than t_0 , we can, in principle, determine from the initial conditions, what $\underline{v}^t$ would have been if $\langle \underline{v} \rangle$ had not been switched on, and can measure both $\underline{v}$ and $\langle \underline{v} \rangle$, and hence $\underline{v}^1$ is well-determined.

For times $t > t_0$ we have:

$$\partial_t v_i + v_j \partial_j v_i = \nu \nabla^2 v_i - \partial_i \pi \quad (1.3)$$

and the pressure equation is:

$$\partial^2 \pi = -(\partial_i v_j)(\partial_j v_i) \quad \partial^2 \equiv \nabla^2 \quad (1.4)$$

where the total velocity and pressure can be decomposed to give:

$$\underline{v} = \langle \underline{v} \rangle + \underline{v}^t + \underline{v}^l ; \quad \pi = \langle \pi \rangle + \pi^t + \pi^l \quad (1.5)$$

Equations (1.5) can be considered to be the definition of $\underline{v}^l$ and π^l .

Upon inserting the expressions (1.5) into (1.3) and using (1.1), we obtain, after taking the ensemble averages:

$$\begin{aligned} \partial_t \langle v_i \rangle + \langle v_j \rangle \partial_j \langle v_i \rangle \\ = - \langle v_j^l \partial_j v_i^l \rangle - \langle v_j^t \partial_j v_i^l + v_j^l \partial_j v_i^t \rangle \\ + \nu \partial^2 \langle v_i \rangle - \partial_i \langle \pi \rangle \end{aligned} \quad (1.6)$$

Initially, since $|\langle \underline{v} \rangle| \ll |\underline{v}^t|$, we expect $|\underline{v}^l| \ll |\underline{v}^t|$, and so the terms non-linear in $\langle \underline{v} \rangle$ and $\underline{v}^l$ can be neglected. This simplifies the above equation, which reduces to:

$$\begin{aligned} \partial_t \langle v_i \rangle + \langle v_j^t \partial_j v_i^l \rangle + \langle v_j^l \partial_j v_i^t \rangle \\ = \nu \partial^2 \langle v_i \rangle - \partial_i \langle \pi \rangle \end{aligned} \quad (1.7)$$

The last two terms on the left-hand side can be obtained from the mixing correlator $\langle \underline{v}^t(\underline{x}) \underline{v}^l(\underline{y}) \rangle$, and so the evolution can be determined if this correlator can be calculated.

Since the mixing velocity is a function of $\underline{v}^t$ and π^t only, and these are assumed to be Gaussian variables, we

can use the Novikov-Furutsu formula^{22,23} to express the mixing correlator in the form:

$$\begin{aligned} \langle v_i^t(1) v_j^t(2) \rangle &= \int \langle v_i^t(1) v_k^t(3) \rangle R_{kj}(2,3) d3 \\ &+ \int \langle v_i^t(1) \pi^t(3) \rangle C_j(2,3) d3 \end{aligned} \quad (1.8)$$

where R and C are the responses of the mixing velocity to a change in the turbulent driving fields, that is:

$$R_{kj}(2,3) = \left\langle \frac{\delta v_k^t(2)}{\delta v_j^t(3)} \right\rangle \quad (1.9)$$

and:

$$C_j(2,3) = \left\langle \frac{\delta v_j^t(2)}{\delta \pi^t(3)} \right\rangle \quad (1.10)$$

where $(1) \equiv (x_1, t_1)$ and similarly for the other arguments, while $\delta/\delta \dots$ indicates a functional derivative.

Equation (1.8) is obtained by writing the function $v^t(\underline{v}^t)$ as a Taylor series in $\underline{v}^t$, and then noting that for Gaussian fields of zero mean, the n^{th} order moment $\langle v^t(x_1) v^t(x_2) \dots v^t(x_n) \rangle$ can be expressed in terms of the second order moments $\langle v^t(x_i) v^t(x_j) \rangle$ only.

We now proceed to calculate the response functions R and C .

Place expressions (1.5) into equation (1.3) and simplify using (1.1). After performing some tedious algebra, we obtain an equation of motion for the mixing velocity:

$$\frac{\partial v_i^t}{\partial t} = -\langle v_j \rangle \partial_j v_i^t - v_j^t \partial_j \langle v_i \rangle + A_i \quad (1.11)$$

where $\underline{A}$ is a known vector function. This can be integrated to give a formal solution:

$$\begin{aligned} v_i'(t) = & - \int_0^t \left[v_k^t(t') \partial_k \langle v_i(t') \rangle \right. \\ & + \langle v_k(t') \rangle \partial_k v_i^t(t') \\ & \left. + A_i(t') \right] dt' \end{aligned} \quad (1.12)$$

in which the (obvious) label $\underline{x}$ has been suppressed.

Using the equation above allows the response functions to be calculated. Taking the functional derivative with respect to the pressure first gives:

$$\left\langle \frac{\delta v_i'(\underline{x}, t)}{\delta \pi^t(\underline{y}, t'')} \right\rangle = \int_0^t \left\langle \frac{\delta A_i(\underline{x}, t')}{\delta \pi^t(\underline{y}, t'')} \right\rangle dt' \quad (1.13)$$

From causality, the interaction term can only respond to a change in the driving field at an earlier time, so that the lower limit in (1.12) can be replaced by t'' . However, since we are only interested in equal time commutation relations, we let $t \rightarrow t''$ in the above to give:

$$\mathcal{G}_i(\underline{x}, t | \underline{y}, t) = 0 \quad (1.14)$$

Taking the functional derivative with respect to the velocity gives:

$$\begin{aligned} R_{ij}(\underline{x}, t | \underline{y}, t'') = & - \int_0^t \left(\left[\langle v_k(t') \rangle \frac{\partial}{\partial x_k} \delta(\underline{x} - \underline{y}) \delta(t - t'') \right] \right. \\ & - \delta(\underline{x} - \underline{y}) \delta(t - t'') \frac{\partial \langle v_i(\underline{x}, t') \rangle}{\partial x_j} \\ & \left. + \left\langle \frac{\delta A_i(\underline{x}, t')}{\delta v_j^t(\underline{y}, t'')} \right\rangle \right) dt' \end{aligned} \quad (1.15)$$

Following the same procedure as before yields the equation:

$$R_{ij}(\underline{x}, t | \underline{y}, t) = -\frac{1}{2} \left[\delta(\underline{x} - \underline{y}) \frac{\partial \langle v_i(\underline{x}, t) \rangle}{\partial x_k} + \delta_{ij} \langle v_k(\underline{x}, t) \rangle \frac{\partial \delta(\underline{x} - \underline{y})}{\partial x_k} \right] \quad (1.16)$$

where the factor of one half arises from the derivative of the Dirac delta function.

The expressions (1.13) and (1.15) can now be inserted into the equation for the mixing correlator and, after performing the integrations, we obtain:

$$\langle v_i^t(\underline{x}) v_j'(\underline{y}) \rangle = -\frac{1}{2} \langle v_i^t(\underline{x}) v_k^t(\underline{y}) \rangle \frac{\partial \langle v_j(\underline{y}) \rangle}{\partial y_k} \quad (1.17)$$

We need now only specify the correlator K , where:

$$K_{ij}(\underline{x}, t | \underline{y}, t + \tau) = \langle v_i^t(\underline{x}, t) v_j^t(\underline{y}, t + \tau) \rangle \quad (1.18)$$

to determine the terms $\langle (v_i^1, \underline{\nabla}) v_j^t + (v_j^t, \underline{\nabla}) v_i^1 \rangle$.

We assume, for simplicity, that this correlator may be decomposed into the product of spatial and temporal components. Moreover, if the turbulence is statistically stationary, the time-dependent function will depend only on the time-separation τ . Thus we have:

$$K_{ij}(\underline{x}, t | \underline{y}, t + \tau) = K_{ij}(\underline{x}, \underline{y}) f(\tau) \quad (1.19)$$

In the above, we have taken $f(\tau)$ to be the δ -function.

It is usually assumed that the turbulence is both

homogeneous and isotropic, so that there is no preferred origin or direction for the axes. In that case, the correlator K is simply a function of $|\underline{r}|$. For this work we retain some weakly homogeneous terms of a particular nature. Having made a particular choice for the origin, we write the correlator as:

$$\begin{aligned} K_{ij}(\underline{R}, \underline{S}) = & \phi(\underline{R}, \underline{S}) \delta_{ij} + \epsilon_{ijk} g(\underline{R}, \underline{S}) R_k \\ & + D(\underline{R}, \underline{S}) R_i R_j \\ & + \lambda \left[\epsilon_{ijk} S_k h(\underline{R}, \underline{S}) \right. \\ & \quad + A(\underline{R}, \underline{S}) (R_i S_j + R_j S_i) \\ & \quad \left. + B(\underline{R}, \underline{S}) (R_i S_j - R_j S_i) \right] \\ & + \lambda^2 C(\underline{R}, \underline{S}) S_i S_j \end{aligned} \quad (1.20)$$

where $\underline{R}$ and $\underline{S}$ are the relative and centroid co-ordinates, namely:

$$\underline{R} = \frac{1}{2}(\underline{x} - \underline{y}) \quad ; \quad \underline{S} = \frac{1}{2}(\underline{x} + \underline{y}) \quad (1.21)$$

and λ is an expansion parameter, which can be taken to be the ratio of the length scale of the inhomogeneities in the turbulent field to the dimension of the system. Terms involving powers of R greater than two have been neglected because they do not contribute to the final equations (at most the first derivative of K with respect to R is required, and the limit as $R \rightarrow 0$ is then taken). Terms of $O(\lambda^3)$ will be neglected, as it is assumed that the expansion parameter is small.

We note that this choice for the correlator is similar to that given by Batchelor for axisymmetric turbulence,

the axis of symmetry being here replaced by the centroid coordinate $\underline{S}$. This places a particular significance on the vector $\underline{S}$, and hence on the choice for the origin. It thus restricts the nature of the inhomogeneities considered.

It is easy to see that the $\phi(R, S)$ gives a measure of the "longitudinal" correlation, whilst $g(R, S)$ and $h(R, S)$ represent partially the "transverse" correlation.

We are not free, however, to choose any functions for the set $\{\phi, A, B, C, D, g, h\}$, since under the transformations:

$$i \rightleftharpoons j \quad ; \quad x \rightleftharpoons y \quad (1.22)$$

the correlator must remain unchanged. This imposes the following set of symmetry conditions:

$$\begin{aligned} Q(\underline{R}, \underline{S}) &= Q(-\underline{R}, \underline{S}) \quad Q \in \{\phi, B, C, D, g\} \\ &= -Q(-\underline{R}, \underline{S}) \quad Q \in \{A, h\} \end{aligned} \quad (1.23)$$

In addition to this, we assume, for simplicity, that:

$$\lim_{R \rightarrow 0} \frac{\partial Q(\underline{R}, \underline{S})}{\partial R_i} = 0 \quad Q \in \{\phi, A, B, C, D, g, h\} \quad (1.24)$$

This is a particularly restrictive assumption, and again constrains the type of inhomogeneities considered. In the limit $\lambda \rightarrow 0$, these conditions reduce to those assumed by Moiseev et al.

It is now possible to obtain the derivatives:

$$\lim_{\substack{R \rightarrow 0 \\ S \rightarrow x}} \frac{\partial K_{ij}}{\partial R_q} = \epsilon_{ijq} g(x) + \lambda A(x) (x_j \delta_{iq} + x_i \delta_{jq}) + \lambda B(x) (x_j \delta_{iq} - x_i \delta_{jq}) \quad (1.25)$$

and:

$$\lim_{\substack{R \rightarrow 0 \\ S \rightarrow x}} \frac{\partial K_{ij}}{\partial S_q} = \frac{\partial \phi}{\partial x_q} \delta_{ij} + \lambda \left[\epsilon_{ijk} x_k \frac{\partial h}{\partial x_q} + \epsilon_{ijq} h \right] + \lambda^2 \left[\frac{\partial c}{\partial x_q} x_i x_j + c(x) (x_j \delta_{iq} + x_i \delta_{jq}) \right] \quad (1.26)$$

where we have used the notation:

$$Q(x) = \lim_{\substack{R \rightarrow 0 \\ S \rightarrow x}} Q(R, S) \quad (1.27)$$

However, from equation (1.20), we have that:

$$Q(Q, S) = -Q(0, S) \quad Q = h, A \quad (1.28)$$

which implies that:

$$h(x) = A(x) = 0 \quad (1.29)$$

Further to this, we have the incompressibility condition, namely:

$$\frac{\partial K_{ij}}{\partial x_i} = \frac{\partial K_{ij}}{\partial y_j} = 0 \quad (1.30)$$

with:

$$\frac{\partial}{\partial x_i} = \frac{1}{2} \left(\frac{\partial}{\partial s_i} + \frac{\partial}{\partial r_i} \right) ; \quad \frac{\partial}{\partial y_i} = \frac{1}{2} \left(\frac{\partial}{\partial s_i} - \frac{\partial}{\partial r_i} \right) \quad (1.31)$$

The condition imposed by equation (1.29) must be true everywhere. If we apply it specifically at the point $\underline{y}=\underline{x}$, we obtain:

$$\underline{\nabla}\phi - 2\lambda B(\underline{x})\underline{x} + \lambda^2 \underline{x} ((\underline{x} \cdot \underline{\nabla})C + 4C) = 0 \quad (1.32)$$

In order to satisfy this equation at every point, we choose to put:

$$\underline{\nabla}\phi = 0 \Rightarrow \phi = \text{const} = \phi_0 ; B(\underline{x}) = 0 \quad (1.33a)$$

and:

$$(\underline{x} \cdot \underline{\nabla})C(\underline{x}) + 4C(\underline{x}) = 0 \quad (1.33b)$$

We note that this particular choice makes all terms of first order in λ zero.

Inserting equations (1.29) and (1.33) into (1.25) and (1.26) and using (1.27) gives:

$$\begin{aligned} \left\langle \frac{\partial v_i^t(\underline{x})}{\partial x_q} v_j^t(\underline{x}) \right\rangle &= \epsilon_{ijq} g(\underline{x}) \\ &+ \lambda^2 \left[\frac{\partial C}{\partial x_q} x_i x_j \right. \\ &\left. + C(\underline{x})(x_i \delta_{jq} + x_j \delta_{iq}) \right] \quad (1.34) \end{aligned}$$

We can use this to obtain different interpretations of the quantity $g(\underline{x})$, since the mean helicity density of the turbulent field is:

$$\langle \underline{v}^t(\underline{x}, t) \cdot \underline{\omega}^t(\underline{x}, t) \rangle = 6g(\underline{x})f(0) \quad (1.35)$$

Likewise the enstrophy density is:

$$\langle \omega^2(\underline{x}, t) \rangle = [6D(\underline{x}) + \lambda^2 \{ \underline{x}^2 \nabla^2 C - 2C \}] f(0) \quad (1.36)$$

and the kinetic energy density is:

$$\langle v^2(\underline{x}, t) \rangle = [3\phi_0 + \lambda^2 \underline{x}^2 C(\underline{x})] f(0) \quad (1.37)$$

From (1.33), we have that:

$$\nabla \cdot [C(\underline{x}) \underline{x}] + C(\underline{x}) = 0 \quad (1.38)$$

which, if we integrate using Gauss's theorem, gives:

$$\int_S C(\underline{x}) \underline{x} \cdot d^2 \underline{S} = - \int_V C(\underline{x}) d^3 \underline{x} \quad (1.39)$$

If there exists a closed surface S_0 , such that on this surface $C(\underline{x})=0$, then within the volume V_0 enclosed by S_0 , there must exist regions of positive and negative $C(\underline{x})$, since:

$$\int_{V_0} C(\underline{x}) d^3 \underline{x} = 0 \quad (1.40)$$

provided that $C(\underline{x})$ is not zero everywhere within V_0 . From equation (1.37), these correspond to regions of enhancement or reduction of turbulent kinetic energy.

