Analysis and Approximation of Incompressible Chemically Reacting non-Newtonian Fluids

Seungchan Ko

Keble College

University of Oxford

A thesis submitted for the degree of

Doctor of Philosophy

Trinity 2018

Abstract

Analysis and Approximation of Incompressible Chemically Reacting non-Newtonian Fluids

Seungchan Ko, Keble College
Doctor of Philosophy, Trinity 2018

In this thesis, we consider a system of nonlinear partial differential equations modelling the motion of an incompressible non-Newtonian fluid, which is chemically reacting. The governing system consists of a convection-diffusion equation for the concentration and the generalized Navier–Stokes equations, where the viscosity coefficient is a power-law-type function of the shear-rate, and the coupling between the equations results from the concentration-dependence of the power-law index. This system of nonlinear partial differential equations arises in mathematical models of the synovial fluid found in the cavities of moving joints. The only established mathematical result for this model is the existence of weak solutions for the stationary model. As a subsequent mathematical study, we first construct a finite element approximation of the model and perform the mathematical analysis of the numerical method in two and three space dimensions. Key technical tools include discrete counterparts of the Bogovskiĭ operator, De Giorgi-type regularity theorem in two dimensions, and the Acerbi–Fusco Lipschitz truncation of Sobolev functions. Then we move to the unsteady problem, and we prove the existence of global weak solutions of the non-stationary model by using the Galerkin method combined with generalized monotone operator theory and parabolic De Giorgi–Nash–Moser theory. As the governing equations involve a nonlinearity with a variable power-law index, our theory exploits the framework of generalized Lebesgue and Sobolev spaces with variable-integrability exponent.

Statement of Originality

I declare that the work in this thesis is, to the best of my knowledge, original and my own work, except where otherwise indicated, cited, or commonly known. This thesis has not been submitted for a degree at another university.

Seungchan Ko
Oxford, 23 May 2018

Acknowledgements

Above all, I would like to express my sincere gratitude to Professor Endre Süli, my doctoral supervisor. It was a great privilege and fortune to be able to learn from and have discussions with such a mathematician. His mathematical knowledge and insight, whose depths stretch beyond my imagination, gave great inspiration for my doctoral research, and his encouragement and advice were indispensable for shaping myself to become an independent researcher in Mathematics.

Furthermore I would like to whole-heartedly thank all members of the Oxford Centre for Nonlinear Partial Differential Equations (OXPDE). It was an absolute honour and sheer joy to learn from and interact with leading experts and excellent peers. I would like especially thank Professor Gui-Qiang Chen and Professor Jan Kristensen for their guidance and support over the whole course of my studies.

I am also indebted to Professor Kyungkeun Kang, who introduced me to the theory of Partial Differential Equations and has been charitable with his guidance and encouragement even to this date. Moreover I am grateful to my teacher Mr. Kyungjae Choi, who first sparked my curiosity and excitement for Mathematics.

I would also like to thank all my friends, colleagues, beer and coffee for lifting me out of the occasional turbulent periods, and enabling my life at Oxford to be a fulfilling and delightful one.

And to my family, who are always unquestioningly supportive and offer me the biggest strength. My father, my mother, my sister and my brother-in-law's faith in me and assistance allowed me to reach the end of this degree. Thank you so much and I would like to say that I love you.

Finally, I would like to express my biggest and most heartfelt gratitude for my wife Minjung, who has sacrificed so much to hold my hand through the difficult times that I've faced in Oxford. I want to tell her that I am incredibly happy and blessed to be her husband, and I would not have been able to finish my thesis if it were not for her. All my achievements in Oxford are indebted entirely to her. I would like her to know that I love her from the bottom of my heart.

Contents

List of Symbols

d	space dimension
$\mathbb{R}$	set of all real numbers
$\mathbb{R}_{\geq 0}$	set of all non-negative real numbers
x	spatial variable
t	time variable
Ω	spatial domain
$\partial\Omega$	boundary of Ω
Q_T	$\Omega \times (0,T)$
S_T	$\partial\Omega \times (0,T)$
Γ_T	$S_T \cup \{(x,t) : x \in \overline{\Omega}, t = 0\}$
supp	support
$\cdot$	scalar product for vectors
$:$	scalar product for tensors
$\times$	Cartesian product
$\otimes$	tensor product
∇	gradient operator
$\boldsymbol{D}$	symmetric gradient operator
div	divergence operator
∂_t	partial time derivative
$\hookrightarrow$	(continuous) embedding
$\hookrightarrow\hookrightarrow$	compact embedding
$\to$	(strong) convergence
$\rightrightarrows$	uniform convergence
$\rightharpoonup$	weak convergence
$\rightharpoonup^*$	weak$-*$ convergence
$\langle \cdot, \cdot \rangle$	duality pairing
$*$	convolution operator
χ_Q	characteristic function of Q
$B_r(x)$	ball of radius r centered at x
δ_{ij}	Kronecker delta with index i and j

$\mathcal{P}(\Omega)$	set of all variable exponents on Ω
$\mathcal{P}^{\log}(\Omega)$	set of all log-Hölder continuous functions on Ω
$L^p(\Omega)$	Lebesgue space of order p on Ω
$W^{k,p}(\Omega)$	Sobolev space of order k and p on Ω
$L^{p(\cdot)}(\Omega)$	variable-exponent Lebesgue space on Ω
$W^{k,p(\cdot)}(\Omega)$	variable-exponent Sobolev space on Ω
$C^{0,\alpha}(\Omega)$	Hölder space with exponent α on Ω
$C^{\alpha,\alpha/2}(Q_T)$	parabolic Hölder space with exponent α on Q_T
$d_p(\cdot,\cdot)$	parabolic distance
$\boldsymbol{u}$	fluid velocity
π	fluid pressure
c	concentration of hyaluronan
$\boldsymbol{S}(\cdot,\cdot)$	Cauchy stress tensor of synovial fluids
$\bar{\boldsymbol{S}}$, $\hat{\boldsymbol{S}}$	limits of approximate stress tensors
$\boldsymbol{q}_c(\cdot,\cdot,\cdot)$	diffusion flux of concentration
$\bar{\boldsymbol{q}}_c$, $\hat{\boldsymbol{q}}_c$	limits of diffusion flux vectors
$\boldsymbol{f}$	external force
$\mathcal{B}$	Bogovskiĭ operator
M	Hardy–Littlewood maximal operator
$\mathcal{G}_n$	partition of $\overline{\Omega}$ with discretization parameter n
$\mathcal{H}_m$	partition of $\overline{\Omega}$ with discretization parameter m
$\mathbb{V}^n$	finite element space for velocity on $\mathcal{G}_n$
$\mathbb{Q}^n$	finite element space for pressure on $\mathcal{G}_n$
$\mathbb{Z}^n$ ($\mathbb{Z}^m$)	finite element space for concentration on $\mathcal{G}_n$ ($\mathcal{H}_m$)
Π^n_{div}	divergence-preserving, locally $W^{1,1}$-stable projection operator into $\mathbb{V}^n$
$\Pi^n_{\mathbb{Q}}$	locally L^1-stable projection operator into $\mathbb{Q}^n$
$\Pi^n_{\mathbb{Z}}$ ($\Pi^m_{\mathbb{Z}}$)	locally $W^{1,1}$-stable projection operator into $\mathbb{Z}^n$ ($\mathbb{Z}^m$)
$\boldsymbol{U}^n$, $\boldsymbol{U}^{n,m}$	approximate velocity
P^n, $P^{n,m}$	approximate pressure
C^n, $C^{n,m}$	approximate concentration
B_u	modified convective term in Navier–Stokes equations
B_c	modified convective term in convection-diffusion equation
$\mathcal{B}^n$	discrete Bogovskiĭ operator

Chapter 1

Introduction

During the past decade, the mathematical study of non-Newtonian fluids has become an active field of research, stimulated by the wide range of scientific and industrial problems in which they arise. Examples of non-Newtonian fluids include biological fluids (such as mucus, blood, and various polymeric solutions), as well as numerous fluids of significance in engineering, food industry, cosmetics, and agriculture. This then motivates fundamental research on non-Newtonian flow models, with special emphasis on its mathematical theory, to understand the behaviour of the fluids and to provide a theoretical background for engineers and applied scientists. In particular, from a practical point of view, developing robust numerical methods which describe the motion of non-Newtonian fluids is of significant importance. Furthermore, the mathematical analysis of the corresponding partial differential equations is also important, as it promotes a deeper insight into the construction and analysis of the numerical methods.

In this thesis, we investigate the mathematical and numerical analysis of certain kinds of non-Newtonian fluid flow models, which describe the motion of synovial fluids: from the mathematical point of view, the synovial fluid is a viscous incompressible fluid where the generalized viscosity depends on the shear-rate and the concentration of the specific molecule. The central goal of this thesis is to establish the existence of weak solutions for the model, and construct a numerical method describing the model properly and perform convergence analysis of the numerical schemes.

The essence of the subject of modelling the synovial fluid is encapsulated in the generalized Navier–Stokes equations with concentration and shear-rate dependent viscosity, coupled with a convection-diffusion equation for the concentration. This system of equations is often referred to as a model for “chemically reacting fluids” to emphasize that the behaviour of the fluids fundamentally depend on chemical properties of the solution.

In this introductory chapter, we discuss some background material on the history of the mathematical study of non-Newtonian fluids and the modelling of synovial fluids. In Section 1.1, we first briefly survey some relevant mathematical results on non-Newtonian fluids with shear-rate dependent viscosity, which are of interest in the thesis. In Section 1.2, we then introduce a model with generalized viscosity, which will be considered throughout this thesis. Since the model we will consider is novel and not very common, we shall highlight the origin and validity of the model, and then explain why this model describes the relevant physical phenomenon accurately. Finally, in Section 1.3, we shall make a comment of the structure of the thesis, and the chapter concludes with an overview of the goals of this thesis.

1.1 Non-Newtonian fluids with shear-rate dependent viscosity

In fluid dynamics, the motion of incompressible fluids in a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$, over the time interval $(0, T)$ of interest, is described by the system of partial differential equations

$$\partial_t \boldsymbol{u} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div} \boldsymbol{S} = \nabla \pi + \boldsymbol{f} \qquad \text{in } Q_T, \tag{1.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } Q_T, \tag{1.2}$$

where $Q_T = (0, T) \times \Omega$. This system of equations is derived from the conservation of momentum and conservation of mass equations, and here the unknown quantities are the velocity field $\boldsymbol{u} : Q_T \to \mathbb{R}^d$ and the pressure $\pi : Q_T \to \mathbb{R}$. The function $\boldsymbol{f} : Q_T \to \mathbb{R}^d$ represents a given external force and $\boldsymbol{S} : Q_T \to \mathbb{R}^{d\times d}_{\text{sym}}$ is the Cauchy stress tensor. See, for instance, [13] for more details about the derivation of the model. In order to describe a particular fluid in this framework, we need a constitutive law relating the stress tensor $\boldsymbol{S}$ to the symmetric velocity gradient

$$\boldsymbol{D}\boldsymbol{u} := \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right)$$

whose modulus $|\boldsymbol{Du}|$ is called *shear-rate*; the rate at which a fluid is sheared during flow. The simplest example is when the relationship is linear, i.e.,

$$\boldsymbol{S} = \boldsymbol{S}(\boldsymbol{Du}) = 2\nu \boldsymbol{Du}, \tag{1.3}$$

where $\nu > 0$ is the viscosity of the fluid. In this case the system (1.1), (1.2) becomes the system of classical Navier–Stokes equations (Newtonian fluid flow model), and the study of these equations is of independent interest. However, this system can only model a limited set of fluids, with simple molecular structure, such as water and air. In order to accurately model the motion of fluids with more complicated structure, one needs to consider generalized models. Those which do not satisfy the linear relation (1.3) are called *non-Newtonian fluids*; see [7] for detailed discussion of non-Newtonian fluid flow models. A special class among these are generalized Newtonian fluids. Here the viscosity is assumed to be a function of the shear-rate $|\boldsymbol{Du}|$ and the constitutive relation is

$$\boldsymbol{S}(\boldsymbol{Du}) = \nu(|\boldsymbol{Du}|)\boldsymbol{Du}. \tag{1.4}$$

For these generalized Newtonian models, two different reactions are possible:

- The fluid becomes thicker (for example, batter): the viscosity of a shear-thickening (dilatant) fluid is an increasing function of the shear-rate.
- The fluid becomes thinner (for example, blood or ketchup): the viscosity of a shear-thinning (pseudoplastic) fluid is a decreasing function of the shear-rate.

One of the popular generalized Newtonian fluid flow models and the one that we are interested in this thesis, is the power-law model in which the viscosity is of the form

$$\nu(|\boldsymbol{Du}|) = \nu_0(\kappa_1 + \kappa_2|\boldsymbol{Du}|^2)^{\frac{p-2}{2}} \tag{1.5}$$

with ν_0, $\kappa_2 > 0$, $\kappa_1 \geq 0$ and $p \in (1, \infty)$ is specified by physical experiments. More details about this model and an extensive list of specific p-values for different fluids can be found in [13]. If $p = 2$, the model becomes the Newtonian fluid flow model. The fluid is shear-thickening if $p > 2$ and is shear-thinning if $p < 2$.

This mathematical model is useful because of its simplicity, and it is widely used in the theory of fluid dynamics. There are other models that better describe the entire flow behaviour of shear-dependent fluids, but they do so at the expense of increased complexity. Therefore, the mathematical study of power-law models is important since, with this model, we can strike a balance between the relative ease of development of a mathematical theory and practical usefulness.

The rigorous mathematical study of power-law fluids began in the late nineteen sixties with the pioneering works of Lions and Ladyzhenskaya in [65] and [60], [61], respectively. In these papers the authors developed the existence theory of weak solutions of the non-stationary model for $p \geq \frac{3d+2}{d+2}$ ($p \geq \frac{3d}{d+2}$ for the stationary model) with the aid of monotone operator theory. The cases they could handle were restrictive for real world situations. Nevertheless, their work was a breakthrough in the theory of partial differential equations. Since then there has been significant progress in the mathematical theory of generalized Newtonian fluids. To control the convective term, some useful approximation techniques were developed such as the so-called L^∞ and Lipschitz truncation techniques, and they were successfully applied in the existence analysis. As a result, the existence of weak solutions was established with a wide range of power-law-type models for $p > \frac{2d}{d+2}$; see [52, 74, 53, 39] in the case of steady, and [67, 81, 43, 16] for unsteady flows.

Once the existence of a weak solution is shown, one of the interesting subsequent questions from the point of view of computational mathematics is to discretize the equations and approximate the weak solution in a stable and accurate way. In [8, 9], the authors derived (in some cases optimal) error estimates for the finite element approximation of steady-state power-law-like models without a convective term. For the power-law fluid flow model with a convective term, in [38, 80], the authors constructed the finite element approximation of a general class of incompressible non-Newtonian fluids where the viscous stress tensor and the rate-of-strain tensor are related by a maximal monotone graph, and performed the convergence analysis of the family of methods under consideration. In particular, in [38], the discrete counterpart of Lipschitz truncation was constructed, which is a useful technique for controlling the convective term at the discrete level.

Interestingly, it was discovered in laboratory experiments that in a number of non-Newtonian fluid flow problems, the power-law index p is not a fixed constant but is, rather, variable-dependent; see, e.g., [72] for a fluid model where the power-law index is of the form $p(\cdot) := p(|\boldsymbol{E}(x)|^2) \sim p(x)$, $\boldsymbol{E}$ being a given electric field.

This fluids are called electro-rheological fluids. When these fluids are exposed to an electro-magnetic field, their viscosity undergoes a significant change. Nowadays, electro-rheological fluids have the quality and potential in a wide range of applications, for example, actuators, clutches, shock absorbers, and rehabilition equipment. Such electro-rheological models were mathematically investigated in [75, 76], where the existence theory was developed by using monotone operator theory and also the so-called higher regularity technique under the assumption that the given $\boldsymbol{E}$, and consequently $p(\cdot)$, is smooth enough. Also, some numerical approximations were considered. Recently, the Lipschitz truncation method was generalized to variable-exponent spaces in [39], where the existence of weak solutions was shown provided that $p(\cdot)$ is sufficiently regular in some sense, $1 < p^- \leq p(x) \leq p^+ < \infty$ and $p^- > \frac{2d}{d+2}$. Since the equations have a variable nonlinearity, the solutions are sought in (variable-exponent) generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$, whose functional-analytic properties are more complicated than those of classical Lebesgue and Sobolev spaces; the theory of variable-exponent spaces will be covered in Chapter 2. Regarding the numerical approximation of solutions, electro-rheological fluid flow models were studied in [25, 12], where a discretization of the problem was constructed and the convergence of a sequence of numerical solutions was established.

In this thesis, we will consider a similar, but more complicated model. As will be explained in more detail in the next section, the governing system consists of a convection-diffusion equation for the concentration and a generalized Navier–Stokes system for the velocity and the pressure of the fluid, where the viscosity coefficient is a power-law-type function of the shear-rate, and the coupling between the equations occurs through a concentration-dependent power-law index. The detailed discussion of the mathematical model we consider in this thesis will be covered in the next section.

1.2 Mathematical model of the synovial fluids

In this thesis, we shall investigate the equations modelling the rheological response of a synovial fluid (a biological fluid found in the moving joint cavities) during a steady shear experiment. From the rheological point of view, synovial fluid is incompressible and is the ultrafiltrate of blood plasma (namely, plasma-free from large proteins) enriched with the locally synthesized polysaccharide molecules, called *hyaluronan*. Though one could model the solution using mixture theory, we shall restrict ourselves to the situation where the solution can be described as a single-constituent fluid. This perspective is fairly reasonable because the mass concentration of hyaluronan is negligible, and even if molecules of hyaluronan are accumulated locally, the mass concentration does not exceed 2% and the solution remains in a practically homogeneous state. Nevertheless, we still need to consider the experimentally observed chemical properties of the fluid.

The presence of hyaluronan in the synovial fluid is important since hyaluronan is the constituent responsible for the non-Newtonian behaviour of the synovial fluid, which determines the functioning of the whole synovial joint system. In fact, it was already seen in viscosimetric experiments performed in the early 1950s that the synovial fluid has a strong shear-thinning property, depending on the concentration of hyaluronan in the solution. Explicitly, the viscosity of the fluid is a function of the concentration as well as of the shear-rate. Higher shear-rates imply higher alignment of the chains and thus a decrease in the viscosity. On the other hand, the influence of concentration works in the opposite way because higher concentration of hyaluronan implies higher enlacement of the chains, which increases the viscosity. In previous studies, the viscous behavior of synovial fluid was mathematically modelled by a shear-thinning fluid with constant concentration (see [73]) or the effects of concentration and the shear-rate on the viscosity were separated (see [69]). This is contrary to the results of laboratory experiments, which show that the concentration influences the shear-rate response, which implies that the shear-rate-index depends on the concentration for the power-law model. The restriction of constant concentration is not appropriate for modelling the behaviour of synovial fluid under physiological conditions. In real joints, the concentration of hyaluronan varies. For example, it was shown in [27] that hyaluronan creates a boundary layer near the synovium with a concentration that is five times higher than in the central parts of the synovial joint cavity. Thus we need to consider a mathamatical model for the generalized viscosity for the synovial fluid that depends on the shear-rate and the varying concentration of hyaluronan. Therefore, from the viewpoint of mathematical modelling, a power-law-like model, where the power-law index is concentration-dependent, seems reasonable.

More precisely, denoting by c the concentration of hyaluronan in the solution, it was observed in laboratory experiments (see [63]) that the effect of concentration and the shear-rate on the viscosity are not separated (in the sense that, $\nu(c, |\boldsymbol{D}\boldsymbol{u}|) \sim f(c)\tilde{\nu}(|\boldsymbol{D}\boldsymbol{u}|)$), but that the concentration of hyaluronan affects the level of shear-thinning. For zero concentration, the viscosity becomes constant, corresponding to the fact that the fluid is composed only of ultrafiltrated blood plasma, exhibiting properties of a Newtonian fluid. If the concentration of hyaluronan increases in the solution, the fluid displays higher apparent viscosity and moreover, it thins the shear more rapidly. Therefore a new power-law-like model of the synovial fluid was proposed in [54], where the power-law index was considered to be a function of the concentration. This new model describes the viscous properties of the synovial fluid more accurately, and it naturally reflects the fact that non-Newtonian effects diminish as the concentration of hyaluronan decreases.

Based on the discussion above, from the mathematical point of view, we shall investigate a system of equations describing the motion of a shear-thinning fluid with a non-standard growth condition on the viscosity. More precisely, we shall investigate the incompressible generalized Navier–Stokes equations with a power-law-like viscosity where the power-law index is not fixed, but depends on the concentration. To close the system, we shall assume that the concentration satisfies a convection-diffusion equation. The resulting system of partial differential equations is therefore fully coupled. Namely, we consider the following system of partial differential equations:

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } Q_T, \tag{1.6}$$

$$\rho\, \partial_t \boldsymbol{u} + \rho \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div} \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u}) = -\nabla \pi + \rho\, \boldsymbol{f} \qquad \text{in } Q_T, \tag{1.7}$$

$$\partial_t c + \operatorname{div}(c\boldsymbol{u}) - \operatorname{div} \boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u}) = 0 \qquad \text{in } Q_T, \tag{1.8}$$

where $Q_T := \Omega \times (0,T)$, $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$ is a bounded open domain, and where $\boldsymbol{u} : Q_T \to \mathbb{R}^d$, $\pi : Q_T \to \mathbb{R}$, $c : Q_T \to \mathbb{R}_{\geq 0}$ are the velocity, pressure and concentration fields respectively. In the present context, $\boldsymbol{f} : Q_T \to \mathbb{R}^d$ represents a given density of the bulk force, ρ is the density of synovial fluid, and $\boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u})$ is the shear stress tensor component of the Cauchy stress tensor. For the shear stress tensor $\boldsymbol{S}$, as we discussed above, the prototypical example we are considering is of the following form:

$$\boldsymbol{S}(c, \boldsymbol{Du}) = \nu(c, |\boldsymbol{Du}|)\boldsymbol{Du},$$

where the generalized viscosity $\nu(c, |\boldsymbol{Du}|)$ depending on the concentration and on the shear-rate is of the form:

$$\nu(c, |\boldsymbol{Du}|) \sim \nu_0(\kappa_1 + \kappa_2 |\boldsymbol{Du}|^2)^{\frac{p(c)-2}{2}},$$

where ν_0, $\kappa_2 > 0$ and $\kappa_1 \geq 0$. We shall assume that the stress tensor $\boldsymbol{S} : \mathbb{R}_{\geq 0} \times \mathbb{R}^{d\times d}_{\text{sym}} \to \mathbb{R}^{d\times d}_{\text{sym}}$ is a continuous function satisfying the following growth, strict monotonicity and coercivity conditions: there exist positive constants C_1, C_2 and C_3 such that

$$|\boldsymbol{S}(\xi, \boldsymbol{B})| \leq C_1(|\boldsymbol{B}|^{p(\xi)-1} + 1), \tag{1.9}$$

$$(\boldsymbol{S}(\xi, \boldsymbol{B_1}) - \boldsymbol{S}(\xi, \boldsymbol{B_2})) \cdot (\boldsymbol{B_1} - \boldsymbol{B_2}) > 0 \text{ for } \boldsymbol{B_1} \neq \boldsymbol{B_2}, \tag{1.10}$$

$$\boldsymbol{S}(\xi, \boldsymbol{B}) \cdot \boldsymbol{B} \geq C_2(|\boldsymbol{B}|^{p(\xi)} + |\boldsymbol{S}|^{p'(\xi)}) - C_3, \tag{1.11}$$

where $p : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a Hölder continuous function with $1 < p^- \leq p(\xi) \leq p^+ < \infty$ and $p'(\xi)$ is defined as $\frac{p(\xi)}{p(\xi)-1}$.

On the other hand, in the convection-diffusion equation (1.8), $\boldsymbol{q}_c(c, \nabla c, \boldsymbol{Du})$ represents the diffusive flux of the concentration c which, analogously to the shear stress tensor $\boldsymbol{S}$, has to be specified by a constitutive equation for the diffusivity of concentration. In our case, we assume that the diffusion is given by the Fick's law

$$\boldsymbol{q}_c(c, \nabla c, \boldsymbol{Du}) = \boldsymbol{K}(c, |\boldsymbol{Du}|)\nabla c, \tag{1.12}$$

where $\boldsymbol{K}$ is the diffusivity, the characteristic of the solute with respect to the solvent. In general, it is a tensorial function. As it was experimentally documented by [73], for instance, the diffusivity is dependent of concentration and the shear-rate. For this concentration flux vector $\boldsymbol{q}_c$, we assume that $\boldsymbol{q}_c(\xi, \boldsymbol{g}, \boldsymbol{B}) : \mathbb{R}_{\geq 0} \times \mathbb{R}^d \times \mathbb{R}^{d\times d}_{\text{sym}} \to \mathbb{R}^d$ is a continuous function, which is linear with respect to $\boldsymbol{g}$, and additionally satisfies the following inequalities: there exist positive constants C_4 and C_5 such that

$$|\boldsymbol{q}_c(\xi, \boldsymbol{g}, \boldsymbol{B})| \leq C_4 |\boldsymbol{g}|, \tag{1.13}$$

$$\boldsymbol{q}_c(\xi, \boldsymbol{g}, \boldsymbol{B}) \cdot \boldsymbol{g} \geq C_5 |\boldsymbol{g}|^2. \tag{1.14}$$

We shall refer to $p(\cdot)$ as the *variable-exponent function* and we will cover the associate mathematical theory in Chapter 2. Since we are considering a specific application in mind, this function should satisfy additional requirements. More precisely, p is assumed to be a continuous strictly monotonic function with the properties of shear-thinning/shear-thickening of the fluid; this means that it is bounded from below and from above as follows: $1 < p(c) < \infty$, where the limiting values 1 and ∞ are attained only for non-physiological and/or non-physical values of c.

Finally, since the concentration of hyaluronan is very low, we further assume that the density ρ is not influenced by the varying concentration c and hence it remains constant.

This is the complete description of the model we are going to consider in this thesis. For further rheological background and results of numerical simulations of the model, see [63, 54].

1.3 Structure of the thesis

We are now in a position to give more details regarding the aims of the thesis. Our objective is to develop the mathematical theory of the model described in the previous section. The only known mathematical results concerning our model are the proofs of the existence of weak solutions for the stationary model (see, for example,[20, 21]). Therefore, the aim of the thesis is to develop the mathematical theory of numerical methods for the approximation of weak solutions for the stationary model, and to show the existence of weak solution for the unsteady model.

The rest of the thesis is organized as follows:

Chapter 2 will include some preliminaries which will be used throughout the whole thesis. For example, the basic theory of function spaces related to the given problem will be covered. More precisely, we present some key results from the theory of variable-exponent Lebesgue and Sobolev spaces. Since the power-law index of the viscosity depends on the concentration, variable-exponent spaces naturally arise in the analysis of the problem under consideration.

In Chapter 3, we will consider the finite element approximation of the stationary model in a two-dimensional domain. We will discretize the given problem with a conforming finite element method, using inf-sup stable finite element pairs and with a discrete concentration space. The main objective of this chapter is to show the convergence of the sequence of finite element solutions to a weak solution of the problem.

Next, in Chapter 4, we extend the analysis developed in Chapter 3 to three space dimensions, and we formulate an analogous convergence result for a finite element method in a three-dimensional domain. Before constructing the approximation of the problem, we shall formulate a regularized problem and perform the finite element convergence analysis of this regularized problem. Finally, we shall show that weak solutions of this regularized problem, in turn, converge to a weak solution of the original problem.

In Chapter 5, we aim to prove the existence of global weak solutions to the non-stationary model by using the Galerkin method combined with generalized monotone operator theory and parabolic De Giorgi–Nash–Moser theory.

Finally, in Chapter 6, we will summarize the contents of the previous chapters. We revisit the description of the problem, recall the main difficulty and how the problem was solved. We shall also discuss how the results we obtained might be improved. We then point to possible future research topics including open problems in related fields, potential improvements of the current results and another related problems that we are also interested in.

Chapter 2

Lebesgue and Sobolev spaces with variable exponent

2.1 Brief description of the chapter

As we discussed in the previous chapter, we consider power-law-type non-Newtonian fluids flow models, where the power-law index depends on the concentration, coupled with a convection-diffusion equation for the concentration. Since the concentration is the function of the spatial and time variables x and t, the model under consideration has variable nonlinearity, which causes numerous technical difficulties. In particular, if we consider a system of partial differential equations with a variable nonlinearity, one of the main problems is the natural choice of function spaces where we can hope to find solutions. A natural energy norm cannot be described within the context of classical Lebesgue and Sobolev spaces. For instance, consider a weak solution $\boldsymbol{u}$ to the following system

$$
\begin{aligned}
-\operatorname{div}\left(\boldsymbol{S}(\boldsymbol{D}\boldsymbol{u})\right) &= \boldsymbol{f} \qquad && \text{in } \Omega, \\
\boldsymbol{u} &= 0 && \text{on } \partial\Omega,
\end{aligned}
$$

with a $p(\cdot)$-Laplacian type nonlinearity $\boldsymbol{S}(\boldsymbol{D}\boldsymbol{u}) = |\boldsymbol{D}\boldsymbol{u}|^{p(x)-2}\boldsymbol{D}\boldsymbol{u}$ where $p : \Omega \to [1, \infty)$ is a measurable function. Then the energy norm naturally induced from the above equations is given by

$$
\int_\Omega \boldsymbol{S}(\boldsymbol{D}\boldsymbol{u}) : \boldsymbol{D}\boldsymbol{u} \,\mathrm{d}x = \int_\Omega |\boldsymbol{D}\boldsymbol{u}|^{p(x)} \,\mathrm{d}x.
$$

However, this information on $\boldsymbol{D}\boldsymbol{u}$ cannot be correctly understood within the context of classical Lebesgue and Sobolev spaces. Therefore, we need to exploit the theory of variable-exponent Lebesgue and Sobolev spaces whose functional-analytic properties are more complicated than those of classical Lebesgue and Sobolev spaces. This chapter is devoted to introducing the theory of these generalized spaces. The results presented in this chapter are mainly quoted from [6, 36]. To present an intuition for variable-exponent spaces, we include the proofs for basic properties of the spaces. For more advanced results (e.g. Theorem 2.3.8 in Section 2.3, all the results in Section 2.4 and Section 2.5) whose proofs are too long to be included, we omit the proofs and make some comments on the quotations. Here we only introduce mathematical results needed for later chapters of this thesis. For further mathematical results on variable-exponent spaces, see [36] as a comprehensive source of information.

Throughout the chapter, we assume that $\Omega \subset \mathbb{R}^d$ is a bounded open domain with the Lipschitz boundary $\partial\Omega$. Also, $X(\Omega)^d$ will denote the space of d-component vector-valued functions with components from $X(\Omega)$. We also define the space of tensor-valued functions $X(\Omega)^{d\times d}$. Furthermore, we shall denote the duality pairing between $f \in X$ and $g \in X^*$ by $\langle g, f\rangle$, and for two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $\boldsymbol{a}\cdot\boldsymbol{b}$ denotes their scalar product; and, similarly, for two tensors $\mathbb{A}$ and $\mathbb{B}$, $\mathbb{A} : \mathbb{B}$ signifies their scalar product. Finally, for any Lebesgue measurable set $Q \subset \mathbb{R}^d$, $|Q|$ denotes the standard Lebesgue measure of the set Q, and C signifies a generic positive constant, which may change at each appearance.

2.2 Definitions and basic properties

Let $\mathcal{P}(\Omega)$ be the set of all measurable functions $p : \Omega \to [1,\infty)$. We shall call the function $p \in \mathcal{P}(\Omega)$ a variable exponent. We also define

$$p^- := \operatorname*{ess\,inf}_{x\in\Omega} p(x) \qquad \text{and} \qquad p^+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x),$$

and for simplicity, we only consider the case

$$1 < p^- \leq p^+ < \infty. \tag{2.1}$$

On the set of measurable functions $f : \Omega \to \mathbb{R}$, we define the functional

$$A_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)}\,\mathrm{d}x.$$

Then it is easy to verify the following properties:

(1) $A_{p(\cdot)}(f) \geq 0$ for every measurable function f;

(2) $A_{p(\cdot)}(f) = 0$ if and only if $f = 0$;

(3) $A_{p(\cdot)}(f) = A_{p(\cdot)}(-f)$ for every measurable function f;

(4) The map $f \mapsto A_{p(\cdot)}(f)$ is convex.

A functional which satisfies the properties (1)–(4) is called (*convex*) *modular*, and hence the functional $f \mapsto A_{p(\cdot)}(f)$ is a (convex) modular.

Now we introduce the space

$$L^{p(\cdot)}(\Omega) := \{f \text{ is measurable on } \Omega \,:\, A_{p(\cdot)}(f) < \infty\},$$

which is obviously a linear space.

Then let us define the functional

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \|f\|_{p(\cdot)} := \inf \left\{ \lambda > 0 : A_{p(\cdot)} \left(\frac{f}{\lambda} \right) \leq 1 \right\}.$$

Then we have the following proposition defining the norm of the space $L^{p(\cdot)}(\Omega)$.

Proposition 2.2.1. *The functional*

$$\| \cdot \|_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to [0, \infty)$$

defines a norm on the space $L^{p(\cdot)}(\Omega)$. *This is called Luxemburg's norm for the variable-exponent space* $L^{p(\cdot)}(\Omega)$.

Proof. It is obvious that $\|f\|_{p(\cdot)} \geq 0$ for all $f \in L^{p(\cdot)}(\Omega)$. Next if $f = 0$, it is clear that $\|f\|_{p(\cdot)} = 0$. Conversely, suppose that $\|f\|_{p(\cdot)} = 0$. Then, by the definition of Luxemburg's norm, for an arbitrary $\varepsilon > 0$, we have

$$\frac{1}{\varepsilon} \int_\Omega |f(x)|^{p(x)} \, \mathrm{d}x \leq \int_\Omega \left| \frac{f(x)}{\varepsilon} \right|^{p(x)} \mathrm{d}x \leq 1.$$

Hence $\int_\Omega |f(x)|^{p(x)} \, \mathrm{d}x < \varepsilon$ for an arbitrary $\varepsilon > 0$, which implies that $f(x) = 0$ a.e. in Ω. Next, for any constant $c \in \mathbb{R}$ and $f \in L^{p(\cdot)}(\Omega)$, we have

$$\begin{aligned}
\|cf\|_{p(\cdot)} &= \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{cf(x)}{\lambda} \right|^{p(x)} \mathrm{d}x \leq 1 \right\} \\
&= \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{f(x)}{\lambda/|c|} \right|^{p(x)} \mathrm{d}x \leq 1 \right\} \\
&= \inf \left\{ |c|\delta > 0 : \int_\Omega \left| \frac{f(x)}{\delta} \right|^{p(x)} \mathrm{d}x \leq 1 \right\} \\
&= |c| \|f\|_{p(\cdot)}.
\end{aligned}$$

Finally we shall verify the triangle inequality. Let us define the set

$$\mathcal{M} = \left\{ f \in L^{p(\cdot)}(\Omega) : A_{p(\cdot)}(f) \le 1 \right\}.$$

Since $A_{p(\cdot)}(\cdot)$ is convex, $\mathcal{M}$ is also convex. For an arbitrary $\varepsilon > 0$ and every f, $g \in L^{p(\cdot)}(\Omega)$, it is easy to see that

$$\frac{f}{\varepsilon + \|f\|_{p(\cdot)}}, \ \frac{g}{\varepsilon + \|g\|_{p(\cdot)}} \in \mathcal{M},$$

and hence

$$\frac{f\theta}{\varepsilon + \|f\|_{p(\cdot)}} + \frac{g(1-\theta)}{\varepsilon + \|g\|_{p(\cdot)}} \in \mathcal{M} \qquad \forall \theta \in (0,1).$$

If we choose $\theta \in (0,1)$ such that $\frac{\theta}{\varepsilon + \|f\|_{p(\cdot)}} = \frac{1-\theta}{\varepsilon + \|g\|_{p(\cdot)}}$, we have

$$\frac{f+g}{\|f\|_{p(\cdot)} + \|g\|_{p(\cdot)} + 2\varepsilon} \in \mathcal{M},$$

which implies that

$$\|f+g\|_{p(\cdot)} \le \|f\|_{p(\cdot)} + \|g\|_{p(\cdot)} + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the assertion follows. $\square$

Similarly, we introduce the following variable-exponent Sobolev space:

$$W^{1,p(\cdot)}(\Omega) := \left\{ u \in W^{1,1}(\Omega) \cap L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}$$

with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{1,p(\cdot)} := \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

From Proposition 2.2.1, it is easy to see that $\| \cdot \|_{W^{1,p(\cdot)}}(\Omega)$ defines a norm on the space $W^{1,p(\cdot)}(\Omega)$.

Next, we shall verify a relation between the norm $\|f\|_{p(\cdot)}$ and the modular $A_{p(\cdot)}(f)$ for $f \in L^{p(\cdot)}(\Omega)$. We begin with the following lemma.

Lemma 2.2.2. *If $p^+ < \infty$, then for all measurable f with $0 < \|f\|_{p(\cdot)} < \infty$, we have*

$$A_{p(\cdot)}\left(\frac{f}{\|f\|_{p(\cdot)}} \right) = 1.$$

In particular, if $\|f\|_{p(\cdot)} = 1$, we have $\|f\|_{p(\cdot)} = A_{p(\cdot)}(f) = 1$.

Proof. By the definition of Luxemburg's norm, it is easy to see that $A_{p(\cdot)}(f/\|f\|_{p(\cdot)}) \leq 1$. Therefore it suffices to check that $A_{p(\cdot)}(f/\|f\|_{p(\cdot)}) < 1$ is impossible. Note that, for any $0 < \lambda \leq \|f\|_{p(\cdot)}$,

$$\begin{aligned} A_{p(\cdot)}\left(\frac{f}{\lambda}\right) &= \int_\Omega \frac{|f(x)|^{p(x)}}{\lambda^{p(x)}}\,\mathrm{d}x = \int_\Omega \left(\frac{|f(x)|}{\|f\|_{p(\cdot)}}\right)^{p(x)} \left(\frac{\|f\|_{p(\cdot)}}{\lambda}\right)^{p(x)} \mathrm{d}x \\ &\leq \left(\frac{\|f\|_{p(\cdot)}}{\lambda}\right)^{p^+} A_{p(\cdot)}\left(\frac{f}{\|f\|_{p(\cdot)}}\right). \end{aligned}$$

If $A_{p(\cdot)}(f/\|f\|_{p(\cdot)}) < 1$, we can choose $\lambda < \|f\|_{p(\cdot)}$ such that $\left(\frac{\|f\|_{p(\cdot)}}{\lambda}\right)^{p^+} A_{p(\cdot)}\left(\frac{f}{\|f\|_{p(\cdot)}}\right) < 1$, which implies that $A_{p(\cdot)}(f/\lambda) \leq 1$. This is a contradiction. □

The explicit relation between the norm $\|f\|_{p(\cdot)}$ and the modular $A_{p(\cdot)}(f)$ is given in the following assertion.

Proposition 2.2.3. *If $1 < p^- \leq p^+ < \infty$, for every $f \in L^{p(\cdot)}(\Omega)$ we have*

$$\min\left\{\|f\|_{p(\cdot)}^{p^-},\ \|f\|_{p(\cdot)}^{p^+}\right\} \leq A_{p(\cdot)}(f) \leq \max\left\{\|f\|_{p(\cdot)}^{p^-},\ \|f\|_{p(\cdot)}^{p^+}\right\},$$

or equivalently,

$$\min\left\{A_{p(\cdot)}^{\frac{1}{p^-}}(f),\ A_{p(\cdot)}^{\frac{1}{p^+}}(f)\right\} \leq \|f\|_{p(\cdot)} \leq \max\left\{A_{p(\cdot)}^{\frac{1}{p^-}}(f),\ A_{p(\cdot)}^{\frac{1}{p^+}}(f)\right\}.$$

Proof. If $\mu := \|f\|_{p(\cdot)} = 0$, we are done. So assume that $\mu \neq 0$, and consider $h(x) = \frac{f(x)}{\mu}$. By Lemma 2.2.2, $h \in L^{p(\cdot)}(\Omega)$ and we have

$$1 = \|h\|_{p(\cdot)} = A_{p(\cdot)}(h) \leq \begin{cases} \mu^{-p^+} A_{p(\cdot)}(f) & \text{if } \mu \leq 1, \\ \mu^{-p^-} A_{p(\cdot)}(f) & \text{if } \mu > 1, \end{cases}$$

from which we have $A_{p(\cdot)}(f) \geq \min\left\{\mu^{p^+},\ \mu^{p^-}\right\}$. Note further that

$$A_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)}\,\mathrm{d}x = \int_\Omega \mu^{p(x)}|h(x)|^{p(x)}\,\mathrm{d}x \leq \begin{cases} \mu^{p^-} A_{p(\cdot)}(f) & \text{if } \mu \leq 1, \\ \mu^{p^+} A_{p(\cdot)}(f) & \text{if } \mu > 1, \end{cases}$$

which yields $A_{p(\cdot)}(f) \leq \max\left\{\mu^{p^-},\ \mu^{p^+}\right\}$. □

Corollary 2.2.4. (Unit ball property) *If $1 < p^- \leq p^+ < \infty$, for every $f \in L^{p(\cdot)}(\Omega)$ we have the following properties:*

- $\|f\|_{p(\cdot)} = 1$ *if and only if* $A_{p(\cdot)}(f) = 1$*;*
- $\|f\|_{p(\cdot)} < 1$ *if and only if* $A_{p(\cdot)}(f) < 1$*;*

- $\|f\|_{p(\cdot)} > 1$ *if and only if* $A_{p(\cdot)}(f) > 1$,

Corollary 2.2.5. *Suppose that* $1 < p^- \leq p^+ < \infty$ *and let* $\{f_n\}_{n=1}^{\infty}$ *be a sequence of functions in* $L^{p(\cdot)}(\Omega)$ *and* $f \in L^{p(\cdot)}(\Omega)$. *Then we have*

$$\|f_n - f\|_{p(\cdot)} \to 0 \qquad \text{if and only if} \qquad A_{p(\cdot)}(f_n - f) \to 0 \qquad \text{as } n \to \infty.$$

Now we shall prove some important inequalities, which are well-known in classical Lebesgue spaces. For $p \in \mathcal{P}(\Omega)$, we denote $p'(x) = \frac{p(x)}{p(x)-1}$ for a.e. $x \in \Omega$.

Lemma 2.2.6. (Hölder's inequality) *Suppose that* $1 < p^- \leq p^+ < \infty$. *For every* $f \in L^{p(\cdot)}(\Omega)$ *and* $g \in L^{p'(\cdot)}(\Omega)$, *we have*

$$\int_{\Omega} |f(x)g(x)| \,\mathrm{d}x \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)} \leq 2\|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}.$$

Proof. Let us denote $\|f\|_{p(\cdot)} = \lambda$, $\|g\|_{p'(\cdot)} = \mu$ and assume that $\lambda \neq 0$, $\mu \neq 0$. Otherwise, the assertion is trivial. By Young's inequality we have, for a.e. $x \in \Omega$,

$$\begin{aligned} |f(x)g(x)| = \lambda\mu \left|\frac{f(x)}{\lambda}\right| \left|\frac{g(x)}{\mu}\right| &\leq \lambda\mu \left(\frac{1}{p(x)}\left|\frac{f(x)}{\lambda}\right|^{p(x)} + \frac{1}{p'(x)}\left|\frac{g(x)}{\mu}\right|^{p'(x)}\right) \\ &\leq \lambda\mu \left(\frac{1}{p^-}\left|\frac{f(x)}{\lambda}\right|^{p(x)} + \frac{1}{(p')^-}\left|\frac{g(x)}{\mu}\right|^{p'(x)}\right). \end{aligned}$$

Integrating both sides over Ω yields

$$\begin{aligned} \int_{\Omega} |f(x)g(x)| \,\mathrm{d}x &\leq \lambda\mu \left(\frac{1}{p^-} A_{p(\cdot)}\left(\frac{f}{\lambda}\right) + \frac{1}{(p')^-} A_{p(\cdot)}\left(\frac{g}{\mu}\right)\right) \\ &= \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}, \end{aligned}$$

where the last equality comes from the fact $A_{p(\cdot)}(f/\lambda) = 1$ and $A_{p(\cdot)}(g/\mu) = 1$ due to Lemma 2.2.2. ☐

Now we shall prove an embedding result for variable-exponent Lebesgue spaces.

Proposition 2.2.7. *Suppose that* p, $q \in \mathcal{P}(\Omega)$, $1 < p^- \leq p(x) \leq p^+ < \infty$, $1 < q^- \leq q(x) \leq q^+ < \infty$ *and* $p(x) \geq q(x)$ *a.e.* $x \in \Omega$. *Then* $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ *with*

$$\|f\|_{q(\cdot)} \leq C(\Omega, p^{\pm}, q^{\pm}) \|f\|_{p(\cdot)} \qquad \forall f \in L^{p(\cdot)}(\Omega).$$

Proof. Let $f \in L^{p(\cdot)}(\Omega)$ with $\|f\|_{p(\cdot)} = \lambda > 0$. Denote

$$\Omega_2 = \{x \in \Omega : p(x) = q(x)\} \qquad \text{and} \qquad \Omega_1 = \Omega \setminus \Omega_2.$$

If we set $h = f/\lambda$, by Lemma 2.2.2 we can easily see that $A_{p(\cdot)}(h) = 1$. Then, by Hölder's inequality, we have

$$\begin{aligned} A_{q(\cdot)}(h) &= \int_{\Omega_1} |h(x)|^{q(x)} \,\mathrm{d}x + \int_{\Omega_2} |h(x)|^{q(x)} \,\mathrm{d}x \leq 2\|1\|_{\frac{p(\cdot)}{p(\cdot)-q(\cdot)}} \left\| |h|^{q(x)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} + A_{p(\cdot)}(h) \\ &\leq 2 \max\left\{ |\Omega|^{1-\frac{q^+}{p^-}}, |\Omega|^{1-\frac{q^-}{p^+}} \right\} \max\left\{ A_{\frac{p(\cdot)}{q(\cdot)}}^{\frac{q^-}{p^+}}(|h|^{q(x)}), A_{\frac{p(\cdot)}{q(\cdot)}}^{\frac{q^+}{p^-}}(|h|^{q(x)}) \right\} + A_{p(\cdot)}(h) \\ &= 2 \max\left\{ |\Omega|^{1-\frac{q^+}{p^-}}, |\Omega|^{1-\frac{q^-}{p^+}} \right\} \max\left\{ A_{p(\cdot)}^{\frac{q^-}{p^+}}(h), A_{p(\cdot)}^{\frac{q^+}{p^-}}(h) \right\} + A_{p(\cdot)}(h) \\ &= 2 \max\left\{ |\Omega|^{1-\frac{q^+}{p^-}}, |\Omega|^{1-\frac{q^-}{p^+}} \right\} + 1. \end{aligned}$$

On the other hand, by Proposition 2.2.3, we have

$$\frac{1}{\lambda}\|f\|_{q(\cdot)} = \|h\|_{q(\cdot)} \leq \max\left\{ A_{q(\cdot)}^{\frac{1}{q^-}}(h), A_{q(\cdot)}^{\frac{1}{q^+}}(h) \right\}.$$

Altogether, we have the desired inequality. □

2.3 Basic function space theory

In this subsection, we introduce some function space theory for variable-exponent spaces, which will be used throughout the thesis. We begin with the dual norm of the space $L^{p(\cdot)}(\Omega)$. For given $f \in L^{p(\cdot)}(\Omega)$, we define the functional

$$|f|_{p(\cdot)} = \sup_{\|g\|_{p'(\cdot)} \leq 1} \int_\Omega f(x)g(x) \,\mathrm{d}x.$$

Proposition 2.3.1. (Dual norm of $L^{p(\cdot)}(\Omega)$) *The functional*

$$|\cdot|_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$$

defines a norm on $L^{p(\cdot)}(\Omega)$.

Proof. It is straightforward to see that $|\mu f|_{p(\cdot)} = |\mu||f|_{p(\cdot)}$ for any $\mu \in \mathbb{R}$. Also, the triangle inequality is trivial. Now let us assume that $|f|_{p(\cdot)} < 0$ and let $g \in L^{p'(\cdot)}(\Omega)$ be an arbitrary function such that $\|g\|_{p'(\cdot)} \leq 1$. By Hölder's inequality,

$$\frac{1}{2}\int_\Omega f(x)g(x) \,\mathrm{d}x \leq |f|_{p(\cdot)} < 0.$$

Since $\|g\|_{p'(\cdot)} = \|-g\|_{p'(\cdot)}$, we further have that

$$0 < -\frac{1}{2}\int_\Omega f(x)g(x)\,\mathrm{d}x = \frac{1}{2}\int_\Omega f(x)\left(-g(x)\right)\,\mathrm{d}x \le |f|_{p'(\cdot)} < 0,$$

which is a contradiction. Next, if $f = 0$ a.e. in Ω, it is obvious that $|f|_{p(\cdot)} = 0$. Conversely, suppose that $|f|_{p(\cdot)} = 0$ but $f \neq 0$ on the subset of Ω with positive measure, in other words, $A_{p(\cdot)}(f) > 0$. Let us introduce the function

$$g(x) = |f(x)|^{p(x)-1}\,\mathrm{sign}\;(f(x)) \in L^{p'(\cdot)}(\Omega).$$

Note that $\|g\|_{p'(\cdot)} > 0$. Then we have

$$0 = |f|_{p(\cdot)} \ge \int_\Omega f(x)\frac{g(x)}{\|g\|_{p'(\cdot)}}\,\mathrm{d}x = \frac{1}{\|g\|_{p'(\cdot)}}\int_\Omega |f(x)|^{p(x)}\,\mathrm{d}x = \frac{1}{\|g\|_{p'(\cdot)}}A_{p(\cdot)}(f) > 0,$$

which is impossible. □

Proposition 2.3.2. *Suppose that* $1 < p^- \le p^+ < \infty$. *For* $f \in L^{p(\cdot)}(\Omega)$, $|f|_{p(\cdot)} \le 1$ *implies that* $A_{p(\cdot)}(f) \le 1$.

Proof. We argue by contradiction. Suppose that $A_{p(\cdot)}(f) > 1$. Then, by Corollary 2.2.4, we have $\|f\|_{p(\cdot)} > 1$. Let us introduce the function

$$g(x) = \left|\frac{f(x)}{\|f\|_{p(\cdot)}}\right|^{p(x)-1}\mathrm{sign}\;(f(x))\,.$$

Then, by Lemma 2.2.2, we have

$$A_{p'(\cdot)}(g) = \int_\Omega \left(\left|\frac{f(x)}{\|f\|_{p(\cdot)}}\right|^{p(x)-1}\right)^{\frac{p(x)}{p(x)-1}}\mathrm{d}x = \int_\Omega \left|\frac{f(x)}{\|f\|_{p(\cdot)}}\right|^{p(x)}\mathrm{d}x = A_{p(\cdot)}\left(\frac{f}{\|f\|_{p(\cdot)}}\right) = 1,$$

which implies $\|g\|_{p'(\cdot)} = 1$ by Corollary 2.2.4. But then,

$$|f|_{p(\cdot)} \ge \int_\Omega f(x)g(x)\,\mathrm{d}x = \|f\|_{p(\cdot)}\int_\Omega \left|\frac{f(x)}{\|f\|_{p(\cdot)}}\right|^{p(x)}\mathrm{d}x = \|f\|_{p(\cdot)} > 1,$$

which is a contradiction. □

Now we are ready to prove the so called *norm-conjugate formula*, which is well-known for classical Lebesgue spaces.

Theorem 2.3.3. (Norm-conjugate formula) *If* $1 < p^- \le p^+ < \infty$, *for all* $f \in L^{p(\cdot)}(\Omega)$, *we have that*

$$\|f\|_{p(\cdot)} \le |f|_{p(\cdot)} \le 2\|f\|_{p(\cdot)} \qquad \forall f \in L^{p(\cdot)}(\Omega).$$

Proof. Let $f \in L^{p(\cdot)}(\Omega)$. Then by Hölder's inequality,

$$\int_\Omega f(x)g(x)\,\mathrm{d}x \leq 2\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)},$$

which implies $|f|_{p(\cdot)} \leq 2\|f\|_{p(\cdot)}$. Next, note that

$$\left|\frac{f}{|f|_{p(\cdot)}}\right|_{p(\cdot)} = \sup_{\|g\|_{p'(\cdot)}\leq 1}\int_\Omega \frac{f(x)}{|f|_{p(\cdot)}}g(x)\,\mathrm{d}x = \frac{|f|_{p(\cdot)}}{|f|_{p(\cdot)}} = 1.$$

Therefore, by Proposition 2.3.2, we have $A_{p(\cdot)}(f/|f|_{p(\cdot)}) \leq 1$. Hence, by Corollary 2.2.4, we finally obtain

$$\left\|\frac{f}{|f|_{p(\cdot)}}\right\|_{p(\cdot)} \leq 1,$$

which implies that $\|f\|_{p(\cdot)} \leq |f|_{p(\cdot)}$. □

Next we shall prove the completeness of the spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$.

Lemma 2.3.4. *If $1 < p^- \leq p^+ < \infty$, the space $L^{p(\cdot)}(\Omega)$ is complete.*

Proof. Let $\{f_k\}_{k=1}^\infty$ be a Cauchy sequence in $L^{p(\cdot)}(\Omega)$. Then, by Theorem 2.3.3, for arbitrary $\varepsilon > 0$, there exists an $N_0 \in \mathbb{N}$ such that

$$\int_\Omega |f_m(x) - f_n(x)||g(x)|\,\mathrm{d}x \leq \frac{\varepsilon}{1+|\Omega|} \qquad \forall m, n \geq N_0$$

for every $g \in L^{p'(\cdot)}(\Omega)$ with $\|g\|_{p'(\cdot)} \leq 1$. Note that we can take $g \equiv \frac{1}{1+|\Omega|}$ as we have

$$\int_\Omega \left(\frac{1}{1+|\Omega|}\right)^{p(x)} \mathrm{d}x \leq \frac{|\Omega|}{1+|\Omega|} \leq 1.$$

Therefore we obtain that, for any $m, n \geq N_0$,

$$\int_\Omega |f_m(x) - f_n(x)|\,\mathrm{d}x \leq \varepsilon,$$

which means that the sequence $\{f_k\}_{k=1}^\infty$ is Cauchy in $L^1(\Omega)$. Hence we can extract a subsequence (not relabelled) such that

$$f_k \to f \qquad \text{a.e. in } \Omega \text{ as } k \to \infty$$

for some $f \in L^1(\Omega)$. Now for $n > N_0$, we apply Fatou's lemma to get

$$\int_\Omega |f_n(x) - f(x)||g(x)|\,\mathrm{d}x \leq \sup_{m \geq N_0}\int_\Omega |f_n(x) - f_m(x)||g(x)|\,\mathrm{d}x \leq \varepsilon$$

for all $g \in L^{p'(\cdot)}(\Omega)$ with $\|g\|_{p'(\cdot)} \leq 1$. Therefore by Theorem 2.3.3, we conclude that

$$\|f_k - f\|_{p(\cdot)} \to 0 \qquad \text{as } k \to \infty.$$

□

Lemma 2.3.5. *If $1 < p^- \leq p^+ < \infty$, the space $W^{1,p(\cdot)}(\Omega)$ is complete.*

Proof. Let $\{f_k\}_{k=1}^\infty$ be a Cauchy sequence in $W^{1,p(\cdot)}(\Omega)$. By Lemma 2.3.4, there exist $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p(\cdot)}(\Omega)^d$ such that

$$\|f_k - f\|_{p(\cdot)} \to 0 \qquad \text{and} \qquad \|\nabla f_k - g\|_{p(\cdot)} \to 0 \qquad \text{as } k \to \infty. \tag{2.2}$$

Let $\psi \in C_0^\infty(\Omega)$. Then we have

$$\int_\Omega f_k \, \nabla\psi \, \mathrm{d}x = -\int_\Omega \psi \, \nabla f_k \, \mathrm{d}x \qquad \forall k \in \mathbb{N}.$$

Then, by (2.2), we obtain

$$\int_\Omega f \, \nabla\psi \, \mathrm{d}x = -\int_\Omega \psi \, g \, \mathrm{d}x,$$

which implies that $\nabla f = g$. Thus it follows that

$$\|f_k - f\|_{1,p(\cdot)} \to 0 \qquad \text{as } k \to \infty.$$

□

Next, we introduce some density theorems which are necessary to prove more advanced properties of variable-exponent spaces. We start with the density of simple functions in $L^{p(\cdot)}(\Omega)$.

Theorem 2.3.6. *Suppose that $p^+ < \infty$. Then the set of all simple functions on Ω is dense in $L^{p(\cdot)}(\Omega)$*

Proof. Note that all simple functions on Ω are contained in $L^{p(\cdot)}(\Omega)$, since $p^+ < \infty$. Then the theorem follows by the standard argument which is similar to the one for the classical Lebesgue spaces based on the Lebesgue dominated convergence theorem. □

Theorem 2.3.7. *If $p^+ < \infty$, then $C_0^\infty(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$*

Proof. By Theorem 2.3.6, simple functions are dense in $L^{p(\cdot)}(\Omega)$. Since a simple function is contained in $L^{p^+}(\Omega) \cap L^{p^-}(\Omega)$, it can be approximated by a sequence of $C_0^\infty(\Omega)$ functions in the same space. Then the claim follows from the fact that $L^{p^+}(\Omega) \cap L^{p^-}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$, which is not difficult to verify. □

As a next step, regarding duality, we introduce the following analogue of the Riesz representation theorem in variable-exponent Lebesgue spaces.

Theorem 2.3.8. *Suppose that* $1 < p^- \leq p^+ < \infty$. *For any linear functional* $F : L^{p(\cdot)}(\Omega) \to \mathbb{R}$, *there exists a unique function* $f \in L^{p'(\cdot)}(\Omega)$ *such that*

$$F(u) = \int_\Omega f(x)u(x)\,\mathrm{d}x \qquad \forall u \in L^{p(\cdot)}(\Omega).$$

The proof can be performed by the usual method for the classical Lebesgue spaces based on the use of Radon–Nikodym theorem and on the density of simple functions. Let us briefly sketch the proof.

Proof. For any measurable set $E \subset \Omega$, define $\lambda(E) = F(\chi_E)$. Then it is easy to see that λ is a finite signed measure provided that $p^+ < \infty$. Also, λ is absolutely continuous with respect to the usual Lebesgue measure since $|E| = 0$ implies $\|\chi_E\|_{p(\cdot)} = 0$. Then by the Radon–Nikodym theorem, there exists an $f \in L^1(\Omega)$ such that for all measurable sets $E \subset \Omega$,

$$F(\chi_E) = \int_\Omega \chi_E f\,\mathrm{d}x.$$

Then it is straightforward to verify that such an f is the desired one by the use of Theorem 2.3.6. The uniqueness is trivial. $\square$

The proof for the problem in a more general setting can be found in [59] or in Section 2.7 of [36].

Theorem 2.3.9. *If* $1 < p^- \leq p^+ < \infty$, *the spaces* $L^{p(\cdot)}(\Omega)$ *and* $W^{1,p(\cdot)}(\Omega)$ *are reflexive and separable Banach spaces.*

Proof. The reflexivity of $L^{p(\cdot)}(\Omega)$ is a direct consequence of Theorem 2.3.8. The separability of $L^{p(\cdot)}(\Omega)$ can be obtained by a simple application of Theorem 2.3.7, since any function belongs to $C_0^\infty(\Omega)$ can be approximated by a sequence of polynomials with rational coefficients.

On the other hand, by the mapping $f \mapsto (f, \nabla f)$, the space $W^{1,p(\cdot)}(\Omega)$ can be regarded as a closed subspace of $L^{p(\cdot)}(\Omega) \times \left(L^{p(\cdot)}(\Omega)\right)^d$. Thus $W^{1,p(\cdot)}(\Omega)$ is a reflexive and separable Banach space if $1 < p^- \leq p^+ < \infty$. $\square$

2.4 Log-Hölder continuity

We have studied the definitions and some basic properties of variable-exponent Lebesgue and Sobolev spaces. However, for the study of partial differential equations in the variable-exponent context, it is necessary to develop more advanced tools for variable-exponent spaces. For instance, we are interested in density theorems, embedding theorems, functional inequalities and the boundedness of some important operators.

