

# Weak Solutions to the Navier-Stokes Equations with Bounded Scale-invariant Quantities

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Gregory A. Seregin\*

## Abstract

The main assumption of the so-called  $\varepsilon$ -regularity theory of suitable weak solutions to the Navier-Stokes equations is uniform smallness of certain scale-invariant quantities, which rules out singularities. One of the best results of  $\varepsilon$ -regularity is the famous Caffarelli-Kohn-Nirenberg theorem. Our goal is to understand what happens if the assumption on smallness of scale-invariant quantities is replaced with their uniform boundedness. The latter makes it possible to use blow-up technique and reduce the local regularity problem to the question of existence or non-existence of “non-trivial” ancient (backward) solutions to the Navier-Stokes equations. There are at least two potential scenarios: the classical Liouville type problem for mild bounded ancient solutions and backward uniqueness for the Navier-Stokes equations. In this survey, we discuss sufficient conditions implying non-existence of “non-trivial” solutions and the corresponding sufficient conditions ensuring local regularity of original weak solutions.

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## 1. Introduction

One of the main problems of the mathematical hydrodynamics can be formulated as follows. Is the Cauchy problem, describing the flow of viscous incompressible fluids, *globally well-posed*? In other words, given a smooth divergence

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\*OxPDE, Mathematical Institute, University of Oxford, UK<sup>1</sup>.

E-mail: gaseregin@googlemail.com.

<sup>1</sup>on leave from POMI, St.Petersburg, Russia

free velocity field  $a$  compactly supported in  $\mathbb{R}^3$ , does the classical Navier-Stokes system

$$\begin{aligned}\partial_t v(x, t) + v(x, t) \cdot \nabla v(x, t) - \nu \Delta v(x, t) + \nabla q(x, t) &= 0, \\ \operatorname{div} v(x, t) &= 0\end{aligned}\tag{1.1}$$

have a unique solution subject to the initial condition

$$v(x, 0) = a(x), \quad x \in \mathbb{R}^3,\tag{1.2}$$

which is defined globally for all  $x \in \mathbb{R}^3$  and for all  $0 < t < +\infty$ ? Here, as usual,  $v$  and  $q$  stand for the velocity field and for the pressure field, respectively. In this paper, we are not going to study the very important issue how solutions depend on the viscosity  $\nu$  and let it equal to 1.

There are two different approaches to attack the above problem. In one of them, the Cauchy problem (1.1) and (1.2) can be reformulated as an integral equation by removing the non-linear term to the right hand side of the first equation in (1.1), by applying Leray's projector  $P$  to both sides of it, and by inverting then the linear part. As a result, the following equation with respect to a function  $v$  of time  $t$  with values in a Banach space appears

$$v(t) = S(t)a - \int_0^t S(t-\tau)P(v(\tau) \cdot \nabla v(\tau))d\tau.$$

Here,  $S(t)$  is the solution operator of the Cauchy problem for the heat equation. Any solution to the above integral equation is called a *mild* solution to the Cauchy problem (1.1) and (1.2). Existence and uniqueness of mild solutions can be proved with the help of contraction mappings. For history, details, and references, we recommend papers [14], [11], [2], and [18]. This approach is quite effective for proving local well-posedness for a wide range of initial data.

Another method gives *energy* solutions called nowadays *weak Leray-Hopf* solutions. They have been introduced by J. Leray in his pioneering paper [22] for the Cauchy problem and in a sense by E. Hopf in [13] for initial boundary value problems. The modern definition of weak solutions can be found, for example, in [19] and includes the following ingredients:

- (i) the velocity field  $v$  has finite kinetic energy and finite dissipation;
- (ii) the Navier-Stokes system is satisfied in the sense of distributions with divergence free test functions;
- (iii) the velocity field  $v$  is a continuous function of time  $t$  with values in the space  $L_2$  equipped with the weak topology;
- (iv) the initial data are fulfilled in the strong  $L_2$ -sense;

- (v) the velocity field  $v$  satisfies the global energy inequality for all possible values of  $t$ .

Not all the above properties are independent each of other. Choosing divergence free test functions in (ii), we exclude the pressure field from the definition completely.

So, we have a global weak Leray-Hopf solution to the classical problem but we do not know whether it is unique or not. However, as it has been observed and proved by J. Leray in [22], *any smooth solution to the classical Cauchy problem is unique in the class of weak solutions*. In other words, the problem of uniqueness of weak solutions can be posed as a more particular problem of their smoothness. The latter has been proposed as one of the seven Millennium problems in [10].

By definition, the space-time point  $z = (x, t)$  is called a *regular point of the velocity field*  $v$  if  $v$  is of class  $L_\infty$  in a parabolic vicinity with the center at  $z$ . The first moment of time  $T$  when singularities occur is called a *blowup time*. Further smoothing in a neighborhood of a regular point is straightforward and a simple consequence of the linear theory.

Our approach to the regularity problem is quite typical for the classical theory of partial differential equations, namely, we are going to study smoothness of weak solutions locally in space-time. Now, let us state the local regularity problem for the Navier-Stokes equations rigorously.

Consider the Navier-Stokes system (1.1) in a canonical domain, say, in the unit parabolic cylinder  $Q$  being the Cartesian product of the unit ball  $B$  of  $\mathbb{R}^3$  with the center at the origin and the time interval  $] - 1, 0[$ . More general cases can be reduced to the canonical one with the help of the space-time shift and the Navier-Stokes scaling

$$\begin{aligned} v^\lambda(y, s) &= \lambda v(x_0 + \lambda y, t_0 + \lambda^2 s), \\ q^\lambda(y, s) &= \lambda^2 q(x_0 + \lambda y, t_0 + \lambda^2 s). \end{aligned} \quad (1.3)$$

Our question is as follows. What are the weakest assumptions on  $v$  and  $q$  that provide regularity of  $v$  at the origin  $z = (x, t) = 0$ ? Ideally, they should be fulfilled for energy solutions but this is unknown and might be not necessary true.

Nowadays, it is well understood that the main object of the local regularity theory of the Navier-Stokes equations is the so-called suitable weak solutions. They were introduced in the middle 70s by V. Scheffer in the series of papers, see for example [25] and [26], where the importance of solutions satisfying the local energy inequality was pointed out and exploited. In the early 80s, the essential contribution to understanding suitable weak solutions in the context of the local regularity theory was made in the celebrated paper [1] by L. Caffarelli, R.-V. Kohn, and L. Nirenberg. However, in this survey, we are going to accept a more

particular but very much convenient version of the definition of suitable weak solutions given by F.-H. Lin in [23].