If we assume that $C(\underline{x}) = C(|\underline{x}|)$, then (1.33) becomes:

$$\frac{1}{r} \frac{dC}{dr} + 4C = 0 \quad r = |\underline{x}| \quad (1.41)$$

which has a solution:

$$C = \frac{6}{r^4} \quad (1.42)$$

We note that this expression is singular at the origin, again illustrating the nature of the inhomogeneity.

Equation (1.42) gives:

$$\langle \underline{\omega}^{t^2}(\underline{x}, t) \rangle = 6D(\underline{x}) + \frac{10\lambda^2 C_0}{r^4} \quad (1.43)$$

As well as being able to obtain a specific form for the enstrophy, it is also possible to obtain an expression for the energy spectrum, since we have that:

$$E(\underline{k}) = \frac{1}{2} \int \langle \underline{v}^{t^2}(\underline{x}) \rangle e^{-i\underline{k} \cdot \underline{x}} d^3 \underline{x} \quad (1.44)$$

which, in this case, becomes:

$$E(\underline{k}) = \frac{1}{2} \int \left[3\phi_0 + \frac{\lambda^2 C_0}{r^2} \right] e^{-i\underline{k} \cdot \underline{x}} d^3 \underline{x} \quad (1.45)$$

which, on integration, yields:

$$E(\underline{k}) = \frac{3\phi_0 \delta(\underline{k})}{2(2\pi)^3} + \frac{\lambda^2 \pi^2 C_0}{k} \quad (1.46)$$

The sign of C_0 determines whether the kinetic energy is enhanced or reduced in the presence of inhomogeneities.

From equations (1.35)+(1.37), we can construct two length scales characteristic of the turbulent field, namely:

$$l_1 = \frac{\langle v^{t^2} \rangle}{\langle \underline{v}^t \cdot \underline{\omega}^t \rangle} = \frac{\phi_0}{2g} \quad (1.47)$$

and:

$$L_2 = \left[\frac{\langle v^2 \rangle}{\langle \omega^2 \rangle} \right]^{\frac{1}{2}} = \left[\frac{\phi_0}{D} \right]^{\frac{1}{2}} \quad (1.48)$$

The implications of these "natural" length scales will be discussed later.

The terms which need to be calculated are:

$$\langle (\underline{v}' \cdot \underline{\nabla}) \underline{v}^t \rangle_i = \lim_{\underline{x} \rightarrow \underline{y}} \frac{\partial}{\partial x_k} \langle v_i^t(\underline{x}) v_k'(\underline{y}) \rangle \quad (1.49)$$

and:

$$\langle (\underline{v}^t \cdot \underline{\nabla}) \underline{v}' \rangle_k = \lim_{\underline{x} \rightarrow \underline{y}} \frac{\partial}{\partial y_i} \langle v_i^t(\underline{x}) v_k'(\underline{y}) \rangle \quad (1.50)$$

Inserting equation (1.17) into equation (1.49) gives:

$$\langle (\underline{v}' \cdot \underline{\nabla}) \underline{v}^t \rangle_i = -\frac{1}{2} \frac{\partial \langle v_k \rangle}{\partial x_j} \lim_{\underline{x} \rightarrow \underline{y}} \frac{\partial \langle v_i^t(\underline{x}) v_j^t(\underline{y}) \rangle}{\partial x_k} \quad (1.51)$$

which, using equation (1.34) yields:

$$\langle (\underline{v}' \cdot \underline{\nabla}) \underline{v}^t \rangle_i = -\frac{1}{2} \frac{\partial \langle v_k \rangle}{\partial x_j} \left\{ \epsilon_{ijk} g(\underline{x}) + \lambda^2 x_i x_j \frac{\partial C}{\partial x_k} + \lambda^2 C(\underline{x}) (x_i \delta_{jk} + x_j \delta_{ik}) \right\} \quad (1.52)$$

or equivalently, in vector form:

$$\begin{aligned} \langle (\underline{v}' \cdot \underline{\nabla}) \underline{v}^t \rangle = & -\frac{1}{2} \left\{ g(\underline{x}) \underline{\nabla} \wedge \langle \underline{v} \rangle \right. \\ & + \lambda^2 \left(\underline{\nabla} \cdot [C(\underline{x}) (\underline{x} \cdot \underline{\nabla}) \langle \underline{v} \rangle] \right) \underline{x} \\ & \left. + \lambda^2 C(\underline{x}) (\underline{x} \cdot \underline{\nabla}) \langle \underline{v} \rangle \right\} \end{aligned} \quad (1.53)$$

By inserting equation (1.17) into equation (1.50), we obtain:

$$\langle \underline{v}^t \cdot \underline{\nabla} \rangle_k = -\frac{1}{2} \left[\frac{\partial \langle v_k \rangle}{\partial x_j} \langle v_i^t(x) \frac{\partial v_j^t}{\partial x_i} \rangle + K_{ij}(x, x) \frac{\partial^2 \langle v_k \rangle}{\partial x_i \partial x_j} \right] \quad (1.54)$$

which, upon using equation (1.34), simplifies to give:

$$\langle (\underline{v}^t \cdot \underline{\nabla}) \underline{v}' \rangle_k = -\frac{1}{2} \frac{\partial^2 \langle v_k \rangle}{\partial x_i \partial x_j} \left[\phi_0 \delta_{ij} + \lambda^2 c(x) x_i x_j \right] \quad (1.55)$$

We note that the trace of the quantity in the brackets is simply the kinetic energy density of the turbulent field.

Hence we can write the (linearised) equation of motion for the mean field $\langle \underline{v} \rangle$ in the form:

$$\begin{aligned} \partial_t \langle \underline{v} \rangle = & \nu' \nabla^2 \langle \underline{v} \rangle + g'(x) \underline{\nabla} \wedge \langle \underline{v} \rangle - \underline{\nabla} \langle \pi \rangle \\ & + \lambda^2 \left\{ c'(x) (\hat{q} + 2\hat{q}^2) \langle \underline{v} \rangle \right. \\ & \left. + [\underline{\nabla} \cdot (c'(x) \hat{q} \langle \underline{v} \rangle)] x \right\} \end{aligned} \quad (1.56)$$

where we have defined the quantities:

$$\begin{aligned} g'(x) &= \frac{g(x)}{2} \quad ; \quad c'(x) = \frac{c(x)}{2} \quad \nu' = \nu + \frac{\phi_0}{2} \\ \hat{q}^n &= x_{i_1} \dots x_{i_n} \frac{\partial}{\partial x_{i_1}} \dots \frac{\partial}{\partial x_{i_n}} \end{aligned} \quad (1.57)$$

As the kinetic energy is positive, we note that the viscosity is enhanced due to the presence of the turbulent driving field.

If we had not chosen the particular form of equation (1.20) for the correlator K_{ij} , we would have obtained the following equation:

$$\begin{aligned} \partial_t \langle v_i \rangle = & \nu \nabla^2 \langle v_i \rangle - \nabla_i \langle \pi \rangle \\ & + \frac{1}{2} K_{jq} \frac{\partial^2 \langle v_i \rangle}{\partial x_j \partial x_q} + \frac{\Lambda_{ijq}}{2} \frac{\partial \langle v_j \rangle}{\partial x_q} \end{aligned} \quad (1.57a)$$

where:

$$\Delta_{ijq}(\underline{x}) = \left\langle \frac{\partial v_i^t}{\partial x_j} v_q^t \right\rangle \quad (1.57b)$$

so that:

$$\epsilon_{qji} \Delta_{ijq} = \langle \underline{v} \cdot \underline{\omega} \rangle \quad (1.57c)$$

In equation (1.57), it can be seen that the operator $\hat{Q}$ can be interpreted as a stretching term, since $(\underline{x} \cdot \nabla) \langle \underline{v} \rangle$ gives the spatial rate of change of $\langle \underline{v} \rangle$ in the direction of $\underline{x}$, and that when this is large, then $\langle \underline{v} \rangle$ will increase more rapidly with time, depending on the sign of $C(\underline{x})$. In regions where energy enhancement of the turbulent field occurs, so that $C(\underline{x}) \gg 0$, the mean field will tend to increase on a time scale $(\lambda^2 C')^{-1}$.

We now proceed to expand all variables as a power series in λ , so that:

$$Q(\underline{x}) = Q_0(\underline{x}) + \lambda Q_1(\underline{x}) + \lambda^2 Q_2(\underline{x}) \dots \quad (1.58)$$

This gives the following equation for the velocity to zeroth order in λ :

$$\begin{aligned} \partial_t \langle \underline{v}_0 \rangle = & v_0' \nabla^2 \langle \underline{v}_0 \rangle + g_0' \nabla \wedge \langle \underline{v}_0 \rangle \\ & - \nabla \langle \pi_0 \rangle \end{aligned} \quad (1.59)$$

This is the result obtained by Moiseev et al, provided that we set $g(\underline{x})$ equal to a constant. The equations are therefore not particularly sensitive to the equation of

state.

From equation (1.58), we can also obtain the first order expression for the velocity, namely:

$$\begin{aligned} \partial_t \langle \underline{v}_1 \rangle &= \nu'_0 \nabla^2 \langle \underline{v}_1 \rangle + \nu'_1 \nabla^2 \langle \underline{v}_0 \rangle \\ &+ g'_0 \nabla \wedge \langle \underline{v}_1 \rangle + g'_1 \nabla \wedge \langle \underline{v}_0 \rangle \\ &- \nabla \langle \pi_1 \rangle \end{aligned} \quad (1.60)$$

By taking the curl of equation (1.59) we obtain:

$$\begin{aligned} \partial_t \langle \underline{\omega}_0 \rangle &= \nu'_0 \nabla^2 \langle \underline{\omega}_0 \rangle + g'_0 \nabla \wedge \langle \underline{\omega}_0 \rangle \\ &- \nabla g'_0 \wedge \langle \underline{\omega}_0 \rangle \end{aligned} \quad (1.61)$$

which, upon Fourier Transforming, assuming g'_0 to be constant, gives the following growth rate for vortices:

$$\gamma(k) = g'_0 k - \nu'_0 k^2 \quad (1.62)$$

There is thus a band of scales $0 \leq k \leq g'_0/\nu'$ for which growth occurs. The maximum growth rate $\gamma_{\max}$ occurs at a scale k_0 , where:

$$\gamma_{\max} = \frac{g'^2_0}{4\nu'_0} ; \quad k_0 = \frac{g'_0}{2\nu'_0} \quad (1.63)$$

We can compare the size of the vortex structures formed with the "typical" length scale of the turbulent field given in equation (1.47), giving:

$$N = \frac{\text{length scale of turbulent field}}{\text{" " " vortices}} = \frac{\Phi_0}{g} k \quad (1.64)$$

so that:

$$0 \leq N \leq \left(\frac{1}{1 + 2\nu/\phi_0} \right) \quad (1.65)$$

The ratio of these two scales is thus dependent on a single number:- the ratio of the "turbulent" to the molecular viscosity. Growth occurs on a range of scales, so there will be some large scale structures formed, whatever the value of this ratio.

To obtain the first order equation for the vorticity, we simply take the curl of equation (1.60), giving:

$$\begin{aligned} \partial_t \langle \underline{\omega}_1 \rangle &= \nu'_0 \nabla^2 \langle \underline{\omega}_1 \rangle + \nu'_1 \nabla^2 \langle \underline{\omega}_0 \rangle \\ &+ g'_0 \nabla \wedge \langle \underline{\omega}_1 \rangle + g'_1 \nabla \wedge \langle \underline{\omega}_0 \rangle \\ &+ \nabla g'_0 \wedge \langle \underline{\omega}_1 \rangle + \nabla g'_1 \wedge \langle \underline{\omega}_0 \rangle \end{aligned} \quad (1.66)$$

Taking the divergence of equations (1.59) and (1.60) gives the zeroth and first order equations for the pressure, namely:

$$\nabla^2 \langle \pi_0 \rangle = \nabla g'_0 \cdot \langle \underline{\omega}_0 \rangle \quad (1.67)$$

and:

$$\nabla^2 \langle \pi_1 \rangle = \nabla g'_0 \cdot \langle \underline{\omega}_1 \rangle + \nabla g'_1 \cdot \langle \underline{\omega}_0 \rangle \quad (1.68)$$

It is interesting to note from equation (1.67) that a helicity gradient can generate pressure and that, in the absence of such a gradient, the pressure obeys Laplace's equation. If $g_0(\underline{x})$ is such that $g_0(\underline{x}) = g_0(|\underline{x}|)$, then

equation (1.67) becomes:

$$\nabla^2 \langle \pi \rangle = -\frac{1}{r} \frac{dg(r)}{dr} \nabla \cdot [\underline{x} \wedge \langle \underline{v} \rangle] \quad (1.69)$$

so that in this case the pressure can be related, through the helicity density, to the mean angular momentum, since:

$$\langle \underline{L} \rangle = \rho \int d^3r (\underline{x} \wedge \langle \underline{v} \rangle) \quad (1.70)$$

We have also the second order equations:

$$\begin{aligned} \partial_t \langle \underline{v}_2 \rangle &= v_0' \nabla^2 \langle \underline{v}_2 \rangle + v_1' \nabla^2 \langle \underline{v}_1 \rangle + v_2' \nabla^2 \langle \underline{v}_0 \rangle \\ &+ g_0' \nabla \wedge \langle \underline{v} \rangle_2 + g_1' \nabla \wedge \langle \underline{v} \rangle_1 + g_2' \nabla \wedge \langle \underline{v} \rangle_0 \\ &- \nabla \langle \pi_2 \rangle + c_0' (\hat{q} + 2\hat{q}^2) \langle \underline{v}_0 \rangle \\ &+ (\nabla \cdot (c_0' \hat{q} \langle \underline{v} \rangle)) \underline{x} \end{aligned} \quad (1.71)$$

and:

$$\begin{aligned} \partial_t \langle \omega_2 \rangle &= v_0' \nabla^2 \langle \omega_2 \rangle + v_1' \nabla^2 \langle \omega_1 \rangle + v_2' \nabla^2 \langle \omega_0 \rangle \\ &+ g_0' \nabla \wedge \langle \omega_2 \rangle + g_1' \nabla \wedge \langle \omega_1 \rangle + g_2' \nabla \wedge \langle \omega_0 \rangle \\ &+ \nabla g_0' \wedge \langle \omega_2 \rangle + \nabla g_1' \wedge \langle \omega_1 \rangle + \nabla g_2' \wedge \langle \omega_0 \rangle \\ &+ \nabla \wedge \{ c_0' (\hat{q} + 2\hat{q}^2) \langle \underline{v}_0 \rangle + [\nabla \cdot (c_0' \hat{q} \langle \underline{v} \rangle)] \underline{x} \} \end{aligned} \quad (1.72)$$

as well as:

$$\begin{aligned} \nabla^2 \langle \pi \rangle &= \nabla g_0' \cdot \langle \omega_2 \rangle + \nabla g_1' \cdot \langle \omega_1 \rangle + \nabla g_2' \cdot \langle \omega_0 \rangle \\ &+ \nabla \cdot [c_0' (4\hat{q} + 2\hat{q}^2) \langle \underline{v}_0 \rangle] \\ &+ \hat{q} \nabla \cdot (c_0' \hat{q} \langle \underline{v}_0 \rangle) \end{aligned} \quad (1.73)$$

It is also possible to determine the zeroth order equations for the kinetic energy, enstrophy and helicity associated with the mean field.