In the study of variable-exponent spaces, it was shown that a certain regularity needs to be assumed on the exponent. A major breakthrough in the theory of variable-exponent spaces was the identification of the condition on the exponent p which guarantees the validity of some advanced function spaces theoretical results: the so-called *log-Hölder continuity.* We begin with the following definition.

Definition 2.4.1. We say that a function $p : \Omega \to \mathbb{R}$ is *locally log-Hölder continuous* if there exists a $C_1 > 0$ such that

$$|p(x) - p(y)| \leq \frac{C_1}{\log\left(e + 1/|x-y|\right)} \qquad \forall x, y \in \Omega.$$

Moreover, we say that p satisfies the *log-Hölder decay condition* if there exists $p_\infty \in \mathbb{R}$ and a constant $C_2 > 0$ such that

$$|p(x) - p_\infty| \leq \frac{C_2}{\log\left(e + |x|\right)} \qquad \forall x \in \Omega.$$

We say that p is *globally log-Hölder continuous* in Ω if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition. The constant $\max\{C_1, C_2\}$ is called the log-Hölder constant of p.

Definition 2.4.2. We define the following class of variable exponents:

$$\mathcal{P}^{\log}(\Omega) := \{p \in \mathcal{P}(\Omega) : \frac{1}{p} \text{ is globally log-Hölder continuous}\}.$$

If $p \in \mathcal{P}(\Omega)$ is bounded, then $p \in \mathcal{P}^{\log}(\Omega)$ if and only if p is globally log-Hölder continuous. This is due to the fact that $p \mapsto \frac{1}{p}$ is a bi-Lipschitz mapping from $[p^-, p^+]$ to $[\frac{1}{p^+}, \frac{1}{p^-}]$. Furthermore, if p is bounded and $|\Omega| < \infty$, the log-Hölder decay condition is automatically satisfied. Note that we are assuming that $\Omega \subset \mathbb{R}^d$ is bounded open Lipschitz domain and $1 < p^- \leq p^+ < \infty$ throughout the thesis. Therefore, in this thesis, we shall use the following alternative definition, which is widely used in the theory of variable-exponent spaces.

Definition 2.4.3. We say that the function $p \in \mathcal{P}(\Omega)$ is log-Hölder continuous on Ω if there exists a constant $C_{\log}(p) > 0$ such that

$$|p(x) - p(y)| \leq \frac{C_{\log}(p)}{-\log|x-y|} \qquad \forall x, y \in \Omega \text{ with } 0 < |x-y| \leq \frac{1}{2}.$$

We denote by $\mathcal{P}^{\log}(\Omega) \subset \mathcal{P}(\Omega)$ the set of all log-Hölder continuous functions defined on Ω.

Now, we can start with the basic density theorem, which is well-known for classical Lebesgue and Sobolev spaces. Density of smooth functions was one of the first questions investigated in the function space setting, as early as 1992, by Edmunds and Rákosník in [46]. Samko [77, 78] and Diening [33] have shown independently, that log-Hölder continuity of the exponent is sufficient for the density of smooth functions. See Section 9.1 of [36] for more detail and the proof of the following theorem.

Theorem 2.4.4. (Density of smooth functions) *Suppose that* $p \in \mathcal{P}^{\log}(\Omega)$ *satisfies* $1 < p^- \leq p^+ < \infty$. *Then, for each* $k \in \mathbb{N}$, $C^\infty(\overline{\Omega})$ *is dense in* $W^{k,p(\cdot)}(\Omega)$.

Next, we can ask a similar question about the generalization of the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $q \leq p^*$, where p^* is the Sobolev conjugate. Let us first define the variable Sobolev conjugate:

$$p^*(x) := \begin{cases} \frac{dp(x)}{d-p(x)} & p(x) < d, \\ \infty & p(x) \geq d, \end{cases}$$

For a bounded Lipschitz domain Ω, in [47], Edmunds and Rákosník proved under the assumption of $p(x)$ being Lipschitz continuous with $\sup_{x \in \Omega} p(x) < d$ that $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$. In fact, they proved the embedding even for $\sup_{x \in \Omega} p(x) = d$; nevertheless, the space $L^{p^*(\cdot)}(\Omega)$ with $p^*(x) = \infty$ needs to be replaced by a more general space of Orlicz–Musielak type. Later in [48], they extended this result for $p(x) \in W^{1,\sigma}(\Omega)$ for some $\sigma > d$, assuming $p^+ < d$. In [34], Diening proved same optimal Sobolev embeddings and the generalization of Rellich–Kondrachov theorem on bounded Lipschitz domains with log-Hölder continuous $p(x)$ satisfying $1 < p^- \leq p^+ < d$. Here we summarize the result in [34].

Theorem 2.4.5. *Let* $p \in \mathcal{P}^{\log}(\Omega)$ *such that* $1 < p^- \leq p^+ < d$, *and assume that* $q \in \mathcal{P}(\Omega)$ *satisfies* $q(x) \leq p^*(x)$ *a.e.* $x \in \Omega$. *Then we have*

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$$

where the corresponding embedding constant depends only on d, $|\Omega|$, p^+ *and* $C_{\log}(p)$. *Moreover, if* $q(x) < p^*(x)$ *a.e.* $x \in \Omega$, *the embedding is compact.*

In the later analysis, we also need some important functional inequalities. For the proofs of these inequalities, see [36] for more detail. We begin with Poincaré's inequality.

Theorem 2.4.6. (Poincaré's inequality) *Suppose that* $p \in \mathcal{P}^{\log}(\Omega)$. *Then, for every* $u \in W_0^{1,p(\cdot)}(\Omega)$, *the inequality*

$$\|u\|_{p(\cdot)} \leq C \operatorname{diam}(\Omega) \|\nabla u\|_{p(\cdot)}$$

holds with a positive constant $C > 0$ *depending only on* d *and* $C_{\log}(p)$.

In the constant-exponent case, there is an obvious connection between modular and norm versions of the inequality, which does not hold in the variable-exponent context. Indeed, the following one-dimensional example shows that the Poincaré inequality cannot, in general, hold in a modular form.

Example 2.4.7. Let $p : (-2, 2) \to [2, 3]$ be a Lipschitz continuous exponent that equals 3 in $(-2, -1) \cup (1, 2)$, 2 in $(-1/2, 1/2)$ and is linear elsewhere. Let u_λ be a Lipschitz function such that $u_\lambda(\pm 2) = 0$, $u_\lambda = \lambda$ in $(-1, 1)$ and $|u'_\lambda| = \lambda$ in $(-2, -1) \cup (1, 2)$. Then,

$$\frac{A_{p(\cdot)}(u_\lambda)}{A_{p(\cdot)}(u'_\lambda)} = \frac{\int_{-2}^{2} |u_\lambda|^{p(x)} \,\mathrm{d}x}{\int_{-2}^{2} |u'_\lambda|^{p(x)} \,\mathrm{d}x} \geq \frac{\int_{-1/2}^{1/2} \lambda^2 \,\mathrm{d}x}{2\int_{-2}^{-1} \lambda^3 \,\mathrm{d}x} = \frac{1}{2\lambda} \to \infty$$

as $\lambda \to 0^+$.

In fact, Fan, Zhao and Zhang [50] have shown that the modular Poincaré inequality $A_{p(\cdot)}(u) \leq C A_{p(\cdot)}(\nabla u)$ does not hold if p is continuous and has a minimum or maximum. But in [4], Allegretto has shown that the inequality holds if there exists a function $\xi \in W^{1,1}(\Omega)$ such that $\nabla p \cdot \nabla \xi \geq 0$ and $\nabla \xi \neq 0$. This holds if p is suitably monotone.

Let us now turn our attention to Korn's inequality. In the context of fluid dynamics and elasticity, the governing partial differential equation only gives control of the symmetric part of the gradient rather of the full gradient itself. For $\boldsymbol{u} \in W^{1,1}_{\mathrm{loc}}(\Omega)^d$, we define the symmetric gradient $\boldsymbol{D}\boldsymbol{u}$ by

$$\boldsymbol{D}\boldsymbol{u} := \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right).$$

In particular, the partial differential equation only ensures that the norm of $\boldsymbol{D}\boldsymbol{u}$ can be controlled. However, from a point of view of Sobolev spaces, it is desirable to have control of $\nabla\boldsymbol{u}$. Although it is not possible to estimate $|\nabla\boldsymbol{u}|$ pointwise by $|\boldsymbol{D}\boldsymbol{u}|$, it is in some cases possible to bound the norm of $\nabla\boldsymbol{u}$ in terms of the norm of $\boldsymbol{D}\boldsymbol{u}$. In particular, for all $\boldsymbol{u} \in W_0^{1,q}(\mathbb{R}^d)$ with $1 < q < \infty$, we have Korn's inequality

$$\|\nabla\boldsymbol{u}\|_q \le C\|\boldsymbol{D}\boldsymbol{u}\|_q.$$

Moreover, it was shown that one can generalize Korn's inequality to the variable-exponent spaces $L^{p(\cdot)}(\Omega)$. First, for bounded domains, we introduce the generalized Sobolev space of functions with zero boundary value.

$$W_0^{1,p(\cdot)}(\Omega) := \left\{u \in W^{1,p(\cdot)}(\Omega) : u = 0 \text{ on } \partial\Omega\right\}.$$

What we need in the subsequent analysis is the following result whose proof can be found in Chapter 14 in [36], which follows the idea of [42]. A completely different proof can be found in [40]. Note that we assume zero boundary values for $\boldsymbol{u}$.

Theorem 2.4.8. (Korn's inequality) *Let $p \in \mathcal{P}^{\log}(\Omega)$ with $1 < p^- \le p^+ < \infty$. Then there exists a positive constant $C > 0$ such that*

$$\|\nabla\boldsymbol{u}\|_{p(\cdot)} \le C\|\boldsymbol{D}\boldsymbol{u}\|_{p(\cdot)}$$

for all $\boldsymbol{u} \in W_0^{1,p(\cdot)}(\Omega)^d$.

Next, we shall investigate another auxiliary result, which is referred to as the *local-to-global result*, which is a generalization of an analogous result in classical L^p spaces. In the classical Lebesgue space setting, the following is easily seen to hold:

$$\|f\|_p = \left(\sum_i \|\chi_{\Omega_i} f\|_p^p\right)^{\frac{1}{p}} = \left\|\sum_i \chi_{\Omega_i}\frac{\|\chi_{\Omega_i} f\|_p}{\|\chi_{\Omega_i}\|_p}\right\|_p,$$

where $\Omega = \cup_i \Omega_i$ and $\Omega_i \cap \Omega_j \neq \emptyset$ for $i \neq j$. This raises the question whether in a variable-exponent space $L^{p(\cdot)}(\Omega)$, one has

$$\|f\|_{p(\cdot)} \approx \left\| \sum_i \chi_{\Omega_i} \frac{\|\chi_{\Omega_i} f\|_{p(\cdot)}}{\|\chi_{\Omega_i}\|_{p(\cdot)}} \right\|_{p(\cdot)} .$$

This statement is indeed true, provided that $p \in \mathcal{P}^{\log}$ and $\{\Omega_i\}$ is locally N-finite in the sense of the following definition.

Definition 2.4.9. Let $N \in \mathbb{N}$. A family $\mathcal{Q}$ of measurable sets $Q \subset \mathbb{R}^d$ is called *locally N-finite* if

$$\sum_{Q \in \mathcal{Q}} \chi_Q \leq N$$

almost everywhere in $\mathbb{R}^d$.

Let us now state the norm-equivalence theorem precisely; for its proof, see Chapter 7 in [36].

Theorem 2.4.10. *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^d)$ and let $\mathcal{Q}$ be a locally N-finite family of cubes or balls $Q \subset \mathbb{R}^d$. Then,*

$$\left\| \sum_{Q \in \mathcal{Q}} \chi_Q f \right\|_{p(\cdot)} \approx \left\| \sum_{Q \in \mathcal{Q}} \chi_Q \frac{\|\chi_Q f\|_{p(\cdot)}}{\|\chi_Q\|_{p(\cdot)}} \right\|_{p(\cdot)}$$

for all $f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^d)$. The constants, not explicitly indicated in this norm-equivalence (henceforth referred to as 'implicit constants'), only depend on $C_{\log}(p)$, d and N.

Theorem 2.4.10 is a very powerful tool. It allows us to extend many estimates and results known for cubes or balls to more complicated domains. To be able to make use of the formula appearing on the right-hand side of the norm-equivalence stated in Theorem 2.4.10, we need to compute the variable-exponent norm $\|\chi_Q\|_{p(\cdot)}$ of the characteristic function χ_Q. Some related results are presented in Chapter 4 of [36]; what we need here is the following theorem stated therein.

Theorem 2.4.11. *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^d)$. Then, for every cube or ball $Q \subset \mathbb{R}^d$,*

$$\|\chi_Q\|_{p(\cdot)} \approx |Q|^{\frac{1}{p(x)}} \qquad \text{if } |Q| \leq 2^d \text{ and } x \in Q.$$

The implicit constants only depend on $C_{\log}(p)$.

2.5 Bounded operators in variable-exponent spaces

Finally, we introduce some important operators and we discuss the boundedness of these operators in variable-exponent spaces.

We start with the divergence equation, which is of great importance in the theory of incompressible fluids. For given f with mean value zero, we seek a solution $\boldsymbol{u}$ with zero boundary values of

$$\operatorname{div} \boldsymbol{u} = f \qquad \text{in } \Omega, \tag{2.3}$$

where Ω is a bounded, open and Lipschitz domain. This problem has been studied by many authors. The L^q-theory in Lipshcitz domain is based on an explicit representation formula that is due to Bogovskiĭ in [14, 15]. Here we introduce the generalized version of the result to the case of Lebesgue and Sobolev spaces with variable exponents. Since we are looking for solutions $\boldsymbol{u}$ of (2.3) with zero boundary values, we need to assume that the right-hand side has a vanishing integral. Let us denote the space of such functions by

$$L_0^{p(\cdot)}(\Omega) := \left\{ f \in L^{p(\cdot)}(\Omega) : \int_\Omega f(x)\,\mathrm{d}x = 0 \right\}.$$

What we need here is the following theorem, which concerns the existence of a certain bounded linear operator, called the *Bogovskiĭ operator*, in variable-exponent spaces.

Theorem 2.5.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open Lipschitz domain and suppose that $p \in \mathcal{P}^{\log}(\Omega)$ with $1 < p^- \leq p^+ < \infty$. Then, there exists a bounded linear operator $\mathcal{B} : L_0^{p(\cdot)}(\Omega) \to W_0^{1,p(\cdot)}(\Omega)^d$ such that for all $f \in L_0^{p(\cdot)}(\Omega)$ we have*

$$\operatorname{div}(\mathcal{B}f) = f,$$
$$\|\mathcal{B}f\|_{1,p(\cdot)} \leq C\|f\|_{p(\cdot)},$$

where C depends on Ω, p^-, p^+, and $C_{\log}(p)$.

As a consequence of Theorem 2.5.1, we can prove an inf-sup condition, which has a crucial role in the mathematical analysis of incompressible fluid flow problems. The following proposition is a direct consequence of the existence of the Bogovskiĭ operator in spaces with fixed exponent, which is a special case of Theorem 2.5.1; see [14, 42] for additional details.

Proposition 2.5.2. *For any s, $s' \in (1,\infty)$, with $\frac{1}{s}+\frac{1}{s'}=1$, there exists a positive constant $\alpha_s > 0$ such that*

$$\alpha_s \|q\|_{s'} \leq \sup_{0\neq \boldsymbol{v}\in W_0^{1,s}(\Omega)^d} \frac{\langle \operatorname{div} \boldsymbol{v}, q\rangle}{\|\boldsymbol{v}\|_{1,s}} \qquad \forall\, q \in L_0^{s'}(\Omega). \tag{2.4}$$

Furthermore, one can prove the following inf-sup condition in spaces with variable-exponent norms, which will play an important role in the subsequent analysis.

Proposition 2.5.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open Lipschitz domain and let $p \in \mathcal{P}^{\log}(\Omega)$ with $1 < p^- \leq p^+ < \infty$. Then, there exists a constant $\alpha_p > 0$ such that*

$$\alpha_p \|q\|_{p'(\cdot)} \leq \sup_{0\neq \boldsymbol{v}\in W_0^{1,p(\cdot)}(\Omega)^d} \frac{\langle \operatorname{div} \boldsymbol{v}, q\rangle}{\|\boldsymbol{v}\|_{1,p(\cdot)}} \qquad \forall\, q \in L_0^{p'(\cdot)}(\Omega).$$

Proposition 2.5.3 is a consequence of Theorem 2.5.1 and the norm-conjugate formula presented in Theorem 2.3.3.

Next, we discuss the properties of the maximal operator. In order to derive more sophisticated results for the spaces $L^{p(\cdot)}(\mathbb{R}^d)$, we have to investigate the Hardy–Littlewood maximal operator M on $L^{p(\cdot)}(\mathbb{R}^d)$. The central property of the maximal operator is that it is a bounded operator from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ for $p \in (1,\infty]$. We introduce the variable-exponent generalization of this result.

For any $f \in L^1(\mathbb{R}^d)$, we define the Hardy–Littlewood maximal operator by

$$(Mf)(x) := \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| \,\mathrm{d}y, \qquad x \in \mathbb{R}^d,$$

where $B_r(x)$ is the open ball in $\mathbb{R}^d$ of radius r centred at $x \in \mathbb{R}^d$. Similarly, for any $f \in W^{1,1}(\mathbb{R}^d)$, we define $M(\nabla f) := M(|\nabla f|)$.

To prove the boundedness of this operator in $L^{p(\cdot)}(\mathbb{R}^d)$, we need a technical tool concerning variable-exponent spaces, which is called the *key estimate.* This will be independently used in our analysis. The motivation for the key estimate comes from the integral version of Jensen's inequality, which states that, for every real-valued convex function ψ defined on $[0,\infty)$, and every cube Q, we have

$$\psi\left(⨍_Q |f(y)| \,\mathrm{d}y\right) \leq ⨍_Q \psi(|f(y)|) \,\mathrm{d}y.$$

This simple but crucial estimate allows, for example, to transfer the L^1-L^∞ estimates for interpolation operators to the setting of Orlicz spaces, see [41]. Therefore, it is necessary for us to find a suitable substitute for Jensen's inequality in the context of variable exponents. Our goal is to control $\left(\fint_Q |f(y)|\,\mathrm{d}y\right)^{p(x)}$ in terms of $\fint_Q |f(x)|^{p(x)}\,\mathrm{d}x$. If p is constant, this is exactly Jensen's inequality and it holds for all $f \in L^p(Q)$. However, for variable p, it is impossible to derive such estimates for all f and we have to restrict ourselves to a certain set of admissible functions f. Moreover, an additional error term appears. The following statement is a special case of Corollary 1 in [44], which is an improvement of Corollary 4.2.5 in [36].

Theorem 2.5.4. (Key estimate). *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^d)$ with $p^+ < \infty$. Then, for every $m > 0$, there exists a constant $C > 0$, which depends only on m, $C_{\log}(p)$ and p^+, such that*

$$\left(\fint_Q |f(y)|\,\mathrm{d}y\right)^{p(x)} \leq C \fint_Q |f(y)|^{p(y)}\,\mathrm{d}y + C|Q|^m \tag{2.5}$$

for every cube (or ball) $Q \subset \mathbb{R}^d$ with $|Q| \leq 1$, all $x \in Q$ and all $f \in L^1(Q)$ with

$$\fint_Q |f|\,\mathrm{d}y \leq |Q|^{-m}.$$

By using the key estimate, we can prove the following boundedness result, whose proof can be found in Chapter 4 in [36].

Theorem 2.5.5. *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^d)$ with $p^- > 1$. Then there exists a constant $K > 0$ depending only on d and $C_{\log}(p)$ such that*

$$\|Mf\|_{p(\cdot)} \leq K(p^-)'\|f\|_{p(\cdot)}$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^d)$.

The proof of Theorem 2.5.5 goes back to many authors. The first version is due to Diening [33], who proved the result for bounded exponents that are constant outside a large ball. This condition was later relaxed by Cruz-Uribe, Fiorenza, Martell and Pérez in [29] to the log-Hölder decay condition and by Nekvinda in [71] to the integral condition $1 \in L^{p(\cdot)}$. The boundedness of the exponent was then removed in [35] and [28]. The proof presented in reference [36] is closest to the one in [35].

Similarly to the case of Poincaré's inequality, the above theorem does not hold in a modular form. For instance, Lerner [64] showed that

$$\int_{\mathbb{R}^d} |Mf|^{p(x)}\,\mathrm{d}x \leq C \int_{\mathbb{R}^d} |f|^{p(x)}\,\mathrm{d}x$$

if and only if $p \in (1, \infty)$ is constant.

Finally, we recall the following generalization of McShane's extension theorem (cf. Corollary 1 in [68]) to variable-exponent spaces and the boundedness of the maximal operator in variable-exponent context.

Lemma 2.5.6. (Variable-index extension [30]) *Let $\Omega \subset \mathbb{R}^d$ be an bounded open Lipschitz domain and suppose that $p \in \mathcal{P}^{\log}(\Omega)$ is arbitrary with $p^- > 1$. Then, there exists an extension $q \in \mathcal{P}^{\log}(\mathbb{R}^d)$ such that $q^- = p^-$ and $q^+ = p^+$, and the Hardy–Littlewood maximal operator M is bounded from $L^{q(\cdot)}(\mathbb{R}^d)$ to $L^{q(\cdot)}(\mathbb{R}^d)$.*

Having discussed the mathematical background concerning variable-exponent function spaces, in the next chapter we shall start to focus on the numerical approximation of the model problem under consideration.

Chapter 3

Finite element approximation of the stationary model in 2D

3.1 Brief description of the chapter

As we described in the Introduction, we shall investigate a system of equations describing the motion of a shear-thinning fluid with a non-standard growth condition on the viscosity. More precisely, we shall consider the incompressible generalized Navier–Stokes equations with a power-law-like viscosity where the power-law index is not fixed, but depends on the concentration, and we shall assume that the concentration satisfies a convection-diffusion equation. We first consider the stationary model. In other words, we consider the following system of PDEs:

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{3.1}$$

$$\operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div} \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u}) = -\nabla \pi + \boldsymbol{f} \qquad \text{in } \Omega, \tag{3.2}$$

$$\operatorname{div}(c\boldsymbol{u}) - \operatorname{div} \boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u}) = 0 \qquad \text{in } \Omega. \tag{3.3}$$

As discussed in the Introduction, we shall further assume suitable structural conditions on $\boldsymbol{S}(\cdot,\cdot)$ and $\boldsymbol{q}_c(\cdot,\cdot,\cdot)$. First, the stress tensor $\boldsymbol{S} : \mathbb{R}_{\geq 0} \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is assumed to be a continuous function satisfying, for some positive constant C_1, C_2 and C_3, that

$$|\boldsymbol{S}(\xi, \boldsymbol{B})| \leq C_1(|\boldsymbol{B}|^{p(\xi)-1} + 1), \tag{3.4}$$

$$(\boldsymbol{S}(\xi, \boldsymbol{B_1}) - \boldsymbol{S}(\xi, \boldsymbol{B_2})) \cdot (\boldsymbol{B_1} - \boldsymbol{B_2}) > 0 \text{ for } \boldsymbol{B_1} \neq \boldsymbol{B_2}, \tag{3.5}$$

$$\boldsymbol{S}(\xi, \boldsymbol{B}) \cdot \boldsymbol{B} \geq C_2(|\boldsymbol{B}|^{p(\xi)} + |\boldsymbol{S}|^{p'(\xi)}) - C_3, \tag{3.6}$$

where : $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a Hölder continuous function with $1 < p^- \leq p(\xi) \leq p^+ < \infty$ and $p'(\xi)$ is defined as its Hölder conjugate, $\frac{p(\xi)}{p(\xi)-1}$. We further assume that the

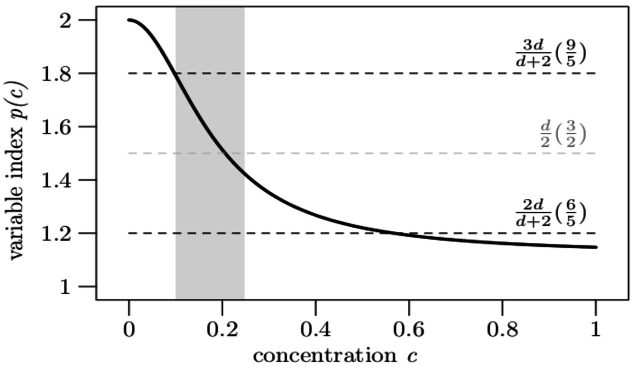

Figure 3.1: Shear-thinning index of viscosity for synovial fluid

concentration flux vector $\boldsymbol{q}_c(\xi, \boldsymbol{g}, \boldsymbol{B}) : \mathbb{R}_{\geq 0} \times \mathbb{R}^d \times \mathbb{R}^{d\times d}_{\text{sym}} \to \mathbb{R}^d$ is a continuous function, which is linear with respect to $\boldsymbol{g}$, and additionally satisfies the following inequalities: there exist positive constants C_4 and C_5 such that

$$|\boldsymbol{q}_c(\xi, \boldsymbol{g}, \boldsymbol{B})| \leq C_4 |\boldsymbol{g}|, \tag{3.7}$$

$$\boldsymbol{q}_c(\xi, \boldsymbol{g}, \boldsymbol{B}) \cdot \boldsymbol{g} \geq C_5 |\boldsymbol{g}|^2. \tag{3.8}$$

To complete the problem, we prescribe the following Dirichlet boundary conditions:

$$\boldsymbol{u} = \boldsymbol{0}, \qquad c = c_d \qquad \text{on } \partial\Omega, \tag{3.9}$$

where $c_d \in W^{1,q}(\Omega)$ for some $q > d$ and $c_d \geq 0$ a.e. on Ω

In this chapter, we consider the construction of a finite element approximation of the system of nonlinear partial differential equations (3.1)–(3.3) and, motivated by the ideas in [21], we develop the convergence analysis of this numerical method in the case of variable-exponent spaces in a two-dimensional domain.

Let us make some comments on the lower bound of p^-. Figure 3.1 presents the shear-thinning index of viscosity for the synovial fluid. The exponent p is plotted as a function of concentration. The physiological values are approximately in the range $(0.1, 0.25)$, in the graph depicted by the gray rectangle. The non-dimensionalized concentration $c = 1$ refers to (non-physical) 100% concentration of the solvent. Dashed lines correspond to lower bounds p^- as required by the mathematical tools employed in the proof of existence of a weak solution; see, for example, [20] and [21]. As a starting point, in this chapter, we perform the finite element analysis with $p^- \geq \frac{3}{2}$ corresponding to $p^- \geq \frac{3d}{d+2}$ in 2D, which can be covered by the application of monotone operator theory, where the use of Lipschitz truncation can be avoided. However, as one can see from Firgure 3.1, it is desirable to perform the analysis with p^- less than $\frac{3d}{d+2}$ for practical reasons. Therefore, instead of just using monotone operator theory, here we develop a more general theory which might be useful for the analysis with p^- lower than $\frac{3d}{d+2}$. For example, if we can prove the estimate (3.23) with p^- lower than $\frac{3}{2}$, with the theory developed here, we can perform the same analysis with lower values of p^-. In that case, monotone operator theory is not applicable and the use of Lipschitz truncation is necessary as developed here. Key technical tools include discrete counterparts of the Bogovskiĭ operator, a De Giorgi-type regularity theorem in two dimensions, and the Acerbi–Fusco Lipschitz truncation of Sobolev functions, in function spaces with variable-integrability exponents. The extension of the results to the case of three space dimensions will be considered in the next chapter as we need to use a different numerical scheme. Nevertheless, at least initially, we shall admit $d \in \{2, 3\}$. Subsequently we shall restrict ourselves to the case of $d = 2$. Also, as no uniqueness result is currently available for weak solutions of the problem under consideration, we can only show that a subsequence of the sequence of numerical approximations converges to *a* weak solution of the problem. First, as mentioned above, we perform the finite element analysis with the general theory using Lipschitz truncation, which is applicable to the problem with p^- lower than $\frac{3}{2}$. And for the completeness, at the end of the chapter, we will include how current result can be also obtained by the use monotone operator theory without using Lipschitz truncation. The content of this chapter is based on the journal paper [57].

3.2 Definition of weak solutions

We begin with the definitions of the following function spaces that are frequently used in connection with mathematical models of incompressible fluids:

$$\begin{aligned} W_0^{1,p(\cdot)}(\Omega)^d &:= \left\{ \boldsymbol{u} \in W^{1,p(\cdot)}(\Omega)^d : \boldsymbol{u} = \mathbf{0} \text{ on } \partial\Omega \right\}, \\ W_{0,\mathrm{div}}^{1,p(\cdot)}(\Omega)^d &:= \left\{ \boldsymbol{u} \in W_0^{1,p(\cdot)}(\Omega)^d : \mathrm{div}\, \boldsymbol{u} = 0 \right\}. \end{aligned}$$

Using this notation, the weak formulation of the problem (3.1)–(3.3), with the nonlinear terms satisfying the assumptions above, is as follows.

Problem (Q). For $\boldsymbol{f} \in (W_0^{1,p^-}(\Omega)^d)^*$, $c_d \in W^{1,q}(\Omega)$, $q > d$, and a Hölder continuous function p, with $1 < p^- \leq p(\xi) \leq p^+ < \infty$ for all $\xi \in \mathbb{R}_{\geq 0}$, find $(c - c_d) \in W_0^{1,2}(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$, $\boldsymbol{u} \in W_0^{1,p(c)}(\Omega)^d$, $\pi \in L_0^{p'(c)}(\Omega)$ such that

$$\int_\Omega (\boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u}) : \nabla \boldsymbol{\psi} - (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{\psi}) \,\mathrm{d}x - \langle \operatorname{div} \boldsymbol{\psi}, \pi \rangle = \langle \boldsymbol{f}, \boldsymbol{\psi} \rangle \quad \forall\, \boldsymbol{\psi} \in W_0^{1,\infty}(\Omega)^d,$$

$$\int_\Omega q \operatorname{div} \boldsymbol{u} \,\mathrm{d}x = 0 \qquad \forall\, q \in L_0^{p'(c)}(\Omega),$$

$$\int_\Omega (\boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u}) \cdot \nabla \varphi - c\boldsymbol{u} \cdot \nabla \varphi) \,\mathrm{d}x = 0 \qquad \forall\, \varphi \in W_0^{1,2}(\Omega).$$

Thanks to Proposition 2.5.3, we can restate **Problem (Q)** in the following (equivalent) divergence-free setting.

Problem (P). For $\boldsymbol{f} \in (W_0^{1,p^-}(\Omega)^d)^*$, $c_d \in W^{1,q}(\Omega)$, $q > d$, and a Hölder continuous function p, with $1 < p^- \leq p(\xi) \leq p^+ < \infty$ for all $\xi \in \mathbb{R}_{\geq 0}$, find $(c - c_d) \in C^{0,\alpha}(\overline{\Omega}) \cap W_0^{1,2}(\Omega)$, $\boldsymbol{u} \in W_{0,\mathrm{div}}^{1,p(c)}(\Omega)^d$, such that

$$\int_\Omega (\boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u}) : \nabla \boldsymbol{\psi} - (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{\psi}) \,\mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{\psi} \rangle \qquad \forall\, \boldsymbol{\psi} \in W_{0,\mathrm{div}}^{1,\infty}(\Omega)^d,$$

$$\int_\Omega (\boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u}) \cdot \nabla \varphi - c\boldsymbol{u} \cdot \nabla \varphi) \,\mathrm{d}x = 0 \qquad \forall\, \varphi \in W_0^{1,2}(\Omega).$$

The existence of a weak solution to problem **(P)** was initially proved in [20] in the case when the variable exponent $x \mapsto p(x)$ is bounded below by $p^- > \max\left\{\frac{d}{2}, \frac{3d}{d+2}\right\}$ with the help of monotone operator theory. The result was improved in [21], where the existence of weak solutions was proved for $p^- > \frac{d}{2}$ by using Lipschitz truncation technique. As mentioned above, the aim of this chapter is to perform the convergence analysis of a finite element approximations of this problem in two dimensions.

3.3 Finite element approximation

In this section, we will construct finite element spaces, which we shall use in this chapter and state the Galerkin approximation of the problem (3.1)–(3.3). The existence of a finite element solution in the discretely divergence-free setting will be established by using Brouwer's fixed point theorem. Next, we shall prove a discrete inf-sup condition to ensure the existence of a discrete pressure. Finally we will state and prove discrete counterparts of some well-known theorems, which will be key tools in the convergence analysis of the finite element approximation of the problem under consideration.

3.3.1 Finite element spaces

Let $\{\mathcal{G}_n\}$ be a family of shape-regular partitions of $\overline{\Omega}$ satisfying the following properties:

- **Affine equivalence**: For every element $E \in \mathcal{G}_n$, there exists a non-singular affine mapping
$$\boldsymbol{F}_E : E \to \hat{E},$$
where $\hat{E}$ is the standard reference d-simplex in $\mathbb{R}^d$.

- **Shape-regularity**: For any element $E \in \mathcal{G}_n$, the ratio of $\operatorname{diam} E$ to the radius of the inscribed ball is bounded below uniformly by a positive constant, with respect to all $\mathcal{G}_n$ and $n \in \mathbb{N}$.

For a given partition $\mathcal{G}_n$, the finite element spaces are defined by

$$\begin{aligned}
\mathbb{V}^n &= \mathbb{V}(\mathcal{G}_n) := \{\boldsymbol{V} \in C(\overline{\Omega})^d : \boldsymbol{V}_{|E} \circ \boldsymbol{F}_E^{-1} \in \hat{\mathbb{P}}_{\mathbb{V}}, E \in \mathcal{G}_n \text{ and } \boldsymbol{V}_{|\partial\Omega} = \boldsymbol{0}\},\\
\mathbb{Q}^n &= \mathbb{Q}(\mathcal{G}_n) := \{Q \in L^\infty(\Omega) : Q_{|E} \circ \boldsymbol{F}_E^{-1} \in \hat{\mathbb{P}}_{\mathbb{Q}}, E \in \mathcal{G}_n\},\\
\mathbb{Z}^n &= \mathbb{Z}(\mathcal{G}_n) := \{Z \in C(\overline{\Omega}) : Z_{|E} \circ \boldsymbol{F}_E^{-1} \in \hat{\mathbb{P}}_{\mathbb{Z}}, E \in \mathcal{G}_n \text{ and } Z_{|\partial\Omega} = 0\},
\end{aligned}$$

where $\hat{\mathbb{P}}_{\mathbb{V}} \subset W^{1,\infty}(\hat{E})^d$, $\hat{\mathbb{P}}_{\mathbb{Q}} \subset L^\infty(\hat{E})$ and $\hat{\mathbb{P}}_{\mathbb{Z}} \subset W^{1,\infty}(\hat{E})$ are finite-dimensional subspaces.

$\mathbb{V}^n$ and $\mathbb{Z}^n$ are assumed to have finite and locally supported bases; for example, in the case of $\mathbb{V}^n$, for each $n \in \mathbb{N}$, there exists an $N_n \in \mathbb{N}$ such that

$$\mathbb{V}^n = \operatorname{span}\{\boldsymbol{V}_1^n, \dots, \boldsymbol{V}_{N_n}^n\}$$

and for each basis function $\boldsymbol{V}_i^n$, $i = 1, \dots, N_n$, we have that if there exists an $E \in \mathcal{G}_n$ with $\boldsymbol{V}_j^n \neq 0$ on E, then

$$\operatorname{supp} \boldsymbol{V}_j^n \subset \bigcup \{E' \in \mathcal{G}_n : E' \cap E \neq \emptyset\} =: S_E.$$

We shall assume that, for each $n \in \mathbb{N}$ and for each (closed) element $E \in \mathcal{G}_n$, either the (closed) patch of elements S_E has empty intersection with $\partial\Omega$, or, if the intersection of S_E with $\partial\Omega$ is nonempty, then $S_E \cap \partial\Omega$ has positive $(d-1)$-dimensional surface measure.

For the pressure space $\mathbb{Q}^n$, we assume that $\mathbb{Q}^n$ has a basis consisting of discontinuous piecewise polynomials; i.e., for each $n \in \mathbb{N}$, there exists an $\tilde{N}_n \in \mathbb{N}$ such that

$$\mathbb{Q}^n = \operatorname{span}\{Q_1^n, \dots, Q_{\tilde{N}_n}^n\}$$

and for each basis function Q_i^n, we have that

$$\operatorname{supp} Q_i^n = E \qquad \text{for some } \ E \in \mathcal{G}_n.$$

We assume further that $\mathbb{V}^n$ contains continuous piecewise linear functions and $\mathbb{Q}^n$ contains piecewise constant functions.

Note further, by shape-regularity, that

$$\exists m \in \mathbb{N} : |S_E| \leq m|E| \ \text{ for all } \ E \in \mathcal{G}_n,$$

where m is independent of n. We denote by h_E the diameter of E.

We also introduce the subspace $\mathbb{V}^n_{\mathrm{div}}$ of discretely divergence-free functions. More precisely, we define

$$\mathbb{V}^n_{\mathrm{div}} := \{\boldsymbol{V} \in \mathbb{V}^n : \langle \operatorname{div} \boldsymbol{V}, Q\rangle = 0 \ \ \forall\, Q \in \mathbb{Q}^n\},$$

and the subspace of $\mathbb{Q}^n$ consisting of vanishing integral mean-value approximations:

$$\mathbb{Q}^n_0 := \{Q \in \mathbb{Q}^n : \int_\Omega Q \,\mathrm{d}x = 0\}.$$

Throughout this chapter, we assume that all finite element spaces introduced above have the following properties.

Assumption 1 (Approximability): For all $s \in [1, \infty)$,

$$\begin{aligned} \inf_{\boldsymbol{V} \in \mathbb{V}^n} \|\boldsymbol{v} - \boldsymbol{V}\|_{1,s} &\to 0 && \forall\, \boldsymbol{v} \in W_0^{1,s}(\Omega)^d \text{ as } n \to \infty, \\ \inf_{Q \in \mathbb{Q}^n} \|q - Q\|_s &\to 0 && \forall\, q \in L^s(\Omega) \text{ as } n \to \infty, \\ \inf_{Z \in \mathbb{Z}^n} \|z - Z\|_{1,s} &\to 0 && \forall\, z \in W_0^{1,s}(\Omega) \text{ as } n \to \infty. \end{aligned}$$

For this, a necessary condition is that the maximal mesh size vanishes, i.e. we have $\max_{E \in \mathcal{G}_n} h_E \to 0$ as $n \to \infty$.

Assumption 2 (Existence of a projection operator Π^n_{div}): For each $n \in \mathbb{N}$, there exists a linear projection operator $\Pi^n_{\mathrm{div}} : W_0^{1,1}(\Omega)^d \to \mathbb{V}^n$ such that:

- Π^n_{div} preserves the divergence structure in the dual of the discrete pressure space, in other words, for any $\boldsymbol{v} \in W^{1,1}_0(\Omega)^d$, we have

$$\langle \operatorname{div} \boldsymbol{v}, Q\rangle = \langle \operatorname{div} \Pi^n_{\mathrm{div}} \boldsymbol{v}, Q\rangle \qquad \forall\, Q \in \mathbb{Q}^n.$$

- Π^n_{div} is locally $W^{1,1}$-stable, i.e., there exists a constant $c_1 > 0$, independent of n, such that for all $\boldsymbol{v} \in W^{1,1}_0(\Omega)^d$ and all $E \in \mathcal{G}_n$,

$$\fint_E \left(|\Pi^n_{\mathrm{div}}\boldsymbol{v}| + h_E|\nabla \Pi^n_{\mathrm{div}}\boldsymbol{v}|\right) \mathrm{d}x \le c_1 \fint_{S_E} \left(|\boldsymbol{v}| + h_E|\nabla \boldsymbol{v}|\right) \mathrm{d}x. \tag{3.10}$$

We claim that (3.10) implies the following inequality: there exists a constant $c > 0$, independent of n, such that

$$\fint_E |\nabla \Pi^n_{\mathrm{div}}\boldsymbol{v}| \,\mathrm{d}x \le c \fint_{S_E} |\nabla \boldsymbol{v}| \,\mathrm{d}x \qquad \forall\, \boldsymbol{v} \in W^{1,1}_0(\Omega)^d \text{ and } \forall\, E \in \mathcal{G}_n. \tag{3.11}$$

The proof of (3.11) proceeds as follows. As, by hypothesis, $\mathbb{V}^n$ contains the set of all d-component continuous piecewise linear functions on $\mathcal{G}_n$ that vanish on $\partial\Omega$, for any (closed) element $E \in \mathcal{G}_n$ for which the (closed) patch of elements S_E has empty intersection with $\partial\Omega$, any d-component vector function $\mathbf{c}$ whose components are constant on S_E can be extended to a d-component continuous piecewise linear function on $\mathcal{G}_n$, contained in $\mathbb{V}^n$. Thus we have, using (3.10) with $\boldsymbol{v} - \mathbf{c} \in \mathbb{V}^n$, that

$$\fint_E h_E |\nabla \Pi^n_{\mathrm{div}}\boldsymbol{v}| \,\mathrm{d}x = \fint_E h_E |\nabla \Pi^n_{\mathrm{div}}(\boldsymbol{v} - \mathbf{c})| \,\mathrm{d}x \le c \fint_{S_E} \left(|\boldsymbol{v} - \mathbf{c}| + h_E|\nabla \boldsymbol{v}|\right) \mathrm{d}x.$$

With $\mathbf{c} = \fint_{S_E} \boldsymbol{v} \,\mathrm{d}x$, Poincaré's inequality gives

$$\fint_{S_E} |\boldsymbol{v} - \mathbf{c}| \,\mathrm{d}x \le c \fint_{S_E} h_E |\nabla \boldsymbol{v}| \,\mathrm{d}x.$$

Combining the last two inequalities and cancelling the factor h_E then yields (3.11) for elements $E \in \mathcal{G}_n$ for which S_E has empty intersection with $\partial\Omega$.

If, on the other hand, $E \in \mathcal{G}_n$ is such that S_E has nonempty intersection with $\partial\Omega$, then, since by hypothesis the intersection of S_E with $\partial\Omega$ has, for such E, positive $(d-1)$-dimensional surface measure, we have, this time by Friedrichs' inequality, that

$$\fint_{S_E} |\boldsymbol{v}| \,\mathrm{d}x \le c \fint_{S_E} h_E |\nabla \boldsymbol{v}| \,\mathrm{d}x.$$

Using this on the right-hand side of (3.10) directly yields (3.11) for any such E. Thus we have shown that (3.10) implies (3.11).

Note further that the local $W^{1,1}(\Omega)^d$-stability of Π^n_{div} implies its local and global $W^{1,s}(\Omega)^d$-stability for $s \in [1,\infty]$. Indeed, since the function $t \mapsto t^s$ is a convex function for $s \in [1,\infty)$, by the equivalence of norms in finite-dimensional spaces, standard scaling arguments and Jensen's inequality, we have that

$$\begin{aligned}
|\Pi^n_{\mathrm{div}}\boldsymbol{v}(x)| + h_E|\nabla\Pi^n_{\mathrm{div}}\boldsymbol{v}(x)| &\leq \|\Pi^n_{\mathrm{div}}\boldsymbol{v}\|_{L^\infty(E)} + h_E\,\|\nabla\Pi^n_{\mathrm{div}}\boldsymbol{v}\|_{L^\infty(E)} \\
&\leq c \fint_E \left(|\Pi^n_{\mathrm{div}}\boldsymbol{v}(x)| + h_E|\nabla\Pi^n_{\mathrm{div}}\boldsymbol{v}(x)|\right)\mathrm{d}x \\
&\leq c \fint_{S_E} \left(|\boldsymbol{v}(x)| + h_E|\nabla\boldsymbol{v}(x)|\right)\mathrm{d}x \\
&\leq c \left(\fint_{S_E} \left(|\boldsymbol{v}(x)|^s + h_E^s|\nabla\boldsymbol{v}(x)|^s\right)\mathrm{d}x\right)^{\frac{1}{s}} \qquad \forall\, E \in \mathcal{G}_n.
\end{aligned}$$

Raising this inequality to the s-th power and integrating over E gives

$$\int_E \left(|\Pi^n_{\mathrm{div}}\boldsymbol{v}|^s + h_E^s|\nabla\Pi^n_{\mathrm{div}}\boldsymbol{v}|^s\right)\mathrm{d}x \leq c \int_{S_E} \left(|\boldsymbol{v}|^s + h_E^s|\nabla\boldsymbol{v}|^s\right)\mathrm{d}x. \tag{3.12}$$

Summing over all elements $E \in \mathcal{G}_n$, considering the locally finite overlap of patches (which is the consequence of the assumed shape-regularity of $\mathcal{G}_n$), and Poincaré's inequality implies, for any $s \in [1,\infty)$, that

$$\|\Pi^n_{\mathrm{div}}\boldsymbol{v}\|_{1,s} \leq c_s\,\|\boldsymbol{v}\|_{1,s} \qquad \forall\,\boldsymbol{v} \in W^{1,s}_0(\Omega)^d, \tag{3.13}$$

with a constant $c_s > 0$ independent of $n > 0$. With a similar argument, the same inequality can be derived for $s = \infty$. Note further that the approximability (**Assumption 1**) and inequality (3.13) imply the convergence of $\Pi^n_{\mathrm{div}}\boldsymbol{v}$ to $\boldsymbol{v}$. In fact, for any $\boldsymbol{V} \in \mathbb{V}^n$,

$$\begin{aligned}
\|\boldsymbol{v} - \Pi^n_{\mathrm{div}}\boldsymbol{v}\|_{1,s} &\leq \|\boldsymbol{v} - \boldsymbol{V}\|_{1,s} + \|\boldsymbol{V} - \Pi^n_{\mathrm{div}}\boldsymbol{v}\|_{1,s} \\
&= \|\boldsymbol{v} - \boldsymbol{V}\|_{1,s} + \|\Pi^n_{\mathrm{div}}(\boldsymbol{v} - \boldsymbol{V})\|_{1,s} \\
&\leq \|\boldsymbol{v} - \boldsymbol{V}\|_{1,s} + c_s\,\|\boldsymbol{v} - \boldsymbol{V}\|_{1,s}.
\end{aligned}$$

Hence we have

$$\|\boldsymbol{v} - \Pi^n_{\mathrm{div}}\boldsymbol{v}\|_{1,s} \leq C \inf_{\boldsymbol{V}\in\mathbb{V}^n} \|\boldsymbol{v} - \boldsymbol{V}\|_{1,s} \to 0 \qquad \forall\,\boldsymbol{v} \in W^{1,s}_0(\Omega)^d \text{ as } n \to \infty. \tag{3.14}$$

Assumption 3 (Existence of a projection operator $\Pi^n_{\mathbb{Q}}$): For each $n \in \mathbb{N}$, there exists a linear projection operator $\Pi^n_{\mathbb{Q}} : L^1(\Omega) \to \mathbb{Q}^n$ such that $\Pi^n_{\mathbb{Q}}$ is locally L^1-stable; i.e., there exists a constant $c_2 > 0$, independent of n, such that

$$\fint_E |\Pi^n_{\mathbb{Q}} q|\,\mathrm{d}x \leq c_2 \fint_{S_E} |q|\,\mathrm{d}x \tag{3.15}$$

for all $q \in L^1(\Omega)$ and all $E \in \mathcal{G}_n$.

Note that with the same argument as above, we have

$$\int_E |\Pi^n_{\mathbb{Q}} q|^{s'} \,\mathrm{d}x \le c_{s'} \int_{S_E} |q|^{s'} \,\mathrm{d}x \qquad \forall\, E \in \mathcal{G}_n, \quad \forall\, q \in L^{s'}(\Omega), \quad \forall\, s' \in (1,\infty), \tag{3.16}$$

and summing over all $E \in \mathcal{G}_n$ yields

$$\|\Pi^n_{\mathbb{Q}} q\|_{s'} \le c_{s'} \|q\|_{s'} \qquad \forall\, q \in L^{s'}(\Omega), \quad \forall\, s' \in (1,\infty). \tag{3.17}$$

Also, the stability of $\Pi^n_{\mathbb{Q}}$ and **Assumption 1** imply that $\Pi^n_{\mathbb{Q}}$ satisfies

$$\|q - \Pi^n_{\mathbb{Q}} q\|_{s'} \to 0, \qquad \text{as } n \to \infty \text{ for all } q \in L^{s'}(\Omega) \text{ and } s' \in (1,\infty). \tag{3.18}$$

Remark 3.3.1. According to [9], the following pairs of velocity-pressure finite element spaces satisfy **Assumptions 1, 2** and **3**, for example:

- The conforming Crouzeix–Raviart Stokes element, i.e., continuous piecewise quadratic plus cubic bubble velocity and discontinuous piecewise linear pressure approximation (compare e.g. with [19]);
- The space of continuous piecewise quadratic polynomials for the velocity and piecewise constant pressure approximation; see, [19].

Our final assumption is the existence of a projection operator for the concentration space.

Assumption 4 (Existence of a projection operator $\Pi^n_{\mathbb{Z}}$): For each $n \in \mathbb{N}$, there exists a linear projection operator $\Pi^n_{\mathbb{Z}} : W^{1,1}_0(\Omega) \to \mathbb{Z}^n$ such that

$$⨍_E \left(|\Pi^n_{\mathbb{Z}} z| + h_E |\nabla \Pi^n_{\mathbb{Z}} z|\right) \mathrm{d}x \le c_3 ⨍_{S_E} \left(|z| + h_E |\nabla z|\right) \mathrm{d}x$$

for all $z \in W^{1,1}_0(\Omega)$ and for all $E \in \mathcal{G}_n$, where c_3 does not depend on n.

Similarly as above, the projection operator $\Pi^n_{\mathbb{Z}}$ is globally $W^{1,s}$-stable for $s \in [1,\infty]$, and thus, by approximability,

$$\|\Pi^n_{\mathbb{Z}} z - z\|_{1,s} \to 0 \qquad \forall\, z \in W^{1,s}_0(\Omega). \tag{3.19}$$

3.3.2 Stability of projection operators in variable-exponent spaces

In this subsection, we shall state and prove some important auxiliary results regarding projection operators in the variable-exponent context. A first key step is to prove stability of the projection operator Π^n_{div}. The main difficulty lies in the fact that we are dealing with variable-exponent spaces, so several classical results are not applicable. The main technical tool is the key estimate presented in Theorem 2.5.4.

Proposition 3.3.2. *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^d)$ with $p^+ < \infty$. Then, there exists a constant $C > 0$, which depends on Ω, $C_{\log}(p)$ and p^+, such that, for all $\boldsymbol{v} \in W^{1,p(\cdot)}_0(\Omega)^d$,*

$$\int_\Omega |\nabla \Pi^n_{\mathrm{div}} \boldsymbol{v}(x)|^{p(x)} \,\mathrm{d}x \leq C \int_\Omega |\nabla \boldsymbol{v}(x)|^{p(x)} \,\mathrm{d}x + C \max_{E \in \mathcal{G}_n} h_E^{d+1}.$$

Proof. For $E \in \mathcal{G}_n$, by equivalence of norms in finite-dimensional spaces and a standard scaling argument,

$$\begin{aligned}
\int_E |\nabla \Pi^n_{\mathrm{div}} \boldsymbol{v}(x)|^{p(x)} \,\mathrm{d}x &\leq C \int_E \left(\fint_E |\nabla \Pi^n_{\mathrm{div}} \boldsymbol{v}(y)| \,\mathrm{d}y \right)^{p(x)} \mathrm{d}x \\
&\leq C \int_E \left(\fint_{S_E} |\nabla \boldsymbol{v}(y)| \,\mathrm{d}y \right)^{p(x)} \mathrm{d}x \\
&\leq C \int_E \left(\fint_{S_E} |\nabla \boldsymbol{v}(y)|^{p(y)} \,\mathrm{d}y + h_E^{d+1} \right) \mathrm{d}x \\
&\leq C \int_E \fint_{S_E} |\nabla \boldsymbol{v}(y)|^{p(y)} \,\mathrm{d}y \,\mathrm{d}x + C\,|E| \max_{E \in \mathcal{G}_n} h_E^{d+1} \\
&= C \int_{S_E} |\nabla \boldsymbol{v}(y)|^{p(y)} \,\mathrm{d}y + C\,|E| \max_{E \in \mathcal{G}_n} h_E^{d+1},
\end{aligned}$$

where we have used (3.11) in the second inequality and the key estimate in the third inequality. Summing up the above inequalities over $E \in \mathcal{G}_n$, we have

$$\int_\Omega |\nabla \Pi^n_{\mathrm{div}} \boldsymbol{v}(x)|^{p(x)} \,\mathrm{d}x \leq C \int_\Omega |\nabla \boldsymbol{v}(x)|^{p(x)} \,\mathrm{d}x + C\,|\Omega| \max_{E \in \mathcal{G}_n} h_E^{d+1}.$$

That completes the proof. □

Next, we shall investigate the stability of the projection operator $\Pi^n_{\mathbb{Q}}$ in variable-exponent Lebesgue spaces. The key technical tool is the local-to-global result, which is presented in Theorem 2.4.10. We begin with some auxiliary results, which are quoted from [12].

Lemma 3.3.3. *Let* $p \in \mathcal{P}^{\log}(\mathbb{R}^d)$ *with* $p^+ < \infty$ *and* $m > 0$*. Then, for every* $Q \subset \mathbb{R}^d$ *with* $|Q| \le 1$*,* $\kappa \in [0,1]$ *and* $t \ge 0$ *that satisfy* $|Q|^m \le t \le |Q|^{-m}$*, one has*

$$(\kappa + t)^{p(x)-p(y)} \le C$$

for all $x, y \in Q$*. The constant* C *only depends on* $C_{\log}(p)$*,* m*,* p^+ *and* d*.*

Proof. Since $\kappa \le |Q|^{-m}$ for $\kappa \in [0,1]$, we have $|Q|^m \le \kappa + t \le 2|Q|^{-m}$. Thus the log-Hölder continuity of p in conjunction with the elementary inequality

$$\left(\frac{1}{s}\right)^{\frac{1}{\log \frac{1}{s}}} \le \mathrm{e} \qquad \forall s \in (0,1)$$

implies that

$$(\kappa + t)^{p(x)-p(y)} \le 2^{p^+} (|Q|^{-|p(x)-p(y)|})^m \le C,$$

where the constant depends on $C_{\log}(p)$, m, p^+ and d (but since $d \in \{2,3\}$ by hypothesis here, any dependence of constants on d will be ignored henceforth). $\square$

Next, we shall prove the following lemma, which will be useful for computing a variable-exponent norm locally. To state it, we define a piecewise constant approximation of a given exponent $p(\cdot)$ by

$$p_{\mathrm{loc}} := \sum_{E \in \mathcal{G}_n} p(x_E)\chi_E = \sum_{E \in \mathcal{G}_n} p_E \chi_E,$$

where $x_E := \arg\min_E p$, i.e., $p_E := p(x_E) \le p(x)$ for all $x \in E$. What we need here is the fact that the norms $\|\cdot\|_{p(\cdot)}$ and $\|\cdot\|_{p_{\mathrm{loc}}(\cdot)}$ are equivalent, which is also quoted from [12].

Lemma 3.3.4. *If* $p \in \mathcal{P}^{\log}(\mathbb{R}^d)$ *with* $p^+ < \infty$*, the norms* $\|\cdot\|_{p_{\mathrm{loc}}(\cdot)}$ *and* $\|\cdot\|_{p(\cdot)}$ *are equivalent on* $\mathbb{Q}^n$*.*

Proof. Let $g^n \in \mathbb{Q}^n$ with $\|g^n\|_{p_{\mathrm{loc}}(\cdot)} \le 1$, which means that $\int_\Omega |g^n|^{p_{\mathrm{loc}}(x)} \,\mathrm{d}x \le 1$. By equivalence of norms in finite-dimensional spaces and a usual scaling argument,

$$\begin{aligned}\|g^n\|_{L^\infty(E)} &\le c \fint_E |g^n| \,\mathrm{d}x \le c \left(\fint_E |g^n|^{p(x_E)} \,\mathrm{d}x\right)^{\frac{1}{p(x_E)}} \\ &\le c \left(\frac{1}{h_E^d} \int_E |g^n|^{p_{\mathrm{loc}}(x)} \,\mathrm{d}x\right)^{\frac{1}{p(x_E)}} \le c h_E^{-\frac{d}{p(x_E)}}.\end{aligned}$$

Thus we can apply Lemma 3.3.3 with $m = \frac{1}{p(x_E)}$, $\kappa = 0$ and $t = 1 + |g^n|$ to deduce that

$$\begin{aligned}\int_\Omega |g^n|^{p(x)}\,\mathrm{d}x &= \sum_{E\in\mathcal{G}_n}\int_E |g^n|^{p(x)}\,\mathrm{d}x \le \sum_{E\in\mathcal{G}_n}\int_E (1+|g^n|)^{p(x)}\,\mathrm{d}x \\ &\le c\sum_{E\in\mathcal{G}_n}\int_E (1+|g^n|)^{p_{\mathrm{loc}}(x)}\,\mathrm{d}x = c\int_\Omega (1+|g^n|)^{p_{\mathrm{loc}}(x)}\,\mathrm{d}x \le c.\end{aligned}$$

On the other hand, if $\|g^n\|_{p(\cdot)} \le 1$, then we have

$$\|g^n\|_{L^\infty(E)} \le c\left(\fint_E |g^n|^{p(x_E)}\,\mathrm{d}x\right)^{\frac{1}{p(x_E)}} \le c\left(\fint_E (1+|g^n|^{p(x)})\,\mathrm{d}x\right)^{\frac{1}{p(x_E)}} \le c h_E^{-\frac{d}{p(x_E)}},$$

and as before, $\int_\Omega |g^n|^{p_{\mathrm{loc}}(x)}\,\mathrm{d}x \le c$. ☐

Now we are ready to prove the stability of $\Pi^n_{\mathbb{Q}}$ in the variable-exponent context. The precise statement of the stability property is encapsulated in the following proposition.

Proposition 3.3.5. *For a sequence of exponents* $\{p^n\}_{n\in\mathbb{N}}$*, assume that* $p^n \to p$ *in* $C^{0,\alpha}(\overline{\Omega})$ *for some* $\alpha \in (0,1)$*. Then, there exists a constant* C*, independent of* n*, such that*

$$\|\Pi^n_{\mathbb{Q}} q\|_{p^n(\cdot)} \le C\|q\|_{p^n(\cdot)} \qquad \forall\, q \in L^{p^n(\cdot)}(\Omega).$$

Proof. Let $q \in L^{p^n(\cdot)}(\Omega)$. Then, by Theorem 2.4.10 and Lemma 3.3.4,

$$\begin{aligned}\|\Pi^n_{\mathbb{Q}} q\|_{p^n(\cdot)} &= \left\|\sum_{E\in\mathcal{G}^n}\chi_E \Pi^n_{\mathbb{Q}} q\right\|_{p^n(\cdot)} \\ &\le C\left\|\sum_{E\in\mathcal{G}^n}\chi_E \frac{\|\chi_E\Pi^n_{\mathbb{Q}} q\|_{p^n(\cdot)}}{\|\chi_E\|_{p^n(\cdot)}}\right\|_{p^n(\cdot)} \le C\left\|\sum_{E\in\mathcal{G}^n}\chi_E \frac{\|\chi_E\Pi^n_{\mathbb{Q}} q\|_{p^n_{\mathrm{loc}}(\cdot)}}{\|\chi_E\|_{p^n(\cdot)}}\right\|_{p^n(\cdot)}.\end{aligned}$$

By the definition of the variable-exponent norm, one has that $\|\chi_E\Pi^n_{\mathbb{Q}} q\|_{p^n_{\mathrm{loc}}(\cdot)} \le \|\chi_E\Pi^n_{\mathbb{Q}} q\|_{p^n_E}$ for each $E \in \mathcal{G}^n$. Therefore, by (3.16),

$$\|\Pi^n_{\mathbb{Q}} q\|_{p^n(\cdot)} \le C\left\|\sum_{E\in\mathcal{G}^n}\chi_E \frac{\|\chi_E\Pi^n_{\mathbb{Q}} q\|_{p^n_E}}{\|\chi_E\|_{p^n(\cdot)}}\right\|_{p^n(\cdot)} \le C\left\|\sum_{E\in\mathcal{G}^n}\chi_E \frac{\|\chi_{S_E} q\|_{p^n_E}}{\|\chi_E\|_{p^n(\cdot)}}\right\|_{p^n(\cdot)}.$$

Here the constant C might depend on p^n_E, but since $1 < p^- \le p(x) \le p^+ < \infty$, we can choose a uniform constant C independent of n and E.