**Definition 1.1.** Functions  $v \in L_{2,\infty}(Q) \cap W_2^{1,0}(Q)$  and  $q \in L_{\frac{3}{2}}(Q)$  are said to be a suitable weak solution to the Navier-Stokes equations in  $Q$  if they satisfy (1.1) in  $Q$  in the sense of distributions and, for a.a.  $t \in ]-1, 0[$ , the local energy inequality

$$\int_B \varphi(x, t) |v(x, t)|^2 dx + 2 \int_{-1}^t \int_B \varphi |\nabla v|^2 dx dt' \leq \int_{-1}^t \int_B \left( |v|^2 (\partial_t \varphi + \Delta \varphi) + v \cdot \nabla \varphi (|v|^2 + 2q) \right) dx dt'$$

holds for any non-negative test function  $\varphi \in C_0^\infty(B \times ]-1, 1[)$ .

Here, the following notation for mixed Lebesgue and Sobolev spaces is used:  $L_{m,n}(Q) = L_n(-1, 0; L_m(B))$ ,  $L_{m,m} = L_m$ ,  $W_{m,n}^{1,0}(Q) = \{v, \nabla v \in L_{m,n}(Q)\}$ , and  $W_{m,m}^{1,0} = W_m^{1,0}$ .

It is not so difficult to show that among weak Leray-Hopf solutions to a given Cauchy problem (1.1) and (1.2), there is at least one with the following property. For any point  $z_0 = (x_0, t_0)$  with  $t_0 > 0$ , this solution  $v$ , together with the associated pressure  $q$ , satisfies all the requirements to be a suitable weak solution to the Navier-Stokes equations in  $z_0 + Q(R)$  for any  $R > 0$  subject to the restriction  $t_0 - R^2 > 0$ . By  $Q(R)$ , we denote a parabolic ball (cylinder) of  $\mathbb{R}^3 \times \mathbb{R}$  with radius  $R$  centered at the origin, i.e.,  $Q(R) = B(R) \times ]-R^2, 0[$  and  $B(R) \subset \mathbb{R}^3$  is a ball of radius  $R$  with the center at the origin.

Another relatively well-understood thing is the role of quantities invariant with respect to the Navier-Stokes scaling (1.3) with  $x_0 = 0$  and  $t_0 = 0$ . By the definition, such quantities are defined on parabolic balls  $Q(r)$  and have the property  $F(v, q; r) = F(v^\lambda, q^\lambda; r/\lambda)$ .

Now, we are in a position to explain the so-called  $\varepsilon$ -regularity theory for suitable weak solutions to the Navier-Stokes equations. There are two types of statements in it and the first one essentially proved in [1], see also [24], and reads:

*Suppose that  $v$  and  $q$  are a suitable weak solution to the Navier-Stokes equations in  $Q$ . There exist universal positive constants  $\varepsilon$  and  $c_k$ ,  $k = 0, 1, 2, \dots$  such that if  $F(v, q; 1) < \varepsilon$  then  $|\nabla^k v(0)| < c_k$ ,  $k = 0, 1, 2, \dots$ . Moreover, the function  $z \mapsto \nabla^k v(z)$  is Hölder continuous (relative to the usual parabolic metric) with any exponent less  $1/3$  in the closure of  $Q(1/2)$ .*

Here, it is an important example of such kind of quantities:

$$F(v, q; r) = \frac{1}{r^2} \int_{Q(r)} \left( |v|^3 + |q|^{\frac{3}{2}} \right) dz.$$

The limited smoothing in time cannot be improved. This can be easily seen from Serrin's example

$$\begin{aligned} v(x, t) &= C(t) \nabla h(x), \\ q(x, t) &= -C'(t)h(x) - 1/2 C(t) |\nabla h(x)|^2, \end{aligned}$$

in which  $h$  is a harmonic function of  $x$  and  $C$  is a given function of  $t$ .

In the other type of statements, it is supposed that our quantity  $F$  is independent of the pressure  $q$ :

*Let  $v$  and  $q$  be a suitable weak solution in  $Q$ . There exist a universal positive constant  $\varepsilon$  with the property: if  $\sup_{0 < r < 1} F(v; r) < \varepsilon$  then  $z = 0$  is a regular point. Moreover, for any  $k = 0, 1, 2, \dots$ , the function  $z \mapsto \nabla^k v(z)$  is Hölder continuous with any exponent less  $1/3$  in the closure of  $Q(r)$  for some positive  $r$ .*

Dependence on the pressure in the above statement is hidden. In fact, the radius  $r$  is determined by the  $L_{\frac{3}{2}}$ -norm of the pressure over the whole parabolic cylinder  $Q$ .

To illustrate the second statement, let us consider several examples. In the first one, we deal with the Ladyzhenskaya-Prodi-Serrin type quantities

$$F(v; r) = M_{s,l}(v; r) = \|v\|_{s,l,Q(r)}^l = \int_{-r^2}^0 \left( \int_{B(r)} |v|^s dx \right)^{\frac{l}{s}} dt$$

provided

$$\frac{3}{s} + \frac{2}{l} = 1$$

and  $s \geq 3$ . Local regularity results connected with those quantities have been proved partially by J. Serrin in [34] and then by M. Struwe in [35] for the velocity field  $v$  having finite energy even with no assumption on the pressure. However, in such a case, we lose Hölder continuity, see the above example.

The second kind of quantities will be called scaled energy quantities. Let us list some of them:

$$\begin{aligned} A(v; r) &= \sup_{-r^2 < t < 0} \frac{1}{r} \int_{B(r)} |v(x, t)|^2 dx, \\ C(v; r) &= \frac{1}{r^2} \int_{Q(r)} |v|^3 dz, \\ E(v; r) &= \frac{1}{r} \int_{Q(r)} |\nabla v|^2 dz, \\ D(q; r) &= \frac{1}{r^2} \int_{Q(r)} |q|^{\frac{3}{2}} dz. \end{aligned}$$

For more examples of scaled energy quantities, we refer to the paper [12]. It is interesting to note that the second statement applied to the scaled dissipation  $E$  is the famous Caffarelli-Kohn-Nirenberg theorem. It gives the best estimate for Hausdorff's dimension of the singular set for a class of weak Leray-Hopf solutions to the Cauchy problem. A sort of logarithmic improvement of the latter result is explained in [6]. A certain generalization of the Caffarelli-Kohn-Nirenberg theorem itself has been proved in [28] and is formulated as follows.