Forming the scalar product of equation (1.60) with $\langle \underline{v} \rangle_0$

and integrating gives:

$$\begin{aligned} \partial_t \frac{1}{2} \int \langle \underline{v}_0 \rangle^2 d^3 \underline{r} &= \nu' \int \langle \underline{v}_0 \rangle \cdot \nabla^2 \langle \underline{v}_0 \rangle d^3 \underline{r} \\ &+ \int g'(\underline{x}) [\langle \underline{v}_0 \rangle \cdot \langle \underline{\omega}_0 \rangle] d^3 \underline{x} \quad (1.74) \end{aligned}$$

The energy associated with the mean field is thus subject to the usual viscous dissipation, (but with an enhanced coefficient of viscosity) and is also affected by the helicity of the mean field. If this helicity is of the same sign everywhere as the mean helicity of the turbulent field, then the energy will tend to increase. This is a result similar to that obtained by Moffatt²⁵. The equation for the enstrophy is obtained by forming the dot product of equation (1.68) with $\langle \underline{\omega} \rangle_0$ and integrating. The result is:

$$\begin{aligned} \partial_t \frac{1}{2} \int \langle \underline{\omega}_0 \rangle^2 d^3 \underline{r} &= \nu' \int \langle \underline{\omega}_0 \rangle \cdot \nabla^2 \langle \underline{\omega}_0 \rangle d^3 \underline{r} \\ &+ \int g'(\underline{x}) [\langle \underline{\omega}_0 \rangle \cdot \nabla \wedge \langle \underline{\omega}_0 \rangle] d^3 \underline{r} \quad (1.75) \end{aligned}$$

Here we have viscous dissipation, but also enstrophy generation caused by the "smoothing" of the turbulent field helicity and the helicity associated with the vorticity (as opposed to the velocity) field.

Finally, the time evolution of the helicity is given by:

$$\begin{aligned} \partial_t \int [\langle \underline{v}_0 \rangle \cdot \langle \underline{\omega}_0 \rangle] d^3 \underline{r} \\ = \int d^3 \underline{r} \left[\partial_t \langle \underline{v}_0 \rangle \cdot \langle \underline{\omega}_0 \rangle + \langle \underline{v}_0 \rangle \cdot \partial_t \langle \underline{\omega}_0 \rangle \right] \quad (1.76) \end{aligned}$$

which, in this case, becomes:

$$\begin{aligned}
 & \partial_t \int \langle \underline{v}_0 \rangle \cdot \langle \underline{\omega}_0 \rangle d^3r \\
 &= \nu_0 \int \left[\langle \underline{\omega}_0 \rangle \cdot \nabla^2 \langle \underline{v}_0 \rangle + \langle \underline{v}_0 \rangle \cdot \nabla^2 \langle \underline{\omega}_0 \rangle \right] d^3r \\
 &+ \int d^3r \left[g_0(x) \langle \omega_0^2(x) \rangle + \frac{1}{2} g_0(x) A_{ij}^2 \right] \quad (1.77)
 \end{aligned}$$

where A_{ij} is the vorticity tensor, defined such that:

$$A_{ij} = \partial_j \langle v_i \rangle - \partial_i \langle v_j \rangle \quad (1.78)$$

and hence:

$$A_{ij} = -\epsilon_{xij} \langle \omega_k \rangle \quad (1.79)$$

Again there is a viscous damping term, helicity generation due to the mixing of the (small scale) turbulent helicity with the (large scale) enstrophy associated with the mean field, and a term occurring due to the skewness of the mean velocity field.

In this chapter, we have developed the closure scheme proposed by Moiseev et al. in order to discuss an incompressible fluid driven by a weakly inhomogeneous, Gaussian-correlated driving field of a definite mean helicity. This model is valid for systems where the turbulent viscosity is small in comparison to the molecular viscosity. We have found an enhancement of viscosity due to the energy of the turbulent field. Vortex production occurs on large scales due to the presence of a mean helicity. The energy associated with the mean field will increase provided that the turbulent

-37_a

and mean field helicity have the same sign.

In the second chapter, we use this technique to investigate the behaviour of passively convected fields.

CHAPTER TWO

THE EFFECT OF HELICITY ON MAGNETIC FIELDS.

Many astrophysical objects possess magnetic moments, the Earth, Sun and pulsars being examples. From elementary considerations, as we shall see later, it would appear that the magnetic fields of these two bodies should have decayed to zero over their lifetime, some 10^9 years. The fact that it has not suggests that there should be some form of self-generation of the magnetic field, and much work has been done on applying dynamo theory to these systems, in order to explain the long-lived nature of the field.

In recent work²⁶, the helicity of the magnetic field has been shown to play a part in the formation of stellar magnetic structures. As a result, the magnetic field $\underline{B}$ is often assumed to be purely helical, so that:

$$\underline{\nabla} \wedge \underline{B} = \alpha \underline{B} \quad (2.1)$$

Such a field is called force-free, since the Lorentz force density $\underline{j} \wedge \underline{B}$ ($\underline{j}$ is the current density) associated with it is zero. It is also the field configuration of minimum magnetic energy. For these fields, a current is produced parallel to the magnetic field, thus creating a poloidal component. This " α effect" is a major ingredient in modern dynamo theory²⁷. If we assume further that α

has a constant value, then it is easy to show that:

$$\nabla^2 \underline{B} + \alpha^2 \underline{B} = 0 \quad (2.2)$$

This is the linear Helmholtz equation, and solutions can be found for given boundary conditions.. The force-free condition has been applied to the solar corona to determine the form of the magnetic flux tubes²⁸. Further work has shown that should such a flux tube be expanded by an external force, then magnetic helicity will be transported to this region²⁹.

The results of chapter one show that a mean kinematic helicity (i.e., $\langle \underline{v} \cdot \underline{\omega} \rangle \neq 0$) causes a self-generation of vorticity. It is to be hoped that the application of the closure scheme developed there might help to explain the phenomenon of self-generating magnetic fields. It is also possible that the results may be applicable to astrophysical objects, since rotational motion encourages the development of kinetic helicity²⁷. Work by De Young³⁰, however, indicates that turbulent generation of magnetic fields in extra-galactic radio sources is unlikely, unless the kinetic energy input is accompanied by helicity flow on all length scales.

In the presence of a magnetic field, the equation of motion for a conducting fluid will become:

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \nabla) \underline{v} = -\nabla \pi + \rho_e \underline{E} + \underline{j} \wedge \underline{B} \quad (2.3)$$

where $\underline{j}$ is the current density, $\underline{B}$ is the magnetic field, $\underline{E}$ the electric field, and ρ_e the charge density; the last two terms on the right-hand side clearly constitute the

Lorentz force.

In addition to this, we have Ohm's law:

$$\underline{j} = \sigma \left(\underline{E} + \frac{\underline{v} \wedge \underline{B}}{c} \right) \quad (2.4)$$

where σ is the (scalar) conductivity, and Maxwell's equations which, neglecting displacement currents, become:

$$\begin{aligned} \underline{\nabla} \cdot \underline{E} &= 4\pi \rho_e & \underline{\nabla} \wedge \underline{E} &= -\frac{1}{c} \frac{\partial \underline{B}}{\partial t} \\ \underline{\nabla} \cdot \underline{B} &= 0 & \underline{\nabla} \wedge \underline{B} &= \frac{4\pi}{c} \underline{j} \end{aligned} \quad (2.5)$$

with the equation of continuity of charge:

$$\frac{\partial \rho_e}{\partial t} + \underline{\nabla} \cdot \underline{j} = 0 \quad (2.6)$$

In the non-relativistic limit, the terms involving ρ_e can be neglected, and after some manipulation we obtain the magneto-hydrodynamic equations for the magnetic field and fluid velocity³¹:

$$\left(\frac{\partial}{\partial t} + \underline{v} \cdot \underline{\nabla} \right) \underline{B} = \eta \nabla^2 \underline{B} + (\underline{B} \cdot \underline{\nabla}) \underline{v} \quad (2.7)$$

and:

$$\left(\frac{\partial}{\partial t} + \underline{v} \cdot \underline{\nabla} \right) \underline{v} = \nu \nabla^2 \underline{v} + \frac{(\underline{B} \cdot \underline{\nabla}) \underline{B}}{4\pi \rho} - \nabla \left(\pi + \frac{B^2}{8\pi \rho} \right) \quad (2.8)$$

where:

$$\eta = \frac{c^2}{4\pi \sigma} \quad (2.9)$$

is the magnetic diffusivity.

An order of magnitude estimate for the lifetime τ of a magnetic field is thus:

$$\tau = \frac{L^2}{\eta} = \frac{4\pi\sigma L^2}{c^2} \quad (2.10)$$

For the case of sunspots, for example, $\eta \approx 10$ and $L \approx 10^7$, so that $\tau \approx 10^{13}$ s, much less than the age of the sun. Some form of dynamo action is therefore expected.

We note that equation (2.7) is similar to the vorticity equation:

$$\left(\frac{\partial}{\partial t} + \underline{v} \cdot \underline{\nabla} \right) \underline{\omega} = \nu \nabla^2 \underline{\omega} + (\underline{\omega} \cdot \underline{\nabla}) \underline{v} \quad (2.11)$$

except that in equation (2.11) we have, instead of equation (2.8) the constraint $\underline{\omega} = \underline{\nabla} \times \underline{v}$.

By analogy with fluid turbulence, we can define a magnetic Reynolds number R_m such that:

$$R_m = \left| \frac{(\underline{B} \cdot \underline{\nabla}) \underline{v}}{\eta \nabla^2 \underline{B}} \right| \sim \frac{uL}{\eta} \quad (2.12)$$

and at extreme values of R_m we have:

$$\frac{\partial \underline{B}}{\partial t} \simeq \eta \nabla^2 \underline{B} \quad R_m \ll 1$$

and:

$$\frac{\partial \underline{B}}{\partial t} \simeq \underline{\nabla} \wedge (\underline{v} \wedge \underline{B}) \quad R_m \gg 1 \quad (2.13)$$

so that for small R_m , there is a decaying magnetic field, but for R_m large, there will be field generation. In astrophysical applications, the length scale involved is quite often large, and so R_m is usually high.

Usually only the case of a perfectly conducting fluid is dealt with, so that $\eta \rightarrow 0$, but in the following treatment it has been retained, since the analysis is not complicated by it, and physically it represents the effects of molecular collisions.

We now define the Alfvén velocity $\underline{b}$ by:

$$\underline{b} = \frac{\underline{B}}{\sqrt{4\pi\rho}} \quad (2.14)$$

and, since the density ρ is constant, equations (2.7) and (2.8) become:

$$\left(\frac{\partial}{\partial t} + \underline{v} \cdot \underline{\nabla}\right) \underline{b} = \eta \nabla^2 \underline{b} + (\underline{b} \cdot \underline{\nabla}) \underline{v} \quad (2.15)$$

and:

$$\left(\frac{\partial}{\partial t} + \underline{v} \cdot \underline{\nabla}\right) \underline{v} = \nu \nabla^2 \underline{v} + (\underline{b} \cdot \underline{\nabla}) \underline{b} - \underline{\nabla} \left(\pi + \frac{\underline{b}^2}{2}\right) \quad (2.16)$$

We proceed in the same manner as in chapter 1, by splitting the velocity, density, and magnetic fields into a turbulent driving, a mean and a mixing term:

$$\begin{aligned} \underline{v} &= \langle \underline{v} \rangle + \underline{v}^t + \underline{v}^i \\ \underline{b} &= \langle \underline{b} \rangle + \underline{b}^t + \underline{b}^i \\ \pi &= \langle \pi \rangle + \pi^t + \pi^i \end{aligned} \quad (2.17)$$

where we assume that the turbulent fields obey Gaussian statistics, and $\underline{v}^t, \underline{b}^t$ are of zero mean, and that $|\langle \underline{b} \rangle|, |\underline{b}^i| \ll |\underline{b}^t|$. This time, to simplify the calculations, we also assume homogeneity.

Inserting equation (2.17) into equations (2.15) and

(2.16) and averaging yields:

$$\begin{aligned} \frac{\partial \langle \underline{b} \rangle}{\partial t} + \langle (\underline{v}' \cdot \underline{\nabla}) \underline{b}^t + (\underline{v}^t \cdot \underline{\nabla}) \underline{b}' \rangle \\ = \eta \nabla^2 \langle \underline{b} \rangle + \langle (\underline{b}' \cdot \underline{\nabla}) \underline{v}^t + (\underline{b}^t \cdot \underline{\nabla}) \underline{v}' \rangle \end{aligned} \quad (2.18)$$

and:

$$\begin{aligned} \frac{\partial \langle \underline{v} \rangle}{\partial t} + \langle (\underline{v}' \cdot \underline{\nabla}) \underline{v}^t + (\underline{v}^t \cdot \underline{\nabla}) \underline{v}' \rangle \\ = \nu \nabla^2 \langle \underline{v} \rangle + \langle (\underline{b}' \cdot \underline{\nabla}) \underline{b}^t + (\underline{b}^t \cdot \underline{\nabla}) \underline{b}' \rangle \\ - \underline{\nabla} (\langle \pi \rangle + \langle \underline{b}' \cdot \underline{b}^t \rangle) \end{aligned} \quad (2.19)$$

We also have equations for both $\underline{b}^1$ and $\underline{v}^1$, namely:

$$\begin{aligned} \frac{\partial \underline{b}_i}{\partial t} = (\langle \underline{b} \rangle \cdot \underline{\nabla}) \underline{v}^t + (\underline{b}^t \cdot \underline{\nabla}) \langle \underline{v} \rangle - (\langle \underline{v} \rangle \cdot \underline{\nabla}) \underline{b}^t \\ - (\underline{v}^t \cdot \underline{\nabla}) \langle \underline{b} \rangle + \underline{G}(\underline{v}', \underline{b}') \end{aligned} \quad (2.20)$$

and:

$$\begin{aligned} \frac{\partial \underline{v}_i}{\partial t} = (\langle \underline{b} \rangle \cdot \underline{\nabla}) \underline{b}^t + (\underline{b}^t \cdot \underline{\nabla}) \langle \underline{b} \rangle - \underline{\nabla} (\langle \underline{b} \rangle \cdot \underline{b}^t) \\ - (\langle \underline{v} \rangle \cdot \underline{\nabla}) \underline{v}^t - (\underline{v}^t \cdot \underline{\nabla}) \langle \underline{v} \rangle + \underline{F}(\underline{v}', \underline{b}') \end{aligned} \quad (2.21)$$

where $\underline{F}$ and $\underline{G}$ are known functions. From the above two equations, by integrating with respect to time, we obtain the following functional derivatives:

$$\left\langle \frac{\delta b_j^1(\underline{y})}{\delta b_k^t(\underline{z})} \right\rangle = \frac{1}{2} \left\{ \delta(\underline{y} - \underline{z}) \frac{\partial \langle v_j^1(\underline{y}) \rangle}{\partial y_k} - \langle v_q(\underline{y}) \rangle \frac{\partial}{\partial y_q} [\delta(\underline{y} - \underline{z}) \delta_{jk}] \right\} \quad (2.22)$$

$$\left\langle \frac{\delta b'_j(\underline{y})}{\delta v_k^t(\underline{z})} \right\rangle = \frac{1}{2} \left\{ \langle b_q(\underline{y}) \rangle \frac{\partial}{\partial y_q} [\delta(\underline{y}-\underline{z}) \delta_{jk}] - \left[\frac{\partial}{\partial y_j} \delta(\underline{y}-\underline{z}) \right] \langle b_k(\underline{y}) \rangle \right\} \quad (2.23)$$

$$\left\langle \frac{\delta v'_j(\underline{y})}{\delta b_k^t(\underline{z})} \right\rangle = \frac{1}{2} \left\{ \langle b_q(\underline{y}) \rangle \frac{\partial}{\partial y_q} [\delta(\underline{y}-\underline{z}) \delta_{jk}] + \frac{\partial \langle b_j(\underline{y}) \rangle}{\partial y_k} \delta(\underline{y}-\underline{z}) - \frac{\partial \langle b_k \rangle}{\partial y_j} \delta(\underline{y}-\underline{z}) - \langle b_k \rangle \frac{\partial}{\partial y_j} [\delta(\underline{y}-\underline{z})] \right\} \quad (2.24)$$

$$\left\langle \frac{\delta v'_j(\underline{y})}{\delta v_k^t(\underline{z})} \right\rangle = \frac{1}{2} \left\{ -\langle v_q \rangle \frac{\partial}{\partial y_q} [\delta(\underline{y}-\underline{z}) \delta_{jk}] - \frac{\partial \langle v_j \rangle}{\partial y_k} \delta(\underline{y}-\underline{z}) \right\} \quad (2.25)$$