At this stage, we claim that

$$\|\chi_{S_E} q\|_{p^n_E} \le \|\chi_{S_E} q\|_{p^n_{\mathrm{loc}}(\cdot)}.$$

Indeed, if this were not the case, then, by the definition of the Luxembourg norm, we would have that

$$\int_\Omega \left| \frac{\chi_{S_E} q}{\|\chi_{S_E} q\|_{p^n_E}} \right|^{p^n_{\mathrm{loc}}(x)} \mathrm{d}x < 1.$$

However, by writing $S_E = E \cup E_1 \cup \cdots \cup E_j$, we have that

$$\int_\Omega \left| \frac{\chi_{S_E} q}{\|\chi_{S_E} q\|_{p^n_E}} \right|^{p^n_{\mathrm{loc}}(x)} \mathrm{d}x = \int_E \left| \frac{\chi_{S_E} q}{\|\chi_{S_E} q\|_{p^n_E}} \right|^{p^n_E} \mathrm{d}x + \sum_{i=1}^{j} \int_{E_i} \left| \frac{\chi_{S_E} q}{\|\chi_{S_E} q\|_{p^n_E}} \right|^{p^n_{\mathrm{loc}}(x)} \mathrm{d}x \geq 1,$$

which is a contradiction. Hence together with Lemma 3.3.4 again, the above claim implies that

$$\|\Pi^n_{\mathbb{Q}} q\|_{p^n(\cdot)} \leq C \left\| \sum_{E \in \mathcal{G}^n} \chi_E \frac{\|\chi_{S_E} q\|_{p^n(\cdot)}}{\|\chi_E\|_{p^n(\cdot)}} \right\|_{p^n(\cdot)}.$$

Next we claim that

$$\|\chi_{S_E}\|_{p^n(\cdot)} \leq C \|\chi_E\|_{p^n(\cdot)}.$$

By Theorem 2.4.11, for any $x \in E$,

$$\|\chi_E\|_{p^n(\cdot)} \geq C|E|^{\frac{1}{p^n(x)}} \geq C|E|^{\frac{1}{p^n_E}} \geq C|S_E|^{\frac{1}{p^n_E}} \geq C|S_E|^{\frac{1}{p^n_{S_E}}} \geq C\|\chi_{S_E}\|_{p^n(\cdot)},$$

and hence the claim is proved. Therefore, together with Theorem 2.4.10 again, we have

$$\|\Pi^n_{\mathbb{Q}} q\|_{p^n(\cdot)} \leq C \left\| \sum_{E \in \mathcal{G}^n} \chi_{S_E} \frac{\|\chi_{S_E} q\|_{p^n(\cdot)}}{\|\chi_{S_E}\|_{p^n(\cdot)}} \right\|_{p^n(\cdot)} \leq C \left\| \sum_{E \in \mathcal{G}_n} \chi_{S_E} q \right\|_{p^n(\cdot)} \leq C\|q\|_{p^n(\cdot)}$$

by the finite overlap property of the patches. Note that the constant C above depends on $C_{\log}(p^n)$, and therefore also on n. However, since $p^n \to p$ in $C^{0,\alpha}(\overline{\Omega})$, this constant can be bounded uniformly by a new constant, which is independent of n. Thus the proof is complete. ☐

3.3.3 Discrete inf-sup condition

The aim of this subsection is to state and prove a discrete inf-sup condition, which plays an important role in our proof of the existence of the discrete pressure and the analysis of its approximation properties. The key technical tools required in the proof of the discrete inf-sup condition are the existence of a Bogovskiĭ operator, stated in Theorem 2.5.1, and the stability property of Π^n_{div} shown in the previous subsection.

Proposition 3.3.6. *Assume that* $1 < p^- \le p^+ < \infty$ *and* $p^n \to p$ *in* $C^{0,\alpha}(\overline{\Omega})$ *for some* $\alpha \in (0,1)$*. Then, there exists a constant* $\beta > 0$*, independent of* n*, such that*

$$\sup_{0 \ne \boldsymbol{V} \in \mathbb{V}^n,\ \|\boldsymbol{V}\|_{1,p^n(\cdot)} \le 1} \langle \operatorname{div} \boldsymbol{V}, Q \rangle \ge \frac{1}{\beta} \|Q\|_{(p^n)'(\cdot)} \qquad \forall\, Q \in \mathbb{Q}_0^n,\ n \in \mathbb{N}.$$

Proof. The assertion follows from the isomorphism between $(L_0^{p^n}(\Omega))^*$ and $L_0^{(p^n)'}(\Omega)$ (with the norm-equivalence constants bounded from above by 2 and from below by 1/2). In fact, it follows from Theorem 2.5.1 and Theorem 2.3.3 that we have

$$\begin{aligned} \|Q\|_{(p^n)'(\cdot)} &\le 2 \sup_{v \in L_0^{p^n(\cdot)},\ \|v\|_{p^n(\cdot)} \le 1} \int_\Omega Q\, v \,\mathrm{d}x \\ &= 2 \sup_{v \in L_0^{p^n(\cdot)},\ \|v\|_{p^n(\cdot)} \le 1} \int_\Omega Q \operatorname{div}(\mathcal{B}v) \,\mathrm{d}x \\ &= 2 \sup_{v \in L_0^{p^n(\cdot)},\ \|v\|_{p^n(\cdot)} \le 1} \int_\Omega Q \operatorname{div}(\Pi_{\mathrm{div}}^n \mathcal{B}v) \,\mathrm{d}x. \end{aligned}$$

Now, by Theorem 2.5.1 and Proposition 3.3.2,

$$\|v\|_{p^n(\cdot)} \le 1 \quad \text{implies} \quad \|\nabla \Pi_{\mathrm{div}}^n \mathcal{B}v\|_{p^n(\cdot)} \le C_1.$$

The constant C_1 depends on $C_{\log}(p^n)$, and therefore also on n. However, since $p^n \to p$ in $C^{0,\alpha}(\overline{\Omega})$, the constant C_1 can be bounded uniformly by a new constant, still denoted by C_1, which is independent of n. Therefore,

$$\begin{aligned} \|Q\|_{(p^n)'(\cdot)} &\le 2 \sup_{\|\Pi_{\mathrm{div}}^n \mathcal{B}v\|_{1,p^n(\cdot)} \le C_1} \int_\Omega Q \operatorname{div}(\Pi_{\mathrm{div}}^n \mathcal{B}v)\ \mathrm{d}x \\ &= 2C_1 \sup_{\|\Pi_{\mathrm{div}}^n \mathcal{B}\frac{v}{C_1}\|_{1,p^n(\cdot)} \le 1} \int_\Omega Q \operatorname{div}(\Pi_{\mathrm{div}}^n \mathcal{B}\frac{v}{C_1})\ \mathrm{d}x \\ &\le \beta \sup_{\boldsymbol{V} \in \mathbb{V}^n,\ \|\boldsymbol{V}\|_{1,p^n(\cdot)} \le 1} \langle \operatorname{div} \boldsymbol{V}, Q \rangle. \end{aligned}$$

That completes the proof of the proposition. □

3.3.4 Discrete Bogovskiĭ operator

In this subsection, we construct a discrete counterpart of the Bogovskiĭ operator in the variable-exponent setting and explore its properties.

Suppose that $1 < p^- \le p^+ < \infty$ and $p^n \to p$ in $C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. For $H \in \operatorname{div} \mathbb{V}^n$, define the linear functional $\mathcal{L}^n : L^{(p^n)'(\cdot)}(\Omega) \to \mathbb{R}$ by

$$\mathcal{L}^n(q) = \int_\Omega H\, \Pi_{\mathbb{Q}}^n q \,\mathrm{d}x, \qquad q \in L^{(p^n)'(\cdot)}(\Omega).$$

Then, thanks to Proposition 3.3.5, $\mathcal{L}^n$ is a bounded linear functional on $L^{(p^n)'(\cdot)}(\Omega)$. Hence, by Theorem 2.3.8, there exists a unique $\mathcal{K}(H) \in L^{p^n(\cdot)}(\Omega)$ such that

$$\mathcal{L}^n(q) = \int_\Omega H\,\Pi^n_{\mathbb{Q}} q \,\mathrm{d}x = \int_\Omega \mathcal{K}(H)\, q \,\mathrm{d}x.$$

Note that since $H \in L_0^{p^n(\cdot)}(\Omega)$ and $\Pi^n_{\mathbb{Q}} c = c$ for all constants c, we have $\mathcal{K}(H) \in L_0^{p^n(\cdot)}(\Omega)$.

Now we define the discrete Bogovskiĭ operator. For $n \in \mathbb{N}$, we consider the linear operator $\mathcal{B}^n : \operatorname{div} \mathbb{V}^n \to \mathbb{V}^n$ by

$$\mathcal{B}^n H := \Pi^n_{\operatorname{div}} \mathcal{B}\mathcal{K}(H) \in \mathbb{V}^n \qquad \text{for } \; H \in \operatorname{div} \mathbb{V}^n, \tag{3.20}$$

where $\mathcal{B}$ is defined in Theorem 2.5.1.

For later use, we require the following bound on $\mathcal{K}(H)$ in a variable-exponent norm:

$$\begin{aligned} \|\mathcal{K}(H)\|_{p^n(\cdot)} &\leq 2 \sup_{q \in L^{(p^n)'(\cdot)}(\Omega),\ \|q\|_{(p^n)'(\cdot)} \leq 1} \int_\Omega \mathcal{K}(H)\, q \,\mathrm{d}x \\ &= 2 \sup_{q \in L^{(p^n)'(\cdot)}(\Omega),\ \|q\|_{(p^n)'(\cdot)} \leq 1} \int_\Omega H\,\Pi^n_{\mathbb{Q}} q \,\mathrm{d}x \\ &\leq C \sup_{Q \in \mathbb{Q}^n,\ \|Q\|_{(p^n)'(\cdot)} \leq 1} \int_\Omega H\, Q \,\mathrm{d}x. \end{aligned} \tag{3.21}$$

Next, we will show a relevant convergence property of the discrete Bogovskiĭ operator. To this end, we need the following lemma, which is quoted from [38].

Lemma 3.3.7. *Let* $\{\boldsymbol{v}_n\}_{n=1}^\infty \subset W_0^{1,s}(\Omega)^d$, $s \in (1, \infty)$, *such that* $\boldsymbol{v}_n \rightharpoonup \boldsymbol{v}$ *weakly in* $W_0^{1,s}(\Omega)^d$ *as* $n \to \infty$. *Then*

$$\Pi^n_{\operatorname{div}} \boldsymbol{v}_n \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } \; W_0^{1,s}(\Omega)^d \; \text{ as } \; n \to \infty.$$

Proof. By the stability property (3.13), there exists a weakly converging subsequence of $\{\Pi^n_{\operatorname{div}} \boldsymbol{v}_n\}_{n=1}^\infty$ in $W_0^{1,s}(\Omega)^d$. By the uniqueness of the weak limit, it suffices to identify the limit of $\{\Pi^n_{\operatorname{div}} \boldsymbol{v}_n\}_{n=1}^\infty$ in $L^s(\Omega)^d$. We deduce by recalling (3.12) that

$$\begin{aligned} \|\boldsymbol{v} - \Pi^n_{\operatorname{div}} \boldsymbol{v}_n\|_s &\leq \|\boldsymbol{v} - \Pi^n_{\operatorname{div}} \boldsymbol{v}\|_s + \|\Pi^n_{\operatorname{div}} (\boldsymbol{v}_n - \boldsymbol{v})\|_s \\ &\leq \|\boldsymbol{v} - \Pi^n_{\operatorname{div}} \boldsymbol{v}\|_s + C\|\boldsymbol{v}_n - \boldsymbol{v}\|_s + C \max_{E \in \mathcal{G}^n} h_E \,\|\nabla(\boldsymbol{v}_n - \boldsymbol{v})\|_s, \end{aligned}$$

The first term vanishes because of (3.14) and the second term tends to zero since $\boldsymbol{v}_n \to \boldsymbol{v}$ strongly in $L^s(\Omega)^d$, thanks to the compact embedding $W_0^{1,s}(\Omega)^d \hookrightarrow\hookrightarrow$

$L^s(\Omega)^d$. The last term vanishes since $\{\boldsymbol{v}_n\}_{n=1}^\infty$ is bounded uniformly in $W_0^{1,s}(\Omega)^d$ and $\max_{E\in\mathcal{G}^n} h_E \to 0$ as $n \to \infty$. $\square$

Now we are ready to prove the desired convergence property of the discrete Bogovskiĭ operator.

Proposition 3.3.8. *Suppose that* $\boldsymbol{V}^n \in \mathbb{V}^n$, $n \in \mathbb{N}$, *and* $\boldsymbol{V}^n \to \boldsymbol{V}$ *weakly in* $W_0^{1,s}(\Omega)^d$ *as* $n \to \infty$. *Then, we have that*

$$\mathcal{B}^n \mathrm{div}\, \boldsymbol{V}^n \rightharpoonup \mathcal{B}\, \mathrm{div}\, \boldsymbol{V} \qquad \text{weakly in } W_0^{1,s}(\Omega)^d \text{ as } n \to \infty.$$

Proof. Let us define $A^n := \mathrm{div}\, \boldsymbol{V}^n$; then, $A^n \rightharpoonup A := \mathrm{div}\, \boldsymbol{V}$ weakly in $L_0^s(\Omega)$ as $n \to \infty$. Therefore, thanks to (3.18), we have, for all $q \in L^{s'}(\Omega)$ by the classical Riesz representation theorem (here we shall use the same notation $\mathcal{K}$ as above, but in this case the constructed $\mathcal{K}(A^n)$ lies in a fixed-exponent space $L_0^s(\Omega)$), and since $\Pi_{\mathbb{Q}}^n q \to q$ strongly in $L^{s'}(\Omega)$ by (3.18), that, for all $q \in L^{s'}(\Omega)$,

$$\int_\Omega \mathcal{K}(A^n) q \,\mathrm{d}x = \int_\Omega A^n \Pi_{\mathbb{Q}}^n q \,\mathrm{d}x \to \int_\Omega A\, q \,\mathrm{d}x \qquad \text{as } n \to \infty.$$

In other words, we have that $\mathcal{K}(A^n) \rightharpoonup A$ weakly in $L_0^s(\Omega)$ as $n \to \infty$. The Bogovskiĭ operator defined in Theorem 2.5.1 is linear and continuous, and hence it is also continuous with respect to the weak topologies of the respective spaces. Therefore, we have $\mathcal{B}\mathcal{K}(A^n) \rightharpoonup \mathcal{B}A$ weakly in $W_0^{1,s}(\Omega)^d$ as $n \to \infty$. Hence, by Lemma 3.3.7, $\mathcal{B}^n A^n := \Pi_{\mathrm{div}}^n \mathcal{B}\mathcal{K}(A^n) \rightharpoonup \mathcal{B}A$ weakly in $W_0^{1,s}(\Omega)^d$ as $n \to \infty$. As $A^n := \mathrm{div}\, \boldsymbol{V}^n$ and $A := \mathrm{div}\, \boldsymbol{V}$ the proof is complete. $\square$

3.3.5 The finite element approximation

We are now ready to construct the finite element approximation of the problem (3.1)–(3.3) and prove that the approximate problem has a solution.

An essential property of the problem (3.1)–(3.3) is that, thanks to the fact that the velocity field $\boldsymbol{u}$ is divergence-free, the convective terms appearing in the equations are skew-symmetric. It is important to ensure that this skew-symmetry is preserved under discretization, even though the finite element approximations to the velocity field are now only discretely (rather than pointwise) divergence-free.

Since we wish to ensure that the discrete counterparts of the convective terms appearing in the system inherit the skew-symmetry of the continuous convective terms, we define the following trilinear forms:

$$\begin{aligned} B_u[\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{h}] &:= \frac{1}{2}\int_\Omega ((\boldsymbol{v} \otimes \boldsymbol{h}) : \nabla \boldsymbol{w} - (\boldsymbol{v} \otimes \boldsymbol{w}) : \nabla \boldsymbol{h}) \,\mathrm{d}x, \\ B_c[b, \boldsymbol{v}, z] &:= \frac{1}{2}\int_\Omega (z\boldsymbol{v} \cdot \nabla b - b\boldsymbol{v} \cdot \nabla z) \,\mathrm{d}x, \end{aligned}$$

for all $\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{h} \in W_0^{1,\infty}(\Omega)^d$, $b, z \in W^{1,\infty}(\Omega)$. These trilinear forms then coincide with the trilinear forms associated with the corresponding convection terms if we are considering pointwise divergence-free functions and also, thanks to their skew symmetry, they now also vanish when $\boldsymbol{w} = \boldsymbol{h}$ and $b = z$, respectively. Explicitly, we have

$$\begin{aligned} B_u[\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{v}] = 0 \quad \text{and} \quad B_c[z, \boldsymbol{v}, z] &= 0 && \forall\, \boldsymbol{v} \in W_0^{1,\infty}(\Omega)^d,\ \ z \in W^{1,\infty}(\Omega), \\ B_u[\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{h}] &= -\int_\Omega (\boldsymbol{v} \otimes \boldsymbol{w}) : \nabla \boldsymbol{h} \,\mathrm{d}x && \forall\, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{h} \in W_{0,\mathrm{div}}^{1,\infty}(\Omega)^d, \\ B_c[b, \boldsymbol{v}, z] &= -\int_\Omega b\boldsymbol{v} \cdot \nabla z \,\mathrm{d}x && \forall\, \boldsymbol{v} \in W_{0,\mathrm{div}}^{1,\infty}(\Omega)^d,\ \ b, z \in W^{1,\infty}(\Omega). \end{aligned} \tag{3.22}$$

Furthermore, the trilinear form $B_u[\cdot,\cdot,\cdot]$ is also bounded in a sense to be discussed below in more detail. Observe that for $\frac{3d}{d+2} \le p^- \le p^+ < d$, we have the Sobolev embedding

$$W^{1,p(\cdot)}(\Omega)^d \hookrightarrow L^{2p'(\cdot)}(\Omega)^d.$$

Then, Hölder's inequality yields that

$$\begin{aligned} \int_\Omega (\boldsymbol{v} \otimes \boldsymbol{w}) : \nabla \boldsymbol{h} \,\mathrm{d}x &\le C\|\boldsymbol{v}\|_{2p'(\cdot)}\|\boldsymbol{w}\|_{2p'(\cdot)}\|\boldsymbol{h}\|_{1,p(\cdot)} \\ &\le C\|\boldsymbol{v}\|_{1,p(\cdot)}\|\boldsymbol{w}\|_{1,p(\cdot)}\|\boldsymbol{h}\|_{1,p(\cdot)}. \end{aligned}$$

In the same way, we have

$$\int_\Omega (\boldsymbol{v} \otimes \boldsymbol{h}) : \nabla \boldsymbol{w} \,\mathrm{d}x \le C\|\boldsymbol{v}\|_{1,p(\cdot)}\|\boldsymbol{h}\|_{1,p(\cdot)}\|\boldsymbol{w}\|_{1,p(\cdot)}.$$

Thus we obtain the bound

$$|B_u[\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{h}]| \le C\|\boldsymbol{v}\|_{1,p(\cdot)}\|\boldsymbol{w}\|_{1,p(\cdot)}\|\boldsymbol{h}\|_{1,p(\cdot)}. \tag{3.23}$$

This is the only part where the condition $p^- \ge \frac{3d}{d+2}$ ($p^- \ge \frac{3}{2}$ in 2D) is needed. As mentioned in Section 3.1, if we can improve this estimate, we can prove the main theorem with lower values of p^-.

Now, for $n \in \mathbb{N}$, we call a triple of functions $(\boldsymbol{U}^n, P^n, C^n) \in \mathbb{V}^n \times \mathbb{Q}_0^n \times (\mathbb{Z}^n + c_d)$ a finite element approximation to a solution of the problem **(Q)** if it satisfies

$$\int_\Omega \boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) : \boldsymbol{D}\boldsymbol{V} \, \mathrm{d}x + B_u[\boldsymbol{U}^n, \boldsymbol{U}^n, \boldsymbol{V}] - \langle \operatorname{div} \boldsymbol{V}, P^n \rangle = \langle \boldsymbol{f}, \boldsymbol{V} \rangle \quad \forall\, \boldsymbol{V} \in \mathbb{V}^n, \tag{3.24}$$

$$\int_\Omega Q \operatorname{div} \boldsymbol{U}^n \, \mathrm{d}x = 0 \qquad \forall\, Q \in \mathbb{Q}^n, \tag{3.25}$$

$$\int_\Omega \boldsymbol{q}_c(C^n, \nabla C^n, \boldsymbol{D}\boldsymbol{U}^n) \cdot \nabla Z \, \mathrm{d}x + B_c[C^n, \boldsymbol{U}^n, Z] = 0 \qquad \forall\, Z \in \mathbb{Z}^n, \tag{3.26}$$

where $c_d \in W^{1,q}(\Omega)$ with $q > d$ and $\boldsymbol{f} \in (W_0^{1,p^-}(\Omega)^d)^*$.

If we restrict the test-functions to $\mathbb{V}^n_{\mathrm{div}}$, the above problem reduces to finding $(\boldsymbol{U}^n, C^n) \in \mathbb{V}^n_{\mathrm{div}} \times (\mathbb{Z}^n + c_d)$ such that

$$\int_\Omega \boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) : \boldsymbol{D}\boldsymbol{V} \, \mathrm{d}x + B_u[\boldsymbol{U}^n, \boldsymbol{U}^n, \boldsymbol{V}] = \langle \boldsymbol{f}, \boldsymbol{V} \rangle \quad \forall\, \boldsymbol{V} \in \mathbb{V}^n_{\mathrm{div}}, \tag{3.27}$$

$$\int_\Omega \boldsymbol{q}_c(C^n, \nabla C^n, \boldsymbol{D}\boldsymbol{U}^n) \cdot \nabla Z \, \mathrm{d}x + B_c[C^n, \boldsymbol{U}^n, Z] = 0 \qquad \forall\, Z \in \mathbb{Z}^n. \tag{3.28}$$

The existence of a solution to the discrete problem (3.27), (3.28) follows by a combination of a fixed point argument and iteration. To prove the existence of a solution, we need the following lemma, which is a consequence of Brouwer's fixed point theorem (cf. [49], Chapter 9).

Lemma 3.3.9. *Suppose that $v : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function, which satisfies*

$$(\exists r > 0) \;\; (\forall\, x \in \mathbb{R}^N \, : \, |x| = r) \qquad v(x) \cdot x \geq 0.$$

Then, there exists an $x \in B_r(0)$ such that $v(x) = 0$.

Let us prove the existence of a solution to the discrete problem (3.27), (3.28). Let $\{\boldsymbol{w}_i\}_{i=1}^m$, $\{z_i\}_{i=1}^\ell$ be bases of $\mathbb{V}^n_{\mathrm{div}}$ and $\mathbb{Z}^n$ respectively, satisfying $\int_\Omega \boldsymbol{w}_i \cdot \boldsymbol{w}_j \, \mathrm{d}x = \delta_{ij}$ for $i, j = 1, \dots, m$ and $\int_\Omega z_i z_j \, \mathrm{d}x = \delta_{ij}$ for $i, j = 1, \dots, \ell$.

We wish to find $\boldsymbol{U}^n \in \mathbb{V}^n_{\mathrm{div}}$, $C^n \in \mathbb{Z}^n + c_d$ of the forms

$$\boldsymbol{U}^n = \sum_{i=1}^m \alpha_i \boldsymbol{w}_i, \qquad C^n = \sum_{i=1}^\ell \beta_i z_i + c_d$$

such that

$$\begin{aligned}\int_\Omega \boldsymbol{S}(C^n,\boldsymbol{D}\boldsymbol{U}^n):\boldsymbol{D}\boldsymbol{V}\,\mathrm{d}x+B_u[\boldsymbol{U}^n,\boldsymbol{U}^n,\boldsymbol{V}]&=\langle \boldsymbol{f},\boldsymbol{V}\rangle \qquad \forall\,\boldsymbol{V}\in\mathbb{V}^n_{\mathrm{div}},\\ \int_\Omega \boldsymbol{q}_c(C^n,\nabla C^n,\boldsymbol{D}\boldsymbol{U}^n)\cdot\nabla Z\,\mathrm{d}x+B_c[C^n,\boldsymbol{U}^n,Z]&=0 \qquad \forall\,Z\in\mathbb{Z}^n.\end{aligned}$$

Equivalently, we can rewrite the above as

$$\begin{aligned}\int_\Omega \boldsymbol{S}(C^n,\boldsymbol{D}\boldsymbol{U}^n):\boldsymbol{D}\boldsymbol{w}_i\,\mathrm{d}x+B_u[\boldsymbol{U}^n,\boldsymbol{U}^n,\boldsymbol{w}_i]&=\langle \boldsymbol{f},\boldsymbol{w}_i\rangle, \qquad i=1,\dots,m,\\ \int_\Omega \boldsymbol{q}_c(C^n,\nabla C^n,\boldsymbol{D}\boldsymbol{U}^n)\cdot\nabla z_j\,\mathrm{d}x+B_c[C^n,\boldsymbol{U}^n,z_j]&=0, \qquad j=1,\dots,\ell.\end{aligned}$$

For a given $n\in\mathbb{N}$, we will construct an iteration scheme; i.e., we will define a sequence of solutions $\{\boldsymbol{U}^n_k,C^n_k\}$ acting on the equations iteratively. As a first step, define $C^n_1:=c_d\in\mathbb{Z}^n+c_d$, and let $\boldsymbol{U}^n_1\in\mathbb{V}^n_{\mathrm{div}}$ be a solution of

$$\int_\Omega \boldsymbol{S}(C^n_1,\boldsymbol{D}\boldsymbol{U}^n_1):\nabla\boldsymbol{w}_i\,\mathrm{d}x+B_u[\boldsymbol{U}^n_1,\boldsymbol{U}^n_1,\boldsymbol{w}_i]=\langle \boldsymbol{f},\boldsymbol{w}_i\rangle, \qquad i=1,\dots,m. \tag{3.29}$$

The existence of such a $\boldsymbol{U}^n_1$ can be established as follows. Define the function $A:\mathbb{R}^m\to\mathbb{R}^m$ by

$$A(\alpha_1,\dots,\alpha_m)=(a_1(\alpha_1,\dots,\alpha_m),\dots,a_m(\alpha_1,\dots,\alpha_m))$$

with

$$a_j(\alpha_1,\dots,\alpha_m)=\int_\Omega \boldsymbol{S}(C^n_1,\boldsymbol{D}\boldsymbol{\alpha}):\nabla\boldsymbol{w}_j\,\mathrm{d}x+B_u[\boldsymbol{\alpha},\boldsymbol{\alpha},\boldsymbol{w}_i]-\langle \boldsymbol{f},\boldsymbol{w}_j\rangle,$$

where

$$\boldsymbol{\alpha}=\sum_{i=1}^m\alpha_i\boldsymbol{w}_i.$$

We note that A is a continuous function on $\mathbb{R}^m$. Then we have, by Sobolev embedding and because the term $B_u[\boldsymbol{\alpha},\boldsymbol{\alpha},\boldsymbol{\alpha}]$ vanishes thanks to the skew-symmetry of the trilinear form B_u, that

$$\begin{aligned}A(\alpha_1,\dots,\alpha_m)\cdot(\alpha_1,\dots,\alpha_m)&=\int_\Omega \boldsymbol{S}(C^n_1,\boldsymbol{D}\boldsymbol{\alpha}):\nabla\boldsymbol{\alpha}\,\mathrm{d}x+B_u[\boldsymbol{\alpha},\boldsymbol{\alpha},\boldsymbol{\alpha}]-\langle \boldsymbol{f},\boldsymbol{\alpha}\rangle\\ &\geq C_1\int_\Omega|\boldsymbol{D}\boldsymbol{\alpha}|^{p(C^n_1)}\,\mathrm{d}x-C_2-|\langle \boldsymbol{f},\boldsymbol{\alpha}\rangle|\\ &\geq C_1\int_\Omega|\boldsymbol{D}\boldsymbol{\alpha}|^{p^-}\,\mathrm{d}x-\|\boldsymbol{f}\|_{(W^{1,p^-}_0(\Omega))^*}\|\boldsymbol{\alpha}\|_{1,p^-}-C_2\\ &\geq C_1\|\boldsymbol{\alpha}\|^{p^-}_{1,p^-}-C(\varepsilon)\|\boldsymbol{f}\|^{(p^-)'}_{(W^{1,p^-}_0(\Omega))^*}-\varepsilon\|\boldsymbol{\alpha}\|^{p^-}_{1,p^-}-C_2\\ &\geq (C_1-\varepsilon)\|\boldsymbol{\alpha}\|^{p^-}_2-C_2\\ &=(C_1-\varepsilon)|(\alpha_1,\dots,\alpha_m)|^{p^-}-C_2.\end{aligned}$$

Hence by Lemma 3.3.9, there exists an m-tuple $(\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$ such that $A(\alpha_1, \dots, \alpha_m) = 0$, which implies the existence of $\boldsymbol{U}_1^n \in \mathbb{V}_{\text{div}}^n$.

Now, multiplying the i-th equation in (3.29) by α_i and taking the sum over $i = 1, \dots, m$, we obtain

$$\int_\Omega \boldsymbol{S}(C_1^n, \boldsymbol{D}\boldsymbol{U}_1^n) : \nabla \boldsymbol{U}_1^n \,\mathrm{d}x + B_u[\boldsymbol{U}_1^n, \boldsymbol{U}_1^n, \boldsymbol{U}_1^n] = \langle \boldsymbol{f}, \boldsymbol{U}_1^n \rangle,$$

where the term $B_u[\boldsymbol{U}_1^n, \boldsymbol{U}_1^n, \boldsymbol{U}_1^n] = 0$ thanks to the skew-symmetry of the trilinear form B_u. Then, because of the coercivity of $\boldsymbol{S}$, we have

$$C_1 \int_\Omega |\boldsymbol{D}\boldsymbol{U}_1^n|^{p(C_1^n)} \,\mathrm{d}x - C_2 \leq \|\boldsymbol{f}\|_{(W_0^{1,p^-}(\Omega)^d)^*} \|\boldsymbol{U}_1^n\|_{1,p^-}.$$

By Young's inequality,

$$\|\boldsymbol{D}\boldsymbol{U}_1^n\|_{p^-}^{p^-} \leq C(\varepsilon) \|\boldsymbol{f}\|_{(W_0^{1,p^-}(\Omega)^d)^*}^{(p^-)'} + \varepsilon \|\boldsymbol{U}_1^n\|_{1,p^-}^{p^-} + C_2.$$

Finally, by Korn's inequality,

$$\|\boldsymbol{U}_1^n\|_{1,p^-} \leq C, \tag{3.30}$$

where the constant C is independent of n.

Now, let $C_2^n \in \mathbb{Z}^n + c_d$ be a solution of the equation

$$\int_\Omega \boldsymbol{q}_c(C_2^n, \nabla C_2^n, \boldsymbol{D}\boldsymbol{U}_1^n) \cdot \nabla z_j \,\mathrm{d}x + B_c[C_2^n, \boldsymbol{U}_1^n, z_j] = 0, \qquad j = 1, \dots, \ell. \tag{3.31}$$

As before, we define a function $B : \mathbb{R}^\ell \to \mathbb{R}^\ell$ by

$$B(\beta_1, \dots, \beta_\ell) = (b_1(\beta_1, \dots, \beta_\ell), \dots, b_\ell(\beta_1, \dots, \beta_\ell))$$

with

$$b_j(\beta_1, \dots, \beta_\ell) = \int_\Omega \boldsymbol{q}_c(\beta, \nabla\beta, \boldsymbol{D}\boldsymbol{U}_1^n) \cdot \nabla z_j \,\mathrm{d}x + B_c[\beta, \boldsymbol{U}_1^n, z_j],$$

where

$$\beta = \sum_{i=1}^{\ell} \beta_i z_i + c_d.$$

We note that B is continuous on $\mathbb{R}^\ell$. Furthermore, we have that

$$\begin{aligned} B(\beta_1, \dots, \beta_\ell) \cdot (\beta_1, \dots, \beta_\ell) &= \int_\Omega \boldsymbol{q}_c(\beta, \nabla\beta, \boldsymbol{D}\boldsymbol{U}_1^n) \cdot \nabla(\beta - c_d) \,\mathrm{d}x + B_c[\beta, \boldsymbol{U}_1^n, \beta - c_d] \\ &=: \mathrm{I} + \mathrm{II}, \end{aligned}$$

with obvious definitions of I and II. Since $\boldsymbol{q}_c$ is linear with respect to its second variable, by (3.7) and (3.8) we have that

$$\begin{aligned}
\mathrm{I} &= \int_\Omega \left(\boldsymbol{q}_c(\beta, \nabla(\beta - c_d), \boldsymbol{D}\boldsymbol{U}_1^n) \cdot \nabla(\beta - c_d) + \boldsymbol{q}_c(\beta, \nabla c_d, \boldsymbol{D}\boldsymbol{U}_1^n) \cdot \nabla(\beta - c_d)\right) \mathrm{d}x \\
&\geq C_5 \|\nabla(\beta - c_d)\|_2^2 - C_4 \int_\Omega |\nabla c_d| |\nabla(\beta - c_d)| \,\mathrm{d}x \\
&\geq C_5 \|\nabla(\beta - c_d)\|_2^2 - C_4 \|\nabla c_d\|_2 \|\nabla(\beta - c_d)\|_2 \\
&\geq C_5 \|\nabla(\beta - c_d)\|_2^2 - C(\varepsilon)\|\nabla c_d\|_2^2 - \varepsilon \|\nabla(\beta - c_d)\|_2^2 \\
&\geq (C_5 - \varepsilon)\|\nabla(\beta - c_d)\|_2^2 - C.
\end{aligned}$$

Also,

$$\begin{aligned}
\mathrm{II} &= \frac{1}{2}\int_\Omega (\beta - c_d)\boldsymbol{U}_1^n \cdot \nabla c_d \,\mathrm{d}x - \int_\Omega c_d \boldsymbol{U}_1^n \cdot \nabla(\beta - c_d)\,\mathrm{d}x \\
&= \mathrm{I}' + \mathrm{II}',
\end{aligned}$$

with obvious definitions of I' and II'. Concerning I', for sufficiently large $t > 2$ (i.e. $t \in (2, \infty)$ when $d = 2$, and $t \in (2, 6]$ when $d = 3$), we have by Sobolev embedding that

$$\begin{aligned}
\left|\int_\Omega (\beta - c_d)\boldsymbol{U}_1^n \cdot \nabla c_d \,\mathrm{d}x\right| &\leq \|\beta - c_d\|_t \|\boldsymbol{U}_1^n\|_{\frac{2t}{t-2}} \|\nabla c_d\|_2 \\
&\leq C\|\beta - c_d\|_{1,2} \|\boldsymbol{U}_1^n\|_{1,p^-} \\
&\leq C\|\nabla(\beta - c_d)\|_2 \|\boldsymbol{U}_1^n\|_{1,p^-} \\
&\leq \varepsilon\|\nabla(\beta - c_d)\|_2^2 + C(\varepsilon)\|\boldsymbol{U}_1^n\|_{1,p^-}^2.
\end{aligned}$$

Hence,

$$\mathrm{I}' \geq -\varepsilon\|\nabla(\beta - c_d)\|_2^2 - C(\varepsilon)\|\boldsymbol{U}_1^n\|_{1,p^-}^2.$$

For the term II', since c_d is bounded, we have that

$$\begin{aligned}
\int_\Omega c_d \boldsymbol{U}_1^n \cdot \nabla(\beta - c_d)\,\mathrm{d}x &\leq C\int_\Omega \boldsymbol{U}_1^n \cdot \nabla(\beta - c_d)\,\mathrm{d}x \\
&\leq C\|\boldsymbol{U}_1^n\|_2 \|\nabla(\beta - c_d)\|_2 \\
&\leq C(\varepsilon)\|\boldsymbol{U}_1^n\|_{1,p^-}^2 + \varepsilon\|\nabla(\beta - c_d)\|_2^2.
\end{aligned}$$

Therefore,

$$\mathrm{II} = \mathrm{I}' + \mathrm{II}' \geq -2\varepsilon\|\nabla(\beta - c_d)\|_2^2 - C(\varepsilon)\|\boldsymbol{U}_1^n\|_{1,p^-}^2,$$

and by (3.30), and choosing $\varepsilon \in \left(0, \frac{1}{6}C_5\right)$, we have that

$$\begin{aligned} B(\beta_1, \dots, \beta_\ell) \cdot (\beta_1, \dots, \beta_\ell) = \mathrm{I} + \mathrm{II} &\geq (C_5 - 3\varepsilon)\|\nabla(\beta - c_d)\|_2^2 - C \\ &\geq \frac{1}{2}C_5 \, |(\beta_1, \dots, \beta_\ell)|^2 - C. \end{aligned}$$

Thus, again, by Lemma 3.3.9, there exists an ℓ-tuple $(\beta_1, \dots, \beta_\ell) \in \mathbb{R}^\ell$ such that $B(\beta_1, \dots, \beta_\ell) = 0$, which implies the existence of $C_2^n \in \mathbb{Z}^n$.

Next, multiplying the j-th equation of (3.31) by β_j and taking the sum over $j = 1, \dots, \ell$, we arrive at

$$\int_\Omega \boldsymbol{q}_c(C_2^n, \nabla C_2^n, \boldsymbol{D}\boldsymbol{U}_1^n) \cdot \nabla(C_2^n - c_d) \, \mathrm{d}x + B_c[C_2^n, \boldsymbol{U}_1^n, C_2^n - c_d] = 0.$$

Hence, by (3.7) and (3.8),

$$\begin{aligned} \|\nabla C_2^n\|_2^2 &\leq C \int_\Omega \boldsymbol{q}_c(C_2^n, \nabla C_2^n, \boldsymbol{D}\boldsymbol{U}_1^n) \cdot \nabla c_d \, \mathrm{d}x - B_c[C_2^n, \boldsymbol{U}_1^n, C_2^n - c_d] \\ &\leq C \int_\Omega |\nabla C_2^n||\nabla c_d| \, \mathrm{d}x - B_c[C_2^n, \boldsymbol{U}_1^n, C_2^n] + B_c[C_2^n, \boldsymbol{U}_1^n, c_d] \\ &\leq \varepsilon\|\nabla C_2^n\|_2^2 + C(\varepsilon)\|\nabla c_d\|_2^2 + B_c[C_2^n, \boldsymbol{U}_1^n, c_d]. \end{aligned}$$

By integration by parts, Sobolev embedding, Hölder's inequality and Young's inequality,

$$\begin{aligned} B_c[C_2^n, \boldsymbol{U}_1^n, c_d] &= \frac{1}{2}\int_\Omega c_d \boldsymbol{U}_1^n \cdot \nabla C_2^n \, \mathrm{d}x - \frac{1}{2}\int_\Omega C_2^n \boldsymbol{U}_1^n \cdot \nabla c_d \, \mathrm{d}x \\ &= \frac{1}{2}\int_\Omega c_d \boldsymbol{U}_1^n \cdot \nabla C_2^n \, \mathrm{d}x + \frac{1}{2}\int_\Omega \mathrm{div}\,(C_2^n \boldsymbol{U}_1^n) c_d \, \mathrm{d}x \\ &= \int_\Omega c_d \boldsymbol{U}_1^n \cdot \nabla C_2^n \, \mathrm{d}x + \frac{1}{2}\int_\Omega C_2^n (\mathrm{div}\, \boldsymbol{U}_1^n) c_d \, \mathrm{d}x \\ &\leq \|c_d\|_\infty \|\boldsymbol{U}_1^n\|_2 \|\nabla C_2^n\|_2 + \frac{1}{2}\|c_d\|_\infty \|C_2^n\|_{\frac{p^-}{p^- - 1}} \|\mathrm{div}\, \boldsymbol{U}_1^n\|_{p^-} \\ &\leq C\|\boldsymbol{U}_1^n\|_{1,p^-} \|\nabla C_2^n\|_2 + C\|\boldsymbol{U}_1^n\|_{1,p^-} \|\nabla C_2^n\|_{\frac{dp^-}{(d+1)p^- - d}} \\ &\leq C(\varepsilon)\|\boldsymbol{U}_1^n\|_{1,p^-}^2 + \varepsilon\|\nabla C_2^n\|_2^2, \end{aligned}$$

provided that $\frac{2d}{d+2} < p^- \leq p^+ < d$. Therefore, by (3.30), we obtain

$$\|\nabla C_2^n\|_2^2 \leq C + C\|\boldsymbol{U}_1^n\|_{1,p^-}^2 \leq C, \tag{3.32}$$

where the constant C does not depend on n.

Now we define $\boldsymbol{U}_2^n \in \mathbb{V}_{\mathrm{div}}^n$ as a solution of the equation

$$\int_\Omega \boldsymbol{S}(C_2^n, \boldsymbol{D}\boldsymbol{U}_2^n) : \nabla \boldsymbol{w}_i \, \mathrm{d}x + B_u[\boldsymbol{U}_2^n, \boldsymbol{U}_2^n, \boldsymbol{w}_i] = \langle \boldsymbol{f}, \boldsymbol{w}_i \rangle, \qquad i = 1, \dots, m,$$

and define $C_3^n \in \mathbb{Z}^n + c_d$ as a solution of the equation

$$\int_\Omega \boldsymbol{q}_c(C_3^n, \nabla C_3^n, \boldsymbol{D}\boldsymbol{U}_2^n) \cdot \nabla z_j \,\mathrm{d}x + B_c[C_3^n, \boldsymbol{U}_2^n, z_j] = 0, \qquad j = 1, \dots, \ell.$$

The existence of such $\boldsymbol{U}_2^n$ and C_3^n can be established by the same argument as above. We continue this process so that we obtain, by iteration, a sequence of solutions $\{\boldsymbol{U}_k^n, C_k^n\} \in \mathbb{V}_{\mathrm{div}}^n \times \mathbb{Z}^n + c_d$ where $\boldsymbol{U}_k^n$ is a solution of the equation

$$\int_\Omega \boldsymbol{S}(C_k^n, \boldsymbol{D}\boldsymbol{U}_k^n) : \nabla \boldsymbol{w}_i \,\mathrm{d}x + B_u[\boldsymbol{U}_k^n, \boldsymbol{U}_k^n, \boldsymbol{w}_i] = \langle \boldsymbol{f}, \boldsymbol{w}_i \rangle, \qquad i = 1, \dots, m, \quad (3.33)$$

and C_{k+1}^n is a solution of the equation

$$\int_\Omega \boldsymbol{q}_c(C_{k+1}^n, \nabla C_{k+1}^n, \boldsymbol{D}\boldsymbol{U}_k^n) \cdot \nabla z_j \,\mathrm{d}x + B_c[C_{k+1}^n, \boldsymbol{U}_k^n, z_j] = 0, \qquad j = 1, \dots, \ell. \quad (3.34)$$

Also, by the same argument as the one we used to derive the bounds (3.30) and (3.32), we have

$$\|\boldsymbol{U}_k^n\|_{1,p^-} \leq C_1 \qquad \text{and} \qquad \|\nabla C_k^n\|_2^2 \leq C_2,$$

where C_1, C_2 are positive constants, independent of $k \in \mathbb{N}$.

Now we consider the spaces $(\mathbb{V}_{\mathrm{div}}^n, \|\cdot\|_{1,p^-})$ and $(\mathbb{Z}^n, \|\cdot\|_{1,2})$. Both spaces are finite-dimensional, hence by the Bolzano–Weierstass theorem there exist subsequences (not relabelled) such that as $k \to \infty$,

$$\begin{aligned} \boldsymbol{U}_k^n &\to \boldsymbol{U}^n && \text{(strongly) in } \mathbb{V}_{\mathrm{div}}^n, \\ C_k^n - c_d &\to C^n - c_d && \text{(strongly) in } \mathbb{Z}^n. \end{aligned}$$

By the equivalence of norms in finite-dimensional spaces, as $k \to \infty$, and for each fixed $n \in \mathbb{N}$,

$$\begin{aligned} \|\boldsymbol{U}_k^n - \boldsymbol{U}^n\|_{1,\infty} &\leq C(n)\|\boldsymbol{U}_k^n - \boldsymbol{U}^n\|_{1,p^-} \to 0, \\ \|C_k^n - C^n\|_{1,\infty} &\leq C(n)\|C_k^n - C^n\|_{1,2} \quad \to 0. \end{aligned}$$

Since convergence of a sequence of functions in the L^∞-norm implies uniform convergence, we deduce that $k \to \infty$,

$$\boldsymbol{U}_k^n \rightrightarrows \boldsymbol{U}^n \quad \boldsymbol{D}\boldsymbol{U}_k^n \rightrightarrows \boldsymbol{D}\boldsymbol{U}^n, \quad C_k^n \rightrightarrows C^n \quad \text{and} \quad \nabla C_k^n \rightrightarrows \nabla C^n \quad \text{a.e. on } \overline{\Omega}.$$

Note further that since $\boldsymbol{S}$ and $\boldsymbol{q}_c$ are continuous, we have

$$\begin{aligned} \boldsymbol{S}(C_k^n, \boldsymbol{D}\boldsymbol{U}_k^n) &\rightrightarrows \boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n), \\ \boldsymbol{q}_c(C_{k+1}^n, \nabla C_{k+1}^n, \boldsymbol{D}\boldsymbol{U}_k^n) &\rightrightarrows \boldsymbol{q}_c(C^n, \nabla C^n, \boldsymbol{D}\boldsymbol{U}^n). \end{aligned}$$

Now if we pass to the limit $k \to \infty$ in (3.33) and (3.34), we have

$$\begin{aligned} \int_\Omega \boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) : \nabla \boldsymbol{w}_i \,\mathrm{d}x + B_u[\boldsymbol{U}^n, \boldsymbol{U}^n, \boldsymbol{w}_i] = \langle \boldsymbol{f}, \boldsymbol{w}_i \rangle, \qquad & i = 1, \dots, m, \\ \int_\Omega \boldsymbol{q}_c(C^n, \nabla C^n, \boldsymbol{D}\boldsymbol{U}^n) \cdot \nabla z_j \,\mathrm{d}x + B_c[C^n, \boldsymbol{U}^n, z_j] = 0, \qquad & j = 1, \dots, \ell. \end{aligned}$$

Therefore, we have established the existence of a solution to the Galerkin approximations (3.27) and (3.28) for a given $n > 0$. The existence of a discrete solution triple for (3.24)–(3.26) then follows by the discrete inf-sup condition stated in Proposition 3.3.6.

Our objective is now to pass to the limit $n \to \infty$. To this end we require two technical tools: a finite element counterpart of the Acerbi–Fusco Lipschitz truncation method in variable-exponent Sobolev spaces, and a finite element counterpart of De Giorgi's regularity theorem for elliptic problems. We shall discuss these in the next two subsections, respectively. The finite element De Giorgi estimate considered here is restricted to the case of two space dimensions ($d = 2$), as our proof rests on a discrete version of Meyers' regularity estimate in conjunction with Morrey's embedding theorem, which, by the nature of the argument, is limited to the case of $d = 2$. A direct proof of a discrete De Giorgi estimate in the case of $d \geq 2$, for Poisson's equation with a source term in $W^{-1,s}(\Omega)$ and $s > d$, is contained in [3], subject to a restriction on the finite element stiffness matrix, analogous to the assumption that is usually made to ensure that the discrete maximum principle holds. It is stated there, without proof, that more general operators may be covered with little or no change, including, for instance, "any uniformly elliptic operator in divergence form with bounded measurable coefficients". Indeed, Casado-Díaz et al. [26] consider linear elliptic problems of the form $-\mathrm{div}(\boldsymbol{A}\nabla u) = f$ with $\boldsymbol{A} \in L^\infty(\Omega)^{d\times d}$ uniformly elliptic and $f \in L^1(\Omega)$, and assume diagonal dominance of the associated finite element stiffness matrix, a condition, which now also involves the bounded measurable matrix function $\boldsymbol{A}$ (cf. (1.17) there). As in our setting the concentration equation is nonlinear, and the diffusion coefficient is a nonlinear function of both the concentration and the Frobenius norm of the velocity gradient, it is unclear how exactly such a diagonal dominance condition on the associated stiffness matrix would translate into a practically verifiable restriction on the sequence of triangulations. We have therefore confined ourselves here to the case of $d = 2$.

3.3.6 Discrete Lipschitz truncation

The Lipschitz truncation method has a crucial role in the proof of our main result, which will be stated in the next section. In this section, we shall introduce a discrete Lipschitz truncation, acting on finite element spaces, following the ideas by Diening et al. in [38], as the composition of a ‘continuous’ Lipschitz truncation and the projection defined in **Assumption 2** for the finite element spaces. For this reason, as a starting point for the construction, we shall first recall a result by Diening et al. [39] concerning Lipschitz truncation in $W_0^{1,1}(\Omega)^d$, which refines the original estimates by Acerbi & Fusco [1]. Note that in the following theorem, the no-slip boundary condition on $\partial\Omega$ is preserved under Lipschitz truncation.

Let $\boldsymbol{v} \in W_0^{1,1}(\Omega)^d$. We can then assume that $\boldsymbol{v} \in W^{1,1}(\mathbb{R}^d)^d$ by extending $\boldsymbol{v}$ by zero outside Ω. For fixed $\lambda > 0$, we define

$$\mathcal{U}_\lambda(\boldsymbol{v}) := \{M(\nabla \boldsymbol{v}) > \lambda\}$$

and

$$\mathcal{H}_\lambda(\boldsymbol{v}) := \mathbb{R}^d \setminus (\mathcal{U}_\lambda(\boldsymbol{v}) \cap \Omega) = \{M(\nabla \boldsymbol{v}) \leq \lambda\} \cup (\mathbb{R}^d \setminus \Omega).$$

As $M(\nabla \boldsymbol{v})$ is lower-semicontinuous, the set $\mathcal{U}_\lambda(\boldsymbol{v})$ is open and the set $\mathcal{H}_\lambda(\boldsymbol{v})$ is closed.

Theorem 3.3.10. *Let $\lambda > 0$ and $\boldsymbol{v} \in W_0^{1,1}(\Omega)^d$. Then there exists a Lipschitz truncation $\boldsymbol{v}_\lambda \in W_0^{1,\infty}(\Omega)^d$ satisfying the following properties:*

(a) $\boldsymbol{v}_\lambda = \boldsymbol{v}$ *on* $\mathcal{H}_\lambda(\boldsymbol{v})$, *i.e.,* $\{\boldsymbol{v} \neq \boldsymbol{v}_\lambda\} \subset \{M(\nabla \boldsymbol{v}) > \lambda\} \cap \Omega$;

(b) $\|\boldsymbol{v}_\lambda\|_s \leq C\|\boldsymbol{v}\|_s$ *for all* $s \in [1, \infty]$, *provided that* $\boldsymbol{v} \in L^s(\Omega)^d$;

(c) $\|\nabla \boldsymbol{v}_\lambda\|_s \leq C\|\nabla \boldsymbol{v}\|_s$ *for all* $s \in [1, \infty]$, *provided that* $\boldsymbol{v} \in W^{1,s}(\Omega)^d$;

(d) $\|\nabla \boldsymbol{v}_\lambda\|_\infty \leq C\lambda$ *almost everywhere in* $\mathbb{R}^d$.

The constant C in the inequalities stated in parts (b), (c) *and* (d) *depends on Ω and d. In* (b) *and* (c), *the constant C additionally depends on s.*

Next, following Diening et al. [38], we modify the ‘continuous’ Lipschitz truncation so that the resulting truncation is again a finite element function.

Since $\mathbb{V}^n \subset W_0^{1,1}(\Omega)^d$ for all $n \in \mathbb{N}$, we can apply Theorem 3.3.10 with arbitrary $\lambda > 0$. Note however that the Lipschitz truncation $\boldsymbol{V}_\lambda$ of $\boldsymbol{V} \in \mathbb{V}^n$ is not contained in $\mathbb{V}^n$ in general. Thus we define the discrete Lipschitz truncation by

$$\boldsymbol{V}_\lambda^n := \Pi_{\text{div}}^n \circ \boldsymbol{V}_\lambda \in \mathbb{V}^n. \tag{3.35}$$

According to the next lemma, which we quote from [38] (cf. Lemma 14 in [38]), the projection operator Π_{div}^n modifies $\boldsymbol{V}_\lambda$ in a neighborhood of $\mathcal{U}_\lambda(\boldsymbol{V})$ only.

Lemma 3.3.11. *Let $\boldsymbol{V} \in \mathbb{V}^n$; then, we have that*

$$\{\boldsymbol{V}_\lambda^n \neq \boldsymbol{V}\} \subset \Omega_\lambda^n(\boldsymbol{V}) := \text{interior}\left(\bigcup \{S_E : E \in \mathcal{G}_n \text{ with } E \cap \mathcal{U}_\lambda(\boldsymbol{V}) \neq \emptyset\}\right).$$

The set $\Omega_\lambda^n(\boldsymbol{v})$ from Lemma 3.3.11 is clearly larger than $\mathcal{U}_\lambda(\boldsymbol{V}) \cap \Omega$. However, according to the following result, we can still control the increase of the set. This is the most important step in the construction of the discrete Lipschitz truncation; Lemma 3.3.12 is, again, quoted from [38].

Lemma 3.3.12. *For $n \in \mathbb{N}$, $\boldsymbol{V} \in \mathbb{V}^n$ and $\lambda > 0$, let $\Omega_\lambda^n(\boldsymbol{V})$ be defined as in* Lemma 3.3.11. *Then, there exists a constant $\kappa \in (0,1)$, only depending on $\hat{\mathbb{P}}_{\mathbb{V}}$ and the shape-regularity of $\mathcal{G}_n$, such that*

$$\mathcal{U}_\lambda(\boldsymbol{V}) \cap \Omega \subset \Omega_\lambda^n(\boldsymbol{V}) \subset \mathcal{U}_{\kappa\lambda}(\boldsymbol{V}) \cap \Omega.$$

Now we are ready to state and prove the discrete Lipschitz truncation theorem, which has a suitable form for our problem. The couple $(\boldsymbol{V}^n, C^n)$ denotes a sequence of approximate solutions and define the variable Lebesgue exponent

$$p^n(x) := (p \circ C^n)(x) \qquad \text{for all } x \in \overline{\Omega}.$$

The following theorem is a generalization of the result stated in Theorem 3.3.10. Here, however, we have the added difficulty that the variable exponent changes with the given sequence.