**Proposition 1.2.** *Let  $v$  and  $q$  be a suitable weak solution to the Navier-Stokes equations in  $Q$ . Given  $M > 0$ , there exists a positive number  $\varepsilon(M)$  having the property: if two inequalities  $\limsup_{r \rightarrow 0} E(r) < M$  and  $\liminf_{r \rightarrow 0} E(r) < \varepsilon(M)$  hold, then  $z = 0$  is a regular point of  $v$ .*

Typical examples of the third group of quantities invariant to the Navier-Stokes scaling are:

$$\begin{aligned} G_1(v; r) &= \sup_{z=(x,t) \in Q(r)} |x| |v(z)|, \\ G_2(v; r) &= \sup_{z=(x,t) \in Q(r)} \sqrt{-t} |v(z)|. \end{aligned}$$

A proof of the corresponding statements has been presented in [32], see also [36], [16], and [5] for similar results.

The question we are interested in is *what happens if we drop the condition on smallness of scale-invariant quantities, assuming their uniform boundedness only*, i.e.,  $\sup_{0 < r < 1} F(v, r) < +\infty$ . For Ladyzhenskaya-Prodi-Serrin type quantities with  $s > 3$ , the answer is still positive, i.e.,  $z = 0$  is a regular point. It follows from scale-invariance and the fact that the assumption  $M_{s,l}(v; 1) = \sup_{0 < r < 1} M_{s,l}(v; r) < +\infty$  implies  $M_{s,l}(v; r) \rightarrow 0$  as  $r \rightarrow 0$  if

$s > 3$ . Although in the marginal case  $s = 3$  and  $l = +\infty$  the answer remains positive, the known proof is more complicated and will be outlined later.

In this review, we shall discuss various approaches to the problem in question. Before going into details, let us recall certain definitions and make some general remarks about relationships between some scale-invariant quantities. Boundedness of  $\sup_{0 < r < 1} G_2(v; r) = G_2(v, 1) = G_{20} < +\infty$  can be rewritten in the form

$$|v(z)| \leq \frac{G_{20}}{\sqrt{-t}}$$

for all  $z = (x, t) \in Q$ . If  $v$  satisfies the above inequality and  $z = 0$  is still a singular point of  $v$ , we say that a *singularity of Type I* or *Type I blowup* takes place at  $t = 0$ . All other singularities are of Type II. The main feature of Type I singularities is that they have the same rate as potential self-similar solutions. The important properties connected with possible singularities of Type I have been proved in [31] and are as follows.

**Proposition 1.3.** *Let functions  $v$  and  $q$  be a suitable weak solution to the Navier-Stokes equations in  $Q$ .*

(i) *If  $\min\{G_1(v; 1), G_2(v; 1)\} < +\infty$ , then*

$$g = \sup_{0 < r < 1} \{A(v; r) + C(v; r) + D(q; r) + E(v; r)\} < +\infty.$$

(ii) *If*

$$g' = \min\left\{\sup_{0 < r < 1} A(v; r), \sup_{0 < r < 1} C(v; r), \sup_{0 < r < 1} E(v; r)\right\} < +\infty,$$

*then  $g < +\infty$ .*

This proposition admits many obvious generalizations.

## 2. Blowup Techniques, Bounded Ancient Solutions

In this section, we always assume that  $z = 0$  is a singular point. Making use of the space-time shift and the Navier-Stokes scaling, we can reduce the general problem of local regularity to a particular one that in a sense mimics the first time singularity.

**Proposition 2.1.** *Let  $v$  and  $q$  be a suitable weak solution to the Navier-Stokes equations in  $Q$  and  $z = 0$  be a singular point of  $v$ . There exist two functions  $\tilde{v}$  and  $\tilde{q}$  having the following properties:*

(i)  *$\tilde{v} \in L_3(Q)$  and  $\tilde{q} \in L_{\frac{3}{2}}(Q)$  obey the Navier-Stokes equations in  $Q$  in the sense of distributions;*

(ii)  $\tilde{v} \in L_\infty(B \times ]-1, -a^2[)$  for all  $a \in ]0, 1[$ ;

(iii) there exists a number  $0 < r_1 < 1$  such that  $\tilde{v} \in L_\infty(\{(x, t) : r_1 < |x| < 1, -1 < t < 0\})$ .

Moreover, functions  $\tilde{v}$  and  $\tilde{q}$  are obtained from  $v$  and  $q$  with the help of the space-times shift and the Navier-Stokes scaling and the origin remains to be a singular point of  $\tilde{v}$ .

The proof of Proposition 2.1 is essentially based on the application of the Caffarelli-Kohn-Nirenberg theorem and given in the paper [31]. In what follows, it is always deemed that such a replacement of  $v$  and  $q$  with  $\tilde{v}$  and  $\tilde{q}$  has been already made. Coming back to the original notation, we assume that functions  $v$  and  $q$  satisfy all the properties listed in Proposition 2.1 and  $z = 0$  is a singular point of  $v$ .

One of the most powerful methods to study potential singularities is a blowup technique based on the Navier-Stokes scaling

$$u^{(k)}(y, s) = \lambda_k v(x, t), \quad p^{(k)}(y, s) = \lambda_k^2 q(x, t)$$

with

$$x = x^{(k)} + \lambda_k y, \quad t = t_k + \lambda_k^2 s,$$

where  $x^{(k)} \in \mathbb{R}^3$ ,  $-1 < t_k \leq 0$ , and  $\lambda_k > 0$  are parameters of the scaling and  $\lambda_k \rightarrow 0$  as  $k \rightarrow +\infty$ . It is supposed that functions  $v$  and  $q$  are extended by zero to the whole  $\mathbb{R}^3 \times \mathbb{R}$ . A particular selection of scaling parameters  $x^{(k)}$ ,  $t_k$ , and  $\lambda_k$  depends upon a problem under consideration.