From the Novikov-Furutsu formula we have that:

$$\begin{aligned} \langle v_i^t(\underline{x}) v'_j(\underline{y}) \rangle &= \int \langle v_i^t(\underline{x}) v_k^t(\underline{z}) \rangle \left\langle \frac{\delta v'_j(\underline{y})}{\delta v_k^t(\underline{z})} \right\rangle d^3 \underline{z} \\ &+ \int \langle v_i^t(\underline{x}) b_k^t(\underline{z}) \rangle \left\langle \frac{\delta v'_j(\underline{y})}{\delta b_k^t(\underline{z})} \right\rangle d^3 \underline{z} \end{aligned} \quad (2.26)$$

which, upon using the functional derivatives and integrating (using the fact that both $\langle \underline{v} \rangle$ and $\langle \underline{b} \rangle$ are solenoidal) gives:

$$\begin{aligned} \langle v_i^t(\underline{x}) v'_j(\underline{y}) \rangle &= -\frac{1}{2} \langle v_i^t(\underline{x}) v_q^t(\underline{y}) \rangle \frac{\partial \langle v_j(\underline{y}) \rangle}{\partial y_q} \\ &+ \frac{1}{2} \langle v_i^t(\underline{x}) b_q^t(\underline{y}) \rangle \frac{\partial \langle b_j(\underline{y}) \rangle}{\partial y_q} \end{aligned} \quad (2.27)$$

Similarly we obtain:

$$\begin{aligned} \langle v_i^t(\underline{x}) b'_j(\underline{y}) \rangle &= \frac{1}{2} \left\{ \langle v_i^t(\underline{x}) b_q^t(\underline{y}) \rangle \frac{\partial \langle v_j(\underline{y}) \rangle}{\partial y_q} \right. \\ &\quad \left. - \langle v_i^t(\underline{x}) v_q^t(\underline{y}) \rangle \frac{\partial \langle b_j(\underline{y}) \rangle}{\partial y_q} \right\} \end{aligned} \quad (2.28)$$

and:

$$\begin{aligned} \langle b_i^t(x) v_j^t(y) \rangle = \frac{1}{2} \left\{ \langle b_i^t(x) b_q^t(y) \rangle \frac{\partial \langle b_j(y) \rangle}{\partial y_q} \right. \\ \left. - \langle b_i^t(x) v_q^t(y) \rangle \frac{\partial \langle v_j(y) \rangle}{\partial y_q} \right\} \end{aligned} \quad (2.29)$$

and finally:

$$\begin{aligned} \langle b_i^t(x) b_j^t(y) \rangle = \frac{1}{2} \left\{ \langle b_i^t(x) b_q^t(y) \rangle \frac{\partial \langle v_j(y) \rangle}{\partial y_q} \right. \\ \left. - \langle b_i^t(x) v_q^t(y) \rangle \frac{\partial \langle b_j(y) \rangle}{\partial y_q} \right\} \end{aligned} \quad (2.30)$$

By analogy with equation (1.18) we now construct the correlators of the driving fields as follows:

$$\begin{aligned} \langle v_i^t(x) v_j^t(y) \rangle = \phi(r) \delta_{ij} + \epsilon_{ijk} g(r) r_k \\ + B(r) r_i r_j \end{aligned} \quad (2.31)$$

$$\begin{aligned} \langle b_i^t(x) b_j^t(y) \rangle = \mu(r) \delta_{ij} + \epsilon_{ijk} f(r) r_k \\ + D(r) r_i r_j \end{aligned} \quad (2.32)$$

$$\begin{aligned} \langle b_i^t(x) v_j^t(y) \rangle = \psi(r) \delta_{ij} + \epsilon_{ijk} h(r) r_k \\ + E(r) r_i r_j \end{aligned} \quad (2.33)$$

where $\underline{r} = \underline{x} - \underline{y}$. We note that incompressibility is satisfied if we choose:

$$\lim_{r \rightarrow 0} \frac{\partial Q(r)}{\partial r_i} = 0 \quad Q \in \{\phi, \mu, \psi, f, g, h, B, D, E\} \quad (2.34)$$

We have also that the kinematic helicity density γ , the magnetic helicity density γ_m and $\langle \underline{v}^t \cdot \underline{j}^t \rangle$ are given by:

$$\gamma = \langle \underline{v}^t \cdot \underline{\omega}^t \rangle = 6g(0) \quad (2.35)$$

$$\gamma_m = \langle \underline{b}^t \cdot \underline{\nabla} \wedge \underline{b}^t \rangle = \frac{4\pi}{c} \langle \underline{b}^t \cdot \underline{j}^t \rangle = 6\mu(0) \quad (2.36)$$

$$\langle \underline{v}^t \cdot \underline{j}^t \rangle = \frac{4\pi}{c} \langle \underline{v}^t \cdot \underline{\nabla} \wedge \underline{b}^t \rangle = 6\gamma(0) \quad (2.37)$$

We note that $\langle \underline{v}^t \cdot \underline{\nabla} \times \underline{b}^t \rangle = \langle \underline{b}^t \cdot \underline{\omega}^t \rangle$. This is consistent with the result obtained by taking the divergence of equation (2.4) (Ohm's Law) in the non-relativistic limit. If the turbulent magnetic field is purely helical, then we have that:

$$\langle \underline{v}^t \cdot \underline{\nabla} \wedge \underline{b}^t \rangle = \left\langle \frac{\gamma_m \gamma_c}{b^2} \right\rangle \quad (2.38)$$

where $\gamma_c = \langle \underline{v}^t \cdot \underline{b}^t \rangle$ is the cross-helicity density. The physical consequence of assuming that $\langle \underline{v}^t \cdot \underline{j}^t \rangle \neq 0$ will be discussed later.

The kinetic and magnetic energies and the cross-helicity γ_c are given by:

$$K.E = \frac{1}{2} \langle v^{t^2} \rangle = \frac{3\phi(0)}{2} \quad (2.39)$$

$$\text{magnetic energy} = \frac{1}{2} \langle b^{t^2} \rangle = \frac{3\mu(0)}{2} \quad (2.40)$$

$$\gamma_c = \langle \underline{v}^t \cdot \underline{b}^t \rangle = \frac{3\gamma(0)}{2} \quad (2.41)$$

We have also that:

$$\lim_{r \rightarrow 0} \frac{\partial C_{ij}}{\partial r_k} = \epsilon_{ijk} a(0) \quad (2.42)$$

where C_{ij} is either $\langle \underline{v}^t(\underline{x}) \underline{v}^t(\underline{x}+\underline{r}) \rangle$, $\langle \underline{b}^t(\underline{x}) \underline{b}^t(\underline{x}+\underline{r}) \rangle$ or $\langle \underline{b}^t(\underline{x}) \underline{v}^t(\underline{x}+\underline{r}) \rangle$ and $a(0)$ then equals $g(0)$, $f(0)$ or $h(0)$ respectively. It is now possible to determine the correlators in equations (2.18) and (2.19). We have:

$$\langle (\underline{v}' \cdot \underline{\nabla}) \underline{b}^t \rangle = \lim_{\underline{x} \rightarrow \underline{y}} \frac{\partial}{\partial x_j} \langle b_i^t(\underline{x}) v_j'(\underline{y}) \rangle \quad (2.43)$$

so that:

$$\begin{aligned} \langle (\underline{v}' \cdot \underline{\nabla}) \underline{b}^t \rangle = \lim_{r \rightarrow 0} \frac{1}{2} \left\{ \frac{\partial \langle b_j \rangle}{\partial x_q} \frac{\partial \langle b_i^t(\underline{x}) b_q^t(\underline{y}) \rangle}{\partial r_j} \right. \\ \left. - \frac{\partial \langle v_j \rangle}{\partial x_q} \frac{\partial \langle b_i^t(\underline{x}) v_q^t(\underline{y}) \rangle}{\partial r_j} \right\} \end{aligned} \quad (2.44)$$

which, using equation (2.42) gives:

$$\langle (\underline{v}' \cdot \underline{\nabla}) \underline{b}^t \rangle = \frac{1}{2} \left\{ f(0) \underline{\nabla} \wedge \langle \underline{b} \rangle - h(0) \underline{\nabla} \wedge \langle \underline{v} \rangle \right\} \quad (2.45)$$

Similarly we obtain:

$$\langle (\underline{v}^t \cdot \underline{\nabla}) \underline{b}' \rangle = \frac{1}{2} \left\{ \mu(0) \nabla^t \langle \underline{v} \rangle - \phi(0) \nabla^t \langle \underline{b} \rangle \right\} \quad (2.46)$$

and:

$$\langle (\underline{b}' \cdot \underline{\nabla}) \underline{v}^t \rangle = \frac{1}{2} \left\{ h(0) \underline{\nabla} \wedge \langle \underline{v} \rangle - g(0) \underline{\nabla} \wedge \langle \underline{b} \rangle \right\} \quad (2.47)$$

and finally:

$$\langle (\underline{b}^t \cdot \underline{\nabla}) \underline{v}' \rangle = \frac{1}{2} \left\{ \mu(0) \nabla^t \langle \underline{b} \rangle - \chi(0) \nabla^t \langle \underline{v} \rangle \right\} \quad (2.48)$$

Inserting equations (2.45) to (2.48) into (2.18) gives:

$$\begin{aligned} \frac{\partial \langle \underline{b} \rangle}{\partial t} = & \left[\eta + \frac{\phi + \mu}{2} \right] \nabla^2 \langle \underline{b} \rangle - \gamma \nabla^2 \langle \underline{v} \rangle \\ & - \frac{1}{2} (g + f) \nabla \wedge \langle \underline{b} \rangle + h \nabla \wedge \langle \underline{v} \rangle \end{aligned} \quad (2.49)$$

where we have omitted the argument (0).

In order to determine the equation of motion for the mean velocity, we need to evaluate the correlators:

$$\langle v_i^t(\underline{x}) v_j'(\underline{y}) \rangle \quad (2.50)$$

and:

$$\langle b_i^t(\underline{x}) b_j'(\underline{y}) \rangle \quad (2.51)$$

Using the correlators (2.32), (2.33) and (2.34) yields:

$$\langle (\underline{v}^t \cdot \nabla) \underline{v}' \rangle = \frac{1}{2} \{ \gamma(0) \nabla^2 \langle \underline{b} \rangle - \phi(0) \nabla^2 \langle \underline{v} \rangle \} \quad (2.52)$$

and:

$$\langle (\underline{v}' \cdot \nabla) \underline{v}^t \rangle = \frac{1}{2} \{ h(0) \nabla \wedge \langle \underline{b} \rangle - g(0) \nabla \wedge \langle \underline{v} \rangle \} \quad (2.53)$$

$$\langle (\underline{b}' \cdot \nabla) \underline{b}^t \rangle = \frac{1}{2} \{ f(0) \nabla \wedge \langle \underline{v} \rangle - h(0) \nabla \wedge \langle \underline{b} \rangle \} \quad (2.54)$$

$$\langle (\underline{b}^t \cdot \nabla) \underline{b}' \rangle = \frac{1}{2} \{ \mu(0) \nabla^2 \langle \underline{v} \rangle - \gamma(0) \nabla^2 \langle \underline{b} \rangle \} \quad (2.55)$$

Inserting these expressions into equation (2.19) gives:

$$\begin{aligned} \frac{\partial \langle \underline{v} \rangle}{\partial t} = & \left[\nu + \frac{\phi + \mu}{2} \right] \nabla^2 \langle \underline{v} \rangle + \left(\frac{f + g}{2} \right) \nabla \wedge \langle \underline{v} \rangle \\ & - \gamma(0) \nabla^2 \langle \underline{b} \rangle - \nabla (\langle \pi \rangle + \langle \underline{b}' \cdot \underline{b}^t \rangle) \\ & - h \nabla \wedge \langle \underline{b} \rangle \end{aligned} \quad (2.56)$$

By taking the curl of equations (2.49) and (2.56), we are in a position to investigate the joint formation of current and vortex structures. We have:

$$\begin{aligned} \frac{\partial}{\partial t} (\nabla \wedge \langle \underline{b} \rangle) &= \eta'' \nabla^2 (\nabla \wedge \langle \underline{b} \rangle) - \nu \nabla^2 \langle \underline{\omega} \rangle \\ &\quad - \left(\frac{g+f}{2} \right) \nabla \wedge (\nabla \wedge \langle \underline{b} \rangle) + h \nabla \wedge \langle \underline{\omega} \rangle \end{aligned} \quad (2.57)$$

and also:

$$\begin{aligned} \frac{\partial}{\partial t} \langle \underline{\omega} \rangle &= \nu'' \nabla^2 \langle \underline{\omega} \rangle - \nu \nabla^2 (\nabla \wedge \langle \underline{b} \rangle) \\ &\quad + \left(\frac{g+f}{2} \right) \nabla \wedge \langle \underline{\omega} \rangle - h \nabla \wedge (\nabla \wedge \langle \underline{b} \rangle) \end{aligned} \quad (2.58)$$

where we have introduced the quantities:

$$\eta'' = \eta + \frac{\phi + \mu}{2} \quad ; \quad \nu'' = \nu + \frac{\phi + \mu}{2} \quad (2.59)$$

Equations (2.57) and (2.58) constitute a pair of linear, coupled differential equations. This coupling occurs due to the non-zero correlation between the turbulent velocity and magnetic fields. Again we note that the transport coefficients are enhanced by the total energy (both magnetic and kinetic) stored in the turbulent field.

The kinematic helicity and the magnetic helicity can be combined to give an effective helicity γ_e , where:

$$\gamma_e = \frac{1}{2}(g+f) \quad (2.60)$$

If we denote the Fourier transform of $\langle \underline{\omega} \rangle$ and $\langle \nabla \times \underline{b} \rangle$ by $\underline{\Omega}$ and $\underline{J}$ respectively, then, upon Fourier transforming equations (2.56) and (2.57), we obtain:

$$\begin{bmatrix} -\kappa^2 & i\eta\kappa & \lambda_1 & -i\gamma_e\kappa \\ -i\eta\kappa & -\kappa^2 & i\gamma_e\kappa & \lambda_1 \\ \lambda_2 & \gamma_e i\kappa & -\kappa^2 & -i\eta\kappa \\ -\gamma_e i\kappa & \lambda_2 & i\eta\kappa & -\kappa^2 \end{bmatrix} \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ J_1 \\ J_2 \end{bmatrix} = \underline{0} \quad (2.61)$$

where we have written:

$$\begin{aligned} \lambda_1 &= -i\omega + \eta''\kappa^2 \\ \lambda_2 &= -i\omega + \nu''\kappa^2 \end{aligned} \quad (2.62)$$

If we assume that there is no correlation between the turbulent velocity and magnetic fields, then the situation becomes particularly simple. For consistency, we would then require that:

$$\begin{vmatrix} 0 & 0 & \lambda_1 & -\gamma_e i\kappa \\ 0 & 0 & +\gamma_e i\kappa & \lambda_1 \\ \lambda_2 & +\gamma_e i\kappa & 0 & 0 \\ -\gamma_e i\kappa & \lambda_2 & 0 & 0 \end{vmatrix} = 0 \quad (2.63)$$

which implies that either:

$$\begin{vmatrix} \lambda_1 & -\gamma_e i\kappa \\ \gamma_e i\kappa & \lambda_1 \end{vmatrix} = 0 \quad (2.64)$$

or:

$$\begin{vmatrix} \lambda_2 & +\gamma_e i\kappa \\ -\gamma_e i\kappa & \lambda_2 \end{vmatrix} = 0 \quad (2.65)$$

which lead to the dispersion relations:

$$\omega = i(\gamma_e\kappa - \eta''\kappa^2) \quad (2.66)$$

and:

$$\omega = i(\gamma_e k^2 - \nu'' k^2) \quad (2.67)$$

The dispersion relation (2.67) reduces, in the non-magnetic limit, to the expression obtained by Moiseev et al.