Theorem 3.3.13. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open Lipschitz domain and suppose that $\{\boldsymbol{V}^n, p^n\}$ is a sequence satisfying $1 < p^- \leq p^n(x) \leq p^+ < \infty$ for all $x \in \overline{\Omega}$ and*

$$\boldsymbol{V}^n \rightharpoonup \boldsymbol{V} \qquad \text{weakly in } W_0^{1,p^-}(\Omega)^d, \tag{3.36}$$

$$p^n \to p \qquad \text{strongly in } C^{0,\alpha}(\overline{\Omega}), \tag{3.37}$$

for some $\alpha \in (0,1)$. Assume further that, for all $n \in \mathbb{N}$,

$$\int_\Omega |\nabla \boldsymbol{V}^n|^{p^n(x)} \,\mathrm{d}x \le C. \tag{3.38}$$

Then, for each $j \in \mathbb{N}$, there exists a sequence $\{\lambda_j^n\}_{n\in\mathbb{N}}$ such that

$$(2^j)^{2^j} \le \lambda_j^n < (2^{j+1})^{2^{j+1}}, \tag{3.39}$$

and a sequence of Lipschitz truncations $\{\boldsymbol{V}_j^n\}_{n\in\mathbb{N}} \subset \mathbb{V}^n \subset W^{1,\infty}(\Omega)^d$ such that for all $n, j \in \mathbb{N}$,

$$\|\nabla \boldsymbol{V}_j^n\|_\infty \le C\lambda_j^n \le C(2^{j+1})^{2^{j+1}}. \tag{3.40}$$

In addition, we can extract a (not relabelled) subsequence with respect to n such that, for each $j \in \mathbb{N}$,

$$\begin{aligned} \boldsymbol{V}_j^n &\to \boldsymbol{V}_j && \text{strongly in } L^\sigma(\Omega)^d \text{ for all } \sigma \in (1,\infty), && (3.41)\\ \boldsymbol{V}_j^n &\rightharpoonup \boldsymbol{V}_j && \text{weakly in } W^{1,\sigma}(\Omega)^d \text{ for all } \sigma \in (1,\infty), && (3.42)\\ \nabla\boldsymbol{V}_j^n &\rightharpoonup^* \nabla\boldsymbol{V}_j && \text{weakly} - * \text{ in } L^\infty(\Omega)^{d\times d}, && (3.43) \end{aligned}$$

where $\boldsymbol{V}_j \in W^{1,\infty}(\Omega)^d$. Moreover,

$$\|\nabla \boldsymbol{V}_j\|_{p(\cdot)} \le C, \tag{3.44}$$

and we can extract a (not relabelled) subsequence so that

$$\boldsymbol{V}_j \rightharpoonup \boldsymbol{V} \qquad \text{weakly in } W^{1,p(\cdot)}(\Omega)^d. \tag{3.45}$$

Furthermore, if we extend $\boldsymbol{V}^n$ outside $\overline{\Omega}$ by zero, we have

$$\{x \in \Omega : \boldsymbol{V}_j^n \ne \boldsymbol{V}^n\} \subset \{x \in \Omega : M(\nabla \boldsymbol{V}^n) > \kappa\lambda_j^n\}, \tag{3.46}$$

where κ is defined in Lemma 3.3.12*, and for all n, j,*

$$\int_\Omega |\nabla \boldsymbol{V}_j^n \chi_{\{\boldsymbol{V}_j^n \ne \boldsymbol{V}^n\}}|^{p^n(x)} \,\mathrm{d}x \le C \int_\Omega |\lambda_j^n \chi_{\{\boldsymbol{V}_j^n \ne \boldsymbol{V}^n\}}|^{p^n(x)} \,\mathrm{d}x \le \frac{C}{2^j}. \tag{3.47}$$

Proof. We first extend each $\boldsymbol{V}^n$ outside $\overline{\Omega}$ by zero and we extend each p^n defined as in Lemma 2.5.6. Then we have

$$\begin{aligned} \boldsymbol{V}^n &\rightharpoonup \boldsymbol{V} && \text{weakly in } W^{1,p^-}(\mathbb{R}^d)^d,\\ p^n &\to p && \text{strongly in } C^{0,\alpha}(\mathbb{R}^d). \end{aligned}$$

By boundedness of the maximal operator (Theorem 2.5.5) for $p^n(x) > 1$, we have that

$$\|M(\nabla \boldsymbol{V}^n)\|_{p^n(\cdot)} \leq C(n)\|\nabla \boldsymbol{V}^n\|_{p^n(\cdot)}.$$

Note that the constant $C(n)$ depends on $C_{\log}(p^n)$, but by the assumption $p^n \to p$ in $C^{0,\alpha}(\overline{\Omega})$, $C(n)$ can be bounded by some uniform constant C independent of $n \in \mathbb{N}$. Thus directly from (3.38), we have

$$\int_{\mathbb{R}^d} |M(\nabla \boldsymbol{V}^n)|^{p^n(x)} \,\mathrm{d}x \leq C. \tag{3.48}$$

Now, for each $j \in \mathbb{N}$, define the sequence $\{\theta_j^i\}_{i=2^j}^{2^{j+1}-1}$ by

$$\theta_j^i := (2^j)^i,$$

and a sequence of subsets $\{U_{j,n}^i\}_{i=2^j}^{2^{j+1}-1}$ as

$$U_{j,n}^i := \{x \in \mathbb{R}^d : \kappa\theta_j^i < M(\nabla \boldsymbol{V}^n)(x) \leq \kappa\theta_j^{i+1}\}.$$

Note that $U_{j,n}^i$ are mutually disjoint bounded sets, and thus

$$\sum_{i=2^j}^{2^{j+1}-1} \int_{U_{j,n}^i} |M(\nabla \boldsymbol{V}^n)|^{p^n(x)} \,\mathrm{d}x \leq \int_{\mathbb{R}^d} |M(\nabla \boldsymbol{V}^n)|^{p^n(x)} \,\mathrm{d}x \leq C.$$

By the pigeon hole principle, there exists an $i^* \in \{2^j, \dots, 2^{j+1}-1\}$ such that

$$\int_{U_{j,n}^{i^*}} |M(\nabla \boldsymbol{V}^n)|^{p^n(x)} \,\mathrm{d}x \leq \frac{C}{2^j}.$$

Then, for this i^*, we set

$$\lambda_j^n := \theta_j^{i^*} = (2^j)^{i^*},$$

and thus (3.39) follows. Therefore we have

$$\int_{\{\kappa\lambda_j^n < M(\nabla \boldsymbol{V}^n) \leq \kappa 2^j \lambda_j^n\}} |M(\nabla \boldsymbol{V}^n)|^{p^n(x)} \,\mathrm{d}x \leq \frac{C}{2^j}. \tag{3.49}$$

Having such a λ_j^n, we can use (3.35) with $\lambda = \lambda_j^n$ applied to $\boldsymbol{V}^n$ and thus we introduce

$$\boldsymbol{V}_j^n := \boldsymbol{V}_{\lambda_j^n}^n.$$

Then, by Theorem 3.3.10, part (d), and the $W^{1,\infty}(\Omega)^d$-stability of Π_{div}^n, we have (3.40). Additionally, combining Lemma 3.3.11 and Lemma 3.3.12 yields (3.46). To prove (3.47), we use (3.40) and (3.49), and thus

$$\begin{aligned}\int_{\{\boldsymbol{V}_j^n \neq \boldsymbol{V}^n\}} |\nabla \boldsymbol{V}_j^n|^{p^n(x)} \, \mathrm{d}x &\leq C \int_{\{\boldsymbol{V}_j^n \neq \boldsymbol{V}^n\}} |\kappa \lambda_j^n|^{p^n(x)} \, \mathrm{d}x \leq C \int_{\{\kappa \lambda_j^n < M(\nabla \boldsymbol{V}^n)\}} |\kappa \lambda_j^n|^{p^n(x)} \, \mathrm{d}x \\ &= C \int_{U_{j,n}^{i*}} |\kappa \lambda_j^n|^{p^n(x)} \, \mathrm{d}x + C \int_{\{\kappa 2^j \lambda_j^n < M(\nabla \boldsymbol{V}^n)\}} |\kappa \lambda_j^n|^{p^n(x)} \, \mathrm{d}x \\ &\leq C \int_{U_{j,n}^{i*}} (M(\nabla \boldsymbol{V}^n))^{p^n(x)} \, \mathrm{d}x + C \int_{\mathbb{R}^d} \left(\frac{M(\nabla \boldsymbol{V}^n)}{2^j} \right)^{p^n(x)} \mathrm{d}x \\ &\leq \frac{C}{2^j} + \frac{C}{(2^j)^{p^-}} \int_{\mathbb{R}^d} (M(\nabla \boldsymbol{V}^n))^{p^n(x)} \, \mathrm{d}x \leq \frac{C}{2^j}.\end{aligned}$$

By compact embedding, (3.40), and the fact that $\boldsymbol{V}_j^n$ are compactly supported in $\mathbb{R}^d$, we can, for arbitrarily fixed $j \in \mathbb{N}$, extract a subsequence satisfying (3.41)–(3.43). Furthermore, by using a diagonal process, we can extract a further subsequence in n such that (3.41)–(3.43) hold for each $j \in \mathbb{N}$. Finally, from (3.36), (3.41), (3.46) and Hölder's inequality, we obtain

$$\begin{aligned}\|\boldsymbol{V}_j - \boldsymbol{V}\|_1 &\leq \lim_{n \to \infty} \int_\Omega |\boldsymbol{V}_j - \boldsymbol{V}_j^n| \, \mathrm{d}x + \lim_{n \to \infty} \int_\Omega |\boldsymbol{V}_j^n - \boldsymbol{V}^n| \, \mathrm{d}x + \lim_{n \to \infty} \int_\Omega |\boldsymbol{V}^n - \boldsymbol{V}| \, \mathrm{d}x \\ &= \lim_{n \to \infty} \int_\Omega |\boldsymbol{V}_j^n - \boldsymbol{V}^n| \, \mathrm{d}x \leq C \limsup_{n \to \infty} |\{\boldsymbol{V}_j^n \neq \boldsymbol{V}^n\}|^{\frac{1}{(p^-)'}} \\ &\leq C \limsup_{n \to \infty} |\{M(\nabla \boldsymbol{V}^n) > \kappa \lambda_j^n\}|^{\frac{1}{(p^-)'}} \\ &\leq C \limsup_{n \to \infty} \left(\int_\Omega \frac{M(\nabla \boldsymbol{V}^n)}{\kappa \lambda_j^n} \, \mathrm{d}x \right)^{\frac{1}{(p^-)'}} \\ &\leq \limsup_{n \to \infty} \frac{C}{(\lambda_j^n)^{\frac{1}{(p^-)'}}} \leq \frac{C}{(2^j)^{\frac{2^j}{(p^-)'}}} \leq \frac{C}{2^j} \qquad \text{for sufficiently large } j \in \mathbb{N}.\end{aligned}$$

Consequently, we have that for a (not relabelled) subsequence, $\boldsymbol{V}_j \to \boldsymbol{V}$ a.e. in Ω as $j \to \infty$. So if we prove (3.44), by the uniqueness of the weak limit, (3.45) follows. To prove (3.44), we note that

$$\begin{aligned}\liminf_{n \to \infty} \int_\Omega |\nabla \boldsymbol{V}_j^n|^{p^n(x)} \, \mathrm{d}x &= \liminf_{n \to \infty} \int_{\{\boldsymbol{V}_j^n = \boldsymbol{V}^n\}} |\nabla \boldsymbol{V}^n|^{p^n(x)} \, \mathrm{d}x \\ &\quad + \liminf_{n \to \infty} \int_{\{\boldsymbol{V}_j^n \neq \boldsymbol{V}^n\}} |\nabla \boldsymbol{V}_j^n|^{p^n(x)} \, \mathrm{d}x \leq C,\end{aligned}$$

which, by weak lower-semicontinuity (for the details, see the argument leading to (3.62)) implies the bound

$$\int_\Omega |\nabla \boldsymbol{V}_j|^{p(x)} \, \mathrm{d}x \leq C.$$

That completes the proof of the theorem. □

3.3.7 Uniform Hölder norm bound in two space dimensions

When studying numerical approximations to nonlinear partial differential equations, it is often the case that, in order to prove convergence of the sequence of numerical approximations to a solution of the original problem, some *a priori* knowledge about the regularity of the discrete solution is helpful. The aim of this section is to summarize some results of this type, whose continuous counterparts are well-known in the context of PDE analysis thanks to, primarily, the work of De Giorgi, Nash and Moser, and which will be required here in order to complete the convergence analysis of the numerical method under consideration. In [18], the authors formulate a Meyers-type regularity estimate for the sequence of approximate solutions to a second-order linear elliptic equation obtained by a finite element method. As a corollary, by Morrey's embedding theorem, in two space dimensions at least, we will obtain a uniform bound on a Hölder norm of the sequence of approximate solutions. We shall discuss the approximation scheme and the associated discrete De Giorgi theorem in more detail.

From the definition of the finite element space we have constructed, we know that $\mathbb{Z}^n \subset W_0^{1,\infty}(\Omega)$. So we can consider a conforming finite element approximation from $\mathbb{Z}^n$ to the weak solution $c \in W_0^{1,2}(\Omega)$ of the problem $-\nabla \cdot (\boldsymbol{A}\nabla c) = \nabla \cdot \boldsymbol{F} + h$, for $\boldsymbol{F} \in L^s(\Omega)^d$, $h \in L^{\frac{ds}{d+s}}(\Omega)$, $s > d$, and $\boldsymbol{A} \in L^\infty(\Omega)^{d\times d}$ uniformly elliptic, with the approximation $W^n \in \mathbb{Z}^n$ defined by: for all $Z^n \in \mathbb{Z}^n$,

$$\int_\Omega \boldsymbol{A}(x)\nabla W^n(x) \cdot \nabla Z^n(x)\,\mathrm{d}x = -\int_\Omega \boldsymbol{F}(x) \cdot \nabla Z^n(x)\,\mathrm{d}x + \int_\Omega h(x) Z^n(x)\,\mathrm{d}x. \quad (3.50)$$

An application of the Lax–Milgram theorem implies the existence of a unique solution to equation (3.50). Moreover, as a direct consequence of Proposition 8.6.2 in [18] and Theorem 5.1 in [55], we have the following result.

Theorem 3.3.14. *Assume that $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, is a bounded open convex polytopal domain and $\boldsymbol{A} \in L^\infty(\Omega)^{d\times d}$ is uniformly elliptic. Then, there exist constants $C > 0$, $n_0 \geq 1$ and $\varepsilon > 0$, such that, for all $n \geq n_0$, $s \in (2, 2+\varepsilon)$, and all $\boldsymbol{F} \in L^s(\Omega)^d$ and $h \in L^{\frac{ds}{d+s}}(\Omega)$, the solution $W^n \in \mathbb{Z}^n$ of (3.50) satisfies*

$$\|W^n\|_{W^{1,s}(\Omega)} \leq C\left(\|\boldsymbol{F}\|_{L^s(\Omega)} + \|h\|_{L^{\frac{ds}{d+s}}(\Omega)}\right).$$

In particular, if $d = 2$, by Morrey's embedding theorem, we have

$$\|W^n\|_{C^{0,\alpha}(\overline{\Omega})} \leq C\left(\|\boldsymbol{F}\|_{L^s(\Omega)} + \|h\|_{L^{\frac{ds}{d+s}}(\Omega)}\right) \quad \text{with } \alpha = 1 - \tfrac{2}{s} \in (0,1).$$

Since we need the second inequality stated in the above theorem in the subsequent analysis, we shall henceforth restrict ourselves to the case of $d = 2$, and will assume that Ω is a bounded open convex polygonal domain in $\mathbb{R}^2$. Obtaining a De Giorgi-type regularity result for the sequence of finite element approximations to (3.50) is a challenging open problem in the case of $d = 3$.

Once we have the above result, by standard boundary reduction argument, we can obtain a similar result for the equation (3.50) with non-homogeneous Dirichlet boundary datum c_d. Indeed, if we consider $Y^n := W^n - c_d$ instead of W^n, we have the following definition of the approximate solution

$$\begin{aligned}\int_\Omega \boldsymbol{A}(x)\,\nabla Y^n(x)\cdot\nabla Z^n(x)\,\mathrm{d}x = -\int_\Omega \boldsymbol{F}(x)\cdot\nabla Z^n(x)\,\mathrm{d}x + \int_\Omega h(x) Z^n(x)\,\mathrm{d}x \\ -\int_\Omega \boldsymbol{A}(x)\,\nabla c_d(x)\cdot\nabla Z^n(x)\,\mathrm{d}x\end{aligned}$$

for all $Z^n \in \mathbb{Z}^n$. We choose q such that $d = 2 < s \leq q < 2 + \varepsilon$, where ε is as in Theorem 3.3.14. Then, if $\boldsymbol{F} \in L^s(\Omega)^d$ and $c_d \in W^{1,q}(\Omega)$, it is easy to show that $\boldsymbol{G} := \boldsymbol{F} + \boldsymbol{A}\nabla c_d \in L^s(\Omega)^d$ again. Therefore, we have the following corollary, which will be used in the subsequent analysis.

Corollary 3.3.15. *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded open convex polygonal domain and that $\boldsymbol{A} \in L^\infty(\Omega)^{2\times 2}$ is uniformly elliptic with ellipticity constant $\lambda > 0$. Then, there exists a $q > 2$ such that the following holds: for any $\boldsymbol{G} \in L^q(\Omega)^2$, $h \in L^{\frac{2q}{q+2}}(\Omega)$ and any $c_d \in W^{1,q}(\Omega)$, there exists a unique $W^n \in \mathbb{Z}^n + c_d$ such that $W^n - c_d \in \mathbb{Z}^n \cap C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, satisfying*

$$\int_\Omega \boldsymbol{A}(x)\nabla W^n(x)\cdot\nabla Z^n(x)\,\mathrm{d}x = -\int_\Omega \boldsymbol{G}(x)\cdot\nabla Z^n(x)\,\mathrm{d}x + \int_\Omega h(x)\,Z^n(x)\,\mathrm{d}x$$

for all $Z^n \in \mathbb{Z}^n$ and fulfilling the uniform bound

$$\|W^n\|_{W^{1,q}(\Omega)\cap C^{0,\alpha}(\overline{\Omega})} \leq C\left(\Omega, \lambda, q, \|\boldsymbol{A}\|_\infty, \|\boldsymbol{G}\|_q, \|h\|_{\frac{2q}{q+2}}, \|c_d\|_{1,q}\right).$$

3.4 The main theorem

We are now ready to state and prove our main theorem. Note that because of the restriction $d = 2$ in Corollary 3.3.15, we only consider a two-dimensional convex polygonal domain Ω. Also, we need a stronger condition on $p(x)$.

Theorem 3.4.1. *Assume that* $\Omega \subset \mathbb{R}^2$ *is a convex polygonal domain, and* $c_d \in W^{1,q}(\Omega)$ *for some* $q > 2$*. Let us assume that* $p : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *is a Hölder continuous function with* $\frac{3}{2} \leq p^- \leq p(\xi) \leq p^+ < 2$ *for all* $\xi \in \mathbb{R}_{\geq 0}$ *and let* $\boldsymbol{f} \in (W_0^{1,p^-}(\Omega)^2)^*$*. Let* $\{\mathbb{V}^n, \mathbb{Q}^n, \mathbb{Z}^n\}_{n=1}^{\infty}$ *be the sequence of finite element space triples from* Section 3.3 *and let* $\{\boldsymbol{U}^n, P^n, C^n\}_{n=1}^{\infty}$ *be a sequence of discrete solution triples defined by the finite element approximation* (3.24)–(3.26)*. Then, there exists a (not relabelled) subsequence* $\{\boldsymbol{U}^n, P^n, C^n\}_{n=1}^{\infty}$*, which converges to a weak solution* $\{\boldsymbol{u}, \pi, c\}$ *of* (3.1)–(3.3) *defined in* **Problem (Q)** *as* $n \in \mathbb{N}$ *tends to* ∞ *in the following sense:*

$$\begin{aligned}
\boldsymbol{U}^n &\rightharpoonup \boldsymbol{u} && \text{weakly in } W_0^{1,p^-}(\Omega)^2,\\
P^n &\rightharpoonup \pi && \text{weakly in } L_0^{(p^+)'}(\Omega),\\
C^n &\rightharpoonup c && \text{weakly in } W^{1,2}(\Omega),\\
C^n &\to c && \text{strongly in } C^{0,\alpha}(\overline{\Omega}) \text{ for some } \alpha \in (0,1).
\end{aligned}$$

3.4.1 Convergence of the finite element approximations

As a first step in the proof of our main theorem, we pass to the limit in the sequence of solution triples and show the existence of weak limits for the sequences in question. First, we test with $\boldsymbol{U}^n$ in (3.24) and then thanks to (3.25) and (3.22), we have

$$\int_\Omega \boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) : \boldsymbol{D}\boldsymbol{U}^n \,\mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{U}^n \rangle.$$

By using (3.6), duality estimates, Young's inequality and Korn's inequality, we obtain

$$\int_\Omega \left(|\nabla \boldsymbol{U}^n|^{p(C^n)} + |\boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n)|^{p'(C^n)} \right) \mathrm{d}x \leq C_1, \tag{3.51}$$

where C_1 is independent of n.

Next, we test with $C^n - c_d$ in (3.26), and note that by (3.22) we have

$$\int_\Omega \boldsymbol{q}_c(C^n, \nabla C^n, \boldsymbol{D}\boldsymbol{U}^n) \cdot \nabla C^n \,\mathrm{d}x = \int_\Omega \boldsymbol{q}_c(C^n, \nabla C^n, \boldsymbol{D}\boldsymbol{U}^n) \cdot \nabla c_d \,\mathrm{d}x + B_c[C^n, \boldsymbol{U}^n, c_d].$$

By (3.7), (3.8), Hölder's inequality and Young's inequality,

$$\begin{aligned}
\|\nabla C^n\|_2^2 &\leq C \int_\Omega |\nabla C^n|\, |\nabla c_d| \,\mathrm{d}x + B_c[C^n, \boldsymbol{U}^n, c_d]\\
&\leq \varepsilon \|\nabla C^n\|_2^2 + C(\varepsilon) \|\nabla c_d\|_2^2 + B_c[C^n, \boldsymbol{U}^n, c_d].
\end{aligned}$$

By integration by parts, Sobolev embedding, Hölder's inequality and Young's inequality,

$$\begin{aligned}
B_c[C^n, \boldsymbol{U}^n, c_d] &= \frac{1}{2}\int_\Omega c_d \boldsymbol{U}^n \cdot \nabla C^n \,\mathrm{d}x - \frac{1}{2}\int_\Omega C^n \boldsymbol{U}^n \cdot \nabla c_d \,\mathrm{d}x \\
&= \frac{1}{2}\int_\Omega c_d \boldsymbol{U}^n \cdot \nabla C^n \,\mathrm{d}x + \frac{1}{2}\int_\Omega \mathrm{div}\,(C^n \boldsymbol{U}^n) c_d \,\mathrm{d}x \\
&= \int_\Omega c_d \boldsymbol{U}^n \cdot \nabla C^n \,\mathrm{d}x + \frac{1}{2}\int_\Omega C^n (\mathrm{div}\,\boldsymbol{U}^n) c_d \,\mathrm{d}x \\
&\leq \|c_d\|_\infty \|\boldsymbol{U}^n\|_2 \|\nabla C^n\|_2 + \frac{\|c_d\|_\infty}{2}\|C^n\|_{\frac{p^-}{p^- -1}} \|\mathrm{div}\,\boldsymbol{U}^n\|_{p^-} \\
&\leq C\|\boldsymbol{U}^n\|_{1,p^-}\|\nabla C^n\|_2 + C\|\boldsymbol{U}^n\|_{1,p^-}\|\nabla C^n\|_{\frac{2p^-}{3p^- -2}} \\
&\leq C(\varepsilon)\|\boldsymbol{U}^n\|_{1,p^-}^2 + \varepsilon\|\nabla C^n\|_2^2.
\end{aligned}$$

Therefore, by (3.7) and (3.51), we have

$$\int_\Omega \left(|\nabla C^n|^2 + |\boldsymbol{q}_c(C^n, \nabla C^n, \boldsymbol{D}\boldsymbol{U}^n)|^2 \right) \,\mathrm{d}x \leq C_2, \tag{3.52}$$

where C_2 is independent of n.

Now, by Sobolev embedding and the uniform estimates (3.51) and (3.52), we have for sufficiently large $t > 0$ and for $q > 2$ sufficiently close to 2, we have

$$\|C^n \boldsymbol{U}^n\|_q \leq \|C^n\|_t \|\boldsymbol{U}^n\|_{\frac{tq}{t-q}} \leq C\|C^n\|_{1,2}\|\boldsymbol{U}^n\|_{1,p^-} \leq C.$$

Also if we set $s := \frac{2q}{q+2}$, for $q > 2$ sufficiently close to 2, we have

$$\|\nabla C^n \cdot \boldsymbol{U}^n\|_s \leq \|\nabla C^n\|_2 \|\boldsymbol{U}^n\|_{\frac{2s}{2-s}} \leq \|C^n\|_{1,2}\|\boldsymbol{U}^n\|_q \leq C\|C^n\|_{1,2}\|\boldsymbol{U}^n\|_{1,p^-} \leq C.$$

Then we can apply Corollary 3.3.15 with $\boldsymbol{G} = C^n \boldsymbol{U}^n$ and $h = \nabla C^n \cdot \boldsymbol{U}^n$. Hence for some $\alpha \in (0,1)$, we obtain the following uniform bound, independent of $n \in \mathbb{N}$:

$$\|C^n\|_{C^{0,\alpha}(\overline{\Omega})} \leq C_3. \tag{3.53}$$

Since $C^{0,\alpha}(\overline{\Omega})$ is compactly embedded in $C^{0,\tilde{\alpha}}(\overline{\Omega})$ for all $\tilde{\alpha} \in (0, \alpha)$, we have that

$$C^n \to c \qquad \text{strongly in } C^{0,\tilde{\alpha}}(\overline{\Omega}),$$

which implies that

$$p \circ C^n \to p \circ c \qquad \text{strongly in } C^{0,\beta}(\overline{\Omega})$$

for some $\beta \in (0,1)$. We can therefore apply Proposition 3.3.6 with $p^n(x) := p \circ C^n(x)$. By (3.24), (3.23) and Hölder's inequality,

$$\begin{aligned}
&\|P^n\|_{(p^n)'(\cdot)} \\
&\leq C \sup_{0\neq \boldsymbol{V}\in\mathbb{V}^n, \|\boldsymbol{V}\|_{1,p^n(\cdot)}\leq 1} \langle \operatorname{div} \boldsymbol{V}, P^n\rangle \\
&\leq C \sup_{0\neq \boldsymbol{V}\in\mathbb{V}^n, \|\boldsymbol{V}\|_{1,p^n(\cdot)}\leq 1} \left| \int_\Omega \boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) : \boldsymbol{D}\boldsymbol{V} \,\mathrm{d}x + B_u[\boldsymbol{U}^n, \boldsymbol{U}^n, \boldsymbol{V}] - \langle \boldsymbol{f}, \boldsymbol{V}\rangle \right| \\
&\leq C \sup_{0\neq \boldsymbol{V}\in\mathbb{V}^n, \|\boldsymbol{V}\|_{1,p^n(\cdot)}\leq 1} \Big(\|\boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n)\|_{(p^n)'(\cdot)} \|\boldsymbol{D}\boldsymbol{V}\|_{p^n(\cdot)} + \|\boldsymbol{U}^n\|^2_{1,p^n(\cdot)} \|\boldsymbol{V}\|_{1,p^n(\cdot)} \\
&\qquad\qquad + \|\boldsymbol{f}\|_{(W_0^{1,p^-}(\Omega)^2)^*} \|\boldsymbol{V}\|_{1,p^n(\cdot)} \Big).
\end{aligned}$$

Therefore, by (3.51), we have

$$\|P^n\|_{(p^n)'(\cdot)} \leq C_4, \tag{3.54}$$

where C_4 is independent of $n \in \mathbb{N}$.

Using the bounds (3.51)–(3.54), thanks to their independence of $n \in \mathbb{N}$, reflexivity of the relevant spaces and compact Sobolev embedding, we can extract (not relabelled) subsequences such that

$$\begin{aligned}
\boldsymbol{U}^n &\rightharpoonup \boldsymbol{u} && \text{weakly in } W_0^{1,p^-}(\Omega)^2, && (3.55)\\
\boldsymbol{U}^n &\to \boldsymbol{u} && \text{strongly in } L^{2(1+\varepsilon)}(\Omega)^2, \text{ (for some } \varepsilon > 0), && (3.56)\\
C^n &\rightharpoonup c && \text{weakly in } W^{1,2}(\Omega), && (3.57)\\
C^n &\to c && \text{strongly in } C^{0,\tilde{\alpha}}(\overline{\Omega}), && (3.58)\\
P^n &\rightharpoonup \pi && \text{weakly in } L^{(p^+)'}(\Omega), && (3.59)\\
\boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) &\rightharpoonup \bar{\boldsymbol{S}} && \text{weakly in } L^{(p^+)'}(\Omega)^{2\times 2}, && (3.60)\\
\boldsymbol{q}_c(C^n, \nabla C^n, \boldsymbol{D}\boldsymbol{U}^n) &\rightharpoonup \bar{\boldsymbol{q}}_c && \text{weakly in } L^2(\Omega)^2. && (3.61)
\end{aligned}$$

Before proceeding, we shall prove that the limit function $\boldsymbol{u}$ is contained in the desired space $W_0^{1,p(c)}(\Omega)^d$. Since $C^n \to c$ in $C^{0,\tilde{\alpha}}(\overline{\Omega})$, and by the continuity of p,

$$\forall\, \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } n \geq N \text{ implies } |p(C^n) - p(c)| < \frac{\varepsilon}{\theta},$$

where $\theta > 1$ is large enough to satisfy $p(c) - \frac{\theta+1}{\theta}\varepsilon > 1$. We can then deduce from the estimate above that

$$\begin{aligned}
C &\geq \int_\Omega |\nabla \boldsymbol{U}^n|^{p(C^n)}\,\mathrm{d}x \geq \int_{\{|\nabla \boldsymbol{U}^n|\geq 1\}} |\nabla \boldsymbol{U}^n|^{p(C^n)}\,\mathrm{d}x \\
&\geq \int_{\{|\nabla \boldsymbol{U}^n|\geq 1\}} |\nabla \boldsymbol{U}^n|^{p(C^n)-p(c)+p(c)-\varepsilon}\,\mathrm{d}x \geq \int_{\{|\nabla \boldsymbol{U}^n|\geq 1\}} |\nabla \boldsymbol{U}^n|^{p(c)-\frac{\theta+1}{\theta}\varepsilon}\,\mathrm{d}x.
\end{aligned}$$

Then, after adding to the inequality the term $\int_{\{|\nabla \boldsymbol{U}^n|<1\}} |\nabla \boldsymbol{U}^n|^{p(c)-\frac{\theta+1}{\theta}\varepsilon}\,\mathrm{d}x$, which is bounded by some constant $\bar{C} \leq |\Omega|$, we obtain

$$C + \bar{C} \geq \int_\Omega |\nabla \boldsymbol{U}^n|^{p(c)-\frac{\theta+1}{\theta}\varepsilon}\,\mathrm{d}x.$$

Again, we can extract a (not relabelled) subsequence such that

$$\boldsymbol{U}^n \rightharpoonup \boldsymbol{u} \text{ weakly in } W_0^{1,p(c)-\frac{\theta+1}{\theta}\varepsilon}(\Omega)^2.$$

Thus by using the weak lower-semicontinuity of the norm function, we see that

$$\int_\Omega |\nabla \boldsymbol{u}|^{p(c)-\frac{\theta+1}{\theta}\varepsilon}\,\mathrm{d}x \leq C,$$

and consequently, Fatou's Lemma with $\varepsilon \to 0$ leads us to

$$\int_\Omega |\nabla \boldsymbol{u}|^{p(c)}\,\mathrm{d}x \leq C, \tag{3.62}$$

which implies that $\boldsymbol{u} \in W_0^{1,p(c)}(\Omega)^2$ by Poincaré's inequality. With the same argument as above we can also show that

$$\int_\Omega |\bar{\boldsymbol{S}}|^{p'(c)} + |\pi|^{p'(c)}\,\mathrm{d}x \leq C. \tag{3.63}$$

Next, we prove that the limit $\boldsymbol{u}$ is also exactly divergence-free. Let us consider an arbitrary but fixed $q \in C_0^\infty(\Omega)$. Then, by (3.25),

$$\begin{aligned}
0 &= \int_\Omega (\Pi_{\mathbb{Q}}^n q)\,\mathrm{div}\,\boldsymbol{U}^n\,\mathrm{d}x \\
&= \int_\Omega (\Pi_{\mathbb{Q}}^n q - q)\,\mathrm{div}\,\boldsymbol{U}^n\,\mathrm{d}x + \int_\Omega q(\mathrm{div}\,\boldsymbol{U}^n - \mathrm{div}\,\boldsymbol{u})\,\mathrm{d}x + \int_\Omega q\,\mathrm{div}\,\boldsymbol{u}\,\mathrm{d}x.
\end{aligned}$$

The first term tends to zero by (3.18), (3.51) and the second term converges to zero by (3.55). Therefore,

$$\int_\Omega q\,\mathrm{div}\,\boldsymbol{u}\,\mathrm{d}x = 0 \qquad \text{for any } q \in C_0^\infty(\Omega),$$

which implies that $\operatorname{div} \boldsymbol{u} = 0$ a.e. on Ω. In this case, we can identify the limit of the convective term $B_u[\cdot,\cdot,\cdot]$ as follows. Let us choose an arbitrary function $\boldsymbol{v} \in W_0^{1,\infty}(\Omega)^2$ for which we define $\boldsymbol{V}^n := \Pi^n_{\operatorname{div}} \boldsymbol{v} \in \mathbb{V}^n$. Then, by (3.14), we have

$$\boldsymbol{V}^n \to \boldsymbol{v} \qquad \text{strongly in } W_0^{1,\sigma}(\Omega)^2 \text{ for } \sigma \in [1,\infty). \tag{3.64}$$

Also, by the restriction $p^- > 1$, we have the continuous embedding $W_0^{1,p^n(\cdot)}(\Omega)^2 \hookrightarrow L^{2(1+\varepsilon)}(\Omega)^2$. Therefore, by (3.51) and (3.56),

$$\boldsymbol{U}^n \otimes \boldsymbol{U}^n \to \boldsymbol{u} \otimes \boldsymbol{u} \qquad \text{strongly in } L^{1+\varepsilon}(\Omega)^2.$$

This then enables us to identify the second part of the convective term

$$-\int_\Omega (\boldsymbol{U}^n \otimes \boldsymbol{U}^n) : \nabla \boldsymbol{V}^n \, \mathrm{d}x \to -\int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{v} \, \mathrm{d}x \qquad \text{as } n \to \infty.$$

On the other hand, for $p^- > \frac{4}{3}$, we have the continuous embedding $W_0^{1,p^n(\cdot)}(\Omega)^2 \hookrightarrow L^{(p^-)'+\varepsilon}(\Omega)^2$; thus $\boldsymbol{U}^n \cdot \boldsymbol{V}^n \to \boldsymbol{u} \cdot \boldsymbol{v}$ strongly in $L^{(p^-)'}(\Omega)^2$. Indeed,

$$\begin{aligned}
\|\boldsymbol{U}^n \cdot \boldsymbol{V}^n - \boldsymbol{u} \cdot \boldsymbol{v}\|_{(p^-)'} &\le \|(\boldsymbol{V}^n - \boldsymbol{v})\boldsymbol{U}^n + (\boldsymbol{U}^n - \boldsymbol{u})\boldsymbol{v}\|_{(p^-)'} \\
&\le \|\boldsymbol{V}^n - \boldsymbol{v}\|_s \|\boldsymbol{U}^n\|_{(p^-)'+\varepsilon} + \|\boldsymbol{U}^n - \boldsymbol{u}\|_{(p^-)'+\varepsilon} \|\boldsymbol{v}\|_s \\
&\le \|\boldsymbol{V}^n - \boldsymbol{v}\|_s \|\boldsymbol{U}^n\|_{1,p^n(\cdot)} + \|\boldsymbol{U}^n - \boldsymbol{u}\|_{\frac{2p^-}{2-p^-}-\varepsilon} \|\boldsymbol{v}\|_s
\end{aligned}$$

for some $s \in (1,\infty)$. The first term tends to zero thanks to (3.14), (3.51) and the second term converges to zero by (3.55) in conjunction with a compact embedding theorem. Therefore, together with $\operatorname{div} \boldsymbol{u} = 0$, we have

$$\begin{aligned}
\int_\Omega (\boldsymbol{U}^n \otimes \boldsymbol{V}^n) : \nabla \boldsymbol{U}^n \, \mathrm{d}x &= -\int_\Omega (\boldsymbol{U}^n \otimes \boldsymbol{U}^n) : \nabla \boldsymbol{V}^n \, \mathrm{d}x + \int_\Omega (\operatorname{div} \boldsymbol{U}^n)\, \boldsymbol{U}^n \cdot \boldsymbol{V}^n \, \mathrm{d}x \\
&\to -\int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{v} \, \mathrm{d}x \qquad \text{as } n \to \infty.
\end{aligned}$$

Collecting these limits, we then deduce that

$$\lim_{n\to\infty} B_u[\boldsymbol{U}^n, \boldsymbol{U}^n, \boldsymbol{V}^n] = -\int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{v} \, \mathrm{d}x. \tag{3.65}$$

Now, we are ready to pass to the limit in the first equation. By linearity of the projection operator $\Pi^n_{\operatorname{div}}$ and by noting (3.24), we obtain that

$$\begin{aligned}
\langle \operatorname{div} \boldsymbol{v}, P^n \rangle &= \langle \operatorname{div} \boldsymbol{V}^n, P^n \rangle + \langle \operatorname{div} (\boldsymbol{v} - \boldsymbol{V}^n), P^n \rangle \\
&= \int_\Omega \boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) : \boldsymbol{D}\boldsymbol{V}^n \, \mathrm{d}x - \langle \boldsymbol{f}, \boldsymbol{V}^n \rangle + B_u[\boldsymbol{U}^n, \boldsymbol{U}^n, \boldsymbol{V}^n] \\
&\quad + \langle \operatorname{div} (\boldsymbol{v} - \boldsymbol{V}^n), P^n \rangle \\
&\to \int_\Omega \left(\bar{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{v} - (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{v} \right) \mathrm{d}x - \langle \boldsymbol{f}, \boldsymbol{v} \rangle,
\end{aligned}$$

where we have used (3.59), (3.60), (3.64) and (3.65). Also, by (3.59) again,

$$\langle \operatorname{div} \boldsymbol{v}, P^n \rangle \to \langle \operatorname{div} \boldsymbol{v}, \pi \rangle.$$

Altogether, we have

$$\int_\Omega \big(\bar{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{v} - (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{v} \big) \,\mathrm{d}x - \langle \operatorname{div} \boldsymbol{v}, \pi \rangle = \langle \boldsymbol{f}, \boldsymbol{v} \rangle \qquad \forall\, \boldsymbol{v} \in W_0^{1,\infty}(\Omega)^2. \tag{3.66}$$

We note that by using the same argument as above we have that

$$\int_\Omega \big(\bar{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{v} - (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{v} \big) \,\mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{v} \rangle \qquad \forall\, \boldsymbol{v} \in W_{0,\operatorname{div}}^{1,\infty}(\Omega)^2. \tag{3.67}$$

Now, let us investigate the limit of the equation for the concentration, (3.26). We fix an arbitrary $z \in W_0^{1,2}(\Omega)$ and define $Z^n := \Pi_{\mathbb{Z}}^n z \in \mathbb{Z}^n$. Thanks to (3.56) and (3.58),

$$\begin{aligned} \|C^n \boldsymbol{U}^n - c\boldsymbol{u}\|_2 &\le \|(C^n - c)\boldsymbol{U}^n\|_2 + \|c(\boldsymbol{U}^n - \boldsymbol{u})\|_2 \\ &\le \|C^n - c\|_\infty \|\boldsymbol{U}^n\|_{2(1+\varepsilon)} + \|c\|_\infty \|\boldsymbol{U}^n - \boldsymbol{u}\|_{2(1+\varepsilon)} \to 0. \end{aligned}$$

Also, by (3.19), (3.56) and Sobolev embedding,

$$\begin{aligned} \|Z^n \boldsymbol{U}^n - z\boldsymbol{u}\|_2 &\le \|(Z^n - z)\boldsymbol{U}^n\|_2 + \|z(\boldsymbol{U}^n - \boldsymbol{u})\|_2 \\ &\le \|Z^n - z\|_{\frac{2(1+\varepsilon)}{\varepsilon}} \|\boldsymbol{U}^n\|_{2(1+\varepsilon)} + \|z\|_{\frac{2(1+\varepsilon)}{\varepsilon}} \|\boldsymbol{U}^n - \boldsymbol{u}\|_{2(1+\varepsilon)} \\ &\le C\|Z^n - z\|_{1,2} \|\boldsymbol{U}^n\|_{2(1+\varepsilon)} + C\|z\|_{1,2} \|\boldsymbol{U}^n - \boldsymbol{u}\|_{2(1+\varepsilon)} \to 0. \end{aligned}$$

In other words,

$$C^n \boldsymbol{U}^n \to c\boldsymbol{u} \qquad \text{strongly in } L^2(\Omega)^2, \tag{3.68}$$

$$Z^n \boldsymbol{U}^n \to z\boldsymbol{u} \qquad \text{strongly in } L^2(\Omega)^2. \tag{3.69}$$

By (3.57) and (3.69),

$$\begin{aligned} &\left| \int_\Omega Z^n \boldsymbol{U}^n \cdot \nabla C^n \,\mathrm{d}x - \int_\Omega z\boldsymbol{u} \cdot \nabla c \,\mathrm{d}x \right| \\ &\qquad \le \int_\Omega |Z^n \boldsymbol{U}^n - z\boldsymbol{u}| |\nabla C^n| \,\mathrm{d}x + \left| \int_\Omega z\boldsymbol{u} \cdot (\nabla C^n - \nabla c) \,\mathrm{d}x \right| \to 0. \end{aligned}$$

Hence, because $\operatorname{div} \boldsymbol{u} = 0$ a.e. on Ω, we have that

$$\int_\Omega Z^n \boldsymbol{U}^n \cdot \nabla C^n \,\mathrm{d}x \to \int_\Omega z\boldsymbol{u} \cdot \nabla c \,\mathrm{d}x = -\int_\Omega c\boldsymbol{u} \cdot \nabla z \,\mathrm{d}x \qquad \text{as } n \to \infty.$$

Additionally, by (3.19) and (3.68),

$$\left|\int_\Omega C^n \boldsymbol{U}^n \cdot \nabla Z^n \,\mathrm{d}x - \int_\Omega c\boldsymbol{u}\cdot\nabla z\,\mathrm{d}x\right| \\ \le \|C^n\boldsymbol{U}^n\|_2 \|Z^n - z\|_{1,2} + \|C^n\boldsymbol{U}^n - c\boldsymbol{u}\|_2\|z\|_{1,2} \to 0.$$

Altogether, we have

$$\lim_{n\to\infty} B_c[C^n, \boldsymbol{U}^n, Z^n] = -\int_\Omega c\boldsymbol{u}\cdot\nabla z\,\mathrm{d}x.$$

Finally, from (3.19) and (3.61), we have

$$\int_\Omega \boldsymbol{q}_c(C^n, \nabla C^n, \boldsymbol{D}\boldsymbol{U}^n)\cdot\nabla Z^n\,\mathrm{d}x \to \int_\Omega \bar{\boldsymbol{q}}_c\cdot\nabla z\,\mathrm{d}x \qquad \text{as } n\to\infty.$$

By collecting the limits of the two terms, we then have that

$$\int_\Omega (\bar{\boldsymbol{q}}_c\cdot\nabla z - c\boldsymbol{u}\cdot\nabla z)\,\mathrm{d}x = 0 \qquad \forall\, z\in W_0^{1,2}(\Omega). \tag{3.70}$$

We see from (3.66) and (3.70) that all that remains to be shown is the identification of the limits:

$$\bar{\boldsymbol{S}} = \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u}) \quad \text{and} \quad \bar{\boldsymbol{q}}_c = \boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u}).$$

3.4.2 Compactness of $\boldsymbol{D}\boldsymbol{U}^n$

Our proof of the identification of the limits begins by showing the compactness of $\boldsymbol{D}\boldsymbol{U}^n$ in the sense that

$$\lim_{n\to\infty}\int_\Omega ((\boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) - \boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{u})) : (\boldsymbol{D}\boldsymbol{U}^n - \boldsymbol{D}\boldsymbol{u}))^{\frac14}\,\mathrm{d}x = 0. \tag{3.71}$$

By (3.4), (3.5), (3.51), (3.62) and Hölder's inequality, we see that

$$0 \le \limsup_{n\to\infty}\int_\Omega ((\boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) - \boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{u})) : (\boldsymbol{D}\boldsymbol{U}^n - \boldsymbol{D}\boldsymbol{u}))^{\frac14}\,\mathrm{d}x = L < \infty. \tag{3.72}$$

Hence, it is enough to show that $L = 0$. For arbitrary fixed $\chi > 0$, define

$$\Omega_\chi := \{x\in\Omega : |\boldsymbol{D}\boldsymbol{u}| > \chi\}.$$

Then by (3.62), we have

$$|\Omega_\chi| \le \int_\Omega \frac{|\boldsymbol{D}\boldsymbol{u}|}{\chi}\,\mathrm{d}x \le \frac{C}{\chi}.$$

Now we decompose the integral

$$\int_\Omega ((\boldsymbol{S}(C^n,\boldsymbol{D}\boldsymbol{U}^n)-\boldsymbol{S}(C^n,\boldsymbol{D}\boldsymbol{u})):(\boldsymbol{D}\boldsymbol{U}^n-\boldsymbol{D}\boldsymbol{u}))^{\frac{1}{4}}\,\mathrm{d}x = A(n,\chi)+B(n,\chi), \quad (3.73)$$

where

$$\begin{aligned}A(n,\chi) &:= \int_{\Omega_\chi} ((\boldsymbol{S}(C^n,\boldsymbol{D}\boldsymbol{U}^n)-\boldsymbol{S}(C^n,\boldsymbol{D}\boldsymbol{u})):(\boldsymbol{D}\boldsymbol{U}^n-\boldsymbol{D}\boldsymbol{u}))^{\frac{1}{4}}\,\mathrm{d}x,\\ B(n,\chi) &:= \int_{\Omega\setminus\Omega_\chi} ((\boldsymbol{S}(C^n,\boldsymbol{D}\boldsymbol{U}^n)-\boldsymbol{S}(C^n,\boldsymbol{D}\boldsymbol{u})):(\boldsymbol{D}\boldsymbol{U}^n-\boldsymbol{D}\boldsymbol{u}))^{\frac{1}{4}}\,\mathrm{d}x.\end{aligned}$$

First, by (3.4), (3.51), (3.62) and Hölder's inequality,

$$A(n,\chi)\le C|\Omega_\chi|^{\frac{1}{2}}\le \frac{C}{\sqrt{\chi}}.$$

Next, we introduce a matrix-truncation function $T_\chi:\mathbb{R}^{2\times2}\to\mathbb{R}^{2\times2}$ as

$$T_\chi(\boldsymbol{M})=\begin{cases}\boldsymbol{M} & \text{for } |\boldsymbol{M}|\le\chi,\\ \chi\frac{\boldsymbol{M}}{|\boldsymbol{M}|} & \text{for } |\boldsymbol{M}|>\chi.\end{cases}$$

Since $T_\chi(\boldsymbol{D}\boldsymbol{u})=\boldsymbol{D}\boldsymbol{u}$ on $\Omega\setminus\Omega_\chi$ and the integrand is positive, we can rewrite $B(n,\chi)$ as

$$\begin{aligned}B(n,\chi) &= \int_{\Omega\setminus\Omega_\chi} ((\boldsymbol{S}(C^n,\boldsymbol{D}\boldsymbol{U}^n)-\boldsymbol{S}(C^n,T_\chi(\boldsymbol{D}\boldsymbol{u}))):(\boldsymbol{D}\boldsymbol{U}^n-T_\chi(\boldsymbol{D}\boldsymbol{u})))^{\frac{1}{4}}\,\mathrm{d}x\\ &\le \int_{\Omega} ((\boldsymbol{S}(C^n,\boldsymbol{D}\boldsymbol{U}^n)-\boldsymbol{S}(C^n,T_\chi(\boldsymbol{D}\boldsymbol{u}))):(\boldsymbol{D}\boldsymbol{U}^n-T_\chi(\boldsymbol{D}\boldsymbol{u})))^{\frac{1}{4}}\,\mathrm{d}x.\end{aligned}$$

Since p is a Hölder continuous function and C^n satisfies (3.58), we can apply Theorem 3.3.13. Therefore, for any $j\in\mathbb{N}$, we can find $\boldsymbol{U}_j^n\in\mathbb{V}^n\subset W_0^{1,\infty}(\Omega)^2$. Let us denote

$$\boldsymbol{S}_{n,\chi}:=(\boldsymbol{S}(C^n,\boldsymbol{D}\boldsymbol{U}^n)-\boldsymbol{S}(C^n,T_\chi(\boldsymbol{D}\boldsymbol{u}))):(\boldsymbol{D}\boldsymbol{U}^n-T_\chi(\boldsymbol{D}\boldsymbol{u})).$$

Then, by Hölder's inequality,

$$\begin{aligned}B(n,\chi) &\le \left(\int_{\{\boldsymbol{U}_j^n=\boldsymbol{U}^n\}}|\boldsymbol{S}_{n,\chi}|\,\mathrm{d}x\right)^{\frac{1}{4}}|\Omega|^{\frac{3}{4}}+\left(\int_{\{\boldsymbol{U}_j^n\ne\boldsymbol{U}^n\}}|\boldsymbol{S}_{n,\chi}|^{\frac{1}{2}}\,\mathrm{d}x\right)^{\frac{1}{2}}|\{\boldsymbol{U}_j^n\ne\boldsymbol{U}^n\}|^{\frac{1}{2}}\\ &=: (B_j(n,\chi))^{\frac{1}{4}}|\Omega|^{\frac{3}{4}}+(\tilde{B}_j(n,\chi))^{\frac{1}{2}}|\{\boldsymbol{U}_j^n\ne\boldsymbol{U}^n\}|^{\frac{1}{2}}.\end{aligned}$$

First, by (3.39), (3.46) and (3.48), we have

$$|\{\boldsymbol{U}_j^n\ne\boldsymbol{U}^n\}|=\|\chi_{\{\boldsymbol{U}_j^n\ne\boldsymbol{U}^n\}}\|_{L^1(\Omega)}\le\int_{\mathbb{R}^2}\frac{M(\boldsymbol{D}\boldsymbol{U}^n)}{\kappa\lambda_j^n}\,\mathrm{d}x\le\frac{C}{(2^j)2^{2^j}},$$

and thus it follows from (3.51), (3.62) and Hölder's inequality that

$$(\tilde{B}_j(n,\chi))^{\frac{1}{2}}|\{\boldsymbol{U}_j^n \neq \boldsymbol{U}^n\}|^{\frac{1}{2}} \leq \frac{C}{2^j}.$$

Next, we can rewrite $B_j(n,\chi)$ as

$$B_j(n,\chi) = \int_\Omega (\boldsymbol{S}(C^n,\boldsymbol{D}\boldsymbol{U}^n) - \boldsymbol{S}(C^n,T_\chi(\boldsymbol{D}\boldsymbol{u}))) : (\boldsymbol{D}\boldsymbol{U}_j^n - T_\chi(\boldsymbol{D}\boldsymbol{u}))\,\mathrm{d}x \tag{3.74}$$

$$- \int_{\{\boldsymbol{U}_j^n \neq \boldsymbol{U}^n\}} (\boldsymbol{S}(C^n,\boldsymbol{D}\boldsymbol{U}^n) - \boldsymbol{S}(C^n,T_\chi(\boldsymbol{D}\boldsymbol{u}))) : (\boldsymbol{D}\boldsymbol{U}_j^n - T_\chi(\boldsymbol{D}\boldsymbol{u}))\,\mathrm{d}x. \tag{3.75}$$

By (3.4), (3.40), (3.47), Hölder's inequality and Young's inequality, we can analyze the second term, appearing in (3.75):

$$\begin{aligned}
&\left|\int_{\{\boldsymbol{U}_j^n \neq \boldsymbol{U}^n\}} (\boldsymbol{S}(C^n,\boldsymbol{D}\boldsymbol{U}^n) - \boldsymbol{S}(C^n,T_\chi(\boldsymbol{D}\boldsymbol{u}))) : (\boldsymbol{D}\boldsymbol{U}_j^n - T_\chi(\boldsymbol{D}\boldsymbol{u}))\,\mathrm{d}x\right| \\
&\leq \int_{\{\boldsymbol{U}_j^n \neq \boldsymbol{U}^n\}} |\boldsymbol{S}(C^n,\boldsymbol{D}\boldsymbol{U}^n) : \boldsymbol{D}\boldsymbol{U}_j^n|\,\mathrm{d}x \\
&\quad + C(\chi)\int_{\{\boldsymbol{U}_j^n \neq \boldsymbol{U}^n\}} (|\boldsymbol{S}(C^n,\boldsymbol{D}\boldsymbol{U}^n)| + |\boldsymbol{D}\boldsymbol{U}_j^n| + 1)\,\mathrm{d}x \\
&\leq C\int_{\{\boldsymbol{U}_j^n \neq \boldsymbol{U}^n\}} |\nabla \boldsymbol{U}^n|^{p^n(x)-1}\lambda_j^n\,\mathrm{d}x + C(\chi)|\{\boldsymbol{U}_j^n \neq \boldsymbol{U}^n\}|^{\frac{1}{p^+}} + \frac{C(\chi)}{2^j} \\
&\leq \frac{C}{(p^+)'}\int_{\{\boldsymbol{U}_j^n \neq \boldsymbol{U}^n\}} |\nabla \boldsymbol{U}^n|^{p^n(x)}\,\mathrm{d}x + \frac{C}{p^-}\int_{\{\boldsymbol{U}_j^n \neq \boldsymbol{U}^n\}} |\lambda_j^n|^{p^n(x)}\,\mathrm{d}x + \frac{C(\chi)}{2^j} \\
&\leq \frac{C(\chi)}{2^j}.
\end{aligned}$$

Now, to analyze the first term (3.74) above, we have to use the weak formulation. Here, however, we cannot use the Lipschitz truncation $\boldsymbol{U}_j^n$ as a test function, as it is not guaranteed to be discretely divergence-free. To overcome this difficulty, we shall define discretely divergence-free approximations with zero trace with the help of the discrete Bogovskiĭ operator; more precisely, let

$$\begin{aligned}
\boldsymbol{\Psi}_j^n &:= \mathcal{B}^n(\operatorname{div}\boldsymbol{U}_j^n), \\
\boldsymbol{\Phi}_j^n &:= \boldsymbol{U}_j^n - \boldsymbol{\Psi}_j^n.
\end{aligned}$$

It is then clear that $\boldsymbol{\Phi}_j^n$ has a zero trace on $\partial\Omega$ and, by construction, $\boldsymbol{\Phi}_j^n \in \mathbb{V}_{\mathrm{div}}^n$. Moreover, from the compact embedding $W_0^{1,\sigma}(\Omega) \hookrightarrow\hookrightarrow L^\sigma(\Omega)$, (3.42) and Lemma 3.3.8, we have

$$\boldsymbol{\Phi}_j^n \rightharpoonup \boldsymbol{U}_j - \mathcal{B}(\operatorname{div}\boldsymbol{U}_j) =: \boldsymbol{\Phi}_j \qquad \text{weakly in } W_0^{1,\sigma}(\Omega)^2, \tag{3.76}$$

$$\boldsymbol{\Phi}_j^n \to \boldsymbol{\Phi}_j \qquad \text{strongly in } L^\sigma(\Omega)^2, \tag{3.77}$$

as $n \to \infty$, where $\sigma \in (1, \infty)$ is arbitrary. We can then rewrite (3.74) above in terms of this approximation to obtain

$$
\begin{aligned}
&\int_\Omega (\boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) - \boldsymbol{S}(C^n, T_\chi(\boldsymbol{D}\boldsymbol{u}))) : (\boldsymbol{D}\boldsymbol{U}^n_j - T_\chi(\boldsymbol{D}\boldsymbol{u}))\,\mathrm{d}x \\
&= \int_\Omega \boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) : (\boldsymbol{D}\boldsymbol{\Phi}^n_j + \boldsymbol{D}\boldsymbol{\Psi}^n_j)\,\mathrm{d}x \\
&\quad - \int_\Omega \boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) : T_\chi(\boldsymbol{D}\boldsymbol{u})\,\mathrm{d}x - \int_\Omega \boldsymbol{S}(C^n, T_\chi(\boldsymbol{D}\boldsymbol{u})) : (\boldsymbol{D}\boldsymbol{U}^n_j - T_\chi(\boldsymbol{D}\boldsymbol{u}))\,\mathrm{d}x \\
&=: B^{n,1}_{\chi,j} - B^{n,2}_{\chi,j} - B^{n,3}_{\chi,j}.
\end{aligned}
$$

Now we use (3.27) with $\boldsymbol{V} = \boldsymbol{\Phi}^n_j \in \mathbb{V}^n_{\mathrm{div}}$ and pass to the limit with (3.56), (3.60), and (3.76); thus we have, by (3.67), that

$$
\begin{aligned}
\lim_{n\to\infty} \int_\Omega \boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) : \boldsymbol{D}\boldsymbol{\Phi}^n_j\,\mathrm{d}x &= -\lim_{n\to\infty} B_u[\boldsymbol{U}^n, \boldsymbol{U}^n, \boldsymbol{\Phi}^n_j] + \lim_{n\to\infty} \langle \boldsymbol{f}, \boldsymbol{\Phi}^n_j \rangle \\
&= \int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{\Phi}_{\boldsymbol{j}}\,\mathrm{d}x + \langle \boldsymbol{f}, \boldsymbol{\Phi}_{\boldsymbol{j}} \rangle \\
&= \int_\Omega \bar{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{\Phi}_{\boldsymbol{j}}\,\mathrm{d}x.
\end{aligned}
$$

Let us now consider the second integral in $B^{n,1}_{\chi,j}$. Using the boundedness of $\boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n)$ in $L^{p'(C^n)}(\Omega)^{2\times 2}$, we can estimate it by Hölder's inequality as follows:

$$
\int_\Omega \boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) : \boldsymbol{D}\boldsymbol{\Psi}^n_j\,\mathrm{d}x \leq C\|\boldsymbol{D}\boldsymbol{\Psi}^n_j\|_{p^n(\cdot)} \leq C\|\Pi^n_{\mathrm{div}}\mathcal{B}\mathcal{K}(\operatorname{div}\boldsymbol{U}^n_j)\|_{1,p^n(\cdot)}. \tag{3.78}
$$

By (3.21), and Theorem 2.5.1,

$$
\|\mathcal{B}\mathcal{K}(\operatorname{div}\boldsymbol{U}^n_j)\|_{1,p^n(\cdot)} \leq C\|\mathcal{K}(\operatorname{div}\boldsymbol{U}^n_j)\|_{p^n(\cdot)} \leq C \sup_{Q\in\mathbb{Q}^n,\ \|Q\|_{(p^n)'(\cdot)}\leq 1} \langle \operatorname{div}\boldsymbol{U}^n_j\, Q\rangle.
$$

We deduce, by Hölder's inequality, that

$$
\begin{aligned}
\langle \operatorname{div}\boldsymbol{U}^n_j, Q\rangle &= \sum_{E\subset\{\boldsymbol{U}^n_j=\boldsymbol{U}^n\}} \langle \operatorname{div}\boldsymbol{U}^n, \chi_E Q\rangle + \sum_{E\cap\{\boldsymbol{U}^n_j\neq\boldsymbol{U}^n\}\neq\emptyset} \langle \operatorname{div}\boldsymbol{U}^n_j, \chi_E Q\rangle \\
&\leq \left\|\operatorname{div}\boldsymbol{U}^n_j \chi_{S_{\{\boldsymbol{U}^n_j\neq\boldsymbol{U}^n\}}}\right\|_{p^n(\cdot)} \left\| \sum_{E\cap\{\boldsymbol{U}^n_j\neq\boldsymbol{U}^n\}\neq\emptyset} \chi_E Q \right\|_{(p^n)'(\cdot)} \\
&\leq \left\|\nabla\boldsymbol{U}^n_j \chi_{S_{\{\boldsymbol{U}^n_j\neq\boldsymbol{U}^n\}}}\right\|_{p^n(\cdot)} \left\| \sum_{E\cap\{\boldsymbol{U}^n_j\neq\boldsymbol{U}^n\}\neq\emptyset} \chi_E Q \right\|_{(p^n)'(\cdot)},
\end{aligned}
$$

where $\chi_{S_{\{\boldsymbol{U}^n_j\neq\boldsymbol{U}^n\}}}$ is the characteristic function of the set

$$
S_{\{\boldsymbol{U}^n_j\neq\boldsymbol{U}^n\}} := \bigcup\left\{S_E : E \in \mathcal{G}_n \text{ such that } E \cap \overline{\{\boldsymbol{U}^n_j \neq \boldsymbol{U}^n\}} \neq \emptyset\right\}.
$$

Then, by Lemma 3.3.12 and (3.47),

$$\left\|\nabla \boldsymbol{U}_j^n \chi_{S_{\{\boldsymbol{U}_j^n \neq \boldsymbol{U}^n\}}}\right\|_{p^n(\cdot)} \leq \frac{C}{2^{j/p^+}}.$$

Also, by Theorem 2.4.10,

$$\begin{aligned}
\left\|\sum_{E\cap\{\boldsymbol{U}_j^n \neq \boldsymbol{U}^n\}\neq\emptyset} \chi_E Q\right\|_{(p^n)'(\cdot)} &\leq C\left\|\sum_{E\cap\{\boldsymbol{U}_j^n \neq \boldsymbol{U}^n\}\neq\emptyset} \chi_E \frac{\|\chi_E Q\|_{(p^n)'(\cdot)}}{\|\chi_E\|_{(p^n)'(\cdot)}}\right\|_{(p^n)'(\cdot)} \\
&\leq C\left\|\sum_{E\in\mathcal{G}_n} \chi_E \frac{\|\chi_E Q\|_{(p^n)'(\cdot)}}{\|\chi_E\|_{(p^n)'(\cdot)}}\right\|_{(p^n)'(\cdot)} \\
&\leq C\left\|\sum_{E\in\mathcal{G}_n} \chi_E Q\right\|_{(p^n)'(\cdot)} \\
&\leq C\|Q\|_{(p^n)'(\cdot)}.
\end{aligned}$$

Therefore, we have

$$\|\mathcal{B}\mathcal{K}(\operatorname{div}\boldsymbol{U}_j^n)\|_{1,p^n(\cdot)} \leq \frac{C}{2^{j/p^+}},$$

which implies, together with Proposition 3.3.2, that

$$\|\Pi_{\operatorname{div}}^n \mathcal{B}\mathcal{K}(\operatorname{div}\boldsymbol{U}_j^n)\|_{1,p^n(\cdot)} \leq \left(\frac{C}{2^{j/p^+}} + C\max_{E\in\mathcal{G}_n} h_E^{d+1}\right)^{\gamma} \tag{3.79}$$

for some $\gamma = \gamma(p^-, p^+) > 0$.