Now, our goal is to describe a universal method that makes it possible to reformulate the local regularity problem as a classical Liouville type problem for the Navier-Stokes equations. To see how things work, let us introduce the function

$$M(t) = \sup_{-1 < \tau \leq t} \|v(\cdot, \tau)\|_{\infty, \overline{B}(r_1)}.$$

It tends to infinity as time  $t$  goes to zero from the left since the origin is a singular point of  $v$ . Thanks to the obvious properties of the function  $M$ , one can choose parameters of the scaling in a particular way letting  $\lambda_k = 1/M_k$ , where a sequence  $M_k$  is defined as

$$M_k = \|v(\cdot, t_k)\|_{\infty, \overline{B}(r_1)} = |v(x^{(k)}, t_k)|.$$

Before discussing what happens if  $k$  tends to infinity, let us introduce a subclass of bounded ancient (backward) solutions playing an important role in the regularity theory of the Navier-Stokes equations.



**Definition 2.2.** A bounded vector field  $u$ , defined on  $\mathbb{R}^3 \times ]-\infty, 0[$ , is called a mild bounded ancient solution to the Navier-Stokes equation if there exists a function  $p$  in  $L_\infty(-\infty, 0; BMO(\mathbb{R}^3))$  such that  $u$  and  $p$  satisfy the Navier-Stokes system

$$\begin{aligned}\partial_t u + \operatorname{div} u \otimes u - \Delta u + \nabla p &= 0, \\ \operatorname{div} u &= 0\end{aligned}$$

in  $\mathbb{R}^3 \times ]-\infty, 0[$  in the sense of distributions.

The notion of mild bounded ancient solutions has been introduced in [17]. It has been proved there that  $u$  has continuous derivatives of any order in both spatial and time variables. Actually, the definition accepted in the present paper is different but equivalent to the one given in [17]. Here, we follow [31].

The statement below that has been proved in [31] shows how mild bounded ancient solutions appear in the regularity theory of the Navier-Stokes equations.

**Proposition 2.3.** There exist a subsequence of  $u^{(k)}$  (still denoted by  $u^{(k)}$ ) and a mild bounded ancient solution  $u$  such that, for any  $a > 0$ , the sequence  $u^{(k)}$  converges uniformly to  $u$  on the closure of the set  $Q(a) = B(a) \times ]-a^2, 0[$ . The function  $u$  has the additional properties:  $|u| \leq 1$  in  $\mathbb{R}^3 \times ]-\infty, 0[$  and  $|u(0)| = 1$ .

Let us demonstrate how this method works in the simplest case of the regular Ladyzhenskaya-Prodi-Serrin quantity  $M_{5,5}$ . Suppose that

$$M_{5,5}(v, 1) = \sup_{0 < r < 1} M_{5,5}(v, r) < +\infty.$$

By the scale-invariance and by the pressure equation, we may assume without loss of generality that

$$\int_{-\infty}^0 \int_{\mathbb{R}^3} (|u|^5 + |p|^{\frac{5}{2}}) dx dt < +\infty.$$

Given  $\varepsilon > 0$ , we can find  $T < 0$  such that

$$\int_{-\infty}^T \int_{\mathbb{R}^3} (|u|^5 + |p|^{\frac{5}{2}}) dx dt < \varepsilon.$$

Then, by Hölder inequality, we have

$$\frac{1}{R^2} \int_{t_0-R^2}^{t_0} \int_{B(x_0, R)} (|u|^3 + |p|^{\frac{3}{2}}) dx dt < c\varepsilon^{\frac{3}{5}}$$

for any  $x_0 \in \mathbb{R}^3$ , any  $R > 0$ , and any  $t_0 \leq T$  with some universal constant  $c$ . In turn, the  $\varepsilon$ -regularity theory ensures the inequality

$$|u(x_0, t_0)| < \frac{c}{R}$$

with another universal constant  $c$ . Tending  $R \rightarrow \infty$ , we get  $u(\cdot, t) = 0$  as  $t \leq T$ . One can repeat more or less the same arguments in order to show that in fact  $u$  is identically zero on  $\mathbb{R}^3 \times ]-\infty, 0]$ , which contradicts non-triviality condition  $|u(0)| = 1$ .

It is worthy to notice that the trivial bounded ancient solution of the form

$$u(x, t) = c(t), \quad p(x, t) = -c'(t) \cdot x,$$

with arbitrary bounded function  $c(t)$ , is going to be a mild bounded ancient solution if and only if  $c(t) \equiv \text{constant}$ . As in [31], this allows us to make the following plausible conjecture.

**Conjecture** *Any mild bounded ancient solution to the Navier-Stokes equations is a constant.*

To explain what consequences of the conjecture could be for regularity theory of the Navier-Stokes equations, let us assume that some “reasonable” scale-invariant quantity for  $v$  is “uniformly” bounded, see Introduction for definitions. By this assumption, together with Proposition 1.3, and by the conjecture, any mild bounded ancient solution must be zero. However, by Proposition 2.3, if  $z = 0$  is a singular point, there must be at least one non-trivial mild bounded ancient solution. So, the origin  $z = 0$  cannot be a singular point of  $v$ . This would be a positive answer to the question raised in the introduction. In particular, according to Proposition 1.3, validity of the conjecture would rule out Type I blowups.

As it has been shown in [17], the conjecture is true at least in two non-trivial cases. One of them is the two-dimensional flow for which regularity of energy solutions is well known, see Ladyzhenskaya’s monograph [19]. In the second case, axial symmetry with respect to  $x_3$ -axis is assumed and the behavior of solutions far away from the axis of symmetry is supposed to respect the property:

$$\sqrt{x_1^2 + x_2^2} |u(x, t)| \leq C < +\infty$$

for any  $x \in \mathbb{R}^3$  and for any  $-\infty < t < 0$ . This result can be exploited to

show that boundedness of  $g'$ , see definition of  $g'$  in Proposition 1.3, implies regularity of axially symmetric solutions with no assumption on the swirl. In the corresponding arguments, the crucial point is that, for axially symmetric solutions, boundedness of  $g'$  for  $v$  provides the required decay of  $u$ , see a proof in [33] or in [31]. A simple consequence of the latter statement is that axially symmetric solutions cannot develop Type I blowups. Indeed, to this end, it is sufficient to apply Proposition 1.3 (i) and get boundedness of  $g'$ .

Smoothness of axially symmetric solutions with no swirl is well known due to O. A. Ladyzhenskaya in [20] and M. R. Ukhovskij and V. L. Yudovich in [37], while the absence of Type I blowups with no assumptions on swirl has been established relatively recently, see details in [3], [4], [17], and [31].

It is interesting to notice that for the non-regular Ladyzhenskaya-Prodi-Serrin condition (so-called  $L_{3,\infty}$ -case), we are still not able to prove this conjecture. In the next sections, we shall discuss other ways of constructing ancient solutions to the Navier-Stokes equations in order to solve  $L_{3,\infty}$ -problem.