It is interesting to note that even when the turbulent magnetic field possesses no magnetic helicity, large scale current structures may still grow, provided that the kinematic helicity is non-zero. Likewise, vortex structures may grow if the magnetic helicity is non-zero.

The equation we have obtained for the large scale magnetic field (for the case when there is no correlation between the turbulent velocity and magnetic fields) is:

$$\frac{\partial \langle \underline{b} \rangle}{\partial t} = \eta'' \nabla^2 \langle \underline{b} \rangle - \gamma_e \nabla \wedge \langle \underline{b} \rangle \quad (2.68)$$

This is the α -dynamo equation (with α replaced by $g(0)$), which has been derived by other workers using different methods. (See, for example, the book by Parker³²). In one method, used by Moffatt²⁷, the electric field, assuming the velocity field to be known, is expanded as:

$$E_i = \alpha_{ij} \langle b_j \rangle + \beta_{ijk} \frac{\partial \langle b_k \rangle}{\partial x_j} + \dots \quad (2.69)$$

Then, by setting:

$$\alpha_{ij} = \alpha \delta_{ij} \quad ; \quad \beta_{ijk} = \beta \epsilon_{ijk} \quad (2.70)$$

he obtains:

$$\frac{\partial \langle \underline{b} \rangle}{\partial t} = (\eta + \beta) \nabla^2 \langle \underline{b} \rangle + \alpha \nabla \wedge \langle \underline{b} \rangle \quad (2.71)$$

The problem with this method is to determine the value of α . In our work we have obtained the expression:

$$\alpha = -\frac{1}{2} (g(0) + f(0)) \quad (2.72)$$

which we can write as:

$$\alpha = -\frac{1}{2} \left[\langle \underline{v}^t \cdot \underline{\omega}^t \rangle \tau_v + \langle \underline{b}^t \cdot \underline{j}^t \rangle \tau_m \right] \quad (2.73)$$

where τ_m and τ_v are the correlation times for the turbulent magnetic and velocity fields respectively. We have, however, that:

$$\langle v^{t^2} \rangle = \frac{3\phi(0)}{\tau_v} ; \quad \langle b^{t^2} \rangle = \frac{3\mu(0)}{\tau_m} \quad (2.74)$$

so that we can write:

$$\alpha = -\frac{1}{2} \left[\frac{\langle \underline{v}^t \cdot \underline{\omega}^t \rangle 3\phi(0)}{\langle v^{t^2} \rangle} + \frac{\langle \underline{b}^t \cdot \underline{j}^t \rangle 3\mu(0)}{\langle b^{t^2} \rangle} \right] \quad (2.75)$$

If we assume equipartition of energy between the turbulent magnetic and velocity fields, then we can write equation (2.75) as:

$$\alpha = -\frac{3}{2} \left[\frac{\langle \underline{v}^t \cdot \underline{\omega}^t \rangle \mu(0) + \langle \underline{b}^t \cdot \underline{j}^t \rangle \mu(0)}{\langle b^{t^2} \rangle} \right] \quad (2.76)$$

The first term in the brackets is the expression for α obtained by Keinigs³³, whilst both terms have been derived as possible expressions for α by Matthaeus *et al*³⁴., with the exception that $3\phi/2$ and $3\mu/2$ replace the molecular viscosity and magnetic diffusivity.

If we set the cross-helicity equal to zero, then the equation for the current density is:

$$\frac{\partial \underline{j}}{\partial t} = \eta'' \nabla^2 \underline{j} - \gamma_e \underline{\nabla} \wedge \underline{j} \quad (2.77)$$

Thus current structures will form in the wavenumber range:

$$0 \leq k \leq \frac{\gamma_e}{\eta''} \quad (2.78)$$

with a maximum growth rate $\gamma_{\max}$ occurring at a wavenumber k_0 , where:

$$k_0 = \frac{\gamma_e}{2\eta''} ; \quad \gamma_{\max} = \frac{\gamma_e^2}{4\eta''} \quad (2.78)$$

The "natural" length scale associated with the turbulent magnetic field is:

$$\frac{\langle b^2 \rangle}{\langle \underline{b} \cdot \underline{\nabla} \wedge \underline{b} \rangle} = \frac{\mu(0)}{2f(0)} \quad (2.79)$$

so that we have:

$$N_{\text{mag}} = \frac{\text{length scale of turbulent field}}{\text{" " " current structures}} = \frac{\mu}{2f} \frac{\gamma_e}{\eta''} \quad (2.80)$$

which is bounded by:

$$N_{mag} \leq \frac{|f+g|}{|f|} \left(\frac{1}{1 + \frac{2\eta}{\mu}} \right) \quad (2.81)$$

We wish this ratio to be as small as possible, in order that the structures are much larger than the scales over which the turbulent field changes. This will occur in systems where the turbulent diffusivity is much less than the molecular magnetic diffusivity, and also for systems where the kinematic and magnetic helicity have the opposite sign.

If we retain the cross-helicity, but set the kinematic and magnetic helicity to zero, and neglect all the damping terms, we obtain the equations:

$$\begin{aligned} \frac{\partial}{\partial t} \nabla \wedge \langle \underline{b} \rangle &= -\gamma \nabla^2 \langle \underline{\omega} \rangle \\ \frac{\partial}{\partial t} \langle \underline{\omega} \rangle &= -\gamma \nabla^2 (\nabla \wedge \langle \underline{b} \rangle) \end{aligned} \quad (2.82)$$

which lead to:

$$\begin{aligned} \frac{\partial^2 \langle \underline{\omega} \rangle}{\partial t^2} &= \gamma^2 \nabla^4 \langle \underline{\omega} \rangle \\ \frac{\partial^2 \langle \underline{j} \rangle}{\partial t^2} &= \gamma^2 \nabla^4 \langle \underline{j} \rangle \end{aligned} \quad (2.83)$$

which gives the dispersion relation:

$$\omega = i\gamma k^2 \quad (2.84)$$

This implies that vorticity can grow on all scales, due to the presence of cross-helicity between the turbulent velocity and magnetic fields. The largest growth rates occur on the smallest scales.

Finally, if we neglect all the damping terms, and retain

only the terms involving $\langle \underline{v}^t \cdot \nabla \times \underline{b}^t \rangle$, then we obtain:

$$\begin{aligned} \frac{\partial}{\partial t} (\nabla \wedge \langle \underline{b} \rangle) &= h \nabla \wedge \langle \underline{\omega} \rangle \\ \frac{\partial \langle \underline{\omega} \rangle}{\partial t} &= -h \nabla \wedge (\nabla \wedge \langle \underline{b} \rangle) \end{aligned} \quad (2.85)$$

These may be combined to give the wave equations:

$$\frac{\partial^2 \langle \underline{\omega} \rangle}{\partial t^2} = h^2 \nabla^2 \langle \underline{\omega} \rangle ; \quad \frac{\partial^2 \langle \underline{j} \rangle}{\partial t^2} = h^2 \nabla^2 \langle \underline{j} \rangle \quad (2.86)$$

We see thus that $h(0)$ has the dimensions of velocity. We have that:

$$h = \langle \underline{v}^t \cdot \nabla \wedge \underline{b}^t \rangle \tau \sim \langle v^t v_a^t \rangle \frac{\tau}{l} \quad (2.87)$$

where τ is the typical correlation time, v_a^t is the Alfven velocity associated with the turbulent magnetic field, and l is the length scale over which the magnetic field changes. An order of magnitude estimate thus suggests that $h(0)$ is approximately the Alfven velocity.

In this chapter we have found that both vortices and current structures may grow in a fluid which is driven by turbulent fields possessing either kinematic or magnetic helicity. The coefficients of viscosity and magnetic diffusivity are enhanced by the presence of the total energy (both kinetic and magnetic) of the turbulent fields.

CHAPTER THREE.

HELICITY AND THE PASSIVE CONVECTION OF A SCALAR FIELD.

We now move on to discuss the convection of a passive scalar field, in this case the temperature, by the fluid.

The flow of heat across a fluid element's surface $d^2\underline{S}$ per unit time is $\underline{h} \cdot d^2\underline{S}$, where $\underline{h}$ is the heat flux vector. This must equal, in the absence of viscosity, the rate of change of heat flowing out of the fluid element, which is $d_t(\rho c \theta)$, where c is the heat capacity of the fluid, θ is the temperature, and d_t is the convective derivative, namely:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underline{v} \cdot \underline{\nabla} \quad (3.1)$$

and signifies differentiation with respect to time in a frame which is co-moving with the fluid.

If we assume that the temperature gradients are not too large, then we can put $\underline{h} = -k \underline{\nabla} \theta$, where k is the thermal conductivity.

Integrating throughout the volume of the fluid gives:

$$\int \frac{d}{dt} (\rho c \theta) d^3\underline{r} = \int k \underline{\nabla} \theta \cdot d^2\underline{S} \quad (3.2)$$

which, using Gauss's theorem, gives:

$$\left(\frac{\partial}{\partial t} + \underline{v} \cdot \underline{\nabla} \right) \theta = \chi \nabla^2 \theta \quad (3.3)$$

where we have introduced the thermal diffusivity χ , defined by:

$$\chi = \frac{\kappa}{\rho c} \quad (3.4)$$

The effect of viscosity is to introduce a heat generation term due to the presence of stresses within the fluid. In the book by Landau and Lifshitz⁷, for example, it is shown that equation (3.3) must be modified so that:

$$\left(\frac{\partial}{\partial t} + \underline{v} \cdot \underline{\nabla} \right) \theta = \chi \nabla^2 \theta + \frac{\nu}{2c} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right)^2 \quad (3.5)$$

We proceed in exactly the same manner as in chapters 1 and 2, and decompose the velocity and temperature fields into three parts, a turbulent Gaussian-correlated part, the mean part and an interaction part, so that, e.g.,

$$\theta = \langle \theta \rangle + \theta^t + \theta' \quad (3.6)$$

By averaging equation (3.3) we obtain an equation for the mean temperature $\langle \theta \rangle$, which is:

$$\begin{aligned} \partial_t \langle \theta \rangle + \langle (\underline{v}^t \cdot \underline{\nabla}) \theta' + (\underline{v}' \cdot \underline{\nabla}) \theta^t \rangle \\ = \chi \nabla^2 \langle \theta \rangle + \frac{2\nu}{c} \left\langle \frac{\partial v_i^t}{\partial x_k} \left(\frac{\partial v_i'}{\partial x_k} + \frac{\partial v_k'}{\partial x_i} \right) \right\rangle \end{aligned} \quad (3.7)$$

We have also the equation for the interaction part of the temperature, obtained by inserting equation (3.6) into equation (3.3) and simplifying using equation (3.7). The resulting expression is:

$$\begin{aligned} \partial_t \theta' = & -v_i^t \partial_i \langle \theta \rangle - \langle v_i \rangle \partial_i \theta^t \\ & + \frac{2\nu}{c} \left(\frac{\partial \langle v_i \rangle}{\partial x_k} + \frac{\partial \langle v_k \rangle}{\partial x_i} \right) \frac{\partial v_i^t}{\partial x_k} \\ & + H(\underline{v}', \theta') \end{aligned} \quad (3.8)$$

where $H(\underline{v}', \theta')$ is a known function.

We can use equation (3.8) to obtain the functional derivatives:

$$\left\langle \frac{\delta \theta_i(\underline{y})}{\delta \theta^t(\underline{z})} \right\rangle = -\frac{1}{2} \langle v_i(\underline{y}) \rangle \frac{\partial}{\partial y_i} \delta(\underline{y} - \underline{z}) \quad (3.9)$$

and:

$$\begin{aligned} \left\langle \frac{\delta \theta'(\underline{y})}{\delta v_q^t(\underline{z})} \right\rangle = & \frac{1}{2} \left\{ -\delta(\underline{y} - \underline{z}) \frac{\partial \langle \theta(\underline{y}) \rangle}{\partial y_q} \right. \\ & \left. + \frac{2\nu}{c} \left(\frac{\partial \langle v_q \rangle}{\partial y_k} + \frac{\partial \langle v_k \rangle}{\partial y_q} \right) \frac{\partial \delta(\underline{y} - \underline{z})}{\partial y_k} \right\} \quad (3.10) \end{aligned}$$

From the Novikov-Furutsu formula we have that:

$$\begin{aligned} \langle v_i^t(\underline{x}) \theta'(\underline{y}) \rangle = & \int \langle v_i^t(\underline{x}) v_q^t(\underline{z}) \rangle \left\langle \frac{\delta \theta'(\underline{y})}{\delta v_q^t(\underline{z})} \right\rangle d^3 \underline{z} \\ & + \int \langle v_i^t(\underline{x}) \theta^t(\underline{z}) \rangle \left\langle \frac{\delta \theta'(\underline{y})}{\delta \theta^t(\underline{z})} \right\rangle d^3 \underline{z} \end{aligned} \quad (3.11)$$

which, using equations (3.9) and (3.10) becomes:

$$\begin{aligned} \langle v_i^t(\underline{x}) \theta'(\underline{y}) \rangle = & -\frac{1}{2} \langle v_i^t(\underline{x}) v_q^t(\underline{y}) \rangle \frac{\partial \langle \theta(\underline{y}) \rangle}{\partial y_q} \\ & + \frac{\nu}{c} \langle v_i^t(\underline{x}) v_q^t(\underline{y}) \rangle \frac{\partial^2 \langle v_q \rangle}{\partial y_k^2} \end{aligned} \quad (3.12)$$

However, we know that:

$$\langle (\underline{v}^t, \underline{\nabla}) \theta' \rangle = \lim_{\underline{y} \rightarrow \underline{x}} \frac{\partial}{\partial y_i} \langle v_i^t(\underline{x}) \theta'(\underline{y}) \rangle \quad (3.13)$$

and so we need only specify the correlator $\langle \underline{v}^t(\underline{x}) \underline{v}^t(\underline{y}) \rangle$ in order to obtain $\langle \underline{v}^t \cdot \underline{\nabla} \theta \rangle$.