Now, note further that by weak lower-semicontinuity and boundedness of $\bar{\boldsymbol{S}}$ in $L^{p'(c)}$,

$$\begin{aligned}
\int_\Omega \bar{\boldsymbol{S}} : \boldsymbol{D}\mathcal{B}(\operatorname{div}\boldsymbol{U}_j)\,\mathrm{d}x &\leq C\|\mathcal{B}(\operatorname{div}\boldsymbol{U}_j)\|_{1,p(c)} \\
&\leq C\limsup_{n\to\infty}\|\mathcal{B}^n(\operatorname{div}\boldsymbol{U}_j^n)\|_{1,p^n(\cdot)} \leq \left(\frac{C}{2^{j/p^+}}\right)^{\gamma}.
\end{aligned} \tag{3.80}$$

For the last two integrals $B_{\chi,j}^{n,2}$ and $B_{\chi,j}^{n,3}$, we use (3.42), (3.58), (3.60) and the boundedness of the truncation T_χ to get

$$\lim_{n\to\infty}\left(B_{\chi,j}^{n,2} + B_{\chi,j}^{n,3}\right) = \int_\Omega \bar{\boldsymbol{S}} : T_\chi(\boldsymbol{D}\boldsymbol{u})\,\mathrm{d}x + \int_\Omega \boldsymbol{S}(c, T_\chi(\boldsymbol{D}\boldsymbol{u})) : (\boldsymbol{D}\boldsymbol{U}_j - T_\chi(\boldsymbol{D}\boldsymbol{u}))\,\mathrm{d}x.$$

Altogether, we have

$$\begin{aligned}
&\lim_{n\to\infty}\left(B_{\chi,j}^{n,1} - B_{\chi,j}^{n,2} - B_{\chi,j}^{n,3}\right) \\
&\quad\leq \int_\Omega \bar{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{\Phi}_j\,\mathrm{d}x + \left(\frac{C}{2^{j/p^+}}\right)^{\gamma} - \lim_{n\to\infty}\left(B_{\chi,j}^{n,2} + B_{\chi,j}^{n,3}\right) \\
&\quad= \int_\Omega \bar{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{U}_j\,\mathrm{d}x - \int_\Omega \bar{\boldsymbol{S}} : \boldsymbol{D}\mathcal{B}(\operatorname{div}\boldsymbol{U}_j)\,\mathrm{d}x + \left(\frac{C}{2^{j/p^+}}\right)^{\gamma} - \lim_{n\to\infty}\left(B_{\chi,j}^{n,2} + B_{\chi,j}^{n,3}\right) \\
&\quad\leq \int_\Omega (\bar{\boldsymbol{S}} - \boldsymbol{S}(c, T_\chi(\boldsymbol{D}\boldsymbol{u}))) : (\boldsymbol{D}\boldsymbol{U}_j - T_\chi(\boldsymbol{D}\boldsymbol{u}))\,\mathrm{d}x + \left(\frac{C}{2^{j/p^+}}\right)^{\gamma}.
\end{aligned}$$

Going back to (3.73), we finally let χ, $j \to \infty$ and $n \to \infty$, and estimate

$$\begin{aligned}
&\lim_{\chi\to\infty}\lim_{j\to\infty}\lim_{n\to\infty}\left(A(n,\chi)+B(n,\chi)\right)\\
&\quad\le \lim_{\chi\to\infty}\lim_{j\to\infty}\lim_{n\to\infty}\left(C\left(B^{n,1}_{\chi,j}-B^{n,2}_{\chi,j}-B^{n,3}_{\chi,j}+\frac{C(\chi)}{2^j}\right)^{\frac{1}{4}}|\Omega|^{\frac{3}{4}}+\frac{C}{\sqrt{\chi}}+\frac{C}{2^j}\right)\\
&\quad\le \lim_{\chi\to\infty} C\left(\left(\int_\Omega(\bar{\boldsymbol{S}}-\boldsymbol{S}(c,T_\chi(\boldsymbol{Du}))):(\boldsymbol{Du}-T_\chi(\boldsymbol{Du}))\,\mathrm{d}x\right)^{\frac{1}{4}}+\frac{C}{\sqrt{\chi}}\right)=0,
\end{aligned}$$

where we have used (3.45) for $j \to \infty$ and the pointwise convergence of $T_\chi(\boldsymbol{Du}) \to \boldsymbol{Du}$ on Ω with the dominated convergence theorem for $\chi \to \infty$. We have thereby completed the proof of the desired compactness of the sequence $\{\boldsymbol{DU}^n\}_{n=1}^\infty$ in the sense that (3.71) holds.

3.4.3 Identification of $\bar{\boldsymbol{S}} = \boldsymbol{S}(c, \boldsymbol{Du})$ and $\bar{\boldsymbol{q}}_c = \boldsymbol{q}_c(c, \nabla c, \boldsymbol{Du})$

In the previous section we showed that

$$\lim_{n\to\infty}\int_\Omega\left((\boldsymbol{S}(C^n,\boldsymbol{DU}^n)-\boldsymbol{S}(C^n,\boldsymbol{Du})):(\boldsymbol{DU}^n-\boldsymbol{Du})\right)^{\frac{1}{4}}\,\mathrm{d}x=0. \tag{3.81}$$

Since the integrand is nonnegative, (3.81) also holds for a set $Q_\gamma \subset \Omega$ where

$$Q_\gamma := \{x \in \Omega : |\boldsymbol{Du}| \le \gamma\},$$

with an arbitrarily fixed constant $\gamma > 0$. From the sequence of integrands of (3.81), we can extract a subsequence (again not relabelled), which converges to zero almost everywhere in Q_γ. Then, by Egoroff's theorem, for arbitrary $\varepsilon > 0$, we can find a set $Q^\varepsilon_\gamma \subset \Omega$ such that $|Q_\gamma \setminus Q^\varepsilon_\gamma| < \varepsilon$, where the sequence of integrands converges uniformly. It is obvious that, thanks to the choice of Q^ε_γ, we have

$$\lim_{\gamma\to\infty}\lim_{\varepsilon\to 0}|\Omega \setminus Q^\varepsilon_\gamma| = 0,$$

and furthermore, from the uniform convergence, we have

$$\lim_{n\to\infty}\int_{Q^\varepsilon_\gamma}(\boldsymbol{S}(C^n,\boldsymbol{DU}^n)-\boldsymbol{S}(C^n,\boldsymbol{Du})):(\boldsymbol{DU}^n-\boldsymbol{Du})\,\mathrm{d}x=0. \tag{3.82}$$

Thanks to the boundedness of $\boldsymbol{Du}$ on Q^ε_γ, by the dominated convergence theorem, we have $\boldsymbol{S}(C^n, \boldsymbol{Du}) \to \boldsymbol{S}(c, \boldsymbol{Du})$ strongly in $L^q(\Omega)^{2\times 2}$ for any $q \in [1, \infty)$. Thus, together with the above L^q-convergence and weak convergence (3.55), from (3.82), we have

$$\lim_{n\to\infty}\int_{Q_\gamma^\varepsilon} \boldsymbol{S}(C^n,\boldsymbol{D}\boldsymbol{U}^n) : (\boldsymbol{D}\boldsymbol{U}^n - \boldsymbol{D}\boldsymbol{u})\,\mathrm{d}x = 0.$$

Hence, by the boundedness of $\boldsymbol{D}\boldsymbol{u}$ on Q_γ^ε and the convergence result (3.59), we have

$$\lim_{n\to\infty}\int_{Q_\gamma^\varepsilon} \boldsymbol{S}(C^n,\boldsymbol{D}\boldsymbol{U}^n) : \boldsymbol{D}\boldsymbol{U}^n\,\mathrm{d}x = \int_{Q_\gamma^\varepsilon} \bar{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{u}\,\mathrm{d}x. \tag{3.83}$$

Now, let $\boldsymbol{B} \in L^\infty(Q_\gamma^\varepsilon)^{2\times 2}$ be arbitrarily fixed. By the monotonicity assumption (3.5),

$$0 \le \int_{Q_\gamma^\varepsilon} (\boldsymbol{S}(C^n,\boldsymbol{D}\boldsymbol{U}^n) - \boldsymbol{S}(C^n,\boldsymbol{B})) : (\boldsymbol{D}\boldsymbol{U}^n - \boldsymbol{B})\,\mathrm{d}x. \tag{3.84}$$

Thus, from (3.83), the L^q-convergence of $\boldsymbol{S}(C^n,\boldsymbol{B}) \to \boldsymbol{S}(c,\boldsymbol{B})$ and the weak convergence (3.55), we have

$$\begin{aligned} 0 &\le \lim_{n\to\infty}\int_{Q_\gamma^\varepsilon} (\boldsymbol{S}(C^n,\boldsymbol{D}\boldsymbol{U}^n) - \boldsymbol{S}(C^n,\boldsymbol{B})) : (\boldsymbol{D}\boldsymbol{U}^n - \boldsymbol{B})\,\mathrm{d}x \\ &= \int_{Q_\gamma^\varepsilon} \bar{\boldsymbol{S}} : (\boldsymbol{D}\boldsymbol{u} - \boldsymbol{B})\,\mathrm{d}x - \int_{Q_\gamma^\varepsilon} \boldsymbol{S}(c,\boldsymbol{B}) : (\boldsymbol{D}\boldsymbol{u} - \boldsymbol{B})\,\mathrm{d}x \\ &= \int_{Q_\gamma^\varepsilon} (\bar{\boldsymbol{S}} - \boldsymbol{S}(c,\boldsymbol{B})) : (\boldsymbol{D}\boldsymbol{u} - \boldsymbol{B})\,\mathrm{d}x. \end{aligned}$$

Now we use Minty's trick. First, choose $\boldsymbol{B} = \boldsymbol{D}\boldsymbol{u} \pm \lambda \boldsymbol{A}(x)$ with $\lambda > 0$ and $\boldsymbol{A} \in L^\infty(Q_\gamma^\varepsilon)^{2\times 2}$. Then, passing to the limit $\lambda \to 0$, the continuity of $\boldsymbol{S}$ gives us

$$\int_{Q_\gamma^\varepsilon} (\bar{\boldsymbol{S}} - \boldsymbol{S}(c,\boldsymbol{D}\boldsymbol{u})) : \boldsymbol{A}(x)\,\mathrm{d}x = 0.$$

Therefore, we have

$$\bar{\boldsymbol{S}} = \boldsymbol{S}(c,\boldsymbol{D}\boldsymbol{u}) \;\text{ a.e. on }\; Q_\gamma^\varepsilon.$$

So now we let $\varepsilon \to 0$ and then $\gamma \to \infty$ to conclude that

$$\bar{\boldsymbol{S}} = \boldsymbol{S}(c,\boldsymbol{D}\boldsymbol{u}) \;\text{ a.e. on }\; \Omega.$$

Finally, since $\boldsymbol{S}$ is strictly monotonic and $C^n \to c$ in $C^{0,\tilde{\alpha}}(\overline{\Omega})$, from (3.81) we have

$$\boldsymbol{D}\boldsymbol{U}^n \to \boldsymbol{D}\boldsymbol{u} \;\text{ a.e. on }\; \Omega. \tag{3.85}$$

As a continuous linear operator preserves weak convergence, by the dominated convergence theorem with (3.57), (3.58) and (3.85), we deduce that

$$\boldsymbol{q}_c(C^n,\nabla C^n,\boldsymbol{D}\boldsymbol{U}^n) \rightharpoonup \boldsymbol{q}_c(c,\nabla c,\boldsymbol{D}\boldsymbol{u}) \qquad \text{weakly in } L^2(\Omega)^2.$$

Therefore, by the uniqueness of the weak limit, we can identify

$$\bar{\boldsymbol{q}}_c = \boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u}),$$

thus completing the proof of the convergence of the finite element method under consideration to a weak solution of the problem.

3.5 Proof of Theorem 3.4.1 using monotone operator theory

As mentioned in Section 3.1, here we present the proof of the main theorem by using monotone operator theory. If $p^- \geq \frac{3}{2}$, we can test with $\boldsymbol{u}$ in (3.67). Indeed,

$$\int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{u} \, \mathrm{d}x \leq \|\boldsymbol{u}\|^2_{2(p^-)'} \|\boldsymbol{u}\|_{1,p^-} \leq C \|\boldsymbol{u}\|^2_{\frac{2p^-}{2-p^-}} \|\boldsymbol{u}\|_{1,p^-} \leq C,$$

and hence, $\boldsymbol{u}$ is an admissible test function. Then by (3.27) and (3.67), we have

$$\lim_{n\to\infty} \int_\Omega \boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) : \boldsymbol{D}\boldsymbol{U}^n \, \mathrm{d}x = \lim_{n\to\infty} \langle \boldsymbol{f}, \boldsymbol{U}^n \rangle = \langle \boldsymbol{f}, \boldsymbol{u} \rangle = \int_\Omega \bar{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{u} \, \mathrm{d}x.$$

Next, by (3.58) and the dominated convergence theorem, we have for arbitrary $\boldsymbol{\phi} \in C^\infty_{0,\mathrm{div}}(\Omega)^2$ that

$$\boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{\phi}) \to \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{\phi}) \text{ strongly in } L^q(\Omega)^{2\times 2} \text{ for all } q \in (1, \infty). \tag{3.86}$$

Then, by the monotonicity (3.5), we deduce for any $\boldsymbol{\phi} \in C^\infty_{0,\mathrm{div}}(\Omega)^2$ that

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \int_\Omega (\boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) - \boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{\phi})) : (\boldsymbol{D}\boldsymbol{U}^n - \boldsymbol{D}\boldsymbol{\phi}) \, \mathrm{d}x \\ &\leq \int_\Omega \big(\bar{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{u} - \bar{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{\phi} - \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{\phi}) : \boldsymbol{D}\boldsymbol{u} + \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{\phi}) : \boldsymbol{D}\boldsymbol{\phi}\big) \, \mathrm{d}x \\ &= \int_\Omega \big(\bar{\boldsymbol{S}} - \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{\phi})\big) : (\boldsymbol{D}\boldsymbol{u} - \boldsymbol{D}\boldsymbol{\phi}) \, \mathrm{d}x. \end{aligned}$$

Now, the density of smooth functions and Minty's trick with $\boldsymbol{\phi} = \boldsymbol{u} \pm \lambda \boldsymbol{v}$ where $\boldsymbol{v} \in C^\infty_{0,\mathrm{div}}(\Omega)^2$ gives the desired identification $\bar{\boldsymbol{S}} = \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u})$.

Next, we shall show the almost everywhere convergence of $\boldsymbol{D}\boldsymbol{U}^n$ to $\boldsymbol{D}\boldsymbol{u}$. Let $\boldsymbol{B} \in \mathbb{R}^{2\times 2}$ be a bounded symmetric function. Then with the same argument as above and the fact $\bar{\boldsymbol{S}} = \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u})$, we obtain that

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \int_\Omega (\boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) - \boldsymbol{S}(C^n, \boldsymbol{B})) : (\boldsymbol{D}\boldsymbol{U}^n - \boldsymbol{B}) \, \mathrm{d}x \\ &\leq \int_\Omega (\boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u}) - \boldsymbol{S}(c, \boldsymbol{B})) : (\boldsymbol{D}\boldsymbol{u} - \boldsymbol{B}) \, \mathrm{d}x. \end{aligned}$$

We set $\boldsymbol{B} = \boldsymbol{D}\boldsymbol{u}\chi_{\{|\boldsymbol{D}\boldsymbol{u}|<\lambda\}}$ for $\lambda > 0$ and let $0 < k < \lambda$. Then we have

$$\begin{aligned}
\limsup_{n\to\infty} & \int_{\{|\boldsymbol{D}\boldsymbol{u}|<k\}} (\boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) - \boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{u})) : (\boldsymbol{D}\boldsymbol{U}^n - \boldsymbol{D}\boldsymbol{u}) \,\mathrm{d}x \\
&= \limsup_{n\to\infty} \int_{\{|\boldsymbol{D}\boldsymbol{u}|<k\}} (\boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) - \boldsymbol{S}(C^n, \boldsymbol{B})) : (\boldsymbol{D}\boldsymbol{U}^n - \boldsymbol{B}) \,\mathrm{d}x \\
&\le \limsup_{n\to\infty} \int_{\Omega} (\boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) - \boldsymbol{S}(C^n, \boldsymbol{B})) : (\boldsymbol{D}\boldsymbol{U}^n - \boldsymbol{B}) \,\mathrm{d}x \\
&= \int_{\Omega} (\boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u}) - \boldsymbol{S}(c, \boldsymbol{B})) : (\boldsymbol{D}\boldsymbol{u} - \boldsymbol{B}) \,\mathrm{d}x \\
&= \int_{\{|\boldsymbol{D}\boldsymbol{u}|>\lambda\}} (\boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u}) - \boldsymbol{S}(c, \boldsymbol{B})) : (\boldsymbol{D}\boldsymbol{u} - \boldsymbol{B}) \,\mathrm{d}x.
\end{aligned}$$

As the first integral is independent of $\lambda > 0$, passing λ to ∞ gives us that

$$\limsup_{n\to\infty} \int_{\{|\boldsymbol{D}\boldsymbol{u}|<k\}} (\boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) - \boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{u})) : (\boldsymbol{D}\boldsymbol{U}^n - \boldsymbol{D}\boldsymbol{u}) \,\mathrm{d}x = 0.$$

Since C^n converges to c uniformly and $\boldsymbol{S}$ is strictly monotone, the only way the above can be true is that $\boldsymbol{D}\boldsymbol{U}^n$ tends to $\boldsymbol{D}\boldsymbol{u}$ almost everywhere in the set $\{|\boldsymbol{D}\boldsymbol{u}| < k\}$. If we pass k to ∞, we finally have

$$\boldsymbol{D}\boldsymbol{U}^n \to \boldsymbol{D}\boldsymbol{u} \;\text{ a.e. in }\; \Omega. \tag{3.87}$$

Now by the dominated convergence theorem,

$$\boldsymbol{K}(C^n, |\boldsymbol{D}\boldsymbol{U}^n|) \to \boldsymbol{K}(c, |\boldsymbol{D}\boldsymbol{u}|) \;\text{ strongly in }\; L^q(\Omega) \;\text{ for all }\; q \in (1, \infty).$$

Then we obtain that

$$\boldsymbol{q}_c(C^n, \nabla C^n, \boldsymbol{D}\boldsymbol{U}^n) \rightharpoonup \boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u}) \;\text{ weakly in }\; L^2(\Omega)^2.$$

Therefore, by the uniqueness of the weak limit, we have

$$\bar{\boldsymbol{q}}_c = \boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u}),$$

which completes the proof of Theorem 3.4.1.

Our aim in the next chapter is to develop a modification of the numerical algorithm considered here in the two-dimensional setting which can be shown to converge to a weak solution in three space dimensions

Chapter 4

Finite element approximation of the stationary model in 3D

4.1 Brief description of the chapter

In the previous chapter, the convergence of a finite element approximation to the system (3.1)–(3.3) was shown in two space dimensions, using a discrete De Giorgi regularity result. Because of the absence of an analogous discrete De Giorgi regularity result in three space dimensions, the analysis in Chapter 3 was restricted to the case of two space dimensions. The aim of this chapter is to extend the analysis developed in Chapter 3 to the physically relevant case of three space dimensions. To this end, we exploit a more complicated limiting process here than in Chapter 3, so as to avoid reliance on a discrete De Giorgi regularity result. In particular, we consider different meshes for the generalized Navier–Stokes system and the concentration equation, resulting in a two-level Galerkin approximation. This enables us to decouple the passage to the limit in the finite element approximation of the concentration equation from the passage to the limit in the finite element approximation of the generalized Navier–Stokes system, and we thus completely circumvent the need for a discrete De Giorgi estimate.

As a first step, in Section 4.2, we define a suitable regularized problem, which enables us to enlarge the range of the power-law index so as to cover the practically relevant range of values of this index. In Sections 4.3 and 4.4, we construct a finite element approximation to the regularized problem and perform a convergence analysis of the numerical method. Finally, in Section 4.5, we prove that weak solutions of the regularized problem converge to a weak solution of the original problem. The content of this chapter is based on the journal paper [58].

4.2 Regularization of the problem

Before constructing the approximation of **Problem (Q)** stated in the previous chapter, we shall formulate a regularized problem; it will then be the regularized problem that will be approximated by a finite element method. We shall show that the sequence of finite element approximations converges to a weak solution of the regularized problem, and that solutions of the regularized problem, in turn, converge to a weak solution of **Problem (Q)**. The reason for proceeding in this way is that direct approximation of **Problem (Q)**, which bypasses the use of the regularized problem, necessitates the imposition of an unnaturally strong condition on the variable exponent p in the convergence analysis of the finite element method. More precisely, if no regularization were used, then the condition $p^- > 2$ would need to be assumed in order to be able to apply Theorem 4.4.1, which would be an overly restrictive hypothesis as it would exclude even the case of a Newtonian fluid, corresponding to p being identically equal to 2. The procedure that we describe below does not suffer from this shortcoming and also allows us to perform the finite element analysis with $\frac{3d}{d+2} > p^- > \frac{d}{2}$ which is physically relevant as described in Section 3.1.

As discussed above, we shall utilize the following regularized problem, involving the regularization parameter $k \in \mathbb{N}$. We choose a sufficiently large $t > 2$, so that $p^- > \frac{3}{2} > \frac{t}{t-2}$. We then seek a weak solution $(\boldsymbol{u}, \pi, c) := (\boldsymbol{u}^k, \pi^k, c^k)$ to

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{4.1}$$

$$\operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div} \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u}) + \frac{1}{k}|\boldsymbol{u}|^{t-2}\boldsymbol{u} = -\nabla \pi + \boldsymbol{f} \qquad \text{in } \Omega, \tag{4.2}$$

$$\operatorname{div}(c\boldsymbol{u}) - \operatorname{div} \boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u}) = 0 \qquad \text{in } \Omega, \tag{4.3}$$

Note that we have added a new regularizing term $\frac{1}{k}|\boldsymbol{u}|^{t-2}\boldsymbol{u}$ in (4.2) for sufficiently large $t > 0$. We consider the following weak formulation for the regularized problem.

Problem (Q*). For $\boldsymbol{f} \in (W_0^{1,p^-}(\Omega)^3)^*$, $c_d \in W^{1,s}(\Omega)$, $s > 3$, and a Hölder continuous function p, with $1 < p^- \leq p(\xi) \leq p^+ < \infty$ for all $\xi \in \mathbb{R}_{\geq 0}$, and $p^- > \frac{3}{2} > \frac{t}{t-2}$, $t > 2$, find $(c - c_d) := (c^k - c_d) \in W_0^{1,2}(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0, 1)$,

$\boldsymbol{u} := \boldsymbol{u}^k \in W_0^{1,p(c^k)}(\Omega)^3$, $\pi := \pi^k \in L_0^{p'(c^k)}(\Omega)$ such that for all $\boldsymbol{\psi} \in W_0^{1,\infty}(\Omega)^3$, $q \in L_0^{p'(c)}(\Omega)$ and $\varphi \in W_0^{1,2}(\Omega)$,

$$\int_\Omega \left(\boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u}) : \nabla \boldsymbol{\psi} - (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{\psi} + \frac{1}{k} |\boldsymbol{u}|^{t-2} \boldsymbol{u} \cdot \boldsymbol{\psi} \right) \mathrm{d}x - \langle \operatorname{div} \boldsymbol{\psi}, \pi \rangle = \langle \boldsymbol{f}, \boldsymbol{\psi} \rangle, \tag{4.4}$$

$$\int_\Omega q \operatorname{div} \boldsymbol{u} \, \mathrm{d}x = 0, \tag{4.5}$$

$$\int_\Omega (\boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u}) \cdot \nabla \varphi - c\boldsymbol{u} \cdot \nabla \varphi) \, \mathrm{d}x = 0. \tag{4.6}$$

By using Proposition 2.5.3, we can restate **Problem (Q*)** in the following (equivalent) divergence-free setting.

Problem (P*). For $\boldsymbol{f} \in (W_0^{1,p^-}(\Omega)^3)^*$, $c_d \in W^{1,s}(\Omega)$, $s > 3$, and a Hölder continuous function p, with $1 < p^- \leq p(\xi) \leq p^+ < \infty$ for all $\xi \in \mathbb{R}_{\geq 0}$, and $p^- > \frac{3}{2} > \frac{t}{t-2}$, $t > 2$, find $(c - c_d) := (c^k - c_d) \in C^{0,\alpha}(\overline{\Omega}) \cap W_0^{1,2}(\Omega)$, $\boldsymbol{u} := \boldsymbol{u}^k \in W_{0,\mathrm{div}}^{1,p(c^k)}(\Omega)^3$, such that for all $\boldsymbol{\psi} \in W_{0,\mathrm{div}}^{1,\infty}(\Omega)^3$ and $\varphi \in W_0^{1,2}(\Omega)$,

$$\int_\Omega \left(\boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u}) : \nabla \boldsymbol{\psi} - (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{\psi} + \frac{1}{k} |\boldsymbol{u}|^{t-2} \boldsymbol{u} \cdot \boldsymbol{\psi} \right) \mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{\psi} \rangle, \tag{4.7}$$

$$\int_\Omega (\boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u}) \cdot \nabla \varphi - c\boldsymbol{u} \cdot \nabla \varphi) \, \mathrm{d}x = 0. \tag{4.8}$$

We shall formulate the finite element approximation of the regularized problem **Problem (Q*)** in a three-dimensional domain; the convergence analysis of the method is presented in Section 4.3 and Section 4.4. In Section 4.5, we will prove that a sequence of weak solution triples $\{(\boldsymbol{u}^k, \pi^k, c^k)\}_{k=1}^\infty$ of the regularized problem converges to a weak solution triple $(\boldsymbol{u}, \pi, c)$ of **Problem (Q)**. The latter result is recorded in our next theorem. As will become clear from the subsequent analysis, the condition $p^- > \frac{3}{2} > \frac{t}{t-2}$ is necessary in order for us to be able to obtain a uniform bound on the sequence of approximate pressures (i.e., to prove (4.39)) and to apply Theorem 4.4.1 in the final step in our passage to the limit (i.e., to prove (4.71)).

Theorem 4.2.1. *Suppose that $\Omega \subset \mathbb{R}^3$ is a polyhedral domain and $c_d \in W^{1,s}(\Omega)$ for some $s > 3$. Let us further assume that $p : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a bounded Hölder continuous function with $p^- > \frac{3}{2} > \frac{t}{t-2}$, $t > 2$, and suppose that $\boldsymbol{f} \in (W_0^{1,p^-}(\Omega)^3)^*$. Let $(\boldsymbol{u}^k, \pi^k, c^k)$ be a weak solution of the regularized problem* (4.1)–(4.3). *Then, as $k \to \infty$, (a subsequence, not indicated, of) the sequence $\{(\boldsymbol{u}^k, \pi^k, c^k)\}_{k=1}^\infty$ converges* *to* $(\boldsymbol{u}, \pi, c)$ *in the following sense:*

$$\begin{aligned}
\boldsymbol{u}^k &\rightharpoonup \boldsymbol{u} && \text{weakly in } W^{1,p^-}_{0,\mathrm{div}}(\Omega)^3, \\
c^k &\rightharpoonup c && \text{weakly in } W^{1,2}(\Omega), \\
c^k &\to c && \text{strongly in } C^{0,\alpha}(\overline{\Omega}) \qquad \text{for some } \alpha \in (0,1), \\
\pi^k &\rightharpoonup \pi && \text{weakly in } L^{j'}(\Omega) \qquad \forall\, j > \max\{p^+, 2\}.
\end{aligned}$$

Furthermore, $(\boldsymbol{u}, \pi, c)$ *is a weak solution of the problem* **Problem (Q)** *stated in* Chapter 3.

4.3 Finite element approximation

4.3.1 Finite element spaces

As discussed before, we shall consider different meshes for the momentum equation and the convection-diffusion equation. For the velocity and pressure, we will use the same spaces $\mathbb{V}^n(\mathcal{G}_n)$ and $\mathbb{Q}^n(\mathcal{G}_n)$ for each $n \in \mathbb{N}$, introduced in Chapter 3. In other words, $\{\mathcal{G}_n\}$ is a family of shape-regular partitions of $\overline{\Omega}$, $\mathbb{V}^n(\mathcal{G}_n)$ is a finite-dimensional subspace of $W^{1,\infty}_0(\Omega)^3$ spanned by a finite number of continuous, locally supported basis functions, and $\mathbb{Q}^n(\mathcal{G}_n)$ is a finite-dimensional subspace of $L^\infty(\Omega)$ spanned by a finite number of discontinuous piecewise polynomials. We shall also use the same notation as in Chapter 3,

$$\mathbb{V}^n_{\mathrm{div}} = \{\boldsymbol{V} \in \mathbb{V}^n : \langle \operatorname{div} \boldsymbol{V}, Q \rangle = 0 \;\; \forall Q \in \mathbb{Q}^n\} \;\text{ and }\; \mathbb{Q}^n_0 = \{Q \in \mathbb{Q}^n : \int_\Omega Q \,\mathrm{d}x = 0\},$$

and we further assume the approximability of $\mathbb{V}^n$ and $\mathbb{Q}^n$ (**Assumption 1**), the existence of a divergence-preserving, locally $W^{1,1}$-stable projection operator Π^n_{div} into $\mathbb{V}^n$ (**Assumption 2**) and the existence of a locally L^1-stable projection operator $\Pi^n_{\mathbb{Q}}$ into $\mathbb{Q}^n$ (**Assumption 3**). As mentioned before, there exist finite element pairs $(\mathbb{V}^n, \mathbb{Q}^n)$ satisfying all the assumptions above: for example, the conforming Crouzeix–Raviart Stokes element. See Chapter 3 for more details.

For the concentration, we introduce another mesh and adopt a different discretization with a new discritization parameter $m \in \mathbb{N}$. As a first step, we define a new triangulation of the domain for the concentration. Let $\{\mathcal{H}_m\}$ be a family of shape-regular partitions of $\overline{\Omega}$ such that the following properties hold:

- **Affine equivalence**: For each element $E \in \mathcal{H}_m$, there exists an invertible affine mapping
$$\bar{\boldsymbol{F}}_E : E \to \hat{E},$$
where $\hat{E}$ is the standard reference 3-simplex in $\mathbb{R}^3$.

- **Shape-regularity**: For any element $E \in \mathcal{H}_m$, the ratio of diam(E) to the radius of the inscribed ball is bounded below uniformly by a positive constant, with respect to all $\mathcal{H}_m$ and $m \in \mathbb{N}$.

For given partition $\mathcal{H}_m$, the finite element space for the concentration is defined by

$$\mathbb{Z}^m = \mathbb{Z}(\mathcal{H}_m) := \{Z \in C(\overline{\Omega}) : Z_{|E} \circ \bar{\boldsymbol{F}}_E^{-1} \in \hat{\mathbb{P}}_{\mathbb{Z}}, E \in \mathcal{H}_m \text{ and } Z_{|\partial\Omega} = 0\},$$

where $\hat{\mathbb{P}}_{\mathbb{Z}} \subset W^{1,\infty}(\hat{E})$ are finite-dimensional linear subspaces.

We assume that $\mathbb{Z}^m$ have finite and locally supported bases; for example, for each $m \in \mathbb{N}$, there exists an $N_m \in \mathbb{N}$ such that

$$\mathbb{Z}^m = \operatorname{span}\{Z_1^m, \dots, Z_{N_m}^m\},$$

and for each basis function Z_j^m, we have that if there exists an $E \in \mathcal{H}_m$, with $Z_j^m \neq 0$ on E, then
$$\operatorname{supp} Z_j^m \subset \bigcup\{E' \in \mathcal{H}_m : E' \cap E \neq \emptyset\} =: T_E.$$
Using the assumed shape-regularity we can easily verify that

$$\exists Y \in \mathbb{N} : |T_E| \leq Y|E| \text{ for all } E \in \mathcal{H}_m,$$

where Y is independent of m. We denote by h_E the diameter of $E \in \mathcal{H}_m$.

Throughout this chapter, we assume that the finite element spaces introduced above have the following minimal approximation property.

Assumption 1* (Approximability) For all $s \in [1, \infty)$,

$$\inf_{Z \in \mathbb{Z}^m} \|z - Z\|_{1,s} \to 0 \qquad \forall\, z \in W_0^{1,s}(\Omega) \text{ as } m \to \infty.$$

For this, a necessary condition is that the maximal mesh size vanishes, i.e., that $\max_{E \in \mathcal{H}_m} h_E \to 0$ as $m \to \infty$.

Assumption 2* (Existence of a projection operator $\Pi_{\mathbb{Z}}^m$) For each $m \in \mathbb{N}$, there exists a linear projection operator $\Pi_{\mathbb{Z}}^m : W_0^{1,1}(\Omega) \to \mathbb{Z}^m$ such that

$$\fint_E (|\Pi_{\mathbb{Z}}^m z| + h_E |\nabla \Pi_{\mathbb{Z}}^m z|) \,\mathrm{d}x \leq c_1 \fint_{T_E} (|z| + h_E |\nabla z|) \,\mathrm{d}x$$

for all $z \in W_0^{1,1}(\Omega)$ and for all $E \in \mathcal{H}_m$ where c_1 does not depend on m.

Similarly as before, the projection operator $\Pi_{\mathbb{Z}}^m$ is globally $W^{1,s}$-stable for $s \in [1, \infty]$, and thus, by approximability,

$$\|\Pi_{\mathbb{Z}}^m z - z\|_{1,s} \to 0 \qquad \text{as } m \to \infty, \text{ for all } z \in W_0^{1,s}(\Omega) \text{ and all } s \in [1, \infty). \quad (4.9)$$

Finally, we state a discrete inf-sup condition, which holds in our finite element setting. It is a direct consequence of (2.4) and the existence of Π_{div}^n; see [9] for further details.

Proposition 4.3.1. *For s, $s' \in (1, \infty)$ satisfying $\frac{1}{s} + \frac{1}{s'} = 1$, there exists a positive constant $\beta_s > 0$, which is independent of n, such that*

$$\beta_s \|Q\|_{s'} \leq \sup_{0 \neq \boldsymbol{V} \in \mathbb{V}^n} \frac{\langle \operatorname{div} \boldsymbol{V}, Q \rangle}{\|\boldsymbol{v}\|_{1,s}} \qquad \forall\, Q \in \mathbb{Q}_0^n \text{ and } \forall\, n \in \mathbb{N}.$$

4.3.2 The finite element approximation

In this subsection, we shall construct the finite element approximation of the problem (4.1)–(4.3). Similarly as we did in the previous chapter, we shall use the modified convective term to ensure that the skew-symmetry is preserved under discretization;

$$\begin{aligned} B_u[\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{h}] &:= \frac{1}{2} \int_\Omega ((\boldsymbol{v} \otimes \boldsymbol{h}) : \nabla \boldsymbol{w} - (\boldsymbol{v} \otimes \boldsymbol{w}) : \nabla \boldsymbol{h}) \,\mathrm{d}x, \\ B_c[b, \boldsymbol{v}, z] &:= \frac{1}{2} \int_\Omega (z \boldsymbol{v} \cdot \nabla b - b \boldsymbol{v} \cdot \nabla z) \,\mathrm{d}x, \end{aligned}$$

for all $\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{h} \in W_0^{1,\infty}(\Omega)^3$, $b, z \in W^{1,\infty}(\Omega)$. As before, these trilinear forms then coincide with the corresponding trilinear forms appearing in the weak formulations of the momentum equation and the concentration equation, provided that we are considering pointwise divergence-free velocity fields. Furthermore, thanks to their skew-symmetry, these two trilinear forms now also vanish for discretely divergence-free functions when $\boldsymbol{w} = \boldsymbol{h}$ and $b = z$, respectively.

Moreover, the trilinear form $B_u[\cdot,\cdot,\cdot]$ is bounded in the following sense. If $\boldsymbol{v},\boldsymbol{w},\boldsymbol{h} \in W_0^{1,\infty}(\Omega)^3$, then, by Hölder's inequality,

$$\int_\Omega (\boldsymbol{v}\otimes\boldsymbol{w}) : \nabla\boldsymbol{h}\,\mathrm{d}x \leq \|\boldsymbol{v}\|_{2(p^-)'}\|\boldsymbol{w}\|_{2(p^-)'}\|\boldsymbol{h}\|_{1,p^-}$$

and

$$\int_\Omega (\boldsymbol{v}\otimes\boldsymbol{h}) : \nabla\boldsymbol{w}\,\mathrm{d}x \leq \|\boldsymbol{v}\|_{2(p^-)'}\|\boldsymbol{h}\|_{2(p^-)'}\|\boldsymbol{w}\|_{1,p^-}.$$

Therefore, we obtain the bound

$$|B_u[\boldsymbol{v},\boldsymbol{w},\boldsymbol{h}]| \leq \|\boldsymbol{v}\|_{2(p^-)'}\|\boldsymbol{w}\|_{2(p^-)'}\|\boldsymbol{h}\|_{1,p^-} + \|\boldsymbol{v}\|_{2(p^-)'}\|\boldsymbol{w}\|_{1,p^-}\|\boldsymbol{h}\|_{2(p^-)'}. \tag{4.10}$$

Now, we first fix the regularization parameter $k \in \mathbb{N}$, and perform the finite element convergence analysis, for a fixed $k \in \mathbb{N}$, of the regularized problem. Recall that we denote by n the discretization parameter associated with the velocity and the pressure, and by m the discretization parameter for the concentration (cf. the discussion at the beginning of Section 4.3), and will pass to the limit, first as $m \to \infty$ and then as $n \to \infty$. Since $k \in \mathbb{N}$ is fixed throughout this section and the next section, it will be omitted from our notation: for example, we shall write $\boldsymbol{U}^{n,m}$ instead of $\boldsymbol{U}^{n,m,k}$, $P^{n,m}$ instead of $P^{n,m,k}$, $C^{n,m}$ instead of $C^{n,m,k}$, and $(\boldsymbol{u},\pi,c)$ instead of $(\boldsymbol{u}^k,\pi^k,c^k)$ for a solution triple of the regularized problem. Having passed to the limits $m,n \to \infty$ with $k \in \mathbb{N}$ fixed, we shall reinstate the index k in our notation at the start of Section 4.5 in preparation for the final passage to the limit $k \to \infty$ with the regularization parameter.

For each $n,m \in \mathbb{N}$, we call a triple $(\boldsymbol{U}^{n,m},P^{n,m},C^{n,m}) \in \mathbb{V}^n \times \mathbb{Q}_0^n \times (\mathbb{Z}^m + c_d)$ a discrete solution to the Galerkin approximation if it satisfies, for all $\boldsymbol{V} \in \mathbb{V}^n$, $Q \in \mathbb{Q}^n$ and $Z \in \mathbb{Z}^m$,

$$\int_\Omega \left(\boldsymbol{S}(C^{n,m},\boldsymbol{D}\boldsymbol{U}^{n,m}) : \boldsymbol{D}\boldsymbol{V} + \frac{1}{k}|\boldsymbol{U}^{n,m}|^{t-2}\boldsymbol{U}^{n,m}\cdot\boldsymbol{V}\right)\mathrm{d}x + B_u[\boldsymbol{U}^{n,m},\boldsymbol{U}^{n,m},\boldsymbol{V}]$$
$$-\langle \operatorname{div}\boldsymbol{V}, P^{n,m}\rangle = \langle \boldsymbol{f},\boldsymbol{V}\rangle, \tag{4.11}$$

$$\int_\Omega Q\operatorname{div}\boldsymbol{U}^{n,m}\,\mathrm{d}x = 0, \tag{4.12}$$

$$\int_\Omega \boldsymbol{q}_c(C^{n,m},\nabla C^{n,m},\boldsymbol{D}\boldsymbol{U}^{n,m})\cdot\nabla Z\,\mathrm{d}x + B_c[C^{n,m},\boldsymbol{U}^{n,m},Z] = 0, \tag{4.13}$$

where $c_d \in W^{1,s}(\Omega)$ with $s > 3$ and $\boldsymbol{f} \in (W_0^{1,p^-}(\Omega)^3)^*$.

If we restrict the test functions $\boldsymbol{V}$ to $\mathbb{V}^n_{\text{div}}$, then the above problem is transformed to the following: find $(\boldsymbol{U}^{n,m}, C^{n,m}) \in \mathbb{V}^n_{\text{div}} \times (\mathbb{Z}^m + c_d)$ such that, for all $\boldsymbol{V} \in \mathbb{V}^n_{\text{div}}$ and $Z \in \mathbb{Z}^m$,

$$\int_\Omega \left(\boldsymbol{S}(C^{n,m}, \boldsymbol{D}\boldsymbol{U}^{n,m}) : \boldsymbol{D}\boldsymbol{V} + \frac{1}{k} |\boldsymbol{U}^{n,m}|^{t-2}\, \boldsymbol{U}^{n,m} \cdot \boldsymbol{V} \right) \mathrm{d}x + B_u[\boldsymbol{U}^{n,m}, \boldsymbol{U}^{n,m}, \boldsymbol{V}] = \langle \boldsymbol{f}, \boldsymbol{V} \rangle, \tag{4.14}$$

$$\int_\Omega \boldsymbol{q}_c(C^{n,m}, \nabla C^{n,m}, \boldsymbol{D}\boldsymbol{U}^{n,m}) \cdot \nabla Z \, \mathrm{d}x + B_c[C^{n,m}, \boldsymbol{U}^{n,m}, Z] = 0. \tag{4.15}$$

If $\frac{3}{2} < p^-$, the existence of the discrete solution pair $(\boldsymbol{U}^{n,m}, C^{n,m}) \in \mathbb{V}^n_{\text{div}} \times (\mathbb{Z}^m + c_d)$ follows from a fixed point argument combined with an iteration scheme as in Chapter 3. However, since we have an additional regularization term here and the proof is slightly different from the one presented in the previous chapter, let us briefly summarize the proof of the existence of the pair $(\boldsymbol{U}^{n,m}, C^{n,m}) \in \mathbb{V}^n_{\text{div}} \times (\mathbb{Z}^m + c_d)$. Let $\{\boldsymbol{w}_i\}_{i=1}^{N_n}$ be a basis of $\mathbb{V}^n_{\text{div}} \subset W^{1,\infty}_0(\Omega)^3$ such that $\int_\Omega \boldsymbol{w}_i \cdot \boldsymbol{w}_j \, \mathrm{d}x = \delta_{ij}$ and let $\{z_j\}_{j=1}^{N_m}$ be a basis of $\mathbb{Z}^m \subset W^{1,2}_0(\Omega)$ such that $\int_\Omega z_i z_j \, \mathrm{d}x = \delta_{ij}$. Then, for fixed $n, m \in \mathbb{N}$, we define the Galerkin approximations.

$$\boldsymbol{U}^{n,m} := \sum_{i=1}^{N_n} \alpha_i^{n,m} \boldsymbol{w}_i, \qquad C^{n,m} := \sum_{i=1}^{N_m} \beta_i^{n,m} z_i + c_d,$$

which satisfy (4.14), (4.15).

First we define $C^{n,m}_1 := c_d \in \mathbb{Z}^m + c_d$. Then, for any $\ell \in \mathbb{N}$, we define $\boldsymbol{U}^{n,m}_\ell \in \mathbb{V}^n_{\text{div}}$ as a solution of the following finite-dimensional problem: for all $\boldsymbol{V} \in \mathbb{V}^n_{\text{div}}$,

$$\int_\Omega \left(\boldsymbol{S}(C^{n,m}_\ell, \boldsymbol{D}\boldsymbol{U}^{n,m}_\ell) : \boldsymbol{D}\boldsymbol{V} + \frac{1}{k} |\boldsymbol{U}^{n,m}_\ell|^{t-2}\, \boldsymbol{U}^{n,m}_\ell \cdot \boldsymbol{V} \right) \mathrm{d}x + B_u[\boldsymbol{U}^{n,m}_\ell, \boldsymbol{U}^{n,m}_\ell, \boldsymbol{V}] = \langle \boldsymbol{f}, \boldsymbol{V} \rangle.$$

Then we define $C^{n,m}_\ell \in \mathbb{Z}^m + c_d$ as a solution of the finite-dimensional problem

$$\int_\Omega \boldsymbol{q}_c(C^{n,m}_\ell, \nabla C^{n,m}_\ell, \boldsymbol{D}\boldsymbol{U}^{n,m}_{\ell-1}) \cdot \nabla Z \, \mathrm{d}x + B_c[C^{n,m}_\ell, \boldsymbol{U}^{n,m}_{\ell-1}, Z] = 0 \qquad \forall\, Z \in \mathbb{Z}^m.$$

For each $\ell \in \mathbb{N}$, the existence of the functions $\boldsymbol{U}^{n,m}_\ell \in \mathbb{V}^n_{\text{div}}$ and $C^{n,m}_\ell \in \mathbb{Z}^m + c_d$ is easily shown by means of Brouwer's fixed point theorem. Furthermore, for each $n, m \in \mathbb{N}$, the sequences of functions $\{\boldsymbol{U}^{n,m}_\ell\}_{\ell=1}^\infty$ and $\{C^{n,m}_\ell\}_{\ell=1}^\infty$ satisfy the following uniform bounds:

$$\|\boldsymbol{U}^{n,m}_\ell\|_{1,p^-} + \|\boldsymbol{U}^{n,m}_\ell\|_t \leq C_1, \qquad \|\nabla C^{n,m}_\ell\|_2 \leq C_2,$$

where C_1 and C_2 are positive constants, independent of ℓ. Thus, by the Bolzano–Weierstrass theorem, we deduce the existence of limits $\boldsymbol{U}^{n,m} \in \mathbb{V}^n_{\text{div}}$ and $C^{n,m} \in \mathbb{Z}^m + c_d$ for $\boldsymbol{U}^{n,m}_\ell$ and $C^{n,m}_\ell$, respectively, as $\ell \to \infty$, and these limits form a solution pair for the Galerkin approximation (4.14), (4.15). This establishes the existence of a solution to the Galerkin approximations (4.14), (4.15) for any fixed pair of integers $n, m \in \mathbb{N}$. The existence of a discrete solution triple for (4.11)–(4.13) then follows by the discrete inf-sup condition stated in Proposition 4.3.1, and we write $P^{n,m} = \sum_{i=1}^{\tilde{N}_n} \gamma_i^{n,m} y_i$, where $\{y_i\}_{i=1}^{\tilde{N}_n}$ is a basis of $\mathbb{Q}^n_0$.

We are now ready to state and prove our main theorem in this section. It asserts that, as $n, m \to \infty$, the sequence of discrete solution triples converges to a weak solution triple of the regularized problem.

Theorem 4.3.2. *Suppose that $\Omega \subset \mathbb{R}^3$ is a polyhedral domain and $c_d \in W^{1,s}(\Omega)$ for some $s > 3$. Let us assume that $p : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a bounded Hölder continuous function with $p^- > \frac{3}{2} > \frac{t}{t-2}$, $t > 2$, and let $\boldsymbol{f} \in (W_0^{1,p^-}(\Omega)^3)^*$. Let $(\boldsymbol{U}^{n,m}, P^{n,m}, C^{n,m}) \in \mathbb{V}^n_{\text{div}} \times \mathbb{Q}^n_0 \times (\mathbb{Z}^m + c_d)$ be a discrete solution triple defined by the finite element approximation* (4.11)–(4.13)*. Then, the following convergence results hold.*

- *At the first level of Galerkin approximation, there exist subsequences (not relabelled) with respect to m such that (as $m \to \infty$),*

$$\begin{aligned}
\boldsymbol{U}^{n,m} &\to \boldsymbol{U}^n && \textit{uniformly on } \overline{\Omega},\\
\boldsymbol{D}\boldsymbol{U}^{n,m} &\to \boldsymbol{D}\boldsymbol{U}^n && \textit{uniformly on } \overline{\Omega},\\
P^{n,m} &\to P^n && \textit{uniformly on } \overline{\Omega},\\
C^{n,m} &\rightharpoonup C^n && \text{weakly in } W^{1,2}(\Omega),
\end{aligned}$$

  where $\boldsymbol{U}^n \in \mathbb{V}^n$, $P^n \in \mathbb{Q}^n_0$.

- *At the second level of Galerkin approximation, there exist subsequences (not relabelled) with respect to n such that (as $n \to \infty$),*

$$\begin{aligned}
\boldsymbol{U}^n &\rightharpoonup \boldsymbol{u} && \text{weakly in } W_0^{1,p^-}(\Omega)^3,\\
P^n &\rightharpoonup \pi && \text{weakly in } L^{j'}(\Omega) \qquad \forall\, j > \max\{p^+, 2\},\\
C^n &\rightharpoonup c && \text{weakly in } W^{1,2}(\Omega),\\
C^n &\to c && \text{strongly in } C^{0,\alpha}(\overline{\Omega}),
\end{aligned}$$

where $(\boldsymbol{u}, \pi, c) = (\boldsymbol{u}^k, \pi^k, c^k)$ *is a weak solution triple of the regularized problem* (4.4)*–*(4.6)*.*

4.4 Proof of Theorem 4.3.2

4.4.1 The limit $m \to \infty$

First, we shall derive some uniform bounds, independent of $m \in \mathbb{N}$, and let m tend to infinity by using the weak compactness properties in the corresponding reflexive spaces. For simplicity, we shall denote $\boldsymbol{S}^{n,m} := \boldsymbol{S}(C^{n,m}, \boldsymbol{D}\boldsymbol{U}^{n,m})$, $\boldsymbol{q}_c^{n,m} := \boldsymbol{q}_c(C^{n,m}, \nabla C^{n,m}, \boldsymbol{D}\boldsymbol{U}^{n,m})$.

We test with $\boldsymbol{U}^{n,m} \in \mathbb{V}^n_{\mathrm{div}}$ in (4.11); then, thanks to the skew symmetry of $B_u[\cdot,\cdot,\cdot]$, we have

$$\begin{aligned}\int_\Omega \left(\boldsymbol{S}^{n,m} : \nabla \boldsymbol{U}^{n,m} + \frac{1}{k}|\boldsymbol{U}^{n,m}|^t \right) \mathrm{d}x &= \int_\Omega \left(\boldsymbol{S}^{n,m} : \boldsymbol{D}\boldsymbol{U}^{n,m} + \frac{1}{k}|\boldsymbol{U}^{n,m}|^t \right) \mathrm{d}x \\ &= \langle \boldsymbol{f}, \boldsymbol{U}^{n,m} \rangle.\end{aligned}$$

By (3.6) and Young's inequality, we have

$$\int_\Omega \left(|\nabla \boldsymbol{U}^{n,m}|^{p(C^{n,m})} + |\boldsymbol{S}^{n,m}|^{p'(C^{n,m})} + |\boldsymbol{U}^{n,m}|^t \right) \mathrm{d}x \le C_1, \tag{4.16}$$

where C_1 is independent of m, n and k.

Next, we test with $C^{n,m} - c_d \in \mathbb{Z}^m$ in (4.13) and deduce that

$$\int_\Omega \boldsymbol{q}_c(C^{n,m}, \nabla C^{n,m}, \boldsymbol{D}\boldsymbol{U}^{n,m}) \cdot \nabla (C^{n,m} - c_d)\, \mathrm{d}x = B_c[C^{n,m}, \boldsymbol{U}^{n,m}, c_d].$$

By (3.7), (3.8), Hölder's inequality and Young's inequality,

$$\begin{aligned}\|\nabla C^{n,m}\|_2^2 &\le \int_\Omega |\nabla C^{n,m}||\nabla c_d|\, \mathrm{d}x + B_c[C^{n,m}, \boldsymbol{U}^{n,m}, c_d] \\ &\le \varepsilon \|\nabla C^{n,m}\|_2^2 + C(\varepsilon)\|\nabla c_d\|_2^2 + B_c[C^{n,m}, \boldsymbol{U}^{n,m}, c_d].\end{aligned}$$

Then, by Sobolev embedding,

$$\begin{aligned}B_c[C^{n,m}, \boldsymbol{U}^{n,m}, c_d] &= \frac{1}{2}\int_\Omega c_d \boldsymbol{U}^{n,m} \cdot \nabla C^{n,m}\, \mathrm{d}x - \frac{1}{2}\int_\Omega C^{n,m}\boldsymbol{U}^{n,m} \cdot \nabla c_d\, \mathrm{d}x \\ &= \int_\Omega c_d \boldsymbol{U}^{n,m} \cdot \nabla C^{n,m}\, \mathrm{d}x + \frac{1}{2}\int_\Omega C^{n,m}(\mathrm{div}\, \boldsymbol{U}^{n,m}) c_d\, \mathrm{d}x \\ &\le \|c_d\|_\infty \|\boldsymbol{U}^{n,m}\|_2 \|\nabla C^{n,m}\|_2 + \frac{1}{2}\|c_d\|_\infty \|C^{n,m}\|_{(p^-)'} \|\mathrm{div}\, \boldsymbol{U}^{n,m}\|_{p^-} \\ &\le C\|\boldsymbol{U}^{n,m}\|_{1,p^-}\|\nabla C^{n,m}\|_2 + C\|\boldsymbol{U}^{n,m}\|_{1,p^-}\|\nabla C^{n,m}\|_{\frac{3p^-}{4p^- - 3}} \\ &\le C(\varepsilon)\|\boldsymbol{U}^{n,m}\|_{1,p^-}^2 + \varepsilon\|\nabla C^{n,m}\|_2^2.\end{aligned}$$

Hence, by (3.7) and (4.16), we have

$$\int_\Omega \left(|\nabla C^{n,m}|^2 + |\boldsymbol{q}_c^{n,m}|^2\right) \,\mathrm{d}x \le C_2, \tag{4.17}$$

where C_2 is independent of m, n and k.

Next, we shall derive a uniform bound on the pressure. By Proposition 4.3.1 together with (4.11), (4.10) and the equivalence of norms in finite-dimensional spaces, we have

$$\begin{aligned}
\beta_p \|P^{n,m}\|_{(p^+)'} &\le \sup_{0\neq \boldsymbol{V}\in \mathbb{V}^n} \frac{\langle \operatorname{div} \boldsymbol{V}, P^{n,m}\rangle}{\|\boldsymbol{V}\|_{1,p^+}} \\
&\le \sup_{0\neq \boldsymbol{V}\in \mathbb{V}^n} \frac{|\int_\Omega \boldsymbol{S}^{n,m} : \boldsymbol{D}\boldsymbol{V} \,\mathrm{d}x|}{\|\boldsymbol{V}\|_{1,p^+}} \\
&\quad + C \sup_{0\neq \boldsymbol{V}\in \mathbb{V}^n} \frac{|B_u[\boldsymbol{U}^{n,m}, \boldsymbol{U}^{n,m}, \boldsymbol{V}] - \langle \boldsymbol{f}, \boldsymbol{V}\rangle|}{\|\boldsymbol{V}\|_{1,p^-}} \\
&\le C \sup_{0\neq \boldsymbol{V}\in \mathbb{V}^n} \frac{\|\boldsymbol{S}^{n,m}\|_{(p^+)'}\|\boldsymbol{D}\boldsymbol{V}\|_{p^+}}{\|\boldsymbol{V}\|_{1,p^+}} \\
&\quad + C(n) \sup_{0\neq \boldsymbol{V}\in \mathbb{V}^n} \frac{\|\boldsymbol{U}^{n,m}\|^2_{2(p^-)'}\|\boldsymbol{V}\|_{1,p^-} + \|\boldsymbol{f}\|_{(W_0^{1,p^-}(\Omega)^3)^*}\|\boldsymbol{V}\|_{1,p^-}}{\|\boldsymbol{V}\|_{1,p^-}},
\end{aligned}$$

where C and $C(n)$ are independent of m. Therefore, by (4.16), we deduce that

$$\|P^{n,m}\|_{(p^+)'} \le C(n). \tag{4.18}$$

Now we are ready to let m tend to infinity. By (4.16) and (4.18) with the equivalence of norms in finite-dimensional spaces, we have $|\boldsymbol{\alpha}^{n,m}| \le C_1(n)$ and $|\boldsymbol{\gamma}^{n,m}| \le C_2(n)$, where $C_1(n)$ and $C_2(n)$ are independent of m. Then, together with the uniform estimates (4.17), we can extract (not relabelled) subsequences such that

$$\boldsymbol{\alpha}^{n,m} \to \boldsymbol{\alpha}^n \qquad \text{strongly in } \mathbb{R}^{N_n}, \tag{4.19}$$

$$\boldsymbol{\gamma}^{n,m} \to \boldsymbol{\gamma}^n \qquad \text{strongly in } \mathbb{R}^{\tilde{N}_n}, \tag{4.20}$$

$$C^{n,m} \rightharpoonup C^n \qquad \text{weakly in } W^{1,2}(\Omega). \tag{4.21}$$

From (4.19), (4.20), (4.21) and compact embedding, we have

$$\boldsymbol{U}^{n,m} \to \boldsymbol{U}^n \qquad \text{uniformly on } \overline{\Omega}, \tag{4.22}$$

$$\boldsymbol{D}\boldsymbol{U}^{n,m} \to \boldsymbol{D}\boldsymbol{U}^n \qquad \text{uniformly on } \overline{\Omega}, \tag{4.23}$$

$$P^{n,m} \to P^n \qquad \text{uniformly on } \overline{\Omega}, \tag{4.24}$$

$$C^{n,m} \to C^n \qquad \text{strongly in } L^2(\Omega). \tag{4.25}$$

By (4.19) and (4.20), note that

$$\boldsymbol{U}^n \in \mathbb{V}^n \qquad \text{and} \qquad P^n \in \mathbb{Q}_0^n.$$

Finally, from (4.25), we can extract a further subsequence (not relabelled) such that

$$C^{n,m} \to C^n \qquad \text{a.e. in } \Omega. \tag{4.26}$$

Note that since $\boldsymbol{S}$ is continuous, by (4.26) and (4.23), we have

$$\boldsymbol{S}(C^{n,m}, \boldsymbol{D}\boldsymbol{U}^{n,m}) \to \boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) \qquad \text{a.e. in } \Omega.$$

Now, by (4.23), we have that, for sufficiently large $m \in \mathbb{N}$,

$$|\boldsymbol{D}\boldsymbol{U}^{n,m}| < 1 + |\boldsymbol{D}\boldsymbol{U}^n| \qquad \text{for a.e. } x \in \Omega.$$

Thus, by (3.4), we have, for sufficiently large $m \in \mathbb{N}$,

$$\begin{aligned} |\boldsymbol{S}(C^{n,m}, \boldsymbol{D}\boldsymbol{U}^{n,m})| &\le C|\boldsymbol{D}\boldsymbol{U}^{n,m}|^{p(C^{n,m})-1} + C \\ &\le C(1 + |\boldsymbol{D}\boldsymbol{U}^n|)^{p(C^{n,m})-1} + C \\ &\le C(1 + |\boldsymbol{D}\boldsymbol{U}^n|)^{p^+-1} + C, \end{aligned}$$

and $C(1 + |\boldsymbol{D}\boldsymbol{U}^n|)^{p^+-1} + C \in L^{(p^+)'}(\Omega)$, where C is independent of m. Therefore, by the Dominated Convergence Theorem, we have

$$\boldsymbol{S}^{n,m} \to \boldsymbol{S}^n := \boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) \qquad \text{strongly in } L^{(p^+)'}(\Omega)^{3\times 3}. \tag{4.27}$$

Furthermore, by (4.26) and (4.23), together with the Dominated Convergence Theorem,

$$\boldsymbol{K}(C^{n,m}, |\boldsymbol{D}\boldsymbol{U}^{n,m}|) \to \boldsymbol{K}(C^n, |\boldsymbol{D}\boldsymbol{U}^n|) \qquad \text{strongly in } L^q(\Omega) \qquad \forall\, q \in (1, \infty).$$

Therefore, together with (4.21), we have

$$\boldsymbol{q}_c^{n,m} \rightharpoonup \boldsymbol{q}_c^n := \boldsymbol{q}_c(C^n, \nabla C^n, \boldsymbol{D}\boldsymbol{U}^n) \qquad \text{weakly in } L^2(\Omega)^3. \tag{4.28}$$

Now we are ready to pass m to infinity in the Galerkin approximation (4.11)–(4.13). First, by (4.22) and (4.23),

$$\begin{aligned} B_u[\boldsymbol{U}^{n,m}, \boldsymbol{U}^{n,m}, \boldsymbol{V}] &\to B_u[\boldsymbol{U}^n, \boldsymbol{U}^n, \boldsymbol{V}] \qquad && \forall\, \boldsymbol{V} \in \mathbb{V}^n, \\ \frac{1}{k}|\boldsymbol{U}^{n,m}|^{t-2}\boldsymbol{U}^{n,m} \cdot \boldsymbol{V} &\to \frac{1}{k}|\boldsymbol{U}^n|^{t-2}\boldsymbol{U}^n \cdot \boldsymbol{V} \qquad && \forall\, \boldsymbol{V} \in \mathbb{V}^n. \end{aligned}$$

Furthermore, from (4.27) and (4.24),

$$\begin{aligned}\int_\Omega \boldsymbol{S}^{n,m} : \boldsymbol{D}\boldsymbol{V}\,\mathrm{d}x \to \int_\Omega \boldsymbol{S}^{n} : \boldsymbol{D}\boldsymbol{V}\,\mathrm{d}x \qquad &\forall\,\boldsymbol{V}\in\mathbb{V}^n,\\ \langle \operatorname{div}\boldsymbol{V}, P^{n,m}\rangle \to \langle \operatorname{div}\boldsymbol{V}, P^{n}\rangle \qquad &\forall\,\boldsymbol{V}\in\mathbb{V}^n.\end{aligned}$$

Therefore, we have for all $\boldsymbol{V}\in\mathbb{V}^n$,

$$\int_\Omega \left(\boldsymbol{S}^n : \boldsymbol{D}\boldsymbol{V} + \frac{1}{k}|\boldsymbol{U}^n|^{t-2}\boldsymbol{U}^n\cdot\boldsymbol{V}\right)\mathrm{d}x + B_u[\boldsymbol{U}^n,\boldsymbol{U}^n,\boldsymbol{V}] - \langle \operatorname{div}\boldsymbol{V}, P^n\rangle = \langle \boldsymbol{f},\boldsymbol{V}\rangle. \tag{4.29}$$

Moreover, from (4.12) and (4.23),

$$\int_\Omega Q\operatorname{div}\boldsymbol{U}^n\,\mathrm{d}x = 0 \qquad \forall\,Q\in\mathbb{Q}^n. \tag{4.30}$$

Next, let us investigate the limit of the concentration equation, (4.13). We fix an arbitrary $Z\in W_0^{1,2}(\Omega)$ and define $Z^m := \Pi_{\mathbb{Z}}^m Z\in\mathbb{Z}^m$. Thanks to (4.22) and (4.25),

$$\begin{aligned}\|C^{n,m}\boldsymbol{U}^{n,m} - C^n\boldsymbol{U}^n\|_2 &\le \|(\boldsymbol{U}^{n,m}-\boldsymbol{U}^n)C^{n,m}\|_2 + \|\boldsymbol{U}^n(C^{n,m}-C^n)\|_2\\ &\le \|\boldsymbol{U}^{n,m}-\boldsymbol{U}^n\|_\infty\|C^{n,m}\|_2 + \|\boldsymbol{U}^n\|_\infty\|C^{n,m}-C^n\|_2 \to 0.\end{aligned}$$

Also, thanks to (4.22) and (4.9),

$$\begin{aligned}\|Z^m\boldsymbol{U}^{n,m} - Z\boldsymbol{U}^n\|_2 &\le \|(\boldsymbol{U}^{n,m}-\boldsymbol{U}^n)Z^m\|_2 + \|\boldsymbol{U}^n(Z^m-Z)\|_2\\ &\le \|\boldsymbol{U}^{n,m}-\boldsymbol{U}^n\|_\infty\|Z^m\|_2 + \|\boldsymbol{U}^n\|_\infty\|Z^m-Z\|_2 \to 0.\end{aligned}$$

In other words, we have

$$C^{n,m}\boldsymbol{U}^{n,m} \to C^n\boldsymbol{U}^n \qquad \text{strongly in } L^2(\Omega)^3, \tag{4.31}$$

$$Z^m\boldsymbol{U}^{n,m} \to Z\boldsymbol{U}^n \qquad \text{strongly in } L^2(\Omega)^3. \tag{4.32}$$

By (4.32) and (4.21),

$$\begin{aligned}&\left|\int_\Omega Z^m\boldsymbol{U}^{n,m}\cdot\nabla C^{n,m}\,\mathrm{d}x - \int_\Omega Z\boldsymbol{U}^n\cdot\nabla C^n\,\mathrm{d}x\right|\\ &\quad\le \int_\Omega |Z^m\boldsymbol{U}^{n,m} - Z\boldsymbol{U}^n||\nabla C^{n,m}|\,\mathrm{d}x + \left|\int_\Omega Z\boldsymbol{U}^n(\nabla C^{n,m}-\nabla C^n)\,\mathrm{d}x\right| \to 0.\end{aligned}$$

Moreover, from (4.31) and (4.9),

$$\begin{aligned}&\left|\int_\Omega C^{n,m}\boldsymbol{U}^{n,m}\cdot\nabla Z^m\,\mathrm{d}x - \int_\Omega C^n\boldsymbol{U}^n\cdot\nabla Z\,\mathrm{d}x\right|\\ &\quad\le \|C^{n,m}\boldsymbol{U}^{n,m}\|_2\|Z^m-Z\|_{1,2} + \|Z\|_{1,2}\|C^{n,m}\boldsymbol{U}^{n,m} - C^n\boldsymbol{U}^n\|_2 \to 0.\end{aligned}$$

Therefore, we have

$$\lim_{m\to\infty} B_c[C^{n,m}, \boldsymbol{U}^{n,m}, Z^m] = B_c[C^n, \boldsymbol{U}^n, Z].$$

Finally, from (4.28),

$$\int_\Omega \boldsymbol{q}_c^{n,m} \cdot \nabla Z^m \,\mathrm{d}x \to \int_\Omega \boldsymbol{q}_c^n \cdot \nabla Z \,\mathrm{d}x \qquad \text{as } m \to \infty.$$

Altogether, we have

$$\int_\Omega \boldsymbol{q}_c^n \cdot \nabla Z \,\mathrm{d}x + B_c[C^n, \boldsymbol{U}^n, Z] = 0 \qquad \forall\, Z \in W_0^{1,2}(\Omega). \tag{4.33}$$

4.4.2 The limit $n \to \infty$

Now we shall derive further uniform estimates, independent of $n \in \mathbb{N}$, and let n pass to infinity. First, we test with $\boldsymbol{U}^n$ in (4.29). Then, by (4.30), we have

$$\int_\Omega \left(\boldsymbol{S}^n : \boldsymbol{D}\boldsymbol{U}^n + \frac{1}{k}|\boldsymbol{U}^n|^t \right) \mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{U}^n \rangle.$$

By using (3.6) and Young's inequality, we have

$$\int_\Omega \left(|\boldsymbol{D}\boldsymbol{U}^n|^{p(C^n)} + |\boldsymbol{S}^n|^{p'(C^n)} + \frac{1}{k}|\boldsymbol{U}^n|^t \right) \mathrm{d}x \le C_1, \tag{4.34}$$

where C_1 is independent of n and k, which leads us to

$$\|\boldsymbol{U}^n\|_{1,p^-}^{p^-} + \|\boldsymbol{S}^n\|_{(p^+)'}^{(p^+)'} + \frac{1}{k}\|\boldsymbol{U}^n\|_t^t \le C_1, \tag{4.35}$$

where C_1 is independent of n and k.