### 3. Backward Uniqueness for Navier-Stokes Equations

In this section, we deal with another subclass of ancient solutions  $u$  possessing the following property: *there exists a function  $p$  defined on  $\mathbb{R}^3 \times ]-\infty, 0[$  such that functions  $u$  and  $p$  are a suitable weak solution to the Navier-Stokes equations in  $\mathbb{R}^3 \times ]-\infty, 0[$ , i.e., they are a suitable weak solution on each parabolic ball of the form  $Q(a) = B(a) \times ]-a^2, 0[$  with  $-\infty < a < +\infty$ .* We call  $u$  a *local energy ancient solution*. Certainly, mild bounded ancient solutions belong to this subclass.

Local energy ancient solutions can be obtained from a given suitable weak solution  $v$  and  $q$  defined in  $Q$  with the help of the scaling mentioned in the previous section provided boundedness of  $g'$  takes place, see the definition of  $g'$  in Proposition 1.3.

**Proposition 3.1.** *Let  $v$  and  $q$  be a suitable weak solution to the Navier-Stokes equations in  $Q$  with  $g' < +\infty$  and let  $u^{(k)}(y, s) = \lambda_k v(\lambda_k y, \lambda_k^2 s)$  and  $p^{(k)}(y, s) = \lambda_k^2 q(\lambda_k y, \lambda_k^2 s)$  with  $\lambda_k \rightarrow 0$  as  $k \rightarrow +\infty$ . Then there exist subsequences of  $u^{(k)}$  and  $p^{(k)}$  still denoted by  $u^{(k)}$  and  $p^{(k)}$  such that, for each  $a > 0$ ,*

$$u^{(k)} \rightarrow u$$

*in  $L_3(Q(a)) \cap C([-a^2, 0]; L_{\frac{9}{8}}(B(a)))$  and*

$$p^{(k)} \rightharpoonup p$$

*in  $L_{\frac{3}{2}}(Q(a))$ , where  $u$  is a local energy ancient solution with the corresponding*

pressure  $p$ . For them, the scaled energy quantities are uniformly bounded, i.e.,

$$\sup_{0 < a < +\infty} \{A(u; a) + C(u; a) + D(p; a) + E(u; a)\} < +\infty.$$

Moreover, if  $z = 0$  is a singular point of the velocity field  $v$ , then

$$\int_{Q(3/4)} |u|^3 dz > c \quad (3.1)$$

with a positive universal constant  $c$ , i.e.,  $u$  is not identically equal to zero.

A proof of this proposition and similar facts can be found in [8], [29], [31], and [30]. Let us comment the last statement of Proposition 3.1. Indeed, if  $z = 0$  is a singular point of  $v$ , the  $\varepsilon$ -regularity theory gives us

$$\frac{1}{r^2} \int_{Q(r)} (|v|^3 + |q|^{\frac{3}{2}}) dz > \varepsilon > 0$$

for all  $0 < r < 1$  and for some universal constant  $\varepsilon$ . Making the inverse change of variables, we find

$$\begin{aligned} \frac{1}{a^2} \int_{Q(a)} (|u^{(k)}|^3 + |p^{(k)}|^{\frac{3}{2}}) dy ds = \\ \frac{1}{\lambda_k^2 a^2} \int_{Q(\lambda_k a)} (|v|^3 + |q|^{\frac{3}{2}}) dx ds > \varepsilon > 0 \end{aligned}$$

for each fixed radius  $a > 0$  and for sufficiently large natural number  $k$ . We cannot simply pass to the limit in the latter identity since it is not clear whether the pressure  $p^{(k)}$  converges strongly. This is quite typical issue when working with sequences of weak solutions to the Navier-Stokes equations. In order to treat this case one can split the pressure  $p^{(k)}$  into two parts. The first part is completely controlled by the velocity field  $u^{(k)}$  while the second one is a harmonic function with respect to the spatial variables. This, together with a certain boundedness of the sequence  $p^{(k)}$ , implies (3.1). For more details, we recommend papers [29] and [30].

We do not know whether local energy ancient solutions with bounded scaled energy quantities are identically equal to zero. However, there are some interesting cases for which the answer is positive. Let us describe them.

Our additional standing assumption of this section can be interpreted as a restriction on the blowup profile of  $v$  and has the form

$$\frac{1}{r^{\frac{15}{8}}} \int_{B(r)} |v(x, 0)|^{\frac{9}{8}} dx \rightarrow 0 \quad (3.2)$$

as  $r \rightarrow 0$ . The most important consequence of (3.2) is that

$$u(\cdot, 0) = 0, \quad (3.3)$$

where  $u$  is a local energy ancient solution generated by the scaling of Proposition 3.1. Indeed, for any  $a > 0$ , we have

$$\begin{aligned} & \frac{1}{a^{\frac{15}{8}}} \int_{B(a)} |u(y, 0)|^{\frac{9}{8}} dy \leq \\ & c \frac{1}{a^{\frac{15}{8}}} \int_{B(a)} |u(y, 0) - u^{(k)}(y, 0)|^{\frac{9}{8}} dy + c \frac{1}{a^{\frac{15}{8}}} \int_{B(a)} |u^{(k)}(y, 0)|^{\frac{9}{8}} dy = \\ & \alpha_k(a) + c \frac{1}{(\lambda_k a)^{\frac{15}{8}}} \int_{B(\lambda_k a)} |v(x, 0)|^{\frac{9}{8}} dx. \end{aligned}$$

Now, by Proposition 3.1 and by (3.2), the right hand side of the latter inequality tends to zero and this completes the proof of (3.3).

In a view of (3.3), one could expect that our local energy ancient solution is identically equal to zero. We call this phenomenon a backward uniqueness for the Navier-Stokes equations. So, if the backward uniqueness takes place or at least our ancient solution is zero on the time interval  $] - 3/4, 0[$ , then (3.1) cannot be true and thus, by Proposition 3.1, the origin  $z = 0$  is not a singular point of the velocity field  $v$ .