We shall again treat the fluid as being inhomogeneous, and hence adopt the same correlator as used in chapter one, namely:

$$\begin{aligned} \langle v_i^t(\underline{x}) v_j^t(\underline{y}) \rangle = & \phi(\underline{R}, \underline{S}) \delta_{ij} + \epsilon_{ijk} g(\underline{R}, \underline{S}) R_k \\ & + D(\underline{R}, \underline{S}) R_i R_j \\ & + \lambda^2 C(\underline{R}, \underline{S}) S_i S_j \end{aligned} \quad (3.14)$$

Using this expression in equation (3.13) yields:

$$\begin{aligned} \langle (\underline{v}^t \cdot \underline{\nabla}) \theta' \rangle = & -\frac{1}{2} \phi_0 \nabla^2 \langle \theta \rangle \\ & + \lambda^2 C(\underline{x}) \left[\frac{\underline{v}}{c} \lambda^2 \underline{x} \cdot \hat{\underline{q}} \nabla^2 \langle \underline{v} \rangle \right. \\ & \left. - \frac{1}{2} \hat{\underline{q}}^2 \langle \theta \rangle \right] \end{aligned} \quad (3.15)$$

If we again employ the Novikov-Furutsu formula, we obtain:

$$\begin{aligned} \langle \theta^t(\underline{x}) v_i^t(\underline{y}) \rangle = & \int \langle \theta^t(\underline{x}) \theta^t(\underline{z}) \rangle \left\langle \frac{\delta v_i^t(\underline{y})}{\delta \theta^t(\underline{z})} \right\rangle d^3 \underline{z} \\ & + \int \langle \theta^t(\underline{x}) v_j^t(\underline{z}) \rangle \left\langle \frac{\delta v_i^t(\underline{y})}{\delta v_j^t(\underline{z})} \right\rangle d^3 \underline{z} \end{aligned} \quad (3.16)$$

However, from the Navier-Stokes equation, we know that:

$$\left\langle \frac{\delta v_i^t(\underline{y})}{\delta \theta^t(\underline{z})} \right\rangle = 0 \quad (3.17)$$

and also that:

$$\begin{aligned} \left\langle \frac{\delta v_i^t(\underline{y})}{\delta v_j^t(\underline{z})} \right\rangle = & -\frac{1}{2} \left\{ \delta(\underline{y} - \underline{z}) \frac{\partial \langle v_i \rangle}{\partial y_j} \right. \\ & \left. + \delta_{ij} \langle v_k \rangle \frac{\partial}{\partial y_k} \delta(\underline{y} - \underline{z}) \right\} \end{aligned} \quad (3.18)$$

We need now to specify the correlator $\langle \theta^t(\underline{x}) v^t(\underline{y}) \rangle$. We again perform a power series expansion and set:

$$\begin{aligned} \langle \theta^t(\underline{x}) v^t(\underline{y}) \rangle = & a(\underline{R}, \underline{S}) R_i + \lambda b(\underline{R}, \underline{S}) S_i \\ & + \lambda \epsilon_{ijk} (R_j S_k - R_k S_j) d(\underline{R}, \underline{S}) \end{aligned} \quad (3.19)$$

We have the incompressibility condition to use, namely:

$$\frac{\partial}{\partial y_i} \langle \theta^t(\underline{x}) v^t(\underline{y}) \rangle = 0 \quad (3.20)$$

and we again restrict the inhomogeneities that are being considered to those for which:

$$\lim_{R \rightarrow 0} \frac{\partial Q(\underline{R}, \underline{S})}{\partial R_i} = 0 \quad Q = \{a, b, d\} \quad (3.21)$$

We must have:

$$3a(\underline{x}) + \lambda(3b + (\underline{x} \cdot \underline{\nabla})b) = 0 \quad (3.22)$$

and again, for simplicity, we make the following choices:

$$a(\underline{x}) = 0 \quad (3.23)$$

and:

$$3b(\underline{x}) + (\underline{x} \cdot \underline{\nabla})b(\underline{x}) = 0 \quad (3.24)$$

We can write this latter condition in the form:

$$\underline{\nabla} \cdot (b(\underline{x}) \underline{x}) = 0 \quad (3.25)$$

so that we must have:

$$b(\underline{x})\underline{x} = \nabla \wedge \underline{M}(\underline{x}) \quad (3.26)$$

for some vector field $\underline{M}$.

If we make the further assumption that $b(\underline{x})$ depends only on the magnitude of $\underline{x}$, then we can solve equation (3.24) to obtain:

$$b(r) = \frac{b_0}{r^3} \quad (3.27)$$

In order to interpret $b(\underline{x})$ and $d(\underline{x})$ physically, we note that:

$$\langle \theta^t(\underline{x}) \underline{v}^t(\underline{x}) \rangle = \lambda b(\underline{x})\underline{x} \quad (3.28)$$

and also:

$$\begin{aligned} \langle \theta^t(\underline{x}) \underline{\omega}^t(\underline{x}) \rangle &= \frac{\lambda}{2} \left[\nabla b \wedge \underline{x} - 3d(\underline{x})\underline{x} \right] \\ &= \frac{\lambda}{2} \left[\nabla \wedge (\nabla \wedge \underline{M}) - 3d(\underline{x})\underline{x} \right] \end{aligned} \quad (3.29)$$

so that $b(\underline{x})$ models the correlation between the temperature and the velocity, whilst $d(\underline{x})$ contributes to the correlation between the temperature and the vorticity. The $d(\underline{x})$ contribution to the correlator (3.29) is along the "line of sight" from the origin to $\underline{x}$, whilst the $b(\underline{x})$ term is perpendicular to it. This latter term will be zero if $b = b(|\underline{x}|)$. The fact that $\langle \theta^t(\underline{x}) \underline{v}^t(\underline{x}) \rangle$ is along the line of sight is a consequence of the assumptions

(3.21), (3.22) and (3.23). It shows the importance of the choice of origin, and the special nature of the inhomogeneity.

It is interesting to note that the one-point correlation between the temperature gradient and the velocity is:

$$\left\langle \frac{\partial \theta^t}{\partial x_j} v_i^t(\underline{x}) \right\rangle = \frac{\lambda}{2} \left[2\epsilon_{ikj} x_k d(\underline{x}) + \frac{\partial b}{\partial x_j} x_i + b \delta_{ij} \right] \quad (3.30)$$

The longitudinal part is given by:

$$\langle (\underline{v}^t \cdot \underline{\nabla}) \theta^t \rangle = 0 \quad (3.31)$$

We can construct a length scale from $\langle \underline{v}^t(\underline{x}) \theta^t(\underline{x}) \rangle$ and $\langle \underline{\omega}^t(\underline{x}) \theta^t(\underline{x}) \rangle$ which is:

$$\frac{|\langle \underline{v}^t(\underline{x}) \theta^t(\underline{x}) \rangle|}{|\langle \underline{\omega}^t(\underline{x}) \theta^t(\underline{x}) \rangle|} = \frac{2}{3} \frac{|b(\underline{x})|}{|d(\underline{x})|} \quad (3.32)$$

and is a length scale typifying the turbulent fields.

If we insert equations (3.17) and (3.18) into equation (3.16) and integrate, we obtain:

$$\langle \theta^t(\underline{x}) v_i^t(\underline{y}) \rangle = -\frac{1}{2} \langle \theta^t(\underline{x}) v_j^t(\underline{y}) \rangle \frac{\partial \langle v_i(\underline{y}) \rangle}{\partial y_j} \quad (3.33)$$

However, we have that:

$$\langle (\underline{v}^t \cdot \underline{\nabla}) \theta^t \rangle = \lim_{\underline{x} \rightarrow \underline{y}} \frac{\partial}{\partial x_i} \langle \theta^t(\underline{x}) v_i^t(\underline{y}) \rangle \quad (3.34)$$

which, upon using equation (3.19) yields:

$$\langle (\underline{v}' \cdot \underline{\nabla}) \theta^t \rangle = \frac{\lambda}{2} \left[\underline{\nabla} b \cdot \hat{q} \langle \underline{v} \rangle - d(\underline{x}) \underline{x} \cdot \langle \underline{\omega} \rangle \right] \quad (3.35)$$

From Chapter 1 (equation (1.17)), we know that:

$$\langle v_i^t(\underline{x}) v_k^1(\underline{y}) \rangle = -\frac{1}{2} \langle v_i^t(\underline{x}) v_j^t(\underline{y}) \rangle \frac{\partial \langle v_k \rangle}{\partial y_j} \quad (3.36)$$

and we can use this to find the correlator:

$$\left\langle \frac{\partial v_i^t}{\partial x_k} \frac{\partial v_j^1}{\partial x_p} \right\rangle = \lim_{\underline{x} \rightarrow \underline{y}} \frac{\partial}{\partial x_k} \frac{\partial}{\partial y_p} \langle v_i^t(\underline{x}) v_j^1(\underline{y}) \rangle \quad (3.37)$$

which we can write as:

$$\left\langle \frac{\partial v_i^t}{\partial x_k} \frac{\partial v_j^1}{\partial x_p} \right\rangle = \lim_{\underline{x} \rightarrow \underline{y}} -\frac{1}{2} \left[\frac{\partial^2 K_{ia}}{\partial x_k \partial y_p} \frac{\partial \langle v_j \rangle}{\partial x_a} + \frac{\partial K_{ia}}{\partial x_k} \frac{\partial^2 \langle v_j \rangle}{\partial y_p \partial y_a} \right] \quad (3.38)$$

This can be expanded using the correlator (3.14). The result is:

$$\begin{aligned} \left\langle \frac{\partial v_i^t}{\partial x_k} \frac{\partial v_j^1}{\partial x_p} \right\rangle = & -\frac{1}{2} \left\{ \left[\left(\epsilon_{iqk} \frac{\partial g}{\partial x_p} - \epsilon_{iap} \frac{\partial g}{\partial x_k} \right) \right. \right. \\ & + \lambda^2 C(\underline{x}) (\delta_{ip} \delta_{qk} + \delta_{ik} \delta_{ap}) \\ & + \lambda^2 \frac{\partial C}{\partial x_k} (\delta_{ip} x_q + x_i \delta_{ap}) \\ & + \lambda^2 \frac{\partial C}{\partial x_q} (\delta_{ik} x_p + x_i \delta_{ap}) \\ & \left. \left. - D(\underline{x}) (\delta_{ip} \delta_{qk} + \delta_{ik} \delta_{ap}) \right] \frac{\partial \langle v_j \rangle}{\partial x_a} \right. \\ & + \left[\epsilon_{iqk} g(\underline{x}) + \lambda^2 \frac{\partial C}{\partial x_k} x_i x_q \right. \\ & \left. \left. + \lambda^2 C (x_i \delta_{kp} + x_p \delta_{ik}) \right] \frac{\partial^2 \langle v_j \rangle}{\partial x_p \partial x_a} \right\} \quad (3.39) \end{aligned}$$

We can put $j=i$, $p=k$ in the above, and then $j=k$, $p=i$, and hence obtain the equation:

$$\left\langle \frac{\partial v_i^t}{\partial x_k} \left(\frac{\partial v_i^1}{\partial x_k} + \frac{\partial v_k^1}{\partial x_i} \right) \right\rangle = -\frac{1}{2} \left\{ \frac{\partial g}{\partial x_k} \langle \omega \rangle_k + \lambda^2 C x_k \nabla^2 \langle v_k \rangle \right. \\ \left. + \lambda^2 \frac{\partial C}{\partial x_k} \left(5 \hat{q} \langle v_k \rangle + 2 x_i (1 + \hat{q}) S_{ki} \right) \right\} \quad (3.40)$$

where we have introduced the dilation tensor, namely:

$$S_{ki} = \frac{\partial \langle v_k \rangle}{\partial x_i} + \frac{\partial \langle v_i \rangle}{\partial x_k} \quad (3.41)$$

We have thus obtained the equation describing the evolution of the mean temperature of the fluid, which is:

$$\left[\partial_t - \left(\chi + \frac{\Phi_0}{2} \right) \right] \nabla^2 \langle \theta \rangle = \frac{\lambda^2}{2} \left[\hat{q}^2 \langle \theta \rangle - 2 \frac{\nu}{\epsilon} \underline{x} \cdot (\hat{q} \nabla^2 \langle \underline{v} \rangle) \right] \\ + \frac{\lambda^2}{2} \left[d(\underline{x}) \underline{x} \cdot \langle \underline{\omega} \rangle - \nabla b \cdot (\hat{q} \langle \underline{v} \rangle) \right] \\ - \frac{\nu}{\epsilon} \left[\nabla g \cdot \langle \underline{\omega} \rangle + \lambda^2 C(\underline{x}) \underline{x} \cdot (\nabla^2 \langle \underline{v} \rangle) \right. \\ \left. + \lambda^2 \nabla C \cdot [5 \hat{q} \langle \underline{v} \rangle + 2 \underline{x} (1 + \hat{q}) \underline{S}] \right] \quad (3.42)$$

We note that again the transport coefficient is enhanced due to the energy stored in the turbulent velocity field.

Equation (3.42) can be simplified if we assume that the viscous heating effect is negligible, and that the turbulent velocity and temperature fields are uncorrelated. We obtain:

$$\partial_t \langle \theta \rangle = \chi' \nabla^2 \langle \theta \rangle + \lambda^2 C(\underline{x}) x_i x_j \frac{\partial^2 \langle \theta \rangle}{\partial x_i \partial x_j} \quad (3.43)$$

$(\chi' = \chi + \Phi_0/2)$

In the homogeneous limit, this reduces to the diffusion equation.

Equation (3.43) is particularly interesting if we assume that $C(\underline{x})$ depends only on $|\underline{x}|$. In this case, we have that:

$$C(\underline{r}) = \frac{C_0}{r^4} \quad (3.44)$$

It is now possible to seek solutions for $\langle \theta(\underline{x}, t) \rangle$ of the form:

$$\langle \theta(\underline{x}, t) \rangle = T(t)R(r) \quad (3.45)$$

where $r = |\underline{x}|$.

Substituting equations (3.44) and (3.45) into equation (3.43) yields:

$$(\chi' r^2 + \lambda^2 C_0) \frac{d^2 R}{dr^2} + 2\chi' r \frac{dR}{dr} + \alpha^2 r^2 R = 0 \quad (3.46)$$

and, the time-dependent term obeys:

$$T = T_0 e^{-\alpha^2 t} \quad (3.47)$$

As is expected, the solution to equation (3.46) is, in the homogeneous limit, the zeroth order spherical Bessel function. The second order (in λ) correction to $R(r)$ is given by the solution to:

$$\frac{d^2 R_2}{dr^2} + \frac{2}{r} \frac{dR_2}{dr} + R_2 = -\frac{\alpha^2 C_0}{\chi'^2 r^2} \frac{d^2}{dr^2} j_0(r) \quad (3.48)$$

We note, however, that we have generated solutions to the mean temperature equation which depend only on r . The physical significance of this is that whilst the underlying turbulent velocity and temperature fields might be inhomogeneous, it is still possible to get isotropic behaviour of the mean fields.

In this chapter we have obtained the equation of motion of the mean temperature of the fluid. The thermal diffusivity is enhanced due to the turbulent motion. In the homogeneous case when viscous damping may be neglected, we obtain the diffusion equation. It is still possible to obtain isotropic behaviour of the mean field even if the underlying turbulent fluid is inhomogeneous.

CHAPTER FOUR

FORCED MHD IN THE PRESENCE OF AN EXTERNAL MAGNETIC FIELD.

In this chapter, we discuss forced MHD in the presence of an external magnetic field. Much work on this subject of a numerical nature has recently been undertaken^{19,20,21}, but as yet a simple interpretation of the main observed features has been lacking.

The Navier-Stokes and magnetic induction equations can be written as:

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \nabla) \underline{v} = \underline{j} \wedge \underline{B} - \nabla \pi + \nu \nabla^2 \underline{v} \quad (4.1)$$

and:

$$\frac{\partial \underline{B}}{\partial t} = \nabla \wedge (\underline{v} \wedge \underline{B}) + \eta \nabla^2 \underline{B} \quad (4.2)$$

If we introduce dimensionless variables, ν and η are the inverse Reynolds and magnetic Reynolds numbers respectively. We shall now consider two-dimensional flow, so that we may assume that there is no dependence on the z coordinate. The fluid velocity is obtained from a stream function ψ , according to:

$$\underline{v} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0 \right) \quad (4.3)$$

Slight care has to be taken with the magnetic induction

B. This comprises of two terms; a constant field along the x-axis, and a turbulent field b, so that:

$$\underline{B} = B_0 \hat{\underline{L}} + \underline{b} \quad (4.4)$$

and the turbulent component is derivable from a vector potential say a, according to:

$$\underline{b} = \underline{\nabla} \wedge (a \hat{\underline{k}}) \quad (4.5)$$

where i and k are unit vectors along the x and z axes respectively.

From (4.3) and (4.5), it can be seen that the vorticity ω and the current j are of the form (0,0,ω) and (0,0,j), where:

$$\omega = -\nabla^2 \psi ; j = -\nabla^2 a \quad (4.6)$$

Taking the curl of (4.1) and "removing" the curl from (4.2) gives:

$$\frac{\partial \omega}{\partial t} + (\underline{v} \cdot \underline{\nabla}) \omega = (\underline{b} \cdot \underline{\nabla}) j + \nu \nabla^2 \omega + B_0 \frac{\partial j}{\partial x} + F \quad (4.7)$$

and:

$$\frac{\partial a}{\partial t} + (\underline{v} \cdot \underline{\nabla}) a = \eta \nabla^2 a + B_0 \frac{\partial \psi}{\partial x} + G \quad (4.8)$$

where stochastic driving forces F and G have been introduced.