Next, we test with $C^n - c_d$ in (4.33), and we obtain

$$\int_\Omega \boldsymbol{q}_c^n \cdot \nabla C^n \,\mathrm{d}x = \int_\Omega \boldsymbol{q}_c^n \cdot \nabla c_d \,\mathrm{d}x + B_c[C^n, \boldsymbol{U}^n, c_d].$$

From (3.7), (3.8), Hölder's inequality and Young's inequality we have

$$\begin{aligned} \|\nabla C^n\|_2^2 &\le C \int_\Omega |\nabla C^n||\nabla c_d| \,\mathrm{d}x + B_c[C^n, \boldsymbol{U}^n, c_d] \\ &\le \varepsilon \|\nabla C^n\|_2^2 + C(\varepsilon)\|\nabla c_d\|_2^2 + B_c[C^n, \boldsymbol{U}^n, c_d]. \end{aligned}$$

Furthermore, by Sobolev embedding,

$$\begin{aligned} B_c[C^n, \boldsymbol{U}^n, c_d] &= \frac{1}{2}\int_\Omega c_d \boldsymbol{U}^n \cdot \nabla C^n \,\mathrm{d}x - \frac{1}{2}\int_\Omega C^n \boldsymbol{U}^n \cdot \nabla c_d \,\mathrm{d}x \\ &= \int_\Omega c_d \boldsymbol{U}^n \cdot \nabla C^n \,\mathrm{d}x + \frac{1}{2}\int_\Omega C^n (\operatorname{div} \boldsymbol{U}^n) c_d \,\mathrm{d}x \\ &\le \|c_d\|_\infty \|\boldsymbol{U}^n\|_2 \|\nabla C^n\|_2 + \frac{\|c_d\|_\infty}{2}\|C^n\|_{(p^-)'}\|\operatorname{div} \boldsymbol{U}^n\|_{p^-} \\ &\le C\|\boldsymbol{U}^n\|_{1,p^-}\|\nabla C^n\|_2 + C\|\boldsymbol{U}^n\|_{1,p^-}\|\nabla C^n\|_{\frac{3p^-}{4p^- - 3}} \\ &\le C(\varepsilon)\|\boldsymbol{U}^n\|_{1,p^-}^2 + \varepsilon\|\nabla C^n\|_2^2. \end{aligned}$$

Hence, from (3.7) and (4.34),

$$\int_\Omega \left(|\nabla C^n|^2 + |\boldsymbol{q}_c^n|^2\right) \,\mathrm{d}x \le C_2, \tag{4.36}$$

where C_2 is independent of n and k. Thus we have

$$\|C^n\|_{1,2}^2 + \|\boldsymbol{q}_c^n\|_2^2 \le C_2, \tag{4.37}$$

where C_2 is independent of n and k.

Now, we need to derive a uniform Hölder estimate for the approximate concentrations. As we can see from (4.33), the concentration equation is at the 'continuous' level (rather than at the 'discrete' level), so we can apply the following well-known result due to De Giorgi and Nash [31, 70]; see also [11] for its application to the system of partial differential equations considered here.

Theorem 4.4.1. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and let $s > d$ be fixed. Suppose that $\boldsymbol{K} \in L^\infty(\Omega)^{d\times d}$ is uniformly elliptic with ellipticity constant $\lambda > 0$. Then, there exists an $\alpha \in (0,1)$ such that, for any $\boldsymbol{F} \in L^s(\Omega)^d$, $g \in L^{\frac{ds}{d+s}}(\Omega)$ and any $c_d \in W^{1,s}(\Omega)$, there exists a unique $c \in W^{1,2}(\Omega)$ such that $c - c_d \in W_0^{1,2}(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$ and*

$$\int_\Omega \boldsymbol{K}\nabla c \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega \boldsymbol{F}\cdot\nabla\varphi\,\mathrm{d}x + \int_\Omega g\varphi\,\mathrm{d}x \qquad \forall\,\varphi \in W_0^{1,2}(\Omega);$$

furthermore, the following uniform bound holds:

$$\|c\|_{W^{1,2}\cap C^{0,\alpha}} \le C\left(\Omega, \lambda, s, \|\boldsymbol{K}\|_\infty, \|\boldsymbol{F}\|_s, \|g\|_{\frac{ds}{d+s}}, \|c_d\|_{1,s}\right).$$

To apply the above theorem, we need uniform estimates for $C^n\boldsymbol{U}^n$ and $\nabla C^n\cdot\boldsymbol{U}^n$. Since $\frac{3}{2} > \frac{t}{t-2}$ for $t > 2$, by Sobolev embedding and the uniform estimates (4.34) and (4.36), for $s > 3$ sufficiently close to 3,

$$\|C^n\boldsymbol{U}^n\|_s \le \|C^n\|_6\|\boldsymbol{U}^n\|_{\frac{6s}{6-s}} \le C\|C^n\|_{1,2}\|\boldsymbol{U}^n\|_t \le C,$$

where C is independent of n. Also, for $s > 3$ sufficiently close to 3, we have

$$\|\nabla C^n\cdot\boldsymbol{U}^n\|_{\frac{3s}{s+3}} \le \|\nabla C^n\|_2\|\boldsymbol{U}^n\|_{\frac{6s}{6-s}} \le C\|C^n\|_{1,2}\|\boldsymbol{U}^n\|_t \le C,$$

where C is independent of n. Note that if we considered the original equation (3.2) instead of the regularized problem (4.2), we would only be able to deduce that

$\|\boldsymbol{U}^n\|_{3p^-/(3-p^-)} \leq C$, which would force us to assume the strong condition $p^- > 2$ so as to be able to prove the above estimates.

Therefore, we can apply Theorem 4.4.1 with $\boldsymbol{F} = C^n\boldsymbol{U}^n$ and $g = \nabla C^n \cdot \boldsymbol{U}^n$. Hence, there exists an $\alpha_1 \in (0,1)$ such that

$$\|C^n\|_{C^{0,\alpha_1}(\overline{\Omega})} \leq C_3, \tag{4.38}$$

where C_3 is independent of n. Since $C^{0,\alpha_1}(\overline{\Omega}) \hookrightarrow\!\!\hookrightarrow C^{0,\tilde{\alpha}_1}(\overline{\Omega})$ for all $\tilde{\alpha}_1 \in (0,\alpha_1)$, we have by (4.38) that

$$C^n \to c \qquad \text{strongly in } C^{0,\tilde{\alpha}_1}(\overline{\Omega}),$$

which implies that

$$p \circ C^n \to p \circ c \qquad \text{strongly in } C^{0,\beta_1}(\overline{\Omega}),$$

for some $\beta_1 \in (0,1)$.

We now apply Proposition 4.3.1. For a given $p^+ > 0$, choose $j > \max\{p^+, 2\}$. Then, since $p^- > \frac{3}{2}$, we have that $W_0^{1,j}(\Omega)^3 \hookrightarrow L^{2(p^-)'}(\Omega)^3$ by Sobolev embedding. Furthermore, since $\frac{t}{t-2} < p^-$, we have that $2(p^-)' < t$. Now, from (4.11) and (4.10), we have

$$\begin{aligned}
\beta\|P^n\|_{j'} &\leq \sup_{0\neq\boldsymbol{V}\in\mathbb{V}^n} \frac{\langle \operatorname{div}\boldsymbol{V}, P^n\rangle}{\|\boldsymbol{V}\|_{1,j}} \\
&\leq \sup_{0\neq\boldsymbol{V}\in\mathbb{V}^n} \frac{\left|\int_\Omega \boldsymbol{S}^n : \boldsymbol{D}\boldsymbol{V}\,\mathrm{d}x + B_u[\boldsymbol{U}^n,\boldsymbol{U}^n,\boldsymbol{V}] - \langle \boldsymbol{f},\boldsymbol{V}\rangle\right|}{\|\boldsymbol{V}\|_{1,j}} \\
&\leq C \sup_{0\neq\boldsymbol{V}\in\mathbb{V}^n} \frac{\|\boldsymbol{S}^n\|_{(p^+)'}\|\boldsymbol{V}\|_{1,p^+}}{\|\boldsymbol{V}\|_{1,p^+}} \\
&\quad + C \sup_{0\neq\boldsymbol{V}\in\mathbb{V}^n} \frac{\|\boldsymbol{U}^n\|_t^2\|\boldsymbol{V}\|_{1,p^-} + \|\boldsymbol{f}\|_{(W_0^{1,p^-}(\Omega)^3)^*}\|\boldsymbol{V}\|_{1,p^-}}{\|\boldsymbol{V}\|_{1,p^-}} \\
&\quad + C \sup_{0\neq\boldsymbol{V}\in\mathbb{V}^n} \frac{\|\boldsymbol{U}^n\|_{2(p^-)'}\|\boldsymbol{V}\|_{2(p^-)'}\|\boldsymbol{U}^n\|_{1,p^-}}{\|\boldsymbol{V}\|_{2(p^-)'}},
\end{aligned}$$

where C is independent of n and k. Hence, by noting (4.34),

$$\|P^n\|_{j'} \leq C_4, \tag{4.39}$$

where C_4 is independent of n.

Now, by (4.34)–(4.39), thanks to the reflexivity of the relevant spaces and by compact embedding, we can extract (not relabelled) subsequences such that

$$\boldsymbol{U}^n \rightharpoonup \boldsymbol{u} \qquad \text{weakly in } W_0^{1,p^-}(\Omega)^3 \cap L^t(\Omega)^3, \tag{4.40}$$

$$\boldsymbol{U}^n \to \boldsymbol{u} \qquad \text{strongly in } L^\sigma(\Omega)^3 \qquad \forall\, \sigma \in [1,t), \tag{4.41}$$

$$|\boldsymbol{U}^n|^{t-2}\boldsymbol{U}^n \rightharpoonup |\boldsymbol{u}|^{t-2}\boldsymbol{u} \qquad \text{weakly in } L^{\frac{t}{t-1}}(\Omega)^3, \tag{4.42}$$

$$C^n \rightharpoonup c \qquad \text{weakly in } W^{1,2}(\Omega), \tag{4.43}$$

$$C^n \to c \qquad \text{strongly in } C^{0,\tilde{\alpha}_1}(\overline{\Omega}), \tag{4.44}$$

$$P^n \rightharpoonup \pi \qquad \text{weakly in } L^{j'}(\Omega) \qquad \forall\, j > \max\{p^+, 2\}, \tag{4.45}$$

$$\boldsymbol{S}^n \rightharpoonup \bar{\boldsymbol{S}} \qquad \text{weakly in } L^{(p^+)'}(\Omega)^{3\times 3}, \tag{4.46}$$

$$\boldsymbol{q}_c^n \rightharpoonup \bar{\boldsymbol{q}}_c \qquad \text{weakly in } L^2(\Omega)^3. \tag{4.47}$$

Before proceeding further, we note that these limits, together with weak lower semicontinuity and (4.34), in conjunction with Korn's inequality, imply that

$$\int_\Omega |\nabla \boldsymbol{u}|^{p(c)} + |\bar{\boldsymbol{S}}|^{p'(c)} \,\mathrm{d}x \leq C, \tag{4.48}$$

where C is independent of k; hence the limit function $\boldsymbol{u}$ is, in fact, contained in the space $W_0^{1,p(c)}(\Omega)^3$; see Chapter 3 for the details of the proof of this.

Next, we shall prove that the limit function $\boldsymbol{u}$ is pointwise divergence-free. For an arbitrary $q \in C_0^\infty(\Omega)$, by (4.30),

$$\begin{aligned} 0 &= \int_\Omega (\Pi_{\mathbb{Q}}^n q) \operatorname{div} \boldsymbol{U}^n \,\mathrm{d}x \\ &= \int_\Omega (\Pi_{\mathbb{Q}}^n q - q) \operatorname{div} \boldsymbol{U}^n \,\mathrm{d}x + \int_\Omega q \,(\operatorname{div} \boldsymbol{U}^n - \operatorname{div} \boldsymbol{u}) \,\mathrm{d}x + \int_\Omega q \operatorname{div} \boldsymbol{u} \,\mathrm{d}x. \end{aligned}$$

The first term tends to zero by (4.34), (3.18) and the second term converges to zero by (4.40). Therefore,

$$\int_\Omega q \operatorname{div} \boldsymbol{u} \,\mathrm{d}x = 0 \qquad \text{for any } q \in C_0^\infty(\Omega),$$

which implies that $\operatorname{div} \boldsymbol{u} = 0$ a.e. on Ω.

Now, we shall identify the limit of the convective term $B_u[\cdot,\cdot,\cdot]$ as follows. For an arbitrary $\boldsymbol{v} \in W_0^{1,\infty}(\Omega)^3$, we define $\boldsymbol{V}^n := \Pi_{\mathrm{div}}^n \boldsymbol{v} \in \mathbb{V}^n$. Then, by (3.14), we have

$$\boldsymbol{V}^n \to \boldsymbol{v} \qquad \text{strongly in } W_0^{1,\sigma}(\Omega)^2 \text{ for } \sigma \in (1,\infty). \tag{4.49}$$

By (4.41),

$$\boldsymbol{U}^n \otimes \boldsymbol{U}^n \to \boldsymbol{u} \otimes \boldsymbol{u} \qquad \text{strongly in } L^{1+\varepsilon}(\Omega)^{3\times 3}.$$

Hence, we can identify the second part of the convective term

$$-\int_\Omega (\boldsymbol{U}^n \otimes \boldsymbol{U}^n) : \nabla \boldsymbol{V}^n \,\mathrm{d}x \to -\int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{v} \,\mathrm{d}x \qquad \text{as } n \to \infty.$$

Also, we assert that $\boldsymbol{U}^n \cdot \boldsymbol{V}^n \to \boldsymbol{u} \cdot \boldsymbol{v}$ strongly in $L^{(p^-)'}(\Omega)$. Indeed,

$$\begin{aligned}\|\boldsymbol{U}^n \cdot \boldsymbol{V}^n - \boldsymbol{u} \cdot \boldsymbol{v}\|_{(p^-)'} &\leq \|(\boldsymbol{V}^n - \boldsymbol{v})\boldsymbol{U}^n + (\boldsymbol{U}^n - \boldsymbol{u})\boldsymbol{v}\|_{(p^-)'} \\ &\leq \|\boldsymbol{V}^n - \boldsymbol{v}\|_\sigma \|\boldsymbol{U}^n\|_{t-\varepsilon} + \|\boldsymbol{U}^n - \boldsymbol{u}\|_{t-\varepsilon} \|\boldsymbol{v}\|_\sigma\end{aligned}$$

for some $\sigma \in (1, \infty)$. The first term tends to zero thanks to (4.49), (4.41) and the second term tends to zero by (4.41). Therefore, since $\operatorname{div} \boldsymbol{u} = 0$, we have

$$\begin{aligned}\int_\Omega (\boldsymbol{U}^n \otimes \boldsymbol{V}^n) : \nabla \boldsymbol{U}^n \,\mathrm{d}x &= -\int_\Omega (\boldsymbol{U}^n \otimes \boldsymbol{U}^n) : \nabla \boldsymbol{V}^n \,\mathrm{d}x + \int_\Omega (\operatorname{div} \boldsymbol{U}^n)\, \boldsymbol{U}^n \cdot \boldsymbol{V}^n \,\mathrm{d}x \\ &\to -\int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{v} \,\mathrm{d}x \qquad \text{as } n \to \infty.\end{aligned}$$

Altogether, we then deduce that

$$\lim_{n\to\infty} B_u[\boldsymbol{U}^n, \boldsymbol{U}^n, \boldsymbol{V}^n] = -\int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{v} \,\mathrm{d}x. \tag{4.50}$$

Now, we are ready to pass n to infinity in the momentum equation. Since Π^n_{div} is linear, by noting (4.29), we have

$$\begin{aligned}\langle \operatorname{div} \boldsymbol{v}, P^n \rangle &= \langle \operatorname{div} \boldsymbol{V}^n, P^n \rangle + \langle \operatorname{div} (\boldsymbol{v} - \boldsymbol{V}^n), P^n \rangle \\ &= \int_\Omega \left(\boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) : \boldsymbol{D}\boldsymbol{V}^n + \frac{1}{k} |\boldsymbol{U}^n|^{t-2}\, \boldsymbol{U}^n \cdot \boldsymbol{V}^n \right) \mathrm{d}x - \langle \boldsymbol{f}, \boldsymbol{V}^n \rangle \\ &\quad + B_u[\boldsymbol{U}^n, \boldsymbol{U}^n, \boldsymbol{V}^n] + \langle \operatorname{div} (\boldsymbol{v} - \boldsymbol{V}^n), P^n \rangle \\ &\to \int_\Omega \left(\bar{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{v} + \frac{1}{k} |\boldsymbol{u}|^{t-2} \boldsymbol{u} \cdot \boldsymbol{v} - (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{v} \right) \mathrm{d}x - \langle \boldsymbol{f}, \boldsymbol{v} \rangle,\end{aligned}$$

where we have used (4.45), (4.46), (4.42), (4.49) and (4.50). Also, by (4.45) again,

$$\langle \operatorname{div} \boldsymbol{v}, P^n \rangle \to \langle \operatorname{div} \boldsymbol{v}, \pi \rangle.$$

Collecting all the limits gives us that, for all $\boldsymbol{v} \in W^{1,\infty}_0(\Omega)^3$,

$$\int_\Omega \left(\bar{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{v} + \frac{1}{k} |\boldsymbol{u}|^{t-2} \boldsymbol{u} \cdot \boldsymbol{v} - (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{v} \right) \mathrm{d}x - \langle \operatorname{div} \boldsymbol{v}, \pi \rangle = \langle \boldsymbol{f}, \boldsymbol{v} \rangle. \tag{4.51}$$

With the same argument as above, we also have that, for all $\boldsymbol{v} \in W^{1,\infty}_{0,\mathrm{div}}(\Omega)^3$,

$$\int_\Omega \left(\bar{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{v} + \frac{1}{k} |\boldsymbol{u}|^{t-2} \boldsymbol{u} \cdot \boldsymbol{v} - (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{v} \right) \mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{v} \rangle. \tag{4.52}$$

Note that by Proposition 2.5.3 and (4.48) we have

$$\pi \in L_0^{p'(c)}(\Omega).$$

Now, let us investigate the limit of the convection-diffusion equation, (4.33). For an arbitrary but fixed $z \in W_0^{1,2}(\Omega)$, we define $Z^n := \Pi^n_{\mathbb{Z}} z \in \mathbb{Z}^n$. Thanks to (4.41) and (4.44),

$$\begin{aligned} \|C^n \boldsymbol{U}^n - c\boldsymbol{u}\|_2 &\leq \|(C^n - c)\boldsymbol{U}^n\|_2 + \|c(\boldsymbol{U}^n - \boldsymbol{u})\|_2 \\ &\leq \|C^n - c\|_\infty \|\boldsymbol{U}^n\|_2 + \|c\|_\infty \|\boldsymbol{U}^n - \boldsymbol{u}\|_2 \to 0. \end{aligned}$$

Moreover, by (4.41), (3.19) and Sobolev embedding,

$$\begin{aligned} \|Z^n \boldsymbol{U}^n - z\boldsymbol{u}\|_2 &\leq \|(Z^n - z)\boldsymbol{U}^n\|_2 + \|z(\boldsymbol{U}^n - \boldsymbol{u})\|_2 \\ &\leq \|Z^n - z\|_6 \|\boldsymbol{U}^n\|_3 + \|z\|_6 \|\boldsymbol{U}^n - \boldsymbol{u}\|_3 \\ &\leq C\|Z^n - z\|_{1,2} \|\boldsymbol{U}^n\|_3 + C\|z\|_{1,2} \|\boldsymbol{U}^n - \boldsymbol{u}\|_3 \to 0. \end{aligned}$$

In other words,

$$C^n \boldsymbol{U}^n \to c\boldsymbol{u} \qquad \text{strongly in } L^2(\Omega)^3, \tag{4.53}$$

$$Z^n \boldsymbol{U}^n \to z\boldsymbol{u} \qquad \text{strongly in } L^2(\Omega)^3. \tag{4.54}$$

From (4.43) and (4.54),

$$\begin{aligned} &\left| \int_\Omega Z^n \boldsymbol{U}^n \cdot \nabla C^n \,\mathrm{d}x - \int_\Omega z\boldsymbol{u} \cdot \nabla c \,\mathrm{d}x \right| \\ &\qquad \leq \int_\Omega |Z^n \boldsymbol{U}^n - z\boldsymbol{u}| |\nabla C^n| \,\mathrm{d}x + \left| \int_\Omega z\boldsymbol{u} \cdot (\nabla C^n - \nabla c) \,\mathrm{d}x \right| \to 0. \end{aligned}$$

Therefore, as $\operatorname{div} \boldsymbol{u} = 0$ a.e. on Ω, we obtain

$$\int_\Omega Z^n \boldsymbol{U}^n \cdot \nabla C^n \,\mathrm{d}x \to \int_\Omega z\boldsymbol{u} \cdot \nabla c \,\mathrm{d}x = -\int_\Omega c\boldsymbol{u} \cdot \nabla z \,\mathrm{d}x \qquad \text{as } n \to \infty.$$

Moreover, by (4.53) and (3.19),

$$\begin{aligned} &\left| \int_\Omega C^n \boldsymbol{U}^n \cdot \nabla Z^n \,\mathrm{d}x - \int_\Omega c\boldsymbol{u} \cdot \nabla z \,\mathrm{d}x \right| \\ &\qquad \leq \|C^n \boldsymbol{U}^n\|_2 \|Z^n - z\|_{1,2} + \|C^n \boldsymbol{U}^n - c\boldsymbol{u}\|_2 \|z\|_{1,2} \to 0. \end{aligned}$$

Altogether, we have

$$\lim_{n\to\infty} B_c[C^n, \boldsymbol{U}^n, Z^n] = -\int_\Omega c\boldsymbol{u}\cdot\nabla z\,\mathrm{d}x.$$

Finally, by (4.47) and (3.19), we have

$$\int_\Omega \boldsymbol{q}_c(C^n, \nabla C^n, \boldsymbol{D}\boldsymbol{U}^n)\cdot\nabla Z^n\,\mathrm{d}x \to \int_\Omega \bar{\boldsymbol{q}}_c\cdot\nabla z\,\mathrm{d}x \qquad \text{as } n\to\infty.$$

By collecting all the limits, we obtain that

$$\int_\Omega (\bar{\boldsymbol{q}}_c\cdot\nabla z - c\boldsymbol{u}\cdot\nabla z)\,\mathrm{d}x = 0 \qquad \forall\, z\in W_0^{1,2}(\Omega). \tag{4.55}$$

As we can see from (4.51) and (4.55), what we now need to prove is the identification of the limits:

$$\bar{\boldsymbol{S}} = \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u}) \quad\text{and}\quad \bar{\boldsymbol{q}}_c = \boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u}).$$

To this end, we require the following lemma which is similar to the compactness result presented in Section 3.4.

Lemma 4.4.2. *The sequences* $\{\boldsymbol{D}\boldsymbol{U}^n\}_{n=1}^\infty$ *and* $\{C^n\}_{n=1}^\infty$ *satisfy the following equality:*

$$\lim_{n\to\infty}\int_\Omega ((\boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) - \boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{u})) : (\boldsymbol{D}\boldsymbol{U}^n - \boldsymbol{D}\boldsymbol{u}))^{\frac{1}{4}}\,\mathrm{d}x = 0. \tag{4.56}$$

The proof of the lemma is similar to that in Chapter 3. We shall briefly summarize the key steps of the proof as we have an additional regularizing term here and we shall require a similar, but more involved, argument in the next section. As in the previous chapter, we introduce the matrix-truncation function $T_\chi : \mathbb{R}^{3\times 3}\to\mathbb{R}^{3\times 3}$ by

$$T_\chi(\boldsymbol{M}) = \begin{cases} \boldsymbol{M} & \text{for } |\boldsymbol{M}|\le\chi,\\ \chi\frac{\boldsymbol{M}}{|\boldsymbol{M}|} & \text{for } |\boldsymbol{M}|>\chi. \end{cases}$$

Then, the most important and most difficult part of the proof of (4.56) is to show that

$$\lim_{\chi\to\infty}\lim_{j\to\infty}\lim_{n\to\infty}\int_\Omega (\boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) - \boldsymbol{S}(C^n, T_\chi(\boldsymbol{D}\boldsymbol{u}))) : (\boldsymbol{D}\boldsymbol{U}^n_j - T_\chi(\boldsymbol{D}\boldsymbol{u}))\,\mathrm{d}x \le 0, \tag{4.57}$$

where $\boldsymbol{U}^n_j$ is the discrete Lipschitz truncation of the function of $\boldsymbol{U}^n$ introduced in Theorem 3.3.13 . To show (4.57), we define the following discretely divergence-free approximations with zero trace on $\partial\Omega$:

$$\boldsymbol{\Psi}^n_j := \mathcal{B}^n(\operatorname{div}\boldsymbol{U}^n_j),$$

$$\boldsymbol{\Phi}^n_j := \boldsymbol{U}^n_j - \boldsymbol{\Psi}^n_j.$$

Here $\mathcal{B}^n$ is the discrete Bogovskiĭ operator defined in the previous chapter. It is then clear that $\boldsymbol{\Phi}_j^n$ has zero trace on $\partial\Omega$ and, by construction, $\boldsymbol{\Phi}_j^n \in \mathbb{V}_{\mathrm{div}}^n$. Moreover, it can be easily verified, by using basic properties of the discrete Lipschitz truncation and the discrete Bogovskiĭ operator, that

$$\boldsymbol{\Phi}_j^n \rightharpoonup \boldsymbol{U}_j - \mathcal{B}(\operatorname{div} \boldsymbol{U}_j) =: \boldsymbol{\Phi}_j \qquad \text{weakly in } W_0^{1,\sigma}(\Omega)^3, \tag{4.58}$$

$$\boldsymbol{\Phi}_j^n \to \boldsymbol{\Phi}_j \qquad \text{strongly in } L^{\sigma}(\Omega)^3, \tag{4.59}$$

as $n \to \infty$, where $\sigma \in (1, \infty)$ is arbitrary. We can then rewrite (4.57) above in terms of this approximation to obtain

$$\begin{aligned}
&\int_\Omega (\boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) - \boldsymbol{S}(C^n, T_\chi(\boldsymbol{D}\boldsymbol{u}))) : (\boldsymbol{D}\boldsymbol{U}_j^n - T_\chi(\boldsymbol{D}\boldsymbol{u}))\,\mathrm{d}x \\
&= \int_\Omega \boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) : (\boldsymbol{D}\boldsymbol{\Phi}_j^n + \boldsymbol{D}\boldsymbol{\Psi}_j^n)\,\mathrm{d}x \\
&\quad - \int_\Omega \boldsymbol{S}(C^n, \boldsymbol{D}\boldsymbol{U}^n) : T_\chi(\boldsymbol{D}\boldsymbol{u})\,\mathrm{d}x - \int_\Omega \boldsymbol{S}(C^n, T_\chi(\boldsymbol{D}\boldsymbol{u})) : (\boldsymbol{D}\boldsymbol{U}_j^n - T_\chi(\boldsymbol{D}\boldsymbol{u}))\,\mathrm{d}x \\
&=: B_{\chi,j}^{n,1} - B_{\chi,j}^{n,2} - B_{\chi,j}^{n,3}.
\end{aligned}$$

Now we use (4.29) with $\boldsymbol{V} = \boldsymbol{\Phi}_j^n \in \mathbb{V}_{\mathrm{div}}^n$ and pass to the limit; thus we have, by (4.52), that as n goes to ∞,

$$\int_\Omega \boldsymbol{S}^n : \boldsymbol{D}\boldsymbol{\Phi}_j^n\,\mathrm{d}x = -B_u[\boldsymbol{U}^n, \boldsymbol{U}^n, \boldsymbol{\Phi}_j^n] + \int_\Omega \frac{1}{k}|\boldsymbol{U}^n|^{t-2}\boldsymbol{U}^n \cdot \boldsymbol{\Phi}_j^n\,\mathrm{d}x - \langle \boldsymbol{f}, \boldsymbol{\Phi}_j^n \rangle \tag{4.60}$$

$$\to \int_\Omega \left((\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{\Phi_j} - \frac{1}{k}|\boldsymbol{u}|^{t-2}\boldsymbol{u} \cdot \boldsymbol{\Phi_j} \right) \mathrm{d}x + \langle \boldsymbol{f}, \boldsymbol{\Phi_j} \rangle \tag{4.61}$$

$$= \int_\Omega \bar{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{\Phi_j}\,\mathrm{d}x. \tag{4.62}$$

Furthermore, with the help of Lipschitz truncation, we can show that

$$\lim_{n\to\infty} \int_\Omega \boldsymbol{S}^n : \boldsymbol{D}\boldsymbol{\Psi}_j^n\,\mathrm{d}x \le \left(\frac{C}{2^{j/p^+}} \right)^{\gamma(p^-,p^+)}, \tag{4.63}$$

$$\int_\Omega \bar{\boldsymbol{S}} : \boldsymbol{D}\mathcal{B}(\operatorname{div} \boldsymbol{U}_j)\,\mathrm{d}x \le \left(\frac{C}{2^{j/p^+}} \right)^{\gamma(p^-,p^+)}. \tag{4.64}$$

For the proofs of the above inequalities, see the arguments leading (3.78), (3.79) and (3.80). Altogether, we have

$$\begin{aligned}
&\lim_{\chi\to\infty} \lim_{j\to\infty} \lim_{n\to\infty} \left(B_{\chi,j}^{n,1} - B_{\chi,j}^{n,2} - B_{\chi,j}^{n,3} \right) \\
&\qquad \le \lim_{\chi\to\infty} \int_\Omega (\bar{\boldsymbol{S}} - \boldsymbol{S}(c, T_\chi(\boldsymbol{D}\boldsymbol{u}))) : (\boldsymbol{D}\boldsymbol{U}_j - T_\chi(\boldsymbol{D}\boldsymbol{u}))\,\mathrm{d}x.
\end{aligned}$$

The last limit is equal to zero by using the Dominated Convergence Theorem. That completes the proof of (4.57), and thereby also of the most technical step in the proof of the lemma.

Having shown (4.56), now we proceed with exactly same argument as in Chapter 3 to identify the limits;

$$\bar{\boldsymbol{S}} = \boldsymbol{S}(c, \boldsymbol{Du}) \qquad \text{and} \qquad \bar{\boldsymbol{q}}_c = \boldsymbol{q}(c, \nabla c, \boldsymbol{Du}).$$

Indeed, what we have shown here is that

$$\bar{\boldsymbol{S}} = \bar{\boldsymbol{S}}^k = \boldsymbol{S}(c^k, \boldsymbol{Du}^k) \qquad \text{and} \qquad \bar{\boldsymbol{q}}_c = \bar{\boldsymbol{q}}_c^k = \boldsymbol{q}(c^k, \nabla c^k, \boldsymbol{Du}^k)$$

for an arbitrary, but fixed $k \in \mathbb{N}$. From now on, we reintroduce the regularizing parameter $k \in \mathbb{N}$, and we will pass k to ∞ in the next section.

Remark 4.4.3. Indeed, in the proof of Theorem 4.3.2, the passage to the limit $n \to \infty$ can be also done by the use of monotone operator theory without using Lipschitz truncation. This is due to the stabilization we used here. More precisely, if $t > 0$ is sufficiently large, we have a higher integrability of $\boldsymbol{U}^n$ and $\boldsymbol{u}$, and hence $\boldsymbol{u}$ is an admissible test function in (4.52). Therefore, we can apply monotone operator theory and the use of Lipschitz truncation can be avoided. However, as mentioned in Section 3.1, here we have used the Lipschitz truncation method as we wish to consider more general case which can not be covered by monotone operator theory.

4.5 Proof of Theorem 4.2.1

4.5.1 Minimum and maximum principles

Having passed to the limits $m, n \to \infty$, we shall reinstate the index k for the regularization parameter in our notation, in preparation for passage to the limit $k \to \infty$.

Before we proceed, let us prove minimum and maximum principles for the concentration. Let $\varphi_1^k = (c^k - \min_{x\in\partial\Omega} c_d)_-$ and $\varphi_2^k = (c^k - \max_{x\in\partial\Omega} c_d)_+$. Since $c^k = c_d$ on $\partial\Omega$, it is clear that $\varphi_1^k, \varphi_2^k \in W_0^{1,2}(\Omega)$, so we can test with φ_1^k and φ_2^k in (4.55). Therefore, we have

$$-\int_\Omega \boldsymbol{u}^k c^k \cdot \nabla \varphi_1^k \,\mathrm{d}x + \int_\Omega \bar{\boldsymbol{q}}_c \cdot \nabla \varphi_1^k \,\mathrm{d}x = 0, \tag{4.65}$$

$$-\int_{\Omega} \boldsymbol{u}^k c^k \cdot \nabla \varphi_2^k \,\mathrm{d}x + \int_{\Omega} \bar{\boldsymbol{q}}_c \cdot \nabla \varphi_2^k \,\mathrm{d}x = 0. \tag{4.66}$$

We first consider (4.65). From (3.8) with integration by parts we obtain

$$\int_{\Omega^-} \boldsymbol{u}^k \cdot \nabla c^k \varphi_1^k \,\mathrm{d}x + \int_{\Omega^-} C|\nabla c^k|^2 \,\mathrm{d}x \leq 0,$$

where $\Omega^- = \{x \in \Omega : \varphi_1^k(x) < 0\}$, since $\operatorname{div} \boldsymbol{u}^k = 0$ and $\boldsymbol{u}^k = 0$ on $\partial\Omega$. By using the fact that $\nabla c^k = \nabla \varphi_1^k$ on Ω^- and the extension of ∇c^k from Ω^- to the whole domain Ω by using the negative part, we have

$$\int_{\Omega} \boldsymbol{u}^k \cdot \nabla \varphi_1^k \varphi_1^k \,\mathrm{d}x + \int_{\Omega} C|\nabla \varphi_1^k|^2 \,\mathrm{d}x \leq 0.$$

Note that

$$\int_{\Omega} \boldsymbol{u}^k \cdot \nabla \varphi_1^k \varphi_1^k \,\mathrm{d}x = \frac{1}{2}\int_{\Omega} \boldsymbol{u}^k \cdot \nabla |\varphi_1^k|^2 \,\mathrm{d}x = -\frac{1}{2}\int_{\Omega} (\operatorname{div} \boldsymbol{u}^k)|\varphi_1^k|^2 \,\mathrm{d}x = 0,$$

and thus,

$$\varphi_1^k = (c^k - \min_{x\in\partial\Omega} c_d)_- = \text{constant a.e. in } \Omega.$$

In the same way, we can also show that

$$\varphi_2^k = (c^k - \max_{x\in\partial\Omega} c_d)_+ = \text{constant a.e. in } \Omega.$$

By combining the above results, we finally obtain that for any $k \in \mathbb{N}$,

$$\min_{x\in\partial\Omega} c_d \leq c^k \leq \max_{x\in\partial\Omega} c_d \qquad \text{a.e. in } \Omega. \tag{4.67}$$

4.5.2 The limit $k \to \infty$

First, note that by weak lower-semicontinuity of the norm-function, and (4.35), (4.37) and (4.39), we obtain the following uniform bounds, independent of $k \in \mathbb{N}$:

$$\|\boldsymbol{u}^k\|_{1,p^-}^{p^-} + \|\boldsymbol{S}(c^k, \boldsymbol{D}\boldsymbol{u}^k)\|_{(p^+)'}^{(p^+)'} + \frac{1}{k}\|\boldsymbol{u}^k\|_t^t \leq C_1, \tag{4.68}$$

$$\|c^k\|_{1,2}^2 + \|\boldsymbol{q}_c(c^k, \nabla c^k, \boldsymbol{D}\boldsymbol{u}^k)\|_2^2 \leq C_2, \tag{4.69}$$

$$\|\pi^k\|_{j'}^{j'} \leq C_3, \tag{4.70}$$

for some positive constants C_1, C_2 and C_3, which are independent of $k \in \mathbb{N}$.

Now, since $p^- > \frac{3}{2}$, by the min/max principle (4.67), Sobolev embedding and the uniform estimate (4.68), for $s > 3$ sufficiently close to 3, we have that

$$\|c^k \boldsymbol{u}^k\|_s \leq \|c^k\|_\infty \|\boldsymbol{u}^k\|_s \leq C\|\boldsymbol{u}^k\|_{1,p^-} \leq C,$$

where C is independent of $k \in \mathbb{N}$.

Therefore, we can again apply Theorem 4.4.1 with $\boldsymbol{F} = c^k \boldsymbol{u}^k$ and $g = 0$. Hence, there exists an $\alpha_2 \in (0, 1)$ such that

$$\|c^k\|_{C^{0,\alpha_2}(\overline{\Omega})} \leq C_4, \tag{4.71}$$

for some positive constant C_4 independent of $k \in \mathbb{N}$. Since $C^{0,\alpha_2}(\overline{\Omega}) \hookrightarrow\hookrightarrow C^{0,\tilde{\alpha}_2}(\overline{\Omega})$ for all $\tilde{\alpha}_2 \in (0, \alpha_2)$, we have

$$c^k \to c \qquad \text{strongly in } C^{0,\tilde{\alpha}_2}(\overline{\Omega}),$$

which implies that

$$p \circ c^k \to p \circ c \qquad \text{strongly in } C^{0,\beta_2}(\overline{\Omega}),$$

for some $\beta_2 \in (0, 1)$.

Therefore, by the reflexivity of the relevant spaces and compact embedding, there exist subsequences (not relabelled) such that

$$\begin{aligned}
\boldsymbol{u}^k &\rightharpoonup \boldsymbol{u} && \text{weakly in } W^{1,p^-}_{0,\mathrm{div}}(\Omega)^3, && (4.72)\\
\boldsymbol{u}^k &\to \boldsymbol{u} && \text{strongly in } L^{2(1+\varepsilon)}(\Omega)^3, && (4.73)\\
c^k &\rightharpoonup c && \text{weakly in } W^{1,2}(\Omega), && (4.74)\\
c^k &\to c && \text{strongly in } C^{0,\tilde{\alpha}_2}(\overline{\Omega}), && (4.75)\\
\pi^k &\rightharpoonup \pi && \text{weakly in } L^{j'}(\Omega) \qquad \forall\, j > \max\{p^+, 2\}, && (4.76)\\
\boldsymbol{S}(c^k, \boldsymbol{D}\boldsymbol{u}^k) &\rightharpoonup \hat{\boldsymbol{S}} && \text{weakly in } L^{(p^+)'}(\Omega)^{3\times 3}, && (4.77)\\
\boldsymbol{q}_c(c^k, \nabla c^k, \boldsymbol{D}\boldsymbol{u}^k) &\rightharpoonup \hat{\boldsymbol{q}}_c && \text{weakly in } L^2(\Omega)^3. && (4.78)
\end{aligned}$$

Again, by the weak lower-semicontinuity of the norm-function, (4.48) and (4.75) together with Korn's inequality, we have that

$$\int_\Omega \left(|\nabla \boldsymbol{u}|^{p(c)} + |\hat{\boldsymbol{S}}|^{p'(c)} \right) \mathrm{d}x \leq C, \tag{4.79}$$

and thus the weak solution $\boldsymbol{u}$ is in the desired space $W^{1,p(c)}_0(\Omega)^3$.

Now we shall let $k \to \infty$ in (4.51), with $\boldsymbol{v} \in W^{1,\infty}_0(\Omega)^3$ chosen arbitrarily. By (4.73),

$$\boldsymbol{u}^k \otimes \boldsymbol{u}^k \to \boldsymbol{u} \otimes \boldsymbol{u} \qquad \text{strongly in } L^{1+\varepsilon}(\Omega)^{3\times 3}.$$

Thus, we can identify the limit of the convective term

$$-\int_\Omega (\boldsymbol{u}^k \otimes \boldsymbol{u}^k) : \nabla \boldsymbol{v} \, \mathrm{d}x \to -\int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{v} \, \mathrm{d}x \qquad \text{as } k \to \infty, \qquad \forall\, \boldsymbol{v} \in W_0^{1,\infty}(\Omega)^3.$$

Next, by (4.68), we have that

$$\frac{1}{k} \|\boldsymbol{u}^k\|_t^{t-1} \to 0 \qquad \text{as } k \to \infty.$$

Therefore, we have

$$\frac{1}{k} \left| \int_\Omega |\boldsymbol{u}^k|^{t-2} \boldsymbol{u}^k \cdot \boldsymbol{v} \, \mathrm{d}x \right| \leq \frac{1}{k} \|\boldsymbol{u}^k\|_t^{t-1} \|\boldsymbol{v}\|_t \to 0 \qquad \text{as } k \to \infty, \qquad \forall\, \boldsymbol{v} \in W_0^{1,\infty}(\Omega)^3.$$

We recall from the identification asserted in the previous section that $\bar{\boldsymbol{S}} = \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u})$ a.e. on Ω; more precisely, with the index k reinstated in our notation, $\bar{\boldsymbol{S}}^k = \boldsymbol{S}(c^k, \boldsymbol{D}\boldsymbol{u}^k)$ a.e. on Ω. Hence, from (4.77) and (4.76), we obtain for all $\boldsymbol{v} \in W_0^{1,\infty}(\Omega)^3$,

$$\langle \operatorname{div} \boldsymbol{v}, \pi^k \rangle \to \langle \operatorname{div} \boldsymbol{v}, \pi \rangle \quad \text{and} \quad \int_\Omega \bar{\boldsymbol{S}}^k : \boldsymbol{D}\boldsymbol{v} \, \mathrm{d}x \to \int_\Omega \hat{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{v} \, \mathrm{d}x \quad \text{as } k \to \infty.$$

Altogether, we have

$$\int_\Omega \Big(\hat{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{v} + (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{v} \Big) \, \mathrm{d}x - \langle \operatorname{div} \boldsymbol{v}, \pi \rangle = \langle \boldsymbol{f}, \boldsymbol{v} \rangle \qquad \forall\, \boldsymbol{v} \in W_0^{1,\infty}(\Omega)^3. \tag{4.80}$$

Furthermore, from above, it is clear that

$$\int_\Omega \Big(\hat{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{v} + (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{v} \Big) \, \mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{v} \rangle \qquad \forall\, \boldsymbol{v} \in W_{0,\mathrm{div}}^{1,\infty}(\Omega)^3. \tag{4.81}$$

Note that by Proposition 2.5.3 and (4.79) we have

$$\pi \in L_0^{p'(c)}(\Omega).$$

Now, let us investigate the limit of the concentration equation (4.55). Let us choose an arbitrary, but fixed, $z \in W_0^{1,2}(\Omega)$. By (4.73) and (4.75),

$$\|c^k \boldsymbol{u}^k - c\boldsymbol{u}\|_2 \leq \|(c^k - c)\boldsymbol{u}^k\|_2 + \|c(\boldsymbol{u}^k - \boldsymbol{u})\|_2 \leq \|c^k - c\|_\infty \|\boldsymbol{u}^k\|_2 + \|c\|_\infty \|\boldsymbol{u}^k - \boldsymbol{u}\|_2 \to 0.$$

In other words,

$$c^k \boldsymbol{u}^k \to c\boldsymbol{u} \qquad \text{strongly in } L^2(\Omega)^3.$$

Hence we have

$$\int_\Omega c^k \boldsymbol{u}^k \cdot \nabla z \, \mathrm{d}x \to \int_\Omega c\boldsymbol{u} \cdot \nabla z \, \mathrm{d}x.$$

Recalling the identification $\bar{\boldsymbol{q}}_c = \boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u})$ and reinstating the index k, we have $\bar{\boldsymbol{q}}_c^k := \boldsymbol{q}_c(c^k, \nabla c^k, \boldsymbol{D}\boldsymbol{u}^k)$; hence, by (4.78), we get

$$\int_\Omega \bar{\boldsymbol{q}}_c^k \cdot \nabla z \,\mathrm{d}x \to \int_\Omega \hat{\boldsymbol{q}}_c \cdot \nabla z \,\mathrm{d}x \qquad \text{as } k \to \infty.$$

By collecting the above limits, we deduce that

$$\int_\Omega (\hat{\boldsymbol{q}}_c \cdot \nabla z - c\boldsymbol{u} \cdot \nabla z)\,\mathrm{d}x = 0 \qquad \forall\, z \in W_0^{1,2}(\Omega). \tag{4.82}$$

As a final step, we need to identify the limits:

$$\hat{\boldsymbol{S}} = \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u}) \quad \text{and} \quad \hat{\boldsymbol{q}}_c = \boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u}).$$

To this end, analogously as before, we need to prove the following equality:

$$\lim_{k\to\infty} \int_\Omega ((\boldsymbol{S}(c^k, \boldsymbol{D}\boldsymbol{u}^k) - \boldsymbol{S}(c^k, \boldsymbol{D}\boldsymbol{u})) : (\boldsymbol{D}\boldsymbol{u}^k - \boldsymbol{D}\boldsymbol{u}))^{\frac{1}{4}}\,\mathrm{d}x = 0. \tag{4.83}$$

The proof is similar to the one presented in the previous section. The only part of the argument that we shall give here in detail is the proof of the analogue of (4.60)–(4.62) since we now have a different weak formulation at this level. The other parts of the proof proceed as in Chapter 3.

First we define a divergence-free approximation with zero trace as follows:

$$\boldsymbol{\Phi}_j^k := \boldsymbol{u}_j^k - \mathcal{B}(\operatorname{div} \boldsymbol{u}_j^k),$$

where $\mathcal{B}$ is the Bogovskiĭ operator introduced in Theorem 2.5.1. Then, as before, we have

$$\boldsymbol{\Phi}_j^k \rightharpoonup \boldsymbol{u}_j - \mathcal{B}(\operatorname{div} \boldsymbol{u}_j) =: \boldsymbol{\Phi}_j \qquad \text{weakly in } W_0^{1,\sigma}(\Omega)^3, \tag{4.84}$$

$$\boldsymbol{\Phi}_j^k \to \boldsymbol{\Phi}_j \qquad \text{strongly in } L^\sigma(\Omega)^3, \tag{4.85}$$

as $k \to \infty$, where $\sigma \in (1, \infty)$ is arbitrary.

Let us further define $\boldsymbol{\chi}_{1,j}^{n,k} := \Pi_{\mathrm{div}}^n \boldsymbol{\Phi}_j^k$. Then, by (3.14),

$$\boldsymbol{\chi}_{1,j}^{n,k} \to \boldsymbol{\Phi}_j^k \qquad \text{strongly in } W_0^{1,\sigma}(\Omega)^3,\ \forall\, \sigma \in (1, \infty).$$

Now, by (4.29),

$$\int_\Omega \boldsymbol{S}^n : \boldsymbol{D}\boldsymbol{\chi}_{1,j}^{n,k}\,\mathrm{d}x = -B_u[\boldsymbol{U}^n, \boldsymbol{U}^n, \boldsymbol{\chi}_{1,j}^{n,k}] - \int_\Omega \frac{1}{k}|\boldsymbol{U}^n|^{t-2}\boldsymbol{U}^n \cdot \boldsymbol{\chi}_{1,j}^{n,k}\,\mathrm{d}x + \langle \boldsymbol{f}, \boldsymbol{\chi}_{1,j}^{n,k}\rangle.$$

If we take $n \to \infty$ in the above equality, we have

$$\int_\Omega \boldsymbol{S}(c^k, \boldsymbol{D}\boldsymbol{u}^k) : \boldsymbol{D}\boldsymbol{\Phi}_j^k \,\mathrm{d}x = \int_\Omega \left((\boldsymbol{u}^k \otimes \boldsymbol{u}^k) : \nabla \boldsymbol{\Phi}_j^k - \frac{1}{k} |\boldsymbol{u}^k|^{t-2} \boldsymbol{u}^k \cdot \boldsymbol{\Phi}_j^k \right) \mathrm{d}x + \langle \boldsymbol{f}, \boldsymbol{\Phi}_j^k \rangle. \tag{4.86}$$

Next, we define $\boldsymbol{\chi}_{2,j}^{n,k} := \Pi_{\mathrm{div}}^n \boldsymbol{\Phi}_j$, and then we have

$$\boldsymbol{\chi}_{2,j}^{n,k} \to \boldsymbol{\Phi}_j \qquad \text{strongly in } W_0^{1,\sigma}(\Omega)^3, \ \forall\, \sigma \in (1, \infty).$$

Again, by (4.29),

$$\int_\Omega \boldsymbol{S}^n : \boldsymbol{D}\boldsymbol{\chi}_{2,j}^{n,k} \,\mathrm{d}x = -B_u[\boldsymbol{U}^n, \boldsymbol{U}^n, \boldsymbol{\chi}_{2,j}^{n,k}] - \int_\Omega \frac{1}{k} |\boldsymbol{U}^n|^{t-2} \boldsymbol{U}^n \cdot \boldsymbol{\chi}_{2,j}^{n,k} \,\mathrm{d}x + \langle \boldsymbol{f}, \boldsymbol{\chi}_{2,j}^{n,k} \rangle.$$

If we take $n \to \infty$, we have

$$\int_\Omega \boldsymbol{S}(c^k, \boldsymbol{D}\boldsymbol{u}^k) : \boldsymbol{D}\boldsymbol{\Phi}_j \,\mathrm{d}x = \int_\Omega (\boldsymbol{u}^k \otimes \boldsymbol{u}^k) : \nabla \boldsymbol{\Phi}_j - \frac{1}{k} |\boldsymbol{u}^k|^{t-2} \boldsymbol{u}^k \cdot \boldsymbol{\Phi}_j \,\mathrm{d}x + \langle \boldsymbol{f}, \boldsymbol{\Phi}_j \rangle.$$

Subsequently, if we pass k to the infinity, we obtain

$$\int_\Omega \hat{\boldsymbol{S}} : \boldsymbol{D} \,\mathrm{d}x = \int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{\Phi}_j + \langle \boldsymbol{f}, \boldsymbol{\Phi}_j \rangle. \tag{4.87}$$

Therefore, from (4.86) and (4.87), we deduce that

$$\begin{aligned} \lim_{k\to\infty} \int_\Omega \boldsymbol{S}(c^k, \boldsymbol{D}\boldsymbol{u}^k) : \boldsymbol{D}\boldsymbol{\Phi}_j^k \,\mathrm{d}x &= \lim_{k\to\infty} \int_\Omega \left((\boldsymbol{u}^k \otimes \boldsymbol{u}^k) : \nabla \boldsymbol{\Phi}_j^k - \frac{1}{k} |\boldsymbol{u}^k|^{t-2} \boldsymbol{u}^k \cdot \boldsymbol{\Phi}_j^k \right) \mathrm{d}x \\ &\quad + \lim_{k\to\infty} \langle \boldsymbol{f}, \boldsymbol{\Phi}_j^k \rangle \\ &= \int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{\Phi}_j \,\mathrm{d}x + \langle \boldsymbol{f}, \boldsymbol{\Phi}_j \rangle \\ &= \int_\Omega \hat{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{\Phi}_j \,\mathrm{d}x, \end{aligned}$$

which is the desired analogue of (4.60)–(4.62) corresponding to the limit $k \to \infty$, and thereby the proof of (4.83) has been completed.

We can then use the same argument as the one we employed in the previous section to identify $\bar{\boldsymbol{S}} = \bar{\boldsymbol{S}}^k = \boldsymbol{S}(c^k, \boldsymbol{D}\boldsymbol{u}^k)$ and $\bar{\boldsymbol{q}}_c = \bar{\boldsymbol{q}}_c^k = \boldsymbol{q}_c(c^k, \nabla c^k, \boldsymbol{D}\boldsymbol{u}^k)$, and thus we can again identify $\hat{\boldsymbol{S}} = \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u})$, $\hat{\boldsymbol{q}}_c = \boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u})$. That completes the proof of the convergence theorem.

In this and the previous chapter, we have explored the convergence of finite element approximations of the steady version of the model problem under consideration. In the next chapter, we turn our attention to the unsteady version of the model problem and show the existence of a weak solution.

Chapter 5

Existence of weak solutions for the non-stationary model

5.1 Brief description of the chapter

We have successfully completed the convergence analysis of the finite element method for the steady problem in both two and three space dimensions. Now we move on to the unsteady version of the problem. We are interested in developing an existence theory for the non-stationary model. In other words, we would like to know whether weak solutions to the following system of equations exist in the domain $Q_T := \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^d$ with $d \geq 2$ is a bounded open Lipschitz domain and $(0,T)$ denotes the time interval of interest:

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } Q_T, \tag{5.1}$$

$$\partial_t \boldsymbol{u} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div} \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u}) = -\nabla\pi + \boldsymbol{f} \qquad \text{in } Q_T, \tag{5.2}$$

$$\partial_t c + \operatorname{div}(c\boldsymbol{u}) - \operatorname{div} \boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u}) = 0 \qquad \text{in } Q_T, \tag{5.3}$$

where $\boldsymbol{u} : Q_T \to \mathbb{R}^d$, $\pi : Q_T \to \mathbb{R}$, $c : Q_T \to \mathbb{R}_{\geq 0}$ are the velocity field, pressure and concentration respectively. Again, $\boldsymbol{f} : Q_T \to \mathbb{R}^d$ represents a given density of the bulk force, $\boldsymbol{D}\boldsymbol{u}$ denotes the symmetric velocity gradient, and $\boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u})$ and $\boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u})$ are the extra stress tensor of the Cauchy stress tensor and the diffusion flux respectively. In this setting, for given functions $\boldsymbol{u}_0$ and c_0 defined in Ω, we prescribe the following initial conditions

$$\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x) \qquad \text{and} \qquad c(x,0) = c_0(x) \qquad \text{in } \Omega, \tag{5.4}$$

where $0 \leq c_0 \leq \tilde{c}_0$ for some positive constant $\tilde{c}_0$.