The crucial point for understanding the backward uniqueness for the Navier-Stokes equations is a similar phenomenon for the heat operator with lower order terms. The corresponding statement for the partial differential inequality involving the backward heat operator with lower order terms has been proved in [8] and reads:

**Theorem 3.2.** *Assume that we are given a function  $\omega$  defined on  $\mathbb{R}_+^n \times ]0, 1[$ , where  $\mathbb{R}_+^n = \{x = (x_i) \in \mathbb{R}^n, x_n > 0\}$ . Suppose further that they have the properties:*

*$\omega$  and the generalized derivatives  $\nabla\omega$ ,  $\partial_t\omega$ , and  $\nabla^2\omega$  are square integrable over any bounded subdomain of  $\mathbb{R}_+^n \times ]0, 1[$ ;*

$$|\partial_t\omega + \Delta\omega| \leq c(|\omega| + |\nabla\omega|) \quad (3.4)$$

*on  $\mathbb{R}_+^n \times ]0, 1[$  with a positive constant  $c$ ;*

$$|\omega(x, t)| \leq \exp\{M|x|^2\} \quad (3.5)$$

*for all  $x \in \mathbb{R}_+^n$ , for all  $0 < t < 1$ , and for some  $M > 0$ ;*

$$\omega(x, 0) = 0 \quad (3.6)$$

*for all  $x \in \mathbb{R}_+^n$ .*

Then  $\omega$  is identically zero in  $\mathbb{R}_+^n \times ]0, 1[$ .

The interesting feature of Theorem 3.2 is that there has been made no assumption on  $\omega$  on the boundary  $x_n = 0$ . In order to prove the theorem, two Carleman's inequalities have been established, see details in [8] and [9]. For the further improvements of the above backward uniqueness result, we refer to the interesting paper [7].

Theorem 3.2 clearly indicates what one should add to (3.3) in order to get the backward uniqueness for ancient solutions to the Navier-Stokes equations. Apparently, we need more regularity for sufficiently large  $x$  and a right decay at infinity. One can hope then to apply Theorem 3.2 to the vorticity equation

$$\partial_t \omega - \Delta \omega = \omega \cdot \nabla u - u \cdot \nabla \omega, \quad \omega = \nabla \wedge u,$$

which could be interpreted as a perturbation of the heat equation by lower order terms. To make it possible, it is sufficient to show boundedness of  $u$  and  $\nabla u$  outside of the Cartesian product of some spatial ball and some time interval. The most of the rest of the paper will be devoted to description of various situations for which it is really true.

Let us assume that

$$|u(x, t)| + |\nabla u(x, t)| \leq c < +\infty \quad (3.7)$$

for all  $|x| > R$ , for all  $-1 < t < 0$ , and for some constant  $c$  and try to figure out what follows from (3.7). It is not difficult to see that (3.3) and (3.7) implies (3.6) and (3.4), (3.5), respectively. At last, the linear theory ensures the validity of first condition in Theorem 3.2, see details in [27]. So, Theorem 3.2 is applicable and by it,  $\omega(x, t) = 0$  for all  $|x| > R$  and for  $-1 < t < 0$ . Using unique continuation across spatial boundaries, see, for instance, [8], we deduce  $\omega(x, t) = \nabla \wedge u(x, t) = 0$  for all  $x \in \mathbb{R}^3$  and, say, for  $-5/6 < t < 0$ . Since  $u$  is divergence free, it is a harmonic function in  $\mathbb{R}^3$  depending on  $t \in ]-5/6, 0[$  as a parameter. Therefore, for any  $a > \sqrt{5/6}$  and for any  $x_0 \in \mathbb{R}^3$ , by the mean value theorem for harmonic functions, we have

$$\begin{aligned} & \sup_{-5/6 < t < 0} |u(x_0, t)|^2 \leq \\ & c \sup_{-5/6 < t < 0} \frac{1}{a^3} \int_{B(x_0, a)} |u(x, t)|^2 dx \leq \\ & c \sup_{-5/6 < t < 0} \frac{1}{a^3} \int_{B(|x_0|+a)} |u(x, t)|^2 dx \leq \\ & c \frac{a + |x_0|}{a^3} A(u, a + |x_0|). \end{aligned}$$

Thanks to boundedness of scaled energy quantities stated in Proposition 3.1,

the right hand side of the latter inequality tends to zero as  $a$  goes to infinity. By arbitrariness of  $x_0$ , we conclude that  $u(x, t) = 0$  for all  $x \in \mathbb{R}^3$  and for  $-5/6 < t < 0$ , which contradicts (3.1). Hence, the origin  $z = 0$  cannot be a singular point of  $v$ .

Let us go back to the marginal case of Ladyzhenskaya-Prodi-Serrin condition, the so-called  $L_{3,\infty}$ -case, and show that it is completely embedded into the above scheme. So, we assume that functions  $v$  and  $q$  are a suitable weak solution to the Navier-Stokes equations in  $Q$  and satisfy the additional condition

$$\|v\|_{3,\infty,Q} < +\infty. \quad (3.8)$$

With the help of Proposition 1.3, it is not so difficult to show that  $g' < +\infty$ . So, for  $v$ , all the assumptions of Proposition 3.1 hold and thus our blowup procedure produces a local energy ancient solution  $u$  with the properties listed in that proposition. Exploited the  $\varepsilon$ -regularity theory once more, we can show further that  $v(\cdot, 0) \in L_3(B(2/3))$ , which in turn implies (3.2). Now, in order to prove regularity of the velocity  $v$  at the point  $z = 0$ , it is sufficient to verify the validity of (3.7). Indeed, by scale-invariance,

$$\|u\|_{3,\infty,\mathbb{R}^3 \times ]-\infty, 0[} < +\infty.$$

Applying Proposition 3.1 once again and taking into account properties of harmonic functions, one can conclude that

$$\|p\|_{\frac{3}{2},\infty,\mathbb{R}^3 \times ]-\infty, 0[} < +\infty.$$

Combining the latter estimates, we show that for any  $T > 0$

$$\int_{-T}^0 \int_{\mathbb{R}^3} \left( |u|^3 + |p|^{\frac{3}{2}} \right) dx dt < +\infty. \quad (3.9)$$

Our further arguments rely upon the  $\varepsilon$ -regularity theory. Indeed, letting, say,  $T = 4$ , one can find  $R > 4$  so that

$$\int_{-4}^0 \int_{\mathbb{R}^3 \setminus B(R/2)} \left( |u|^3 + |p|^{\frac{3}{2}} \right) dx dt < \varepsilon.$$

The rest of the proof of (3.7) is easy.