We assume that the induced magnetic field b is much

smaller in magnitude than the external field B_0 , and that all velocities are small as compared to the Alfvén velocity. In dimensionless units, these conditions become:

$$\left| \frac{v}{B_0} \right| \lesssim \left| \frac{b}{B_0} \right| \ll 1 \quad (4.9)$$

By considering the relative magnitudes of the terms in (4.7) and (4.9), great simplification can be achieved, since:

$$\left| \frac{(\underline{b} \cdot \nabla) j}{B_0 \partial j / \partial x} \right| \sim \frac{b j / L}{B_0 j / L} \sim \frac{b}{B_0} \ll 1 \quad (4.10)$$

and:

$$\left| \frac{(\underline{v} \cdot \nabla) a}{B_0 \partial a / \partial x} \right| \sim \frac{a / L}{B_0} \sim \frac{b}{B_0} \ll 1 \quad (4.11)$$

as well as:

$$\left| \frac{(\underline{v} \cdot \nabla) \omega}{B_0 \partial j / \partial x} \right| \sim \frac{v^2}{B_0 b} \sim \left(\frac{B_0}{b} \right) \left(\frac{v}{B_0} \right)^2 \ll 1 \quad (4.12)$$

where it has been tacitly assumed that the length scales over which the magnetic and velocity fields change are of the same order of magnitude. If this assumption is not made, the condition:

$$\frac{L_{\text{mag}}}{L_{\text{vel}}} \lesssim 1$$

needs to be imposed to preserve (4.12)

Using (4.10), (4.11) and (4.12) in (4.7) and (4.8) gives:

$$\frac{\partial \omega}{\partial t} = \nu \nabla^2 \omega + B_0 \frac{\partial j}{\partial x} + F \quad (4.13)$$

and:

$$\frac{\partial a}{\partial t} = \eta \nabla^2 a + B_0 \frac{\partial \psi}{\partial x} + G \quad (4.14)$$

These two coupled linear equations can be solved in terms of the properties of the driving fields. By introducing the Fourier transform of a dynamical variable $Q(\underline{x}, t)$ by:

$$Q(\underline{k}, t) = \int e^{-i\underline{k} \cdot \underline{x}} Q(\underline{x}, t) d^2 \underline{x} \quad (4.15)$$

equations (4.13) and (4.14) become (using a dot to denote partial derivative with respect to time):

$$\dot{\omega}_k = -\nu k^2 \omega_k + ik_x B_0 j_k + F_k \quad (4.16)$$

with:

$$\dot{a}_k = -\eta k^2 a_k + ik_x B_0 \psi_k + G_k \quad (4.17)$$

By Fourier transforming (4.6) however, we have:

$$j_k = k^2 a_k ; \quad \omega_k = k^2 \psi_k$$

We obtain, therefore, the equations:

$$\dot{a}_k = -\eta k^2 a_k + ik_x B_0 \psi_k + G_k \quad (4.18)$$

and:

$$\dot{\psi}_k = -\nu k^2 \psi_k + i k_x B_0 a_k + H_k \quad (4.19)$$

for the two potentials, with $H_k = k^{-2} F_k$.

By introducing the (complex) vector $\underline{x}$ by:

$$\underline{x} = \begin{pmatrix} a \\ \psi \end{pmatrix} \quad (4.20)$$

it is possible to cast the above equations in the following form:

$$\dot{x}_i = L_{ij} x_j + S_i \quad (4.21)$$

where:

$$\underline{S} = \begin{pmatrix} G \\ H \end{pmatrix} ; \quad \underline{L} = \begin{bmatrix} -\eta k^2 & i k_x B_0 \\ i k_x B_0 & -\nu k^2 \end{bmatrix} \quad (4.22)$$

If we note that $\underline{S}$ is a stochastic driving term and L is a constant matrix, it can be seen that (4.21) is nothing more than the vectorial form of the Langevin equation. It is possible to write this in a Fokker-Planck form³⁵.

Equation (4.21) suggests a method to account (partially) for the neglected non-linear terms, by the introduction of an "eddy viscosity". This would mean that $v \rightarrow v + v_1(t)$, so that (4.21) becomes a stochastic differential equation, whose solution depends on the autocorrelation time, i.e. the time τ such that $\langle v(t) v(t'+\tau) \rangle \approx 0$ for $t-t' \gg \tau$. Physical intuition would suggest that the eddy viscosity would be auto-correlated on quite a short timescale, so that

standard methods might be used to obtain a series solution for $\underline{x}(t)$ in terms of $\tau^{3/2}$.

In order to solve the simpler problem where the Reynolds number is time independent, we introduce the eigenvalues and eigenvectors of the matrix L through:

$$\underline{L} \underline{y}_n = -\lambda_n \underline{y}_n \quad (4.23)$$

and then expand all variables as:

$$\underline{x} = \sum C_n(t) \underline{y}_n ; \quad \underline{s} = \sum \sigma_n(t) \underline{y}_n \quad (4.24)$$

We note that since L is not Hermitian, the eigenvalues are not real and the eigenvectors are not orthogonal. The expansion may be performed, however, since the eigenvectors form a linearly independent complete set.

Substitution of (4.23) and (4.24) in (4.21) gives:

$$\frac{dC_n}{dt} = -\lambda_n C_n + \sigma_n \quad (4.25)$$

which has the solution:

$$C_n(t) = C_n(0) e^{-\lambda_n t} + e^{-\lambda_n t} \int_0^t e^{\lambda_n t'} \sigma_n(t') dt' \quad (4.26)$$

To restrict our attention to systems in which the velocity and magnetic potentials are initially zero (which is the case in most computer experiments), we require $C_n(0)=0$.

The one-time correlation function is:

$$\begin{aligned} \langle C_n(t) C_m^*(t) \rangle &= e^{-(\lambda_n + \lambda_m^*)t} \\ &\times \left\{ \int_0^t \int_0^t e^{\lambda_n t'} e^{\lambda_m^* t''} \langle \sigma_n(t') \sigma_m^*(t'') \rangle dt' dt'' \right\} \end{aligned} \quad (4.27)$$

Should the external field fluctuate rapidly, then we can assume that:

$$\langle \sigma_n(t') \sigma_m^*(t'') \rangle = \langle \sigma_n \sigma_m^* \rangle \delta(t' - t'') \quad (4.28)$$

which gives:

$$\langle C_n(t) C_m^*(t) \rangle = \frac{\langle \sigma_n \sigma_m^* \rangle}{\lambda_n + \lambda_m^*} \left[1 - e^{-(\lambda_n + \lambda_m^*)t} \right] \quad (4.29)$$

However, from (4.24), we know that:

$$\langle x_i(t) x_j^*(t) \rangle = \langle C_n(t) C_m^*(t) \rangle \Lambda_{ij}^{nm} \quad (4.30)$$

where:

$$\Lambda_{ij}^{nm} = (y_i)_n (y_j^*)_m \quad (4.31)$$

From (4.29) and (4.30) it is possible to see that saturation must occur since:

$$\langle x_i(0) x_j^*(0) \rangle = 0 \quad (4.32)$$

and:

$$\langle x_i(\infty) x_j^*(\infty) \rangle = \frac{\langle \sigma_n \sigma_m^* \rangle}{\lambda_n + \lambda_m^*} \Lambda_{ij}^{nm} \quad (4.33)$$

Moreover, it is now possible to find an analytic

expression for the power spectrum $A_{ij}(\underline{k}, \omega)$. By definition, we have that:

$$A_{ij}(\underline{k}, \omega) = \int_0^\infty dt \langle X_i(t) X_j^*(t) \rangle e^{-i\omega t}$$

so inserting (4.30) gives:

$$A_{ij}(\underline{k}, \omega) = [\langle \sigma_n \sigma_m^* \rangle \Delta_{ij}^{nm}] \times \left[\frac{e^{-i\omega t}}{-i\omega} + \frac{e^{-(\lambda_n + \lambda_m^* + i\omega)t}}{\lambda_n + \lambda_m^* + i\omega} \right]_0^\infty \quad (4.34)$$

In the presence of either magnetic diffusivity or kinematic viscosity, λ_n will have a real part, so we have that:

$$A_{ij}(\underline{k}, \omega) = \frac{i\alpha(\omega)}{\omega} + \frac{\langle \sigma_n \sigma_m^* \rangle \Delta_{ij}^{nm}}{\lambda_n + \lambda_m^* + i\omega} \quad (4.35)$$

where:

$$\alpha(\omega) = \langle \sigma_n \sigma_m^* \rangle \Delta_{ij}^{nm} \lim_{T \rightarrow \infty} \left[e^{-i\omega t} \right]_0^T \quad (4.36)$$

so that the power spectrum has a peak at $\omega=0$ and $\omega = i(\lambda_n + \lambda_m^*)$.

To be more specific, we now proceed to evaluate the eigenvalues of L:

$$\begin{bmatrix} -\eta k^2 & i\omega_A \\ i\omega_A & -\nu k^2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = -\lambda \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad (4.37)$$

where ω_A is the Alfvén frequency ($= \underline{k} \cdot \underline{E}_0$).

This gives:

$$\lambda_{\pm} = \frac{(\eta + \nu)k^2 \pm \sqrt{(\eta - \nu)^2 k^2 - 4\omega_A^2}}{2} \quad (4.38)$$

which we write as:

$$\lambda_{\pm} = \frac{(\eta + \nu)k^2 \pm i2\mu(k)}{2} \quad (4.39)$$

The (damped) peaks in the spectrum are therefore at:

$$\omega = i(\nu + \eta)k^2 \quad (4.40)$$

$$\omega = i(\nu + \eta)k^2 \pm 2\mu(k)$$

For the case of unit Prandtl number, that is, when $\nu = \eta$, we have $\mu(k) = \pm 2\omega_A$ so that the poles will occur at $\omega = 0$ and $\omega = \pm 2\omega_A$, which is in excellent agreement with the numerical experiments of Hussain et al²¹.

The eigenvectors are given by:

$$(\lambda_{\pm} - \eta k^2) y_{1\pm} = -i\omega_A y_{2\pm} \quad (4.41)$$

so that:

$$y_{\pm} = \beta_{\pm} \begin{pmatrix} -i\omega_A \\ \lambda_{\pm} - \eta k^2 \end{pmatrix} \quad (4.42)$$

where the $\beta_{\pm}$ are normalisation constants.

We can now evaluate $\sigma_{\pm}$, since:

$$\underline{S} = \sigma_n \underline{y}_n$$

and so

$$\begin{pmatrix} G \\ H \end{pmatrix} = \sigma_+ \beta_+ \begin{pmatrix} -i\omega_A \\ \lambda_+ - \eta k^2 \end{pmatrix} + \sigma_- \beta_- \begin{pmatrix} -i\omega_A \\ \lambda_- - \eta k^2 \end{pmatrix} \quad (4.43)$$

Define ζ by:

$$\zeta = \frac{(\nu - \eta)k^2}{2} \quad (4.44)$$

so that we can write:

$$\begin{pmatrix} G \\ H \end{pmatrix} = \sigma_+ \beta_+ \begin{pmatrix} -i\omega_A \\ \zeta + \mu \end{pmatrix} + \sigma_- \beta_- \begin{pmatrix} -i\omega_A \\ \zeta - \mu \end{pmatrix} \quad (4.45)$$

which gives:

$$\beta_{\pm} \sigma_{\pm} = \frac{\mp [(\zeta \mp \mu)G + i\omega_A H]}{2\omega_A \mu} \quad (4.46)$$

From now on, we shall consider only the case for which $\nu = \eta$ which is used in the numerical experiments^{19,20,21}.

This means that:

$$\sigma_{\pm} \beta_{\pm} = \frac{i(G \mp H)}{2\omega_A} \quad (4.47)$$

Equation (4.20) defines:

$$\langle |a(k, t)|^2 \rangle = \langle |x_1(k, t)|^2 \rangle \quad (4.48)$$

which, upon expanding gives:

$$\begin{aligned} \langle |a(k, t)|^2 \rangle = & \frac{\langle |\sigma_+|^2 \rangle}{2\nu k^2} [1 - e^{-2\nu k^2 t}] \Delta_{11}^{++} \\ & + \frac{\langle |\sigma_-|^2 \rangle}{2\nu k^2} [1 - e^{-2\nu k^2 t}] \Delta_{11}^{--} \\ & + 2 \operatorname{Re} \left\{ \frac{\langle \sigma_+ \sigma_-^* \rangle}{2(\nu k^2 + i\omega_A)} (1 - e^{-2\nu k^2 t}) e^{i2\omega_A t} \Delta_{11}^{+-} \right\} \end{aligned} \quad (4.49)$$

Inserting (4.47) in (4.49) yields:

$$\begin{aligned} \langle |a(\underline{k}, t)|^2 \rangle = & \frac{\langle |G|^2 + |H|^2 \rangle}{2} \left\{ \frac{1 - e^{-2\nu k^2 t}}{2\nu k^2} \right. \\ & \left. + \frac{\nu k^2 - e^{-2\nu k^2 t} (\nu k^2 \cos 2\omega_A t + \omega_A \sin 2\omega_A t)}{(\nu k^2)^2 + \omega_A^2} \right\} \end{aligned} \quad (4.50)$$

To obtain an expression for $\langle |\psi|^2 \rangle$, we simply exchange H and G in the above. It can thus be seen that $\langle |\psi|^2 \rangle = \langle |a|^2 \rangle$, and hence equipartition of energy applies at all times. This is a consequence of having a Prandtl number exactly equal to unity. The equilibrium value obtained for the magnetic potential is:

$$\langle |a|^2 \rangle_{eqm} = \frac{\langle |G|^2 + |H|^2 \rangle}{2} \left\{ \frac{1}{2\nu k^2} + \frac{\nu k^2}{(\nu k^2)^2 + \omega_A^2} \right\} \quad (4.51)$$

which is clearly bounded by:

$$\frac{1}{4\nu k^2} \langle |G|^2 + |H|^2 \rangle \leq \langle |a|^2 \rangle_{eqm} \leq \frac{3 \langle |G|^2 + |H|^2 \rangle}{4\nu k^2} \quad (4.52)$$

so that the equilibrium value is not particularly sensitive to the external magnetic field, a fact born out in the computer experiments.

By making use of Parseval's theorem, it is possible to obtain an instantaneous value for the total energy, since:

$$\langle u \rangle = \frac{1}{2} \int d^3 \underline{x} (\nu^2 + B^2) = \frac{1}{2} \int d^3 \underline{k} k^2 \langle |a|^2 + |\psi|^2 \rangle \quad (4.53)$$

which gives the following bounds for the energy:

$$\int d^2k \frac{\langle |G|^2 + |H|^2 \rangle}{8\nu} \leq \langle u \rangle \leq \int d^2k \frac{3\langle |G|^2 + |H|^2 \rangle}{8\nu} \quad (4.54)$$

It is interesting to note that in the inviscid Navier-Stokes limit, we have:

$$\langle |v(k,t)|^2 \rangle \rightarrow \frac{3}{2} \langle |H|^2 \rangle k^2 t \quad (4.55)$$

Finally in this chapter, we consider the effect of a broad-band (in wave-number space) driving force, on a two-dimensional non-conducting fluid by putting:

$$\begin{aligned} \langle |F(k)|^2 \rangle &= k^4 \langle |H|^2 \rangle = F_0^2 \quad (k_0 \leq k \leq k_1) \quad (4.56) \\ &= 0 \text{ elsewhere} \end{aligned}$$

The kinetic energy is simply:

$$E(t) = \frac{3\pi F_0^2}{4\nu} \int_{k_0}^{k_1} \frac{1 - e^{-2\nu k^2 t}}{k^4} d(k^2) \quad (4.57)$$

By transforming to the new coordinate:

$$u = 2\nu k^2 t \quad (4.58)$$

we can write (4.57) in terms of a standard integral, namely:

$$E(t) = \frac{3\pi F_0^2}{4\nu k_0^2} \left\{ [1 - E_2(u_0)] - \frac{k_0^2}{k_1^2} [1 - E_2(u_1)] \right\} \quad (4.59)$$

where:

$$E_2(z) = \int_1^\infty \frac{e^{-zt}}{t^2} dt \quad (4.60)$$

However, we also have that

$$E_2(z) = e^{-z} - zE_1(z) \quad (4.61)$$

where:

$$E_1(z) = \int_1^{\infty} \frac{e^{-zt}}{t} dt \quad (4.62)$$

is the exponential integral.