Furthermore, we set $S_T := \partial\Omega \times (0,T)$ and $\Gamma_T := S_T \cup \{(x,t) : x \in \overline{\Omega}, t = 0\}$, and we prescribe the homogeneous boundary conditions

$$\boldsymbol{u}(x,t) = \mathbf{0} \qquad \text{and} \qquad c(x,t) = 0 \qquad \text{on } S_T. \tag{5.5}$$

Finally, as before, we assume again that the extra stress tensor $\boldsymbol{S} : \mathbb{R}_{\geq 0} \times \mathbb{R}^{d\times d}_{\text{sym}} \to \mathbb{R}^{d\times d}_{\text{sym}}$ is a continuous mapping with the following non-standard growth, strict monotonicity and coercivity: there exist positive constants C_1, C_2 and C_3 such that

$$|\boldsymbol{S}(\xi, \boldsymbol{B})| \leq C_1(|\boldsymbol{B}|^{p(\xi)-1} + 1), \tag{5.6}$$

$$(\boldsymbol{S}(\xi, \boldsymbol{B_1}) - \boldsymbol{S}(\xi, \boldsymbol{B_2})) : (\boldsymbol{B_1} - \boldsymbol{B_2}) > 0 \ \text{ for } \boldsymbol{B_1} \neq \boldsymbol{B_2}, \tag{5.7}$$

$$\boldsymbol{S}(\xi, \boldsymbol{B}) \cdot \boldsymbol{B} \geq C_2(|\boldsymbol{B}|^{p(\xi)} + |\boldsymbol{S}|^{p'(\xi)}) - C_3, \tag{5.8}$$

where $p : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a Hölder continuous function such that $1 < p^- \leq p(\xi) \leq p^+ < \infty$ and $p'(\xi)$ is defined as $\frac{p(\xi)}{p(\xi)-1}$. Furthermore, we assume that the diffusion flux vector $\boldsymbol{q}_c(\xi, \boldsymbol{g}, \boldsymbol{B}) : \mathbb{R}_{\geq 0} \times \mathbb{R}^d \times \mathbb{R}^{d\times d}_{\text{sym}} \to \mathbb{R}^d$ is a continuous mapping, which is linear with respect to $\boldsymbol{g}$, and satisfies the following inequalities: there exist positive constants C_4 and C_5 such that

$$|\boldsymbol{q}_c(\xi, \boldsymbol{g}, \boldsymbol{B})| \leq C_4|\boldsymbol{g}|, \tag{5.9}$$

$$\boldsymbol{q}_c(\xi, \boldsymbol{g}, \boldsymbol{B}) \cdot \boldsymbol{g} \geq C_5|\boldsymbol{g}|^2. \tag{5.10}$$

In this chapter, we shall develop the existence theory for the above model with the help of a generalized Galerkin method, together with the use of monotone operator theory in variable-exponent spaces and the parabolic De Giorgi–Nash–Moser regularity theory. As a consequence, we can also obtain the existence result for a class of unsteady electro-rheological fluid flow models as a special case, which is completely new. This by-product will be discussed in Chapter 6. Also, we will discuss a possibility to improve our main result. The content of this chapter is based on the paper [56].

5.2 Definition of the weak solution and the existence theorem

In this section, we introduce our main theorem together with some preliminaries required for the precise statement of the theorem. First, we introduce some concepts and notations for parabolic problems. The main difference compared to previous chapters is that we are considering both the space and time variable, so we are working with the cylinder Q_T rather than with the spatial domain Ω. We shall write $z = (x, t) \in \mathbb{R}^{d+1}$, where $x \in \mathbb{R}^d$ denotes the spatial variable and $t \in \mathbb{R}_{\geq 0}$ is the time variable. Let $\mathcal{P}$ be the set of all measurable functions $p : Q_T \to [1, \infty]$; again, we call a function $p \in \mathcal{P}(Q_T)$ a variable exponent. Then we define $p^- :=$ ess $\inf_{z \in Q_T} p(z)$, $p^+ :=$ ess $\sup_{z \in Q_T} p(z)$. Throughout the chapter, we only deal with the case

$$1 < p^- \leq p^+ < \infty. \tag{5.11}$$

Then we define generalized Lebesgue spaces on Q_T, which are equipped with the corresponding Luxembourg norms:

$$\begin{aligned} L^{p(\cdot)}(Q_T) &:= \left\{ u \in L^1_{\mathrm{loc}}(Q_T) : \int_{Q_T} |u(x,t)|^{p(x,t)} \,\mathrm{d}x \,\mathrm{d}t < \infty \right\}, \\ \|u\|_{L^{p(\cdot)}(Q_T)} &:= \inf \left\{ \lambda > 0 : \int_{Q_T} \left| \frac{u(x,t)}{\lambda} \right|^{p(x,t)} \mathrm{d}x \,\mathrm{d}t \leq 1 \right\}. \end{aligned}$$

Some basic inequalities also hold for variable-exponent Lebesgue spaces on Q_T. For example, if p, q, $s \in \mathcal{P}(Q_T)$ are such that $\frac{1}{s(z)} = \frac{1}{p(z)} + \frac{1}{q(z)}$ for all $z \in Q_T$, then for any $f \in L^{p(\cdot)}(Q_T)$ and $g \in L^{q(\cdot)}(Q_T)$, we have

- ∘ (Hölder's inequality) $\|fg\|_{L^{s(\cdot)}(Q_T)} \leq 2\|f\|_{L^{p(\cdot)}(Q_T)} \|g\|_{L^{q(\cdot)}(Q_T)}$,
- ∘ (Young's inequality) $\int_{Q_T} |fg|^{s(x,t)} \,\mathrm{d}x \,\mathrm{d}t \leq \int_{Q_T} |f|^{p(x,t)} \,\mathrm{d}x \,\mathrm{d}t + \int_{Q_T} |g|^{q(x,t)} \,\mathrm{d}x \,\mathrm{d}t$.

Next we define the following anisotropic Sobolev space with variable exponent:

$$\begin{aligned} W_{p(\cdot)}(Q_T) &:= \left\{ \boldsymbol{u} \in L^1(0, T; W^{1,1}_{0,\mathrm{div}}(\Omega)^d) : |\nabla \boldsymbol{u}| \in L^{p(\cdot)}(Q_T) \right\}, \\ \|\boldsymbol{u}\|_{W_{p(\cdot)}(Q_T)} &:= \|\boldsymbol{u}\|_{L^2(Q_T)} + \|\nabla \boldsymbol{u}\|_{L^{p(\cdot)}(Q_T)}. \end{aligned}$$

If we follow the argument used in Chapter 2 to prove Theorem 2.3.9, it is easy to check that all of the above spaces are Banach spaces, and thanks to (5.11), they are all separable and reflexive. We also define the dual space $(L^{p(\cdot)}(Q_T))^* = L^{p'(\cdot)}(Q_T)$ where the dual variable exponent $p' \in \mathcal{P}(Q_T)$ is defined by $\frac{1}{p(z)} + \frac{1}{p'(z)} = 1$.

Furthermore, in order to study the temporal regularity of functions, we introduce the following parabolic spaces: for a real Banach space X,

$$\begin{aligned} C([0,T]; X) &:= \{u \in L^\infty(0,T;X) : u : [0,T] \to X \text{ is continuous}\}, \\ C_{\mathrm{w}}([0,T]; X) &:= \{u \in L^\infty(0,T;X) : u : [0,T] \to X \text{ is weakly continuous}\}, \end{aligned}$$

where both spaces are equipped with the $L^\infty(0,T;X)$ norm.

Finally, we need to introduce the parabolic Hölder space, which has an important role in our analysis. We consider the parabolic metric $d_p(\cdot,\cdot)$ defined by

$$d_p(z_1,z_2) := |x_1-x_2| + |t_1-t_2|^{\frac{1}{2}},$$

where $z_1=(x_1,t_1)$, $z_2=(x_2,t_2)\in\mathbb{R}^{d+1}$. We then define the parabolic Hölder space using the above parabolic distance: for some $\alpha\in(0,1)$,

$$C^{\alpha,\alpha/2}(Q_T) := \left\{ f\in C(Q_T) : \exists C>0 \text{ such that } \frac{|f(z_1)-f(z_2)|}{d_p(z_1,z_2)^\alpha} \le C \ \ \forall z_1,z_2\in Q_T \right\},$$

with the norm

$$\|f\|_{C^{\alpha,\alpha/2}(Q_T)} := \|f\|_{L^\infty(Q_T)} + \sup_{z_1,z_2\in Q_T,\, z_1\neq z_2,} \frac{|f(z_1)-f(z_2)|}{d_p(z_1,z_2)^\alpha}.$$

Then, our main result concerning the existence of weak solutions is as follows.

Theorem 5.2.1. *Let $\Omega\subset\mathbb{R}^d$ with $d\ge 2$ be a bounded open Lipschitz domain, and assume that $\boldsymbol{f}\in L^2(Q_T)^d$, $\boldsymbol{u}_0\in L^2(\Omega)^d$ and $c_0\in\{f\in C^{0,\alpha_0}(\overline{\Omega})$ for some $\alpha_0\in(0,1) : f=0$ on $\partial\Omega\}$. If p is a bounded Hölder continuous function with $p^->d$, then there exists a weak solution pair $(\boldsymbol{u},c)$ to the system of equations* (5.1)–(5.3) *and some $\alpha\in(0,1)$ such that*

$$\begin{aligned}\boldsymbol{u} &\in C([0,T];L^2(\Omega)^d)\cap L^{p^-}(0,T;W^{1,p^-}_{0,\operatorname{div}}(\Omega)^d)\cap W_{p(c)}(Q_T),\\ c &\in C([0,T];L^2(\Omega))\cap L^2(0,T;W^{1,2}_0(\Omega))\cap C^{\alpha,\alpha/2}(\overline{Q}_T),\end{aligned}$$

satisfying

$$\begin{aligned}\int_{Q_T}\boldsymbol{S}(c,\boldsymbol{D}\boldsymbol{u}):\boldsymbol{D}\boldsymbol{\psi}\,\mathrm{d}x\,\mathrm{d}t &= \int_{Q_T}(\boldsymbol{f}\cdot\boldsymbol{\psi}+(\boldsymbol{u}\otimes\boldsymbol{u}):\boldsymbol{D}\boldsymbol{\psi}+\boldsymbol{u}\cdot\partial_t\boldsymbol{\psi})\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_\Omega \boldsymbol{u}_0(x)\cdot\boldsymbol{\psi}(x,0)\,\mathrm{d}x\end{aligned}$$

for all $\boldsymbol{\psi}\in C^\infty_{0,\operatorname{div}}(\Omega\times[0,T))^d$ and

$$\int_{Q_T}\boldsymbol{q}_c(c,\nabla c,\boldsymbol{D}\boldsymbol{u})\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T}(c\partial_t\varphi + c\boldsymbol{u}\cdot\nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega c_0(x)\varphi(x,0)\,\mathrm{d}x$$

for all $\varphi\in C^\infty_0(\Omega\times[0,T))$.

5.3 Auxiliary tools and results

In this section, we introduce the necessary technical tools, especially some results from function space theory, that we use throughout the chapter. We begin with some embedding theorems. The following proposition concerns the compact embedding of parabolic Hölder spaces.

Proposition 5.3.1. *Let* $Q_T \subset \mathbb{R}^{d+1}$ *be such that* $\overline{Q}_T$ *is compact in* $\mathbb{R}^{d+1}$ *and let* $0 < \alpha < \beta < 1$. *Then the inclusion map* $i : C^{\beta,\beta/2}(\overline{Q}_T) \hookrightarrow C^{\alpha,\alpha/2}(\overline{Q}_T)$ *is compact.*

Proof. We first note that $\overline{Q}_T$ is also compact with respect to the topology induced by the parabolic distance $d_p(\cdot,\cdot)$. Let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence in $C^{\beta,\beta/2}(\overline{Q}_T)$. Then, trivially $\{f_n\}_{n=1}^{\infty}$ is a uniformly bounded and equicontinuous class of functions. By the Arzelá–Ascoli theorem, there exists a subsequence $\{f_n\}_{n=1}^{\infty}$ (not relabelled) such that $f_n \to f$ in $C(\overline{Q}_T)$ for some $f \in C^{\beta,\beta/2}(\overline{Q}_T)$.

Let $g_n := f - f_n \in C^{\beta,\beta/2}(\overline{Q}_T)$. Then $\|g_n\|_{C^{\beta,\beta/2}(\overline{Q}_T)} \leq C$ for some constant $C > 0$ and $\|g_n\|_{C(\overline{Q}_T)} \to 0$ as $n \to \infty$. To complete the proof, it suffices to show that

$$[g_n]_\alpha := \sup_{z_1,\, z_2 \in \overline{Q}_T,\, z_1 \neq z_2} \frac{|g_n(z_1) - g_n(z_2)|}{d_p(z_1,\, z_2)^\alpha} \to 0.$$

For arbitrarily small $\delta > 0$,

$$\begin{aligned}
[g_n]_\alpha &\leq \sup_{z_1 \neq z_2,\, d_p(z_1,z_2) \leq \delta} \frac{|g_n(z_1) - g_n(z_2)|}{d_p(z_1,\, z_2)^\alpha} + \sup_{d_p(z_1,z_2) > \delta} \frac{|g_n(z_1) - g_n(z_2)|}{d_p(z_1,\, z_2)^\alpha} \\
&\leq \sup_{z_1 \neq z_2,\, d_p(z_1,z_2) \leq \delta} \frac{|g_n(z_1) - g_n(z_2)|}{d_p(z_1,\, z_2)^\beta} d_p(z_1,\, z_2)^{\beta-\alpha} + \sup_{d_p(z_1,z_2) > \delta} \frac{|g_n(z_1) - g_n(z_2)|}{d_p(z_1,\, z_2)^\alpha} \\
&\leq \delta^{\beta-\alpha}[g_n]_\beta + 2\delta^{-\alpha}\|g_n\|_{C(\overline{Q}_T)} \\
&\leq C\delta^{\beta-\alpha} + 2\delta^{-\alpha}\|g_n\|_{C(\overline{Q}_T)}.
\end{aligned}$$

Therefore,

$$\limsup_{n\to\infty} [g_n]_\alpha \leq C\delta^{\beta-\alpha} + 2\delta^{-\alpha} \limsup_{n\to\infty} \|g_n\|_{C(\overline{Q}_T)} \leq C\delta^{\beta-\alpha}.$$

Since $\delta > 0$ is arbitrary, we conclude that $[g_n]_\alpha \to 0$ as $n \to \infty$, which completes the proof. □

Also, we will use the following parabolic version of the Sobolev embedding theorem, which comes from the use of standard Sobolev embedding and an interpolation inequality. For the proof, see Proposition 3.1 in [32].

Proposition 5.3.2. *If* $1 \leq p < \infty$*, there exists a constant* $C > 0$*, depending only on* d *and* p*, such that*

$$L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \hookrightarrow L^{\frac{p(d+2)}{d}}(Q_T), \tag{5.12}$$

with

$$\int_{Q_T} |v|^{\frac{p(d+2)}{d}} \,\mathrm{d}x\,\mathrm{d}t \leq C \left(\sup_{0 \in (0,T)} \int_\Omega |v|^2 \,\mathrm{d}x \right)^{\frac{p}{d}} \int_{Q_T} |\nabla v|^p \,\mathrm{d}x\,\mathrm{d}t$$

for every $v \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$.

Additionally, we introduce another important embedding theorem, which is classical in the study of time-dependent problems. See, for example, [79] or Lemma 6.3 in [51].

Lemma 5.3.3. (Aubin–Lions lemma) *Let* V_1*,* V_2 *and* V_3 *be reflexive, separable Banach spaces such that*

$$V_1 \hookrightarrow\hookrightarrow V_2 \qquad \text{and} \qquad V_2 \hookrightarrow V_3,$$

and let $1 < p < \infty$*,* $1 \leq q \leq \infty$*. Then* $\{v \in L^p(0,T;V_1) : \partial_t v \in L^q(0,T;V_3)\}$ *is compactly embedded into* $L^p(0,T;V_2)$*.*

We also need the following existence theorem for the system of ordinary differential equations. Before we state the theorem, we introduce some notation in the context of ordinary differential equations. We want to find a function $\boldsymbol{C} : (t_0 - \delta, t_0 + \delta) \to \mathbb{R}^N$ such that it solves the following ordinary differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{C}(t) = \boldsymbol{G}(t, \boldsymbol{C}(t)), \qquad \forall t \in (t_0 - \delta, t_0 + \delta) \tag{5.13}$$

$$\boldsymbol{C}(t_0) = \boldsymbol{C}_0, \tag{5.14}$$

where $\boldsymbol{C}_0 \in \mathbb{R}^N$. Here we consider $\boldsymbol{G} : (t_0 - \delta, t_0 + \delta) \times B_\varepsilon(\boldsymbol{C}_0) \to \mathbb{R}^N$ for some $\varepsilon > 0$. Then we have the following existence theorem. See [82] for more details.

Theorem 5.3.4. (Carathéodory's theorem) *Suppose that* $\boldsymbol{G} : (t_0 - \delta, t_0 + \delta) \times B_\varepsilon(\boldsymbol{C}_0) \to \mathbb{R}^N$ *satisfies the following properties: if we denote* $\boldsymbol{G} = \{G_1, \ldots, G_N\}$*,*

- $G_i(\cdot, \boldsymbol{C})$ *is measurable for all* $i = 1, \ldots, N$ *and for all* $\boldsymbol{C} \in B_\varepsilon(\boldsymbol{C}_0)$*.*

- $G_i(t, \cdot)$ *is continuous for a.e.* $t \in (t_0 - \delta, t_0 + \delta)$*.*

- *$|\boldsymbol{G}(t,\boldsymbol{C})| \leq F(t)$ for all $(t,\boldsymbol{C}) \in (t_0-\delta, t_0+\delta) \times B_\varepsilon(\boldsymbol{C}_0)$ and $F(t)$ is integrable on $(t_0-\delta, t_0+\delta)$.*

Then there exists a $\delta_1 \in (0,\delta)$ and an absolutely continuous function $\boldsymbol{C} : (t_0-\delta, t_0+\delta) \to \mathbb{R}^N$ such that $\boldsymbol{C}$ solves (5.13)–(5.14) for a.e. $t \in (t_0-\delta_1, t_0+\delta_1)$.

Next, in order to ensure the Hölder continuity of the approximate concentrations in the later analysis, we will use the the following result, which is mainly due to De Giorgi, Nash and Moser; see [62] for more details.

Theorem 5.3.5. *Suppose that for $\boldsymbol{K} \in \mathbb{R}^{d\times d}$ with measurable components, there exist positive constants $C_1, C_2 > 0$ such that*

$$\|\boldsymbol{K}\|_{L^\infty(Q_T)^{d\times d}} \leq C_1, \qquad \boldsymbol{K}\boldsymbol{b}\cdot\boldsymbol{b} \geq C_2|\boldsymbol{b}|^2 \text{ for all } \boldsymbol{b} \in \mathbb{R}^d \text{ a.e. in } Q_T.$$

Assume further that $\boldsymbol{g} \in L^r(0,T;L^q(\Omega)^d)$ where q and r are arbitrary constants satisfying

$$\frac{2}{r} + \frac{d}{q} = 1 - \varepsilon,$$

with

$$q \in \left[\frac{d}{1-\varepsilon}, \infty\right], \quad r \in \left[\frac{2}{1-\varepsilon}, \infty\right] \quad \text{and} \quad 0 < \varepsilon < 1.$$

Let $c \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W_0^{1,2}(\Omega))$ with $\partial_t c \in L^2(0,T;W^{-1,2}(\Omega))$ be a weak solution of $\partial_t c - \operatorname{div}(\boldsymbol{K}\nabla c) = \operatorname{div}\boldsymbol{g}$ in the sense that

$$\langle \partial_t c, \varphi\rangle + \int_\Omega \boldsymbol{K}\nabla c \cdot \nabla\varphi = \int_\Omega \boldsymbol{g}\cdot\nabla\varphi \,\mathrm{d}x \qquad \forall \varphi \in W_0^{1,2}(\Omega).$$

If $c_{|\Gamma_T} \in C^{\beta,\beta/2}(\Gamma_T)$ for some $\beta \in (0,1)$, then there exists an $\alpha \in (0,\beta]$ depending on d, C_1, C_2, β, $\boldsymbol{g}$, q and r such that

$$c \in C^{\alpha,\alpha/2}(\overline{Q}_T),$$

with

$$\|c\|_{C^{\alpha,\alpha/2}} \leq C(d, C_1, C_2, \beta, \|\boldsymbol{g}\|_{r,q}).$$

Now we move to function space theory for variable-exponent spaces. First, we introduce a subset $\mathcal{P}^{\log}(Q_T) \subset \mathcal{P}(Q_T)$ as the class of log-Hölder continuous functions on the cylinder Q_T, satisfying

$$|p(z_1) - p(z_2)| \leq \frac{C_{\log}(p)}{-\log|z_1 - z_2|} \quad \forall z_1, z_2 \in Q_T : 0 < |z_1 - z_2| \leq \frac{1}{2}. \tag{5.15}$$

The following lemma shows that a parabolic Hölder continuous function on Q_T automatically belongs to this class.

Lemma 5.3.6. *Let z_1, $z_2 \in Q_T$ with $|z_1 - z_2| < \frac{1}{8}$. Then, for any $\alpha \in (0, 1)$, there exists a constant $C > 0$ such that*

$$d_p(z_1, z_2)^\alpha \leq \frac{C}{-\log|z_1 - z_2|}.$$

Proof. From the definition of the parabolic distance, we have $d_p(z_1, z_2) < \frac{1}{2}$. Furthermore, by an elementary calculation, we can easily verify the following inequalities:

$$d_p(z_1, z_2) \leq 2|z_1 - z_2|^{\frac{1}{2}} \text{ for } |z_1 - z_2| < 1 \qquad \text{and} \qquad x^\alpha \leq \frac{C}{-\log x} \text{ for } 0 < x < \frac{1}{2}.$$

Therefore, we have

$$d_p(z_1, z_2)^\alpha \leq 2^\alpha |z_1 - z_2|^{\frac{\alpha}{2}} \leq \frac{C2^\alpha}{-\log|z_1 - z_2|}.$$

□

What we need here is a density result for the function space $W_{p(\cdot)}(Q_T)$ defined in Section 5.2. First, from [66], we infer the existence of a sequence $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{R}$ and a sequence of functions $\{\boldsymbol{w}_j\}_{j=1}^\infty \subset W_{0,\,\mathrm{div}}^{\ell,2}(\Omega)^d$, $\ell \in \mathbb{N}$, satisfying

- each $\boldsymbol{w}_j$ is an eigenfunction associated with the eigenvalue λ_j in the following sense: for all $j \in \mathbb{N}$,
$$\langle \boldsymbol{w}_j, \boldsymbol{\phi} \rangle_{W_0^{\ell,2}} = \lambda_j \int_\Omega \boldsymbol{w}_j \cdot \boldsymbol{\phi} \,\mathrm{d}x \qquad \forall \boldsymbol{\phi} \in W_{0,\mathrm{div}}^{\ell,2}(\Omega)^d,$$
- $\int_\Omega \boldsymbol{w}_i \cdot \boldsymbol{w}_j \,\mathrm{d}x = \delta_{ij}$ for all i, $j \in \mathbb{N}$,
- $1 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ and $\lambda_j \to \infty$,
- $\langle \frac{\boldsymbol{w}_i}{\sqrt{\lambda_i}}, \frac{\boldsymbol{w}_j}{\sqrt{\lambda_j}} \rangle_{W_0^{\ell,2}} = \delta_{ij}$ for all i, $j \in \mathbb{N}$,
- $\{\boldsymbol{w}_j\}_{j=1}^\infty$ is a basis of $W_{0,\,\mathrm{div}}^{\ell,2}(\Omega)^d$.

We choose $\ell > 1 + \frac{d}{2}$ so that $W^{\ell,2}_{0,\,\mathrm{div}}(\Omega)^d \hookrightarrow W^{1,\infty}_{0,\,\mathrm{div}}(\Omega)^d$. Then we have the following lemma, which is quoted from [5].

Lemma 5.3.7. *Let $p \in \mathcal{P}^{\log}(Q_T)$. For each $\boldsymbol{u} \in W_{p(\cdot)}(Q_T)$, there exists a sequence $\{d_j(t)\}_{j=1}^{\infty} \subset C^1([0,T])$, satisfying*

$$\left\| \boldsymbol{u} - \sum_{j=1}^{m} d_j\, \boldsymbol{w}_j \right\|_{W_{p(\cdot)}} \to 0 \qquad \text{as } m \to \infty.$$

In fact, unlike the space used in [5], here we are dealing with the function space of solenoidal functions. However, that does not affect the proof of the lemma as here the basis $\{\boldsymbol{w}_j\}_{j=1}^{\infty}$ consists of solenoidal functions.

5.4 Proof of the main theorem

5.4.1 Galerkin approximation

Let $\{\boldsymbol{w}_j\}_{j=1}^{\infty}$ be a basis of $W^{\ell,2}_{0,\mathrm{div}}(\Omega)^d$, constructed in the previous section, and let $\{z_k\}_{k=1}^{\infty}$ be a basis of $W^{1,2}_0(\Omega)$ which is orthonormal in $L^2(\Omega)$. We are looking for a sequence of approximate solutions $\{\boldsymbol{u}^{N,M},\, c^{N,M}\}_{N,M=1}^{\infty}$ of the form

$$\boldsymbol{u}^{N,M} = \sum_{i=1}^{N} a_i^{N,M} \boldsymbol{w}_i, \qquad c^{N,M} = \sum_{i=1}^{M} b_i^{N,M} z_i,$$

where $\boldsymbol{a}^{N,M} = \{a_j^{N,M}\}_{j=1}^{N} : (0,T) \to \mathbb{R}^N$ and $\boldsymbol{b}^{N,M} = \{b_k^{N,M}\}_{k=1}^{M} : (0,T) \to \mathbb{R}^M$, which solve the following system of ordinary differential equations:

$$\int_\Omega \partial_t \boldsymbol{u}^{N,M} \cdot \boldsymbol{w}_j \,\mathrm{d}x - \int_\Omega (\boldsymbol{u}^{N,M} \otimes \boldsymbol{u}^{N,M}) : \boldsymbol{D}\boldsymbol{w}_j \,\mathrm{d}x + \int_\Omega \boldsymbol{S}^{N,M} : \boldsymbol{D}\boldsymbol{w}_j \,\mathrm{d}x \tag{5.16}$$

$$= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w}_j \,\mathrm{d}x \qquad \forall j = 1, \dots, N$$

$$\boldsymbol{u}^{N,M}(\cdot, 0) = \boldsymbol{u}_0^{N,M} = P^N \boldsymbol{u}_0 = \sum_{j=1}^{N} a_{0,j}^{N} \boldsymbol{w}_j, \tag{5.17}$$

$$\int_\Omega \partial_t c^{N,M}\, z_k \,\mathrm{d}x - \int_\Omega \boldsymbol{u}^{N,M} c^{N,M} \cdot \nabla z_k \,\mathrm{d}x + \int_\Omega \boldsymbol{q}_c^{N,M} \cdot \nabla z_k \,\mathrm{d}x = 0 \;\; \forall k = 1, \dots M, \tag{5.18}$$

$$c^{N,M}(\cdot,0) = c_0^{N,M} = P^M c_0 = \sum_{j=1}^{M} b_{0,k}^M z_k, \tag{5.19}$$

where

$$\boldsymbol{S}^{N,M} = \boldsymbol{S}(c^{N,M}, \boldsymbol{Du}^{N,M}), \qquad \boldsymbol{q}_c^{N,M} = \boldsymbol{q}_c(c^{N,M}, \nabla c^{N,M}, \boldsymbol{Du}^{N,M}),$$

and P^N, P^M denote the L^2-orthogonal projections onto the finite-dimensional linear spaces $A_N = \operatorname{span}\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_N\}$, $B_M = \operatorname{span}\{z_1, \dots, z_M\}$ respectively.

Let us define $\mathbf{C}(t) := (a_1^{N,M}(t), \dots, a_N^{N,M}(t), b_1^{N,M}(t), \dots, b_M^{N,M}(t))$ and we shall write $\mathbf{C}_0 := (a_{0,1}^N, \dots a_{0,N}^N, b_{0,1}^M, \dots, b_{0,M}^M)$. Then the system (5.16) and (5.18) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{C}(t) = \boldsymbol{G}(t, \mathbf{C}(t)), \tag{5.20}$$

$$\mathbf{C}(0) = \mathbf{C}_0. \tag{5.21}$$

It is easy to see that $\boldsymbol{G}$ is measurable with respect to t and is continuous with respect to $\mathbf{C}$. Moreover, it is straightforward to show that there exists an integrable function F such that

$$|\boldsymbol{G}(t, \mathbf{C}(t))| \le F(t)$$

at least for all $(t, \mathbf{C}(t)) \in (-T, T) \times D$ where D is $(M+N)$-dimensional cube with radius $R := 2 \max_i |C_0^i|$. Therefore, Theorem 5.3.4 then guarantees the existence of solutions to (5.16)–(5.19) at least for a short time interval. The uniform estimates that we will derive below enable us to extend the solution onto the whole time interval $(0, T)$.

5.4.2 Uniform estimates independent of M

We first derive some uniform estimates that are independent of M. Multiplying the jth equation in (5.16) by $a_j^{N,M}$, followed by taking the sum over $j = 1, \dots, N$ and integrating the result over $(0, t)$, we obtain

$$\|\boldsymbol{u}^{N,M}(t)\|_2^2 + 2\int_0^t \int_\Omega \boldsymbol{S}^{N,M} : \boldsymbol{Du}^{N,M} \,\mathrm{d}x \,\mathrm{d}\tau = \|\boldsymbol{u}_0^{N,M}\|_2^2 + 2\int_0^t \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u}^{N,M} \,\mathrm{d}x \,\mathrm{d}\tau.$$

Applying (5.8) and Poincaré's inequality and using standard duality estimates together with Young's inequality, we conclude that

$$\sup_{t \in (0,T)} \|\boldsymbol{u}^{N,M}(t)\|_2^2 + \int_0^T \int_\Omega \left(|\nabla \boldsymbol{u}^{N,M}|^{p^-} + |\boldsymbol{S}^{N,M}|^{(p^+)'} \right) \mathrm{d}x \,\mathrm{d}t \le C, \tag{5.22}$$

and hence, by Proposition 5.3.2,

$$\|\boldsymbol{u}^{N,M}\|_{L^{\frac{p^-(d+2)}{d}}(Q_T)} \leq C. \tag{5.23}$$

Similarly, multiplying the kth equation in (5.18) by $b_k^{N,M}$, taking the sum over $k = 1, \dots, M$ and integrating the result over $(0,t)$, we arrive at

$$\|c^{N,M}(t)\|_2^2 + 2\int_0^t \int_\Omega \boldsymbol{q}_c^{N,M} \cdot \nabla c^{N,M} \,\mathrm{d}x \,\mathrm{d}\tau = \|c_0^{N,M}\|_2^2.$$

By using (5.9) and (5.10), we have

$$\sup_{t\in(0,T)} \|c^{N,M}(t)\|_2^2 + \int_0^T \int_\Omega \left(|\nabla c^{N,M}|^2 + |\boldsymbol{q}_c^{N,M}|^2\right) \mathrm{d}x \,\mathrm{d}t \leq C. \tag{5.24}$$

Next, we also need to estimate the time derivative of $\boldsymbol{a}^{N,M}$. We therefore multiply the jth equation in (5.16) by $\frac{\mathrm{d}}{\mathrm{d}t} a_j^{N,M}$, sum over $j = 1, \dots, N$, and integrate over time. It is obvious that the first term becomes $\int_0^T |\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{a}^{N,M}|^2 \,\mathrm{d}t$. For the convective term, by (5.22) we have

$$\begin{aligned}
&\int_0^T \int_\Omega (\boldsymbol{u}^{N,M} \otimes \boldsymbol{u}^{N,M}) : \left(\sum_{j=1}^N \frac{\mathrm{d}}{\mathrm{d}t} a_j^{N,M} \nabla \boldsymbol{w}_j\right) \mathrm{d}x \,\mathrm{d}t \\
&\leq \int_0^T \int_\Omega |\boldsymbol{u}^{N,M} \otimes \boldsymbol{u}^{N,M}| \left(\sum_{j=1}^N \left|\frac{\mathrm{d}}{\mathrm{d}t} a_j^{N,M}\right|^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^N |\nabla \boldsymbol{w}_j|^2\right)^{\frac{1}{2}} \mathrm{d}x \,\mathrm{d}t \\
&\leq \int_0^T \left(\sum_{j=1}^N \left|\frac{\mathrm{d}}{\mathrm{d}t} a_j^{N,M}\right|^2\right)^{\frac{1}{2}} \int_\Omega |\boldsymbol{u}^{N,M} \otimes \boldsymbol{u}^{N,M}| \left(\sum_{j=1}^N |\nabla \boldsymbol{w}_j|^2\right)^{\frac{1}{2}} \mathrm{d}x \,\mathrm{d}t \\
&\leq \left(\int_0^T \left|\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{a}^{N,M}\right|^2 \mathrm{d}t\right)^{\frac{1}{2}} \left(\int_0^T \left(\int_\Omega |\boldsymbol{u}^{N,M} \otimes \boldsymbol{u}^{N,M}| \left(\sum_{j=1}^N |\nabla \boldsymbol{w}_j|^2\right)^{\frac{1}{2}} \mathrm{d}x\right)^2 \mathrm{d}t\right)^{\frac{1}{2}} \\
&\leq C(N) \left(\int_0^T \left|\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{a}^{N,M}\right|^2 \mathrm{d}t\right)^{\frac{1}{2}} \left(\int_0^T \left(\int_\Omega |\boldsymbol{u}^{N,M} \otimes \boldsymbol{u}^{N,M}| \,\mathrm{d}x\right)^2 \mathrm{d}t\right)^{\frac{1}{2}} \\
&\leq C(N) \left(\int_0^T \left|\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{a}^{N,M}\right|^2 \mathrm{d}t\right)^{\frac{1}{2}} \sup_{t\in(0,T)} \|\boldsymbol{u}^{N,M}(t)\|_2^2 \\
&\leq C(N) \left(\int_0^T \left|\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{a}^{N,M}\right|^2 \mathrm{d}t\right)^{\frac{1}{2}}.
\end{aligned}$$

For the term involving the stress tensor, note that

$$\begin{aligned}
&\int_0^T \int_\Omega \boldsymbol{S}^{N,M} : \left(\sum_{j=1}^N \frac{\mathrm{d}}{\mathrm{d}t} a_j^{N,M} \nabla \boldsymbol{w}_j \right) \mathrm{d}x \, \mathrm{d}t \\
&\leq \int_0^T \int_\Omega |\boldsymbol{S}^{N,M}| \left(\sum_{j=1}^N \left| \frac{\mathrm{d}}{\mathrm{d}t} a_j^{N,M} \right|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^N |\nabla \boldsymbol{w}_j|^2 \right)^{\frac{1}{2}} \mathrm{d}x \, \mathrm{d}t \\
&\leq \int_0^T \left(\sum_{j=1}^N \left| \frac{\mathrm{d}}{\mathrm{d}t} a_j^{N,M} \right|^2 \right)^{\frac{1}{2}} \int_\Omega |\boldsymbol{S}^{N,M}| \left(\sum_{j=1}^N |\nabla \boldsymbol{w}_j|^2 \right)^{\frac{1}{2}} \mathrm{d}x \, \mathrm{d}t \\
&\leq \left(\int_0^T \left| \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{a}^{N,M} \right|^2 \mathrm{d}t \right)^{\frac{1}{2}} \left(\int_0^T \left(\int_\Omega |\boldsymbol{S}^{N,M}| \left(\sum_{j=1}^N |\nabla \boldsymbol{w}_j|^2 \right)^{\frac{1}{2}} \mathrm{d}x \right)^2 \mathrm{d}t \right)^{\frac{1}{2}} \\
&\leq C(N) \left(\int_0^T \left| \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{a}^{N,M} \right|^2 \mathrm{d}t \right)^{\frac{1}{2}} \left(\int_0^T \left(\int_\Omega |\boldsymbol{S}^{N,M}| \, \mathrm{d}x \right)^2 \mathrm{d}t \right)^{\frac{1}{2}}.
\end{aligned}$$

By equivalence of norms in finite-dimensional spaces,

$$\int_\Omega |\boldsymbol{S}^{N,M}| \, \mathrm{d}x \leq \int_\Omega \left(C_1 + C_2 |\boldsymbol{D}\boldsymbol{u}^{N,M}|^{p^+-1} \right) \mathrm{d}x \leq C + C(N) \|\boldsymbol{u}^{N,M}\|_2^{p^+-1}.$$

Hence, by (5.22), we have

$$\begin{aligned}
&\int_0^T \int_\Omega \boldsymbol{S}^{N,M} : \left(\sum_{j=1}^N \frac{\mathrm{d}}{\mathrm{d}t} a_j^{N,M} \nabla \boldsymbol{w}_j \right) \mathrm{d}x \, \mathrm{d}t \\
&\leq C(N) \left(\int_0^T \left| \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{a}^{N,M} \right|^2 \mathrm{d}t \right)^{\frac{1}{2}} \left(\int_0^T \left(\int_\Omega |\boldsymbol{S}^{N,M}| \, \mathrm{d}x \right)^2 \mathrm{d}t \right)^{\frac{1}{2}} \\
&\leq C(N) \left(\int_0^T \left| \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{a}^{N,M} \right|^2 \mathrm{d}t \right)^{\frac{1}{2}} \left(C + C(N) \int_0^T \|\boldsymbol{u}^{N,M}\|_2^{2(p^+-1)} \, \mathrm{d}t \right)^{\frac{1}{2}} \\
&\leq C(N) \left(\int_0^T \left| \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{a}^{N,M} \right|^2 \mathrm{d}t \right)^{\frac{1}{2}} \left(C + C(N) \sup_{t \in (0,T)} \|\boldsymbol{u}^{N,M}\|_2^{2(p^+-1)} \right)^{\frac{1}{2}} \\
&\leq C(N) \left(\int_0^T \left| \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{a}^{N,M} \right|^2 \mathrm{d}t \right)^{\frac{1}{2}}.
\end{aligned}$$

Finally for the external force term,

$$\begin{aligned}
&\int_0^T \int_\Omega \boldsymbol{f} \cdot \left(\sum_{j=1}^N \frac{\mathrm{d}}{\mathrm{d}t} a_j^{N,M} \boldsymbol{w}_j \right) \mathrm{d}x \, \mathrm{d}t \\
&\leq \int_0^T \int_\Omega |\boldsymbol{f}| \left(\sum_{j=1}^N \left| \frac{\mathrm{d}}{\mathrm{d}t} a_j^{N,M} \right|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^N |\boldsymbol{w}_j|^2 \right)^{\frac{1}{2}} \mathrm{d}x \, \mathrm{d}t \\
&\leq C(N) \int_0^T \left(\sum_{j=1}^N \left| \frac{\mathrm{d}}{\mathrm{d}t} a_j^{N,M} \right|^2 \right)^{\frac{1}{2}} \|\boldsymbol{f}\|_2 \, \mathrm{d}t \\
&\leq C(N) \|\boldsymbol{f}\|_{L^2(Q_T)} \left(\int_0^T \left| \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{a}^{N,M} \right|^2 \mathrm{d}t \right)^{\frac{1}{2}}.
\end{aligned}$$

Collecting all the terms above gives us the following uniform estimate, independent of M:

$$\int_0^T \left| \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{a}^{N,M} \right|^2 \mathrm{d}t \leq C(N). \tag{5.25}$$

As a last step, we shall derive an estimate for the time derivative of the approximate concentration. Let P_1^M be the orthogonal projection onto B_M with respect to the $W_0^{1,2}(\Omega)$ inner product. Then, for any $\varphi \in W_0^{1,2}(\Omega)$, we have from (5.18) that

$$\begin{aligned}
\left| \int_\Omega \partial_t c^{N,M} \varphi \, \mathrm{d}x \right| &= \left| \int_\Omega \partial_t c^{N,M} P_1^M \varphi \, \mathrm{d}x \right| \\
&\leq \int_\Omega |\boldsymbol{u}^{N,M} c^{N,M}| |\nabla P_1^M \varphi| \, \mathrm{d}x + \int_\Omega |\boldsymbol{q}_c^{N,M}| |\nabla P_1^M \varphi| \, \mathrm{d}x \\
&\leq \|\boldsymbol{u}^{N,M} c^{N,M}\|_2 \|P_1^M \varphi\|_{1,2} + \|\boldsymbol{q}_c^{N,M}\|_2 \|P_1^M \varphi\|_{1,2} \\
&\leq C \left(\|\boldsymbol{u}^{N,M} c^{N,M}\|_2 + \|\boldsymbol{q}_c^{N,M}\|_2 \right) \|\varphi\|_{1,2}.
\end{aligned}$$

Integrating over the time interval $(0, T)$ together with (5.22), (5.24), Sobolev embedding and norm equivalence in finite-dimensional spaces leads us to

$$\begin{aligned}
\int_0^T \|\partial_t c^{N,M}\|_{W^{-1,2}(\Omega)}^2 \, \mathrm{d}t &\leq C \int_0^T (\|\boldsymbol{u}^{N,M} c^{N,M}\|_2^2 \, \mathrm{d}t + \|\boldsymbol{q}_c^{N,M}\|_2^2) \, \mathrm{d}t \\
&\leq C \int_0^T \|\boldsymbol{u}^{N,M}\|_3^2 \|c^{N,M}\|_6^2 \, \mathrm{d}t + C \\
&\leq C(N) \int_0^T \|\boldsymbol{u}^{N,M}\|_2^2 \|\nabla c^{N,M}\|_2^2 \, \mathrm{d}t + C \\
&\leq C(N).
\end{aligned}$$

Therefore, we finally have

$$\|\partial_t c^{N,M}\|_{L^2(0,T;W^{-1,2}(\Omega))} \leq C(N). \tag{5.26}$$

5.4.3 The limit $M \to \infty$

Having shown the uniform estimates (5.22)–(5.26), we can establish some necessary convergence results for a selected (not relabelled) subsequence, as $M \to \infty$. First, by (5.25),

$$\boldsymbol{a}^{N,M} \rightharpoonup \boldsymbol{a}^N \text{ weakly in } W^{1,2}(0,T) \text{ and thus strongly in } C([0,T]), \tag{5.27}$$

from which we obtain the following uniform convergences results:

$$\boldsymbol{u}^{N,M} \to \boldsymbol{u}^N \text{ uniformly and } \boldsymbol{Du}^{N,M} \to \boldsymbol{Du}^N \text{ uniformly.} \tag{5.28}$$

Next, by (5.24) and (5.26),

$$c^{N,M} \rightharpoonup c^N \text{ weakly in } \left\{ z \in L^2(0,T; W_0^{1,2}(\Omega)),\ \partial_t z \in L^2(0,T; W^{-1,2}(\Omega)) \right\}. \tag{5.29}$$

Therefore by applying the Aubin–Lions lemma (Lemma 5.3.3), we deduce that

$$c^{N,M} \to c^N \text{ strongly in } L^2(Q_T), \tag{5.30}$$

which implies, for a (relabelled) subsequence, that

$$c^{N,M} \to c^N \text{ almost everywhere in } Q_T. \tag{5.31}$$

Next, from (5.28) and (5.31), we have that

$$\boldsymbol{S}^{N,M} \to \boldsymbol{S}^N := \boldsymbol{S}(c^N, \boldsymbol{Du}^N) \text{ almost everywhere in } Q_T.$$

Also, by (5.28) again, for sufficiently large $M \in \mathbb{N}$, we have that

$$|\boldsymbol{Du}^{N,M}| < 1 + |\boldsymbol{Du}^N| \text{ almost everywhere in } Q_T.$$

Therefore, by (5.6), we have, for sufficiently large $M \in \mathbb{N}$, that

$$\begin{aligned} |\boldsymbol{S}^{N,M}| &\le C|\boldsymbol{Du}^{N,M}|^{p(c^{N,M})-1} + C \\ &\le C(1 + |\boldsymbol{Du}^N|)^{p(c^{N,M})-1} + C \\ &\le C(1 + |\boldsymbol{Du}^N|)^{p^+-1} + C, \end{aligned}$$

and hence,

$$\|\boldsymbol{S}^{N,M}\|_{L^{(p^+)'}(Q_T)^{d\times d}} \le C.$$

Thus, by the dominated convergence theorem, we have

$$\boldsymbol{S}^{N,M} \to \boldsymbol{S}^N \text{ strongly in } L^{(p^+)'}(Q_T)^{d\times d}. \tag{5.32}$$

Finally, concerning $\boldsymbol{q}_c^{N,M} = \boldsymbol{K}(c^{N,M}, |\boldsymbol{D}\boldsymbol{u}^{N,M}|)\nabla c^{N,M}$, by (5.31), (5.28) and the dominated convergence theorem,

$$\boldsymbol{K}(c^{N,M}, |\boldsymbol{D}\boldsymbol{u}^{N,M}|) \to \boldsymbol{K}(c^N, |\boldsymbol{D}\boldsymbol{u}^N|) \text{ strongly in } L^2(Q_T).$$

Therefore, together with (5.29),

$$\boldsymbol{q}_c^{N,M} \rightharpoonup \boldsymbol{q}_c^N := \boldsymbol{q}_c(c^N, \nabla c^N, \boldsymbol{D}\boldsymbol{u}^N) \text{ weakly in } L^2(Q_T)^d. \tag{5.33}$$

The convergence results obtained above allow us to take the limit in our Galerkin approximation. First, by using (5.27), (5.28) and (5.32) we have

$$\int_\Omega \left(\partial_t \boldsymbol{u}^N \cdot \boldsymbol{w}_j - (\boldsymbol{u}^N \otimes \boldsymbol{u}^N) : \boldsymbol{D}\boldsymbol{w}_j + \boldsymbol{S}^N : \boldsymbol{D}\boldsymbol{w}_j\right) \mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w}_j \,\mathrm{d}x \;\; \forall j = 1, \dots, N. \tag{5.34}$$

Regarding the initial condition for the velocity, thanks to (5.27) and the fact that $\boldsymbol{a}^{N,M}(0) = \boldsymbol{a}_0^N$ for all M, if we denote $\lim_{M\to\infty} \boldsymbol{u}_0^{N,M} = \boldsymbol{u}_0^N = P^N \boldsymbol{u}_0$, we have

$$\boldsymbol{u}^N(\cdot, 0) = \boldsymbol{u}_0^N.$$

Next, from (5.28), (5.29), (5.33) and a density argument, we have for a.e. $t \in (0,T)$ that

$$\langle \partial_t c^N, \varphi\rangle - \int_\Omega \left(\boldsymbol{u}^N c^N \cdot \nabla\varphi - \boldsymbol{q}_c^N \cdot \nabla\varphi\right) \mathrm{d}x = 0 \qquad \forall \varphi \in W_0^{1,2}(\Omega). \tag{5.35}$$

We shall verify the attainment of the initial condition for the concentration. First, by (5.24) and (5.26), we have

$$c^N \in C([0,T]; L^2(\Omega)). \tag{5.36}$$

Next, we integrate (5.18) over $t \in (0,\tau)$, and pass M to the limit to get

$$\int_\Omega c^N(\tau) z_k \,\mathrm{d}x - \int_0^\tau \int_\Omega \left(\boldsymbol{u}^N c^N \cdot \nabla z_k - \boldsymbol{q}_c^N \cdot \nabla z_k\right) \mathrm{d}x\,\mathrm{d}t = \int_\Omega c_0 z_k \,\mathrm{d}x,$$

where we have used that $\|c_0^{N,M} - c_0\|_2 = \|P^M c_0 - c_0\|_2 \to 0$ as $M \to \infty$. Since $\int_\Omega \boldsymbol{u}^N c^N \cdot \nabla z_k \,\mathrm{d}x$ and $\int_\Omega \boldsymbol{q}_c^N \cdot \nabla z_k \,\mathrm{d}x$ are integrable in time, as $\tau \to 0$, we have

$$\int_\Omega c^N(\tau) z_k \,\mathrm{d}x \to \int_\Omega c_0 z_k \,\mathrm{d}x.$$

Hence, together with (5.36), we finally obtain

$$\lim_{t\to 0+}\|c^N(t)-c_0\|_2=0.$$

By (5.36) again, we deduce that

$$c^N(0)=c_0 \text{ in } L^2(\Omega).$$

5.4.4 Maximum and minimum principles

Before proceeding further, we shall derive maximum and minimum principles for the concentration by using standard arguments for parabolic problems.

We begin with the maximum principle. Let $\varphi:=\chi_{[0,\tau]}(t)\max\{0,c^N-\tilde{c}_0\}$ where $\tilde{c}_0$ is the upper bound on the initial datum c_0. Clearly $\varphi\in W_0^{1,2}(\Omega)$ for a.e. $t\in(0,T)$, so we test by φ in (5.35). Integrating over $t\in(0,T)$ yields

$$0=\int_0^T\langle\partial_t c^N,\varphi\rangle\,\mathrm{d}t-\int_0^T\int_\Omega\big(\boldsymbol{u}^N c^N\cdot\nabla\varphi-\boldsymbol{q}_c^N\cdot\nabla\varphi\big)\,\mathrm{d}x\,\mathrm{d}t=:\mathrm{I}+\mathrm{II}+\mathrm{III},$$

with an obvious numbering. We can then compute I, II and III as follows:

$$\begin{aligned}
\mathrm{I}&=\int_0^T\langle\partial_t c^N,\varphi\rangle\,\mathrm{d}t=\int_0^\tau\langle\partial_t\varphi,\varphi\rangle\,\mathrm{d}t=\frac{1}{2}\|\varphi(\tau)\|_2^2-\frac{1}{2}\|\varphi(0)\|_2^2,\\
\mathrm{II}&=-\int_0^T\int_\Omega\boldsymbol{u}^N c^N\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_\Omega\boldsymbol{u}^N\cdot\nabla c^N\varphi\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_0^T\int_\Omega\boldsymbol{u}^N\cdot\nabla\varphi\,\varphi\,\mathrm{d}x\,\mathrm{d}t=\frac{1}{2}\int_0^T\int_\Omega\boldsymbol{u}^N\cdot\nabla\,|\varphi|^2\,\mathrm{d}x\,\mathrm{d}t=0,\\
\mathrm{III}&=\int_0^T\int_\Omega\boldsymbol{q}_c^N\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_\Omega\boldsymbol{K}(c^N,|\boldsymbol{D}\boldsymbol{u}^N|)\nabla c^N\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_0^T\int_\Omega\boldsymbol{K}(c^N,|\boldsymbol{D}\boldsymbol{u}^N|)\nabla\varphi\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t\geq 0.
\end{aligned}$$

Altogether, we have $\|\varphi(\tau)\|_2\leq\|\varphi(0)\|_2=0$, which implies that

$$c^N(x,t)\leq\tilde{c}_0 \text{ a.e. in } Q_T. \tag{5.37}$$

In the case of the minimum principle, we choose $\varphi:=\chi_{[0,\tau]}(t)\min\{0,c^N\}$, which obviously belongs to $W_0^{1,2}(\Omega)$ for a.e. $t\in(0,T)$. If we proceed as above, we obtain that

$$0\leq c^N(x,t) \text{ a.e. in } Q_T. \tag{5.38}$$

Note that since convex sets are weakly closed and the set $\{x\in\mathbb{R}:x\geq 0\}$ is convex, weak and strong limits of c^N are still non-negative.

5.4.5 Uniform estimates independent of N

Now we shall derive some uniform estimates independent of N. We first multiply the jth equation in (5.34) by a_j^N, and then take the sum over $j = 1, \dots, N$. If we integrate the result over $(0, t)$, we have

$$\|\boldsymbol{u}^N(t)\|_2^2 + 2\int_0^t \int_\Omega \boldsymbol{S}^N : \boldsymbol{D}\boldsymbol{u}^N \,\mathrm{d}x\,\mathrm{d}\tau = \|\boldsymbol{u}_0^N\|_2^2 + 2\int_0^t \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u}^N \,\mathrm{d}x\,\mathrm{d}\tau. \qquad (5.39)$$

As before, a straightforward argument leads us to the following uniform estimate:

$$\sup_{t\in(0,T)} \|\boldsymbol{u}^N(t)\|_2^2 + \int_0^T \int_\Omega \left(|\nabla \boldsymbol{u}^N|^{p(c^N)} + |\boldsymbol{S}^N|^{p'(c^N)}\right) \mathrm{d}x\,\mathrm{d}t \le C. \qquad (5.40)$$

By Proposition 5.3.2, we have

$$\|\boldsymbol{u}^N\|_{L^{\frac{p^-(d+2)}{d}}(Q_T)} \le C, \qquad (5.41)$$

and thus

$$\|\boldsymbol{u}^N \otimes \boldsymbol{u}^N\|_{L^{\frac{p^-(d+2)}{2d}}(Q_T)} \le C. \qquad (5.42)$$

Furthermore, by weak and weak$-*$ lower semicontinuity of norms, we have from (5.24) that

$$\|c^N\|_{L^\infty(0,T;L^2(\Omega))} + \|c^N\|_{L^2(0,T;W_0^{1,2}(\Omega))} + \|\boldsymbol{q}_c^N\|_{L^2(0,T;L^2(\Omega)^d)} \le C. \qquad (5.43)$$

Now we shall derive a uniform Hölder estimate for the approximate concentrations c^N. Since $c_0 \in \{f \in C^{0,\alpha_0}(\overline{\Omega})$ for some $\alpha_0 \in (0,1) : f = 0$ on $\partial\Omega\}$ and $c^N(\cdot, 0) = c_0$ a.e. in Ω for all $N \in \mathbb{N}$, we have that for some $\alpha_0 \in (0,1)$ independent of $N \in \mathbb{N}$, $c^N(\cdot, 0) \in C^{0,\alpha_0}(\overline{\Omega})$ after possibly being redefined on a set of measure zero. Therefore we have $c^N_{|\Gamma_T} \in C^{\beta,\beta/2}(\Gamma_T)$ for some $\beta \in (0,1)$ independent of $N \in \mathbb{N}$. Moreover, since $p^- > d$, we can choose a positive integer m sufficiently close to $2+d$ so that $\frac{p^-(d+2)}{d} > m > 2+d$ and

$$\frac{2}{m} + \frac{d}{m} = 1 - \varepsilon_1 \text{ and } m > \frac{d}{1-\varepsilon_1} \text{ for some small } \varepsilon_1 > 0.$$

Next, as we see from (5.37) and (5.38) that c^N is uniformly bounded. Hence we have from (5.41) that

$$\|\boldsymbol{u}^N c^N\|_{L^m(Q_T)} \le C\|\boldsymbol{u}^N\|_{L^{\frac{p^-(d+2)}{d}}(Q_T)} \le C.$$

We then apply Theorem 5.3.5 to (5.35) with $m = r = q$ and $\boldsymbol{g} = \boldsymbol{u}^N c^N$. Therefore we have the following estimate:

$$\|c^N\|_{C^{\alpha_1,\alpha_1/2}(\overline{Q}_T)} \leq C, \tag{5.44}$$

where $\alpha_1 \in (0,1)$ and $C > 0$ is independent of N.

Finally, we shall derive a uniform estimate for $\partial_t \boldsymbol{u}^N$. For simplicity, we shall denote $\boldsymbol{H}^N := -\boldsymbol{S}^N + \boldsymbol{u}^N \otimes \boldsymbol{u}^N$ and let $q_0 := \min\left\{\frac{p^-(d+2)}{2d}, (p^+)'\right\} > 1$. Then, from (5.40) and (5.42), we obtain

$$\|\boldsymbol{H}^N\|_{L^{q_0}(Q_T)} \leq C.$$

Now let $\boldsymbol{\psi} \in W^{\ell,2}_{0,\mathrm{div}}(\Omega)^d$ and let P^N_ℓ denote the orthogonal projection into A_N with respect to the $W^{\ell,2}_0(\Omega)$-inner product. Then, from (5.34), we have

$$\begin{aligned}\left|\int_\Omega \partial_t \boldsymbol{u}^N \cdot \boldsymbol{\psi}\,\mathrm{d}x\right| &= \left|\int_\Omega \partial_t \boldsymbol{u}^N \cdot P^N_\ell \boldsymbol{\psi}\,\mathrm{d}x\right| \\ &\leq \int_\Omega |\boldsymbol{H}^N||\boldsymbol{D}P^N_\ell \boldsymbol{\psi}|\,\mathrm{d}x + \int_\Omega |\boldsymbol{f}||P^N_\ell \boldsymbol{\psi}|\,\mathrm{d}x \\ &\leq \|\boldsymbol{H}^N\|_{q_0}\|P^N_\ell \boldsymbol{\psi}\|_{1,q_0'} + \|\boldsymbol{f}\|_2\|P^N_\ell \boldsymbol{\psi}\|_2 \\ &\leq C\|\boldsymbol{H}^N\|_{q_0}\|P^N_\ell \boldsymbol{\psi}\|_{\ell,2} + \|\boldsymbol{f}\|_2\|P^N_\ell \boldsymbol{\psi}\|_{\ell,2} \\ &\leq C\left(\|\boldsymbol{H}^N\|_{q_0} + \|\boldsymbol{f}\|_2\right)\|\boldsymbol{\psi}\|_{\ell,2}.\end{aligned}$$

Therefore we have that

$$\begin{aligned}\int_0^T \|\partial_t \boldsymbol{u}^N\|^{q_0}_{W^{-\ell,2}_{\mathrm{div}}(\Omega)}\,\mathrm{d}t &\leq C\int_0^T \|\boldsymbol{H}^N\|^{q_0}_{q_0}\,\mathrm{d}t + \int_0^T \|\boldsymbol{f}\|^{q_0}_2\,\mathrm{d}t \\ &\leq C\left(\|\boldsymbol{H}^N\|^{q_0}_{L^{q_0}(Q_T)} + \|\boldsymbol{f}\|^2_{L^2(Q_T)}\right) \leq C,\end{aligned}$$

which means that

$$\|\partial_t \boldsymbol{u}^N\|_{L^{q_0}(0,T;W^{-\ell,2}_{\mathrm{div}}(\Omega)^d)} \leq C. \tag{5.45}$$

5.4.6 The limit $N \to \infty$

By using the uniform estimates derived in the previous subsection, we can now pass N to ∞. First of all, from (5.40) we have

$$\boldsymbol{u}^N \rightharpoonup \boldsymbol{u} \text{ weakly in } L^{p^-}(0,T;W^{1,p^-}_{0,\mathrm{div}}(\Omega)^d). \tag{5.46}$$

Furthermore, from (5.40) and (5.45) together with the Aubin–Lions compactness lemma (Lemma 5.3.3), we have $\boldsymbol{u}^N \to \boldsymbol{u}$ strongly in $L^2(Q_T)^d$. Therefore, using (5.41) with an interpolation inequality yields

$$\boldsymbol{u}^N \to \boldsymbol{u} \text{ strongly in } L^s(Q_T)^d \qquad \forall s \in [1, p^-(d+2)/d). \tag{5.47}$$

From (5.45), we also have

$$\partial_t \boldsymbol{u}^N \rightharpoonup \partial_t \boldsymbol{u} \text{ weakly in } L^{q_0}(0,T; W^{-\ell,2}_{\mathrm{div}}(\Omega)^d). \tag{5.48}$$

By (5.40) again, we obtain

$$\boldsymbol{S}^N \rightharpoonup \bar{\boldsymbol{S}} \text{ weakly in } L^{(p^+)'}(Q_T)^{d\times d}, \tag{5.49}$$

for some $\bar{\boldsymbol{S}} \in \mathbb{R}^{d\times d}$.

Next, we move to the convergence of the sequence of approximate concentrations. First, by (5.43), we have

$$c^N \rightharpoonup c \text{ weakly in } L^2(0,T; W^{1,2}_0(\Omega)). \tag{5.50}$$

Furthermore, from (5.44) with Proposition 5.3.1, we obtain

$$c^N \to c \text{ strongly in } C^{\alpha_2, \alpha_2/2}(\overline{Q}_T) \tag{5.51}$$

for some $\alpha_2 \in (0, \alpha_1)$. Using (5.43) again yields

$$\boldsymbol{q}_c^N \rightharpoonup \bar{\boldsymbol{q}}_c \text{ weakly in } L^2(Q_T)^d \tag{5.52}$$

for some $\bar{\boldsymbol{q}}_c \in \mathbb{R}^d$.