## 4. How Does $L_3$ -norm Approach Potential Blowup?

Let  $v$  be a weak Leray-Hopf solution to the classical Cauchy problem. Assume that it has a finite blowup time  $T$ . As it has been already shown by J. Leray,

for any  $3 < s \leq +\infty$ , there is a positive constant  $c_s$  such that

$$\|v(\cdot, t)\|_{s, \mathbb{R}^3} \geq \frac{c_s}{(T-t)^{\frac{s-3}{2s}}}$$

for  $t < T$ .

However, in the limit case  $s = 3$ , according to what has been discussed in the previous section, we just have

$$\limsup_{t \rightarrow T-0} \|v(\cdot, t)\|_{3, \mathbb{R}^3} = +\infty.$$

It would be natural to ask whether

$$\lim_{t \rightarrow T-0} \|v(\cdot, t)\|_{3, \mathbb{R}^3} = +\infty \quad (4.1)$$

is true or not? An answer to this question is still unknown. In this section, we shall seek either some additional conditions providing the positive answer or a weaker version of (4.1). Regarding to the first goal, we could formulate the following statement.

**Theorem 4.1.** *Let  $v$  be a weak Leray-Hopf solution to the Cauchy problem (1.1) and (1.2) and let  $T$  be a finite time blowup. Assume that, for some  $3 < s \leq +\infty$ , there exists a positive constant  $C_s$  such that*

$$\|v(\cdot, t)\|_{s, \mathbb{R}^3} \leq \frac{C_s}{(T-t)^{\frac{s-3}{2s}}}$$

for  $t < T$ . Then (4.1) is true.

Let us outline the proof of Theorem 4.1 following [30]. Without loss of generality, we always may assume  $s = +\infty$ , which means that we restrict ourselves to the case of Type I blowups. Indeed, this is a simple consequence of the  $\varepsilon$ -regularity theory. So, after application of Proposition 1.3, we can conclude that  $g' < +\infty$ .

It is known that, at the blowup time, all singular points of any weak Leray-Hopf solution  $v$  belong to a bounded ball whose radius in a way depends upon  $v$ . So, just by shift in space-time, we may assume that the origin  $z = 0$  is a singular point of  $v$ . Suppose further that (4.1) is wrong. Then, there exists an increasing sequence  $\{t_k\}_{k=1}^\infty$  tending to zero such that

$$\sup_k \|v(\cdot, t_k)\|_{3, \mathbb{R}^3} = M < +\infty. \quad (4.2)$$

Making use of Proposition 3.1, we may construct a local energy ancient solution  $u$ . The sequence  $\lambda_k$  will be specified later. By partial regularity theory and by



(4.2), one can assert that  $\|v(\cdot, 0)\|_{3, \mathbb{R}^3} < +\infty$ , which in turn implies identity (3.3) for  $u$ . In addition, by scale-invariance, our local energy ancient solution satisfies the estimate

$$\|u(\cdot, s)\|_{\infty, \mathbb{R}^3} \leq \frac{C_\infty}{\sqrt{-s}} \quad (4.3)$$

for any  $-\infty < s < 0$ .

Now, we need to provide the validity of decay estimate (3.7). To this end, we choose  $\lambda_k$  in a special way

$$\lambda_k = \frac{\sqrt{-t_k}}{2}.$$

In the rest of this section, we shall show why such a choice of  $\lambda_k$  gives (3.7). The first argument in support of it is as follows:  $u$  obeys the important global property

$$u(\cdot, -4) \in L_3(\mathbb{R}^3). \quad (4.4)$$

Our next step is to construct a solution to the Cauchy problem for the Navier-Stokes equations with the velocity field  $u(\cdot, -4)$  as the initial datum, i.e., to solve the following initial value problem

$$\begin{aligned} \partial_t w(x, t) + w(x, t) \cdot \nabla w(x, t) - \Delta w(x, t) + \nabla r &= 0, \\ \operatorname{div} w(x, t) &= 0 \end{aligned} \quad (4.5)$$

for  $x \in \mathbb{R}^3$  and  $-4 < t < 1$ ,

$$w(x, -4) = u(x, -4)$$

for  $x \in \mathbb{R}^3$ . With such initial data, one can find a mild solution (in Kato's sense) but in general it is local in time and thus does not necessary cover the whole time interval  $] -4, 1[$ . On the other hand, we cannot ensure the existence of a weak Leray-Hopf solution since  $u(\cdot, -4)$  is not necessary in  $L_2(\mathbb{R}^3)$ .

The way out is to use an interesting conception of local energy solutions introduced by P. G. Lemarié-Rieusset in [21]. This is an important generalization of the notion of weak Leray-Hopf solutions to the Cauchy problem. The phase space for local energy solutions is defined with the help of a particular Morrey space

$$L_{2,unif} = \left\{ \|u\|_{L_{2,unif}} = \sup_{x \in \mathbb{R}^3} \|u\|_{2, B(x, 1)} < +\infty \right\}.$$

To proceed with our definitions, let us find the completion of  $C_0^\infty(\mathbb{R}^3)$  in  $L_{2,unif}$  and denote it by  $E_2$ . It is not so difficult to check that the space  $L_3(\mathbb{R}^3)$  is embedded into the space  $E_2$ . Finally, the phase (energy) space  $\mathring{E}_2$  consists of

all divergence free vector fields belonging to  $E_2$ . Having such an energy space in hands, one can give a complete definition of local energy solutions. In our version, we follow paper [15]. It is a little modification of the original definition adopted in the monograph [21].

**Definition 4.2.** Functions  $w \in L_\infty(-4, 1; \mathring{E}_2)$  and  $r \in L_{\frac{3}{2}}(-4, 1; L_{\frac{3}{2}, \text{loc}}(\mathbb{R}^3))$  are said to be a local energy weak Leray-Hopf solution or simply local energy solution to the Cauchy problem (4.5) if the following conditions hold:

$$\sup_{x_0 \in \mathbb{R}^3} \int_{-4}^1 \int_{B(x_0, 1)} |\nabla w|^2 dz < +\infty,$$