This leads to the following expression for the kinetic energy:

$$E(t) = \frac{3\pi F_0^2}{4\nu K_0^2} \left\{ \left[1 - e^{-t/\tau_0} + \frac{t}{\tau_0} E_1\left(\frac{t}{\tau_0}\right) \right] - \frac{K_0^2}{K_1^2} \left[1 - e^{-t/\tau_1} + \frac{t}{\tau_1} E_1\left(\frac{t}{\tau_1}\right) \right] \right\} \quad (4.63)$$

with:

$$\tau_0 = 2\nu K_0^2 \quad ; \quad \tau_1 = 2\nu K_1^2 \quad (4.64)$$

being the eddy turnover times for the largest and smallest driven eddies.

For times large as compared to τ_1 , we can employ the asymptotic expansion of $E_1(z)$, namely:

$$E_1(z) \sim \frac{e^{-z}}{z} \left(1 - \frac{1}{z} + O\left(\frac{1}{z^2}\right) \right) \quad (4.64)$$

to obtain:

$$E(t) \sim \frac{3\pi F_0^2}{4\nu K_0^2} \left\{ \left(1 - \frac{K_0^2}{K_1^2}\right) + \frac{\tau_0}{t} \left(e^{-t/\tau_1} - e^{-t/\tau_0}\right) \right\} \quad (4.65)$$

We can write equation (4.63) in the form:

$$E(t) = \frac{3\pi F_0^2}{4\nu K_0^2} \left\{ \left[1 - E_2\left(\frac{t}{\tau_0}\right)\right] - \delta \left[1 - E_2\left(\frac{\delta t}{\tau_0}\right)\right] \right\} \quad (4.66)$$

where we have introduced a measure of the bandwidth of the driving force, namely:

$$\delta = \frac{K_0^2}{K_1^2} = \frac{\tau_0^2}{\tau_1^2} \quad (4.67)$$

The equilibrium value for the energy is thus:

$$E_{eqm} = \frac{3\pi F_0^2}{4\nu K_0^2} (1 - \delta) \quad (4.68)$$

For broad-band driving, when $f \rightarrow 0$, we have that:

$$E = \frac{3\pi F_0^2}{4\nu K_0^2} \left\{ 1 - E_2\left(\frac{t}{\tau_0}\right) \right\} \quad (4.69)$$

This is the same expression as would have been obtained by setting:

$$\langle |F(k)|^2 \rangle = F_0^2 H(k - k_0) \quad (4.70)$$

where $H(k - k_0)$ is the Heaviside step function.

If we put:

$$F_0^2 K_0^2 (1 - \delta) = 1 \quad (4.71)$$

into equation (4.66) we obtain:

$$E(t) = \frac{3\pi}{4\nu K_0^4 (1 - \delta)} \left\{ (1 - \delta) - E_2(u_0) + \delta E_2(\delta u_0) \right\} \quad (4.72)$$

Taking the limit as $\delta \rightarrow 1$ then gives the expression which corresponds to putting:

$$\langle |F(k)|^2 \rangle = \delta(k - k_0) \quad (4.73)$$

which is:

$$E = \frac{3\pi}{4\nu K_0^4} \left(1 - e^{-2\nu K_0^2 t} \right) \quad (4.74)$$

This is of the same form as both the analytical and numerical expressions obtained by Hossain et al³⁹, although in their numerical experiments the value of δ was set to be 55/70.

Where this result differs from Hossain et al is in the value of the time constant. Their expression is:

$$E = E_\infty \left(1 - e^{-t/\tau} \right) \quad (4.75)$$

with a numerical fit of $\tau = 124$, corresponding to a wavenumber given by $k^2 \approx 2$, as opposed to the value of $k^2 = 55$ which the above theory would predict (as their forcing band is in the region $55 \leq k^2 \leq 70$). As the above theory is linear, it is interesting to speculate that the

presence of non-linear interactions preserve the shape of the kinetic energy, but shift the dominating wavenumber to larger scales.

The enstrophy $\Omega(t)$ is given by:

$$\Omega(t) = \int d^2k \quad k^2 \langle |v(k)|^2 \rangle \quad (4.76)$$

and so, using (4.56) and the Fourier transform of (4.6) we obtain:

$$\Omega(t) = \frac{3\pi F_0^2}{4\nu} \int_{k_0}^{k_1} \frac{1 - e^{-2\nu k^2 t}}{k^2} d(k^2) \quad (4.77)$$

which, when integrated, yields:

$$\Omega(t) = \frac{3\pi F_0^2}{4\nu} \left\{ \ln\left(\frac{k_0}{k_1}\right)^2 + E_1\left(\frac{t}{\tau_1}\right) - E_1\left(\frac{t}{\tau_0}\right) \right\} \quad (4.78)$$

where $E_1(z)$ is the same as in equation (4.62).

In terms of δ , this equation can be written as:

$$\Omega(t) = \frac{3\pi F_0^2}{4\nu} \left\{ \ln \delta + E_1(\delta u_0) - E_1(u_0) \right\} \quad (4.79)$$

so that for broad band driving we have:

$$\Omega(t) = \text{constant} = \lim_{\delta \rightarrow 0} \left(\frac{3\pi F_0^2}{4\nu} \ln \delta \right) \quad (4.80)$$

whilst when $\langle |F(k^2)| \rangle$ is the δ -function, we have:

$$\Omega(t) = \frac{3\pi}{4\nu k_0^2} \left\{ 1 - e^{-2\nu k_0^2 t} \right\} \quad (4.81)$$

Here we remark that in Hossain et al³⁹, when deriving a semi-analytical expression for the energy, assume that a time exists such that the enstrophy has reached a constant value whilst the energy continues to grow. Comparison of equations (4.74) and (4.81) show that this is not the case for this particular form of stirring.

In equilibrium, we have the expressions:

$$\Omega_{eqm} = \frac{3\pi F_0^2}{4\nu} \ln \delta \quad (4.74)$$

which will be zero for stirring at a single wavenumber, but unbounded for broad-band stirring.

In this chapter, we have used a linear model of two-dimensional MHD turbulence to explain the observed features of recent numerical work, namely the saturation of the magnetic potential, the insensitivity of the equilibrium value of the energy to the external magnetic field, and the prominent peaks in the power spectrum. One of the features which this model does not explain is the observed concentration at the smallest scales permitted in the computer experiments. This, we note, is hardly surprising, since linearisation involves the neglect of the non-linear terms, and it is precisely these which are mainly responsible for energy transfer between the various modes.

It also does not account for the rapid saturation followed by a gentle decay of the enstrophy which has been reported by some workers.

It was suggested that these terms could possibly be

accounted for by introducing an eddy viscosity. More work could also be done on the particular form of the driving force, to see whether the peaks in the power spectrum are driver-independent or not.

CONCLUSIONS.

In this thesis, the effect of a non-zero mean helicity on turbulent fluid systems has been discussed. The model used is based on the assumption that the fluid velocity, the magnetic field or the fluid temperature may be decomposed into a turbulent, mean and interaction component. For the velocity field, the turbulent driving component is δ -correlated in time, obeys Gaussian statistics and has weak inhomogeneities of a particular character. It is incompressible and has non-zero mean helicity. It has been shown that large scale (in comparison to the scale of the turbulent field) vortices will form, provided that the turbulent velocity field possesses a mean helicity. The viscosity is enhanced by the turbulent motion. In the homogeneous limit, the equation for the large scale vortex structures reduces to that obtained by Moiseev et al¹⁸. The difference between the length scale of the structures and that of the turbulent velocity field is largest when the viscosity enhancement is small.

For an electrically conducting fluid, the joint formation of vortices and current structures will occur, provided that one of the kinetic or magnetic helicities of the turbulent fields is non-zero. If there is no correlation between the driving velocity and magnetic fields, then the equation for the mean magnetic field is simply the α -dynamo equation, with a value of α consistent

with recent work. Enhancement of magnetic diffusivity occurs due to the total (both magnetic and kinetic) energy stored in the turbulent fields.

When the magnetic and kinetic helicities are zero, we obtain the wave equation for both the vorticity and current density. The speed of propagation depends on the correlation between the velocity and current density of the turbulent fields.

For the case of the temperature of the fluid, we find that the thermal diffusivity is enhanced. If there is no correlation between the driving temperature and velocity fields, we find that it is possible to obtain isotropic solutions for the mean temperature, even when the driving fields are inhomogeneous. In the homogeneous limit, the equation reduces to the diffusion equation.

It would appear that the closure scheme considered is quite successful and should be developed further. The three most important improvements that could be made are to generalise the nature of the inhomogeneities considered, account for the non-linear terms, and to consider the effect of non-Gaussian and non- δ -correlated driving fields. Unfortunately there appears to be no simple way to achieve the second objective, though it may be possible to make some progress with the last two. In order to do so, it may be best to write the equations of motion in terms of the characteristic functional of the turbulent fields. For the velocity field $\underline{v}^t$, this would obey (since it satisfies the Navier-Stokes equation) the equation obtained by Hopf³⁸. The approach of Furutsu may

then be followed. With a non-Gaussian correlator, it should be possible to construct a correlator such that the mean helicity of the turbulent field is zero, and yet the mean-square helicity to be non-zero. It may then be possible to investigate further the conjecture of Levich and Tsinober, namely that large scale vortices may still be formed even if the mean helicity is zero.

One way of generalising the nature of the inhomogeneities considered would be to remove the important restrictions of equations (1.24) and (3.21). More general expressions for $K_{ij}(\underline{x}, \underline{x})$ and $\Lambda_{ijq}(\underline{x})$ can then be used in equation (1.57a). This equation can also be used to describe the case of axisymmetric turbulence.

Another problem that still remains is that whilst we have found that large scale vortex and current structures may form, we are still unable to determine their lifetimes and show that they are long-lived. It is tempting to assume that, like many non-linear dynamical systems, saturation will occur, but this is not necessarily the case. This may only be resolved if a better approximation for the non-linear terms is found.

There are other questions which this work raises. Since we suppose that helicity produces coherent vortex patterns, if we are given a distribution of purely helical regions of fluid at some time t_0 , what will their distribution be at some later time, and what vortex pattern will they have generated? Before this can be answered, a model will have to be constructed to account for the interaction between two regions of different

helicity. This interaction will clearly be largest for regions of opposite helicity, and will cause a shear across the surfaces of these helical structures.

A model for fluid turbulence could then be developed in terms of an interacting ensemble of soliton-like vortex structures of a definite helicity.

The results of the fourth chapter are in very good agreement with experiment. The calculations predict saturation of both the magnetic and kinetic energy densities. These are of the same form as those obtained by Hossain et al²¹. We have also equipartition of energy. Our theory predicts the observed peaks in the power spectrum of the magnetic potential at twice the Alfvén frequency. It does not explain, however, the observed concentration of energy at the largest scales allowed. The explanation will not be forthcoming until the non-linear terms are taken into account, since it is precisely these terms which are responsible for the energy transfer between modes. The theory does not predict the observed decay of enstrophy. This may well be due to the fact that the driving forces considered in this work are different to those used, for simplicity, in numerical experiments

It is possible to develop the model by writing the non-linear terms as an "eddy viscosity". Then, by using the techniques for solving stochastic differential equations, expressions for the magnetic and kinetic energies may be found, in terms of the auto-correlation time of the eddy viscosity.

TABLE ONE.

SOME TYPICAL VALUES FOR THE REYNOLDS NUMBER.

For flow in a pipe, the transition to turbulence usually occurs in the range $2000 \leq Re \leq 2700$.

<u>SYSTEM.</u>	<u>u (ms⁻¹)</u>	<u>L (m)</u>	<u>ν (m²s⁻¹)</u>	<u>Re</u>
Aircraft wing	3×10^2	1×10^1	1.5×10^{-5}	2×10^7
Cricket ball	2×10^1	7×10^{-2}	1.5×10^{-5}	1×10^5
Ship	8×10^0	1×10^2	1.1×10^{-6}	7×10^8
Bacterium in water	2×10^0	1×10^{-4}	1.1×10^{-6}	2×10^2
Blood flow in aorta	1×10^0	2×10^{-2}	4.0×10^{-6}	5×10^3
Airflow around volcano ¹	1×10^1	2×10^4	1.5×10^{-5}	1×10^{10}

¹ The volcano, part of the Aleutian Island group, is 2518km high, and approximately 18km wide.

REFERENCES

1. See, for example, Landau L.D. and Lifshitz E.M. *Fluid Mechanics* (Pergamon Press 1959).
2. For both derivations, see Huang K. *Statistical Mechanics* (Wiley 1963).
3. Prandtl L. *Essentials of Fluid Mechanics* (Blackie and Son 1952).
4. For a discussion of magneto-static equilibria, see Bateman G. *MHD Instabilities* (M.I.T. Press 1978).
5. Lamb H. *Hydrodynamics* (Dover 1879)
6. Rayleigh Lord *Phil Mag* 6 xxvi 776 (1913)
7. Leslie D.C. *Developments in the Theory of Turbulence* (Clarendon Press 1973).
8. See Monin A.S. and Yaglom A.M. *Statistical Fluid Mechanics* (M.I.T. Press 1965).
9. Lovejoy S. in *Twentieth Conference on Radar Meteorology Am. Met. Soc.* 476-483 (1981)
10. Hussain A.K.M.F. *Phys. Fluids* 26 2816-2850 (1983).
11. Levich E. and Tsinober A. *Phys. Lett.* 93A 293-297 (1983)
13. Levich E. and Tzvetkov E. to appear in *Phys. Rep.*
14. Larsen H.F., Kelley H.C., and Cage K.S. *J. Atmos. Sci* 39 929-1126 (1982)
15. Levich E. and Tsinober A. *Phys Lett* 96A 292-297 (1983)
16. Tsuge S. *Phys. Fluids* 27 1370-1376 (1984)
17. Tsinober A. and Levich E. *Phys Lett* 99A 299-321 (1983)
18. Moiseev S.S., Sagdeev R.Z., Tur A.V., Homenko G.A. and Yanovski V.V. *Zh.E.T.F.* 85 1979-1987 (1983) in Russian.
19. Novikov E.A. *J.E.T.P.* 20 1290-1294 (1965)

20. Furutsu K. *J.O.S.A.* 62 240-254 (1972)
21. Levich E. *private communication.*
22. ter Haar D. *private communication.*
23. Moffatt H.K. to be published.
24. Tandberg-Hanssen E. *Solar Prominences* (Reidel 1974)
25. Moffatt H.K. *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge University Press 1978)
26. Browning P.K. and Priest E.R. *Ap. J.* 266 848-865 (1983)
27. Parker E.N. *Ap. J.* 191 245-254 (1974)
28. De Young D.S. *Ap. J.* 241 81-97 (1980)
29. For a full derivation, see Roberts P.H. *An Introduction to Magnetohydrodynamics* (Longmans 1967)
30. Kambe T. *Phys. Fluids* 28 2321-2323
31. Parker E.N. *Ap. J.* 162 665-673 (1970)