Now we are ready to pass N to ∞ in the second level of our Galerkin approximations (5.34) and (5.35). First, we fix $\boldsymbol{v} = g\boldsymbol{w}_j$ where $g \in C^1_0([0,T))$. Then, by (5.47) and (5.49), we have

$$\int_{Q_T} \bar{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{v} \,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} \left(\boldsymbol{f}\cdot\boldsymbol{v} + (\boldsymbol{u}\otimes\boldsymbol{u}) : \boldsymbol{D}\boldsymbol{v} + \boldsymbol{u}\cdot\partial_t\boldsymbol{v}\right) \mathrm{d}x\,\mathrm{d}t \tag{5.53}$$

$$+ \int_\Omega \boldsymbol{u}_0(x)\cdot\boldsymbol{v}(x,0)\,\mathrm{d}x, \tag{5.54}$$

where we have used

$$\int_{Q_T} \partial_t \boldsymbol{u}^N \cdot \boldsymbol{v} \,\mathrm{d}x\,\mathrm{d}t = -\int_\Omega \boldsymbol{u}_0^N \cdot \boldsymbol{v}(x,0)\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} \boldsymbol{u}^N \cdot \partial_t \boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}t$$

and $\boldsymbol{u}_0^N \to \boldsymbol{u}_0$ strongly in $L^2(\Omega)^d$. Since the class of all test functions which are finite linear combinations of functions that can be factorized in space and time is dense in the corresponding Bochner space, we have

$$\int_{Q_T} \bar{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{\psi} \,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} (\boldsymbol{f} \cdot \boldsymbol{\psi} + (\boldsymbol{u} \otimes \boldsymbol{u}) : \boldsymbol{D}\boldsymbol{\psi} + \boldsymbol{u} \cdot \partial_t \boldsymbol{\psi}) \,\mathrm{d}x\,\mathrm{d}t \tag{5.55}$$

$$+ \int_{\Omega} \boldsymbol{u}_0(x) \cdot \boldsymbol{\psi}(x, 0) \,\mathrm{d}x, \tag{5.56}$$

for all $\boldsymbol{\psi} \in C^{\infty}_{0,\mathrm{div}}(\Omega \times [0, T))^d$.

With the same argument, (5.35) together with (5.47), (5.51) and (5.52) implies

$$\int_{Q_T} \bar{\boldsymbol{q}}_c \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} (c\,\partial_t\varphi + c\boldsymbol{u} \cdot \nabla\varphi) \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega} c_0(x)\varphi(x, 0) \,\mathrm{d}x$$

for all $\varphi \in C_0^{\infty}(\Omega \times [0, T))$.

Finally, we shall show that the limit function $\boldsymbol{u}$ is contained in the desired solution space $W_{p(c)}(Q_T)$. By (5.51) and the continuity of p,

$$\forall\varepsilon > 0,\ \exists\hat{N} \in \mathbb{N} \text{ such that } N \geq \hat{N} \text{ implies } |p(c^N) - p(c)| < \frac{\varepsilon}{\theta},$$

where $\theta > 1$ is sufficiently large so that $p(c) - \frac{\theta+1}{\theta}\varepsilon > 1$. Then we have from (5.40) that

$$\begin{aligned} C &\geq \int_{Q_T} |\nabla\boldsymbol{u}^N|^{p(c^N)} \,\mathrm{d}x\,\mathrm{d}t \geq \int_{\{|\nabla\boldsymbol{u}^N| \geq 1\}} |\nabla\boldsymbol{u}^N|^{p(c^N)} \,\mathrm{d}x\,\mathrm{d}t \\ &\geq \int_{\{|\nabla\boldsymbol{u}^N| \geq 1\}} |\nabla\boldsymbol{u}^N|^{p(c^N) - p(c) + p(c) - \varepsilon} \,\mathrm{d}x\,\mathrm{d}t \geq \int_{\{|\nabla\boldsymbol{u}^N| \geq 1\}} |\nabla\boldsymbol{u}^N|^{p(c) - \frac{\theta+1}{\theta}\varepsilon} \,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Thus, we obtain that

$$\begin{aligned} \int_{Q_T} |\nabla\boldsymbol{u}^N|^{p(c) - \frac{\theta+1}{\theta}\varepsilon} \,\mathrm{d}x\,\mathrm{d}t &= \int_{\{|\nabla\boldsymbol{u}^N| \geq 1\}} |\nabla\boldsymbol{u}^N|^{p(c) - \frac{\theta+1}{\theta}\varepsilon} \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_{\{|\nabla\boldsymbol{u}^N| < 1\}} |\nabla\boldsymbol{u}^N|^{p(c) - \frac{\theta+1}{\theta}\varepsilon} \,\mathrm{d}x\,\mathrm{d}t \leq C. \end{aligned}$$

We can then further extract a (not relabelled) subsequence such that

$$\nabla\boldsymbol{u}^N \rightharpoonup \nabla\boldsymbol{u} \text{ weakly in } L^{p(c) - \frac{\theta+1}{\theta}\varepsilon}(Q_T)^{d \times d}.$$

Then, weak lower-semicontinuity leads us to

$$\int_{Q_T} |\nabla\boldsymbol{u}|^{p(c) - \frac{\theta+1}{\theta}\varepsilon} \,\mathrm{d}x\,\mathrm{d}t \leq C,$$

and thus Fatou's Lemma with $\varepsilon \to 0$ yields

$$\int_{Q_T} |\nabla \boldsymbol{u}|^{p(c)} \,\mathrm{d}x\,\mathrm{d}t \le C. \tag{5.57}$$

Similarly, we can also deduce that

$$\int_{Q_T} |\bar{\boldsymbol{S}}|^{p'(c)} \,\mathrm{d}x\,\mathrm{d}t \le C. \tag{5.58}$$

To apply monotone operator theory in the next subsection, the limit function $\boldsymbol{u}$ should be an admissible test function in (5.55). This is where the control of the convective term is needed, and thus the value of p^- should be restricted. Let us fix $\boldsymbol{v} = g\boldsymbol{w}$ where $g \in C^1([0,T])$ and $\boldsymbol{w} \in \{\boldsymbol{w}_j\}_{j=1}^{\infty}$. Then, from (5.34), we have

$$\int_{Q_T} \partial_t \boldsymbol{u}^N \cdot \boldsymbol{v} \,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} \big((\boldsymbol{u}^N \otimes \boldsymbol{u}^N) : \nabla \boldsymbol{v} - \boldsymbol{S}^N : \nabla \boldsymbol{v} + \boldsymbol{f} \cdot \boldsymbol{v} \big) \,\mathrm{d}x\,\mathrm{d}t.$$

From (5.48), (5.47) and (5.49), we have

$$\int_0^T \langle \partial_t \boldsymbol{u}, \boldsymbol{v} \rangle = \int_{Q_T} \big((\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{v} - \bar{\boldsymbol{S}} : \nabla \boldsymbol{v} + \boldsymbol{f} \cdot \boldsymbol{v} \big) \,\mathrm{d}x\,\mathrm{d}t. \tag{5.59}$$

Now let $\boldsymbol{\phi} \in W_{p(c)}(Q_T)$. Then, by Lemma 5.3.7 and (5.59), there exists a subset $\{\boldsymbol{\phi}_k\}_{k=1}^{\infty} \subset \operatorname{span}\{g\boldsymbol{w} : g \in C^1([0,T]) \text{ and } \boldsymbol{w} \in \{\boldsymbol{w}_j\}_{j=1}^{\infty}\}$ such that $\|\boldsymbol{\phi}_k - \boldsymbol{\phi}\|_{W_{p(c)}(Q_T)} \to 0$ as $k \to \infty$ and

$$\int_0^T \langle \partial_t \boldsymbol{u}, \boldsymbol{\phi}_k \rangle = \int_{Q_T} \big((\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{\phi}_k - \bar{\boldsymbol{S}} : \nabla \boldsymbol{\phi}_k + \boldsymbol{f} \cdot \boldsymbol{\phi}_k \big) \,\mathrm{d}x\,\mathrm{d}t. \tag{5.60}$$

Since $p^- > d \ge \frac{3d+2}{d+2}$ for $d \ge 2$, we have from (5.41) that

$$\|\boldsymbol{u} \otimes \boldsymbol{u}\|_{L^{p'(c)}(Q_T)} \le \|\boldsymbol{u}\|^2_{L^{2p'(c)}(Q_T)} \le C\|\boldsymbol{u}\|^2_{L^{2(p^-)'}(Q_T)} \le C\|\boldsymbol{u}\|^2_{L^{\frac{p^-(d+2)}{d}}(Q_T)} \le C,$$

and hence we can pass the first term on the right-hand of (5.60) to the limit. Also, by (5.58), we may pass k to the limit in the second term and the convergence of the third term is trivial. This means that the left-hand side also has a limit as $k \to \infty$. Therefore, we obtain that

$$\int_0^T \langle \partial_t \boldsymbol{u}, \boldsymbol{\phi} \rangle = \int_{Q_T} \big((\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{\phi} - \bar{\boldsymbol{S}} : \nabla \boldsymbol{\phi} + \boldsymbol{f} \cdot \boldsymbol{\phi} \big) \,\mathrm{d}x\,\mathrm{d}t \qquad \forall \boldsymbol{\phi} \in W_{p(c)}(Q_T), \tag{5.61}$$

and

$$\partial_t \boldsymbol{u} \in (W_{p(c)}(Q_T))^*. \tag{5.62}$$

Next, we claim that $\boldsymbol{u} \in C_{\mathrm{w}}([0,T]; W^{-1,(p^+)'}(\Omega)^d)$. Since we know the fact that

$$\sup_{t\in(0,T)} \|\boldsymbol{u}\|_{W^{-1,(p^+)'}} \le C \sup_{t\in(0,T)} \|\boldsymbol{u}(t)\|_2 \le C,$$

it is enough to show that $\int_\Omega (\boldsymbol{u}(t_1) - \boldsymbol{u}(t_0)) \cdot \boldsymbol{\phi} \,\mathrm{d}x \to 0$ as $t_1 \to t_0$ for arbitrary $\boldsymbol{\phi} \in W^{1,p^+}(\Omega)^d$. Note that

$$\begin{aligned}
\int_\Omega (\boldsymbol{u}(t_1) - \boldsymbol{u}(t_0)) \cdot \boldsymbol{\phi} \,\mathrm{d}x &= \int_{t_0}^{t_1} \langle \partial_t \boldsymbol{u}, \boldsymbol{\phi} \rangle \,\mathrm{d}t \\
&= \int_{t_0}^{t_1} \int_\Omega \big((\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{\phi} - \bar{\boldsymbol{S}} : \nabla \boldsymbol{\phi} + \boldsymbol{f} \cdot \boldsymbol{\phi} \big) \,\mathrm{d}x \,\mathrm{d}t \\
&\le \int_{t_0}^{t_1} \Big(\|\boldsymbol{u}\|_{2(p^-)'}^2 + \|\bar{\boldsymbol{S}}\|_{(p^+)'} + \|\boldsymbol{f}\|_2 \Big) \|\boldsymbol{\phi}\|_{1,p^+} \,\mathrm{d}t \\
&\le \Big(\|\boldsymbol{u}\|_{L^{2(p^-)'}(Q_T)}^2 + \|\bar{\boldsymbol{S}}\|_{L^{(p^+)'}(Q_T)} + \|\boldsymbol{f}\|_{L^2(Q_T)} \Big) \\
&\quad \times |t_1 - t_0|^{\frac{1}{p^+}} \|\boldsymbol{\phi}\|_{1,p^+}.
\end{aligned}$$

Since the last term goes to 0 as $t_1 \to t_0$, we deduce that $\boldsymbol{u} \in C_{\mathrm{w}}([0,T]; W^{-1,(p^+)'}(\Omega)^d)$. From this claim together with the fact $\|\boldsymbol{u}\|_{L^\infty(0,T;L^2(\Omega)^d)} \le C$, we further deduce that

$$\boldsymbol{u} \in C_{\mathrm{w}}([0,T]; L^2(\Omega)^d). \tag{5.63}$$

Now, for fixed $t_0 \in [0,T]$, we have from (5.61) that

$$\begin{aligned}
\|\boldsymbol{u}(t) - \boldsymbol{u}(t_0)\|_2^2 &= \|\boldsymbol{u}(t)\|_2^2 + \|\boldsymbol{u}(t_0)\|_2^2 - 2\int_\Omega \boldsymbol{u}(t) \cdot \boldsymbol{u}(t_0) \,\mathrm{d}x \\
&= \|\boldsymbol{u}(t)\|_2^2 - \|\boldsymbol{u}(t_0)\|_2^2 - 2\int_\Omega (\boldsymbol{u}(t) - \boldsymbol{u}(t_0)) \cdot \boldsymbol{u}(t_0) \,\mathrm{d}x \\
&= \int_{t_0}^{t} \int_\Omega (\boldsymbol{f} \cdot \boldsymbol{u} - \bar{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{u}) \,\mathrm{d}x \,\mathrm{d}t - 2\int_\Omega (\boldsymbol{u}(t) - \boldsymbol{u}(t_0)) \cdot \boldsymbol{u}(t_0) \,\mathrm{d}x.
\end{aligned}$$

Since $\int_\Omega (\boldsymbol{f} \cdot \boldsymbol{u} - \bar{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{u}) \,\mathrm{d}x$ is integrable on $[0,T]$ and $\boldsymbol{u}(t_0) \in L^2(\Omega)^d$, we can conclude that

$$\|\boldsymbol{u}(t) - \boldsymbol{u}(t_0)\|_2^2 \to 0 \ \text{ as } \ t \to t_0,$$

which means that

$$\boldsymbol{u} \in C([0,T]; L^2(\Omega)^d). \tag{5.64}$$

We next verify the regularity for the time derivative of the approximate concentrations. From (5.35), we have

$$|\langle \partial_t c^N, \varphi \rangle| = \left| \int_\Omega (\boldsymbol{u}^N c^N \cdot \nabla \varphi - \boldsymbol{q}_c^N \cdot \nabla \varphi) \,\mathrm{d}x \right| \le (\|\boldsymbol{u}^N c^N\|_2 + \|\boldsymbol{q}_c^N\|_2) \|\varphi\|_{1,2}.$$

By using (5.37), (5.38) and the fact $\frac{2d}{d+2} < \frac{3d+2}{d+2} \le p^-$, we obtain

$$\int_0^T \|\partial_t c^N\|^2_{W^{-1,2}(\Omega)} \, \mathrm{d}t \le C \int_0^T (\|\boldsymbol{u}^N c^N\|_2^2 \, \mathrm{d}t + \|\boldsymbol{q}_c^N\|_2^2) \, \mathrm{d}t \le C \int_0^T \|\boldsymbol{u}^N\|_2^2 \, \mathrm{d}t + C \le C.$$

Therefore, we finally have

$$\|\partial_t c^N\|_{L^2(0,T;W^{-1,2}(\Omega))} \le C,$$

which implies

$$\partial_t \, c \in L^2(0,T;W^{-1,2}(\Omega)).$$

Hence, we can now conclude that

$$c \in C([0,T];L^2(\Omega)). \tag{5.65}$$

As a final step, to complete the proof, it remains to show that

$$\bar{\boldsymbol{S}} = \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u}) \qquad \text{and} \qquad \bar{\boldsymbol{q}}_c = \boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u}).$$

These equalities will be verified in the next subsection.

5.4.7 Identification of $\bar{\boldsymbol{S}} = \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u})$ and $\bar{\boldsymbol{q}}_c = \boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u})$

In this subsection, we shall identify the limit functions to complete the proof. We first note that from (5.34) we have

$$\begin{aligned}\int_{Q_T} \boldsymbol{f} \cdot \boldsymbol{u}^N \, \mathrm{d}x \, \mathrm{d}t &= \int_{Q_T} \big(\partial_t \boldsymbol{u}^N \cdot \boldsymbol{u}^N - (\boldsymbol{u}^N \otimes \boldsymbol{u}^N) : \boldsymbol{D}\boldsymbol{u}^N + \boldsymbol{S}^N : \boldsymbol{D}\boldsymbol{u}^N\big) \, \mathrm{d}x \, \mathrm{d}t \\ &= \frac{1}{2} \int_\Omega |\boldsymbol{u}^N(T)|^2 \, \mathrm{d}x - \frac{1}{2} \int_\Omega |\boldsymbol{u}^N(0)|^2 \, \mathrm{d}x + \int_{Q_T} \boldsymbol{S}^N : \boldsymbol{D}\boldsymbol{u}^N \, \mathrm{d}x \, \mathrm{d}t,\end{aligned}$$

where we have used that $\operatorname{div} \boldsymbol{u}^N = 0$. Also, by (5.62), we can test by $\boldsymbol{u}$ in (5.61). Therefore, as $\operatorname{div} \boldsymbol{u} = 0$, we have that

$$\begin{aligned}\int_{Q_T} \boldsymbol{f} \cdot \boldsymbol{u} \, \mathrm{d}x \, \mathrm{d}t &= \int_0^T \langle \partial_t \boldsymbol{u}, \boldsymbol{u} \rangle \, \mathrm{d}t - \int_{Q_T} \big((\boldsymbol{u} \otimes \boldsymbol{u}) : \boldsymbol{D}\boldsymbol{u} - \bar{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{u}\big) \, \mathrm{d}x \, \mathrm{d}t \\ &= \frac{1}{2} \int_\Omega |\boldsymbol{u}(T)|^2 \, \mathrm{d}x - \frac{1}{2} \int_\Omega |\boldsymbol{u}(0)|^2 \, \mathrm{d}x + \int_{Q_T} \bar{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{u} \, \mathrm{d}x \, \mathrm{d}t.\end{aligned}$$

Therefore, it is straightforward to see that

$$\begin{aligned}&\int_{Q_T} \boldsymbol{S}^N : \boldsymbol{D}\boldsymbol{u}^N \, \mathrm{d}x \, \mathrm{d}t - \int_{Q_T} \bar{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{u} \, \mathrm{d}x \, \mathrm{d}t \\ &\qquad = \int_{Q_T} \boldsymbol{f} \cdot \boldsymbol{u}^N \, \mathrm{d}x \, \mathrm{d}t - \int_{Q_T} \boldsymbol{f} \cdot \boldsymbol{u} \, \mathrm{d}x \, \mathrm{d}t \\ &\qquad\quad - \frac{1}{2} \int_\Omega |\boldsymbol{u}^N(T) - \boldsymbol{u}(T)|^2 \, \mathrm{d}x - \int_\Omega \boldsymbol{u}^N(T) \cdot \boldsymbol{u}(T) \, \mathrm{d}x \\ &\qquad\quad + \int_\Omega |\boldsymbol{u}(T)|^2 \, \mathrm{d}x + \frac{1}{2} \int_\Omega |P^N \boldsymbol{u}_0|^2 \, \mathrm{d}x - \frac{1}{2} \int_\Omega |\boldsymbol{u}_0|^2 \, \mathrm{d}x.\end{aligned}$$

It is obvious that $\int_{Q_T} \boldsymbol{f} \cdot \boldsymbol{u}^N \,\mathrm{d}x\,\mathrm{d}t \to \int_{Q_T} \boldsymbol{f} \cdot \boldsymbol{u} \,\mathrm{d}x\,\mathrm{d}t$ and $\|P^N \boldsymbol{u}_0 - \boldsymbol{u}_0\|_2 \to 0$. Also, (5.63) implies that $\boldsymbol{u}^N(T) \rightharpoonup \boldsymbol{u}(T)$ weakly in $L^2(\Omega)$. Thus we have

$$\limsup_{N\to\infty} \int_{Q_T} \boldsymbol{S}^N : \boldsymbol{D}\boldsymbol{u}^N \,\mathrm{d}x\,\mathrm{d}t \le \int_{Q_T} \bar{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{u} \,\mathrm{d}x\,\mathrm{d}t. \tag{5.66}$$

Next, by (5.51) and the dominated convergence theorem, we obtain for arbitrary but fixed $\boldsymbol{\phi} \in C^\infty([0,T]; C_0^\infty(\Omega)^d)$ that

$$\boldsymbol{S}(c^N, \boldsymbol{D}\boldsymbol{\phi}) \to \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{\phi}) \text{ strongly in } L^q(Q_T)^{d\times d} \text{ for all } q \in (1,\infty). \tag{5.67}$$

Then, by the monotonicity (5.7), we deduce for any $\boldsymbol{\phi} \in C^\infty([0,T]; C_0^\infty(\Omega)^d)$ that

$$\begin{aligned}
0 &\le \limsup_{N\to\infty} \int_{Q_T} \big(\boldsymbol{S}(c^N, \boldsymbol{D}\boldsymbol{u}^N) - \boldsymbol{S}(c^N, \boldsymbol{D}\boldsymbol{\phi})\big) : (\boldsymbol{D}\boldsymbol{u}^N - \boldsymbol{D}\boldsymbol{\phi}) \,\mathrm{d}x\,\mathrm{d}t \\
&= \limsup_{N\to\infty} \int_{Q_T} (\boldsymbol{S}(c^N, \boldsymbol{D}\boldsymbol{u}^N) : \boldsymbol{D}\boldsymbol{u}^N - \boldsymbol{S}(c^N, \boldsymbol{D}\boldsymbol{u}^N) : \boldsymbol{D}\boldsymbol{\phi} \\
&\quad - \boldsymbol{S}(c^N, \boldsymbol{D}\boldsymbol{\phi}) : \boldsymbol{D}\boldsymbol{u}^N + \boldsymbol{S}(c^N, \boldsymbol{D}\boldsymbol{\phi}) : \boldsymbol{D}\boldsymbol{\phi}) \,\mathrm{d}x\,\mathrm{d}t \\
&\le \int_{Q_T} \big(\bar{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{u} - \bar{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{\phi} - \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{\phi}) : \boldsymbol{D}\boldsymbol{u} + \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{\phi}) : \boldsymbol{D}\boldsymbol{\phi}\big) \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{Q_T} \big(\bar{\boldsymbol{S}} - \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{\phi})\big) : (\boldsymbol{D}\boldsymbol{u} - \boldsymbol{D}\boldsymbol{\phi}) \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Now we use Minty's trick. Since $C^\infty([0,T]; C_0^\infty(\Omega)^d)$ is dense in $W_{p(c)}(Q_T)$ (see, for example, Lemma 1.15 in [6] where the mollification argument was used to show this fact), we can set $\boldsymbol{\phi} = \boldsymbol{u} \pm \lambda \boldsymbol{v}$ with $\lambda > 0$ and $\boldsymbol{v} \in C^\infty([0,T]; C_0^\infty(\Omega)^d)$. By passing to the limit $\lambda \to 0$ and using the continuity of $\boldsymbol{S}$, we deduce the desired identification $\bar{\boldsymbol{S}} = \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u})$.

Next, we shall show the almost everywhere convergence of $\boldsymbol{D}\boldsymbol{u}^N$ to $\boldsymbol{D}\boldsymbol{u}$. Let $\boldsymbol{B} \in \mathbb{R}^{d\times d}$ be a bounded symmetric function. Then with the same procedure as above and the identification $\bar{\boldsymbol{S}} = \boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u})$, we have that

$$\begin{aligned}
0 &\le \limsup_{N\to\infty} \int_{Q_T} \big(\boldsymbol{S}(c^N, \boldsymbol{D}\boldsymbol{u}^N) - \boldsymbol{S}(c^N, \boldsymbol{B})\big) : (\boldsymbol{D}\boldsymbol{u}^N - \boldsymbol{B}) \,\mathrm{d}x\,\mathrm{d}t \\
&\le \int_{Q_T} (\boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u}) - \boldsymbol{S}(c, \boldsymbol{B})) : (\boldsymbol{D}\boldsymbol{u} - \boldsymbol{B}) \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

We set $\boldsymbol{B} = \boldsymbol{D}\boldsymbol{u}\chi_{\{|\boldsymbol{D}\boldsymbol{u}|<\lambda\}}$ for $\lambda > 0$ and let $0 < k < \lambda$. Then we have

$$\begin{aligned}
\limsup_{N\to\infty} \int_{\{|\boldsymbol{D}\boldsymbol{u}|<k\}} & \big(\boldsymbol{S}(c^N, \boldsymbol{D}\boldsymbol{u}^N) - \boldsymbol{S}(c^N, \boldsymbol{D}\boldsymbol{u})\big) : (\boldsymbol{D}\boldsymbol{u}^N - \boldsymbol{D}\boldsymbol{u}) \,\mathrm{d}x\,\mathrm{d}t \\
&= \limsup_{N\to\infty} \int_{\{|\boldsymbol{D}\boldsymbol{u}|<k\}} \big(\boldsymbol{S}(c^N, \boldsymbol{D}\boldsymbol{u}^N) - \boldsymbol{S}(c^N, \boldsymbol{B})\big) : (\boldsymbol{D}\boldsymbol{u}^N - \boldsymbol{B}) \,\mathrm{d}x\,\mathrm{d}t \\
&\le \limsup_{N\to\infty} \int_{Q_T} \big(\boldsymbol{S}(c^N, \boldsymbol{D}\boldsymbol{u}^N) - \boldsymbol{S}(c^N, \boldsymbol{B})\big) : (\boldsymbol{D}\boldsymbol{u}^N - \boldsymbol{B}) \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{Q_T} (\boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u}) - \boldsymbol{S}(c, \boldsymbol{B})) : (\boldsymbol{D}\boldsymbol{u} - \boldsymbol{B}) \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{\{|\boldsymbol{D}\boldsymbol{u}|>\lambda\}} (\boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u}) - \boldsymbol{S}(c, \boldsymbol{B})) : (\boldsymbol{D}\boldsymbol{u} - \boldsymbol{B}) \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Since the first integral is independent of $\lambda > 0$, passing λ to ∞ gives us that

$$\limsup_{N\to\infty} \int_{\{|\boldsymbol{D}\boldsymbol{u}|<k\}} \big(\boldsymbol{S}(c^N, \boldsymbol{D}\boldsymbol{u}^N) - \boldsymbol{S}(c^N, \boldsymbol{D}\boldsymbol{u})\big) : (\boldsymbol{D}\boldsymbol{u}^N - \boldsymbol{D}\boldsymbol{u}) \,\mathrm{d}x\,\mathrm{d}t = 0.$$

Since c^N converges to c uniformly and $\boldsymbol{S}$ is strictly monotone, the only way the above can be true is that $\boldsymbol{D}\boldsymbol{u}^N$ tends to $\boldsymbol{D}\boldsymbol{u}$ almost everywhere in the set $\{|\boldsymbol{D}\boldsymbol{u}| < k\}$. If we pass k to ∞, we finally have

$$\boldsymbol{D}\boldsymbol{u}^N \to \boldsymbol{D}\boldsymbol{u} \;\text{ a.e. in }\; Q_T. \tag{5.68}$$

Now by (5.51), (5.68) and the dominated convergence theorem,

$$\boldsymbol{K}(c^N, |\boldsymbol{D}\boldsymbol{u}^N|) \to \boldsymbol{K}(c, |\boldsymbol{D}\boldsymbol{u}|) \;\text{ strongly in }\; L^q(Q_T) \;\text{ for all }\; q \in (1, \infty).$$

Finally, together with (5.50), we obtain that

$$\boldsymbol{q}_c^N \rightharpoonup \boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u}) \;\text{ weakly in }\; L^2(Q_T)^d.$$

Therefore, by the uniqueness of the weak limit, we have

$$\bar{\boldsymbol{q}}_c = \boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u}),$$

which completes the proof of Theorem 5.2.1.

5.5 Regularized problem

In this section, we shall discuss a possibility to improve our result. Since the condition $p^- > d$ is the minimum requirement to apply the parabolic De Giorgi–Nash–Moser regularity theorem, we need to find another approach, which avoids the use of

De Giorgi–Nash–Moser theory. One possibility is to consider a regularized problem. More precisely, we consider the standard mollifier $\eta(z) = \eta(x,t)$ in $\mathbb{R}^{d+1}$ and define

$$\eta_\varepsilon(z) = \frac{1}{\varepsilon^{d+1}}\eta\left(\frac{z}{\varepsilon}\right)$$

for each $\varepsilon > 0$. Then we define the stress tensor with a regularized power-law exponent

$$\boldsymbol{S}_\varepsilon(c, \boldsymbol{Du}) = \nu_0(\kappa_1 + \kappa_2|\boldsymbol{Du}|^2)^{\frac{p(\eta_\varepsilon * c)-2}{2}}\boldsymbol{Du},$$

where $*$ is a convolution operator with respect to both space and time variables. Then we proceed using a similar argument to the one in the previous section. We begin with a Galerkin approximation of the regularized problem. Since z_j has zero trace, we can define extensions by zero so that $z_j \in W^{1,2}(\mathbb{R}^d)$ for all $j = 1, \ldots, M$. For sufficiently small $\varepsilon > 0$, we then look for a sequence of approximate solutions $\{\boldsymbol{u}^{N,M},\, c^{N,M}\}_{N,M=1}^\infty$ of the form

$$\boldsymbol{u}^{N,M} = \sum_{i=1}^{N} a_i^{N,M}\boldsymbol{w}_i, \qquad c^{N,M} = \sum_{i=1}^{M} b_i^{N,M} z_i,$$

where $\boldsymbol{a}^{N,M} = \{a_j^{N,M}\}_{j=1}^N : (0,T) \to \mathbb{R}^N$ and $\boldsymbol{b}^{N,M} = \{b_k^{N,M}\}_{k=1}^M : (0,T) \to \mathbb{R}^M$, which solve the following system of ordinary differential equations:

$$\int_\Omega \left(\partial_t \boldsymbol{u}^{N,M} \cdot \boldsymbol{w}_j - (\boldsymbol{u}^{N,M} \otimes \boldsymbol{u}^{N,M}) : \boldsymbol{Dw}_j + \boldsymbol{S}_\varepsilon^{N,M} : \boldsymbol{Dw}_j\right) \mathrm{d}x$$
$$= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w}_j \,\mathrm{d}x \ \ \forall j = 1, \ldots, N,$$

$$\boldsymbol{u}^{N,M}(\cdot, 0) = \boldsymbol{u}_0^{N,M} = P^N \boldsymbol{u}_0,$$

$$\int_\Omega \partial_t c^{N,M}\, z_k \,\mathrm{d}x - \int_\Omega \boldsymbol{u}^{N,M} c^{N,M} \cdot \nabla z_k \,\mathrm{d}x + \int_\Omega \boldsymbol{q}_c^{N,M} \cdot \nabla z_k \,\mathrm{d}x = 0 \ \ \forall k = 1, \ldots M,$$

$$c^{N,M}(\cdot, 0) = c_0^{N,M} = P^M c_0,$$

where

$$\boldsymbol{S}_\varepsilon^{N,M} = \boldsymbol{S}_\varepsilon(c^{N,M}, \boldsymbol{Du}^{N,M}), \qquad \boldsymbol{q}_c^{N,M} = \boldsymbol{q}_c(c^{N,M}, \nabla c^{N,M}, \boldsymbol{Du}^{N,M}).$$

The existence of approximate solutions is implied by Theorem 5.3.4.

When we pass M to ∞, the only part different from the previous section is the limit of regularized concentrations. As we have strong L^2 convergence of $c^{N,M}$ to c^N, by Young's convolution inequality, we have

$$\begin{aligned}\|\eta_\varepsilon * c^{N,M} - \eta_\varepsilon * c^N\|_{L^\infty(Q_T)} &\le \|\eta_\varepsilon\|_{L^2(Q_T)}\|c^{N,M} - c^N\|_{L^2(Q_T)} \\ &\le C(\varepsilon)\|c^{N,M} - c^N\|^2_{L^2(Q_T)} \to 0 \qquad \text{as } M \to \infty,\end{aligned}$$

which implies the uniform convergence

$$\eta_\varepsilon * c^{N,M} \rightrightarrows \eta_\varepsilon * c^N,$$

and hence,

$$\boldsymbol{S}_\varepsilon^{N,M} \rightrightarrows \boldsymbol{S}_\varepsilon^N := \boldsymbol{S}_\varepsilon(c^N, \boldsymbol{D}\boldsymbol{u}^N).$$

Next, when we let $N \to \infty$, instead of using parabolic De Giorgi–Nash–Moser theory to deduce the uniform convergence of concentrations, we apply Lemma 5.3.3 once more to obtain

$$c^N \to c \text{ strongly in } L^2(Q_T),$$

which again implies

$$\eta_\varepsilon * c^N \rightrightarrows \eta_\varepsilon * c.$$

For the remaining part of the proof, since $\eta_\varepsilon * c \in C^{\alpha,\alpha/2}(Q_T)$ for each $\varepsilon > 0$, we can proceed with the same argument as the one used in the previous section, and hence we can prove the existence of weak solutions $(\boldsymbol{u}, c) := (\boldsymbol{u}_\varepsilon, c_\varepsilon)$ for the regularized problem.

To complete the analysis, an ideal scenario would be when the solution of the regularized problem $(\boldsymbol{u}_\varepsilon, c_\varepsilon)$ converges to the weak solution $(\boldsymbol{u}, c)$ of the original problem as $\varepsilon \to 0$. In that case, we could apply a similar argument to the one used earlier in the chapter to pass ε to 0 without exploiting parabolic De Giorgi–Nash–Moser theory, so we could prove the existence of weak solutions to the original problem provided that $p^- \ge \frac{3d+2}{d+2}$. This, however, remains an open problem.

Chapter 6

Conclusions

6.1 Summary of the thesis

In this thesis, we have investigated a system of nonlinear partial differential equations modelling the motion of the synovial fluid, which is a biological fluid found in the cavities of moving joints. From the rheological viewpoint, the synovial fluid consists of ultrafiltrated blood plasma diluting a particular polysaccharide, called hyaluronan. It was already observed in viscosimetric experiments performed in the early 1950s that the synovial fluid has a strong shear-thinning property, depending on the concentration of hyaluronan in the solution. Explicitly, the viscosity of the fluid is a function of the concentration as well as of the shear-rate. Denoting by c the concentration of hyaluronan in the solution, it was observed in laboratory experiments that the effect of concentration is not a scaling factor of the viscosity (as, for instance, $\nu(c, |\boldsymbol{Du}|) \sim f(c)\tilde{\nu}(|\boldsymbol{Du}|)$, but that the concentration of hyaluronan affects the level of shear-thinning. Therefore, a new power-law model of the synovial fluid was proposed in [54], where the power-law index was considered to be a function of the concentration.

More precisely, we considered the incompressible generalized Navier–Stokes equations with a power-law-like viscosity, where the power-law index depends on the concentration. To close the system, the concentration was assumed to satisfy a convection-diffusion equation:

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } Q_T, \tag{6.1}$$

$$\partial_t \boldsymbol{u} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div} \boldsymbol{S}(c, \boldsymbol{Du}) = -\nabla \pi + \boldsymbol{f} \qquad \text{in } Q_T, \tag{6.2}$$

$$\partial_t c + \operatorname{div}(c\boldsymbol{u}) - \operatorname{div} \boldsymbol{q}_c(c, \nabla c, \boldsymbol{Du}) = 0 \qquad \text{in } Q_T, \tag{6.3}$$

where $\boldsymbol{S}(c, \boldsymbol{D}\boldsymbol{u}) = \nu_0(\kappa_1 + \kappa_2|\boldsymbol{D}\boldsymbol{u}|^2)^{\frac{p(c)-2}{2}}\boldsymbol{D}\boldsymbol{u}$ with some positive constants ν_0, $\kappa_2 > 0$, $\kappa_1 \geq 0$ and $\boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}\boldsymbol{u}) = \boldsymbol{K}(c, |\boldsymbol{D}\boldsymbol{u}|)\nabla c$.

We began by studying the stationary problem. The existence of a weak solution to the elliptic problem was proved in [20, 21] in the cases when the variable exponent $p(x)$ is bounded below by $p^- > \max\left\{\frac{d}{2}, \frac{3d}{d+2}\right\}$, $p^- > \frac{d}{2}$ respectively. The main question of interest in Chapters 3 and 4 was the construction of a sequence of computable approximate solutions, that belong to finite-dimensional spaces spanned by piecewise polynomial functions, and which converges to a weak solution of the problem.

In Chapter 3, the construction of a conforming finite element approximation of the stationary model was considered and the convergence analysis of the numerical method was developed in the case of variable-exponent spaces in a two dimensional domain. As no uniqueness result is currently available for weak solutions of the problem under consideration, one can only show that a subsequence of the sequence of numerical approximations converges to a weak solution of the problem.

The key technical tools in the proof included a discrete counterpart of De Giorgi's regularity theorem (cf. [18]) and discrete Lipschitz truncation (cf. [38]). Furthermore, the stability of the projection operator for the pressure was proved without an error term, in the context of variable-exponent spaces. As a consequence, a discrete Bogovskiĭ operator was constructed in variable-exponent spaces, which played a crucial role in the proof. Because of the absence of a discrete De Giorgi regularity result in three space dimensions, the analysis was restricted to the case of two space dimensions.

As a special case of the above result, one obtains the convergence of finite element approximations of electro-rheological fluids, which was not shown before. See, for instance, [12] where (optimal) error bounds were obtained, but the convective term was excluded.

Next, the analysis was extended to three space dimensions in Chapter 4. To avoid the use of a discrete De Giorgi theorem, different meshes were considered for the momentum equation and the concentration equation. This enabled separating the passages to the limits with respect to the discretization parameters in the two equations, thus avoiding the need for a discrete De Giorgi regularity result in three space dimensions.

To this end, a regularized problem was considered; then we approximated the regularized problem by a finite element method. It was shown that the sequence of finite element approximations converges to a weak solution of the regularized problem. We also showed that the solutions of the regularized problem actually converge to a weak solutions of the original problem. The reason for considering the regularized problem is that direct approximation of the original problem required us to assume an unnaturally strong condition on the variable exponent p in the convergence analysis of the finite element method; the procedure we adopted in Chapter 4 does not suffer from this shortcoming.

The proof is based on the arguments used in our earlier 2D analysis. However, adding a new limiting process associated with the regularization parameter creates new technical difficulties: We had to prove a maximum/minimum principle for the sequence of approximate concentrations in order to apply De Giorgi's theorem and had to introduce another limiting process with a new subscript, resulting from the use of Lipschitz truncation, necessitated by the presence of the convective term.

Having successfully completed the convergence analysis of the finite element method for the steady problem in both two and three space dimensions, we moved on to the unsteady version of the problem. We have established the existence of global weak solutions to the unsteady model under consideration. Our main technical tools included generalized monotone operator theory and parabolic De Giorgi–Nash–Moser regularity theory from which we obtained the Hölder continuity of the approximate concentrations. Thus the restriction $p^- > d$ was essential in our analysis since $p^- > d$ was the minimum requirement to ensure the Hölder continuity of the concentration, which then guaranteed the validity of necessary function space theory in variable-exponent spaces.

However, as was mentioned in Section 5.1, we can also apply our method to the electro-rheological fluid flow problem which is of independent interest. In this case, we can prove the existence of weak solutions with lower values of p^-. More precisely, we have the following result as a corollary of Theorem 5.2.1.

Corollary 6.1.1. *Let $\Omega \subset \mathbb{R}^d$ with $d \geq 2$ be a bounded open Lipschitz domain, and assume that $\boldsymbol{f} \in L^{(p^+)'}(Q_T)^d$ and $\boldsymbol{u}_0 \in L^2(\Omega)^d$. If p is a bounded Hölder continuous function with $p^- > \frac{3d+2}{d+2}$, then there exists a weak solution $\boldsymbol{u} \in C([0,T]; L^2(\Omega)^d) \cap$*

$L^{p^-}(0,T;W^{1,p^-}_{0,\mathrm{div}}(\Omega)^d) \cap W_{p(\cdot)}(Q_T)$ *satisfying*

$$\int_{Q_T} \boldsymbol{S}(\boldsymbol{D}\boldsymbol{u}) : \boldsymbol{D}\boldsymbol{\psi} \,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} (\boldsymbol{f}\cdot\boldsymbol{\psi} + (\boldsymbol{u}\otimes\boldsymbol{u}) : \boldsymbol{D}\boldsymbol{\psi} + \boldsymbol{u}\cdot\partial_t\boldsymbol{\psi}) \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \boldsymbol{u}_0(x)\cdot\boldsymbol{\psi}(x,0)\,\mathrm{d}x$$

for all $\boldsymbol{\psi} \in C^\infty_{0,\mathrm{div}}(\Omega\times[0,T))^d$ *where*

$$\boldsymbol{S}(\boldsymbol{D}\boldsymbol{u}) = \nu_0(\kappa_1 + \kappa_2|\boldsymbol{D}\boldsymbol{u}|^2)^{\frac{p(x,t)-2}{2}}\boldsymbol{D}\boldsymbol{u},$$

for some positive constants ν_0, $\kappa_2 > 0$ *and* $\kappa_1 \geq 0$.

Unlike the model considered in this thesis, such an electro-rheological fluid flow problem is free from a coupled concentration equation, so there is a possibility to improve the above result. For example, if one could develop parabolic L^∞ or Lipschitz truncation techniques for variable-exponent spaces, we could control the convective term with lower values of p^-, and hence we could prove the existence of weak solutions with a weaker assumption on p^-. Therefore, an interesting direction for future research is to generalize the solenoidal parabolic Lipschitz truncation technique introduced in [16] to the case of variable-exponent spaces, so that we can prove the existence of weak solutions for $p^- > \frac{2d}{d+2}$.

6.2 Future research topics

6.2.1 Existence of weak solutions

One of the interesting future research topics is the extension of results regarding existence of weak solutions for power-law-like fluid flow models. For example, it is still an open problem whether there exists a weak solution in the case of $1 < p \leq \frac{2d}{d+2}$ for unsteady power-law fluid flow models. In the stationary case, $p > \frac{2d}{d+2}$ is thought to be optimal since it is the minimum requirement to define the convective term in the weak formulation. However unlike the stationary case, the convective term for the parabolic problem is contained in $L^1(Q_T)$ since we always have $\boldsymbol{u} \in L^\infty(0,T;L^2(\Omega))^d$. It is not clear, though, how to obtain the compactness of the sequence of approximate velocities in $L^1(Q_T)$; understanding this open problem would be a challenging research project.

Furthermore, the analysis of the existence of solutions for more general models would be another interesting research topic. For instance, in [22, 23], the authors considered incompressible fluids with power-law-like rheology given by an implicit constitutive equation relating the stress tensor $\boldsymbol{S}$ and symmetric velocity gradient $\boldsymbol{Du}$ in terms of a maximal monotone graph. Such a framework includes many classical models such as standard Navier–Stokes and power-law fluids, Bingham fluids and even shear-rate dependent fluids with discontinuous viscosities. Additionally, the analysis of fluid flow equations coupled with other equations (such as a temperature equation, or an equation describing the variation of the concentration of chemicals) in a non-trivial way would also be exciting. For example, in [24], the authors established the long-time existence of weak solutions for a system of partial differential equations representing the balance of mass, the balance of linear momentum, the balance of energy, and the equation for the entropy production where the viscosity and thermal conductivity depend on the pressure, temperature, and the shear-rate.

Finally, as mentioned in Section 6.1, it would be also interesting to prove the existence of weak solutions for non-stationary electro-rheological fluids with $\frac{2d}{d+2} < p^-$. This could be achieved if we could extend the result in [16] (the construction of parabolic solenoidal Lipschitz truncation) to the case of variable-exponent spaces. However, the method used in [16] is very technical and consists of a number of localization arguments, so it is not immediately clear how it could be extended to variable-exponent spaces.

6.2.2 Finite element methods

A further interesting topic for future research is to approximate weak solutions of nonlinear partial differential equations considered in the previous chapter by using finite element methods and to perform the mathematical analysis of the resulting numerical methods. In [12], the authors studied the finite element approximation of incompressible electro-rheological fluids and derived certain error bounds for the approximations of the velocity and the pressure. However, they only considered the case of a slow flow and neglected the convective term. Therefore, deriving optimal error bounds for electro-rheological fluids including a convective term is a natural subsequent research question.

Furthermore, the analysis of time-stepping in finite element methods for parabolic problems is also intriguing. In [17], an error bound for the finite element approximations of an elliptic $p(x)$-Laplace system was derived. It would be a natural extension to consider the analysis of a space-time finite element approximation of a parabolic $p(x,t)$-Laplace system, as well as unsteady electro-rheological fluids.

Another interesting topic regarding numerical methods is to design an adaptive finite element methods for the given PDEs and perform a mathematical analysis of the numerical methods, in particular, an a posteriori error analysis in order to provide a good mesh refinement strategy. This problem is theoretically interesting, but it is also very important from a practical point of view, as it helps us to reduce computational costs. For a p-Laplace equation, the convergence of adaptive finite element methods was shown in [37] and optimality of the method was proved in [10]. Extending these results to equations with variable nonlinearity would be an interesting topic.

6.2.3 Regularity theory

A regularity theory for partial differential equations with variable nonlinearity would be also an interesting topic for future study, especially for electro-rheological fluids. As a starting point, in [45], the authors established a nonlinear Calderón–Zygmund type estimate for a $p(x)$-Laplace system and thus obtained higher integrability of the solution. One possible future research direction is to extend this result to the case of stationary electro-rheological fluids and develop a nonlinear Calderón–Zygmund theory for these.

Another interesting regularity issue is how to obtain (partial) Hölder continuity of weak solutions. In [2], the authors obtained partial $C^{1,\alpha}$-regularity for weak solutions to a system modelling electro-rheological fluids in the stationary case. Obtaining analogous regularity results for the unsteady problem would be a further exciting direction for future research.

Bibliography

[1] E. Acerbi and N. Fusco. An approximation lemma for $W^{1,p}$ functions. In *Material instabilities in continuum mechanics (Edinburgh, 1985–1986)*, Oxford Sci. Publ., pages 1–5. Oxford Univ. Press, New York, 1988.

[2] E. Acerbi and G. Mingione. Regularity results for stationary electro-rheological fluids. *Arch. Ration. Mech. Anal.*, 164(3):213–259, 2002.

[3] N. E. Aguilera and L. A. Caffarelli. Regularity results for discrete solutions of second order elliptic problems in the finite element method. *Calcolo*, 23(4):327–353 (1987), 1986.

[4] W. Allegretto. Form estimates for the $p(x)$-Laplacean. *Proc. Amer. Math. Soc.*, 135(7):2177–2185, 2007.

[5] S. Antontsev and S. Shmarev. Anisotropic parabolic equations with variable nonlinearity. *Publ. Mat.*, 53(2):355–399, 2009.

[6] S. Antontsev and S. Shmarev. *Evolution PDEs with nonstandard growth conditions*, volume 4 of *Atlantis Studies in Differential Equations*. Atlantis Press, Paris, 2015. Existence, uniqueness, localization, blow-up.

[7] G. Astarita and G. Marrucci. *Principles of non-Newtonian fluid mechanics*. McGraw-Hill, London-New York, 1974.

[8] J. W. Barrett and W. B. Liu. Quasi-norm error bounds for the finite element approximation of a non-Newtonian flow. *Numer. Math.*, 68(4):437–456, 1994.

[9] L. Belenki, L. C. Berselli, L. Diening, and M. Růžička. On the finite element approximation of p-Stokes systems. *SIAM J. Numer. Anal.*, 50(2):373–397, 2012.

[10] L. Belenki, L. Diening, and C. Kreuzer. Optimality of an adaptive finite element method for the p-Laplacian equation. *IMA J. Numer. Anal.*, 32(2):484–510, 2012.

[11] A. Bensoussan and J. Frehse. *Regularity results for nonlinear elliptic systems and applications*, volume 151 of *Applied Mathematical Sciences*. Springer-Verlag, Berlin, 2002.

[12] L. C. Berselli, D. Breit, and L. Diening. Convergence analysis for a finite element approximation of a steady model for electrorheological fluids. *Numer. Math.*, 132(4):657–689, 2016.

[13] R. Bird, R. Armstrong, and O. Hassager. *Dynamics of polymeric liquids, volume 1: fluid mecanics.* John Wiley, 1987.

[14] M. E. Bogovskiĭ. Solution of the first boundary value problem for an equation of continuity of an incompressible medium. *Dokl. Akad. Nauk SSSR*, 248(5):1037–1040, 1979.

[15] M. E. Bogovskiĭ. Solutions of some problems of vector analysis, associated with the operators div and grad. In *Theory of cubature formulas and the application of functional analysis to problems of mathematical physics*, volume 1980 of *Trudy Sem. S. L. Soboleva, No. 1*, pages 5–40, 149. Akad. Nauk SSSR Sibirsk. Otdel., Inst. Mat., Novosibirsk, 1980.

[16] D. Breit, L. Diening, and S. Schwarzacher. Solenoidal Lipschitz truncation for parabolic PDEs. *Math. Models Methods Appl. Sci.*, 23(14):2671–2700, 2013.

[17] D. Breit, L. Diening, and S. Schwarzacher. Finite element approximation of the $p(.)$-Laplacian. *SIAM J. Numer. Anal.*, 53(1):551–572, 2015.

[18] S. C. Brenner and L. R. Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer, New York, third edition, 2008.

[19] F. Brezzi and M. Fortin. *Mixed and hybrid finite element methods*, volume 15 of *Springer Series in Computational Mathematics*. Springer-Verlag, New York, 1991.

[20] M. Bulíček and P. Pustějovská. On existence analysis of steady flows of generalized Newtonian fluids with concentration dependent power-law index. *J. Math. Anal. Appl.*, 402(1):157–166, 2013.

[21] M. Bulíček and P. Pustějovská. Existence analysis for a model describing flow of an incompressible chemically reacting non-Newtonian fluid. *SIAM J. Math. Anal.*, 46(5):3223–3240, 2014.

[22] M. Bulíček, P. Gwiazda, J. Málek, and A. Świerczewska-Gwiazda. On steady flows of incompressible fluids with implicit power-law-like rheology. *Adv. Calc. Var.*, 2(2):109–136, 2009.

[23] M. Bulíček, P. Gwiazda, J. Málek, and A. Świerczewska-Gwiazda. On unsteady flows of implicitly constituted incompressible fluids. *SIAM J. Math. Anal.*, 44(4):2756–2801, 2012.

[24] M. Bulíček, J. Málek, and K. R. Rajagopal. Mathematical analysis of unsteady flows of fluids with pressure, shear-rate, and temperature dependent material moduli that slip at solid boundaries. *SIAM J. Math. Anal.*, 41(2):665–707, 2009.

[25] E. Carelli, J. Haehnle, and A. Prohl. Convergence analysis for incompressible generalized Newtonian fluid flows with nonstandard anisotropic growth conditions. *SIAM J. Numer. Anal.*, 48(1):164–190, 2010.

[26] J. Casado-Díaz, T. Chacón Rebollo, V. Girault, M. Gómez Mármol, and F. Murat. Finite elements approximation of second order linear elliptic equations in divergence form with right-hand side in L^1. *Numer. Math.*, 105(3):337–374, 2007.

[27] P. Coleman, D. Scott, R. Mason, and J. Levick. Characterization of the effect of high molecular weight hyaluronan on trans-synovial flow in rabbit knees. *Journal of Physiology*, 514, no 1:265–282, 1999.

[28] D. Cruz-Uribe, L. Diening, and A. Fiorenza. A new proof of the boundedness of maximal operators on variable Lebesgue spaces. *Boll. Unione Mat. Ital. (9)*, 2(1):151–173, 2009.

[29] D. Cruz-Uribe, A. Fiorenza, J. M. Martell, and C. Pérez. The boundedness of classical operators on variable L^p spaces. *Ann. Acad. Sci. Fenn. Math.*, 31(1):239–264, 2006.

[30] D. Cruz-Uribe, A. Fiorenza, and C. J. Neugebauer. The maximal function on variable L^p spaces. *Ann. Acad. Sci. Fenn. Math.*, 28(1):223–238, 2003.

[31] E. De Giorgi. Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3)*, 3:25–43, 1957.

[32] E. DiBenedetto. *Degenerate parabolic equations*. Universitext. Springer-Verlag, New York, 1993.

[33] L. Diening. Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$. *Math. Inequal. Appl.*, 7(2):245–253, 2004.

[34] L. Diening. Riesz potential and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$. *Math. Nachr.*, 268:31–43, 2004.

[35] L. Diening, P. Harjulehto, P. Hästö, Y. Mizuta, and T. Shimomura. Maximal functions in variable exponent spaces: limiting cases of the exponent. *Ann. Acad. Sci. Fenn. Math.*, 34(2):503–522, 2009.

[36] L. Diening, P. Harjulehto, P. Hästö, and M. Růžička. *Lebesgue and Sobolev spaces with variable exponents*, volume 2017 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011.

[37] L. Diening and C. Kreuzer. Linear convergence of an adaptive finite element method for the p-Laplacian equation. *SIAM J. Numer. Anal.*, 46(2):614–638, 2008.

[38] L. Diening, C. Kreuzer, and E. Süli. Finite element approximation of steady flows of incompressible fluids with implicit power-law-like rheology. *SIAM J. Numer. Anal.*, 51(2):984–1015, 2013.

[39] L. Diening, J. Málek, and M. Steinhauer. On Lipschitz truncations of Sobolev functions (with variable exponent) and their selected applications. *ESAIM Control Optim. Calc. Var.*, 14(2):211–232, 2008.

[40] L. Diening and M. Růžička. Calderón-Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics. *J. Reine Angew. Math.*, 563:197–220, 2003.

[41] L. Diening and M. Růžička. Interpolation operators in Orlicz-Sobolev spaces. *Numer. Math.*, 107(1):107–129, 2007.

[42] L. Diening, M. Růžička, and K. Schumacher. A decomposition technique for John domains. *Ann. Acad. Sci. Fenn. Math.*, 35(1):87–114, 2010.

[43] L. Diening, M. Růžička, and J. Wolf. Existence of weak solutions for unsteady motions of generalized Newtonian fluids. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 9(1):1–46, 2010.

[44] L. Diening and S. Schwarzacher. On the key estimate for variable exponent spaces. *Azerb. J. Math.*, 3(2):62–69, 2013.

[45] L. Diening and S. Schwarzacher. Global gradient estimates for the $p(\cdot)$-Laplacian. *Nonlinear Anal.*, 106:70–85, 2014.

[46] D. E. Edmunds and J. r. Rákosní k. Density of smooth functions in $W^{k,p(x)}(\Omega)$. *Proc. Roy. Soc. London Ser. A*, 437(1899):229–236, 1992.

[47] D. E. Edmunds and J. Rákosník. Sobolev embeddings with variable exponent. *Studia Math.*, 143(3):267–293, 2000.

[48] D. E. Edmunds and J. Rákosník. Sobolev embeddings with variable exponent. II. *Math. Nachr.*, 246/247:53–67, 2002.

[49] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[50] X. Fan, Q. Zhang, and D. Zhao. Eigenvalues of $p(x)$-Laplacian Dirichlet problem. *J. Math. Anal. Appl.*, 302(2):306–317, 2005.

[51] E. Feireisl. *Dynamics of viscous compressible fluids*, volume 26 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2004.

[52] J. Frehse, J. Málek, and M. Steinhauer. An existence result for fluids with shear dependent viscosity—steady flows. In *Proceedings of the Second World Congress of Nonlinear Analysts, Part 5 (Athens, 1996)*, volume 30, pages 3041–3049, 1997.

[53] J. Frehse, J. Málek, and M. Steinhauer. On analysis of steady flows of fluids with shear-dependent viscosity based on the Lipschitz truncation method. *SIAM J. Math. Anal.*, 34(5):1064–1083 (electronic), 2003.

[54] J. Hron, J. Málek, P. Pustějovská, and K. R. Rajagopal. On the modeling of the synovial fluid. *Advances in Tribology*, 2010.

[55] B. Hubbard, editor. *Numerical solution of partial differential equations. III.* Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1976. SYNSPADE 1975.

[56] S. Ko. Existence of global weak solutions for unsteady motions of incompressible chemically reacting generalized newtoinan fluids. *arXiv:1803.08020 [math.AP]*, 2018.

[57] S. Ko, P. Pustějovská, and E. Süli. Finite element approximation of an incompressible chemically reacting non-Newtonian fluid. *ESAIM M2AN: Mathematical Modelling and Numerical Analysis.* (Accepted for publication), 2017. Available from arXiv:1703.04766 [math.NA]: `https://arxiv.org/abs/1703.04766`.

[58] S. Ko and E. Süli. Finite element approximation of steady flows of generalized newtonian fluids with concentration-dependent power-law index. *arXiv:1708.07830v1 [math.NA]*, 2017.

[59] O. Kováčik and J. Rákosník. On spaces $L^{p(x)}$ and $W^{k,p(x)}$. *Czechoslovak Math. J.*, 41(116)(4):592–618, 1991.

[60] O. A. Ladyzhenskaya. New equations for the description of the motions of viscous incompressible fluids, and global solvability for their boundary value problems. *Trudy Mat. Inst. Steklov.*, 102:85–104, 1967.

[61] O. A. Ladyzhenskaya. *The mathematical theory of viscous incompressible flow.* Second English edition, revised and enlarged. Translated from the Russian by Richard A. Silverman and John Chu. Mathematics and its Applications, Vol. 2. Gordon and Breach, Science Publishers, New York-London-Paris, 1969.

[62] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural′ceva. *Linear and quasilinear equations of parabolic type.* Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968.

[63] W. Lai, S. Kuei, and V. Mow. Rheological equations for synovial fluids. *J. Biomech. Eng*, 100:169–186, 1978.

[64] A. K. Lerner. Some remarks on the Hardy-Littlewood maximal function on variable L^p spaces. *Math. Z.*, 251(3):509–521, 2005.

[65] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod; Gauthier-Villars, Paris, 1969.

[66] J. Málek, J. Nečas, M. Rokyta, and M. Růžička. *Weak and measure-valued solutions to evolutionary PDEs*, volume 13 of *Applied Mathematics and Mathematical Computation.* Chapman & Hall, London, 1996.

[67] J. Málek, J. Nečas, and M. Růžička. On weak solutions to a class of non-Newtonian incompressible fluids in bounded three-dimensional domains: the case $p \geq 2$. *Adv. Differential Equations*, 6(3):257–302, 2001.

[68] E. J. McShane. Extension of range of functions. *Bull. Amer. Math. Soc.*, 40(12):837–842, 1934.

[69] E. Morris, D. Rees, and E. Welsh. Conformation and dynamic interatctions in hyaluronate solutions. *Journal of Molecular Biology*, 138(2):383–400, 1980.

[70] J. Nash. Continuity of solutions of parabolic and elliptic equations. *Amer. J. Math.*, 80:931–954, 1958.

[71] A. s. Nekvinda. Hardy-Littlewood maximal operator on $L^{p(x)}(\mathbb{R})$. *Math. Inequal. Appl.*, 7(2):255–265, 2004.

[72] K. R. Rajagopal, M. Růžicka, and A. R. Srinivasa. On the Oberbeck-Boussinesq approximation. *Math. Models Methods Appl. Sci.*, 6(8):1157–1167, 1996.

[73] N. Rudraiah, S. Kasiviswanathan, and P. Kaloni. Generalized dispersion in a synovial fluid of human joints. *Biorheology*, 28(3-4):207–219, 1991.

[74] M. Růžička. A note on steady flow of fluids with shear dependent viscosity. In *Proceedings of the Second World Congress of Nonlinear Analysts, Part 5 (Athens, 1996)*, volume 30, pages 3029–3039, 1997.

[75] M. Růžička. *Electrorheological fluids: modeling and mathematical theory*, volume 1748 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000.

[76] M. Růžička. Modeling, mathematical and numerical analysis of electrorheological fluids. *Appl. Math.*, 49(6):565–609, 2004.

[77] S. Samko. Denseness of $C_0^\infty(\mathbf{R}^N)$ in the generalized Sobolev spaces $W^{M,P(X)}(\mathbf{R}^N)$. In *Direct and inverse problems of mathematical physics (Newark, DE, 1997)*, volume 5 of *Int. Soc. Anal. Appl. Comput.*, pages 333–342. Kluwer Acad. Publ., Dordrecht, 2000.

[78] S. G. Samko. Density of $C_0^\infty(\mathbf{R}^n)$ in the generalized Sobolev spaces $W^{m,p(x)}(\mathbf{R}^n)$. *Dokl. Akad. Nauk*, 369(4):451–454, 1999.

[79] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 146:65–96, 1987.

[80] E. Süli and T. Tscherpel. Fully discrete finite element approximation of unsteady flows of implicitly constituted incompressible fluids. *arXiv:1804.02264 [math.NA]*, 2018.

[81] J. Wolf. Existence of weak solutions to the equations of non-stationary motion of non-Newtonian fluids with shear rate dependent viscosity. *J. Math. Fluid Mech.*, 9(1):104–138, 2007.

[82] E. Zeidler. *Nonlinear functional analysis and its applications. II/B.* Springer-Verlag, New York, 1990. Nonlinear monotone operators, Translated from the German by the author and Leo F. Boron.