$w$  and  $r$  meet (4.5) in the sense of distributions;  
the function  $t \mapsto \int_{\mathbb{R}^3} w(x, t) \cdot \tilde{w}(x) dx$  is continuous on  $[-4, 1]$  for any compactly supported function  $\tilde{w} \in L_2(\mathbb{R}^3)$ ;  
for any compact  $K$ ,

$$\|w(\cdot, t) - u(\cdot, -4)\|_{L_2(K)} \rightarrow 0$$

as  $t \rightarrow -4 + 0$ ;  
for a.a.  $t \in ]-4, 1[$ , the local energy inequality

$$\begin{aligned} \int_{\mathbb{R}^3} \varphi |w(x, t)|^2 dx + 2 \int_{-4}^t \int_{\mathbb{R}^3} \varphi |\nabla w|^2 dx dt' &\leq \int_{-4}^t \int_{\mathbb{R}^3} \left( |w|^2 (\partial_t \varphi + \Delta \varphi) \right. \\ &\quad \left. + w \cdot \nabla \varphi (|w|^2 + 2r) \right) dx dt' \end{aligned}$$

is valid for all nonnegative functions  $\varphi \in C_0^\infty(\mathbb{R}^3 \times ]-4, 2[)$ ;  
for each point  $x_0 \in \mathbb{R}^3$ , there exists a function  $c_{x_0} \in L_{\frac{3}{2}}(-4, 1)$  such that

$$r_{x_0}(x, t) \equiv r(x, t) - c_{x_0}(t) = r_{x_0}^1(x, t) + r_{x_0}^2(x, t),$$

for  $(x, t) \in B(x_0, 3/2) \times ]-4, 1[$ , where

$$r_{x_0}^1(x, t) = -\frac{1}{3}|w(x, t)|^2 + \frac{1}{4\pi} \int_{B(x_0, 2)} K(x - y) : w(y, t) \otimes w(y, t) dy,$$

$$r_{x_0}^2(x, t) = \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B(x_0, 2)} (K(x - y) - K(x_0 - y)) : w(y, t) \otimes w(y, t) dy,$$

and  $K(x) = \nabla^2(1/|x|)$ .

As in the case of weak Leray-Hopf solutions, uniqueness of local energy solutions to the Cauchy problem (4.5) is an open problem. However, bound (4.3) makes it possible to show that our ancient solution  $u$  and any local energy solution  $w$ , defined by the Cauchy problem (4.5) with the help of  $u$ , coincide in the time interval  $] - 4, 0[$ , see a proof, for example, in [30]. On the other hand, we would like to remind that local energy solutions satisfy the local energy inequality and thus the  $\varepsilon$ -regularity theory is applicable to them. Making use of the  $\varepsilon$ -regularity theory, together with a certain decay of some integral norms over unit balls with respect to their centers, see [21] or [15] for exact statements and for a proof, one can show required point-wise estimate (3.7) for  $w$  and thus for  $u$ . The rest of the proof is more or less the same as in the local  $L_{3,\infty}$ -case, which means that  $z = 0$  is actually not a singular point.

Now we are going to discuss a weaker version of (4.1).

**Theorem 4.3.** *Let  $v$  be a weak Leray-Hopf solution to the Cauchy problem (1.1) and (1.2) and let  $T$  be a finite time blowup. Then*

$$\lim_{t \rightarrow T-0} \frac{1}{T-t} \int_t^T \|v(\cdot, \tau)\|_{3, \mathbb{R}^3}^3 d\tau = +\infty.$$

Apparently, Theorem 4.3 can be easily deduced from its local version, see [29], which reads:

**Proposition 4.4.** *Let  $v$  and  $q$  be a suitable weak solution to the Navier-Stokes equations in  $Q$ . Assume, in addition, that*

$$\liminf_{t \rightarrow -0} \frac{1}{-t} \int_t^0 \|v(\cdot, \tau)\|_{3, B}^3 d\tau < +\infty. \quad (4.6)$$

*Then  $z = 0$  is a regular point of  $v$ .*

Let us outline a proof of Proposition 4.4. There are two simple consequences of (4.6):

$$M = \sup_k \frac{1}{-t_k} \int_{t_k}^0 \int_B |v(x, \tau)|^3 dx d\tau < +\infty$$

for some increasing sequence  $\{t_k\}$  tending to zero and

$$v(\cdot, 0) \in L_3(Q(5/6)). \quad (4.7)$$

As to estimates for the pressure field, one can split it into two parts so that

$q = q^1 + q^2$  where  $q^1$  has a “good” bound

$$\frac{1}{-t_k} \int_{t_k}^0 \int_B |q^1(x, \tau)|^{\frac{3}{2}} dx d\tau \leq cM$$

with a universal constant  $c$  while the second counter-part is a harmonic function satisfying the estimate

$$\sup_{x \in B(2/3)} |q^2(x, t)|^{\frac{3}{2}} \leq c \left( \int_B |q(x, t)|^{\frac{3}{2}} dx + \int_B |v(x, t)|^3 dx \right).$$

We cannot apply Proposition 3.1 directly since we do not know whether  $g'$  is bounded or not. So, we have to prove the existence of a non-trivial blowup solution by hands.

We can choose  $\lambda_k = \sqrt{-t_k/10}$  and then, just by scaling, pick up subsequences, denoted by the same symbols, such that, for any positive number  $a$ ,

$$u^{(k)} \rightharpoonup u$$

in  $L_3(B(a) \times ]-10, 0[)$ ,

$$p^{1(k)} \rightharpoonup p$$

in  $L_{\frac{3}{2}}(B(a) \times ]-10, 0[)$ , and

$$p^{2(k)} \rightarrow 0$$

in  $L_{\frac{3}{2}}(B(a) \times ]-10, 0[)$ . Here,  $p^{i(k)}(y, s) = \lambda_k^2 q^i(\lambda_k y, \lambda_k^2 s)$ ,  $i = 1, 2$ . Then by the linear theory and by the usual compactness arguments, one can show that, for any  $a > 0$ ,

$$u^{(k)} \rightarrow u$$

in  $L_3(B(a) \times ]-10, 0[) \cap C([-10, 0]; L_{\frac{9}{8}}(B(a)))$ . Moreover, function  $u$  and  $p$  are a suitable weak solution to the Navier-Stokes equations in  $\mathbb{R}^3 \times ]-10, 0[$  that has the properties  $u \in L_3(\mathbb{R}^3 \times ]-10, 0[)$  and  $p \in L_{\frac{3}{2}}(\mathbb{R}^3 \times ]-10, 0[)$ . This solution is non-trivial since it satisfies (3.1). The further arguments are similar to those which have been used in Section 3. Finally, (4.7) implies (3.3). Repeating the end of the proof in  $L_{3,\infty}$ -case, one can state that, in fact,  $u$  is identically equal to zero, say, on  $\mathbb{R}^3 \times ]-1, 0[$ , which contradicts (3.1). So, the origin is a regular point of  $v$ .

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